Chapter VI
Harmonic Analysis on p-adic Groups

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1 Representation theory

1.1 Representations of the group of finite adeles and of global Hecke algebras.

In Chapter III we encountered admissible representations of the adele group

$$\rho : G(\mathbb{A}_f) \to \mathrm{Gl}(V)$$

which are defined by the property that for any open compact subgroup $K_f \subset G(\mathbb{A}_f)$ the space of invariants V^{K_f} has finite dimension and V is the union of all these subspaces of invariants, which become bigger and bigger if K_f becomes smaller and smaller. In other words

$$V = \lim_{K_f} V^{K_f}.$$

Any such finite dimensional vector space is a module under the Hecke algebra $\mathcal{H}_{K_f} = \mathcal{C}_c(G(\mathbb{A}_f//K_f)$, this are the compactlly supported functions on $G(\mathbb{A}_f)$, which are biinvariant under K_f. We say that K_f is a level subgroup for V if $V^{K_f} \neq \{0\}$. We indicated that a $G(\mathbb{A}_f)$- representation V, for which K_f is a level subgroup, is absolutely irreducible, if and only if V^{K_f} is an absolutely irreducible $\mathcal{H}_{K_f}$ -module. If our level subgroup $K_f = \prod_p K_p$ then an absolutely irreducible $\mathcal{H}_{K_f}$-module V^{K_f} is a tensor product

$$V^{K_f} \xrightarrow{\sim} \bigotimes_p V_p.$$

where the V_p are absolutely irreducible $\mathcal{H}_{K_p}$-modules. Recall that we have seen that for almost all primes the local Hecke algebra is of commutative and hence $\dim V^{K_p} = 1$ for almost all p

This then implies that the representation of the group $G(\mathbb{A}_f)$ is also a tensor product

$$V \xrightarrow{\sim} \bigotimes{}' \tilde{V}_p.$$

where the prime indicates that the space is generated by tensors $v = \otimes_p v_p$ where almost all $v_p \in V^{K'_p}$ for some $K'_f = \prod_p K'_p$. We need some level of understanding of the local theory, and we will give an informal outline of it in this section.

1.2 Admissible representations

This leads us to the following definition. Let F be a field of charcteristic zero, let V be an F-vector space. An admissible representation of the group $G(\mathbb{Q}_p)$ is an action of $G(\mathbb{Q}_p)$ on V which has the following two properties

(i) For any open compact subgroup $K_p \subset G(\mathbb{Q}_p)$ the space V^{K_p} of K_p invariant vectors is finite dimensional.

(ii) For any vector $v \in V$ we can find an open compact subgroup K_p so that $v \in V^{K_p}$ in other words $V = \lim_{K_p} V^{K_p}$.

Then is is clear that the vector spaces V^{K_p} are modules for the Hecke algebra $\mathcal{H}_{K_p}$. An admissible $G(\mathbb{Q}_p)$ -module V is irreducible if it does not contain an invariant proper submodule. Given such an irreducible module $V \neq (0)$, we can find a K_p such that $V^{K_p} \neq (0)$. I claim that then V^{K_p} is an irreducible $\mathcal{H}_{K_p}$-module. To see this we take the identity element e_{K_p} in our Hecke algebra, it induces a projector on V and a decomposition $V = V^{K_p} \oplus V' = e_{K_p}V \oplus (1 - e_{K_p})V$. Let assume we have a proper $\mathcal{H}_{K_p}$-invariant submodule $W \subset V^{K_p}$ Now we convince ourselves that the $G(\mathbb{Q}_p)$-invariant subspace $\tilde{W}$ generated by the elements gw is a proper subspace. We compute the integral

$$\int_{K_p} kgwdk = \int_{K_p \times K_p} k_1 g k_2 w dk_2 dk_1.$$

The first integral gives us the projection to V^{K_p}, the second integral is the Hecke operator, hence the result is in W. We conclude that $e_{K_p}\tilde{W} \subset W$ and tis shows that $(0) \neq \tilde{W} \neq V$.

Now it is not hard to see, that the assignment

$$V \to V^{K_p}$$

from irreducible admissble $G(\mathbb{Q}_p)$-modules with $V^{K_p} \neq (0)$ to finite dimensional irreducible $\mathcal{H}_{K_p}$-modules induces an bijection between the isomorphism classes of the respective types of modules. If we start from V^{K_p} we can reconstruct V by a appropiate form of induction.

1.3 Characters

The local Hecke algebra $\mathcal{C}_c(G(\mathbb{Q}_p)//K_p)$ depends on the choice of the level K_p, we can define the Hecke algebra of all compactly supported smooth functions.

$$\mathcal{C}_c(G(\mathbb{Q}_p)) = \lim_{K_p} \mathcal{C}_c(G(\mathbb{Q}_p)//K_p)$$

this the algebra of compctly supported functions, which are locally constant. It is clear that an admissible $G(\mathbb{Q}_p)$ module V is also a module for $\mathcal{C}_c(G(\mathbb{Q}_p))$. For an element $h \in \mathcal{C}_c(G(\mathbb{Q}_p))$ we can find an open compact subgroup K_p such that $h \in \mathcal{C}_c(G(\mathbb{Q}_p)//K_p)$. Then we can decompose $V = V^{K_p} \oplus V'$ and the endomorphism

$$T_h(v) = \int_{G(\mathbb{Q}_p)} h(x)xvdx$$

is zero on V' and induces an endomorphism also called T_h on V^{K_p}. We define the trace

$$\mathrm{tr}(T_h|V) = \mathrm{tr}(T_h|V^{K_p}).$$

This gives us a linear form

$$\mathrm{tr}_V : \mathcal{C}_c(G(\mathbb{Q}_p)) \to L,\ h \mapsto \mathrm{tr}_V(h|V)$$

which is called the character of the module V.

Of course this character depends on the choice of a measure.

We can modify this definition slightly. An admissible irreducible representation π_p on a vector space V has a central character $\omega_p = \omega_{\pi_p}$. We define a modified space of functions $\mathcal{C}_{c,\omega_p^{-1}}(G(\mathbb{Q}_p))$ which of those functions which satisfy $h(zg) = \omega_p^{-1}(z)h(g)$ and which have compact support modulo the center. For $h \in \mathcal{C}_{c,\omega_p^{-1}}(G(\mathbb{Q}_p))$ we define as before

$$T_h(v) = \int_{G(\mathbb{Q}_p)/Z(\mathbb{Q}_p)} h(x)xvdx.$$

Again we can define the trace of such an operator and the linear form

$$\mathrm{tr}_{\pi_p} : \mathcal{C}_{c,\omega^{-1}}(G(\mathbb{Q}_p)) \to L,\ h \mapsto \mathrm{tr}_{\pi_p}(h|V)$$

It this point we want to make a technical remark. Let V be an absolutely irreducible admissible module for $G(\mathbb{Q}_p)$ with central character ω. We have seen in Chap III.1.1 that we have the isogeny μ from the central torus $C/\mathbb{Q}_p$ to the quotient torus $C' = G/G^{(1)}/\mathbb{Q}_p$. On the group of rational points this gives a homomorphism also called μ which has finite kernel and finite kokernel. let $d : G(\mathbb{Q}_p) \to C'(\mathbb{Q}_p)$ be the projection. If we consider any admissible character $\eta : C'(\mathbb{Q}_p) \to \mathbb{C}^\times$ then we can consider the twist of V by η, this is the module V but the group acts by $v \mapsto \eta(d(g))gv$. We denote the twisted module by $V(\eta)$. Then this changes the central character into $\omega \cdot \eta \circ \mu$. Most of the interesting properties of the module V are invariant under such twists. Therefore we can try to get the central character trivial, this is possible if and only if ω restricted to the kernel of μ is trivial. If we are in the special case, that $F = \mathbb{C}$, then we can find a twist, so that $\omega \cdot \eta \circ \mu$ becomes unitary, i.e. takes values in the unit circle. It is sometimes useful to have this assumption.

2.1.2.1 Intermission: Measures

At this point a few remarks concerning the choice of measures are in order. Before we always chose dx so that K_p has volume one with respect to this measure. But now we consider K_p as variable and then this becomes awkward. We describe another way of constructing measures, which depend on the choice of a differential form of highest degree.

Let $V/\mathbb{Q}_p$ be any smooth affine variety, or more generally a smooth scheme of finite type. Let us also assume that $V/\mathbb{Q}_p$ is irreducible, hence it has a dimension $n = \dim(V)$. Let ω be a non zero differential form of highest degree on V. On $V(\mathbb{Q}_p)$ we have the analytic p-adic topology. For any point $a \in V(\mathbb{Q}_p)$ we can find an open neighborhood U_a and an analytic isomorphism (p-adic version of implicit functions)

$$F : U_a \xrightarrow{\sim} U$$

where U is a compact open neighborhood of $F(a)$ in $\mathbb{Q}_p^n$. On $\mathbb{Q}_p^n$ we have the standard translation invariant measure $|dx_1dx_2 \dots dx_n|$ which gives volume 1 to $\mathbb{Z}_p^n$. We find a form $\omega' = f(x_1, x_2, \dots, x_n)dx_1 \wedge dx_2 \wedge \dots dx_n$ on U such that $\omega = F^*(\omega')$. Then we define

$$\int_{U_a} h(y)|\omega|(dy) = \int_U (h \circ F^{-1})(x)|f(x_1, x_2, \dots, x_n)|_p dx_1dx_2 \dots dx_n$$

These measures have all the properties of the measures in real analysis. We have the transformation rule. The advantage here is that we do not need an orientation.

If our variety over $\mathbb{Q}_p$ is a linear algebraic group $G/\mathbb{Q}_p$ then this group has a Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ and this Lie algebra is identified to the tangent space of G at the identity. The group acts by right translations on itself, we use this to identify the tangent at any point to $\mathfrak{g}$. Hence any non zero element $\omega \in \mathrm{Hom}(\Lambda^n(\mathfrak{g}), \mathbb{Q}_p)$ yields an invariant nowhere vanishing form on G/Q_p, which is also called ω. Hence we can define the right invariant measure $|\omega|$ on $G(\mathbb{Q}_p)$. This measure is also left invariant if $G/\mathbb{Q}_p$ is reductive.

At this point it seems to be appropriate to mention a slight generalization of this construction. Let us assume that $H/\mathbb{Q}_p \subset G/\mathbb{Q}_p$. We can form the quotient space $H\backslash G$, we have the projection

$$G \xrightarrow{\pi} H\backslash G.$$

If $\bar{e} = \pi(e)$ is the image of the identity, then we can identify the tangent space $T_{H\backslash G,\bar{e}} = \mathfrak{g}/\mathfrak{h} = \mathrm{Lie}(G)/\mathrm{Lie}(H)$. The we have with $n = \dim(G), r = \dim(H)$

$$\Lambda^n(\mathfrak{g}) = \Lambda^r(\mathfrak{h}) \otimes \Lambda^{n-m}(\mathfrak{g}/\mathfrak{h}).$$

Hence we see that any choice of non zero alternating forms of highest degree ω_G, ω_H defines an alternating form $\omega_{H\backslash G}$ in the point $\bar{e}$ such that

$$\omega_G = \omega_H \otimes \omega_{H\backslash G}.$$

If we now assume that the adjoint action of H on $\Lambda^{n-m}(\mathfrak{g}/\mathfrak{h})$ is trivial, then we can transport it by right translation to any point in $H\backslash G$. We get an invariant non zero form also called $\omega_{H\backslash G}$ on $H\backslash G$.

This defines a measure $|\omega_{H\backslash G}|$ on $H\backslash G(\mathbb{Q}_p)$ we will write

$$|\omega_{H\backslash G}| = \frac{|\omega_G|}{|\omega_H|}$$

We now assume that our character is defined with respect to some measure $|\omega_G|$. The character only depends on the isomorphism type of the module. For an absolutely irreducible module V over L whose isomorphism type is π_p we denote its character by ch_{π_p}.

We have the following result by Harish-Chandra

The character ch_{π_p} of an absolutely irreducible module of $G(\mathbb{Q}_p)$ is given by integration (with respect to $|\omega_G|$) against a function which is locally constant on the open subset of regular semi simple elements.

Hence we call this function also ch_{π_p} and then we get

$$\mathrm{tr}(h|V) = \int_{G(\mathbb{Q}_p)} h(g)\mathrm{ch}_{\pi_p}(g)|\omega_G|(dg).$$

This also holds for functions in $\mathcal{C}_{c,\omega_{\pi_p}^{-1}}$ if we replace the domain of integration by $G(\mathbb{Q}_p)/Z(\mathbb{Q}_p))$. It is also clear that the function ch_{π_p}, does not depend on the choice of the measure.

These characters are the natural generalizations of the characters of representations of finite or compact groups. To see this we choose a decreasing family of open compact subgroups $K_p^{(1)} \supset K_p^{(2)} \supset \dots$ which converges to Id, this means that for any open compact subgroup K_p we find an index ν such that $K_p \supset K_p^{(\nu)}$ Then we can decompose our module

$$V = V^{K_p^{(1)}} \oplus V_1 = V^{K_p^{(1)}} \oplus V_1^{K_p^{(2)}} \oplus V_2 = \dots$$

and we choose "increasing" basis

$$< v_1, v_2, \dots, v_i, \dots >$$

where increasing means, that the first vectors form a basis of $V^{K_p^{(1)}}$, resp.$W^{K_p^{(1)}}$, the next few vectors form a basis of $V_1^{K_p^{(2)}}$, resp.$W_1^{K_p^{(2)}}$ and so on. We also introduce the dual basis $< \phi_1, \phi_2, \dots >$ on the contragredient spaces $V^\vee$.

Now I want to justify

$$\mathrm{ch}_{\pi_p}(g) = \sum < gv_i, \phi_i > .$$

This sum is of course infinite and if we evaluate at $g = \mathrm{Id}$ we get ∞. The definition of the distribution ch_{π_p} says

$$\int_{G(\mathbb{Q}_p)} \mathrm{ch}_{\pi_p}(g)h(g)dg = \mathrm{tr}(\int_{G(\mathbb{Q}_p)} h(g)gdg)$$

The function h is biinvariant under the action of a a suitably small group $K_p^{(\nu)}$ and we noticed already that

$$T_h : v \mapsto \int_{G(\mathbb{Q}_p)} h(g)gv$$

induces an endomorphism of $V^{K_p^{(\nu)}}$ and is zero on the complement. Hence we defined

$$\mathrm{tr}(T_h|V) = \sum_{i\in I^{(\nu)}} \int_{G(\mathbb{Q}_p)} < gv_i, \phi_i > h(g)dg = \int_{G(\mathbb{Q}_p)} \sum_{i\in I^{(\nu)}} < gv_i, \phi_i > h(g)dg$$

But in the sum in the middle we may extend the summation to all indices because the integral $\int_{G(\mathbb{Q}_p)} < gv_i, \phi_i > h(g)dg$ is zero for $i \notin I^{(\nu)}$.

1.4 An overview over the irreducible representations, cuspidal and supercuspidal representations

1.4.1 Irreducible admissible representations of $T(\mathbb{Q}_p)$

If we have an absolutely irreducible admissible representation $G(\mathbb{Q}_p) \to \mathrm{Gl}(V)$, then it follows from Schurs lemma, that the center $Z(\mathbb{Q}_p)$, which is by definition the group of rational points of a torus, acts by a character ω, this means that $\omega : Z(\mathbb{Q}_p) \to F^\times$ is a homomorphism. Of course it may be that our group is itself a torus $T(\mathbb{Q}_p)$, then the absolutely irreducible admissible representations are such characters. We know that as an abstract topological group $T(\mathbb{Q}_p) \xrightarrow{\sim} K_p^T \times \mathbb{Z}^r$

where K_p^T is the maximal compact subgroup and r is the split rank of the torus. Then it is clear that any character is of the form $\chi = \chi_c \times \chi_s$, where χ_c is a character of finite order on the compact factor, and $\chi_s : \mathbb{Z}^r \to F^\times$. The module of rational characters $X^*(T) = \mathrm{Hom}(T, \mathcal{G}_m)$ is a module for the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$. If $T_c/\mathbb{Q}_p \subset T/\mathbb{Q}_p$ is the maximal anisotropic subtorus the the quotient $T/T_c = T_{\text{split}}$ is split and $X^*(T_{\text{split}}) \xrightarrow{\sim} \mathbb{Z}^r$ with trivial action of the Galois group. Any rational character $\gamma \in X^*(T_{\text{split}})$, induces a homomorphism $\gamma : T(\mathbb{Q}_p) \to \mathbb{Q}_p^\times$ and this defines a character $t \mapsto |\gamma|_p(t) \in F^\times$. Now we embed F into $\mathbb{C}$, then we get a homomorphism

$$X^*(T_{\text{split}}) \otimes \mathbb{C} \to \mathrm{Hom}(T(\mathbb{Q}_p), \mathbb{C}^\times)$$

which is defined by

$$\sum \gamma_i \otimes z_i \mapsto \{t \mapsto \prod |\gamma_i|^{z_i}\}.$$

These are the *unramified* characters on $T(\mathbb{Q}_p)$, because they vanish on the maximal compact subgroup $T(\mathbb{Z}_p) \subset T(\mathbb{Q}_p)$. Clearly we can find a product decomposition

$$T(\mathbb{Q}_p) = T(\mathbb{Z}_p) \times \mathbb{Z}^r,$$

hence we see that any admissible character can be written as $\chi_c \times (\sum \gamma_i \otimes z_i)$, the z_i are not unique, but it is easy to see when $\sum \gamma_i \otimes z_i$ gives a trivial character.

Therefore we have a complete description of the irreducible representations of $T(\mathbb{Q}_p)$.

1.4.2 Induced representations

Let $P/\mathbb{Q}_p \subset G/\mathbb{Q}_p$ be a parabolic subgroup, let $U/\mathbb{Q}_p$ be its unipotent radical, let $M = P/U$ be its reductive quotient. The group $M/\mathbb{Q}_p$ acts on U and hence on the Lie algebra $\mathfrak{u}$ by the adjoint action.Therefore it acts on $\Lambda^{\dim(U)}\mathfrak{u}$ by a rational character which is called $2\rho_P$. Then this defines a rational character $|\rho_P| : M(\mathbb{Q}_p) \to \mathbb{F}^\times$, where it may be necessary to adjoin the $\sqrt{p}$ to F.

Let $\sigma : M(\mathbb{Q}_p) \to \mathrm{Gl}(W)$ be an irreducible admissible representation, it defines also a representation of $P(|Q_p)$. We define

$$\mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\sigma = \{f : G(\mathbb{Q}_p) \to W | f(pg) = \sigma(p)|\rho_P|(p)f(g)\},$$

where the group $G(\mathbb{Q}_p)$ acts by translations from the right. We have to specify some restriction on the functions f. To do this we recall that we can find a maximal compact subgroup K_p such that $G(\mathbb{Q}_p) = P(\mathbb{Q}_p)K_p$. Then a function f as above is determined by its restriction to K_p. If our field $F = \mathbb{C}$ then we may allow for f all functions in $L^2(K_p)$. Then we get a Hilbert space, but this will not be admissible. Therefore it is better at this point to allow only functions whose restriction to K_p will be locally constant. Therefore we make this requirement and then it is easy to see that we get an admissible representation $\mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\sigma$.

These representations have a finite composition series and in general they are even irreducible. What happens when is a difficult problem.

We consider the special case $\mathrm{Gl}_2/\mathbb{Q}_p$. In this case we choose for our parabolic subgroup the Borel subgroup $B/\mathbb{Q}_p$, the group $M/\mathbb{Q}_p$ is the maximal torus, we identify it to $\{\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$. Then our character χ, which assume to be $\mathbb{C}$ valued- can be written as

$$\chi : \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \to \mu_1(t_1)\mu_2(t_2),$$

and then we get the induced representation

$$I_\chi = \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \chi = \{f : G(\mathbb{Q}_p) \to C^\times | f(bg) = \chi(b)|\rho_B|(p)f(g)\}.$$

Here we will sometimes pass to a different notation and write $\chi(b) = b^\chi$, so we find $\chi(b)|\rho_B|(b) = b^{\chi+|\rho_B|} = \mu_1(t_1)\mu_2(t_2)|\frac{t_1}{t_2}|_p^{1/2}$. This means that we use the additive notation for the group of characters.

The central character ω_χ of I_χ is $\mu_1(t)\mu_2(t) = t^{\mu_1+\mu_2}$. If we twist I_χ by a character $g \mapsto \eta(\det(g))$ then the central character changes to $t \mapsto t^{\mu_1+\mu_2-2\eta}$.

Now it is easy to see that we have two cases where our induced representation is not irreducible. Of course we notice that reducibily is invariant under twists, and twisting by η changes $\mu_i \to \mu_i + \eta$. If it now happens that

$$\frac{\mu_1}{\mu_2}(t)|t|_p = 1 \qquad (sub)$$

then we see we can find a character $\eta : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ such that

$$\mu_1(t_1)\mu_2(t_2)|\frac{t_1}{t_2}|_p^{1/2} = \eta(t_1 t_2)$$

and then it is clear that $\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\chi$ contains the one dimensional space spanned by the function $g \mapsto \eta(\det(g))$ and this subspace is invariant. We get an exact sequence of $G(\mathbb{Q}_p)$-modules

$$0 \to \mathbb{C}\chi \circ \det \to I_\chi \to \mathrm{St}_\chi \to 0,$$

the quotient module is irreducible and it is called the Steinberg module. Perhaps we should denote it by St_η.

Let $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, this is the non trivial element in the Weyl group of T. Conjugating by it exchanges t_1 and t_2. Given χ we define $\chi^w(b) = \mu_2(t_1)\mu_1(t_2)$, then it is well known that we have an pairing

$$I_\chi \times I_{\chi^w}(-\omega_\chi) \to \mathbb{C},$$

which is defined by ($K_p = \mathrm{Gl}_2(\mathbb{Z}_p)$)

$$(f, g) \mapsto \int_{K_p} f(k)g(k)dk.$$

(One has to verify that this is invariant (See[Cass], [Go], ...).

If we have for our character χ the relation

$$\frac{\mu_2}{\mu_1}(t)|t|_p = 1 \qquad (quot),$$

then we see that χ^w satisfies (sub) and hence we get the sequence

$$0 \to \mathbb{C}\chi^w \circ \det \to I_{\chi^w} \to \mathrm{St}_{\chi^w} \to 0.$$

If we now use the pairing then we get a sequence

$$0 \to \mathrm{St}_\chi^\vee \to I_\chi \to \mathbb{C}\eta' \circ \det \to 0.$$

It turns out that

The induced representation I_χ is irreducible unless we are in the case (sub) or (quot).
The modules St_χ and $\mathrm{St}_{\chi^w}^\vee$ are irreducible and isomorphic

The Steinberg-module also exists for general reductive groups, then the situation is more complicated.

1.4.3 Unitary representations

Let us still assume that $F = \mathbb{C}$. An admissible module V for $G(\mathbb{Q}_p)$ is called *unitary or unitarylisible* if we can find a positive definite hermitian form, which is invariant under the action of $G(\mathbb{Q}_p)$. In our example above the modules I_χ are unitary, if the character χ is unitary. It is clear that for unitary characters χ we have

$$I_{\bar{\chi}} = I_{\chi^{-1}} = I_{\chi^w}(-\omega_\chi)$$

and then the above pairing yields the hermitian form. These representations are the representations of the *unitary principal series*.

This is a special case of a more general result, we refer to [Cass].

1.4.4 Supercuspidal representations

We do not get all the irreducible admissible representation by induction from smaller parabolic subgroups. We consider the category of admissible $G(\mathbb{Q}_p)$-modules of finite length. For any parabolic subgroup $P/\mathbb{Q}_p$ with unipotent radical $U/\mathbb{Q}_p$ we consider the Jacquet functor $V \to V_U$ where V_U is the quotient of V divided by the subspace, which is generated by all elements of the form $\{v - uv | v \in V, u \in U(\mathbb{Q}_p)\}$. If again $M = P/U$, then it is clear that V_U is in fact a $M(\mathbb{Q}_p)$-module. If X is any $M(\mathbb{Q}_p)$-module and if we consider it as $P(\mathbb{Q}_p)$-module, then we can define V_U by

$$\mathrm{Hom}_{P(\mathbb{Q}_p)}(V, X) = \mathrm{Hom}_{M(\mathbb{Q}_p)}(V_U, X).$$

We have the following theorem by Jacquet

Theorem:*If V is a admissible $G(\mathbb{Q}_p)$-module of finite length and if P is a parabolic subgroup, then V_U is an admissible $M(\mathbb{Q}_p)-$ module of finite length.*

For a proof and a more detailed discussion of the theory of Jacquet functors I refer to [Cass].

I want to consider the case of the representations I_χ for $\mathrm{Gl}_2(\mathbb{Q}_p)$. Let us assume that $X = \mathbb{C}\lambda$ is a one dimensional $T(\mathbb{Q}_p)$-module on which $T(\mathbb{Q}_p)$ acts by the character λ. Then we look at

$$\mathrm{Hom}_{B(\mathbb{Q}_p)}(V, \mathbb{C}\lambda) = \mathrm{Hom}_{T(\mathbb{Q}_p)}(V_U, \mathbb{C}\lambda).$$

Let us forget the character λ for a moment, we look at $U(\mathbb{Q}_p)$- invariant linear maps $\Psi : I_\chi \to \mathbb{C}$. Of course it is clear that the evaluation of $f \in I_\chi$ at the identity element gives us such a map

$$\Psi_e : f \mapsto f(e).$$

To get a second map, we restrict f to the open cell $B(\mathbb{Q}_p)wU(\mathbb{Q}_p)$. It is clear this gives us a space of functions on $U(\mathbb{Q}_p)$ on which $U(\mathbb{Q}_p)$ acts by translations. The only possible way to get a $U(\mathbb{Q}_p)$ invariant linear map to $\mathbb{C}$ is to take the integral

$$\Psi_w : f \mapsto \int_{U(\mathbb{Q}_p)} f(wu)du.$$

Then it is an easy to see that Ψ_e is a $T(\mathbb{Q}_p)$ linear map to $\mathbb{C}(\chi + |\rho_B|)$ and Ψ_w is a map to $\mathbb{C}(\chi^w + |\rho_B|)$. We can reformulate and say

$$(I_\chi)_U \xrightarrow{\sim} \mathbb{C}(\chi + |\rho_B|) \oplus \mathbb{C}(\chi^w + |\rho_B|)$$

we do not discuss what happens if the integral is not convergent and what happens if $\chi = \chi^w$ (See Chap.V ???).

We call an irreducible admissible $G(\mathbb{Q}_p)-$ module V *supercuspidal* if $V_U = 0$ for all proper parabolic subgroups $P/\mathbb{Q}_p \subset G/\mathbb{Q}_p$. If $G/\mathbb{Q}_p = T/\mathbb{Q}_p$ is a torus then all admissible modules are supercuspidal. More precisely we can say that all irreducible $G(\mathbb{Q}_p)$-modules are supercuspidal of the semi simple component $G^{(1)}/\mathbb{Q}_p$ is anisotropic.

Now we do the following: Given V we look for a parabolic subgroup $P/\mathbb{Q}_p$ such that $V_U \neq 0$. If we do not find any then V is supercuspidal. If not then we know that V_U is an $M(\mathbb{Q}_p)$ of finite length. We look at its irreducible subquotients in a Jordan Hölder series. If there is any subquotient, which is not supercuspidal we proceed and eventually we will find a reductive quotient $M_1(\mathbb{Q}_p)$ and a supercuspidal $\sigma : M_1(\mathbb{Q}_p) \to \mathrm{Gl}(W)$. Then we have the following theorem, which is also due to Jacquet:

Theorem: *If the irreducible $G(\mathbb{Q}_p)$-module V is not supercuspidal, then we can find a parabolic subgroup $P_1/\mathbb{Q}_p$ and an irreducible $M_1(\mathbb{Q}_p)-$ module $\sigma : M_1(\mathbb{Q}_p) \to \mathrm{Gl}(W)$ such that V occurs as a subquotient in $\mathrm{Ind}_{P_1(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\sigma$*

This tells us that we have two tasks:

Understand the supercuspidal representations for any given reductive group $G/\mathbb{Q}_p$ and understand the composition series of the induced $\mathrm{Ind}_{P_1(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\sigma$ representations, where σ is supercuspidal.

Coming back to the group $\mathrm{Gl}_2/\mathbb{Q}_p$ we have tackled successfully the the second task, but so far we have not seen any supercuspidal representation of $\mathrm{Gl}_2(\mathbb{Q}_p)$. We will see later that there must be some.

1.4.5 Asymptotic behavior of matrix coefficients

Let Vbe an admissible module for $G(\mathbb{Q}_p)$, let K_p be a maximal compact subgroup in $G(\mathbb{Q}_p)$. We have the sequence $\{K_p^{(\nu)}\}_{\nu=1,2,\dots}$ of congruence subgroups, where the entries are congruent $\mod p^\nu$. They are normal subgroups in K_p. Let $\hat{K}_p$ the set of isomorphism classes of irreducible representations of K_p. For any $\theta \in \hat{K}_p$ we say we have a largest ν such that $\theta|K_p^{(\nu)}$ is trivial. We get a partial order on the set $\hat{K}_p$, we say that $\theta \leq \theta'$ if the level of θ is less or equal to the level of θ'.

We have a direct sum decomposition into isotypical components under K_p

$$V = \bigoplus_{\theta \in \hat{K}_p} V(\theta)$$

and we know that $\dim V(\theta) < \infty$.

We may choose an "increasing" basis $< v_1, v_2, \dots, v_i, \dots >$ of V. This is a basis where all basis vectors lie in an isotypical component, and where we fill up the $V(\theta)$ according to their size.

We may also define the dual $V^\vee$ of our admissible module. First we consider the "full" dual space $V^* = \mathrm{Hom}_F(V, F) = \prod_\theta \mathrm{Hom}_F(V(\theta), F)$. We have the usual pairing $(v, \phi) \mapsto < v, \phi >$ between these spaces. On V^* we have an action of $G(\mathbb{Q}_p)$ for which $< gv, g\phi > = < v, \phi >$. Inside this space we consider the K_p finite vectors $\phi = (\dots, \phi_\theta, \dots)_{\theta \in \hat{K}_p}$. This subspace is our $V^\vee$, it is invariant under $G(\mathbb{Q}_p)$. For this space we have the dual basis $< \phi_1, \phi_2, \dots, \phi_i, \dots >$.

We notice that the central characters of V and $V^\vee$ are inverse to each other.

If we pick a v_i and a $g \in G(\mathbb{Q}_p)$ then we can write by definition

$$gv_i = \sum_i < gv_i, \phi_j > v_j,$$

the sum is finite. The functions $c_{i,j} : G(\mathbb{Q}_p) \to F, c_{i,j}(g) = < gv_i, \phi_j >$ are called the matrix coefficients of the module V. The space of functions generated by these matrix coefficients is independent of the choice of the basis. We are interested to understand the "asymptotic" behavior of the matrix coefficients if $g \to \infty$, of course we have to say first, what we mean by that.

Let us consider the case $G/\mathbb{Q}_p = \mathrm{Gl}_2/\mathbb{Q}_p$. The group $K_p = \mathrm{Gl}_2(\mathbb{Z}_p)$. We consider a matrix coefficient $g \mapsto < gv_i, \phi_j >$, where we assume that $v_i \in V(\theta)$ and $\phi_j \in V^\vee(\theta')$. Then it is clear that for $k_1, k_2 \in K_p$ we have

$$< k_1 g k_2 v_i, \phi_j > = < g k_1 v_i, k_2^{-1} \phi_j > = \sum_{\nu,\mu} b_{i,\nu}(k_1) c_{\mu,j}(k_2^{-1}) < g v_\nu \phi_\mu >$$

where V_ν, ϕ_μ run over the finite set of basis elements in $V(\theta), V(\theta'$.

We think, that we understand the coefficients $b_{i,\nu}(k_1) c_{\mu,j}(k_2^{-1})$. Since we have the well known Cartan decomposition (elementary divisor theorem)

$$\mathrm{Gl}_2(\mathbb{Q}_p) = K_p A K_p \text{ where } A = \{ \begin{pmatrix} p^a & 0 \\ 0 & p^b \end{pmatrix} | a \geq b \},$$

we denote the elements in A by t and we want to understand the functions

$$t \mapsto < tv_i, \phi_j > .$$

If our module is absolutely irreducible, then it has a central character ω and we can restrict our attention to $t = \begin{pmatrix} p^a & 0 \\ 0 & 1 \end{pmatrix}$ with $a \geq 0$. We are interested to know what happens if $a \to \infty$, this means $|t|_p \to 0$. Since it is so easy I prove the following theorem for our special case

Theorem: *The matrix coefficients of an absolutely irreducible supercuspidal representation have compact support modulo the center.*

Proof: (For Gl_2) Let us look at $< tv_i, \phi_j >$. Since our module is supercuspidal we can write v_i as a finite sum

$$v_i = \sum (\mathrm{Id} - \begin{pmatrix} 1 & u_\nu \\ 0 & 1 \end{pmatrix}) v_\mu,$$

and then

$$< t \sum (\mathrm{Id} - \begin{pmatrix} 1 & u_\nu \\ 0 & 1 \end{pmatrix}) v_\mu, \phi_j >=< \sum (\mathrm{Id} - \begin{pmatrix} 1 & tu_\nu \\ 0 & 1 \end{pmatrix}) tv_\mu, \phi_j >=$$

$$\sum < tv_\mu, (\mathrm{Id} - \begin{pmatrix} 1 & -tu_\nu \\ 0 & 1 \end{pmatrix}) \phi_j > .$$

But now tu_μ converges to zero, hence $(\mathrm{Id} - \begin{pmatrix} 1 & -tu_\nu \\ 0 & 1 \end{pmatrix}) \phi_j >= 0$ for $|t|_p << 1$. This proves the assertion.

Actually we can do better, we illustrate this again in the case $\mathrm{Gl}_2/\mathbb{Q}_p$, for the detailed argument I refer to [Cass]. Our module is I_χ. We send our element v_i into V_U then it will be a linear combination $\bar{v}_i = c_1 e_\chi + c_w e_{\chi^w}$, where the $e_?$ are basis vectors of the eigenspaces. Then $t\bar{v}_i = c_1 e_\chi t^{\chi+|\rho|} + c_w e_{\chi^w} t^{\chi^w+|\rho|}$ and this suggests that we get by a similar argument as above: There are constants $\tilde{c}_?$ such that

$$< tv_i, \phi_j >= \tilde{c}_1 t^{\chi+|\rho|} + \tilde{c}_w t^{\chi^w+|\rho|}$$

for all t with $|t|_p << 1$. This is indeed true and shown in [Cass].

We are interested in this asymptotic behavior, because we want to know whether for two irreducible admissible modules V, W, with equal central characters, expressions of the form

$$\int_{G(\mathbb{Q}_p)/Z(\mathbb{Q}_p)} < gv_i, \phi_j >< w_\nu, g\psi_\mu > dg \qquad (scalprod)$$

make sense. This is of course clear provided one of the two modules is supercuspidal.

If neither of them is supercuspidal, then we assume that our field F is $\mathbb{C}$. We assume that the central character is unitary, we can achieve this by twisting. Let us assume in addition that our character χ is unitary, i.e. it takes values in the unit circle S^1. Then we see that the absolute value $| < tv_i, \phi_j > |_\mathbb{C} \leq C_1 |t|_p^{1/2}$, this gives us the rate of decay if $|t|_p \to 0$. This gets worse if χ is not unitary, in other words if $|\chi(\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix})|_\mathbb{C} \neq 1$, because one of the terms decays slower. We will come back to that further down.

If we consider the surjective map $m : K_p \times A \times K_p \to G(\mathbb{Q}_p)$ then the inverse of the Haar measure on $G(\mathbb{Q}_p)$ is

$$m^*(dg) = a^{-2|\rho_B|} dk_1 d^* a dk_2,$$

where dk_i are the Haar measures on K_p and d^*a is the multiplicatively invariant measure. In simple terms we get that the volume of a double coset $K_p \begin{pmatrix} p^a & 0 \\ 0 & 1 \end{pmatrix} K_p$ is p^a. Hence we see that the integral of the product of two matrix coefficients as above is given by

$$\int_{K_p \times K_p} C(k_1,k_2) dk_1 dk_2 \sum_{a=0}^{\infty} p^a < \begin{pmatrix} p^a & 0 \\ 0 & 1 \end{pmatrix} v_i, \phi_j >< w_\nu, \begin{pmatrix} p^a & 0 \\ 0 & 1 \end{pmatrix} \psi_\mu >,$$

where the function $C(k_1,k_2)$ is computed from the product of matrix coefficients of θ and θ' and does not depend on a. Hence if our two modules $V = I_\chi, W = I_{\chi'}$ then the rate of decay of the product of the matrix coefficients and the factor p^a cancel and we should not expect convergence. If χ, χ' are unitary, then we get $\sum_{a=0}^{\infty} 1 = \infty$. If one of the characters is not unitary the situation becomes worse.

But there is one important exception. We saw that besides the supercuspidal representations and the irreducible I_χ we have the case that χ satisfies (*sub*) or (*quot*) and then we get a one dimensional module $\mathbb{C}\eta \circ \det$ and the Steinberg module St_η. If we now look at the asymptotic behavior of the matrix coefficients or what amounts to the same at the values of the Jacquet functor then we see, that

$$\mathbb{C}\eta \circ \det_U = \mathbb{C}\eta \circ \det, \mathrm{St}_U = \mathbb{C}(\eta \circ \det + 2|\rho_B|).$$

Now one checks easily, that under the assumption that the central character is unitary, it follows that η is unitary, i.e. we have $|\eta(p^a)|_{\mathbb{C}} = 1$, the matrix coefficient $\mathbb{C}\eta \circ \det$ does not decay at infinity. But in turn we see that $|\eta(p^a)|2|\rho_B|(\begin{pmatrix} p^a & 0 \\ 0 & 1 \end{pmatrix})|_p = p^{-a}$ decays faster.

It is not fast enough to give that the matrix coefficients of a unitary Steinberg module is integrable. But it is fast enough to show that the scalar product in (*scalprod*) converges, if one of the factors is a Steinberg module and the other one is in the unitary principal series. If the first factor is the matrix coefficient of a Steinberg module, then the only case, where we do not have convergence of (*scalprod*) is when the second factor is the matrix coefficient of a one dimensional representation. There is a remedy in this case, we can replace the matrix coefficient by the Euler-Poincare function (See 2.5)

1.5 Orthogonality relations for matrix coefficients

In the theory of representations of a finite (or more generally compact group) G the orthogonality relations play a fundamental role. They can be derived rather easily from the lemma of Schur. Since they involve the integration over the group they can not be true word for word here.

But they are also valid in the theory of admissible representations of $G(\mathbb{Q}_p)$ if take some precautions and we restrict ourselves to a certain subclass of representations, namely the representations of the discrete series (see further down). The supercuspidal representation belong to this class, but there will be more. For instance the Steinberg modules St_χ belong to this class.

Let V, W be two absolutely irreducible, admissible $G(\mathbb{Q}_p)$-modules over some field L of characteristic zero, let us assume, that the have the same central character. We try to write down an intertwining operator $L : V \to W$, i.e. a linear map, which is compatible with the action of the group. To do this we write down any linear map $\Psi : V \to W$ and then we consider

$$\int_{G(\mathbb{Q}_p)/Z(\mathbb{Q}_p)} g^{-1}\Psi(gv)dg,$$

if this integral makes any sense, then this will be an element

$$\int \Psi \in \mathrm{Hom}_{G(\mathbb{Q}_p)}(V, W).$$

We want to find out under what conditions this integral makes sense. We choose "increasing" basises on both spaces

$$< v_1, v_2, \ldots, v_i, \cdots >, < w_1, w_2, \ldots, w_i, \cdots > .$$

We also introduce the dual basises $< \phi_1, \phi_2, \cdots >, < \psi_1, \psi_2, \cdots >$ on the contragredient spaces $V^\vee, W^\vee$. Now we choose our starting linear map such that it has finite support, it vanishes on basis vectors with a high index.

Then we have $gv_i = \sum_j < gv_i, \phi_j > v_j$ and $\Psi(gv_i) = \sum_j < gv_i, \phi_j > \Psi(v_j)$ where the last sum is finite. Then

$$g^{-1}\Psi(gv_i) = \sum_j < gv_i, \phi_j > g^{-1}\Psi(v_j) = \sum_{j,\nu} < gv_i, \phi_j >< g^{-1}\Psi(v_j), \psi_\nu > w_\nu =$$

$$\sum_{j,\nu} < gv_i, \phi_j >< \Psi(v_j), g\psi_\nu > w_\nu$$

The central character drops out, we see that we can perform the integration, if one of the modules is supercuspidal and therefore has matrix coefficients with compact support, or if both matrix coefficients are in L^2. We write down the integral and get $gv_i = \sum_j < gv_i, \phi_j > v_j$ and $\Psi(gv_i) = \sum_j < gv_i, \phi_j > \Psi(v_j)$ where the last sum is finite. Then

$$\int_{G(\mathbb{Q}_p)/Z(\mathbb{Q}_p)} g^{-1}\Psi(gv_i)dg = \sum_{j,\nu} \int_{G(\mathbb{Q}_p)/Z(\mathbb{Q}_p)} < gv_i, \phi_j >< \Psi(v_j), g\psi_\nu > dg\; w_\nu.$$

Now we have two possibilities: The two modules may be non isomorphic, then the operator $\int \Psi$ must be zero for any choice of Ψ. We can choose our $\Psi = E_{j,\mu}$ where this linear map sends v_j to w_μ and all other basis elements to zero. Then we get

Let V, W be two non isomorphic admissible absolutely irreducible $G(\mathbb{Q}_p)$ modules, one of them is supercuspidal or both are in L^2 and if our basises are chosen as above, then

$$\int_{G(\mathbb{Q}_p)/Z(\mathbb{Q}_p)} < gv_i, \phi_j >< w_\mu, g\psi_\nu > dg = 0 \textit{ for all } i, j, \mu, \nu.$$

If now V, W are isomorphic, then we choose an isomorphism, this means we assume that $V = W$. Of course the we also choose the $w_i = v_i$, $\phi_i = \psi_i$.

The same reasoning yields that $\int \Psi(v_i) = l(\psi)v_i$ for all i, because the operator must be a scalar(Schurs lemma). Again we apply this to $\Psi = E_{j,\mu}$ and get

$$\int_{G(\mathbb{Q}_p)/Z(\mathbb{Q}_p)} < gv_i, \phi_j >< v_\mu, g\phi_\nu > dg = l(E_{j,\mu})\delta_{i,\nu}.$$

But now we can rewrite the integral, we move the g inside the $<\ ,\ >$ to the other side and use the invariance of the measure, i.e. $dg = dg^{-1}$. Then we see that our integral is also equal to $l(E_{i,\nu})\delta_{j,\mu}$. We conclude that $l(E_{j,\mu}) = 0$ if $j \neq \mu$ and $l(E_{j,j})$ is independent of j. This constant depends only on the isomorphism class π_p of our irreducible module V and the choice of the measure, it is non zero and its inverse is called the formal degree $d(\pi_p)$. Hence we get in the case $V = W$ the relation

$$\int_{G(\mathbb{Q}_p)/Z(\mathbb{Q}_p)} < gv_i, \phi_j >< v_\mu, g\phi_\nu > dg = \frac{1}{d(\pi_p)}\delta_{i,\nu}\delta_{j,\mu}.$$

Here we always have to be aware that the formal degree depends on the choice of a measure on $G(\mathbb{Q}_p)$. If compare these orthogonality relations to the orthogonality relations for finite (or compact groups) G then we have a natural choice of such a measure, namely the one that gives volume 1 to G. In that case we have that all irreducible representations are finite dimensional, we get exactly the same relations and the formal degree turns out to be the dimension of the irreducible representation.

But here we do not have an obvious candidate for a normalized measure. We may choose a maximal compact subgroup $\bar{K}_p \subset G(\mathbb{Q}_p)/Z(\mathbb{Q}_p)$ in give volume one to it. But in general these maximal compact subgroups are not necessarily conjugate, so there is still a choice. (See section on Bruhat-Tits buildings).

1.5.1 The formal degree of the Steinberg module

It is not so difficult to compute the formal degree of the Steinberg module

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1.5.2 Characters of principal series representations

They can be written rather explicitely in terms of the inducing data, we can verify Harish-Chandras theorem directly

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....

2 Orbital integrals

2.1 Conjugacy classes

We consider an algebraic group G/k over an arbitrary field k of characteristic zero. For any $g \in G(\bar{k})$ we can consider the set $\{x^{-1}gx | x \in G(\bar{k})\}$. It is known that this set is the set of $\bar{k}$ valued points of a locally closed subvariety $C[g]$ which is defined over $\bar{k}$. If this set $C[g](\bar{k})$ is invariant under the action of the

Galois group $\mathrm{Gal}(\bar{k}/k)$ then $C[g]$ is called a rational conjugacy class and the locally closed subvariety is defined over k. Sometimes we denote such a rational conjugacy class simply by C/k, i.e. we drop the reference to the point g. We should be aware that it can happen that $C(k) = \emptyset$ and in general the action of $G(k)$ on $C(k)$ will not be transitive (See also Chap V).

I suggest to call these classes geometric conjugacy classes, if such class C is rational then we call the set $C(k)$ of rational points a geometric conjugacy class, even if it is empty. Such a geometric conjugacy class decomposes into several conjugacy classes under the action of $G(k)$. These conjugacy classes will be called arithmetic conjugacy classes, they are of course the $G(k)$ conjugacy classes of the abstract group $G(k)$.

If we consider the special case of the group Gl_2/k then we can consider the trace and the determinant, this gives us a map

$$(\det, \mathrm{tr}) : \mathrm{Gl}_2 \to \mathbb{G}_m \times \mathbb{A}^1.$$

This map is constant on the geometric conjugacy classes. In our special situation the central elements map to the subset $\mathcal{Z} \subset \mathbb{G}_m \times \mathbb{A}^1$ which is defined by the equation $\mathcal{Z} = \{(y, x) | x^2 = 4y\}$. The open subset $\mathbb{G}_m \times \mathbb{A}^1 \setminus \mathcal{Z} = (\mathbb{G}_m \times \mathbb{A}^1)_{\mathrm{reg}}$ is called the set of regular classes. The inverse image of this set is the Zariski-open dense subset $\mathrm{Gl}_{2\mathrm{ss,reg}}$ set of semi simple regular elements. For a regular point $a \in \mathbb{G}_m \times \mathbb{A}^1(\bar{k})$ the fiber $C_a = (\det, \mathrm{tr})^{-1}(a)$ consists of semi-simple elements and this is a conjugacy class. The inverse image of these conjugacy classes are closed, such a class C_a is defined over k if and only if $a \in k$. We can say that $(\mathbb{G}_m \times \mathbb{A}^1)_{\mathrm{reg}}$ is the quotient of $\mathrm{Gl}_{2\mathrm{ss,reg}}$ by the action of conjugation. So the set of semi simple, regular classes has the structure of an affine algebraic variety.

For a central element $z \in \mathcal{Z}(\bar{k})$ the fiber $(\det, \mathrm{tr})^{-1}(a)$ is not a a conjugacy class. If $z = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$ is an element in the center then

$$(\det, \mathrm{tr})^{-1}((z^2, 2z)) = C[\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}] \cup C[\begin{pmatrix} z & 1 \\ 0 & z \end{pmatrix}]$$

is the union of two conjugacy classes. The second class is called a unipotent class and it is not closed. Its closure is the fiber.

We can define the subset $\mathrm{Gl}_{2\mathrm{reg}}$ as the subset of elements whose centralizer has dimension one, in this case this is simply the complement of the center. Then we see that we get a smooth morphism

$$(\det, \mathrm{tr}) : \mathrm{Gl}_{2\mathrm{reg}} \to \mathbb{G}_m \times \mathbb{A}^1 \ .$$

We return to the case of a general reductive group G/k. Inside G we have the Zariski open subset of regular semi-simple elements $G_{\mathrm{ss,reg}}$. It is true in general that the quotient by the adjoint action $G_{\mathrm{ss,reg}}/\sim_{\mathrm{conj}} = \mathcal{C}_{\mathrm{reg}}$ is an affine variety. The fibers of

$$\pi : G_{\mathrm{ss,reg}} \to \mathcal{C}_{\mathrm{reg}}$$

are the semi-simple regular conjugacy classes.

To see this in general we may start from a maximal torus T/k. Then it is well known that any semi simple regular element $x \in G(\bar{k})$ is conjugate to a regular element $t \in T(\bar{k})$, two elements in $T(\bar{k})$ are conjugate if they are conjugate by an element in the Weyl group W. This suggests the definion of a morphism

$$\pi : T \times_k T\backslash G \to G, \pi : (t,x) \mapsto x^{-1}tx.$$

Inside T/k we have the open subset of regular elements T_{reg}. Inside G we have the regular semi-simple elements $G_{\text{ss,reg}}$. It is well known (and a basic fact in linear algebra for Gl_n) that the restriction

$$T_{\text{reg}} \times_k T\backslash G \to G_{\text{ss,reg}}$$

is an etale covering whose Galois group is the Weyl group W. We may reformulate this by saying that $\pi_T : T_{\text{reg}} \to \mathcal{C}_{\text{reg}}$ is an etale covering with Galois group W. We take the fiber product of π_T and the morphism p, then our morphism π as provides an isomorphism

$$T_{\text{reg}} \times_{\mathcal{C}_{\text{reg}}} G_{\text{ss,reg}} \xrightarrow{\sim} T_{\text{reg}} \times_k T\backslash G,$$

we have a trivialization of the G bundle $G_{\text{ss,reg}} \to \mathcal{C}_{\text{reg}}$ after taking the pull back over π_T.

Finally I want to remark that the center of the group acts by translations on the group and induces an action of the center on the set of conjugacy classes. Any connected reductive group G/k has a finite covering

$$\pi : S \times_k G^{(1)} \to G,$$

where S/k is the connected component of the center, and where $G^{(1)}$ is a simply connected cover over the derived group of the center. The kernel μ of π is a finite group scheme of multiplicative type. Then we get a morphism

$$\pi^{(1)} : S \times_k G^{(1)} \to S \times \mathbb{A}^r,$$

where the first component is the identity and the second component is given by the traces of the fundamental representations (we have to assume that G/k is an inner form). This gives us also a morphism

$$\pi : G \to S \times \mathbb{A}^r/\mu.$$

We denote the quotient by $\mathcal{C} = \mathcal{C}_G$, inside this quotient we have the open subset $\mathcal{C}_{\text{reg}}$ of regular elements, the fibers $\pi^{-1}(s)$ for s regular are the regular semi simple conjugacy classes. So we can not say that $\mathcal{C}$ is the variety of conjugacy classes, this object does not exist. But we can say that $\mathcal{C}_{\text{reg}}$ is the variety of semi simple conjugacy classes.

If our group $G/k = \mathrm{Gl}_n/k$ then the covering is

$$\pi : G_m \times_k \mathrm{Sl}_n,$$

which sends $(z,g) \mapsto \mathrm{diag}(z)g$ and

$$\pi : g \mapsto (\det(g), a_1(g), \ldots, a_{n-1}(g))$$

where the a_i are the coefficients of the characteristic polynomial.

In this case we know that for a regular element $s \in \mathcal{C}(k)$ the fiber $\pi^{-1}(s)$ is a conjugacy class under $G(k)$.

2.2 Orbital integrals

A conjugacy class $C[g]$ is always smooth, its tangent space in the point g can be identified to the quotient of Lie algebras $\mathfrak{a} = \mathrm{Lie}(G)/\mathrm{Lie}(Z_g)$, where Z_g is the centralizer of g. The centralizer Z_g acts by the adjoint action on $\mathfrak{a}$. If we now assume that our group G/k is semi-simple then it is known that Z_g acts trivially on the highest exterior power $\Lambda^{\dim \mathfrak{a}}\mathfrak{a}$. This implies that we can find a non zero invariant form ω_C of highest degree on $C = C[g]$. If now our ground field is $\mathbb{Q}_p$ (or any p-adic field) then a $\mathbb{Q}_p$ rational conjugacy class is C provides a p-adic manifold $C(\mathbb{Q}_p)$. If we select a non zero ω_C on C, then this defines a measure $|\omega_C|$ on $C(\mathbb{Q}_p)$. For any function h in the Hecke algebra we can consider the integral

$$O(h, C) = \int_{C(\mathbb{Q}_p)} h(x)|\omega_C|(dx).$$

This is a so called *stable orbital integral.* It is clear that this integral is convergent if the conjugacy class is closed, because we assumed that h has a compact support. We will discuss the convergence problem for non closed orbits later.

We want to consider these integrals as functions in the variable C. This requires to put some structure on the set of conjugacy classes and then to make a consistent choice of the measures ω_C, which is compatible with this structure. We return to the considerations in 2.1. and consider the morphism π.

For any geometric point (t, x) the derivative induces an isomorphism of tangent spaces

$$D_\pi : \mathcal{T}_{T,t} \oplus \mathcal{T}_{T\backslash G,x} \xrightarrow{\sim} \mathcal{T}_{G,x^{-1}tx}$$

If $g = x^{-1}tx$ then we can identify $\mathcal{T}_{G,g} = \mathfrak{g}$ by using the right translation by g. On the left hand side we also have an action of $T \times_k G$ from the right and we get an identification

$$\mathfrak{t} \oplus \mathfrak{a} = \mathrm{Lie}(T) \oplus \mathrm{Lie}(G)/\mathrm{Lie}(T) \xrightarrow{\sim} \mathcal{T}_{T,t} \oplus \mathcal{T}_{T\backslash G,x} \xrightarrow{\sim} \mathcal{T}_{G,x^{-1}tx},$$

This identification of $\mathfrak{a}$ with $\mathcal{T}_{T\backslash G,x}$ depends on the the first variable t.

On $T \times_k T\backslash G$ we get a $T \times_k G$ invariant form, if we choose top degree forms $\omega_T, \omega_{T\backslash G}$ on $\mathfrak{t}$,$\mathfrak{a}$ and extend them by translations to invariant forms on T and $T\backslash G$, which we can pull back to forms on $T \times_k T\backslash G$. On G we get an invariant form in top degree if we choose a top degree form ω_G on $\mathfrak{g}$. We assume that we adapted these forms, this means on the Lie algebra we have $\omega_T \otimes \omega_{T\backslash G} = \omega_G$. Then we will get a formula on the set of regular, semi simple elements $\pi^*(\omega_G) = f\omega_T \wedge \omega_{T\backslash G}$, where f is a regular function on $T \times_k T\backslash G$. A formula for this function is easy to find. The element t defines via the adjoint action an automorphism $\mathrm{Ad}(t) : \mathfrak{a} \to \mathfrak{a}$. Since t is regular this automorphism does not have the eigenvalue 1, in other words $-\mathrm{Ad}(t) + \mathrm{Id}$ is an isomorphism of $\mathfrak{a}$. Then we get Weyls formula for the invariant differential forms

$$\pi^*(\omega_G) = \pm \det(-\mathrm{Ad}(t) + \mathrm{Id})\omega_T \wedge \omega_{T\backslash G}.$$

We introduce the notation

$$D(t) = \det(-\mathrm{Ad}(t) + \mathrm{Id}).$$

Now our ground field is $\mathbb{Q}_p$ we have our reductive group $G/\mathbb{Q}_p$ and a maximal torus $T/\mathbb{Q}_p$ in this group, then we can consider the map on the $\mathbb{Q}_p$ valued points

$$T(\mathbb{Q}_p) \times T(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p) \to G(\mathbb{Q}_p).$$

In the regular points this map is a local isomorphism for the p-adic topology these are the points $(t,x) \in T_{\mathrm{reg}}(\mathbb{Q}_p) \times T(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)$.

The image consists of those regular semi simple elements, which are conjugate under $G(\mathbb{Q}_p)$ to an element in $T_{\mathrm{reg}}(\mathbb{Q}_p)$. This is an open subset $T_{\mathrm{reg}}(\mathbb{Q}_p)^G \subset G(\mathbb{Q}_p)$. The number of elements in the fiber over a point in $T_{\mathrm{reg}}(\mathbb{Q}_p)^G$ is $[N_T(\mathbb{Q}_p) : T(\mathbb{Q}_p)]$, where N_T is the normalizer of T. The choice of a top degree form $\omega_{T\backslash G}$ on $\mathfrak{a}$ and the resulting invariant form on $T\backslash G$, which we denote by the same letter yields an invariant measure$|\omega_{T\backslash G}|$ on $T\backslash G(\mathbb{Q}_p)$. One knows- and we will later in the sections on stabilization- that $T(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)$ is an open subset in $(T\backslash G)(\mathbb{Q}_p)$. (Later I will remove the brackets $T\backslash G = (T\backslash G)$.)

We get Weyls formula: For any locally constant compactly supported function h on $G(\mathbb{Q}_p)$

$$\int_{T_{\mathrm{reg}}(\mathbb{Q}_p)^G} h(g)|\omega_G|(dg) =$$
$$\frac{1}{[N_T(\mathbb{Q}_p) : T(\mathbb{Q}_p)]} \int_{T(\mathbb{Q}_p)} |D(t)|_p \big(\int_{T(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)} h(x^{-1}tx)|\omega_{T\backslash G}|(dx)\big)|\omega_T|(dt).$$

Here we have to observe a very subtle point. I mentioned already that $G(\mathbb{Q}_p)$ does not act transitively on $T\backslash G(\mathbb{Q}_p)$, we will see in Chap.V that we have a finite number of (open) orbits.

If we pick any $t \in T_{\mathrm{reg}}(\mathbb{Q}_p)$ then we get an isomorphism of algebraic varieties

$$T\backslash G \xrightarrow{\sim} C_t$$

where C_t is the semi simple regular conjugacy class containing t. Our differential form $\omega_{T\backslash G}$ yields an differential form of highest degree $\tilde{\omega}_{T\backslash G}$ on C_t. If d is the dimension of C_t, then the object $\tilde{\omega}_{T\backslash G}$ is actually a relative invariant form in $\Omega^d_{G_{\mathrm{ss,reg}}/\mathcal{C}_{\mathrm{reg}}}(G_{\mathrm{ss,reg}})$. By definition the linear map

$$(-\mathrm{Id} + \mathrm{Ad}(t)) : \mathrm{Lie}(G)/\mathrm{Lie}(T) \to \mathcal{T}_{\mathcal{C}_t,t}$$

is an isomorphism and

$$\omega_{T\backslash G}(X_1, X_2, \ldots, X_d) = \tilde{\omega}_{T\backslash G}(t)((-\mathrm{Id} + \mathrm{Ad}(t))(X_1), \ldots, (-\mathrm{Id} + \mathrm{Ad}(t))(X_d))$$

This relative differential form defines an invariant measure $|\tilde{\omega}_{T\backslash G}|$ on $C_t(\mathbb{Q}_p)$ and now we look again at the integral

$$O(h, C_t) = \int_{C_t(\mathbb{Q}_p)} h(x)|\tilde{\omega}_{T\backslash G}|(dx).$$

This is again the stable orbital integral, but now the measure $|\tilde{\omega}_{T\backslash G}|$ depends in a consistent way on t, it is clear that we get a locally constant function in the variable $t \in T_{\mathrm{reg}}(\mathbb{Q}_p)$. Finally we observe that these orbital integrals are invariant under the action of the center. If we translate h by an element $z \in Z(\mathbb{Q}_p)$, i.e. $L_z(h)(g) = h(zg)$, then

$$O(L_z(h), t) = O(h, zt)$$

we say that these orbital integrals are homogenous.

We can go one step further. The top degree alternating form ω_G on $\mathfrak{g}$ can be chosen once for all. If we pick a maximal torus $T/\mathbb{Q}_p$ we also choose ω_T. We say that two maximal tori $T, T_1/\mathbb{Q}_p$ are in the same *inner splitting class*, if we can find an element $g \in G(\bar{\mathbb{Q}}_p)$, which conjugates T into T_1 in such a way that the isomorphism $\mathrm{Ad}(g) : T \xrightarrow{\sim} T_1$ is defined in $\mathbb{Q}_p$.

If now $x \in T\backslash G(\mathbb{Q}_p)$ then the stabilizer of x in $G/\mathbb{Q}_p$ is a maximal torus $T_x/\mathbb{Q}_p$. This torus T_x is not necessarily conjugate to $T/\mathbb{Q}_p$ by an element in $G(\mathbb{Q}_p)$. But we can find an element $g \in G(\bar{\mathbb{Q}}_p)$, which maps to x under the projection $G \to T\backslash G$. Then clearly $T_x = g^{-1}Tg$ and the isomorphism $\mathrm{Ad}(g) : T \xrightarrow{\sim} T_x$ is defined over $\mathbb{Q}_p$. Hence we see, that the maximal tori over $\mathbb{Q}_p$ which in the same inner class as a given torus $T/\mathbb{Q}_p$ are all of the form T_x with $x \in T\backslash G(\mathbb{Q}_p)$.

But the correspondence $x \to T_x$ is not one-to-one. We consider the Weyl group $W_T = N(T)/T$, this is a finite algebraic group over $\mathbb{Q}_p$. Let $W(\mathbb{Q}_p)$ be its group of rational points. Here is a caveat: In general we may have strict inclusions $N(T)(\mathbb{Q}_p)/T(\mathbb{Q}_p) \subset W_T(\mathbb{Q}_p) \subset W_T(\bar{\mathbb{Q}}_p)$. If we have chosen ω_T and if $T_1/\mathbb{Q}_p$ is in the same inner splitting class, then choose a $g \in G(\bar{\mathbb{Q}}_p)$ which defines a $\mathbb{Q}_p$ isomorphism $\mathrm{Ad}(g) : T \xrightarrow{\sim} T_1$ and get a corresponding ω_{T_1}. This ω_{T_1}, is unique up to a sign(the Weyl group $W_T(\bar{\mathbb{Q}}_p)$ has a non trivial sign homomorphism whose restriction to $W_T(\mathbb{Q}_p)$ can be non trivial. This means that for all $T_1/\mathbb{Q}_p$ in a given inner splitting class we can choose forms ω_{T_1} of top degree, which are determined up to a sign. Therefore we also get a choice of quotient forms $\omega_{T_1\backslash G}$, which are unique up to a sign.

This implies

If we choose an ω_T on one of the tori in an inner splitting class, then we get a consistent family of measures $|\omega_{T_1\backslash G}|$ on the quotients $T_1\backslash G(\mathbb{Q}_p)$ for all T_1 inner the same inner splitting class. By consistent we mean: Let $t \in T_1(\mathbb{Q}_p)$ be a regular element. It defines a regular conjugacy class $C = C(t)$. We get G-isomorphism

$$T_1\backslash G \xrightarrow{\sim} C.$$

Via this isomorphism we get a measure $|\tilde{\omega}_{T_1\backslash G}|$ on $C(\mathbb{Q}_p)$, this measure only depends on the choice of ω_T and not on the choice of T_1 or $t_1 \in T_1(\mathbb{Q}_p)$

If we pick a system of representatives $y_1, \dots, y_r$ for the orbits of $G(\mathbb{Q}_p)$ on $T\backslash G(\mathbb{Q}_p)$ then we can write

$$O(h, C_t) = \int_{C_t(\mathbb{Q}_p)} h(x)|\tilde{\omega}_{T\backslash G}|(dx) = \sum_{y_i} \int_{T_{y_i}(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)} h(x^{-1}tx)|\omega_{T\backslash G}|(dx))$$

our stable orbital integral is a sum over orbital integrals.

2.3 Measures on tori

If we have a torus $T/\mathbb{Q}_p$, then have two natural choices for an invariant measure on $T(\mathbb{Q}_p)$. First of all we recall, that $T(\mathbb{Q}_p)$ contains a unique maximal compact subgroup, which is open, we call it $T(\mathbb{Z}_p)$. This suggests to choose a form $\omega_T^{\mathrm{arith}}$ in such a way that

$$\int_{T(\mathbb{Z}_p)} |\omega_T^{\mathrm{arith}}|(dt) = 1.$$

This is not always the optimal choice. Another construction uses the concept of schemes. It is well known, that we can extend $T/\mathbb{Q}_p$ to a flat group scheme $\mathcal{T}/\mathbb{Z}_p$ such that $\mathcal{T}(\mathbb{Z}_p) = T(\mathbb{Z}_p)$. This extension is unique, and and we have its Lie-algebra $\mathfrak{t} = \mathrm{Lie}(\mathcal{T})$. The extension is smooth, if the torus splits over a tamely ramified extension. In general it may be not smooth, this means that $\mathfrak{t}$ may have torsion. If we divide by the torsion, then we get a free $\mathbb{Z}_p$ module of rank $r = \dim(T)$. Then $\Lambda^r(\mathfrak{t})$ is free of rank 1, we choose a form ω_T which has value one on a generator. Since this form is unique up to a unit in $\mathbb{Z}_p$, we get a second measure $|\omega_T|$, which is also canonically defined. If the torus splits over an unramified extension, then the measure ω_T behaves well under base extension.

It is not difficult to compare these two measures. In the simple case, where $T/\mathbb{Q}_p$ splits over an unramified extension $F/\mathbb{Q}_p$, the extension $\mathcal{T}/\mathbb{Z}_p$ is still a torus. Then the reduction $\mathcal{T} \times \mathbb{F}_p$ is a torus over the finite field $\mathbb{F}_p$ and we have simple formulas for the number of points of $\mathcal{T}(\mathbb{F}_p)$ in terms of the action the action of the Frobenius on the character module $X^*(T)$. We find

$$\int_{\mathcal{T}(\mathbb{Z}_p)} |\omega_T|(dt) = \frac{\#\mathcal{T}(\mathbb{F}_p)}{p^d}$$

In the other case we compute the orders the finite groups $\mathcal{T}(\mathbb{Z}_p)/(p^s))$. It is not difficult to see that the ratio $\#\mathcal{T}(\mathbb{Z}_p/(p^s))/p^{sd}$ becomes constant for $s >> 0$ and then we have

$$\int_{\mathcal{T}(\mathbb{Z}_p)} |\omega_T|(dt) = \frac{\#\mathcal{T}(\mathbb{Z}_p/(p^s))}{p^{sd}}.$$

If the torus splits over a tamely ramified extension, then we may take $s = 1$.

If we now fix a rule to choose a measure on any maximal torus, we decide for the second option. Then we have a choice of an invariant measures on all the regular semi simple conjugacy classes. The complement of the set of regular semi simple elements in $G(\mathbb{Q}_p)$ is a set of measure zero.

We can of course write down a very "natural" measure on $\mathcal{C}(\mathbb{Q}_p)$. We go back to 2.1. where we identified $\mathcal{C}$ to $S \times \mathbb{A}^r/\mu$. First of all we have can write the form

$$\omega_{\mathcal{C}} = \omega_S \wedge dx_1 \wedge \ldots dx_r$$

on $S \times \mathbb{A}^r$, there is still some arbitrariness in the choice of ω_S. This form is not invariant under the action of μ, but it is clear that is is clear that $|\omega_{\mathcal{C}}|$ descends to a unique measure also called $|\omega_{\mathcal{C}}|$ on $\mathcal{C}$. Now we have of course the morphism

$\pi_T : T \to \mathcal{C}$ which is finite. I claim that there is a constant a_T such that at a point $t \in T(\mathbb{Q}_p)$ we get

$$\pi_T^*(|\omega_{\mathcal{C}}|)(t) = a_T |D(t)|_p^{-1/2} |\omega_T|(t).$$

To any $c \in \mathcal{C}(\mathbb{Q}_p)_{\mathrm{reg}}$ corresponds an inner splitting class of tori, we define $a(c) = a_T$ where $T/\mathbb{Q}_p$ is any torus in this splitting class.

Then Weyls formula becomes

$$\int_{G(\mathbb{Q}_p)} h(g)|\omega_G|(dg) =$$

$$\sum_T \frac{1}{[N_T(\mathbb{Q}_p) : T(\mathbb{Q}_p)]} \int_{T(\mathbb{Q}_p)} |D(t)|_p \big(\int_{T(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)} h(x^{-1}tx)|\omega_{T\backslash G}|(dx) \big) |\omega_T|(dt) =$$

$$\int_{\mathcal{C}(\mathbb{Q}_p)_{\mathrm{reg}}} a(c)|D(c)|_p^{1/2} \int_{\pi^{-1}(c)(\mathbb{Q}_p)} h(x)|\tilde{\omega}_{T\backslash G}|(dx)|\omega_{\mathcal{C}}|(dc)$$

A want to discuss the constants for the case of tori, which are attached to quadratic extensions $F/\mathbb{Q}_p$. For simplicity we assume, that $p > 2$. We have the torus $T = R_{F/\mathbb{Q}_p}(G_m)$, essentially it is given by the rule $T(L) = G_m(F \otimes L)$ for any field extension $L/\mathbb{Q}_p$. Then we have the norm map $N_{F/\mathbb{Q}_p} : T \to G_m$ and the kernel gives us a one dimensional subtorus $T^{(1)} \subset T$.

The quadratic extension can be written as $F = \mathbb{Q}_p[\sqrt{u}]$ where $u \in \mathbb{Z}_p, \mathrm{ord}_p(u) = 0$ or 1. Then $\mathbb{Z}_p[\sqrt{u}]$ is the ring of integers in that field, this is also called the maximal order. We consider the matrix $A_u = \begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix} \in M_2(\mathbb{Z}_p)$ and then our torus $T/\mathbb{Q}_p$ (resp. $T^{(1)}/\mathbb{Q}_p$ is the centralizer of this matrix in $\mathrm{Gl}_2/\mathbb{Q}_p$ (resp $\mathrm{Sl}_2/\mathbb{Q}_p$). The Lie algebra of $T/\mathbb{Q}_p$ is generated by the identity matrix Z_0 and A_u, the Lie algebra of $T^{(1)}$ is generated by A_ω alone. It is clear that we get the flat extension of our two tori, if we take the centralizer of the same matrix in $\mathrm{Gl}_2/\mathbb{Z}_p$ (resp. $\mathrm{Sl}_2/\mathbb{Z}_p$), then the respective Lie algebras are free $\mathbb{Z}_p$ modules generated by Z_0, A_u resp. A_u alone. Hence we may choose for our differential form ω_T the form that takes value 1 on (Z_0, A_u) and $\omega_{T(1)}$ the form that has value 1 on A_u. We can make a short list for the values of the volumes of the maximal compact subgroups

$$\int_{T(\mathbb{Z}_p)} |\omega_T|(dt) = \begin{cases} \frac{(p-1)^2}{p^2} & \text{if } u \text{ is a square} \\ \frac{p^2-1}{p^2} & \text{if } u \text{ is a unit and not a square} \\ \frac{p-1}{p} & \text{if } \mathrm{ord}_p(u) = 1 \end{cases}$$

and

$$\int_{T^{(1)}(\mathbb{Z}_p)} |\omega_{T^{(1)}}|(dt) = \begin{cases} \frac{p-1}{p} & \text{if } u \text{ is a square} \\ \frac{p+1}{p} & \text{if } u \text{ is a unit and not a square} \\ 2 & \text{if } \mathrm{ord}_p(u) = 1 \end{cases}.$$

I propose to call this number v_T.

2.3.1 The arithmetic measures

We can also work with the arithmetic measures. If do this we start from extension of $G/\mathbb{Q}_p$ to a smooth semi simple group scheme $\mathcal{G}/\mathbb{Z}_p$. This may not be always possible, but in any case we can extend $G/\mathbb{Q}_p$ to a smooth Bruhat-Tits group scheme. At this point I do not explain what this means, for our group $\mathrm{Gl}_2/\mathbb{Q}_p$ we have the first option. If we have a maximal torus $T \subset G/\mathbb{Q}_p$ then we always can replace it by another torus such that the flat extension $\mathcal{T}$ is a subtorus of $\mathcal{G}$, especially we have $\mathcal{T}(\mathbb{Z}_p) = \mathcal{G}(\mathbb{Z}_p) \cap T(\mathbb{Q}_p)$.

I point out that under these assumptions the Lie algebra $\mathrm{Lie}(\mathcal{G})$ is free of rank $\dim(G)$ over $\mathbb{Z}_p$ and we may choose for our form ω_G a generator

$$\omega_{\mathcal{G}} \in \mathrm{Hom}(\Lambda^{\dim(G)}, \mathbb{Z}_p),$$

this gives us a well defined measure on $G(\mathbb{Q}_p)$. Basically we followed the same rule when we selected the form $\omega_{\mathcal{T}}$. Hence these data provide a well defined family of measures

$$|\tilde{\omega}_{T\backslash G}|$$

on the regular semi simple conjugacy classes covered by $T/\mathbb{Q}_p$. It is not difficult to see that these measures do not depend on the choice of $\mathcal{G}/\mathbb{Z}_p$ and $T/\mathbb{Q}_p$.

The choice of $\mathcal{G}/\mathbb{Q}_p$ provides a second family of measures. We choose $|\omega_G^{\mathrm{arith}}|$ so that $\mathrm{vol}_{|\omega_G^{\mathrm{arith}}|}(\mathcal{G}(\mathbb{Z}_p)) = 1$. Then we get a quotient measure $\tilde{\omega}^{\mathrm{arith}} = \omega_{T\backslash G}^{\mathrm{arith}}$. This pair of measures also provides a quotient measure $|\omega_{\mathcal{T}\backslash\mathcal{G}}^{\mathrm{arith}}|$ and we get

$$|\omega_{\mathcal{T}\backslash\mathcal{G}}| = \frac{\#\mathcal{G}(\mathbb{F}_p)}{\#\mathcal{T}(\mathbb{F}_p)} p^{-d} |\omega_{\mathcal{T}\backslash\mathcal{G}}^{\mathrm{arith}}|$$

Here $d = \dim(G) - \dim(T)$, here we assume that our torus splits over a tamely ramified extension (see above).

2.3.2 The geometric family of measures

Now we discuss another choice of a family of consistent measures on the conjugacy classes, which in some sense is better, it is obtained from a geometric construction. We discuss this construction for the group Gl_2 (and in principle also for Sl_2).

We consider the subset $G_{\mathrm{reg}} \subset \mathrm{Gl}_2$ The projection

$$p : G_{\mathrm{reg}} \to \mathbb{G}_m \times \mathbb{A}^1 = \mathcal{C}$$

is surjective and smooth. This morphism has now a section, this is the Steinberg morphism

$$\mathrm{St} : \mathcal{C} \to G_{\mathrm{reg}}$$

which is defined by

$$\mathrm{St} : (x, y) \mapsto \begin{pmatrix} y & -1 \\ x & 0 \end{pmatrix}.$$

We have a basis of the Lie algebra of G :

$$\mathrm{Lie}(G) =< Z_0, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} >=< Z_0, H, E_+, E_- >,$$

it is also the tangent space of G at the identity. For any point $c = (y, x) \in \mathcal{C}(\mathbb{Q}_p)$ the tangent space is generated by $y\frac{\partial}{\partial y}, \frac{\partial}{\partial x}$. In the point $a = \mathrm{St}(y, x)$ the tangent space of G at a is identified to the Lie algebra $\mathrm{Lie}(G)$ by the derivative of the right translation $g \mapsto ga$. It decomposes into two subspaces:

The vertical subspace, which is the tangent space $T_{C_a,a}$ of the conjugacy class C_a at the point a, and the horizontal part, which we define as the subspace spanned by $Z_0, E_.$ An easy calculation shows that the vertical subspace $\mathcal{T}_{C_a,a} = \mathbb{Q}_p(H - \frac{x}{y}E_+) \oplus \mathbb{Q}_p(yE_- + E_+)$ under our identification with the Lie algebra. We define an invariant form $\omega_{G/\mathcal{C}}(c)$ by

$$\omega_{G/\mathcal{C}}(a)(H - \frac{x}{y}E_+, yE_- + E_+) = 1.$$

On G_{reg} we have the sheaf $\Omega^2_{G_{\mathrm{reg}}/\mathcal{C}}$ of relative differential forms. We define a sheaf $\mathcal{L}$ of invariant sections on $\mathcal{C}$. Its sections over any Zariski open set $U \subset \mathcal{C}$ are the invariant global sections $\Omega^2_{G_{\mathrm{reg}}/\mathcal{C}}(p^{-1}(U))$. This is clearly a line bundle. Then $\omega_{G/\mathcal{C}}$ is a global section, which trivializes $\mathcal{L}$. Note that this form is defined without any reference to a form ω_G.

Now this form $\omega_{G/\mathcal{C}}$ provides another family of measures on the conjugacy classes in G_{reg}: For any $c \in \mathcal{C}(\mathbb{Q}_p)$ we have the measure $|\omega_{G/\mathcal{C}}(c)|$.

This measure has its advantages over the previous ones. Firstly it extends to a measure on all regular conjugacy classes. Secondly we observe that the horizontal tangent vectors $Z_0, -E_-$ map to $y\frac{\partial}{\partial y}, \frac{\partial}{\partial x}$. If we choose ω_G so that $\omega_G(Z_0, H, E_+, E_-) = 1$, then

$$\omega_G = \omega_{G/\mathcal{C}} \wedge p^*(\frac{dy}{y} \wedge dx),$$

The second factor is the "natural" measure on $\mathcal{C}(\mathbb{Q}_p)$.

Hence we get a new family of measures $|\omega_{G/\mathcal{C}}|$ on the regular conjugacy classes. For regular values of c we define modified orbital integrals

$$\tilde{O}(h, c) = \int_{p^{-1}(c)(\mathbb{Q}_p)} h(w)|\omega_{G/\mathcal{C}}|(dw).$$

This integral can also be considered for singular c if we define more generally

$$\tilde{O}(h, c) = \int_{p^{-1}(c)(\mathbb{Q}_p)_{\mathrm{reg}}} h(w)|\omega_{G/\mathcal{C}}|(dw).$$

But for singular c we have to show convergence. We will do this later. It is important to understand the behavior of the function $c \mapsto \tilde{O}(h, c)$ if c approaches the singular set.

Our above formula for the differential forms yield another version of Weyls formula

$$\int_{G(\mathbb{Q}_p)} h(g)|\omega_G|(dg) = \int_{\mathcal{C}(\mathbb{Q}_p)} \tilde{O}(h, c)|\frac{dy}{y} \wedge dx|(dc).$$

We want to relate the new orbital integrals to our previous definition of orbital integrals. Let T_a be the centralizer of $\mathrm{St}(c) = \mathrm{St}(y,x) = a = \begin{pmatrix} x & -1 \\ y & 0 \end{pmatrix}$, on this torus we choose in accordance with the rules in 2.3 a form ω_{T_a}. It is unique up to a unit in $\mathbb{Z}_p^\times$. To get this form we have to find the element u in 2.3. It is easily seen that in the matrix ring

$$\big(x\mathrm{Id} + 2\begin{pmatrix} x & -1 \\ y & 0 \end{pmatrix}\big)^2 = \begin{pmatrix} x^2-4y & 0 \\ 0 & x^2-4y \end{pmatrix}$$

and hence with $m = \big[\frac{\mathrm{ord}_p(x^2-4y)}{2}\big]$ we may choose

$$u = p^{-m}(x\mathrm{Id} + 2\begin{pmatrix} x & -1 \\ y & 0 \end{pmatrix}).$$

Hence we see that the Lie algebra of the flat extension of T_a is generated by $Z_0, p^{-m}(xH - 2E_+ + 2yE_-)$, therefore $\omega_{T_a}(Z_0, p^{-m}(xH - 2E_+ + 2yE_-)) = 1$.

The Lie algebra $\mathrm{Lie}(G)$ has the basis $(Z_0, xH - 2E_+ + 2yE_-, H, E_-)$ and $\omega_G(Z_0, xH-2E_++2yE_-, H, E_-) = 2$. The first two entries span the Lie algebra of T_a, the second two entries form a basis of $\mathrm{Lie}(G)/\mathrm{Lie}(T_a)$. Hence we get $\omega_{T_a}(Z_0, xH - 2E_+ + 2yE_-)\omega_{T_a\backslash G}(H, E_-) = 2$. We conclude that

$$\omega_{T_a\backslash G}(H, E_-) = p^{-m}/2.$$

On the other hand we find easily that

$$(-\mathrm{Id}+\mathrm{Ad}(a))(H) = -2H + \frac{2x}{y}E_+, (-\mathrm{Id}+\mathrm{Ad}(a))(E_-) = x(-H+\frac{x}{y}E_+) - E_+ - yE_),$$

and this yields

$$\omega_{T_a\backslash G}(H, E_) = \tilde{\omega}_{T_a\backslash G}(-2H + \frac{2x}{y}E_+, x(-H+\frac{x}{y}E_+) - E_+ - yE_-) =$$

$$\tilde{\omega}_{T_a\backslash G}(-2H + \frac{2x}{y}E_+, E_+ + yE_-) = p^{-m}/2.$$

Now we consider the resulting measures, we assumed $p > 2$. Let $\epsilon(a) = 0$ if the torus T_a splits over an unramified extension, otherwise $\epsilon(a) = 1$. Then we have by definition $|D(c)|_p^{1/2} = p^{-m}p^{-\epsilon(a)/2}$ and we get

$$|\omega_{G/\mathcal{C}}(c)| = p^{-\epsilon(a)/2}|D(c)|_p^{1/2}|\omega_{T_a\backslash G}|$$

and hence we get for our orbital integrals

$$O(h,c) = p^{\epsilon(a)/2}|D(c)|_p^{-1/2}|\tilde{O}(h,c)$$

We can define the same orbital integrals for functions h which invariant under the action of the center $Z(\mathbb{Q}_p)$, and which have compact support modulo the center. For such a function we may also consider the integral $\int_{G(\mathbb{Q}_p)/Z(\mathbb{Q}_p)} h(g)|\omega_G|(dg)$, where dg is now a quotient measure. The orbital integrals of h are defined by the same expression. Now the integrals $O(h,c), \tilde{O}(h,c)$ are invariant under the action of $Z(\mathbb{Q}_p)$ on $\mathcal{C}(\mathbb{Q}_p)$. Hence we get a modified Weyls formula

$$\int_{G(\mathbb{Q}_p)/Z(\mathbb{Q}_p)} h(g)|\omega_G|(dg) = \int_{\mathcal{C}(\mathbb{Q}_p)/Z(\mathbb{Q}_p))}\int_{p^{-1}(c)(\mathbb{Q}_p)} h(w)|\omega_{G/\mathcal{C}}|(dw)|\frac{dy}{y}\wedge dx/dz|(dc)$$

It is an important questions to understand the behavior of the functions $\tilde{O}(h,c)$ as functions in the variable $c \in \mathcal{C}(\mathbb{Q}_p)$. It is rather clear, that the restriction of $\tilde{O}(h,c)$ to $\mathcal{C}_{\mathrm{reg}}(\mathbb{Q}_p)$ is locally constant. The interesting question is what happens if we approach elements in $Z(\mathbb{Q}_p)$.

We can summarize. We have three different ways of defining the orbital integrals. The two definitions

$$O^{\mathrm{arith}}(h,c) = \int_{p^{-1}(c)(\mathbb{Q}_p)_{\mathrm{reg}}} h(w)|\omega^{\mathrm{arith}}_{\mathcal{T}\backslash\mathcal{G}}|(dw), O(h,c) = \int_{p^{-1}(c)(\mathbb{Q}_p)_{\mathrm{reg}}} h(w)|\omega_{\mathcal{T}\backslash\mathcal{G}}|(dw),$$

depend on the choice of an extension of $G/\mathbb{Q}_p$ to a semi-simple group scheme and a suitable choice of T, and they differ by a multiplicative constant depending on the isomorphism class of the torus T, it is determined by c and let us call this isomorphism class $< c >$. We denote the constant by $A(< c >)$. Its value can be read of from our table above. Hence we get

$$O^{\mathrm{arith}}(h,c) = A(< c >)O(h,c).$$

Note that $< c >$ is the same object as a quadratic extension $F/\mathbb{Q}_p$. The third one

$$\tilde{O}(h,c) = \int_{p^{-1}(c)(\mathbb{Q}_p)_{\mathrm{reg}}} h(w)|\omega_{G/\mathcal{C}}|(dw)$$

is the most geometric definition. It differs from the two definition by a multiplicative constant times the regularizing factor $|D(c)|_p^{1/2}$. This orbital integral can, as we will see in section 2.6, be extended to a reasonable function on $\mathcal{C}(\mathbb{Q}_p)$.

We discuss some special cases.

2.4 The orbital integrals on hyperbolic regular elements

We recall the situation from Chap. III 1.2.2. For any $h_p \in \mathcal{H}_p$ in the unramified Hecke (???) algebra we have its Fourier transform defined by

$$\int_{G(\mathbb{Q}_p)} \phi_\lambda(gx)h_p(x)dx = \hat{h}_p(\lambda)\phi_\lambda(g),$$

we want to relate this Fourier transform to orbital integrals on hyperbolic elements. We compute the above integral by starting from the Iwasawa decomposition

$$U(\mathbb{Q}_p) \times T(\mathbb{Q}_p) \times K_p \to G(\mathbb{Q}_p),$$

under this map the bi-invariant measure on $G(\mathbb{Q}_p)$ becomes the measure $t_p^{-2\rho}du_p \times dt_p \times dk_p$, where the volumes of $\mathcal{U}(\mathbb{Z}_p), \mathcal{T}(\mathbb{Z}_p), K_p$ are equal 1. Hence we have to compute

$$\int_{U(\mathbb{Q}_p)\times T(\mathbb{Q}_p)\times K_p} t_p^{-2|\rho|}\phi_\lambda(u_pt_pk_p)du_p \times dt_p \times dk_p.$$

The integrand does not depend on k_p and we now the value of ϕ_λ, hence we have to compute

$$\int_{U(\mathbb{Q}_p)\times T(\mathbb{Q}_p)} t_p^{-|\rho|} h(u_p t_p) du_p \times dt_p.$$

In the variable t_p these functions only depend on $T(\mathbb{Q}_p)/T(\mathbb{Z}_p) = \mathbb{Z}\times\mathbb{Z}$, we can write

$$\sum_{a,b\in\mathbb{Z}} p^{(a-b)/2}(\alpha'_p)^a(\beta'_p)^b \int h(u_p \begin{pmatrix} p^a & 0 \\ 0 & p^b \end{pmatrix}) du_p =$$

$$\sum_{a,b\in\mathbb{Z}} p^{(a-b)/2}\alpha_p^a\beta_p^b p^{-(a+b)/2} \int h(u_p \begin{pmatrix} p^a & 0 \\ 0 & p^b \end{pmatrix}) du_p =$$

$$\sum_{a,b\in\mathbb{Z}} \alpha_p^a\beta_p^b\, p^{-b} \int h(u_p \begin{pmatrix} p^a & 0 \\ 0 & p^b \end{pmatrix}) du_p.$$

Now the terms for $a \neq b$ can be expressed in terms of orbital integrals. In this case the centralizer of $\begin{pmatrix} p^a & 0 \\ 0 & p^b \end{pmatrix}$ is the maximal torus $T/\mathbb{Q}_p$ itself and we can consider the orbital integral

$$\int_{T(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)} h(\bar{g}_p^{-1} \begin{pmatrix} p^a & 0 \\ 0 & p^b \end{pmatrix} \bar{g}_p) d\bar{g}_p = O^{\text{arith}}(h_p, \begin{pmatrix} p^a & 0 \\ 0 & p^b \end{pmatrix})$$

The Iwasawa decomposition yields

$$T(\mathbb{Q}_p)\times U(\mathbb{Q}_p)\times K_p \to G(\mathbb{Q}_p),$$

this time the invariant measure becomes $dt_p \times du_p \times dk_p$. On $T(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)$ we get the measure $du_p \times dk_p$, the variable k_p drops out again and we get

$$O^{\text{arith}}(h_p, \begin{pmatrix} p^a & 0 \\ 0 & p^b \end{pmatrix}) = \int h(u_p^{-1} \begin{pmatrix} p^a & 0 \\ 0 & p^b \end{pmatrix} u_p) du_p.$$

Now recall that $u_p = \begin{pmatrix} 1 & u_p \\ 0 & 1 \end{pmatrix}$ and hence our integral is

$$\int h(\begin{pmatrix} 1 & (p^{a-b}-1)u_p \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^a & 0 \\ 0 & p^b \end{pmatrix}) du_p =$$

$$|p^{a-b}-1|_p^{-1} \int h(\begin{pmatrix} 1 & u_p \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^a & 0 \\ 0 & p^b \end{pmatrix}) du_p.$$

Hence we get

$$\int h(\begin{pmatrix} 1 & u_p \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^a & 0 \\ 0 & p^b \end{pmatrix}) du_p = |p^{a-b}-1|_p O^{\text{arith}}(h_p, \begin{pmatrix} p^a & 0 \\ 0 & p^b \end{pmatrix})$$

The absolute value of the factor in front is 1 if $a > b$ and p^{b-a} if $a < b$. Hence we define $m(a,b) = -b$ if $a > b$ and $m(a,b) = -a$ if $a < b$ then we find

$$\hat{h}_p(\lambda) = \sum_{a,b\in\mathbb{Z}, a\neq b} \alpha_p^a\beta_p^b\, p^{m(a,b)} O^{\text{arith}}(h_p, \begin{pmatrix} p^a & 0 \\ 0 & p^b \end{pmatrix}) + \sum_{a\in\mathbb{Z}} \alpha_p^a\beta_p^a \int h_p(u_p \begin{pmatrix} p^a & 0 \\ 0 & p^a \end{pmatrix}) du_p$$

Of course this is always a finite sum.

For any $t_p = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$, where t_1, t_2 have different valuations, the orbital integral $O(h,t)$ only depends on these two valutions and is equal up to a power of p to a Fourier coefficient in the expansion of $\hat{h}(\lambda)$. If $t_p = \begin{pmatrix} up^a & 0 \\ 0 & vp^a \end{pmatrix}$ with $u, v \in \mathbb{Z}_p^\times, u \neq v$ then the same calculation yields

$$\int h(\begin{pmatrix} 1 & u_p \\ 0 & 1 \end{pmatrix} \begin{pmatrix} up^a & 0 \\ 0 & vp^a \end{pmatrix}) du_p = |\frac{u}{v} - 1|_p O^{\mathrm{arith}}(h_p, t_p),$$

the term on the let hand side does not depend on u, v and hence we get

$$\int h(\begin{pmatrix} 1 & u_p \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^a & 0 \\ 0 & p^a \end{pmatrix}) du_p = |\frac{u}{v} - 1|_p O^{\mathrm{arith}}(h_p, t_p)$$

The ratio u/v is equal to the value of the simple root α on t_p. Hence we get the formula: For $t_p = \begin{pmatrix} up^a & 0 \\ 0 & vp^a \end{pmatrix}$ we get (germ expansion???)

$$|\alpha(t_p) - 1|_p^{-1} \int h(\begin{pmatrix} 1 & u_p \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^a & 0 \\ 0 & p^a \end{pmatrix}) du_p = O^{\mathrm{arith}}(h_p, t_p)$$

We can conclude that a function h_p in the Hecke algebra is determined by its orbital integral on regular hyperbolic elements. On the other hand we can prescribe these values of orbital integrals, if we take the following precautions: On the elements t_p with $\alpha(t_p)$ not a unit they only depend on $t_p \mod \mathcal{T}(\mathbb{Z}_p)$, and on the elements where $\alpha(t_p)$ is a unit the number $|\alpha(t_p) - 1|_p O^{\mathrm{arith}}(h_p, t_p)$ only depends on $|\det(t_p)|_p$. Of course almost all of them in the appropriate sense must be zero.

Finally I want to mention that this computation also allows us to compute the orbital integrals for regular p-hyperbolic elements in higher dimensional groups.

We observe that orbital integrals $O^{\mathrm{arith}}(h_p, t_p)$ go to infinity if t_p tends to a central element, but the asymptotic behavior is very simple. The function $t_p \mapsto \Delta(t_p) = (\frac{u}{v} - 1)(\frac{v}{u} - 1)$ has an interpretation in terms of the adjoint action: If $\mathfrak{t}$, resp.$\mathfrak{g}$ are the Lie-algebras of the torus T resp. the group G then $\Delta(t_p)$ is the determinant of the endomorphism $\mathrm{Ad}(t_p) - \mathrm{Id}$ on $\mathfrak{g}/\mathfrak{t}$ (Ref ???). This quantity can also be defined for other semi simple regular elements and the function $t_p \to |\Delta(t_p)|^{-1/2}$ plays an inportant role for the desription of the behavior of orbital integrals if $t_p \to$ central element

If we take also for our function h_p simply the characteristic function of the maximal compact subgroup $K_p = \mathrm{Gl}_2(\mathbb{Z}_p)$ then our formula for the orbital integral becomes

$$O^{\mathrm{arith}}(h_p, t_p) = \begin{cases} 0 & \text{if } |\Delta(t_p)| > 1 \\ |\Delta(t_p)|^{-1/2} & \text{else} \end{cases}$$

where for the second line we use that

$$\int h(\begin{pmatrix} 1 & u_p \\ 0 & 1 \end{pmatrix} du_p = 1$$

We may also choose for the function h the characteristic function of another open compact subgroup $K_p \subset \mathrm{Gl}_2(\mathbb{Z}_p)$, this belongs to a larger Hecke algebra. We can write $\mathrm{Gl}_2(\mathbb{Z}_p) = \bigcup_\xi \xi K_p$.

Then our orbital integral is zero if $t_p \notin T(\mathbb{Z}_p)$. If $t_p \in T(\mathbb{Z}_p)$ is regular then

$$O^{\mathrm{arith}}(h_p, t_p) = \frac{1}{[\mathrm{Gl}_2(\mathbb{Z}_p) : K_p]} \sum_\xi |\Delta(t_p)|^{-1/2} \int h^\xi (t_p \begin{pmatrix} 1 & u_p \\ 0 & 1 \end{pmatrix}) du_p$$

where h^ξ is the characteristic function of $\xi K_p \xi^{-1} \subset \mathrm{Gl}_2(\mathbb{Z}_p)$.

We may for instance choose $K_p = K_{0,p}(p) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Gl}_2(\mathbb{Z}_p) | c \equiv 0 \mod p\}$, then we can take is system of representatives the matrices

$$\Xi = \{\{\begin{pmatrix} 1 & 0 \\ \nu & 1 \end{pmatrix}\}_{\nu=1,\dots,p-1}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\}$$

Now we see that for $\xi = \mathrm{Id}$, i.e. $\nu = 0$, we have

$$h^\xi (t_p \begin{pmatrix} 1 & u_p \\ 0 & 1 \end{pmatrix}) = 1 \text{ if and only if } u_p \in \mathbb{Z}_p$$

and in for the other cases we have

$$h^\xi (t_p \begin{pmatrix} 1 & u_p \\ 0 & 1 \end{pmatrix}) = 1 \text{ if and only if } u_p \in p\mathbb{Z}_p.$$

Hence we get for this special choice of h_p as characteristic function of $K_{p,0}(p)$

$$O^{\mathrm{arith}}(h_p, t_p) = \frac{2}{p+1} |\Delta(t_p)|^{-1/2}.$$

We may consider the difference of characteristic functions

$$h_p^B = \chi_{\mathrm{Gl}_2(\mathbb{Z}_p)} - \frac{p+1}{2} \chi_{K_{p,0}(p)},$$

then we see that this function has vanishing orbital intergals for all p-hyperbolic elements and has the non zero constant value $\frac{p-1}{2}$ in a neigborhood of the identity.

2.5 The Bruhat-Tits building

A decisive tool for the computation of orbital integrals is provided by the Bruhat-Tits building. The Bruhat-Tits building for the group $\mathrm{Gl}_2(\mathbb{Q}_p)$ is a tree whose vertices are the rank 2 free $\mathbb{Z}_p$ submodules $e = M \subset \mathbb{Q}_p^2$ up to homothethy, the edges are pairs $\sigma = M \supset M'$ such that $M/M' = \mathbb{Z}/p\mathbb{Z}$ again up to homothethy. The vertices of an edge σ are the two free lattices M and M'. The edges at a vertex e are the $p+1$ submodules M' of index p. The edges can be identified to the intervall $[0, 1]$.

A vertex M defines a maximal compact subgroup $\mathrm{Gl}_2(M) = K_e$, namely the stabilizer of M in $\mathrm{Gl}_2(\mathbb{Q}_p)$, it can be vied as the group of $\mathbb{Z}_p$-valued points of a semi-simple group scheme extension of $\mathrm{Gl}_2/\mathbb{Q}_p$. If we denote this semi simple group scheme by $\mathcal{G}/\mathbb{Z}_p$, then the edges ending in M are in one to one correspondence to the Borel subgroups $\mathcal{B} \subset \mathcal{G} \times_{\mathbb{Z}_p} \mathbb{F}_p$. Is $\sigma = (M, M')$ an edge ending at e then we define the open compact subgroup K_σ as the invers image of $\mathcal{B}(\mathbb{F}_p)$ in K_e. It is a so called Iwahori subgroup.

The group $G(\mathbb{Q}_p)$ acts on this building, the action on the set of vertices is extended linearily to the edges. It acts transitively on the vertices and the edges. The stabilizer of a vertex e is K_e, the stabilizer of an edge σ is a subgroup $\tilde{K}_\sigma$, which contains K_σ as a subgroup of index 2. If we take as a standard edge the module $M_0 = \mathbb{Z}_p \oplus \mathbb{Z}_p \subset \mathbb{Q}_p \oplus \mathbb{Q}_p$, then the stabilizer is $\mathrm{Gl}_2(\mathbb{Z}_p)$. A standard submodule of index p is $M' = \mathbb{Z}_p \oplus p\mathbb{Z}_p$ and then for $\sigma_0 = (M_0, M')$ is the congruence subgroup

$$K_\sigma = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} | c \equiv 0 \mod p\}.$$

The group $\tilde{K}_\sigma$ contains the element $\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$ which conjugates M' into $p\mathbb{Z}_p \oplus \mathbb{Z}_p = pM'$.

The fundamental fact is

This complex $\mathbb{BT}(\mathrm{Gl}_2/\mathbb{Q}_p)$ is i t a tree, it has no non trivial closed loops. Any two points are joined by a unique shortes path. It is a perfect p-adic analog of the symmetric space attached to a semi-simple group over $\mathbb{R}$.

Any p elliptic element $\gamma \in Gl_2(\mathbb{Q}_p)$ can be is conjugate to an element in $\mathrm{Gl}_2(\mathbb{Z}_p)$ hence it has a fixed point in $\mathbb{BT}(\mathrm{Gl}_2/\mathbb{Q}_p)$. From the assertion above it is clear

The fixed point set $F(\gamma)$ of a p-elliptic element is contractible

The above element $\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$ fixed the edge (M_0, M') but not pointwise. Its fixed point is the central point of the edge.

On the groups $\tilde{K}_\sigma$ we define the sign homomorphism $\mathrm{sgn}_\sigma : \tilde{K}_\sigma \to \pm 1$ which has value 1 on K_σ and -1 the complement.

Following Kottwitz we define the Euler-Poincare function in the Hecke algebra. We consider the characteristic functions $\chi_{K_{e_0}} = \chi_{\mathrm{Gl}_2(\mathbb{Z}_p)}$ and $\chi_{\tilde{K}_{\sigma_0}}$ and put

$$h_p^{EP} = \chi_{K_{e_0}} - \frac{p+1}{2} \chi_{\tilde{K}_{\sigma_0}} \mathrm{sgn}_{\sigma_0}.$$

This is almost the function we looked at in the previous section. We notices that the factor in front can be interpretet as the index of $\tilde{K}_{\sigma_0}$ in K_{e_0}.

This function has very nice orbital integrals. First of all it is clear

The orbital integrals of h_p^{EP} on regular p-hyperbolic elements are still zero.

Any regular p-elliptic element t_p can be conjugated into a torus $T/\mathbb{Q}_p$ such that $T(\mathbb{Q}_p) \subset Z(\mathbb{Q}_p)K_{e_0}$. The $T(\mathbb{Q}_p)$ is the centralizer of t_p. We normalize the measure dg_p on $\mathrm{Gl}_2(\mathbb{Q}_p)$ such that K_{e_0} has volume one and the maesure on $T(\mathbb{Q}_p)$ such that $T(\mathbb{Z}_p)$ has volume one. With respect to these measures we define the quotient measures dx_p^T on $T(\mathbb{Q}_p)\backslash \mathrm{Gl}_2(\mathbb{Q}_p)$ and with respect to these measures we have:

For all regular p elliptic elements we have

$$O^{\mathrm{arith}}(h_p^{EP}, t_p) = 1$$

We leave the proof to the reader. It follows by interpretation that the value of the orbital integral is exactly the Euler characteristic of the fixed point set of t_p on the Bruhat-Tits building. Since this fixed point set is contractible this Euler characteristic is 1.(See [Kott])

We can modify these functions slightly. We observe that they have a constant value on $Z(\mathbb{Z}_p)$. We can translate them by an element $\begin{pmatrix} p^a & 0 \\ 0 & p^a \end{pmatrix}$ in the center and take the sum

$$\tilde{h}_p^{EP} = \sum_{a\in\mathbb{Z}} \begin{pmatrix} p^a & 0 \\ 0 & p^a \end{pmatrix} h_p^{EP}.$$

This function may now be twisted by an (admissible) character $\eta : \mathbb{Q}_p^\times \to \mathbb{C}^\times$, we define

$$\tilde{h}_{p,\eta}^{EP}(g) = \sum_{a\in\mathbb{Z}} h_p^{EP}(g \begin{pmatrix} p^a & 0 \\ 0 & p^a \end{pmatrix})\eta(\det(g)).$$

This function satisfies

$$\tilde{h}_{p,\eta}^{EP}(zg) = \tilde{h}_{p,\eta}^{EP}(g)\eta(\det(z)) \text{ for all } z \in Z(\mathbb{Q}_p), g \in G(\mathbb{Q}_p).$$

It is clear that $O^{\mathrm{arith}}(h_p^{EP}, t_p) = O^{\mathrm{arith}}\tilde{(}h_p^{EP}, t_p)$.

2.5.1 Orbital integrals of supercuspidal matrix coefficients

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2.6 Shalikas germ expansion

Now we come and investigate the asymptotic behavior of orbital integrals $\tilde{O}(h, c)$ if c approaches the singular set. We discuss the special case $\mathrm{Gl}_2/\mathbb{Q}_p$. I think that in this situation it is best to work with the geometric measure $\omega_{G/\mathcal{C}}$ because it extends to all regular elements.

We distinguish two cases, at first we assume that h vanishes on $Z(\mathbb{Q}_p)$. Then h has compact support in $\mathcal{C}(\mathbb{Q}_p)_{\mathrm{reg}}$ and we can write down the integral defining $\tilde{O}(h, x)$ for all x. Simple arguments with the p-adic implicit function theorem and measures show that $\tilde{O}(h, x)$ extends to a locally constant function from $\mathcal{C}(\mathbb{Q}_p)_{\mathrm{reg}}$ to $\mathcal{C}(\mathbb{Q}_p)$. Hence we see that for $c = (y, x) \in \mathcal{C}(\mathbb{Q}_p)$, for which $|x^2 - 4y|_p$ is sufficiently small, we get

$$\tilde{O}(h, c) = \tilde{O}(h, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}) = \tilde{O}(h, (1, 1))$$

where we allowed ourselves to represent a conjugacy class by an element.

Now we drop the assumption that h has support in $G(\mathbb{Q}_p)_{\mathrm{reg}}$. I briefly recall the discussion at the beginning of this section. An element $c = (y, x) \in \mathcal{C}(\mathbb{Q}_p)_{\mathrm{reg}}$ defines a maximal torus and this torus defines an inner splitting class. In our specific case this inner splitting class is simply determined by the extension $F = \mathbb{Q}_p(\sqrt{x^2 - 4y})$. This defines a partition of $\mathcal{C}(\mathbb{Q}_p)_{\text{reg}}$ into subsets $\mathcal{C}(\mathbb{Q}_p)_F$. Let us denote by χ_F the characteristic function of $\mathcal{C}(\mathbb{Q}_p)_F$.

We pick an element $z_0 \in Z(\mathbb{Q}_p)$, we choose an integer a large enough and we consider the congruence subgroup $K^{(a)}$ of elements in $G(\mathbb{Z}_p)$ which are congruent to the identity $\mod p^a$. Then $K^{(a)}z_0$ is an open neighborhood of z_0. The image of $K^{(a)}z_0 \cap Z(\mathbb{Q}_p)$ under $(\det, \mathrm{tr})$ is denoted by $W(z_0, a)$. We consider the characteristic function $\chi_{z_0,a}$ of $K^{(a)}z_0$. We observe that for any function h in the Hecke algebra we can find z_i, a_i and coefficients c_{z_i,a_i} such that $h - \sum_{z_i,a_i} c_{z_i,a_i}\chi_{a_i,z_i}$ vanishes on the center. Hence we have to study the integrals

$$\tilde{O}(\chi_{z_0,a}, c)$$

and see what happens if c approaches the singular set $W(z_0, a)$.

This integral - for $c \in \mathcal{C}(\mathbb{Q}_p)_{\text{reg}}$- is of course equal to the volume of $X(c) = p^{-1}(c)(\mathbb{Q}_p) \cap K^{(a)}z_0$, with respect to the measure $|\omega_{\mathcal{H}\backslash G}|$. This fiber $X(c)$ is a compact analytic manifold as long as we stay away from the singular set. But for $c = (z^2, 2z) \in W(z_0, a)$ it has an isolated singularity, this is the point $\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$. If we remove this point we still have an analytic variety, we call it $X(c)^0$, it is not compact anymore.

Now we have to compute some p-adic integrals, the computation of these integrals comes down to the counting of the number of solutions of diophantine equations. I will discuss this in more detail in the section on the fundamental lemma.

First of all we observe that

$$\tilde{O}(\chi_{z_0,a}, \begin{pmatrix} z & 1 \\ 0 & z \end{pmatrix}) = \mathrm{vol}(X(c)^0)$$

is finite, it is given by a convergent geometric series. It does not depend on z of course.

Then we compute the difference

$$\tilde{O}(\chi_{z_0,a}, \begin{pmatrix} z & 1 \\ 0 & z \end{pmatrix}) - \tilde{O}(\chi_{z_0,a}, c)$$

if $c \to (z^2, 2z)$. This is in principle very elementary, as I said before we have count numbers of solutions of diophantine equations. The result depends the inner splitting class of c, i.e. on the F for which $c \in \mathcal{C}(\mathbb{Q}_p)_F$. We get :

For any F there is an explicitly computable constant c_F such that for $a > 0$ (perhaps $a > 1$ for $p = 2$) and $c \in \mathcal{C}(\mathbb{Q}_p)_F$ we have

$$\tilde{O}(\chi_{z_0,a}, \begin{pmatrix} z & 1 \\ 0 & z \end{pmatrix}) - \tilde{O}(\chi_{z_0,a}, c) = c_F |D(c)|_p^{1/2}$$

We will show this and compute the constants in the section on the fundamental lemma.

Putting everything together we get the germ expansion of Shalika: For any function h in the Hecke algebra we have for all c close enough to an element $(z^2, 2z)$ in the singular set (close enough depends on h)

$$\tilde{O}(h, c) = \tilde{O}(h, (z^2, 2z)) + |D(c)|_p^{1/2} \sum_F c_F \chi_F(c) h(\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}).$$

This tells us that $\tilde{O}(h, c)$ is a sum locally constant function + a function that has a simple asymptotic behavior if c approaches the singular set.

Of course we can rewrite this for the original orbital integrals and get

$$O(h, c) = p^{\epsilon(c)/2} |D(c)|_p^{-1/2} \; O(h, (z^2, 2z)) + \sum_F c_F p^{\epsilon(c)/2} \chi_F(c) h(\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}).$$

for $c \to (z^2, 2z)$.

We can apply this to the Euler-Poincare function h^{EP} , it has the following properties

(i) For $z \in Z(\mathbb{Q}_p)$ we have $h(z) = \frac{1-p}{2}$ for $z \in Z(\mathbb{Z}_p)$ and zero else.

(ii) Its orbital integrals on hyperbolic and on regular unipotent elements are zero.

(iii) For any quadratic extension $F/\mathbb{Q}_p$ - in other words any non split splitting class - we have

$$O^{\text{arith}}(h^{EP}, c) = 1$$

This gives us the values of the constants c_F for the different quadratic extensions. We consider the germ expansion of $\tilde{O}^{\text{arith}}(h^{EP}, c)$, since the unipotent orbital integrals vanish we get

$$O^{\text{arith}}(h^{EP}, c) = \sum_F A(< c >) c_F \chi_F(c) h(\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix})$$

and we conclude that $A(< c >) c_F = \frac{1-p}{2}$. The values of $A(< c >)$ have been computed in 2.3

In this context the arithmetic orbital integrals have the nicest behavior.

At this point I want to keep attention, that the situation is different, if we consider other groups. Already for the case $\mathrm{Sl}_2/\mathbb{Q}_p$ there is some change. We can not construct an Euler-Poincare function, which takes different values at the two central elements, so the above procedure does not give the constants. We discuss this in the section on the fundamental lemma.

2.7 Orthogonality relations for characters of the discrete series

2.7.1 Intermission: A useful global result

Given our group $G/\mathbb{Q}_p$ we can find a discrete, torsion free subgroup $\Gamma \subset G(\mathbb{Q}_p)$, such that $\Gamma \backslash G(\mathbb{Q}_p)$ is compact. This is easy to see for a torus, hence we may assume, that G/Q_p is semi simple. Then we look for a suitable totally real number field $K/\mathbb{Q}_p$, which has a finite place $\mathfrak{p}$ above p, such that $K_{\mathfrak{p}} = \mathbb{Q}_p$. Now we construct a group scheme $\mathcal{G}_\infty/\mathcal{O}_K$ such that $\mathcal{G}_\infty \times (K \otimes \mathbb{R})$ is compact and such that $G \times K_{\mathfrak{p}} \xrightarrow{\sim} G/\mathbb{Q}_p$. (To do this construction we may have to take a field which is larger than $\mathbb{Q}$). We define the group

$$\Gamma = \{\gamma \in G(K) | \gamma \in \mathcal{G}(\mathcal{O}_{\mathfrak{q}}) \text{ for all finite places } \mathfrak{q} \neq \mathfrak{p}\}.$$

Then it follows from strong approximation and reduction theory that Γ has the required properties, possibly up to torsion freeness. Here we observe that we can construct subgroups of finite index of Γ if we put some congruence conditions outside $\mathfrak{p}$. This makes it clear that we cam get Γ torsion free.

If we now choose any compact open subgroup $K_p \subset G(\mathbb{Q}_p)$ then the double coset space $\Gamma \backslash G(\mathbb{Q}_p)/K_p$ is finite. Since Γ is torsion free we also know that $\Gamma \cap g^{-1} K_p g = e$ for all $g \in G(\mathbb{Q}_p)$. Let π be the projection map $G(\mathbb{Q}_p) \to \Gamma \backslash G(\mathbb{Q}_p)/K_p$.

We choose a field F of characteristic zero, at this moment we assume that it is a finite extension of $\mathbb{Q}$. We assume that the functions in the Hecke algebra take values in F. We get a representation of the Hecke algebra $\mathcal{H}(G(\mathbb{Q}_p)//K_p)$ on the space $\mathcal{A}(\Gamma \backslash G(\mathbb{Q}_p)/K_p)$ of F-valued functions on $\Gamma \backslash G(\mathbb{Q}_p)/K_p$. We choose the measure that gives volume 1 to K_p. Now we apply a simple form of the trace formula. We derive this simple version here, the general version of the (topological) trace formula uses the same principles.

Let us assume that h is the characteristic function of the double coset $K_p a K_p$. We choose representing elements g_i for the finitely many elements $x_1, \dots x_m$ in $\Gamma \backslash G(\mathbb{Q}_p)/K_p$. Let δ_{x_i} be the Dirac delta function at x_i. Then

$$T_h(\delta_{x_i}) = \sum_{\xi \in a K_p a^{-1} \cap K_p} \delta_{x_i}(\pi(g_i \xi a) = \sum_j T_h(\delta_{x_i}(x_j)\delta_{x_j}.$$

Then we get for the trace of T_h

$$\operatorname{tr}(h | \mathcal{A}(\Gamma \backslash \mathcal{G}(\mathbb{Q}_p)/K_p) = \sum_i T_h(\delta_{x_i})(x_i).$$

For a given index i we have to compute the number of elements ξ such that we can find a $\gamma \in \Gamma$ such that $\gamma g_i \in g_i \xi a K_p$. It is rather easy to see that in case of such a solution the element γ and the element ξ determine each other. Since $\cup_\xi \xi K_p = K_p a K_p$ we get that the trace is equal to

$$\sum_i \#\{\gamma \,| g_i^{-1} \gamma g_i \in K_p a K_p\} = \sum_i \sum_{\gamma \in \Gamma} h(g_i^{-1} \gamma g_i).$$

This is clearly a finite sum. Now we want to have a more canonical expression. If we replace g_i by another representative, then we have to replace γ by a conjugate element. Therefore the summation over Γ can be replaced by a summation over conjugacy classes and the summation over i is an summation over $\Gamma \backslash G(\mathbb{Q}_p)/K_p$, we get for the trace

$$\sum_{\gamma \in \Gamma/\sim} \int_{G(\mathbb{Q}_p)} h(g^{-1} \gamma g) dg.$$

Finally we introduce the centralizer Z_γ of γ and we put $\Gamma_\gamma = \Gamma \cap Z_\gamma(\mathbb{Q}_p)$. Then $Z_\gamma(\mathbb{Q}_p)/\Gamma_\gamma$ is compact. Now the function $g \mapsto h(g^{-1}\gamma g)$ is invariant under $g \mapsto zg$ if $z \in Z_\gamma(\mathbb{Q}_p)$. We choose a measure dz on $Z_\gamma(\mathbb{Q}_p)$ and define the quotient measure $d\bar{g}$ on $Z_\gamma(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)$. Then we get the final form of the trace formula

$$\mathrm{tr}(h|\mathcal{A}(\Gamma\backslash\mathcal{G}(\mathbb{Q}_p)/K_p) = \sum_{\gamma\in\Gamma/\sim} \mathrm{vol}_{dz}(\Gamma_\gamma\backslash Z_\gamma(\mathbb{Q}_p)) \int_{Z_\gamma(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)} h(\bar{g}^{-1}\gamma\bar{g})d\bar{g}.$$

2.7.2 Applications of the useful global result

We apply this trace formula to a matrix coefficient $h =< gv_i, \phi_i >$ (resp. the Euler-Poincare function $h = h_\chi^{EP}$) of a supercuspidal (resp. in the case Gl_2 Steinberg representation St_χ.)

Since our group Γ does not contain any elliptic non central element, we get by Selbergs principle or the properties of the Euler-Poincare function that

$$\textit{For any } \gamma \in \Gamma \setminus Z(\mathbb{Q}_p) \textit{we have } \int_{Z_\gamma(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)} h(\bar{g}^{-1}\gamma\bar{g})d\bar{g} = 0.$$

We can conclude

$$\mathrm{tr}(h|\mathcal{A}(\Gamma\backslash\mathcal{G}(\mathbb{Q}_p)/K_p) = h(e)\mathrm{vol}_{dg}(\Gamma\backslash G(\mathbb{Q}_p)).$$

On the other hand for a supercuspidal representation π_p the trace of h on π_p is equal to $\frac{1}{d(\pi_p)}$. If $m_\Gamma(\pi_p)$ is the multiplicity of π_p in $\mathcal{A}(\Gamma\backslash\mathcal{G}(\mathbb{Q}_p)/K_p)$, we get

$$h(e)\mathrm{vol}_{dg}(\Gamma\backslash G(\mathbb{Q}_p)) = \frac{m(\pi_p)}{d(\pi_p)}.$$

The number $h(e) = 1$ if π_p is supercuspidal.

(i) If π_p is a supercuspidal representation with $\pi_p^{K_p} \neq \{0\}$ then it occurs with strictly positive multiplicity in $\mathcal{A}(\Gamma\backslash G(\mathbb{Q}_p)/K_p)$. Especially it follows that the number of discrete series representations of level K_p (this means of course $\pi_p^{K_p} \neq \{0\}$) is finite.

(ii) The formal degrees $d(\pi_p)$ are integer multiples of

$$\frac{1}{h(e)\mathrm{vol}_{dg}(\mathcal{A}(\Gamma\backslash G(\mathbb{Q}_p))}$$

In the case of the Steinberg representation we get for the two sides

$$1 - m(\mathrm{St}_\chi) = \frac{1-p}{2}$$

because the trivial representation sits in our space.

If π_p is not supercuspidal, then we consider only the case Gl_2. We give a second application. We choose an embedding of F into $\mathbb{C}$. On the space $\mathcal{A}(\Gamma\backslash\mathcal{G}(\mathbb{Q}_p)/K_p)_\mathbb{C}$ of $\mathbb{C}$ valued functions a positive definite hermitian scalar product:

$$< f,g >= \int_{\Gamma\backslash G(\mathbb{Q}_p)} f(x)\overline{g(x)}dx.$$

If we pass to a subgroup Γ' of finite index, then we get a covering $\Gamma'\backslash G(\mathbb{Q}_p)/K_p \to \Gamma\backslash \mathcal{G}(\mathbb{Q}_p)/K_p$ and an inclusion $\mathcal{A}(\Gamma\backslash G(\mathbb{Q}_p)/K_p)_{\mathbb{C}} \hookrightarrow \mathcal{A}(\Gamma'\backslash G(\mathbb{Q}_p)/K_p)_{\mathbb{C}}$ and for $f,g \in \mathcal{A}(\Gamma'\backslash G(\mathbb{Q}_p)/K_p)_{\mathbb{C}}$ we define the scalar product

$$< f,g >== \frac{1}{[\Gamma:\Gamma']} \int_{\Gamma'\backslash G(\mathbb{Q}_p)} f(x)\overline{g(x)}dx$$

which gives us the same value for functions in the smaller space as before. Let us denote by $[\Gamma]$ the directed family of congruence subgroups.

Therefore we can even pass to the projective limit

$$\lim_{\leftarrow\ \Gamma\in[\Gamma]} \Gamma'\backslash G(\mathbb{Q}_p)/K_p,$$

this is a profinite set and on this space we have the locally constant functions

$$\mathcal{A}_\infty([\Gamma]\backslash G(\mathbb{Q}_p)/K_p)_{\mathbb{C}} = \lim_{\Gamma'} \mathcal{A}(\Gamma'\backslash G(\mathbb{Q}_p)/K_p).$$

Clearly we have a positive definite scalar product on this space, we can complete it to a Hilbert space $\mathcal{A}_2([\Gamma]\backslash G(\mathbb{Q}_p)/K_p)_{\mathbb{C}}$. It is also clear that the the Hecke algebra acts on both spaces by convolution. Furthermore it is also easy to see that the action on the two spaces is faithful, i.e. the homomorphism

$$\mathcal{H}(G(\mathbb{Q}_p)//K_p) \to \mathrm{End}(\mathcal{A}_\infty([\Gamma], G(\mathbb{Q}_p)/K_p)_{\mathbb{C}})$$

is an injection. Finally it follows by a standard argument, that the operators on the Hilbert space are in fact compact.

We can define an involution on the Hecke algebra $h(x) \mapsto \overline{h(x^{-1})} = h^*(x)$ so that $< T_h f, g >=< f, T_{h^*} g >$. Hence we see that our Hecke algebra can be identify to a subalgebra of compact operators on $\mathcal{A}_2(G(\mathbb{Q}_p)/K_p)$. This algebra is closed under adjunction and this implies by a standard argument in functional analysis that our Hilbert space decomposes into a discrete direct sum of irreducible modules, where each isomorphism type occurs with finite multiplicity. We write

$$\mathcal{A}_2([\Gamma], G(\mathbb{Q}_p)/K_p)_{\mathbb{C}} = \overline{\bigoplus_{\pi_p} H_{\pi_p}^{m(\pi_p)}}.$$

The π_p denote the isomorphism types, let us call the set of isomorphism types $\hat{\mathcal{H}}_{[\Gamma]}(G(\mathbb{Q}_p)/K_p)$. We may decompose this set into a discrete part and a part coming from principal series representations (or irreducible representations which are induced from a supercuspidal representation):

$$\hat{\mathcal{H}}_{[\Gamma]}(G(\mathbb{Q}_p)/K_p) = \hat{\mathcal{H}}_{[\Gamma],\mathrm{princ}} \cup \hat{\mathcal{H}}_{[\Gamma],\mathrm{disc}}.$$

The discrete part is finite, it occurs already for any choice of $\Gamma \in [\Gamma]$, provided it is torsion free. We have projectors to the discrete and the non discrete part, accordingly we can decompose

$$\mathcal{H}(G(\mathbb{Q}_p)//K_p)_{\mathbb{C}} = \mathcal{H}(G(\mathbb{Q}_p)//K_p)_{\mathbb{C},\mathrm{princ}} \bigoplus \mathcal{H}(G(\mathbb{Q}_p)//K_p)_{\mathbb{C},\mathrm{disc}}$$

The second summand is a finite dimensional semi simple algebra, which is a direct sum of full matrix algebras. The summands correspond to the elements $\pi_p \in \mathcal{H}_{[\Gamma],\mathrm{disc}}$. We have a system of orthogonal idempotents $e_{\pi_p} \in \mathcal{H}(G(\mathbb{Q}_p)//K_p)_{\mathbb{C}})_{\mathrm{disc}}$ so that we get a decomposition into full matrix algebras

$$\mathcal{H}(G(\mathbb{Q}_p)//K_p)_{\mathbb{C},\mathrm{disc}} = \bigoplus_{\pi_p \in \hat{\mathcal{H}}_{[\Gamma],\mathrm{disc}}} \mathcal{H}(G(\mathbb{Q}_p)//K_p)_{\mathbb{C}} e_{\pi_p}.$$

(Here it may be necessary to discuss discrete series representations, which are not supercuspidal more carefully. For $\mathrm{Gl}_2/\mathbb{Q}_p$ we only have the Steinberg module, which is easy to understand in this case.)

2.7.3 Characters and orbital integrals

Now we consider the case $\mathrm{Gl}_2/\mathbb{Q}_p$, our consideration are also true in general, but the discussion becomes more complicated, since we have to consider the composition series of induced representations more carefully.

We can give a different description of the discrete summand. We consider the linear map given by the orbital integral

$$\tilde{O} : \mathcal{H}(G(\mathbb{Q}_p)//K_p) \to \mathcal{C}(\mathbb{Q}_p),$$

we may restrict the functions $\tilde{O}(h,t)$ to the regular elements and then to the different regions, which are given by the conjugacy classes of maximal tori. Let $\tilde{O}(h,t)_{\mathrm{split}}$ be the restriction to the split regular semisimple classes.

Let us assume that we have an element $h \in \mathcal{H}(G(\mathbb{Q}_p)//K_p)$ for which $\tilde{O}(h,t)_{\mathrm{split}} = 0$. Then for any $\pi_p \in \hat{\mathcal{H}}_{[\Gamma],\mathrm{princ}}$ we get

$$\mathrm{tr}(h|\pi_p) = \int_{\mathcal{C}(\mathbb{Q}_p)} \tilde{O}(h,c)\mathrm{ch}_{\pi_p}(c)|dx|(dc) = 0,$$

because we have seen, that the character ch_{π_p} vanishes on non elliptic regular semi simple elements. Hence we can conclude, that $h \in \mathcal{H}(G(\mathbb{Q}_p)//K_p)_{\mathbb{C},\mathrm{disc}}$.

We return to the characters. An irreducible admissible representation π_p of $G(\mathbb{Q}_p)$ has a central character $\omega_{\pi_p} = \omega_p$. In the following we fix the central character of our representations, let us denote it by ω_p. We consider the space $\mathcal{C}_{c,\omega_p^{-1}}$. For any $h \in \mathcal{C}_{c,\omega_p^{-1}}$ we defined the trace $\mathrm{tr}(h|\pi_p)$. We apply Harish-Chandra theorem and get

$$\mathrm{tr}(h|\pi_p) = \int_{G(\mathbb{Q}_p)/Z(\mathbb{Q}_p)} h(g)\mathrm{ch}_{\pi_p}(g)dg.$$

Now we have to show a little bit of courage and apply Weyls formula and get

$$\mathrm{tr}(h|\pi_p) = \int_{G(\mathbb{Q}_p)/Z(\mathbb{Q}_p)} h(g)\mathrm{ch}_{\pi_p}(g)dg = \int_{\mathcal{C}(\mathbb{Q}_p)/Z(\mathbb{Q}_p)} \tilde{O}(h,c)\mathrm{ch}_{\pi_p}(c)|dx|(dc).$$

We want to plug in suitable functions h into this formula. Such functions h will be a contragredient matrix coefficients $h(g) =< v_i, g\phi_j >_{\pi'_p}$ of supercuspidal representations π'_p or suitable Euler-Poincare function h^{EP}_χ.

We have the following fact

The hyperbolic orbital integrals and the regular unipotent orbital integrals of these functions h vanish.

Then we can conclude from the Shalika germ expansion that the orbital integrals of our functions h are locally constant on $(\mathcal{C}(\mathbb{Q}_p)/Z(\mathbb{Q}_p))_{\mathrm{reg}}$. They are even locally constant with compact support if we restrict them to the closure of any $\mathcal{C}(\mathbb{Q}_p)_F$.

From here it is not so difficult any more to derive the following facts

(i) For all h in our space of functions we have

$$\mathrm{tr}(h|\pi_p) = 0 \textit{ if } \pi_p \textit{ is in the principal series}$$

(ii) If $h =< v_i, g\phi_j >_{\pi'_p}$ for a supercuspidal representation π'_p then $\mathrm{tr}(h|\pi_p) = 0$ *if π_p is not equivalent to π'_p or if $i \neq j$.*

(iii) If $\pi_p = \pi'_p$ and $i = j$ then $\mathrm{tr}(h|\pi_p) = \frac{1}{d(\pi_p)}$

(iv) If $h = h_\chi^{EP}$ then $\mathrm{tr}(h|\pi_p) = 0$ *if $\pi_p \neq St_\chi$ and equal to 1 else*

This implies that for a supercuspidal π_p

$$O(< v_i, g\phi_j >_{\pi'_p}, c) = \frac{1}{d(\pi_p)}\mathrm{ch}_{\pi_p}(c) \text{ for all } c \in \mathcal{C}(\mathbb{Q}_p)_{\mathrm{reg}}$$

and similarly

$$O(h_\chi^{EP}, c) = \mathrm{ch}_{\mathrm{St},\chi}(c).$$

This proves orthogonality relations for characters in the discrete series:
For two discrete series representations π_p, π'_p we have

$$\int_{\mathcal{C}(\mathbb{Q}_p)_{\mathrm{reg}}/Z(\mathbb{Q}_p)} ch_{\pi_p^\vee}(c) ch_{\pi'_p}(c)|dx|(dc) = \begin{cases} 1 & \textit{for } \pi_p = \pi'_p \\ 0 & \textit{else} \end{cases}$$

2.7.4 The orbital integrals of the identity element

We now choose the measure that has volume one on $\mathrm{Gl}_2(\mathbb{Z}_p)$, then the characteristic function h_0 of $\mathrm{Gl}_2(\mathbb{Z}_p)$ is the identity in the unramified Hecke algebra.

We represent the conjugacy classes of semi simple elements by matrices $\gamma = \gamma_{a,b} = \begin{pmatrix} a & -1 \\ b & 0 \end{pmatrix}$. Since we want to have non zero orbital integrals we assume that $b \in \mathbb{Z}_p^\times$ and $a \in \mathbb{Z}_p$. Our local integral is

$$\int_{Z_\gamma(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)} \chi_{\mathrm{Gl}_2(\mathbb{Z}_p)}(\bar{g}_p^{-1}\gamma\bar{g}_p) d\bar{g}_p =$$
$$\sum_{\bar{g}_p \in Z_\gamma(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)/K_p} [K_{p,\max}^Z : K_p^Z(g_p)]\chi_{\mathrm{Gl}_2(\mathbb{Z}_p)}(\bar{g}_p^{-1}\gamma\bar{g}_p) = O^{\mathrm{arith}}(h_0, \gamma).$$

Now we observe that the condition $\chi_{\mathrm{Gl}_2(\mathbb{Z}_p)}(\bar{g}_p^{-1}\gamma\bar{g}_p) = 1$ means that for a representative g_p of $\bar{g}_p$ the free $\mathbb{Z}_p$ module $g_p\mathbb{Z}_p^2 \subset \mathbb{Q}_p^2$ is in fact a $\mathbb{Z}_p[\gamma]$ module.

Two such $\mathbb{Z}_p[\gamma]$ modules $g_p\mathbb{Z}_p^2, g'_p\mathbb{Z}_p^2$ are the same if and only if $g_p \in g'_p\mathrm{Gl}_2(\mathbb{Z}_p)$ and the are isomorphic if and only if $g_p \in Z_\gamma(\mathbb{Q}_p)g'_p$. Hence we see that our orbital integral is equal to the sum

$$\sum_{[M_p]} \mathrm{vol}_{dz_p}(\mathrm{Aut}(M_p))^{-1}$$

over the isomorphism classes of $\mathbb{Z}_p[\gamma]$ submodules $M \subset \mathbb{Q}_p^2$ which are $\mathbb{Z}_p$ modules of rank 2.

It is a tedious computation to evaluate these sums. I want to give a brief outline how this can be done. We have to consider the case of p-elliptic elements. In this case we can conjugate our torus into a torus $T/\mathbb{Q}_p$ such that $\mathcal{T}(\mathbb{Z}_p) \subset \mathrm{Gl}_2(\mathbb{Z}_p)$ and γ_a gets conjugated into an element $\gamma \in \mathcal{T}(\mathbb{Z}_p)$.

Now it is clear, that the fixed point set of $\mathcal{T}(\mathbb{Z}_p)$ on the Bruhat-Tits building is

$$\mathbb{BT}(\mathrm{Gl}_2/\mathbb{Q}_p)^{\mathcal{T}(\mathbb{Z}_p)} = \text{the module } \mathbb{Z}_p^2,$$

if the torus splits over the unramified extension. If the torus splits over a ramified extension, then we should remember that we can identify $\mathbb{Z}_p^2 = \mathcal{O}_F$ and the modules that are fixed by $\mathcal{T}(\mathbb{Z}_p)$ are the ideals. Hence we see two fixed points in the building namely $\mathcal{O}_F$ and the maximal prime ideal, which defines a sublattice of index p. This two fixed points are joined by an edge, which also consists of fixed points. This edge is the fixed point set.

Now our element $\gamma \in \mathcal{T}(\mathbb{Z}_p)$ has a set of fixed points on $\mathbb{BT}(\mathrm{Gl}_2/\mathbb{Q}_p))$, the vertices in this fixed point set are exactly the modules M_p which are $\mathbb{Z}_p[\gamma]$ modules.

Hence it becomes clear that

$$\sum_{[M_p]} \mathrm{vol}_{dz_p}(\mathrm{Aut}(M_p))^{-1} = \#\mathbb{BT}(\mathrm{Gl}_2/\mathbb{Q}_p)^\gamma$$

and to compute the orbital integral we have to count the number of fixed points.

A similar reasoning can be applied in the the case of a split torus. In this case the fixed point set of $\mathcal{T}(\mathbb{Z}_p)$ is an apartment in $\mathbb{BT}(\mathrm{Gl}_2/\mathbb{Q}_p)$ this means in concrete terms it is the set of lattices

$$\{\mathbb{Z}_p \oplus p^\nu \mathbb{Z}_p\}_{\nu\in\mathbb{Z}}.$$

On this fixed point set we have the action of $T(\mathbb{Q}_p)/\mathcal{T}(\mathbb{Z}_p) = \mathbb{Z}^2$, where the central elements act trivially. Hence we can argue as before, but we have to divide the fixed point set $\mathbb{BT}(\mathrm{Gl}_2/\mathbb{Q}_p)^\gamma$ by the action of $\mathbb{Z}$ and then we have to count the fixed points. This also explains the vanishing of the Euler characteristic in 2.7. in this case.

We say that γ is *maximal* at p if $\mathbb{Z}_p[\gamma]$ is a maximal order. For γ maximal at p the value of the orbital integral is equal to 1. If $p \neq 2$ then the element γ is maximal at p if p^2 does not divide $a^2 - 4b$.

We conclude that for a maximal element γ we have

$$O^{\mathrm{arith}}(h, \gamma) = 1$$

in all cases(???).

In general let $p^{d(\gamma)}$ be the exact power of p dividing $a^2 - 4b$ and let $s(\gamma) = [\frac{d(\gamma)}{2}]$ the Gauss bracket.

We begin with the case $p > 2$ then we find for the orbital integrals

$$\frac{p^{s(\gamma)+1} + p^{s(\gamma)} - 2}{p-1} \text{ if } d(\gamma) \text{ is even and } \mathbb{Q}_p(\sqrt{a^2 - 4b}) \text{ is a field}$$

$$p^{s(\gamma)} \text{ if } d(\gamma) \text{ is even and } \mathbb{Q}_p(\sqrt{a^2 - 4b}) \text{ is } \mathbb{Q}_p \oplus \mathbb{Q}_p$$

$$\frac{p^{s(\gamma)+1} - 1}{p-1} \text{ if } d(\gamma) \text{ is odd}$$

For $p = 2$ we get

$$1 \text{ if } a \text{ is odd } (a^2 - 4b \text{ is odd })$$

Now we consider the case that $d(\gamma)$ is even, then we get the values

$$2^{s(\gamma)+1} + 2^{s(\gamma)} - 2 \text{ if } \mathbb{Q}_2(\sqrt{a^2 - 4b}) \text{ is an unramified field extension}$$

$$2^{s(\gamma)} - 1 \text{ if } \mathbb{Q}_2(\sqrt{a^2 - 4b}) \text{ is a ramified field extension}$$

$$2^{s(\gamma)} \text{ if } \mathbb{Q}_2(\sqrt{a^2 - 4b}) \text{ is split}$$

Finally we find

$$2^{s(\gamma)} - 1 \text{ if } m(a, 2) \text{ is odd}$$

Let me just observe that the orbital integral at p becomes large if the eigenvalues of γ are p- adically close to each other. We also check easily that the extension $\mathbb{Q}_p(\sqrt{a^2 - 4b})$ can only split if $d(\gamma)$ is even and then it is also clear that $p^{s(\gamma)} = |\alpha(\gamma) - 1)|_p$ and in the split case we get our previous results. (germ expansion)

2.8 The case $h = t_{p^m}$

Recall that t_{p^m} is the characteristic function of $G(\mathbb{Z}_p) \begin{pmatrix} p^m & 0 \\ 0 & 1 \end{pmatrix} G(\mathbb{Z}_p)$. In this case we put $\gamma = \begin{pmatrix} a & - \\ 1 & p^m \end{pmatrix} 0$. We want to compute

$$\mathcal{O}(t_{p^m}, \gamma) = \int_{Z_\gamma(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)} t_{p^m}(\bar{g}_p \gamma \bar{g}_p^{-1}) d\bar{g}_p$$

Let p^t be the highest power of p dividing a. Let us assume that $2t < m$, this is the easy case. Then $\sqrt{a^2 - 4p^m} = a\sqrt{1 - 4p^m/a^2}$ and p^m/a^2 is still divisible by p. This implies that the square root exists in $\mathbb{Q}_p$ and hence γ splits. It is easy to see that the two eigenvalues have p adic order t and $m - t$, where we know $m - t > t$. Hence 3.2.1 applies and we get under our assumption $2t < m$

$$O(t_{p^m}, \gamma) = \begin{cases} 1 & \text{if } t = 0 \\ p^t(1 - \frac{1}{p}) & \text{if } t > 0 \end{cases}$$

I have not yet checked the other cases, but I think I know the answer for $p > 3$.

We define

$$\chi(a,p) = \begin{cases} 1 & \text{If the extension } \mathbb{Q}_p(\sqrt{a^2 - 4p^m}) \text{ is split} \\ 0 & \text{If the extension } \mathbb{Q}_p(\sqrt{a^2 - 4p^m}) \text{ is ramified} \\ -1 & \text{If the extension } \mathbb{Q}_p(\sqrt{a^2 - 4p^m}) \text{ is unramified non split} \end{cases}.$$

Then let $\nu(a)$ be the highest power of p dividing a. We set $\nu(a) = [m/2]$ for $a = 0$. Then at least in the cases $p > 3$ we have

$$O(t_{p^m}, \gamma) = p^{\nu(a)-1}(p - \chi(a,p))$$

2.9 The fundamental lemma

This fundamental lemma is an assertion, which tells us that we have a lot of cancellations between the expressions, which give us the two orbital integrals which sum up to the stable orbital integral. We go back to Chap. V 3.2.3., where we considered the group $\mathrm{Gl}_2/\mathbb{Q}_p$ and gave the values for the orbital integrals of the characteristic function of $\mathrm{Gl}_2(\mathbb{Z}_p)$ at the matrix $\gamma_a = \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix}$. We observe that this orbital is stable and equal to the stable orbital integral for the characteristic function h_1 of $\mathrm{Sl}_2(\mathbb{Z}_p)$. The centralizer of γ_a is a maximal torus $T^{(1)}/\mathbb{Q}_p$. Let us consider the case that $T^{(1)} \subset \mathrm{Sl}_2/\mathbb{Q}_p$ is an anisotropic torus, which corresponds to the unramified quadratic extension $F/\mathbb{Q}_p$. We assume that $T^{(1)}(\mathbb{Z}_p) \subset \mathrm{Sl}_2(\mathbb{Z}_p)$. Then we have $D(\gamma_a)) = a^2 - 4$ and $|D(\gamma_a)|_p = p^{-d(\gamma_a))} = p^{-2s(\gamma_a))}$. l.

In the previous section we stated that the value of the orbital integral $O^{\mathrm{arith}}(h, \gamma_a)$ is given by

$$\frac{p^{s(\gamma)+1} + p^{s(\gamma)} - 2}{p-1}.$$

This is the sum of two instable orbital integrals, the stable class decomposes into two $G(\mathbb{Q}_p)$ conjugacy classes. This decomposition is given by

$$\frac{p^{s(\gamma)+1}}{p-1} - \frac{1}{p-1} \text{ and } \frac{p^{s(\gamma)}}{p-1} - \frac{1}{p-1}.$$

If we now take the difference, then this is the κ-orbital integral and we get simply

$$\frac{p^{s(\gamma)+1} - p^{s(\gamma)}}{p-1} = p^{s(\gamma)}.$$

This values has to be divided by $|D(\gamma_a)|_p^{-1/2}$ and the value of the κ-orbital becomes constant, it does not depend on a. This is the assertion b) ii in the statement of the fundamental lemma for the special case of the function h_1. We observe that we get a remarkable simplification.

To get a better understanding of these simplification we have to discuss the Shalika germ expansion for the group Sl_2. In this group we have several conjugacy classes of regular unipotent elements, they are represented by matrices

$$\left\{ \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} \mid \xi \in \mathbb{Q}_p^\times, \xi \mod (\mathbb{Q}_p^\times)^2 \right\}.$$

If we now take a quadratic extension $F/\mathbb{Q}_p$, then we know that $p^{-1}(\mathcal{C}(\mathbb{Q}_p)_F$ decomposes into two sets: We pick a $c \in \mathcal{C}(\mathbb{Q}_p)_F$ and a $\gamma \in p^{-1}(c)(\mathbb{Q}_p)$. Its centralizer is a maximal torus $T/\mathbb{Q}_p$, we have a second torus $T'/\mathbb{Q}_p$ in the same inner class, but which is not conjugate to $T/\mathbb{Q}_p$. Then we get the decomposition

$$p^{-1}(\mathcal{C}(\mathbb{Q}_p)_F = C \cup C'$$

where C (resp. C') consists of the elements which are conjugate to an element in $T(\mathbb{Q}_p)$(resp.$T'(\mathbb{Q}_p)$). At this point the roles of C, C' and the roles of these two tori are totally symmetric, we can not distinguish between them.

The closure of C (resp. C') contain conjugacy classes of regular unipotent elements, but each of the closures only contains one half of the regular unipotent elements. We need to understand this decomposition into two subsets.

I recall that we also have chosen a semi simple integral structure $\mathcal{G}/\mathbb{Z}_p = \mathrm{Sl}_2/\mathbb{Z}_p$, we also recall that up to conjugation we have two such choices. We also selected a pair of maximal tori $T/\mathbb{Q}_p, T'/\mathbb{Q}_p$, for any inner splitting class, which is given by a field extension $F/\mathbb{Q}_p$. We always make our choice so that $T(\mathbb{Z}_p) \subset \mathcal{G}(\mathbb{Z}_p)$. To this second torus corresponds the non trivial class in $\mathbb{Q}_p^\times/N_{F/\mathbb{Q}_p}F^\times$. To see what happens we have to distinguish between the two cases

A) The torus $T/\mathbb{Q}_p$ splits over the unramified quadratic extension. In this case it is clear that $T'(\mathbb{Z}_p) \not\subset \mathcal{G}(\mathbb{Z}_p)$. This tells us that in this case the choice of the integral structure $\mathcal{G}$ gives us a rule to distinguish between C and C'. If we choose a conjugacy class of regular unipotent elements, which we represent by an element $\begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}$, then we get regular semi simple elements which are closed to the given unipotent element if we write down

$$\begin{pmatrix} 1 & 0 \\ \eta & 1 \end{pmatrix} \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \xi \\ \eta & \eta\xi + 1 \end{pmatrix},$$

where $\mathrm{ord}_p(\eta) >> 0$. This element has trace $2 + \xi\eta = c$, which is close to 2 and it is regular if $\eta \neq 0$. Now we ask, when are two such elements

$$\begin{pmatrix} 1 & \xi \\ \eta & \eta\xi + 1 \end{pmatrix}, \begin{pmatrix} 1 & \xi' \\ \eta' & \eta'\xi' + 1 \end{pmatrix}$$

with the same trace are actually conjugate in $\mathrm{Sl}_2(\mathbb{Q}_p)$. It is clear that they are conjugate in $\mathrm{Gl}_2(\mathbb{Q}_p)$ and the conjugating matrix is $\begin{pmatrix} \xi/\xi' & 0 \\ 0 & 1 \end{pmatrix}$. It is clear that we may choose for our torus $T/\mathbb{Q}_p$ the centralizer of any element $\begin{pmatrix} 1 & 1 \\ \eta & \eta + 1 \end{pmatrix}$, provided it corresponds to the unramified extension. If we replace this element by $\begin{pmatrix} 1 & \xi \\ \eta\xi^{-1} & \eta + 1 \end{pmatrix}$, then the conjugacy class of its centralizer corresponds to $\xi \in \mathbb{Q}_p^\times/N_{F/\mathbb{Q}_p}F^\times = \mathbb{Z}/2\mathbb{Z}$, the non trivial class is given by the prime element $\{p\}$. This tells us that in this case

The the unipotent classes in the closure of C are represented by $\begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}$ where $\xi \in \mathbb{Z}_p^\times$. The the unipotent classes in the closure of C' are represented by $\begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}$ where $\xi \in p\mathbb{Z}_p^\times$.

The other case is

B) The torus $T/\mathbb{Q}_p$ splits over a ramified extension $F/\mathbb{Q}_p$. Actually we may proceed in the same way, we choose our torus $T/\mathbb{Q}_p$ so that $T(\mathbb{Z}_p) \subset \mathcal{G}(\mathbb{Z}_p)$. Again we know that the other torus is obtained from the non trivial class in $\mathbb{Q}_p^\times/N_{F/\mathbb{Q}_p}F^\times$. But now we can represent this non trivial class by a unit $\xi_0 \in Z_p^\times$. This implies that our second torus $T'/\mathbb{Q}_p$ can also be chosen in such a way that $T(\mathbb{Z}_p) \subset \mathcal{G}(\mathbb{Z}_p)$. (This means that the integral structure $\mathcal{G}$ does not provide a rule to distinguish between T and T' and this is more or less the reason for the assertion (ii) in the fundamental lemma.)

We find a unit $u \in \mathbb{Z}_p t$, such that up is a norm and then

The the unipotent classes in the closure of C are represented by $\begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}$ where $\xi \in \mathbb{Z}_p t^2$ or $\xi \in up\mathbb{Z}_p t^2$. The the unipotent classes in the closure of C' are represented by $\begin{pmatrix} 1 & \xi\xi_0 \\ 0 & 1 \end{pmatrix}$ where $\xi \in \mathbb{Z}_p t^2$ or $\xi \in up\mathbb{Z}_p t^2$.

Now we are in the position to discuss the Shalika expansion for the group $\mathrm{Sl}_2/\mathbb{Z}_p$. For any anisotropic torus $T_1/\mathbb{Q}_p$ and any regular unipotent conjugacy class $\mathcal{O}$ we define

$$\chi(T_1, \mathcal{O}) = \begin{cases} 1 & \text{if } \mathcal{O} \subset \bar{C}_1 \\ 0 & \text{else} \end{cases}$$

where C_1 is the class which is determined by the conjugacy class of $T_1/\mathbb{Q}_p$.

Then the Shalika germ expansion will give us for any h in the Hecke algebra an asymptotic expansion: A regular anisotropic element defines a torus $T = T(\gamma)$ and if γ is close to one of the central elements $z \in Z(\mathbb{Q}_p)$ then

$$\tilde{O}(h, \gamma) = \sum_{\mathcal{O}} \chi(T, \mathcal{O}) O(h, \mathcal{O}) + |D(\gamma)|_p^{1/2} c_T h(z)$$

with some constant c_T.

If we replace our torus by the other torus in the inner class we get

$$\tilde{O}(h, \gamma') = \sum_{\mathcal{O}} \chi(T', \mathcal{O}) O(h, \mathcal{O}) + |D(\gamma')|_p^{1/2} c_{T'} h(z)$$

The $\kappa-$ orbital integral is then the difference of these two terms multiplied by $|D(\gamma')|_p$, hence we get for this κ-orbital integral the asymptotic expansion

$$\sum_{\mathcal{O}} \chi(T, \mathcal{O}) \tilde{O}(h, \mathcal{O}) - \sum_{\mathcal{O}} \chi(T', \mathcal{O}) \tilde{O}(h, \mathcal{O}) + |D(\gamma)|_p^{1/2} (c_T - c_{T'}) h(z).$$

The first term does not depend on γ. Hence it becomes quite clear that the continuity in the assertion of the fundamental lemma is equivalent to $c_T = c_{T'}$.

To get this equality of the constants and therefore the continuity it suffices to compute these constants for any function, which is non zero at the central elements. We consider the congruence subgroup

$$K^{(1)} = \{\gamma = \begin{pmatrix} 1+px & py \\ pz & 1+pu \end{pmatrix} | x, y, z, u \in \mathbb{Z}_p\}$$

of $\mathrm{Sl}_2(\mathbb{Z}_p)$ let h be its characteristic function. We want to compute the orbital integrals $O(h,\gamma)$ for $\gamma \in K^{(1)}$. We assume that $p>2$.The determinant condition allows us to eliminate one of the variables, we get

$$u = \frac{-x+pyz}{1+px},$$

then the Tamagawa measure on $K^{(1)}$ is

$$\frac{|dx\wedge dy\wedge dz|}{p^3}.$$

The trace maps $K^{(1)}$ to $2+p\mathbb{Z}_p$, we put

$$u+x = \frac{-x+pyz}{1+px} + x = p\frac{x^2+yz}{1+px} = pv$$

and hence

$$v = \frac{x^2+yz}{1+px}.$$

Then the trace of the matrix is $2+p^2v$ and $|D(\gamma)|_p = p^{-2-\mathrm{ord}_p(v)}$. We get the relation among differentials

$$(1+px)dv = (-vp+2x)dx + zdy + ydz.$$

Then except for the singular point $(x,y,z)=(0,0,0)$ one of the factors on the right hand side is non zero hence on G_{reg} we can eliminate locally one of the differentials dx, dy, dz and then it becomes clear that outside the set $y=0$, which is of measure zero, we get

$$|\omega_{\mathcal{H}\backslash G}| = \frac{1}{p^3|y|_p}dx\wedge dy$$

Hence we see: If we fix the value of $v\in\mathbb{Z}_p$, then the value of the stable orbital integral is (we forget the factor $1/p^3$ for a while)

$$\int_{\{(x,y,z)|v(1+px)=x^2+yz\}} \frac{|dx\wedge dy|}{|y|_p}.$$

The domain of integration needs some explanation. It contains the variable z, but it is quite clear that z is uniquely determined by the algebraic relation. Hence we are integrating over x, y, but we must be able to solve the relation in z. This means that we have to compute the volume of the domain

$$\{(x,y)|\mathrm{ord}_p(v(1+px)-x^2)\geq \mathrm{ord}_p(y)\}.$$

We distinguish cases. If $v=0$, then we compute the volume of the regular unipotent orbit intersected with $K^{(1)}$, this regular unipotent orbit is given by $yz+x^2=0$, where the central elements are removed. We write the integral over y as an infinite sum, we group according to the order of y. The volume of any annulus $\{y|\mathrm{ord}_p(y)=\nu\}$ with respect to the measure $\frac{dy}{|y|_p}$ is $1-\frac{1}{p}$ hence the summation gives us

$$(1-\frac{1}{p})\sum_{\nu=0}^{\infty}\int_{\mathbb{Z}_p}\{x|2\mathrm{ord}_p(x)\geq\nu\}|dx|(dx)$$
$$=(1-\frac{1}{p})(1+\frac{1}{p})\sum_{\nu=0}^{\infty}p^{-\nu}=1+\frac{1}{p}.$$

This is the stable orbital integral over the regular unipotent elements.

We compute the instable orbital integrals over the unipotent classes. As we know the set of regular unipotent elements in $G(\mathbb{Q}_p)$ decomposes into orbits under $G(\mathbb{Q}_p)$ these correspond to the classes in $\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2$, a unipotent element

$$\gamma=\begin{pmatrix}1+px & py\\ pz & 1-px\end{pmatrix}$$

lies in the class py or pz, one of the two elements is non zero, if both are non zero then their product is x^2p^2, hence they represent the same class. Therefore to get the orbital integrals of h over the individual conjugacy classes $\xi\in\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2$ we have to pick a ν and to replace the factor $(1-\frac{1}{p})$ by $\mathrm{vol}_{|dy|/|y|_p}(\{y\equiv\xi\})$. We can define $\mathrm{ord}_p(\xi) \mod 2$, then this factor is zero if $\mathrm{ord}_p(\xi)\not\equiv\nu+1 \mod 2$ and $(1-\frac{1}{p})/2$ else. Hence we get for the orbital integrals over the different unipotent classes (we still assume $p>2$)

$$\begin{cases}\frac{1}{2} & \text{if } \mathrm{ord}_p(\xi)\equiv 1\\ \frac{1}{2p} & \text{else}\end{cases}.$$

We look at the case $v\neq 0$ we go back to the stable integrals. We consider the stable integral first. We have

$$\mathrm{ord}_p(v(1+px)-x^2)=\begin{cases}\min(\mathrm{ord}_p(v(1+px)),\mathrm{ord}_p(-x^2)) & \text{if the orders are different}\\ \mathrm{ord}_p(v-\frac{x^2}{1+px}) & \text{if these orders are equal}\end{cases}.$$

The second case can only happen if $\mathrm{ord}_p(v)$ is even, and then only if we can solve the quadratic equation $x^2-(1+px)v=0$, i.e. if v is a square, and this means that the torus splits.

We assume that the torus is not split. Then we see that our summation gives us

$$(1-\frac{1}{p})\sum_{\nu=0}^{\mathrm{ord}_p((v)}\mathrm{vol}_{|dx|}\{x|2\mathrm{ord}_p(x)\geq\nu\}=$$
$$(1-\frac{1}{p})\sum_{\nu=0}^{\infty}\mathrm{vol}_{|dx|}\{x|2\mathrm{ord}_p(x)\geq\nu\}-$$
$$(1-\frac{1}{p})\sum_{\nu=\mathrm{ord}_p(v)+1}^{\infty}\mathrm{vol}_{|dx|}\{x|2\mathrm{ord}_p(x)\geq\nu\}$$

The first term is again the unipotent orbital integral. To compute the second term we observe that

$$\mathrm{vol}_{|dx|}\{x|2\mathrm{ord}_p(x) \geq \nu\} = \begin{cases} p^{-\frac{\nu}{2}} & \text{if } \nu \text{ is even} \\ p^{-\frac{\nu+1}{2}} & \text{if } \nu \text{ is even} \end{cases}$$

If $\mathrm{ord}_p(v) + 1$ is even, then the infinite sum is

$$(1 - \frac{1}{p})p^{-\frac{\mathrm{ord}_{p(v)+1}}{2}}(1 + 2(\frac{1}{p} + \frac{1}{p^2} + \dots)) = (1 + \frac{1}{p})p^{-\frac{\mathrm{ord}_{p(v)+1}}{2}}$$

If $\mathrm{ord}_p(v) + 1$ is odd, then we get

$$2(1 - \frac{1}{p})p^{-\frac{\mathrm{ord}_{p(v)+2}}{2}}(1 + \frac{1}{p} + \frac{1}{p^2} + \dots)) = 2p^{-\frac{\mathrm{ord}_{p(v)+2}}{2}}$$

The torus defined by v is unramified if and only if v is even. Recall that for a matrix γ with a given trace $\mathrm{tr}(\gamma) = 2 + p^2 v$ we have

$$|D(\gamma)|_p^{1/2} = p^{-1-\mathrm{ord}_p(v)/2}.$$

Therefore we define for an anisotropic torus $T/\mathbb{Q}_p \subset \mathrm{Sl}_2/\mathbb{Q}_p$ the constant

$$c_T = \begin{cases} 2 & \text{if the torus is unramified} \\ (1 + \frac{1}{p})p^{1/2} & \text{else} \end{cases}$$

The factor in front depends only on the inner splitting class defined by v. Hence we proved the Skalika germ expansion for the stable orbital integrals and we computed the constants.

If we consider the instable orbital integrals, then we have to divide the regions

$$\{x|2\mathrm{ord}_p(x) \geq \nu\}$$

into the two parts corresponding to the two conjugacy classes (v is given). If the extension is unramified, then this division is according to the parity of $\mathrm{ord}_p(x)$. But if we look at the infinite sum which gives us the second term, we see that all powers of p occur twice, and each power of p is obtained once from an x with $\mathrm{ord}_p(x)$ even and from another x with $\mathrm{ord}_p(x)$ odd. We conclude that the two constants for the two conjugacy classes of unramified tori are equal to one.

For the ramified tori we consider the quadratic character χ_F attached to the field. We find a unit $u_0 \in \mathbb{Z}_p^\times$ such that $\chi_F(u_0 p) = 1$. For any $x \in \mathbb{Z}_p \setminus \{0\}$ we look at the value

$$cc_F(x) = \chi_F(x/(u_0 p)^{\mathrm{ord}_p(x)})$$

which is ± 1 and decides in which conjugacy class the element γ lies. Hence it is clear that any of our sets $\{x|2\mathrm{ord}_p(x) \geq \nu\}$ is divided in two classes of equal volume. This means that the constants for the two conjugacy classes of an inner splitting class are just half of the constants for the stable germ expansion.

This gives us the fundamental lemma for the characteristic function of $K^{(1)}$, more precisely we get it asymptotically for γ approaching central elements. But it is clear that it is valid for all elements $\gamma \in K^{(1)}$.

We may also do our computation for the split torus, actually it is not necessary because we know that in this case the value of the trace determines the conjugacy class. But the computation is a little bit amusing. We know that $\mathrm{ord}_p(v)$ must be even. We have that for $\nu \leq \mathrm{ord}_p(v)$ we have the equality of the two sets

$$\{x|\mathrm{ord}_p(v(1+px)-x^2) \geq \nu\} = \{x|2\mathrm{ord}_p(x) \geq \nu\}.$$

If $\nu > \mathrm{ord}_p(v)$, then in the case of a non split torus, the set on the left hand side is empty, therefore the summation above stopped at $\nu = \mathrm{ord}_p(v)$. But in the split case we can find two element x_1, x_x which solve the equation $v(1+px_i)-x_i^2 = 0$, these elements satisfy $\mathrm{ord}_p(x_i) = \mathrm{ord}_p(v)/2$. Then we get for $\nu > \mathrm{ord}_p(v)$

$$\{x|\mathrm{ord}_p(v(1+px)-x^2) \geq \nu\} = \{x = x_i + p^\nu z|i = 1,2, z \in \mathbb{Z}_p\}.$$

The volume of this set is $p^{-\mu}$. and we see that the terms in the summation are exactly the same as the summations for the unipotent orbital integrals, we find that the constant is 0.

Hence it remains to show the assertion concerning the values of the κ-orbital integrals for the elements h in the unramified Hecke algebra. Here we check them for elements $\gamma \in T(\mathbb{Q}_p)$ which are close to central elements, we get

$$\tilde{O}(h,\gamma,\kappa) = \sum_{\mathcal{O}} \chi(T,\mathcal{O})\tilde{O}(h,\mathcal{O}) - \sum_{\mathcal{O}} \chi(T',\mathcal{O})\tilde{O}(h,\mathcal{O})$$

and this value is constant. To compute this value we have to compute the orbital integrals $\tilde{O}(h, \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix})$ for the different choices of ξ. To do this we recall the computation in Chap III. (ref). Let $T_0/\mathbb{Q}_p$ be the standard split maximal torus. We recall the Iwasawa decomposition

$$U(\mathbb{Q}_p) \times T_0(\mathbb{Q}_p) \times K_p \to G(\mathbb{Q}_p),$$

under this map the bi-invariant measure on $G(\mathbb{Q}_p)$ becomes the measure $|t_p^{-2\rho}|du_p \times dt_p \times dk_p$, where the volumes of $\mathcal{U}(\mathbb{Z}_p), \mathcal{T}(\mathbb{Z}_p), K_p$ are equal 1. This pair of measures induces the measure $\tilde{\omega}^{\mathrm{arith}}_{T_0\backslash\mathcal{G}}$ on the set of regular split semi simple conjugacy classes. Recall that $2\rho = \alpha$, this is the positive root. This measure relates to our geometric measure by

$$\tilde{\omega}^{\mathrm{arith}}_{T_0\backslash\mathcal{G}}(c) = |D(c)|_p^{-1/2}\frac{p+1}{p}\omega_{G/\mathcal{C}}(c).$$

We write $g \in G(\mathbb{Q}_p)$ as $g = utk$ then $g^{-1}\begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} g = k^{-1}t^{-1}\begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} tk$. As usual we write $t = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ and we find for our orbital integral

$$\tilde{O}(h, \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}) = |\xi|_p \int_{T^{(1)}(\mathbb{Q}_p)} \int_K h(k^{-1}t^{-1}\begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} tk)|t|_p^{-2}dkdt$$

$$= |\xi|_p \int_{T^{(1)}(\mathbb{Q}_p)} h(t^{-1}\begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} t)|t|_p^{-2}dt = |\xi|_p \int_{T^{(1)}(\mathbb{Q}_p)} h(\begin{pmatrix} 1 & \alpha^{-1}(t)\xi \\ 0 & 1 \end{pmatrix})|t|_p^{-2}dt.$$

Now it is clear that the value $h(\begin{pmatrix} 1 & \alpha^{-1}(t)\xi \\ 0 & 1 \end{pmatrix})$ only depends on $|\alpha^{-1}(t)\xi|_p$, hence our integral becomes an infinite sum

$$p^{-\mathrm{ord}_p(\xi)} \sum_{\nu\in\mathbb{Z}} h(\begin{pmatrix} 1 & p^{-2\nu}\xi \\ 0 & 1 \end{pmatrix})p^{2\nu}.$$

Now we assume that our function h is the characteristic function of the double coset $\mathrm{Sl}_2(\mathbb{Z}_p)\begin{pmatrix} p^m & 0 \\ 0 & p^{-m} \end{pmatrix}\mathrm{Sl}_2(\mathbb{Z}_p)$. We have to find out when

$$\begin{pmatrix} 1 & p^{-2\nu}\xi \\ 0 & 1 \end{pmatrix} \in \mathrm{Sl}_2(\mathbb{Z}_p)\begin{pmatrix} p^m & 0 \\ 0 & p^{-m} \end{pmatrix}\mathrm{Sl}_2(\mathbb{Z}_p)$$

If $m = 0$ then this means that $\mathrm{ord}_p(\xi) \geq 2\nu$ and our sum is

$$p^{-\mathrm{ord}_p(\xi)} \sum_{2\nu\leq\mathrm{ord}_p(\xi)} p^{2\nu} = \begin{cases} \frac{p^2}{p^2-1} & \text{if } \mathrm{ord}_p(\xi) \text{ is even} \\ \frac{p}{p^2-1} & \text{else} \end{cases}.$$

In the ramified case we have $\chi(T, \mathcal{O}) = 1$ for exactly one of the classes with $\mathrm{ord}_p(\xi) = 0$ and exactly one of the classes with $\mathrm{ord}_p(\xi) = 1$. For the remainig two classes the other torus contributes, i.e. we have $\chi(T', \mathcal{O}) = 1$. Hence we see that

$$\tilde{O}(h, \gamma, \kappa) = \sum_{\mathcal{O}} \chi(T, \mathcal{O})\tilde{O}(h, \mathcal{O}) - \sum_{\mathcal{O}} \chi(T', \mathcal{O})\tilde{O}(h, \mathcal{O}) = 0$$

In the unramified case we have $\chi(T, \mathcal{O}) = 1$ for the two classes where $\xi \in \mathbb{Z}_p^\times$, and $\chi(T', \mathcal{O})$ for the two classes in $p\mathbb{Z}_p^\times$. We get

$$\tilde{O}(h, \gamma, \kappa) = \sum_{\mathcal{O}} \chi(T, \mathcal{O})\tilde{O}(h, \mathcal{O}) - \sum_{\mathcal{O}} \chi(T', \mathcal{O})\tilde{O}(h, \mathcal{O}) = \frac{p}{p+1}$$

This yields

$$|D(\gamma)|_p^{1/2}(O^{\mathrm{arith}}(h, \gamma) - O^{\mathrm{arith}}(h, \gamma')) = 1$$

at least if γ is close to one. This is the asymptotic assertion of the fundamental lemma for the identity element. But it is clear that the asymptotic version is good enough for $\gamma \in K^{(1)}$, the other elements γ are maximal and then the assertion is also proved.

If $m > 0$, then it is an easy calculation that our condition is satisfied if and only if $-\mathrm{ord}_p(\xi) + 2\nu = m$ hence (I will do this later)