Weights on Etale Sheaves

§1 Recollection on $\overline{\mathbb{Q}}_\ell$-sheaves

- $\ell \neq p$ two primes
- $\overline{\mathbb{Q}}_\ell$ fixed algebraic closure of $\mathbb{Q}_\ell$
  (note: $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$ as "abstract" fields)

X scheme, for simplicity assume X Noetherian. (later: $X/\mathbb{F}_q$ of finite type)

Def: • An étale sheaf of $\overline{\mathbb{Q}}_\ell$-vector spaces $\mathcal{G}$ on X is <u>lisse</u> if étale-locally it is constant finite-dim'l $\overline{\mathbb{Q}}_\ell$-vector space.

• $\mathcal{G}$ is <u>constructible</u> if $\exists\, X = \coprod X_i$ finite partition into loc. closed subschemes s.t. $\mathcal{G}|_{X_i}$ is lisse.

<u>Etale cohomology</u>: Assume ℓ invertible on X.

Want: Reasonable bounded derived category of constr. $\overline{\mathbb{Q}}_\ell$-sheaves.

But: A priori only works well with torsion coefficients.

Classical approach: $D^b_c(X, \overline{\mathbb{Q}}_\ell) = \left(\varprojlim D^b_c(X, \mathbb{Z}/\ell^n\mathbb{Z})\right) \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}}_\ell$.

Bhatt-Scholze: use pro-étale topology instead.

Then $D^b_c(X, \overline{\mathbb{Q}}_\ell)$ supports six functor formalism: $(- \otimes -, R\mathrm{Hom})$; (f^*, Rf_*); $(Rf_!, Rf^!)$

satisfying e.g. ◦ proper / smooth base change

• Poincaré duality

<u>Frobenius action</u>

Notation: $\kappa = \mathbb{F}_q$, $k = \overline{\mathbb{F}}_q$ an alg. closure

$\sigma_k : k \to k$, $a \mapsto a^q$ the (arithmetic) Frobenius

$F = \sigma_k^{-1}$ the geometric Frobenius.

X_0/κ finite type scheme, then $X = X_0 \otimes_\kappa k$.

$\sigma_{X/k} : X \to X$ absolute Frobenius (identity on top. spaces, $a \mapsto a^q$ on structure sheaf)

Also have $\mathrm{Fr}_X : X \to X$ defined by

$$\begin{tikzcd} X \arrow[dr, dashed, "\mathrm{Fr}_X"] \arrow[drr, bend left, "\sigma_{X/k}"] \arrow[ddr, bend right] & & \\ & X \arrow[r] \arrow[d] & X \arrow[d] \\ & k \arrow[r, "\sigma_k"'] & k \end{tikzcd}$$

"Frobenius endomorphism" (relative Frobenius)

→ map of k-schemes, finite & universal homeo

and $F_X := \mathrm{id}_{X_0} \times F$ the base change of Galois action of geom. Frobenius "Frobenius automorphism"

Note: 1) for $\mathcal{G}_0$ an étale $\bar{\mathbb{Q}}_\ell$-sheaf on X_0 have canonical iso

$$\mathrm{Fr}_{\mathcal{G}}^* : \quad \mathrm{Fr}_X^* \mathcal{G} \xrightarrow{\sim} \mathcal{G}.$$

[Constructed as follows: note: for U/X étale the relative Frob $\mathrm{Fr}_{U/X}$ is an iso. $\Rightarrow$ $\mathcal{G} \cong \mathrm{Fr}_{X,*} \mathcal{G}$,

and hence $\mathrm{Fr}_X^* \mathcal{G} \cong \mathrm{Fr}_X^* \mathrm{Fr}_{X,*} \mathcal{G} \xrightarrow{\cong} \mathcal{G}$ is an iso. (as Fr_X is universal homeo)]

As Fr_X is moreover finite, hence proper, get induced map

$$\begin{tikzcd} H_c^i(X, \mathcal{G}) \arrow[r] \arrow[rr, bend right, "F"'] & H_c^i(X, \mathrm{Fr}_X^* \mathcal{G}) \arrow[r, "\mathrm{Fr}_{\mathcal{G}}^*"'] & H_c^i(X, \mathcal{G}). \end{tikzcd}$$

2) $x \in |X_0|$ closed point. Then $\mathrm{Gal}(k/\kappa(x))$ acts on on $\mathcal{G}_{0,x}$, in particular $F_x \in \mathrm{Gal}(k/\kappa(x))$ acts.

Def: A Weil sheaf $\mathcal{G}_0$ on X_0 is a constructible $\bar{\mathbb{Q}}_\ell$-sheaf $\mathcal{G}$ on X together with $F^* : F_X^* \mathcal{G} \cong \mathcal{G}$.

together with $F^*: F_x^* \mathcal{G} \cong \mathcal{G}$.

Point 1) + 2) above also carry over to Weil sheaves.

$\to$ Can define L-functions for Weil sheaves as for étale $\overline{\mathbb{Q}}_\ell$-sheaves.

From now on: sheaves = Weil sheaves.

We can now state:

<u>Thm</u> (Grothendieck trace formula): As always, X_0/κ of finite type, $\mathcal{G}_0/X_0$ Weil sheaf.

$$L(X_0, \mathcal{G}_0, t) = \prod_{i=0}^{2\dim X} \det\left(1 - tF, H_c^i(X, \mathcal{G})\right)^{(-1)^{(i+1)}}.$$

Note:

$\Rightarrow$ Rationality of L-functions. (!)

§ 2 Weights & Main Theorem

As before, X_0/κ f.t. scheme, $\mathcal{G}_0$ Weil sheaf on X_0.

Fix $\tau: \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$.

For closed point $x \in |X_0|$ denote by $d(x) = [\kappa(x) : \kappa]$ and $N(x) = \# \kappa(x)$.
residue field at x

Def: 1) $\mathcal{G}_0$ is <u>τ-pure</u> of weight $\beta \in \mathbb{R}$ if f.a. $x \in |X_0|$ and all eigenvalues $\alpha \in \overline{\mathbb{Q}}_\ell$ of $F_x: \mathcal{G}_{0\bar{x}} \xrightarrow{\sim} \mathcal{G}_{0\bar{x}}$

$$|\tau(\alpha)|^2 = N(x)^\beta$$

2) $\mathcal{G}_0$ is <u>τ-mixed</u> if there exists finite filtration

$$0 = \mathcal{G}_0^{(0)} \subset \mathcal{G}_0^{(1)} \subset \dots \subset \mathcal{G}_0^{(r)} = \mathcal{G}_0$$

s.t. successive quotients $\mathcal{G}_0^{(i+1)}/\mathcal{G}_0^{(i)}$ are τ-pure.

3) $\mathcal{G}_0$ is <u>pure</u> of weight β if it is τ-pure of weight β for all τ.

4) $\mathcal{G}_0$ is <u>mixed</u> if $\exists$ filtration as in 2) s.t. quotients are pure.

Permanence properties: Let $\mathcal{G}_0$ be τ-pure of weight β.

The following are τ-pure of weight β:

1) $f_0^* \mathcal{G}_0$ for $f_0 : Y_0 \to X_0$ ("$\Leftarrow$" also true for f surjective)

2) $f_{0*} \mathcal{G}_0$ for $f_0 : X_0 \to Y_0$ finite

Moreover,

3) $\mathcal{G}_0^\vee$ is τ-pure of weight $-\beta$

4) $\mathcal{G}_0 \otimes \mathcal{G}_0'$ — " — $\beta + \beta'$ for $\mathcal{G}_0'$ τ-pure of weight β'.

Similar statements also for "mixed".

Example (Tate twist): $\overline{\mathbb{Q}}_\ell(1)$ is pure of weight -1.

Def The <u>maximal weight</u> of $\mathcal{G}_0$ is

$$\omega(\mathcal{G}_0) = \sup_{x \in |X_0|} \sup_{\substack{\alpha \text{ eigenvalue} \\ \text{of } F_x}} \frac{\log(|\tau(\alpha)|^2)}{\log(N(x))}.$$

<u>Main Theorem</u> $f : X_0 \to S_0$ separated & f.t., $\mathcal{G}_0$ sheaf on X_0, τ-mixed of weights $\leq \beta$.

Then $R^n f_! \mathcal{G}_0$ is τ-mixed of weights $\leq \beta + n$.

Corollary (Riemann hypothesis): X_0 / κ smooth projective. Then $H^n(X, \overline{\mathbb{Q}}_\ell)$ is pure of weight n.

§ 3 Semicontinuity theorem

$\hookrightarrow$ [illegible]

§ 3 Semicontinuity theorem

↳ First step in the proof of main theorem

<u>Thm</u>: $\mathcal{G}_0$ lisse sheaf on X_0, $j_0 : U_0 \hookrightarrow X_0$ dense open.

Then 1) $w(\mathcal{G}_0) = w(j_0^* \mathcal{G}_0)$

2) $j^* \mathcal{G}_0$ is τ-pure of weight β $\Rightarrow$ $\mathcal{G}_0$ is τ-pure of weight β.

3) X_0 normal + irreducible, $\mathcal{G}_0$ irreducible, $j^* \mathcal{G}_0$ τ-mixed

$\Rightarrow$ $\mathcal{G}_0$ τ-pure

4) X_0 connected, $j_0^* \mathcal{G}_0$ τ-mixed, $\mathcal{G}_0$ τ-pure of weight β at point x.

$\Rightarrow$ $\mathcal{G}_0$ τ-pure of weight β.

For proof: reduce to case of curves.

For curve case study L-functions:

<u>Lemma</u>: $\mathcal{G}_0$ Weil sheaf on X_0 st $w(\mathcal{G}_0) \le \beta$.

Then $\tau(L(X_0, \mathcal{G}_0, t)) \in \mathbb{C}[[t]]$ converges for all

$|t| \le q^{-\beta/2 - \dim(X_0)}$ and has no zeros / poles in this region.

<u>Proof</u>: (Suppressing τ from notation)

Take logarithmic derivative:

$$\frac{d}{dt} \log L(X_0, \mathcal{G}_0, t) = \sum_{n \ge 1} \sum_{\substack{x \in |X_0| \\ d(x) | n}} d(x) \, \mathrm{Tr}(F_x^{\, n/d(x)}) \, t^{n-1}$$

$$\le \sum_{n \ge 1} \Big(\sum_{x \ldots} d(x) \cdot \underbrace{r}_{\text{max'l dim of stalk of } \mathcal{G}_0} \cdot q^{n\beta/2} \Big) \, t^{n-1}$$

$$= \sum_{n \ge 1} \underbrace{|X_0(\mathbb{F}_{q^n})|}_{\mathcal{O}(q^{n \dim X_0})} \cdot r \cdot q^{n\beta/2} \, t^{n-1}$$

$\leadsto$ geometric series, converging for $|t| \le q^{-\beta/2 - \dim X}$ ☒

Important case of thm is following:

<u>Lemma</u> X_0/k smooth irreducible curve. $j_0 : U_0 \hookrightarrow X_0$ non-empty open.

$S_0 = X_0 \setminus U_0$. $\mathcal{G}_0$ sheaf on X_0 s.t. $j_0^* \mathcal{G}_0$ smooth, $H^0_S(X, \mathcal{G}) = 0$.

Then $w(j_0^* \mathcal{G}_0) \leq \beta \implies w(\mathcal{G}_0) \leq \beta$.