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1 Plan

Hello, welcome to the first talk of the Weil conjectures. In this talk we will do the following:

- Give the statement of the 5 Weil conjectures: rationality, integrality, functional equation, Riemann hypothesis and Betti numbers.
- We then sketch the proofs of 3 of them: rationality, Poincaré duality and Betti numbers.
  - For rationality and Betti numbers we introduce the cohomological interpretation of L-functions, which follows from the Lefschetz trace formula (given as a black box).
  - For the functional equation we also recall (as black box) Poincaré duality.

Then in the next 4 talks, the remaining Riemann hypothesis Weil conjecture will be proven, which will also imply the integrality conjecture.

2 History

(Taken from [1]) We will start by giving some history of the Weil conjectures:

- In 1949, Weil formulated the Weil conjectures: formulated the rationality, integrality, functional equation, Betti numbers and Riemann hypothesis Weil conjectures.
- In 1960, Dwork proved the Rationality conjecture.
- In 1965, Artin, Grothendieck and Verdier proved the rationality, Betti numbers and functional equation part of the Weil conjectures by defining ℓ-adic cohomology and proving the Lefschetz trace formula. This we will sketch in the first talk.
- In 1974 Deligne proved the Riemann hypothesis and the integrality conjectures in his Weil I article.
- In 1980 Deligne improved the results in his Weil II article.
- In 1987 Laumon introduced the Fourier transform which simplifies a step in Deligne's proof. This last proof is the one we will follow in the next 4 talks.

[1] https://en.wikipedia.org/wiki/Weil_conjectures#Background_and_history

3 The 5 conjectures

Let q be a power of a prime p and $X/\mathbb{F}_q$ a smooth projective variety (variety = finite type geometrically integral scheme). Then the zeta function of X is defined as

$$\begin{aligned} \zeta_X(t) &:= \exp\left(\sum_{n\in\mathbb{Z}_{\geq 1}} \frac{\nu_n(X)t^n}{n}\right) \in \mathbb{Z}[[t]] \\ \nu_n(X) &:= \#X(\mathbb{F}_{q^n}). \end{aligned} \tag{1}$$

3.1 The conjectures

The 5 Weil conjectures are as follows: let $d := \dim X$,

1. Rationality: $\zeta_X(t) \in \mathbb{Q}(T)$. More precisely, there is a characteristic 0 field K and $P_i(t) \in K[t]$ for $i = 0, \dots, 2d$ such that
$$\begin{aligned} \zeta_X(t) &= \frac{P_1(t)\cdot\dots\cdot P_{2d-1}(t)}{P_0(t)\cdot\dots\cdot P_{2d}(t)}, \\ P_0(t) &= 1-t, \\ P_{2d}(t) &= 1-q^d t. \end{aligned} \tag{2}$$
2. Integrality:
$$P_i(t) \in \mathbb{Z}[t] \text{ for } i = 0, \dots, 2d. \tag{3}$$
3. Functional equation:
$$\begin{aligned} \zeta_X(q^{-d}t^{-1}) &= \pm q^{\frac{d\cdot\chi}{2}} t^{\chi}\zeta_X(t) \\ \chi &:= \sum_{i=0}^{2d}(-1)^i \deg P_i. \end{aligned} \tag{4}$$
4. Riemann hypothesis: for $i = 0, \dots, 2d$,
$$\begin{aligned} &\text{write } P_i(t) = \prod_{j=1}^{\deg P_i}(1-\alpha_{i,j}t) \quad \text{for some } \alpha_{i,j} \in \overline{K} \hookrightarrow \mathbb{C}, \\ &\text{then } |\alpha_{i,j}| = q^{\frac{i}{2}} \quad \text{for } j = 1, \dots, \deg P_i. \end{aligned} \tag{5}$$
5. Betti numbers: If there is a $Y/\mathbb{Z}_{(p)}$ a smooth projective good reduction mod p variety of X, so $X = Y \times_{\mathbb{Z}_{(p)}} \mathbb{F}_q$, then
$$\deg P_i = \dim_{\mathbb{C}} H^i_{sing}(Y^{cx}_{\mathbb{C}}, \mathbb{C}). \tag{6}$$
where $Y^{cx}_{\mathbb{C}}$ is the complex manifold associated to a variety.

3.2 Example: Calculation of $\zeta_{\mathbb{P}^n_{\mathbb{F}_q}}(t)$

We can write

$$\mathbb{P}^n_{\mathbb{F}_q} = \coprod_{i=0}^{n} \mathbb{A}^i_{\mathbb{F}_q} = \coprod_{i=0}^{n}\prod_{j=1}^{i} \mathbb{A}^1_{\mathbb{F}_q} \tag{7}$$

and

$$\#\mathbb{A}^1_{\mathbb{F}_q}(\mathbb{F}_{q^m}) = \#\operatorname{Mor}_{\mathbb{F}_q}(\mathbb{F}_q[x], \mathbb{F}_{q^m}) = q^m \tag{8}$$

so we get

$$\#\mathbb{P}^n_{\mathbb{F}_q}(\mathbb{F}_{q^m}) = \sum_{i=0}^{n} q^{im} \tag{9}$$

Then

$$\begin{aligned} \zeta_{\mathbb{P}^n_{\mathbb{F}_q}}(t) &= \exp\left(\sum_{m\in\mathbb{Z}_{\geq 1}} \sum_{i=0}^{n} q^{im} \frac{t^m}{m}\right) \\ &= \exp\left(\sum_{i=0}^{n} \sum_{m\in\mathbb{Z}_{\geq 1}} \frac{(q^i t)^m}{m}\right) \\ &= \exp\left(\sum_{i=0}^{n} \log(\frac{1}{1-q^i t})\right) \\ &= \prod_{i=0}^{n} \frac{1}{1-q^i t} \end{aligned} \tag{10}$$

so $P_{2i}(t) = 1 - q^i t$, $P_{2i-1}(t) = 1$.

4 The cohomological interpretation of L-functions

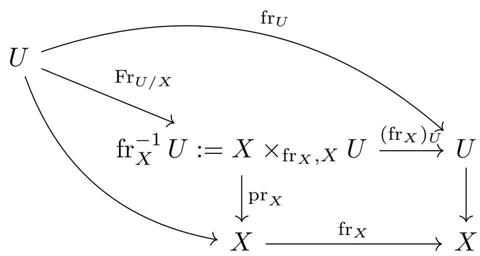
4.1 Frobenius endomorphisms

(Reference: [7, XV §2]) Let (X, E) be a pair where $X/\mathbb{F}_q$ is a scheme and $E \in \mathrm{Sh}(X_{\text{ét}})$.

If we denote $\overline{X} := X \times_{\mathbb{F}_q} \overline{\mathbb{F}}_p$ and $\overline{E} := E \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_p$, then we want to define Frobenius endomorphisms in $\mathrm{End}(H^i_c(\overline{X}, \overline{E}))$ and $\mathrm{End}(i^*_{\overline{x}}\overline{E})$ for $\overline{x} \in \overline{X}^0$, to be able to state the definition of L-functions and its cohomological interpretation.

We start by giving an isomorphism $\mathrm{Fr}_{E/X} : \mathrm{fr}_X^* E \xrightarrow{\cong} E$ where $\mathrm{fr}_X : X \to X$ is the absolute Frobenius.

We do this as follows: For U/X étale, we have the relative Frobenius $\mathrm{Fr}_{U/X}$ defined by the following diagram:



(11)

then we have:

- $\mathrm{Fr}_{U/X}$ etale: as pr_X is the basechange of $U \to X$ étale, it is étale, thus $U \to X$ and pr_X being étale imply $\mathrm{Fr}_{U/X}$ étale.
- $\mathrm{Fr}_{U/X}$ universally bijective: as $(\mathrm{fr}_X)_U$ is the base change of fr_X which is universally bijective, it also is universally bijective. Thus $(\mathrm{fr}_X)_U$ and fr_U being universally bijective imply $\mathrm{Fr}_{U/X}$ universally bijective.

So $\mathrm{Fr}_{U/X}$ is etale and universally bijective so an isomorphism.

So this defines an isomorphism of sections

$$E(\mathrm{Fr}_{U/X}) : (\mathrm{fr}_{X,*} E)(U) = E(\mathrm{fr}_X^{-1}(U)) \to E(U) \tag{12}$$

functorial in U, so an isomorphism $E(\mathrm{Fr}_{\bullet/X}) : \mathrm{fr}_{X,*} E \xrightarrow{\cong} E$. By adjunction we get an isomorphism

$$\mathrm{Fr}_{E/X} : \mathrm{fr}_X^* E \xrightarrow{\cong} E. \tag{13}$$

We can base change the maps fr_X and $\mathrm{Fr}_{E/X}$ to $\overline{\mathbb{F}}_p$ to get maps

$$\begin{aligned} \overline{\mathrm{fr}}_X &:= (\mathrm{fr}_X \times_{\mathbb{F}_q} \mathrm{id}_{\overline{\mathbb{F}}_p}) : \overline{X} \to \overline{X} \\ \overline{\mathrm{Fr}}_{E/X} &:= (\mathrm{Fr}_{E/X} \otimes_{\mathbb{F}_q} \mathrm{id}_{\overline{\mathbb{F}}_p}) : \overline{\mathrm{fr}}_X^* \overline{E} \to \overline{E} \end{aligned} \tag{14}$$

We can now apply it as follows:

- On cohomology: we have a pullback map $H^i_c(\overline{X}, \overline{E}) \to H^i_c(\overline{X}, \overline{\mathrm{fr}}_X^* \overline{E})$ which we can compose with $H^*_c(\overline{X}, \overline{\mathrm{Fr}}_{E/X})$ to get

$$F : H^i_c(\overline{X}, \overline{E}) \to H^i_c(\overline{X}, \overline{\mathrm{fr}}_X^* \overline{E}) \to H^i_c(\overline{X}, \overline{E}) \in \mathrm{End}(H^i_c(\overline{X}, \overline{E})). \tag{15}$$

- On stalks: if we take stalks at $\overline{x} \in \overline{X}^0$ of $\overline{\mathrm{Fr}}_{E/X}$, we get a map

$$i^*_{\overline{x}} \overline{\mathrm{Fr}}_{E/X} : i^*_{\overline{\mathrm{fr}}_X(\overline{x})} \overline{E} \to i^*_{\overline{x}} \overline{E}. \tag{16}$$

Now this is not yet an endomorphism of $i_{\overline{x}}^*\overline{E}$. as $\overline{\mathrm{fr}}_X(\overline{x}) \neq \overline{x}$ in general. But if $x \in X^0$ is the image of $\overline{x} \in \overline{X}^0$, then if we take the $\deg(x) := [\kappa(x) : \mathbb{F}_q]$'th power of this map, we get $\overline{\mathrm{fr}}_X^{\deg(x)}(\overline{x}) = \overline{x}$. So we define

$$F_x := (i_{\overline{x}}^*\overline{\mathrm{Fr}}_{E/X})^{\deg(x)} : i_{\overline{x}}\overline{E} \to i_{\overline{x}}\overline{E} \in \mathrm{End}(i_{\overline{x}}\overline{E}). \tag{17}$$

4.2 Comparison with geometric Frobenius

Let $\sigma_q \in \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_q)$ be the Frobenius map, recall the geometric and arithmetic Frobenius

$$\begin{aligned} 1 \times \sigma_q &:= \mathrm{id}_X \times_{\mathbb{F}_q} \mathrm{Spec}(\sigma_q) = \mathrm{id}_X \times_{\mathbb{F}_q} \mathrm{fr}_{\overline{\mathbb{F}}_p} : \overline{X} \to \overline{X} \quad \text{(arithmetic)} \\ 1 \times \sigma_q^{-1} &:= \mathrm{id}_X \times_{\mathbb{F}_q} \mathrm{Spec}(\sigma_q^{-1}) = \mathrm{id}_X \times_{\mathbb{F}_q} \mathrm{fr}_{\overline{\mathbb{F}}_p}^{-1} : \overline{X} \to \overline{X} \quad \text{(geometric).} \end{aligned} \tag{18}$$

If we compose $\overline{\mathrm{fr}}_X$ and $\overline{\mathrm{Fr}}_{E/X}$ with the arithmetic frobenius, we get the absolute Frobenius of $\overline{X}$:

$$\begin{aligned} (\mathrm{fr}_X \times_{\mathbb{F}_q} \mathrm{id}_{\overline{\mathbb{F}}_p}) \circ (\mathrm{id}_X \times_{\mathbb{F}_q} \mathrm{fr}_{\overline{\mathbb{F}}_p}) &= \mathrm{fr}_{\overline{X}} \\ (\mathrm{Fr}_{E/X} \otimes_{\mathbb{F}_q} \mathrm{id}_{\overline{\mathbb{F}}_p}) \circ (\mathrm{id}_E \otimes_{\mathbb{F}_q} F_{\overline{\mathbb{F}}_p/\mathbb{F}_q}) &= \mathrm{Fr}_{\overline{E}/\overline{X}}\,. \end{aligned} \tag{19}$$

Then we claim that the morphism of sections induced by the absolute Frobenius of $\overline{E}$ is the identity. This follows by functoriality, let $U/\overline{X}$ étale. Then we have

$$\varphi_{\overline{E},U} : H^0(U, \overline{E}|_U) \to H^0(U, \mathrm{fr}_{\overline{X}}^* \overline{E}|_U) \to H^0(U, \overline{E}|_U) \tag{20}$$

is a morphism functorial in $\overline{E}$, so if take any section $s \in \overline{E}(U)$, which we consider as a map $s : h_U \to \overline{E}$, we get the commutative diagram

$$\begin{array}{ccc} H^0(U, h_U) & \xrightarrow{\varphi_{h_U,U}} & H^0(U, h_U) \\ \downarrow{\scriptstyle H^0(U,s)} & & \downarrow{\scriptstyle H^0(U,s)} \\ H^0(U, \overline{E}|_U) & \xrightarrow{\varphi_{\overline{E},U}} & H^0(U, \overline{E}|_U) \end{array} \tag{21}$$

Then because $H^0(U, h_U) = \{*\}$, we get that $\varphi_{h_U,U} = \mathrm{id}_{\{*\}}$ and so $\varphi_{\overline{E},U} \circ H^0(U,s) = H^0(U,s)$ so $\varphi_{\overline{E},U}(s) = s$. Thus we conclude $\varphi_{\overline{E},U} = \mathrm{id}_{H^0(U,\overline{E}|_U)}$.

- As $U/\overline{X}$ was chosen arbitrarily, this implies that the absolute Frobenius acts as the identity on stalks as well. So we can identify F_x with the geometric Frobenius $1 \times \sigma_q^{-1}$ as elements of $\mathrm{End}(\overline{E}_{\overline{x}})$.
- Furthermore by properties of derived functors it can be shown that the absolute Frobenius acts as the identity on $H^i(\overline{X}, \overline{E})$ for $i \geq 1$ as well, so we can identify F with the geometric frobenius $1 \times \sigma_q^{-1}$ as elements of $\mathrm{End}(H_c^i(\overline{X}, \overline{E}))$.

So under this identification, in particular when $E = \mathbb{Q}_\ell(a)$ for some $a \in \mathbb{Z}$, we have that F_x acts on $\mathbb{Q}_\ell(a)_{\overline{x}} = \mathbb{Q}_\ell(a)$ by $\sigma_q^{-a\deg(x)}$ with $\sigma \in \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_q)$, so by multiplication by $q^{-a\deg(x)}$. In particular when $a = 0$, F_x acts as the identity on $(\mathbb{Q}_\ell)_{\overline{x}}$.

4.3 L-functions

Definition 1 ([1, p.80 Def 1.6]). For $X/\mathbb{F}_q$ a scheme of finite type and E a $\mathbb{Q}_\ell$-sheaf, the L-function is defined as

$$\begin{aligned} L(X,E) &:= \prod_{x\in X^0} \frac{1}{\det(1-F_x^{\deg(x)}t^{\deg(x)}, \overline{E}_{\overline{x}})} \\ \text{where } X^0 &:= \{\text{closed points of } X\} \\ \deg(x) &:= [\kappa(x):\mathbb{F}_q]. \end{aligned} \tag{22}$$

The cohomological interpretation of L-functions is then:

Theorem 1 ([1, p.86 Thm 3.1]). *Let $X/\mathbb{F}_q$ be a separated scheme of finite type, E a constructible $\mathbb{Q}_\ell$-sheaf, then*

$$L(X,E) = \prod_{i=0}^{2\dim X} \det(1-Ft, H_c^i(\overline{X},\overline{E}))^{(-1)^{i+1}} \tag{23}$$

This identity is obtained from the following trace formula

Theorem 2 ([1, p.86 Thm 3.2]). *Let $X/\mathbb{F}_q$ a separated scheme of finite type and E a constructible $\mathbb{Q}_\ell$-sheaf on X. Then for $n\in\mathbb{Z}_{\geq 1}$:*

$$\sum_{x\in X^{\mathrm{fr}_X^n}} \operatorname{tr}(F_x^n, \overline{E}_{\overline{x}}) = \sum_{i=0}^{2\dim X} (-1)^i \operatorname{tr}(F^n, H_c^i(\overline{X},\overline{E})) \tag{24}$$

Proof of Thm 2 $\Rightarrow$ Thm 1. [1, p.86,87] Apply Lemma 1 with $K=\mathbb{Q}_\ell$. So we get

$$\begin{aligned} t\frac{d}{dt}\log L(X,E) &= \sum_{x\in X^0}\sum_{n\in\mathbb{Z}_{\geq 1}} \deg(x)\operatorname{tr}(F_x^{n\deg(x)})t^{n\deg(x)} \\ &= \sum_{n\in\mathbb{Z}_{\geq 1}}\sum_{x\in X^0} \deg(x)\operatorname{tr}(F_x^{n\deg(x)})t^{n\deg(x)} \\ &= \sum_{m\in\mathbb{Z}_{\geq 1}}\sum_{\substack{x\in X^0\\ \deg(x)\mid m}} \deg(x)\operatorname{tr}(F_x^m)t^m \end{aligned} \tag{25}$$

Then we use the following combinatorial identity:

$$\sum_{\substack{x\in X^0\\ \deg(x)\mid m}} \deg(x)\operatorname{tr}(F_x^m)t^m = \sum_{x\in X(\mathbb{F}_{q^m})} \operatorname{tr}(F_x^m)t^m \tag{26}$$

which follows because if we consider $x\in X^0$, then there are $\mathrm{Gal}(\kappa(x)/\mathbb{F}_q)$-many $\mathbb{F}_{q^m}\supset\kappa(x)\supset\mathbb{F}_q$ inclusions and we must have $\deg(x)\mid m$ for the inclusion $\mathbb{F}_{q^m}\supset\kappa(x)$ to hold. So we get

$$\begin{aligned} &= \sum_{m\in\mathbb{Z}_{\geq 1}}\sum_{x\in X(\mathbb{F}_{q^m})} \operatorname{tr}(F_x^m)t^m \\ &= \sum_{m\in\mathbb{Z}_{\geq 1}}\sum_{x\in \overline{X}^{\mathrm{fr}_X^n}} \operatorname{tr}(F_x^m)t^m \end{aligned} \tag{27}$$

and

$$
\begin{aligned}
&t\frac{d}{dt}\log\prod_{i=0}^{2\dim X}\det(1-Ft,H_c^i(\overline{X},\overline{E}))^{(-1)^{i+1}} \\
&=\sum_{i=0}^{2\dim X}(-1)^i\sum_{m\in\mathbb{Z}_{\geq 1}}\operatorname{tr}(F^m,H_c^i(\overline{X},\overline{E}))t^m \\
&=\sum_{m\in\mathbb{Z}_{\geq 1}}\sum_{i=0}^{2\dim X}(-1)^i\operatorname{tr}(F^m,H_c^i(\overline{X},\overline{E}))t^m.
\end{aligned} \tag{28}
$$

Then we have equality on the coefficients of t^m by the Lefschetz trace formula. □

5 Proof of rationality

We start with the following combinatorial identity:

$$\begin{aligned}\sum_{m\in\mathbb{Z}_{\geq 1}} \#X(\mathbb{F}_{q^m})t^m &= \sum_{x\in X^0}\sum_{k\in\mathbb{Z}_{\geq 1}} |\operatorname{Gal}(\kappa(x)/\mathbb{F}_q)| t^{k\deg(x)} \\ &= \sum_{x\in X^0}\sum_{k\in\mathbb{Z}_{\geq 1}} \deg(x) t^{k\deg(x)}.\end{aligned} \tag{29}$$

We see this as follows: if $x \in X^0$ then the elements of $X(\mathbb{F}_{q^m})$ which have underlying set-point x are parametrised by maps

$$\begin{array}{ccccc} \mathbb{F}_{q^m} & \longleftarrow & \kappa(x) & \longleftarrow & \mathbb{F}_q \\ & \nwarrow & \big\downarrow \sigma & \swarrow & \\ & & \kappa(x) & & \end{array} \tag{30}$$

which implies in particular that $\deg(x) \mid m$, so $m = k\deg(x)$ and that any two $\mathbb{F}_q$-linear maps $\mathbb{F}_{q^m} \leftarrow \kappa(x)$ differ by an automorphism $\sigma \in \operatorname{Gal}(\kappa(x)/\mathbb{F}_q)$.

So we get

$$\begin{aligned} t\frac{d}{dt}\log\zeta_X(t) &= \sum_{m\in\mathbb{Z}_{\geq 1}} \nu_m(X)t^m \\ t\frac{d}{dt}\prod_{x\in X^0}\frac{1}{1-t^{\deg(x)}} &= \sum_{x\in X^0}\sum_{k\in\mathbb{Z}_{\geq 1}} \deg(x)t^{\deg(x)k}\end{aligned} \tag{31}$$

which implies

$$\zeta_X(t) = \prod_{x\in X^0}\frac{1}{1-t^{\deg(x)}}. \tag{32}$$

We apply Theorem 1 with $E := \mathbb{Q}_\ell$. We have by definition that the $\operatorname{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_q)$-action on $\mathbb{Q}_\ell = \mathbb{Q}_\ell(0)$ is trivial, so $F_x = \mathrm{id}_{\mathbb{Q}_\ell}$ on $(\mathbb{Q}_\ell)_{\overline{x}} = \mathbb{Q}_\ell$. We then obtain

$$\begin{aligned}\zeta_X(t) &= \exp\left(\sum_{m\in\mathbb{Z}_{\geq 0}}\frac{\nu_m(X)}{m}t^m\right) \\ &\overset{(32)}{=} \prod_{x\in X^0}\frac{1}{1-t^{\deg(x)}} \\ &\overset{F_x=\mathrm{id}_{\mathbb{Q}_\ell}}{=} \prod_{x\in X^0}\frac{1}{\det(1-F_xt^{\deg(x)},\mathbb{Q}_\ell)} \\ &\overset{\text{Thm 1}}{=} \prod_{i=0}^{2\dim X}\det(1-Ft, H^i_c(\overline{X},\mathbb{Q}_\ell))^{(-1)^{i+1}} \\ &= \prod_{i=0}^{2\dim X}\det(1-Ft, H^i(\overline{X},\mathbb{Q}_\ell))^{(-1)^{i+1}}.\end{aligned} \tag{33}$$

Then if we take $K = \mathbb{Q}_\ell$ and

$$P_i(t) := \det(1-Ft, H^i(\overline{X},\mathbb{Q}_\ell)) \in \mathbb{Q}_\ell[t] \tag{34}$$

we have shown rationality. Also by triviality of action of F on $\mathbb{Q}_\ell$, we have that F is the identity on $H^0(\overline{X},\mathbb{Q}_\ell)$ so $P_0(t) = 1-t$.

6 Proof of Betti numbers

We have

$$\deg P_i = \deg\det(1-Ft, H^i(\overline{X},\mathbb{Q}_\ell)) = \dim_{\mathbb{Q}_\ell} H^i(\overline{X},\mathbb{Q}_\ell). \tag{35}$$

Let $Y/\mathbb{Z}_{(p)}$ be the smooth projective good reduction mod p variety such that $Y\times_{\mathbb{Z}_{(p)}}\mathbb{F}_q = X$. Let $\pi : Y \to \mathrm{Spec}(\mathbb{Z}_{(p)})$ be the structure map. Let

$$\begin{aligned} x_0 &:= (p) \in \mathrm{Spec}(\mathbb{Z}_{(p)}),\\ x_1 &:= (0) \in \mathrm{Spec}(\mathbb{Z}_{(p)}),\\ \overline{x_0} &:= \mathrm{Spec}(\overline{\mathbb{F}}_p)\\ \overline{x_1} &:= \mathrm{Spec}(\mathbb{C}),\\ x_0 &\in \overline{\{x_1\}} \text{ so } x_1 \rightsquigarrow x_0. \end{aligned} \tag{36}$$

By an inclusion of neighbourhood systems, we obtain a specialisation map

$$(R^i\pi_*\mathbb{Q}_\ell)_{\overline{x_0}} \to (R^i\pi_*\mathbb{Q}_\ell)_{\overline{x_1}} \tag{37}$$

then by a Leray spectral sequence argument, using the smooth base change theorem we get that this map is an isomorphism. So combined with the proper base change theorem we get

$$\begin{aligned} H^i(X_{\overline{\mathbb{F}}_p},\mathbb{Q}_\ell) &= H^i(Y_{\overline{\mathbb{F}}_p},\mathbb{Q}_\ell)\\ &\overset{proper}{\cong} (R^i\pi_*\mathbb{Q}_\ell)_{\overline{x}_0}\\ &\overset{smooth+proper}{\cong} (R^i\pi_*\mathbb{Q}_\ell)_{\overline{x}_1}\\ &\overset{proper}{\cong} H^i(Y_{\mathbb{C}},\mathbb{Q}_\ell). \end{aligned} \tag{38}$$

Then by the comparison theorem of cohomology:

Theorem 3 ([3, p.132 Thm 21.5])**.** *Let Y be a connected nonsingular variety over C. For any locally constant sheaf $\mathcal{F}$ on X_{et} with finite stalks, $H^r(X_{et},\mathcal{F}) \cong H^r(Y_{cx},\mathcal{F}^{cx})$ for all $r\in\mathbb{Z}_{\geq 0}$*

applies to Y with $\mathcal{F}=\mathbb{Q}_\ell$ because Y is smooth and connected. So we have

$$H^i(Y_{\mathbb{C}},\mathbb{Q}_\ell)\otimes_{\mathbb{Q}_\ell}\mathbb{C} \cong H^i(Y_{\mathbb{C}},\mathbb{C}) \cong H^i_{sing}(Y(\mathbb{C})^{cx},\mathbb{C}). \tag{39}$$

So we get

$$\begin{aligned} \deg P_i &= \dim_{\mathbb{Q}_\ell} H^i(\overline{X},\mathbb{Q}_\ell)\\ &= \dim_{\mathbb{Q}_\ell} H^i(X_{\overline{\mathbb{F}}_p},\mathbb{Q}_\ell)\\ &= \dim_{\mathbb{Q}_\ell} H^i(Y_{\overline{\mathbb{F}}_p},\mathbb{Q}_\ell)\\ &= \dim_{\mathbb{Q}_\ell} H^i(Y_{\mathbb{C}},\mathbb{Q}_\ell)\\ &= \dim_{\mathbb{C}} H^i_{sing}(Y(\mathbb{C})^{cx},\mathbb{C}). \end{aligned} \tag{40}$$

7 Proof of functional equation

Recall the Poincaré duality theorem:

Theorem 4 ([5, p. 4 Thm 2.6])**.** *Let X/K smooth proper and equidimensional of dimension d. There is a* $\mathrm{Gal}(K^{sep}/K)$*-equivariant homomorphism*

$$\mathrm{tr} : H^{2d}(\overline{X}, \mathbb{Q}_\ell) \to \mathbb{Q}_\ell(-d) \tag{41}$$

so that the cup product pairing

$$H^i(\overline{X}, \mathbb{Q}_\ell) \times H^{2d-i}(\overline{X}, \mathbb{Q}_\ell) \xrightarrow{\cup} H^{2d}(\overline{X}, \mathbb{Q}_\ell) \xrightarrow{\mathrm{tr}} \mathbb{Q}_\ell(-d) \tag{42}$$

is perfect.

We apply this theorem with $K = \mathbb{F}_q$ and X our smooth projective variety as before. Denote by $\langle a, b\rangle := \mathrm{tr}(a \cup b)$ the pairing. As F is the geometric frobenius $1 \times \sigma_q^{-1}$, it acts as multiplication by q^{-1} on $\mathbb{Q}_\ell(1)$, so it acts by multiplication by q^d on $\mathbb{Q}_\ell(-d)$. So we have by $\mathrm{Gal}(K^{sep}/K)$-equivariance of the pairing that

$$\langle Fa, Fb\rangle = F\langle a, b\rangle = q^d \langle a, b\rangle \tag{43}$$

Side remark: We noted in Rationality that F is the identity on $H^0(\overline{X}, \mathbb{Q}_\ell)$. Now by this perfect pairing we have that F is q^d on $H^{2d}(\overline{X}, \mathbb{Q}_\ell)$, so we conclude that $P_{2d} = 1 - q^d t$.

Let $\beta_i := \dim_{\mathbb{Q}_\ell} H^i = \dim_{\mathbb{Q}_\ell} H^{2d-i}$.

- We can choose a basis $e_1, \ldots, e_{\beta_i}$ of $H^i \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}_\ell}$ such that F is upper triangular:

$$F(e_i) = \sum_{j=1}^{\beta_i} a_{i,j} e_j \quad q_{i,j} = 0 \text{ for } i > j. \tag{44}$$

- By perfectness of the pairing, we can choose a basis $f_1, \ldots, f_{\beta_i}$ of $H^{2d-i} \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}_\ell}$ such that $\langle e_i, f_j\rangle = \delta_{i,j}$.
- If we write F^{-1} on H^{2d-i} as a matrix with respect to the basis $f_1, \ldots, f_{\beta_i}$

$$F^{-1}(f_i) = \sum_{j=1}^{\beta_i} b_{i,j} f_j \tag{45}$$

  then we have

$$\begin{aligned} b_{i,j} &= \langle e_j, \sum_{k=1}^{\beta_i} b_{i,k} f_k\rangle \\ &= \langle e_j, F^{-1}(f_i)\rangle \\ &= F^{-1}\langle Fe_j, f_i\rangle \\ &= q^{-d}\langle \sum_{k=1}^{\beta_i} a_{j,k} e_k, f_i\rangle \\ &= q^{-d} a_{j,i} = 0 \text{ if } j > i \end{aligned} \tag{46}$$

  so F^{-1} has lower-triangular matrix with respect to the basis f_i.

- $$\begin{aligned} \det(F;H^i) &= \prod_{i=1}^{\beta_i} a_{i,i} \\ &= \prod_{i=1}^{\beta_i} \langle F(e_i), f_i\rangle \\ &= \prod_{i=1}^{\beta_i} F\langle e_i, F^{-1}(f_i)\rangle \\ &= \prod_{i=1}^{\beta_i} q^d b_{i,i} \\ &= q^{d\beta_i}\det(F^{-1};H^{2d-i}) \end{aligned} \tag{47}$$

  so
  $$\det(F;H^i)\det(F;H^{2d-i}) = q^{d\beta_i}. \tag{48}$$

- $$\begin{aligned} \det(1-\frac{F}{q^d t};H^i) &= \left(\frac{-1}{q^d t}\right)^{\beta_i}\det(F;H^i)\det(1-F^{-1}q^d t;H^i) \\ &= \left(\frac{-1}{q^d t}\right)^{\beta_i}\det(F;H^i)\prod_{i=1}^{\beta_i}(1-a_{i,i}^{-1}q^d t) \\ &= \left(\frac{-1}{q^d t}\right)^{\beta_i}\det(F;H^i)\prod_{i=1}^{\beta_i}(1-b_{i,i}^{-1}t) \\ &= \left(\frac{-1}{q^d t}\right)^{\beta_i}\det(F;H^i)\det(1-Ft;H^{2d-i}) \end{aligned} \tag{49}$$

  Then using the cohomological interpretation of L-functions we get

$$\begin{aligned} \zeta_X(\frac{1}{q^d t}) &= \prod_{i=0}^{2d}\det(1-\frac{F}{q^d t},H^i)^{(-1)^{i+1}} \\ &= \left(\prod_{i=0}^{2d}(-q^d t)^{(-1)^{i+2}\beta_i}\det(F;H^i)^{(-1)^{i+1}}\right)\cdot\prod_{i=0}^{2d}\det(1-Ft;H^{2d-i})^{(-1)^{i+1}} \\ &= \left(\prod_{i=0}^{2d}(-q^d t)^{(-1)^{i+2}\beta_i}\det(F;H^i)^{(-1)^{i+1}}\right)\zeta_X(t). \end{aligned} \tag{50}$$

Where the factor is

$$\begin{aligned} &(-q^d t)^\chi\cdot\prod_{i=0}^{d-1}(\det(F;H^i)\det(F;H^{2d-i}))^{(-1)^{i+1}}\cdot\det(F;H^d)^{(-1)^{d+1}} \\ &= (-q^d t)^\chi\cdot\pm q^{-\frac{d\chi}{2}} \\ &= \pm q^{\frac{d\chi}{2}}t^\chi. \end{aligned} \tag{51}$$

Here $\chi = \sum_{i=0}^{2d}(-1)^i\beta_i$, so the proof is complete.

8 Linear algebra lemma's

Lemma 1 ([2, p.186 Lem 2.7]). *Let $\alpha \in End_K(V)$ for V a finite dimensional K-vector space. Then*

$$t\frac{d}{dt}\log\det(1-\alpha t)^{-1} = \sum_{n\in\mathbb{Z}_{\geq 1}} \mathrm{tr}(\alpha^n)t^n. \tag{52}$$

Proof. Let $\overline{K}$ an algebraic closure, and set $\overline{V} := V \otimes_K \overline{K}$. Any identity we find on $\overline{V}$ will be valid in V, by faithfulness of the tensor functor $-\otimes_K \overline{K}$. So we may assume $K = \overline{K}$. As K is algebraically closed, we can find a Jordan normal basis of α, so in terms of this basis we can write α as an upper-triangular matrix $[\alpha]$ with diagonal entries $\alpha_1, \ldots, \alpha_n$ with $n := \dim_K V$.

Then $\det(1-\alpha t) = \prod_{i=1}^n (1-\alpha_i t)$ and $\mathrm{tr}(\alpha^k) = \sum_{i=1}^n \alpha_i^k$ so we have

$$\begin{aligned} t\frac{d}{dt}\log\det(1-\alpha t)^{-1} &= t\frac{d}{dt}\left(-\sum_{i=1}^n \log(1-\alpha_i t)\right) \\ &= \sum_{i=1}^n \frac{\alpha_i t}{1-\alpha_i t} \\ &= \sum_{i=1}^n \sum_{k\in\mathbb{Z}_{\geq 1}} (\alpha_i t)^k \\ &= \sum_{k\in\mathbb{Z}_{\geq 1}} \mathrm{tr}(\alpha^k)t^k. \end{aligned} \tag{53}$$

□

Lemma 2 ([4, p.33 Lem 4.15], [6, p.456 Lem 4.3]). *Let $\langle\cdot,\cdot\rangle : V \times W \to K$ be a perfect pairing of vector spaces V, W of dimension r over K. Let $\lambda \in K, \varphi \in \mathrm{End}_K(V), \psi \in \mathrm{End}_K(W)$ such that*

$$\langle\varphi(v), \psi(w)\rangle = \lambda\langle v, w\rangle \quad \forall v \in V, w \in W. \tag{54}$$

Then

$$\begin{aligned} \det(1-\psi t; W) &= \frac{(-1)^r\lambda^r t^r}{\det(\varphi; V)} \det\left(1 - \frac{\varphi}{\lambda t}; V\right), \\ \det(\psi; W) &= \frac{\lambda^r}{\det(\varphi; V)}. \end{aligned} \tag{55}$$

Proof. Let $\overline{K}$ an algebraic closure, and set $\overline{V} := V \otimes_K \overline{K}$. Any identity we find on $\overline{V}$ will be valid in V, by faithfulness of the tensor functor $-\otimes_K \overline{K}$. So we may assume $K = \overline{K}$.

- Choose a basis $v_1, \ldots, v_r$ of V such that φ is upper-triangular with respect to this basis, so $\varphi(v_i) = \sum_{j=1}^r a_{i,j}e_j$ with $a_{i,j} = 0$ for $i > j$.
- By the pairing being perfect, we can choose a basis $w_1, \ldots, w_r$ of W such that $\langle v_i, w_j\rangle = \delta_{i,j}$.
- We have that ψ is injective, thus an isomorphism: if $\psi(w) = 0$, then for all $v \in V$:

$$0 = \langle\varphi(v), \psi(w)\rangle = \lambda\langle v, w\rangle \tag{56}$$

  so by perfectness of the pairing we have $w = 0$.
- Then w_i is a basis for which ψ^{-1} is lower-triangular: if we write $\psi^{-1}(w_j) = \sum_{i=1}^r b_{j,i}w_\ell$, for $i > j$:

$$b_{j,i} = \langle v_i, \psi^{-1}(w_j)\rangle = \langle\varphi(v_i), w_j\rangle = a_{i,j} = 0. \tag{57}$$

- Thus we have

$$
\begin{aligned}
\det(\varphi; V) &= \prod_{i=1}^{r} a_{i,i} \\
&= \prod_{i=1}^{r} \langle \varphi(v_i), w_i \rangle \\
&= \prod_{i=1}^{r} \lambda \langle v_i, \psi^{-1}(w_i) \rangle \\
&= \lambda^r \det(\psi^{-1}; W) \\
&= \frac{\lambda^r}{\det(\psi; W)}
\end{aligned} \tag{58}
$$

  and

$$
\begin{aligned}
\det(1 - \psi t; W) &= \det(\psi; W) \det(\psi^{-1} - t; W) \\
&= \frac{\lambda^r}{\det(\varphi; V)} \cdot \prod_{i=1}^{r} (b_{i,i} - t) \\
&= \frac{\lambda^r}{\det(\varphi; V)} \cdot \prod_{i=1}^{r} (\frac{a_{i,i}}{\lambda} - t) \\
&= \frac{(-1)^r \lambda^r t^r}{\det(\varphi; V)} \cdot \prod_{i=1}^{r} (1 - \frac{a_{i,i}}{\lambda t}) \\
&= \frac{(-1)^r \lambda^r t^r}{\det(\varphi; V)} \det(1 - \frac{\varphi}{\lambda t}; V).
\end{aligned} \tag{59}
$$

□

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