Talk 4 - Kleine AG

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Notation:

We always let $\kappa = \mathbb{F}_q$ be a finite field with $k = \overline{\kappa}$ fixed. We fix a prime $\ell \neq p$ and an isomorphism $\iota : \overline{\mathbb{Q}_\ell} \to \mathbb{C}$. If something is defined over κ, we will put a subscript $_0$ (e.g. X_0 is a κ-scheme) and when the subscript is dropped, this denotes the base-change to k. $\mathcal{F}_0$ will be a Weil-sheaf over X_0. For a closed point $a \in |X_0|$, we will use $\mathcal{F}_{\overline{a}}$ to denote the stalk of $\mathcal{F}$ at a geometric point $\overline{a}$ over a.

Goal

Reducing the proof of Deligne's purity theorem

Theorem 1.1

Let $f_0 : X_0 \to Y_0$ be a morphism of alg. varieties $/ k$, $\mathcal{F}_0$ be a τ-mixed sheaf on X_0 with largest weight β. Then for all i, $R^i(f_0)_! \mathcal{F}_0$ is τ-mixed with weights at most $\beta + i$.

Via Poincaré duality, we get:

Corollary 1.2

Let $f_0 : X_0 \to Y_0$ be a smooth and proper morphism and $\mathcal{F}_0$ a lisse τ-pure sheaf of weight β, then all the lisse sheaves $R^i(f_0)_! \mathcal{F}_0$ are τ-pure of weight $\beta + i$.

This is a generalization of Weil's Riemann hypothesis:

Let $X_0 / k \overset{=}{} \mathbb{F}_q$ be a variety.

Then we can assign to it the zeta function

$$Z(X_0, t) = \exp\left(\sum_{n>0} \frac{\nu_n(X_0) t^n}{n} \right)$$

$$= \prod_{x \in |X_0|} \frac{1}{1 - t^{d(x)}}$$

where $\nu_n(X_0)$ is the number of $\mathbb{F}_{q^n}$-points.

We can generalize this for an arbitrary $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}_0$ on X_0:

$$L(X_0, \mathcal{F}_0, t) = \prod_{x \in |X_0|} \det(1 - t^{d_x} F_x, \mathcal{F}_{0\bar{x}})^{-1}$$

It is easy to check that

$$L(X_0, \mathbb{Q}_\ell, t) = Z(X_0, t).$$

Now the Grothendieck trace formula says that

$$L(X_0, \mathcal{F}_0, t) = \prod_{i=0}^{2\cdot \dim X} \det(1 - tF, H_c^i(X, \mathcal{F}))^{(-1)^{i+1}}$$

For a smooth, proper variety $X_0 \xrightarrow{f_0} \operatorname{Spec}(\mathbb{F}_q)$ Corollary 1.2 now says that $R^i(f_0)_! \mathbb{Q}_\ell$ is pure of weight i.

This shows that for

$$Z(X_0, t) = \prod_{i=0}^{2\cdot \dim X} P_i(X, t)^{(-1)^{i+1}}$$

with
$$P_i(X, t) = \det(1 - tF, H_c^i(X, \mathbb{Q}_\ell)),$$
$$= \prod_r (1 - a_{i,r} t),$$

the $a_{i,r}$ have absolute value $q^{\frac{i}{2}}$.

Reduction I

Our first goal is to reduce Th. 1.1. to

Theorem 2.1

Let $\mathcal{F}_0$ be a ι-pure lisse sheaf of weight β on a smooth, geom. irred. curve X_0 / κ.
Then, for all i, the weights of $H^i_c(X, \mathcal{F})$ are at most $\beta + i$.

Proof of 1.1., using 2.1

We note down the following reductions:

(1) If f_0 is quasi-finite, the assertion is trivial, since $R^i(f_0)_! \mathcal{F}_0 = 0$ for $i > 0$ and the case $i = 0$ is trivial.

(2) We can replace X_0 by any non-empty open $U_0 \subseteq X_0$:

If $Z_0 = X_0 \setminus U_0$, the excision sequence shows that it suffices to prove the theorem for $U_0 \to Y_0$, $Z_0 \to Y_0$ but it holds for Z_0 via Noeth. induction on X_0.

(3) Via the Leray spectral sequence, we deduce that if the assertion holds for g_0, h_0, it also holds for $g_0 \circ h_0$.

(4) We can replace f_0 by $f_0|_{f_0^{-1}(U_0)} : f_0^{-1}(U_0) \to U_0$ for a non-empty open $U_0 \subseteq Y_0$:

1. If $f(X_0) \subseteq Y_0$ is not dense, we can replace Y_0 by a closed subscheme and use Noeth. ind. on Y_0.
2. If $f_0(X_0) \subseteq Y_0$ is dense, $f_0^{-1}(U_0)$ is non-empty and by (2) it suffices to prove the assertion for

$$\begin{tikzcd} f_0^{-1}(U_0) \arrow[r] \arrow[d] & Y_0 \\ U_0 \arrow[ur, "\text{quasi-finite}"'] & \end{tikzcd}$$

①+③ $\Rightarrow$ suffices to show the assertion for $f_0^{-1}(U_0) \to U_0$

Claim: These reductions allow us to assume that

$f_0 : X_0 \to Y_0$ is surjective, affine, smooth and its fibers are geom. irred. curves. Moreover, we can assume that $\mathcal{F}_0$ is lisse and ι-pure with weight β.

By ② + ④, we can assume that X_0, Y_0 are affine, which implies that f_0 is affine.

Taking a generic point $\eta \to X_0$, we take the generic fiber $(X_0)_\eta \to \mathrm{Spec}(\kappa(\eta))$ which is affine.

By Noether Normalization we can find a factorization

$$\begin{tikzcd} (X_0)_\eta \arrow[r, "\text{finite}"] \arrow[dr] & \mathbb{A}^n_{k(\eta)} \arrow[d] \\ & \mathrm{Spec}(\kappa(\eta)) \end{tikzcd}.$$

By spreading out, we get

$$\begin{tikzcd} f_0^{-1}(U_0) \arrow[r, hook, "\text{finite}"] \arrow[dr] & \mathbb{A}^n_{U_0} \arrow[d] \\ & U_0 \end{tikzcd}$$

for an open affine $U_0 \subseteq Y_0$.

By ② + ④ we can replace Y_0 by U_0 and X_0 by $f_0^{-1}(U_0)$, so we have

$$\begin{tikzcd} X_0 \arrow[r, hook, "\text{finite}"] \arrow[dr] & \mathbb{A}^n_{Y_0} \arrow[d] \\ & Y_0 \end{tikzcd}.$$

By ① + ③ we can now assume that

$f_0 : X_0 = \mathbb{A}^1_{Y_0} \to Y_0$ for Y_0 affine

There exist a stratification of X_0 s.t. $\mathcal{F}_0$ is lisse, when restricted to it.

In particular, there is a nonempty open affine $U_0 \to X_0$

s.t. $\mathcal{F}_U|_{U_U}$ is lisse. By replacing X_U with U_U, we can assume that $\mathcal{F}_U$ is lisse.

Since X_U is open in $\mathbb{A}^1_{Y_U}$, f_U is smooth, affine and by 4 we can replace Y_U by $f(X_U)$, so it is also surjective. Since X_U is open in $\mathbb{A}^1_{Y_U}$, the fibers of f_U are non-empty open in $\mathbb{A}^1$ and hence are geometrically irred. curves. Via the long exact sequence associated to $R^\bullet(f_U)_!$, we can pass to the irred. subquotients of $\mathcal{F}_U$. Hence we can assume that $\mathcal{F}_U$ is $\bar{\iota}$-pure.

If we have a geom. pt. $\bar{y} \to Y$ with underlying pt. $y \in Y$, we denote $C_U = f_U^{-1}(y)$. By the above C_U is a affine, smooth and geom. irred. curve.

By the estimates of Th. 2.1, the weights of

$$(R^i(f_U)_! \mathcal{F}_U)_{\bar{y}} \simeq H^i_c(C, \mathcal{F}|_C)$$ are at most $\beta + i$.

We can assume that $\mathcal{F}_U$ is τ-real and since τ-real sheaves are mixed, it suffices to show that $R^i(f_U)_! \mathcal{F}_U$ is τ-real.

Since C is affine, $H^0_c(C, \mathcal{F}|_C) = 0$ and via Grothendieck-Lefschetz we get that

$$\iota L(C_U, \mathcal{F}_U, t) = \frac{\iota \det(1 - tF \mid H^1_c(C, \mathcal{F}|_C))}{\iota \det(1 - tF \mid H^2_c(C, \mathcal{F}|_C))} \qquad (*)$$

has real coefficients since $\mathcal{F}_U$ is ι-real.

By Poincaré duality and $\mathcal{F}_U$ being ι-pure of weight β, we get that every root α of satisfies

$$|\alpha| = N(y)^{-(\beta+2)/2}$$

and by the above weight estimate, we see that every root α' of satisfies

$$|\alpha'| \geq N(y)^{-(\beta+1)/2}$$

$\Rightarrow$ Right-hand side in $(*)$ is in "lowest terms".

This shows that and have real coefficients since both have leading coeff. 1.

$\square$ of 1.1, using 2.1

Reduction II

We will now reduce Th. 2.1 to

Theorem 3.1
Let $\overline{X_0}$ be a smooth, geom. irred. proj. curve / κ and let $j_0 : X_0 \to \overline{X_0}$ be a non-empty open subscheme. Let $\mathcal{F}_0$ be a τ-pure lisse sheaf of weight β on X_0. Then, for all i, $H^i(\overline{X}, j_* \mathcal{F})$ is τ-pure of weight $\beta + i$.

To perform the reduction, let X_0 be the affine curve, satisfying the conditions of Th. 2.1 and $\overline{X_0}$ be its regular completion (which is smooth since κ is perfect) with inclusion

$j_0 : X_0 \to \overline{X_0}$. The reduction follows now from the following lemma:

Lemma:

Let $\overline{X_0}$ be a smooth, geom. irreducible projective curve/X with a non-empty open $j_0 : X_0 \to \overline{X_0}$. Let $\mathcal{F}_0$ be a ι-pure lisse sheaf of weight β on X_0.

Then, for all i TFAE:

a) the weights of $H^i(\overline{X}, j_* \mathcal{F})$ are $\leq \beta + i$.

b) the weights of $H^i_c(X, \mathcal{F})$ are $\leq \beta + i$.

Proof:

We take $\mathcal{H}_0 = (j_0)_* \mathcal{F}_0 / (j_0)_! \mathcal{F}_0$, which is concentrated on the finite complement

$$S_0 = \overline{X_0} \setminus X_0$$

We get an exact sequence

$$0 \to (j_0)_! \mathcal{F}_0 \to (j_0)_* \mathcal{F}_0 \to \mathcal{H}_0 \to 0$$

$$\Big\downarrow H^\bullet(\overline{X}, -)$$

$$H^{i-1}(\bar{X}, \mathcal{H}) \to H^i_c(X, \mathcal{F}) \to H^i(\bar{X}, j_* \mathcal{F}) \to H^i(\bar{X}, \mathcal{H})$$

Since $\mathcal{H}_0$ is concentrated on S_0, we have

$$H^i(\bar{X}, \mathcal{H}) = 0 \quad , \quad i > 0$$

and

$$H^0(\bar{X}, \mathcal{H}) \simeq \bigoplus_{s \in S_0} ((j_0)_* \mathcal{F}_0)_{\bar{s}}$$

Since $(j_0)_* \mathcal{F}_0$ is lisse, we can apply semi-continuity of weights to show that the weights of $\mathcal{H}_0$ are at most β.

Then the equivalence follows from the upper exact sequence.

$\square$

Reduction III

We now want to reduce Th. 3.1 to

Theorem 4.1

In the situation of Th. 3.1., let α be an Eigenvalue of the geometric Frobenius

$$F : H^1(\overline{X}, j_* \mathcal{F}) \to H^1(\overline{X}, j_* \mathcal{F}) .$$

Then $$|\iota(\alpha)| \leq q^{\beta+1} .$$

Before proceeding, we can assume that $\mathcal{F}_0$ is actually a $\overline{\mathbb{Q}_\ell}$-sheaf by twisting with a quasi-character χ_b of $W(\bar{k}/k)$ to turn $\mathcal{F}_0$ into a $\overline{\mathbb{Q}_\ell}$-sheaf. Since χ_b is geom. trivial, it doesn't change anything at the computation.

For a lisse $\mathbb{Q}_\ell$-sheaf $\mathcal{F}_0$, we denote

$$j_* \mathcal{F}_0^\vee := j_* (\mathcal{H}om(\mathcal{F}_0, \overline{\mathbb{Q}_\ell})) .$$

We cite the following (far from easy!) result:

Proposition 4.2:

Let X_0 be a smooth projective curve $/k$, $j_0 : X_0 \hookrightarrow \overline{X_0}$ an open dense subset, and $\mathcal{F}_0$ a lisse $\overline{\mathbb{Q}_\ell}$-sheaf on X_0. Then the cap product induces a perfect pairing

$$H^i(\overline{X}, j_* \mathcal{F}^\vee(1)) \otimes H^{2-i}(\overline{X}, j_* \mathcal{F}) \to H^2(\overline{X}, \overline{\mathbb{Q}_\ell}(1)) \cong \overline{\mathbb{Q}_\ell}$$

Moreover, we can identify $j_*(\mathcal{F}^\vee) = (j_* \mathcal{F})^\vee$

For details of the proof look at the references!

Proof of Th. 3.1 using Th. 4.1:

We can assume that $\mathcal{F}_0$ is a lisse $\overline{\mathbb{Q}_\ell}$-sheaf on U_0.

For $i > 2$, H^i vanishes so the claim is true.

<u>$i = 0$:</u>

$$H^0(\overline{X}, j_* \mathcal{F}) = \Gamma(\overline{X}, j_* \mathcal{F}) = \Gamma(X, \mathcal{F})$$

but every Eigenvalue of the latter must be an Eigenvalue at the stalks

$\Rightarrow H^0(\overline{X}, j_* \mathcal{F})$ inherits ι-purity with weight β from $\mathcal{F}_0$.

$\underline{i=2}$:

Using Prop. 4.2, we have

$$H^2(\bar{X}, j_* \mathcal{F}) = (H^0(\bar{X}, j_* \mathcal{F}(1)))^\vee$$

and since $\mathcal{F}_0$ is pure of weight β, $\mathcal{F}_0^\vee$ is ι-pure of weight $-\beta$. By applying $\underline{i=0}$ case, we get that $H^2(\bar{X}, j_* \mathcal{F})$ is ι-pure of weight $-(-\beta-2) = \beta+2$

$\underline{i=1}$:

Prop. 4.2 implies

$$H^1(\bar{X}, j_* \mathcal{F}) = (H^1(\bar{X}, j_* \mathcal{F}^\vee(1)))^\vee .$$

Applying Th. 4.1. to $\mathcal{F}_0^\vee(1)$ shows that the Eigenvalues α of geom. Frob. on satisfy

$$|\iota(\alpha)|^{-2} \leq q^{-\beta-2+1} \iff |\iota(\alpha)|^2 \geq q^{\beta+1} .$$

Applying Th. 4.1 to $\mathcal{F}_0$, we get the opposite equality which shows ι-purity of $H^1(X, j_* \mathcal{F})$ of weight $\beta+1$. $\square$

References :

·) Weil Conjectures, Perverse Sheaves and l'adic Fourier transform by Reinhardt Kiehl and Rainer Weissauer

·) Notes from Brian Conrad's Seminar on Weil II (can be found online)

·) Étale Cohomology by James Milne