Chapter V
The topological trace formula

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Contents

1 The general purpose of the (topological) trace formula

We choose a reductive group $G/\mathbb{Q}$, let $G^{(1)}/\mathbb{Q}$ be its derived group and let $Z/\mathbb{Q}$ be its centre. We also choose a suitable subgroup $K_\infty \subset G(\mathbb{R})$, which should be of the form $K_\infty^{(1)} \times Z^0(\mathbb{R})$ where $K_\infty^{(1)}$ is the connected component of the identity of a maximal compact subgroup of $G^{(1)}(\mathbb{R})$ and where the second factor is the connected component of the identity of the group of real points of the centre. (As a standard example we can take $SO(2) \cdot \mathbb{R}^* \subset \mathrm{Gl}_2(\mathbb{R})$). The space $X = G(\mathbb{R})/K_\infty$ is a finite union of symmetric spaces, we allow ourselves to call this still a symmetric space despite of the fact that it may be disconnected. Let $\mathbb{A}$ be the ring of adeles we decompose it into its finite and its infinite part: $\mathbb{A} = \mathbb{R} \times \mathbb{A}_f$. We have the group of adeles $G(\mathbb{A}) = G(\mathbb{R}) \times G(\mathbb{A}_f)$ and let K_f be a (variable) open compact subgroup of $G(\mathbb{A}_f)$. We always assume that this group is a product of local groups $K_f = \prod_p K_p$. We may choose an integral structure $\mathcal{G}/\mathrm{Spec}(\mathbb{Z})$, then we know that we have $K_p = \mathcal{G}(\mathbb{Z}_p)$ for almost all p. We select a finite set of finite primes Σ which contains the primes p where $\mathcal{G}/\mathbb{Z}_p$ is not reductive and those where K_p is not equal to $\mathcal{G}(\mathbb{Z}_p)$. Readers who are not so familiar with this language may think of the simple example where $G/(\mathbb{Q}) = GSp_n/\mathbb{Q}$ is the group of symplectic similitudes on $V = \mathbb{Q}^{2n} = \mathbb{Q}e_1 \oplus$

$\cdots \oplus \mathbb{Q}e_n \oplus \mathbb{Q}f_1 \oplus \cdots \oplus \mathbb{Q}f_n$ with the standard symplectic form $< e_i, f_i >= 1$ and all other products zero. The vector space contains the lattice $L = \mathbb{Z}^{2n} = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n \oplus \mathbb{Z}f_1 \oplus \cdots \oplus \mathbb{Z}f_n$ and there is a unique integral structure $\mathcal{G}/\mathbb{Z}$ on $G/\mathbb{Q}$ for which $\mathcal{G}(\mathbb{Z}_p) = \{g \in G(\mathbb{Q}_p) | g(X \otimes \mathbb{Z}_p) = (X \otimes \mathbb{Z}_p)\}$. In this case the group scheme is reductive over $\mathrm{Spec}(\mathbb{Z})$.

The space $(G(\mathbb{R})/K_\infty) \times (G(\mathbb{A}_f)/K_f)$ can be seen as a product of the symmetric space and an infinite discrete set, on this space we have a discontinuous action of $G(\mathbb{Q})$ and we get a quotient

$$\pi : G(\mathbb{R})/K_\infty \times G(\mathbb{A}_f)/K_f \to S^G_{K_f} = G(\mathbb{Q}) \setminus (G(\mathbb{R})/K_\infty \times G(\mathbb{A}_f)/K_f).$$

This space is a finite union of locally symmetric spaces $\Gamma_i \setminus X$ where $X = G(\mathbb{R})/K_\infty$ and the Γ_i are varying arithmetic congruence subgroups.

Finally we choose a (irreducible) rational representation

$$\rho : G/\mathbb{Q} \to \mathrm{Gl}(\mathcal{M}).$$

This representation ρ provides a sheaf $\tilde{\mathcal{M}}$ on $\mathcal{S}^G_K$ whose sections on an open subset V are given by

$$\tilde{\mathcal{M}}(V) = \{s : \pi^{-1}(V) \to \mathcal{M} | s \text{ locally const and } s(\gamma v) = \gamma s(v)\}.$$

We consider the cohomology groups

$$H^\bullet(S^G_{K_f}, \tilde{\mathcal{M}}) = \oplus H^i(S^G_{K_f}, \tilde{\mathcal{M}}).$$

Sometimes we want the representation to be absolutely irreducible, then it may be necessary to extend the field of scalars and consider representations of $G \times \mathbb{Q}L \to \mathrm{Gl}(\mathcal{M})$ where $\mathcal{M}$ is an L vector space. Then our cohomology groups are also L-vector spaces.

On these cohomology groups we have an action of a big algebra the so called Hecke algebra (of level K_f):

$$\mathcal{H} = \bigotimes_p{}' \mathcal{H}_p = \bigotimes_p{}' \mathbb{C}_c(G(\mathbb{Q}_p)//K_p).$$

The fundamental problem is to "understand" the cohomology as a module under the action of this algebra. I mention a few general results.

We introduce the so called inner cohomology $H^\bullet_!(S^G_{K_f}, \mathcal{M})$ which is defined as the image of the cohomology with compact supports in the cohomology, i.e.

$$H^\bullet_!(S^G_{K_f}, \mathcal{M}) = \mathrm{Im}(H^\bullet_c(S^G_{K_f}, \mathcal{M}) \to H^\bullet(S^G_{K_f}, \mathcal{M})).$$

This is a submodule under the Hecke algebra. It can be proved that this submodule is semi-simple. (This requires Hilbert space arguments). In general we should have a filtration on the cohomology

$$H^\bullet_!(S^G_{K_f}, \mathcal{M}) = F^0 H^\bullet(S^G_{K_f}, \mathcal{M}) \subset F^1 H^\bullet(S^G_{K_f}, \mathcal{M}) \ldots \subset F^d H^\bullet(S^G_{K_f}, \mathcal{M}) = H^\bullet(S^G_{K_f}, \mathcal{M})$$

such that on each successive quotient we have an isotypical decomposition

$$F^\nu H^\bullet / F^{\nu+1} H^\bullet = \oplus_{\pi_f} F^\nu H^\bullet / F^{\nu+1} H^\bullet(\pi_f),$$

where the π_f are indecomposable modules for the Hecke algebra. This filtration should be understood in terms of the so called Eisenstein cohomology.

Since the Hecke algebra decomposes into a product of local algebras the modules π_f also decompose into a tensor product of local factors, we write $\pi_f = \otimes_p \pi_p$. For p not in the exceptional set Σ the local component is simply a homomorphism $\pi_p : \mathcal{H}_p \to \bar{\mathbb{Q}}$, this homomorphism is given by its Satake parameter λ_p.

One of the aspects of the problem above is to understand the arithmetic meaning of the irreducible Hecke-modules which occur in the cohomology. It is a general belief that such an isotypical constituent should correspond to a more arithmetic object (a motive $M(\pi_f)$?). The Satake parameters of the local components π_p for $p \notin \Sigma$ should be related to the eigenvalues of Frobenii at p on ℓ-adic the cohomology of this motive.

1.1 The general form of the topological trace formula

We denote by $\mathcal{H}^\vee_{\mathrm{coh}}(\mu) = \mathcal{H}^\vee_{\mathrm{coh}}(K_f)(\mu)$ the cohomological spectrum in the μ-th filtration step, it consists of those isomorphism classes of irreducible modules under the Hecke-algebra which occur in the cohomology $F^\nu H^\bullet / F^{\nu+1} H^\bullet$. We sometimes write $\mathcal{H}^\vee_{\mathrm{coh}}(0) = \mathcal{H}^\vee_{\mathrm{coh},!}$

We choose a Hecke operator h_f which is the product of operators $h_p \in \mathcal{H}_p$, any operator is a linear combination of operators of this type. Then

$$\mathrm{tr}(h_f | H^\bullet(\mathcal{S}, \mathcal{M})) = \sum_\mu \sum_{\pi_f \in \mathcal{H}^\vee_{\mathrm{coh}}(\mu)} \sum (-1)^\nu \sum_{\pi_f \in \mathcal{H}^\vee_{\mathrm{coh}}} m^{(\nu)}(\pi_f) \prod_p \mathrm{tr}(h_p | \pi_p)$$

where for $p \notin \Sigma$ the local module has dimension one and $\mathrm{tr}(h_p|\pi_p) = \pi_p(h_p) = \hat{h}_p(\lambda(\pi_p))$. The number $m^{(\nu)}(\pi_f)$ is the multiplicity of π_f in the cohomology group $H^\nu(\mathcal{S}, \mathcal{M})$.

This is the so called χ **-expansion**. The topological trace formula gives a different expression for this trace. It is obtained by summing over the local contributions at the fixed points of the Hecke operators. Here we are facing the difficulty that the space $\mathcal{S}$ is not compact. There are several ways of compactifying it, we simply write $i : \mathcal{S} \to \bar{\mathcal{S}}$ for one of these compactifications. We get a stratified space, where the strata correspond to the conjugacy classes of $\mathbb{Q}$-rational parabolic subgroups. On this space we have to analyse the fixed point components of the Hecke operators. The fixed points in the interior $\mathcal{S}$ come from the conjugacy classes of elements in $G(\mathbb{Q})$ which are elliptic at the place ∞ of $\mathbb{Q}$, this means that they are conjugate to an element in K_∞ in $G(\mathbb{R})$. This set of elliptic elements is called $G(\mathbb{Q})_{\mathrm{ell}}$. The fixed points at the boundary create some problems, they will not be discussed here. Essentially they can be expressed in terms of traces of of Hecke operators on smaller reductive groups. This follows from the work of J. Bewersdorff, M. Goresky and B. MacPherson on the topological trace formula.

We write

$$\mathrm{tr}(h_f | H^\bullet(\mathcal{S}, \mathcal{M})) = \text{elliptic terms} + \text{terms at infinity } = tr_{\mathrm{ell}}(h_f) + tr_\infty(h_f)$$

and the elliptic contribution is

$$tr_{\mathrm{ell}}(h_f) = \sum_{\gamma \in G(\mathbb{Q})_{\mathrm{ell}}/\sim} \chi(Z_\gamma) \cdot \mathrm{tr}(\gamma|\mathcal{M}) \cdot \prod_p \int_{Z_\gamma(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)} h_p(x_p^{-1}\gamma x_p) dx_p$$

Here $\sim$ is conjugation under $G(\mathbb{Q})$, Z_γ is the centralizer of our element γ, it is again a reductive group. The factor $\chi(Z_\gamma)$ is the Euler characteristic of a locally symmetric space attached to Z_γ, i.e. a space of the form $\mathcal{S}^{Z_\gamma}_{K_f^{Z_\gamma}}$. The Euler characteristic can be computed by means of the Gauss-Bonnet formula. Up to a sometimes nasty normalizing factor this is equal to the Tamagawa number $\tau(Z_\gamma(\mathbb{A})/Z_\gamma(\mathbb{Q}))$ (We are still in a heuristic discussion, the precise form of the elliptic contribution wil be discussed in 4.3.. It involves some unpleasant questions concerning the normalization of measures). The Tamagawa number is a global term, the normalizing factor is local and depends on a comparison of measures. It is an important feature of the trace formula, that the other factors in an individual term are products of local factors at the different places: The factor $\mathrm{tr}(\gamma|\mathcal{M})$ corresponds to the place infinity. The other factors are orbital integrals. The measure dx_p is a quotient of a measure dg_p on $G(\mathbb{Q}_p)$ and a measure dz_p on the centralizer $Z_\gamma(\mathbb{Q}_p)$. (See Chap III,???). For later reference we introduce the notation

$$\mathcal{O}(h_p, \gamma) = \int_{Z_\gamma(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)} h_p(x_p^{-1}\gamma x_p) dx_p$$

This is the **$\mathcal{O}$-expansion** for the trace of a Hecke operator.

One of the most important features of the (topological) trace formula is that it allows comparisons:

(A) We may compare traces of Hecke operators acting on the cohomology of different groups.

(B) We may also compare traces of Hecke operators and operators of the type (Hecke-operator outside p times a power of Frobenius at p) acting on the ℓ-adic cohomology of of the reduction of the Shimura varieties $\mod p$.

It is the second kind of application that opens a possibility to attack the question concerning the arithmetic nature of π_f which was mentioned above.

1.2 The general strategy to apply the topological trace formula

Furtherdown I give some simple examples for the applications of type (A) and (B). Here I want to illustrate the general strategy how this application works in case (B). Before I enter a discussion of the general concept of a Shimura variety, I choose the specifiic example of the group of symplectic similitudes $G = \mathrm{Sp}_g/\mathrm{Spec}(\mathbb{Z})$ (See Chap. IV.7). I choose a congruence subgroup $K_f = K_f^{(0)}(N)$ which is the full congruence group $\mod p^{n_p}$ for the primes p dividing N. Then we consider the space

$$S^G_{K_f} = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K_f,$$

where X is the $G(\mathbb{R})$-conjugacy class of the homomorphism

$$h_0 : \mathbb{C}^\times \to G(\mathbb{R})$$

which is defined by

$$h_0 : a + bi \mapsto \begin{cases} e_\nu \mapsto a e_\nu - b f_\nu \\ f_\nu \mapsto b e_\nu + a f_\nu \text{ for } \nu = 1 \dots g \end{cases}$$

Hence we can say that such an h is of the form $g h_0 h^{-1}$ for some $g \in G(\mathbb{R})$. Notice that X has two connected components and notice also that giving such an h is the same as giving a complex structure on $M \otimes \mathbb{R}$ (notation Chap. IV) which is an isometry for the pairing and for which the associated hermitian form is definite.

It has been explained in the [book] volume I, Chap. V that X is a complex space with an action of $G(\mathbb{R})$ on it. I also explained that our space $S^G_{K_f}$ can be seen as the moduli space of principally polarized abelian varieties of dimension g with level N structure ($N \geq 3$).

It is now a consequence of the general philosophy that we should have a moduli scheme of principally polarized abelian varieties with N-level stucture which is defined over the "smallest" scheme over which the functor makes sense, i.e. over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{N}])$. In other words we expect have a diagram

$$\begin{array}{c} (\mathcal{A}_g, \mathcal{L}, e_1, \dots e_g, f_g, \dots, f_1) \\ \downarrow \pi \\ \mathcal{S}^G_{K_f} \qquad\qquad , \\ \downarrow f \\ \mathrm{Spec}(\mathbb{Z}[\frac{1}{N}]), \end{array}$$

where $\mathcal{S}^G_{K_f}$ is a smooth quasiprojective scheme over the base, where $\mathcal{A}_g$ is an abelian scheme of dimension g, where $\mathcal{L}$ is a principal polarization and where $e_1, \dots e_g, f_g, \dots, f_1$ form a system of generators of the group of N-division points and with value of the Weil pairing $e_N(e_\nu, f_\nu) = \zeta_N$.

Now we have for the $\mathbb{C}$-valued points

$$\mathcal{S}^G_{K_f}(\mathbb{C}) = S^G_{K_f}.$$

On our complex variety we have the local system $R^1\pi_*(\mathbb{Q})$ it is also the local system obtained from the tautological representation $\mathcal{M}_{\omega_1}$ of $G = \mathrm{Sp}_g$ (twisted by ?). If we take exterior and symmetric powers of this representation and decompose the resulting representations into irreducibles and then take tensor products and decompose again we finally get all irreducible representations together with α generate the finite dimensional representations of Sp_g. We also allow twistings by powers of α.

For any irreducible module $\mathcal{M}$ we get a local system $\tilde{\mathcal{M}}$ over the complex manifold $S^G_{K_f}$ and we can consider as before the cohomology groups $H^\bullet_c(S^G_{K_f}, \tilde{\mathcal{M}})$ and $H^\bullet(S^G_{K_f}, \tilde{\mathcal{M}})$ as modules over the Hecke algebra.

But now we also have the ℓ-adic sheaves $\tilde{\mathcal{M}}_\ell$ on our scheme $\mathcal{S}^G_{K_f}$ and we can also consider the $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \times \mathcal{H}$ modules $H^\bullet_c(\mathcal{S}^G_{K_f} \times \bar{\mathbb{Q}}, \tilde{\mathcal{M}}_\ell)$ and $H^\bullet(\mathcal{S}^G_{K_f} \times \bar{\mathbb{Q}}, \tilde{\mathcal{M}}_\ell)$.

Of course we have the comparison isomorphism

$$H^\bullet_c(\mathcal{S}^G_{K_f} \times \bar{\mathbb{Q}}, \tilde{\mathcal{M}}_\ell) \xrightarrow{\sim} H^\bullet_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}) \otimes \mathbb{Q}_\ell \ , \ H^\bullet(\mathcal{S}^G_{K_f} \times \bar{\mathbb{Q}}, \tilde{\mathcal{M}}_\ell) \xrightarrow{\sim} H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}) \otimes \mathbb{Q}_\ell.$$

We now want to generalize Theorem 3 which was stated at the end of Chap. II. This turns out to be rather difficult and the answer is much more subtle than one might think in the beginning. We believe in the filtration of the cohomology that was mentioned in the beginning of this chapter and we believe that this filtration induces a filtration in ℓ-adic cohomology. We consider an absolutely irreducible isotypical summand

$$F^\nu H^i / F^{\nu+1} H^i(\pi_f) \subset F^\nu H^i / F^{\nu+1} H^i,$$

then we know that this gives us a $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \times \mathcal{H}$ module $W(\pi_f) \otimes H_{\pi_f}$, here H_{π_f} is a realization of the isomorphism class π_f. It is a finite dimensional vector space over a finite extension $\mathbb{Q}(\pi_f)/\mathbb{Q}$. We extend the prime ℓ to a place $\mathfrak{l}$ of $\mathbb{Q}(\pi_f)$, then $W(\pi_f)$ will be a $\mathbb{Q}(\pi_f)_\mathfrak{l}$ vector space with a linear action

$$\rho^{(\nu,i)}(\pi_f) : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gl}_{\mathbb{Q}(\pi_f)_\mathfrak{l}}(W(\pi_f)).$$

The fundamental problem is to describe this Galois module $W(\pi_f)$ in terms of the representation π_f and the numbers i, ν. (We have to take into account, that an absolutely irreducible module H_{π_f} may occur in different degrees i and also in different steps of the filtration.)

The representations π_f which occur in higher steps of the filtration are related to the "compactification" of our scheme. The theory of Eisenstein cohomology is designed to understand this part and these π_f are in a certain sense coming from smaller reductive groups. Hence we concentrate on the lowest step in the filtration, i.e. on summands occuring in $H^\bullet_!(\mathcal{S}^G_{K_f} \times \bar{\mathbb{Q}}, \tilde{\mathcal{M}}_\ell)$. But also here are certain contributions that come from "smaller" groups $H/\mathbb{Q}$. Here smaller means that the dual group ${}^LH$ can be embedded into the dual group ${}^LG$. These π_f are called endoscopic contributions. They become visible if we apply the trace formula.

Now we tentatively call a π_f "genuine" if it occurs in $H^\bullet_!(\mathcal{S}^G_{K_f} \times \bar{\mathbb{Q}}, \tilde{\mathcal{M}}_\ell)$ and if it is not endoscopic. For such a genuine constituent we hope for the following

It occurs in the middle degree $g(g+1)/2$ and $H^\bullet_!(\mathcal{S}^G_{K_f} \times \bar{\mathbb{Q}}, \tilde{\mathcal{M}}_\ell)(\pi_f)$ has multiplicity 2^g.

Therefore the representation

$$\rho(\pi_f) : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gl}_{\mathbb{Q}(\pi_f)_\mathfrak{l}}(W(\pi_f))$$

has dimension 2^g. It is unramified outside N.

The Langlands dual group is $\mathrm{GO}(g, g+1)$ and this has the 2^g dimensional spin representation

$$r : \mathrm{GO}(g, g+1) \to \mathrm{Gl}(W(\pi_f) \otimes_{\mathbb{Q}(\pi_f)_\mathfrak{l}} \mathbb{C})$$

and for primes $p \nmid N$ we have

$$\rho(\pi_f)(\Phi_p^{-1}) \sim r(\lambda_p(\pi_f))$$

This last statement can be formulated in a different form

For all positive integers $m > 0$ we have

$$\mathrm{tr}(\Phi_p^{-m}|W(\pi_f)) = \mathrm{tr}(r(\lambda_p(\pi_f)^m))$$

Why can we hope that this should be true? We are now entering the realm of "thin air speculation" and we want to explain how the trace formula can help us to prove such a result.

We start from the congruence relations. I recall the situation in Chap. II where we discuss Theorem 3. There I briefly mention that we have to look at the reduction of the Hecke operators T_p modulo p. The congruence relation gives us an expression for this reduction in terms of the Frobenius:

$$T_p \mod p = T_p \times \mathbb{F}_p = \Phi_p^{-1} +^t \Phi_p^{-1}.$$

This is a classical result. Here we have to be careful and to take the "normalized " Hecke operator T_p which acts on the integral cohomology (see Chap. II). The congruence relation implies

$$(\Phi_p^{-1})^2 - T_p\Phi_p^{-1} + \omega_f(p)p^{k+1} = 0,$$

this relation holds on the cohomolgy with fixed central character ω_f. Hence we can conclude:

Assume $g = 1$ and $\mathcal{M} = \mathcal{M}_k$. If we have an isotypical component $H^1_!(\mathcal{S}^G_{K_f(N)} \times \bar{\mathbb{Q}}, \tilde{\mathcal{M}}_{\mathfrak{l}})(\pi_f)$ and if

$$\lambda(\pi_p) \sim \begin{pmatrix} \alpha_p & 0 \\ 0 & \beta_p \end{pmatrix}$$

then the eigenvalues of $\rho(\Phi_p^{-1})$ are members of the list $\{\alpha_p, \beta_p\}$.

This is certainly not enough to prove $\rho(\Phi_p^{-1}) \sim r(\lambda_p(\pi_f))$. But we have in addition the duality pairing

$$H^1_!(\mathcal{S}^G_{K_f(N)} \times \bar{\mathbb{Q}}, \tilde{\mathcal{M}}_{\mathfrak{l}}) \times H^1_!(\mathcal{S}^G_{K_f(N)} \times \bar{\mathbb{Q}}, \tilde{\mathcal{M}}_{\mathfrak{l}}) \to$$

$$H^2_c(\mathcal{S}^G_{K_f(N)} \times \bar{\mathbb{Q}}, \mathbb{Q}(k)) = \bigoplus_{\chi} \mathbb{Q}(k+1) \otimes \chi.$$

From this we can derive that:

If α_p is eigenvalue of $\rho(\Phi_p^{-1})$ then $\beta_p = \omega_f(p)p^{k+1}/\alpha_p$ is also an eigenvalue

and this is now good enough to show $\rho(\Phi_p^{-1}) \sim r(\lambda_p(\pi_f))$.

We notice that this duality argument only applies to the isotypical components in $H^1_!(\mathcal{S}^G_{K_f(N)} \times \bar{\mathbb{Q}}, \tilde{\mathcal{M}}_\mathfrak{l})$ but not to those in the one of the subquotients in the filtration, in this case to a subquotient of

$$H^1(\mathcal{S}^G_{K_f(N)} \times \bar{\mathbb{Q}}, \tilde{\mathcal{M}}_\mathfrak{l})/H^1_!(\mathcal{S}^G_{K_f(N)} \times \bar{\mathbb{Q}}, \tilde{\mathcal{M}}_\mathfrak{l})$$

And indeed we will see that in this case the Frobenius picks only one of the possible eigenvalues, we will discuss what happens in the example further down.

What happens for higher genus g. We still have the congruence relations. We form the characteristic polynomial

$$\det(T - r(\lambda_p)|V) = T^{2^g} - \sigma_1(\lambda_p)T^{2^g-1} + \cdots + \sigma_{2^g}(\lambda_p)$$

where we view the elementary symmetric functions σ_i as elements in $\mathcal{H}(G(\mathbb{Q}_p)/\mathcal{G}(\mathbb{Z}_p))$.

Then the congruence relation says that this gives the zero endomorphism of the cohomology if we substitute Φ_p for T, i.e.

$$\det(\Phi_p^{-1} - r(\lambda_p)|V) = (\Phi_p^{-1})^{2^g} - \sigma_1(\lambda_p)(\Phi_p^{-1})^{2^g-1} + \cdots + \sigma_{2^g}(\lambda_p)$$

is the trivial endomorphism on cohomology. (In all degrees and independently of support conditions)

Hence we can draw the same conclusion as in the case $g = 1$:

The eigenvalues of $\rho(\pi_f)(\Phi_p^{-1})$ are contained in the set of eigenvalues of $r(\lambda_p)$, i.e. in the set of zeroes of $T^{2^g} - \sigma_1(\lambda_p)T^{2^g-1} + \cdots + \sigma_{2^g}(\lambda_p)$. To state it differently: The maximal squarefree factor of the characteristic polynomial of $\rho(\pi_f)(\Phi_p^{-1})$ on $W(\pi_f)$ divides $\det(T - r(\lambda_p)|V)$.

This is certainly not the best answer.

A much better and also natural answer for a given π_f would be that for all non ramified primes p we have

$$\det(T - \rho(\pi_f)(\Pi_p^{-m}|W(\pi_f)) = \det(T - r(\lambda_p)|V)$$

If this is the case then we know in addition that $m(\pi_f) = 2^g$. We certainly could also live with a weaker statement that $m(\pi_f) = N2^g$ and

$$\det(T - \rho(\pi_f)(\Pi_p^{-m}|W(\pi_f)) = \det(T - r(\lambda_p)|V)^N$$

If this holds for a given π_f we call it*genuine*.

We have seen that this answer is correct if we assume $g = 1$ and π_f occurs in the inner cohomology $H^1_!(\mathcal{S}^G_{K_f(N)} \times \bar{\mathbb{Q}}, \tilde{\mathcal{M}}_\mathfrak{l})$. It is not true for the Eisenstein cohomology which therefore is not genuine.

For $g > 1$ this can fail even if we restrict ourselves to the bottom level $H^\bullet_!(\mathcal{S}^G_{K_f(N)} \times \bar{\mathbb{Q}}, \tilde{\mathcal{M}}_\mathfrak{l})$.

What can we do to get such a statement? For a moment we consider λ_p as a variable. then it follows from the Satake isomorphism, that there is a unique function $h_p^{(m)} \in \mathcal{H}(G(\mathbb{Q}_p//\mathcal{G}(\mathbb{Z}_p))$, which satisfies (Chap. III.1.2.1)

$$\hat{h}_p^{(m)}(\lambda_p) = \mathrm{tr}(r(\lambda_p)^m)$$

We take a prime to p Hecke operator $h_f^{(p)} \in \mathcal{H}^{(p)} = \bigotimes'_{\ell:\ell\neq p}\mathcal{H}_\ell = \bigotimes'_{\ell:\ell\neq p}\mathbb{C}_c(G(\mathbb{Q}_p)//K_p)$ and multiply this together to a Hecke operator $h_p^{(m)} \times h_f^{(p)}$. It yields an endomorphism

$$h_f = h_p^{(m)} \times h_f^{(p)} : H_c^\bullet(S^G_{K_f(N)}, \tilde{\mathcal{M}}) \to H_c^\bullet(\mathcal{S}^G_{K_f(N)}, \tilde{\mathcal{M}})$$

Then we consider the reduction $\mod p$ of our Shimura variety. Here we can consider the operators $\mathrm{F}_p^m \times h_f^{(p)}$ this gives an endomorphism

$$\mathrm{F}_p^m \times h_f^{(p)} : H_c^\bullet(\mathcal{S}^G_{K_f(N)} \times \bar{\mathbb{F}}_p, \tilde{\mathcal{M}}_{\mathfrak{l}}) \to H_c^\bullet(\mathcal{S}^G_{K_f(N)} \times \bar{\mathbb{F}}_p, \tilde{\mathcal{M}}_{\mathfrak{l}}).$$

If we make the false assumption that all constituents π_f are genuine then we get

$$\mathrm{tr}(h_p^{(m)} \times h_f^{(p)} \mid H_c^\bullet(S^G_{K_f(N)}, \tilde{\mathcal{M}}_{\mathfrak{l}}) - 2^g \mathrm{tr}(\mathrm{F}_p^m \times h_f^{(p)} \mid H_c^\bullet(\mathcal{S}^G_{K_f(N)} \times \bar{\mathbb{F}}_p, \tilde{\mathcal{M}}_{\mathfrak{l}}) = 0.$$

But this is almost never true. We ignore this fact and we compute both terms by means of a trace formula. For the topological side we had the expression

$$\mathrm{tr}(h_p^{(m)} \times h_f^{(p)} | H^\bullet(S_{K_f(N)}, \mathcal{M})) =$$
$$\sum_{\gamma \in G(\mathbb{Q})_{\mathrm{ell}}/\sim} \chi(Z_\gamma)\cdot \mathrm{tr}(\gamma|\mathcal{M}) \int_{Z_\gamma(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)} h_p^{(m)}(x_p^{-1}\gamma x_p) dx_p \times$$
$$\prod_{\ell:\ell\neq p} \int h_\ell((x_\ell^{-1}\gamma x_\ell) dx_\ell + tr_\infty(h_f).$$

To compute the second term we apply the Grothendieck-Lefschetz fixed point formula. We have to compute the fixed points of $\Phi_p^m \times h_f^{(p)}$ on $\mathcal{S}^G_{K_f(N)}(\bar{\mathbb{F}}_p)$, and we have to add a contribution from infinity. We have the theorem of Pink which says that for fixed part $h_f^{(p)}$ we have no contribution from infinity provided $m >> 0$.

Again we get a sum over elliptic elements in some different groups. But we can try to compare the two expressions, beginning with the comparison of the index sets in the summation. Then we can compare the terms. We will see that they do not match exactly. We have to explain the discrepancies in terms of non genuine contributions to the χ expansion. This hopefully leads to a complete determination of the Galois module structure.

This is carried out in some detail in the example 2) below.

2 The examples

These examples are meant to explain the general mechanisms of the applications of the (topological) trace formulae. Therefore some of the details of computations of volumes and orbital integrals are omitted. They will be filled in section 3 and the following sections.

2.1 Example 1

(Jacquet-Langlands):

The first and very famous example is the comparison between the groups

$$G = \mathrm{Gl}_2/\mathbb{Q}, G^*/\mathbb{Q}$$

where $G^*/\mathbb{Q}$ is the $\mathbb{Q}$-form of $G/\mathbb{Q}$ attached to a quaternion algebra which is anisotropic at ∞ (does not split over $\mathbb{R}$)(see Chap II.1.1.2). Let Σ be a finite set of primes, we assume that D splits outside Σ. Let $\Sigma_0 \subset \Sigma$ be the set of primes where $D/\mathbb{Q}$ does not split, it contains an odd number of elements. We extend G, G^* to semisimple group schemes $\mathcal{G}/(\mathrm{Spec}(\mathbb{Z}) \setminus \Sigma), \mathcal{G}^*/(\mathrm{Spec}(\mathbb{Z}) \setminus \Sigma)$. As always we choose $K_f = \prod K_p, K_f^* = \prod K_p^*$ where $K_p = \mathcal{G}(\mathbb{Z}_p), K_p^* = \mathcal{G}^*(\mathbb{Z}_p)$ for $p \notin \Sigma$.

For the the symmetric spaces we get a union of two upper half planes $X = \mathbb{H}_+ \cup \mathbb{H}_- = \mathrm{Sl}_2(\mathbb{R})/\mathrm{SO}(2)$ in the first case and a point in the second case. Hence we get for our spaces

$$\begin{array}{c} S^G_{K_f} = \bigcup_i \Gamma_i \setminus H \\ S^{G^*}_{K_f^*} = G^*(\mathbb{Q}) \setminus \big(\mathrm{point} \times G^*(\mathbb{A}_f)\big) K_f^*\big) = \text{finite set.} \end{array}$$

We choose one of our standard representations $\rho : \mathrm{Gl}_2/\mathbb{Q} \to \mathrm{Gl}(\mathcal{M}_k)$ and construct te resulting sheaf $\tilde{\mathcal{M}}_k$ on $S^G_{K_f}$. If we choose any quadratic extension $L/\mathbb{Q}$ which splits $D/\mathbb{Q}$, then we get a corresponding representation of $G^* \times_{\mathbb{Q}} L$ which provides a sheaf on $S^{G^*}_{K_f^*}$ which we denote by the same letter.

For all $p \notin \Sigma_0$ we can choose an isomophism $G \times_{\mathbb{Q}} \mathbb{Q}_p \xrightarrow{\sim} G^* \times_{\mathbb{Q}} \mathbb{Q}_p$ where we require that for $p \notin \Sigma$ this isomorphism induces an isomorphism between the group scheme structures over $\mathrm{Spec}(\mathbb{Z}_p)$. At the places $p \in \Sigma \setminus \Sigma_0$ we asuume that this isomorphism identifies K_p to K_p^*. This induces isomorphisms $\mathcal{H}_p = \mathcal{H}_p^*$ for all $p \notin \Sigma_0$ and for $p \notin \Sigma$ the algebras $\mathcal{H}_p, \mathcal{H}_p^*$ are the unramifed Hecke algebras.

We introduce some notation: If we have a product over all finite places then superscript something $^{(\Sigma)}$ means the product over the factors outside Σ and the subscript denotes the product over the factors in Σ. Hence we get for our Hecke algebra

$$\begin{array}{ll} \mathcal{H} & = \bigotimes_{p \notin \Sigma_0} \mathcal{H}_p \otimes \bigotimes_{p \in \Sigma_0} \mathcal{H}_p^* = \mathcal{H}^{(\Sigma_0)} \otimes \mathcal{H}_{\Sigma_0} \\ \mathcal{H}^* & = \bigotimes_{p \notin \Sigma_0} \mathcal{H}_p \otimes \bigotimes_{p \in \Sigma_0} \mathcal{H}_p^* = \mathcal{H}^{(\Sigma_0)} \otimes \mathcal{H}^*_{\Sigma_0}. \end{array}$$

We choose Hecke operators

$$h_f = \prod_{p \notin \Sigma_0} h_p \times \prod_{p \in \Sigma_0} h_p$$

and

$$h_f^* = \prod_{p \notin \Sigma_0} h_p \times \prod_{p \in \Sigma_0} h_p^*.$$

This means that we take the same local component at those places where we can compare the two local components and we will adjust the choice of h_p, h_p^* later.

We compare

$$\mathrm{tr}(h_f | H^\bullet(S^G_{K_f}, \mathcal{M})) \text{ and } \mathrm{tr}(h_f^* | H^\bullet(\mathcal{S}^{G^*}_{K_f^*}, \mathcal{M})).$$

We have to say what we mean by "comparing": We have identity elements $1_{\Sigma_0} \in \mathcal{H}_{\Sigma_0}$ and $1^*_{\Sigma_0} \in \mathcal{H}^*_{\Sigma_0}$ and therefore we can consider $\mathcal{H}^{(\Sigma_0)}$ as subalgebra of $\mathcal{H}$ and of $\mathcal{H}^*$. This means that $H^\bullet(S^G_{K_f}, \mathcal{M})$ and $H^\bullet(\mathcal{S}^{G^*}_{K^*_f}, \mathcal{M})$ are $\mathcal{H}^{(\Sigma_0)}$ modules and we may compare them. For instance we may ask for a relation between the multiplicities of isotypical $\mathcal{H}^{(\Sigma_0)}$-modules in the summands. Then we also get that these isotypical components are still modules for $\mathcal{H}_{\Sigma_0}$ and $\mathcal{H}^*_{\Sigma_0}$ and we will establish a correspondence between irreducible modules of these two algebras.

Now we have to take into account that on the cohomology of S we will get a filtration $H^\bullet_!(S, \mathcal{M}) \subset H^\bullet(S, \mathcal{M})$ where the quotient is the Eisenstein cohomology $H_{\mathrm{Eis}}(S, \mathcal{M})$. Here the filtration of 1.1 has exactly two steps, namely $\mu = 0$, this is the ! cohomomology and $\mu = 1$, which is the Eisenstein cohomology. Hence we have the two χ-expansions

$$\mathrm{tr}(h_f | H^\bullet(S^G_{K_f}, \mathcal{M})) = \mathrm{tr}(h_f | H^\bullet_!(\mathcal{S}^G_{K_f}, \mathcal{M})) + \mathrm{tr}(h_f | H^\bullet_{\mathrm{Eis}}(\mathcal{S}^G_{K_f}, \mathcal{M})) =$$

$$\sum_{\pi_f \in \mathcal{H}^\vee_{\mathrm{coh},!}} \sum_\nu (-1)^\nu \, m^{(\nu)}(\pi_f) \prod_p \mathrm{tr}(h_p|\pi_p) + \sum_{\pi_f \in \mathcal{H}^\vee_{\mathrm{coh,Eis}}} \sum_\nu (-1)^\nu \, m^{(\nu)}(\pi_f) \prod_p \mathrm{tr}(h_p|\pi_p)$$

and for the other space all cohomology sits in degree zero and

$$\mathrm{tr}(h^*_f | H^\bullet(\mathcal{S}^{G^*}_{K^*_f}, \mathcal{M})) =$$

$$\sum_{\pi^*_f \in \mathcal{H}^\vee_{\mathrm{coh},!}} m^{(0)}(\pi^*_f) \prod_p \mathrm{tr}(h^*_p|\pi^*_p)$$

Since the multiplicities are positive and since (most of) the cohomology in the first trace sits in degree one, hence it comes with a minus sign, we consider a difference (in the Grothendieck group) between $\mathcal{H}^{\Sigma_0}$ modules

$$H^\bullet(S^G_{K_f}, \mathcal{M}) + a(G, G^*) H^\bullet(\mathcal{S}^{G^*}_{K^*_f}, \mathcal{M})$$

and we hope for the existence of a positive constant $a(G, G^*)$ such that this difference can be described in a simple way.

We compare the corresponding $\mathcal{O}$-expansions. We consider the regular, i.e. the non central terms. Then Z_γ, Z_{γ^*} are tori, whose $\mathbb{Q}$ rational points are the multiplicative group of an imaginary quadratic extension $\mathbb{Q}(\gamma)$ (resp. $\mathbb{Q}(\gamma^*)$). We say that γ corresponds to γ^*, if these quadratic extensions are isomorphic and one of the two isomorphisms sends γ to γ^*. We know from the theory of quaternion algebras that an imaginary extension $E/\mathbb{Q}$ can be embedded into D if it is non split at all places where $D/\mathbb{Q}$ is not split. Hence we get an inclusion of conjugacy classes $(G^*(\mathbb{Q})_{\mathrm{ell}}/\sim) \subset (G(\mathbb{Q})_{\mathrm{ell}}/\sim)$ and the subset $G^*(\mathbb{Q})_{\mathrm{ell}}/\sim$ consists of those conjugacy classes which do not split at the places where $D/\mathbb{Q}$ does not split.

In 4.3 we give the precise contribution of an elliptic element to the trace. At this point of my exposition it is not so relevant to understand the individual factors in the product. You should only believe that for regular elements the formula simplifies because Z_γ, Z_{γ^*} are tori, there is no semi-simple component. This also implies that $C'_\gamma = Z_\gamma$ and $C'_{\gamma^*} = Z_{\gamma^*}$ The value $c_\infty(\gamma) = 2$ because γ has a fixed point in the upper and in the lower half plane and $c_\infty(\gamma^*) = 1$

because the symmetric space is simply one point. Then regular terms in the the $\mathcal{O}$-expansions of the trace formula are

$$2\sum_{\gamma\in G(\mathbb{Q})_{\mathrm{ell}}/\sim}\mathrm{tr}(\gamma|\mathcal{M})h(Z_\gamma)\cdot\prod_p\frac{1}{\mathrm{vol}(\omega^{\mathrm{Tam}}_{G,p}(K_p))})\int_{Z_\gamma(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)}h_p(\bar g_p^{-1}\gamma\bar g_p)\omega^{\mathrm{Tam}}_{Z_\gamma\backslash G,f}(d\bar g_p)$$

and

$$\sum_{\gamma^*\in G^*(\mathbb{Q})_{\mathrm{ell}}/\sim}\mathrm{tr}(\gamma^*|\mathcal{M})h(Z_{\gamma^*})\cdot\prod_p\frac{1}{\mathrm{vol}(\omega^{\mathrm{Tam}}_{G^*,p}(K^*_p))}\int_{Z_{\gamma^*}(\mathbb{Q}_p)\backslash G^*(\mathbb{Q}_p)}h^*_p(\bar g_p^{-1}\gamma^*\bar g_p)\omega^{\mathrm{Tam}}_{Z_{\gamma^*}\backslash G^*,f}(d\bar g_p)$$

Hence we choose $a(G,G^*)=2$ and consider the resulting difference of $\mathcal{O}$-expansions. The point is that we still have some freedom to choose the functions $h_f=\prod_p h_p, h^*_f=\prod_p h^*_p$ and we will do this in such a way that for all p the local factors become equal.

For the factors $p\notin\Sigma_0$ we always choose $h_p=h^*_p$, for the primes $p\in\Sigma_0$ we assume that h_p,h^*_p have *matching orbital integrals* and we have to explain hat we mean by that.

First of all we notice, that for an element $\gamma\in G(\mathbb{Q})_{\mathrm{ell}}/\sim$ we have a unique corresponding γ^* if and only if the extension $Q(\gamma)$ is non split at all $p\in\Sigma_0$. This dictates a first assumption on the choice of our elements h_p for $p\in\Sigma$: We require that

(i) For $p\in\Sigma_0$ and all regular split elements $t_p\in G(\mathbb{Q}_p)$ we have

$$\int_{Z_{t_p}(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)}h_p(\bar g_p^{-1}t_p\bar g_p)\omega^{\mathrm{Tam}}_{Z_{t_p}\backslash G,p}(d\bar g_p)=0$$

Our second requirement is

(ii) For $p\in\Sigma_0$ and all regular anisotropic corresponding pairs of elements $t_p\in G(\mathbb{Q}_p), t^*_p\in G^*(\mathbb{Q}_p)$ we have

$$\frac{1}{\mathrm{vol}(\omega^{\mathrm{Tam}}_{G,p}(K_p))})\int_{Z_{t_p}(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)}h_p(\bar g_p^{-1}t_p\bar g_p)\omega^{\mathrm{Tam}}_{Z_{t_p}\backslash G,p}(d\bar g_p)=$$
$$-\frac{1}{\mathrm{vol}(\omega^{\mathrm{Tam}}_{G^*,p}(K^*_p))}\int_{Z_{t^*_p}(\mathbb{Q}_p)\backslash G^*(\mathbb{Q}_p)}h^*_p(\bar g_p^{-1}t^*_p\bar g_p)\omega^{\mathrm{Tam}}_{Z_{t_p}\backslash G,p}(d\bar g_p)$$

We say that the collection of functions and levels $\{h_p,K_p,h^*_p,K^*_p\}_{p\in\Sigma_0}$ have matching orbital integrals at regular elements if (i) and (ii) are satisfied. If we have two functions $h_f=h_f^{(\Sigma_0)}\times\prod_{p\in\Sigma_0}h_p, h^*_f=h_f^{(\Sigma_0)}\times\prod_{p\in\Sigma_0}h^*_p$ and the collection of functions at $\{h_p,h^*_p\}_{p\in\Sigma_0}$ have matching orbital them h_f,h^*_f have matching orbital integrals at regular elements.

Since the cardinality of Σ_0 is odd and since the orbital integrals outside Σ_0 are the same we conclude

If h_f,h^*_f have matching orbital integrals at regular elements, then the regular terms in the $\mathcal{O}$-expansion in

$$\mathrm{tr}(h_f|H^\bullet(S^G_{K_f},\mathcal{M}))+2\mathrm{tr}(h^*_f|H^\bullet(\mathcal{S}^{G^*}_{K^*_f},\mathcal{M}))$$

cancel.

We have to find out what the consequences of this fact for the structure of the module
$$H^\bullet(S^G_{K_f},\mathcal{M})+2H^\bullet(\mathcal{S}^{G^*}_{K^*_f},\mathcal{M})$$
will be.

2.1.1 The terms at infinity and the Eisenstein cohomology

In this section I use some results from my paper on Eisenstein cohomology of Gl_2. We still have the terms coming from the fixed points at infinity. I will explain later that we have a formula

$$\mathrm{tr}_\infty(h_f)=\mathrm{tr}(h_f^{(\mathrm{contr})}\mid H^\bullet(\partial S^G_{K_f},\tilde{\mathcal{M}}_k))$$

where $h_f^{(\mathrm{contr})}$ is a Hecke operator acting on the cohomology of the boundary and it is the "contracting" part of h_f. (For an example see 3.2). It follows easily from the rules how this operator is constructed, that it is of the form

$$\prod_{p\in\Sigma_0} h_p\times h_f^{(\mathrm{contr})\Sigma_0}.$$

Now it is an relatively easy result that the cohomology of the boundary is a direct sum

$$\bigoplus_\chi \pi(\chi_f)$$

Here the χ are algebraic Hecke characters on the torus $T=B/U$. If we define K_f^T as the image of $K_f\cap B(\mathbb{A}_f)$ in $T(\mathbb{A}_f)$ then they are homomorphisms

$$\chi: T(\mathbb{Q})\backslash T(\mathbb{A})/K_f^T\to\mathbb{C}^\times,$$

which have a type at infinity that depends on $\mathcal{M}$ and on the degree in which we take the cohomology. The $\pi(\chi_f)$ are induced modules and it is an easy fact that

$$\mathrm{tr}(h_p|\pi(\chi_p))=0$$

if the regular hyperbolic orbital integrals of h_p vanish, i.e. the assumption (i) is fulfilled.

This implies that under our assumption (i) we have $\mathrm{tr}_\infty(h_f)=0$

On the χ-expansion side we still have the Eisenstein cohomology. It follows from the definition of the Eisenstein cohomology that we have an inclusion

$$H^\bullet_{\mathrm{Eis}}(\mathcal{S}^G_{K_f},\mathcal{M}))\hookrightarrow H^\bullet(\partial S^G_{K_f},\tilde{\mathcal{M}}_k))\xrightarrow{\sim}\bigoplus_\chi\pi(\chi_f).$$

Now the Eisenstein cohomology gives us an answer what the image of the inclusion is. If our coefficient system $\mathcal{M}_k$ is non trivial, i.e. $k>0$ then the inclusion is an isomorphism in degree one and zero in degree zero. In this case our previous argument shows that under the assumption (i) we again get

$$\mathrm{tr}(h_f|H^\bullet_{\mathrm{Eis}}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_k))=0.$$

We summarize:

If $k > 0$ and if h_f and h_f^* have matching orbital integrals at regular elements, then

$$\mathrm{tr}(h_f | H_!^\bullet(S_{K_f}^G, \mathcal{M})) + 2\mathrm{tr}(h_f^* | H^\bullet(\mathcal{S}_{K_f^*}^{G^*}, \mathcal{M})) =$$

the difference of the central terms in the $\mathcal{O}$-expansion.

If $k = 0$ then the situation is a little bit different. In this case we certain characters $\chi : T(\mathbb{Q})\backslash T(\mathbb{A})/K_f^T \to \mathbb{C}^\times$ for which the modules $\pi(\chi_f)$ become reducible. To get these characters we observe that we have the determinant character $\det : T \to G_m$. Hence a Dirichlet character

$$\psi : G_m(\mathbb{Q})\backslash G_m(\mathbb{A})/G_m(\mathbb{R})^0 \times \det(K_f^T) \to \bar{\mathbb{Q}}^\times$$

yields a Dirichlet character $\chi = \psi \circ \det$ on T and since $k = 0$ the induced module $\pi(\chi_f)$ occurs as a summand in $H^0(\partial S_{K_f}^G, \bar{\mathbb{Q}})$). If we twist ψ by the Tate character

$$\alpha : G_m(\mathbb{Q})\backslash G_m(\mathbb{A}) \to \mathbb{Q}^\times, \alpha : \underline{t} \mapsto |\underline{t}|$$

then we can consider $\chi = \psi\alpha \circ \det$ and these modules $\pi(\chi_f)$ occur as direct summands in $H^0(\partial S_{K_f}^G, \bar{\mathbb{Q}})$).

Now these special modules are not irreducible: If $\chi = \psi \circ \det$ then we have an exact sequence

$$0 \to \bar{\mathbb{Q}}\chi_f \to \pi(\chi_f) \to \pi'(\chi_f) \to 0$$

where $\bar{\mathbb{Q}}\chi$ is the one dimensional $\mathrm{Gl}_2(\mathbb{A}_f)-$module on which $\mathrm{Gl}_2(\mathbb{A}_f)$ acts by $\psi \circ \det$ and this is also a module for the Hecke algebra. A dual assertion holds for $\chi = \psi\alpha \circ \det$, in this case we have

$$0 \to \pi'((\psi\alpha \circ \det)_f) \to \pi((\psi\alpha \circ \det)_f) \to \bar{\mathbb{Q}}(\psi\alpha \circ \det)_f \to 0.$$

Now we may define for characters χ which contribute to cohomology in degree 0

$$E(\chi_f) = \begin{cases} \bar{\mathbb{Q}}\chi & \text{if } \chi = \psi \circ \det \\ 0 & \text{if not} \end{cases}$$

and for characters contributing to cohomology in degree 1

$$\pi_E(\chi_f) = \begin{cases} \pi'(\chi_f) & \text{if } \chi = \psi\alpha \circ \det \\ \pi(\chi_f) & \text{if not} \end{cases}$$

Then one of the results in my paper says that

$$H^0(\partial S_{K_f}^G, \bar{\mathbb{Q}})) \xrightarrow{\sim} \bigoplus_\chi E(\chi_f)$$

$$H^1(\partial S_{K_f}^G, \bar{\mathbb{Q}})) \xrightarrow{\sim} \bigoplus_\chi \pi_E(\chi_f)$$

Of course our assumption (i) still implies that $\mathrm{tr}(h_f | \pi(\chi_f)) = 0$ but this only gives us

$$\mathrm{tr}(h_f|E(\chi_f)) = -\mathrm{tr}(h_f|\pi'(\chi_f))$$

for $\chi = \psi \circ \det$. Putting everything together we obtain under the assumption (i): For $k = 0$

$$\mathrm{tr}(h_f|H^\bullet_{\mathrm{Eis}}) = 2\sum_\psi \mathrm{tr}(h_f|E(\chi_f).$$

If we adopt the terminology of 1.2. this is a non "genuine" contribution to the cohomology and the interesting point is that we see a "matching" contribution in the cohomology on the other side. The reduced norm $N : G^* \to G_m$ gives a surjective map from our space to a finite abelian group

$$S^{G^*}_{K^*_f} \xrightarrow{N} G_m(\mathbb{Q})^+ N(K^*_f)\backslash G_m(\mathbb{A}_f)$$

and hence an inclusion in cohomology

$$H^0(G_m(\mathbb{Q})^+ N(K^*_f)\backslash G_m(\mathbb{A}_f), \mathbb{Q}) \hookrightarrow H^0(S^{G^*}_{K^*_f}, \mathbb{Q})$$

and for the left hand side we have the decomposition

$$H^0(G_m(\mathbb{Q})^+ N(K^*_f)\backslash G_m(\mathbb{A}_f), \bar{\mathbb{Q}}) = \bigoplus_\psi \bar{\mathbb{Q}}\psi.$$

This is also a non "genuine" contribution to the cohomology of $S^{G^*}_{K^*_f}$ and it will follow from the considerations in the following section, that this contribution cancels the Eisenstein part on the left hand side. We define the *genuine* part of the cohomology as the complement of the non genuine part.

Hence we can conclude

If $k = 0$ and if h_f and h^*_f have matching orbital integrals at regular elements, then

$$\mathrm{tr}(h_f|H^\bullet_!(S^G_{K_f}, \tilde{\mathcal{M}})) + 2\mathrm{tr}(h^*_f|H^\bullet_{\mathrm{genuine}}(\mathcal{S}^{G^*}_{K^*_f}, \tilde{\mathcal{M}})) =$$

the difference of the central terms in the $\mathcal{O}$-expansion.

The reader should notice that for a non trivial coefficient system $N_*(\tilde{\mathcal{M}}) = 0$. and therefore $H^\bullet_{\mathrm{genuine}}(\mathcal{S}^{G^*}_{K^*_f}, \mathcal{M}) = H^\bullet(\mathcal{S}^{G^*}_{K^*_f}, \mathcal{M})$.

2.1.2 Matching orbital integrals

We have to discuss the question whether we find enough collections of matching orbital integrals. We have the two maps

$$(\mathrm{tr}, \det) : G \to \mathbb{A}^1 \times \mathrm{G_m},\ (\mathrm{tr}, N) : G^* \to \mathbb{A}^1 \times \mathrm{G_m}$$

The central elements map to

$$\mathcal{Z} = \{c = (x, y) \mid x^2 - 4y \ = 0\}.$$

For any $c \notin \mathcal{Z}(\mathbb{Q}_p)$ the fibers $\mathcal{C}_c = (\mathrm{tr}, \det)^{-1}(c), \mathcal{C}^*_c = (\mathrm{tr}, N)^{-1}(c)$ consist of semi simple elements. The fibers $\mathcal{C}_c(\mathbb{Q}_p)$ are always non empty and form a semi simple conjugacy class under $G(\mathbb{Q}_p)$. (See section on stabilization). Any element $t \in \mathcal{C}_c(\mathbb{Q}_p)$ is regular, its centralizer is a torus $T = Z_t/\mathbb{Q}_p$. This torus is split, then we say $c \in (\mathbb{A}^1 \times \mathrm{G_m})(\mathbb{Q}_p)_{\mathrm{split}}$ otherwise we say that $c \in (\mathbb{A}^1 \times \mathrm{G_m})(\mathbb{Q}_p)_{\mathrm{nonsplit}}$. We will also use the terminology that elements or conjugacy classes in $c \in (\mathbb{A}^1 \times \mathrm{G_m})(\mathbb{Q}_p)_{\mathrm{split}}$ are called *hyperbolic at p* or *p-hyperbolic* and the classes in $c \in (\mathbb{A}^1 \times \mathrm{G_m})(\mathbb{Q}_p)_{\mathrm{nonsplit}}$ are called *elliptic at p* or *p-elliptic*. The fibers of (tr, N) over points in $(\mathbb{A}^1 \times \mathrm{G_m})(\mathbb{Q}_p)_{\mathrm{split}}$ do not have a $\mathbb{Q}_p$-rational point and the fibers (tr, N) over points in $(\mathbb{A}^1 \times \mathrm{G_m})(\mathbb{Q}_p)_{\mathrm{nonsplit}}$ are non empty and form a $G^*(\mathbb{Q}_p)$ conjugacy class. Then a $t^* \in (\mathrm{tr}, \det)^{-1}(c)(\mathbb{Q}_p)$ defines a torus $T^* \subset G^*/\mathbb{Q}_p$, which is isomorphic to the corresponding torus in $G/\mathbb{Q}_p$. The set of isomorphism classes of these tori is finite, they are in one to one correspondence to quadratic extensions of $\mathbb{Q}_p$, where the split torus corresponds to $\mathbb{Q}_p \oplus \mathbb{Q}_p$.

To any $c \in (\mathbb{A}^1 \times \mathrm{G_m})(\mathbb{Q}_p)_{\mathrm{reg}} = (\mathbb{A}^1 \times \mathrm{G_m})(\mathbb{Q}_p) \setminus \mathcal{Z}(\mathbb{Q}_p)$ and t(resp. t^*) in the fiber (if it exists) we can find a p-adic neighborhood $V_c \subset (\mathbb{A}^1 \times \mathrm{G_m})(\mathbb{Q}_p)_{\mathrm{reg}}$ and an open compact subgroup $K_p(V_c)$ (resp.$K_p^*(V_c)$) such that we get surjections

$$(\mathrm{tr}, \det) : t_p K_p(V_c) \to V_c, \quad (\mathrm{tr}, N) : t_p K_p^*(V_c) \to V_c$$

and bijections

$$(\mathrm{tr}, \det) : t_p(K_p(V_c) \cap T(\mathbb{Q}_p)) \xrightarrow{\sim} V_c, \quad (\mathrm{tr}, N) : t_p(K_p^*(V_c) \cap T^*(\mathbb{Q}_p)) \xrightarrow{\sim} V_c.$$

(The existence of these groups is a little bit technical and will be discussed later)

Let $\mathrm{ch}_{t_p(K_p(V_c)}, \mathrm{ch}_{t_p(K_p^*(V_c)}$ be the characteristic functions of these two open subsets then it is not so hard to check, that for V_c suitably small, the orbital integrals

$$\frac{1}{\mathrm{vol}(\omega_{G,p}^{\mathrm{Tam}}(K_p))}) \int_{Z_{t_p}(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)} \mathrm{ch}_{t_p(K_p(V_c))}(\bar{g}_p^{-1} t_p \bar{g}_p) \omega_{Z_{t_p}\backslash G,f}^{\mathrm{Tam}}(d\bar{g}_p)$$

$$\frac{1}{\mathrm{vol}(\omega_{G,p}^{\mathrm{Tam}}(K_p^*))}) \int_{Z_{t_p^*}(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)} \mathrm{ch}_{t_p(K_p^*(V_c))}(\bar{g}_p^{-1} t_p^* \bar{g}_p) \omega_{Z_{t_p^*}\backslash G,f}^{\mathrm{Tam}}(d\bar{g}_p)$$

have constant non zero values on V_c and they are zero outside this set. We can multiply these characteristic functions by suitable constants and conclude

For any regular element $c \in (\mathbb{A}^1 \times \mathrm{G_m})(\mathbb{Q}_p)_{\mathrm{reg}}$ we can find arbitrarily small open neighborhoods $V(c)$ and functions h_p, h_p^* in the Hecke algebras whose supports is in the fibres of $V(c)$ such that the orbital integrals are ± 1 the characteristic function on V_c. If c is split, we take $h_p = 0, h_p^* = 0$..

We should be aware that these function h_p, h_p^* have levels K_p, K_p^* which depend on c, the closer we come to the central elements, the smaller we have to choose K_p, K_p^*. Hence we can conclude

For any locally constant function F on on $\mathbb{Q}_p \times \mathbb{Q}_p^\times$ which has compact support, vanishes on $\mathcal{Z}(\mathbb{Q}_p)$ and on the hyperbolic classes, we can find elements h_p, h_p^* in the Hecke algebras such that $O(h_p, t_p) = O(h_p^*, t_p) = F$.

We notice that the central classes are in the closure of the hyperbolic classes, hence the vanishing on $\mathcal{Z}(\mathbb{Q}_p)$ of F follows from the vanishing on the hyperbolic regular classes. Or to say it differently: If we want that h_p, h_p^* have matching orbital integrals and that these orbital integrals are locally constant function on $\mathbb{Q}_p \times \mathbb{Q}_p^\times$, then the support of the orbital integrals should not meet the central classes.

In the neighborhood of the element $(2,1) = (\mathrm{tr}(\mathrm{Id}), \det(\mathrm{Id})$ we define a discontinous function F_0 which takes value 1 on regular elliptic elements with determinant in $\mathbb{Z}_p^\times$, which is zero on the regular hyperbolic classes and takes value $-\frac{p-1}{2}$ in the central elements of $\mathrm{Gl}_2(\mathbb{Z}_p)$. In section 3.2.2 we will introduce the Euler-Poincare element h_p^{EP} in the Hecke algebra of $\mathrm{Gl}_2(\mathbb{Q}_p)$ this function has the remarkable property $O(h_p^{EP}, t_p) = F_0$.

If on the other side we take the characteristic function h_p^{*EP} of the maximal compact subgroup of $G^*(\mathbb{Q}_p)$ then the orbital integrals of this function have value 1 on all elements in this maximal compact subgroup. Hence we see that h_p^{EP} and $h_p^{*,EP}$ have matching orbital integrals on all regular elements.

This implies

For any element h_p in the Hecke algebra of $\mathrm{Gl}_2(\mathbb{Q}_p)$ we can find h_p^* such that these two function have matching orbital integrals on regular elements and vice versa

Let $\{h_p, h_p^*\}_{p\in\Sigma_0}$ be a collection of functions with matching orbital integrals at regular elements. Let us assume that for $p \in \Sigma_0$ they are of level K_p, K_p^*. We choose a common level subgroup $K_f^{(\Sigma_0)}$ and put

$$K_f = \prod_{p\in\Sigma_0} K_p \times K_f^{(\Sigma_0)}, K_f^* = \prod_{p\in\Sigma_0} K_p^* \times K_f^{(\Sigma_0)}$$

We compare the two $\mathcal{O}$- expansion above. An analysis of the normalizing factors shows $\chi(Z_\gamma) = 2\chi(Z_{\gamma^*})$ because γ has a fixed point in either connected component. Then we see that in the two $\mathcal{O}$-expansions all regular elliptic terms cancel. Before we proceed we have a brief look at the central contributions. Again we have to invoke 4.3 We have to compute

$$\sum_{z\in Z(\mathbb{Q})} \mathrm{tr}(z|\mathcal{M})(\chi(S^G_{K_f})h_f(z) + 2\chi(S^{G^*}_{K_f^*})h_f^*(z)).$$

We compute these Euler characteristics $\chi(S^G_{K_f}), \chi(S^{G^*}_{K_f^*})$in 4.2. and the following sections. The general formula is

$$\chi_{\mathrm{orb}}(S^G_{K_f}) = \frac{c_\infty(\mathfrak{g}^{(1)}{}_{\mathbb{Z}})}{[\pi_0(C'(\mathbb{R}) : \pi_0(G(\mathbb{R}))]} \frac{1}{\mathrm{vol}_{\omega^{\mathrm{Tam}}_{G,f}}(K_f)} \tau(G^{(1)})h(C').$$

We observe that in both cases the factor $h(C') = 1$ since the class number of $\mathbb{Z}$ is one. The term $[\pi_0(C'(\mathbb{R}) : \pi_0(G(\mathbb{R}))]$ is also equal to one in both cases, this follows from the definition. If we now invoke the definition and computation of the constants $A_\infty(\mathrm{Sl}_2/\mathbb{Z}), A_\infty(\mathcal{G}^*)$ then we find

$$\mathrm{tr}(h_f|H^\bullet_!(S^G_{K_f}, \mathcal{M})) + 2\mathrm{tr}(h_f^*|H^\bullet_{\mathrm{genuine}}(\mathcal{S}^{G^*}_{K_f^*}, \mathcal{M})) =$$

$$\sum_{z\in Z(\mathbb{Q})} \mathrm{tr}(z|\mathcal{M})\big(-\tau(G^{(1)}))\frac{1}{12}h_f(z) + \tau(G^{*(1)})\frac{1}{12}\prod_{p\notin U}(1+\frac{1}{p})h_f^*(z)\big).$$

We also have the set $\Sigma \supset \Sigma_0$ for all $\ell \notin \Sigma$ the local component $K_\ell = \mathrm{Gl}_2(\mathbb{Z}_\ell)$.

We may pick a prime $\ell \notin \Sigma$. At this prime we consider the Hecke operators $h_\ell^{(m)}$ which on a spherical representation π_ℓ have the eigenvalue

$$\mathrm{tr}(h_\ell^{(m)}|\pi_\ell) = \alpha_\ell^m + \beta_\ell^m$$

where $(\alpha_\ell, \beta_\ell) = (\alpha_\ell(\pi_\ell), \beta_\ell(\pi_\ell))$ are the Satake parameters. It follows from the computation in 3.2.1 that for $m > 0$ the function $h_\ell^{(m)}$ vanishes on central elements. We consider the χ-expansions on both sides. We write the irreducible representations of the Hecke algebra as

$$\pi_f = \pi_{f,\Sigma} \bigotimes \pi_f^{(\Sigma)}, \pi_f^* = \pi_{f,\Sigma}^* \bigotimes \pi_f^{*(\Sigma)},$$

where $\pi_{f,\Sigma}$ (resp. $\pi_f^{(\Sigma)}$) is the tensor product of the local components π_p with $p \in \Sigma$ (resp. $p \notin \Sigma$). At a finite number of places outside Σ we pick a Hecke operator $h_{\ell_i}^{(m_i)}$, we take enough places so that the collections of Satake parameters

$$\{(\alpha_{\ell_1}(\pi_{\ell_1}), \beta_{\ell_1}(\pi_{\ell_1})), (\alpha_{\ell_2}(\pi_{\ell_2}), \beta_{\ell_2}(\pi_{\ell_2})), \ldots, (\alpha_{\ell_r}(\pi_{\ell_r}), \beta_{\ell_r}(\pi_{\ell_r}))$$

separate the $\pi_f^{(\Sigma)}, \pi_f^{*(\Sigma)}$ which actually occur in the cohomology. We form a Hecke operators

$$h_f = h_{f,\Sigma_0} \times h_f^{(\Sigma_0)} = \prod_{p \in \Sigma_0} h_p \times \prod_{i=1}^r h_{\ell_i}^{(m_i)} \times h'_f, h_f^* = \prod_{p \in \Sigma_0} h_p^* \times \prod_{i=1}^r h_{\ell_i}^{(m_i)} \times h'_f$$

where the last factor is just the the product of the characteristic functions of the standard maximal compact subgroups over the remaining primes. We assume that for $p \in \Sigma_0$ we have matching orbital intergrals, for the other p the local factors $h_p = h_p^*$.

Then a π_f or π_f^* contributes to the χ-expansion of the left hand side of our formula above by a term

$$\mathrm{tr}(h_{f,\Sigma_0}|\pi_{f,\Sigma_0})\mathrm{tr}(h_f^{(\Sigma_0)}|\pi_f^{(\Sigma_0)}) = \mathrm{tr}(h_{f,\Sigma_0}|\pi_{f,\Sigma_0}) \prod_{i=1}^r (\alpha_{\ell_i}(\pi_{\ell_i})^{m_i} + \beta_{\ell_i}(\pi_{\ell_i})^{m_i}),$$

$$\mathrm{tr}(h_{f,\Sigma_0}^*|\pi_{f,\Sigma_0}^*)\mathrm{tr}(h_f^{(\Sigma_0)}|\pi_f^{*(\Sigma_0)}) = \mathrm{tr}(h_{f,\Sigma_0}^*|\pi_{f,\Sigma_0}^*) \prod_{i=1}^r (\alpha_{\ell_i}^*(\pi_{\ell_i})^{m_i} + \beta_{\ell_i}^*(\pi_{\ell_i})^{m_i})$$

and we get

$$- \sum_{\pi_f \in \mathrm{Coh}(\tilde{\mathcal{M}})_!} m(\pi_f) \prod_{p \in \Sigma_0} \mathrm{tr}(h_p|\pi_p) \prod_{i=1}^r (\alpha_{\ell_i}(\pi_{\ell_i})^{m_i} + \beta_{\ell_i}(\pi_{\ell_i})^{m_i}) +$$

$$2 \sum_{\pi_f^* \in \mathrm{Coh}(\tilde{\mathcal{M}})_{\mathrm{genuine}}} m(\pi_f^*) \prod_{p \in \Sigma_0} \mathrm{tr}(h_p^*|\pi_p^*) \prod_{i=1}^r (\alpha_{\ell_i}(\pi_{\ell_i}^*)^{m_i} + \beta_{\ell_i}(\pi_{\ell_i}^*)^{m_i}) =$$

$$\sum_{z \in Z(\mathbb{Q})} \mathrm{tr}(z|\mathcal{M})\big(-\tau(G^{(1)})\frac{1}{12} h_f(z) + \tau(G^{*(1)})\frac{1}{12} \prod_{p \notin U} (1 + \frac{1}{p}) h_f^*(z)\big)$$

We rearrange the summation and get

$$-\sum_{\pi_f^{(\Sigma_0)}}(\sum_{\pi_{f,\Sigma_0}} m(\pi_f^{(\Sigma_0)}\times\pi_{\Sigma_0})\mathrm{tr}(h_\Sigma|\pi_\Sigma))\prod_{i=1}^r(\alpha_{\ell_i}(\pi_{\ell_i})^{m_i}+\beta_{\ell_i}(\pi_{\ell_i})^{m_i})+$$

$$2\sum_{\pi_f^{*(\Sigma_0)}}(\sum_{\pi^*_{f,\Sigma_0}} m(\pi_f^{*(\Sigma_0)}\times\pi^*_{\Sigma_0})\mathrm{tr}(h^*_{\Sigma_0}|\pi^*_{\Sigma_0}))\prod_{i=1}^r(\alpha^*_{\ell_i}(\pi_{\ell_i})^{m_i}+\beta^*_{\ell_i}(\pi_{\ell_i})^{m_i})=$$

$$\sum_{z\in Z(\mathbb{Q})}\mathrm{tr}(z|\mathcal{M})\big(-\tau(G^{(1)})\frac{1}{12}h_f(z)+\tau(G^{*(1)})\frac{1}{12}\prod_{p\notin U}(1+\frac{1}{p})h^*_f(z)\big)$$

Now we observe that on the left hand side we have still variables namely the exponents m_i. I mentioned above that if one of the $m_i>0$ we have $h_f(z)=h^*_f(z)=0$ for central elements z. Now it is an easy excercise to show, that this formula can only be true if on the left hand side the terms with a fixed second component

$$\pi_f=\pi_{f,\Sigma_0}\times\pi_f^{(\Sigma_0)},\quad \pi^*_f=\pi^*_{f,\Sigma_0}\times\pi_f^{(\Sigma_0)}$$

cancel, hence we get for any choice of $\pi_f^{(\Sigma_0)}$

$$-\sum_{\pi_{f,\Sigma_0}} m(\pi_{f,\Sigma_0}\times\pi_f^{(\Sigma_0)})\prod_{p\in\Sigma_0}\mathrm{tr}(h_p|\pi_p)+2\sum_{\pi^*_{f,\Sigma_0}} m(\pi^*_{f,\Sigma_0}\times\pi_f^{(\Sigma_0)})\prod_{p\in\Sigma}\mathrm{tr}(h^*_p|\pi^*_p)=0.$$

From this formula we get some conclusions.

If we fix a $\pi^{(\Sigma_0)}$. If now for all $\pi_{\Sigma_0}=\prod_{p\in\Sigma_0}\pi_p$ for which $m(\pi_{f,\Sigma_0}\times\pi_f^{(\Sigma_0)})\neq 0$ at least one of the factors π_p is a principal series representation, then we get for all these factors $\mathrm{tr}(h_{\Sigma_0}|\pi_{\Sigma_0})=\prod_{p\in\Sigma_0}\mathrm{tr}(h_p|\pi_p)=0$. Then we can conclude that $\sum_{\pi^*_{f,\Sigma_0}} m(\pi^*_{f,\Sigma_0}\times\pi_f^{(\Sigma_0)})\prod_{p\in\Sigma}\mathrm{tr}(h^*_p|\pi^*_p)=0$. But since on this side we can choose the h^*_p arbitrarily the only way out is that $\pi^{(*,\Sigma_0)}$ does not occur in $H^\bullet_{\mathrm{genuine}}(\mathcal{S}^{G^*}_{K^*_f},\mathcal{M}))$

This can be turned backwards

If we have an isotypical component $H^\bullet_{\mathrm{genuine}}(\mathcal{S}^{G^*}_{K^*_f},\mathcal{M}))(\pi^{(*,\Sigma_0)})$ then this module $\pi^{(*,\Sigma_0)}=\pi^{(\Sigma_0)}$ also occurs with positive multiplicity in $H^\bullet_!(S^G_{K_f},\mathcal{M}))$ and there exists a $\pi_{\Sigma_0}=\prod_{p\in\Sigma_0}$ for which all π_p are cuspidal and such that $\pi_{\Sigma_0}\times\pi^{(\Sigma_0)}$ occurs in $H^\bullet_!(S^G_{K_f},\mathcal{M}))$

This is actually a very weak consequence, if we exploit the theory of characters we get much more precise results.

Local consequences

We pick any prime p and consider the two groups $\mathrm{Gl}_2(\mathbb{Q}_p)$ and $D(\mathbb{Q}_p)^\times$, where $D/\mathbb{Q}_p$ is the non split quaternion algebra over $\mathbb{Q}_p$. Then we will derive from the trace formula

We have a one to one correspondence between discrete series representations π_p on $\mathrm{Gl}_2(\mathbb{Q}_p)$ and irreducible (finite dimensional) representations π_p^* of $D(\mathbb{Q}_p)^\times$. This correspondence is defined by the character relation

$$char_{\pi_p}(t_p) = -char_{\pi_p^*}(t_p) \text{ for all regular elliptic elements}$$

Global consequences

We also invoke the theorem of strong multiplicity one: It implies that for given $\pi^{(\Sigma)}$ there is at most one π_Σ such that $\pi_\Sigma \times \pi^{(\Sigma)}$ occurs in $H^\bullet_!(S^G_{K_f}, \mathcal{M}))$ and if this is so then $m(\pi_\Sigma \times \pi^{(\Sigma)}) = 2$.

Then we will derive from the trace formula

(A)*For any $\pi_{f,\Sigma_0} \times \pi_f^{(\Sigma_0)}$ which occurs in $H^\bullet_!(S^G_{K_f}, \mathcal{M}))$ for which for all $p \in \Sigma_0$ the local components π_p are in the discrete series the corresponding representation $\prod_{p\in\Sigma_0} \pi_p^* \times \pi_f^{\Sigma_0}$ occurs with multiplicity one in $H^\bullet_{\mathrm{genuine}}(\mathcal{S}^{G^*}_{K_f^*}, \mathcal{M})$*

If conversely $\prod_{p\in\Sigma_0} \pi_p^* \times \pi_f^{\Sigma_0}$ occurs in $H^\bullet_{\mathrm{genuine}}(\mathcal{S}^{G^*}_{K_f^*}, \mathcal{M})$, then $\prod_{p\in\Sigma_0} \pi_p \times \pi_f^{(\Sigma_0)}$ occurs in $H^\bullet_!(S^G_{K_f}, \mathcal{M}))$.

(B)*The Tamagawa numbers of the semi simple derived groups $\tau(G^{(1)})$ and $\tau(G^{*,(1)})$ are equal*

To get the local information we assume that $\Sigma_0 = \{p\}$ consists of one element. Then we know that there is a unique quaternion algebra $D/\mathbb{Q}$ which is anisotropic at infinity and at p and split at all other places.

Then it not to difficult to show: For any discrete series representation π_p of $\mathrm{Gl}_2(\mathbb{Q}_p)$ or any representation π_p^* we can find a level $K_f^{(\{p\})}$ such that in $H^\bullet_!(S^G_{K_f}, \mathcal{M}))$ (resp. $H^\bullet_{\mathrm{genuine}}(\mathcal{S}^{G^*}_{K_f^*}, \mathcal{M})$) we find a constituent $\pi_p \times \pi_f^{(\{p\})}$ (resp. $\pi_p^* \times \pi_f^{(\{p\})}$). (A little bit tricky, choose for h_p a matrix coefficient at p or an Euler Poincare function. Then choose $K_f^{(\{p\})}$ very small, so that only a central element contributes.Then consider growth See ???)

Now let π_p be a discrete series representation which is a local factor at p in some $\pi_f = \pi_p \times \pi_f^{(p)}$ which occurs with positive multiplicity in $H^\bullet_!(S^G_{K_f}, \mathcal{M}))$.. Then this multiplicity is 2. Then comparing the traces for all pairs h_p, h_p^* with matching orbital integrals give a relation among characters

$$-m(\pi_p \times \pi_f^{(\Sigma)})\mathrm{char}_{\pi_p} + 2\sum_{\pi_p^*} m(\pi_p^* \times \pi_f^{(\Sigma)})\mathrm{tr}(h_p^*|\pi_p^*)\mathrm{char}_{\pi_p^*} = 0.$$

We mentioned already that $m(\pi_p \times \pi_f^{(\Sigma)}) = 2$. Such a relation holds for all π_p in the discrete series. For any π_p we can define $S(\pi_p) = \{\pi_p^* | m(\pi_p^* \times \pi_f^{(p)}) \neq 0\}$. This set is always non empty. But the orthogonality relations imply that for two different representations we have $S(\pi_p) \cap S(\pi_p') = \emptyset$, because the multiplicities are positive.

Next step is to show that the sets $S(\pi_p)$ must consist of one element π_p^*. If this is not the case then we can produce a non zero linear combination

$\sum_{\pi_p^* \in S(\pi_p)} n_{\pi_p^*} \mathrm{char}_{\pi_p^*}$ which is orthogonal to all π'_p. This is impossible. Hence our relation is

$$-2\mathrm{char}_{\pi_p} + 2m(\pi_p^* \times \pi_f^{(p)})\mathrm{char}_p^* = 0$$

Now it follows from the character relations that $m(\pi_p^* \times \pi_f^{(p)}) = 1$ and we get the character relation

$$\mathrm{chi}_{\pi_p}(c) = -\mathrm{chi}_{\pi_p^*}(c) \text{ for all } c \in \mathcal{C}(\mathbb{Q}_p)_{\mathrm{reg}}$$

Now we discuss the global consequences. We take again a general set Σ_0 and consider the division algebra $D/\mathbb{Q}$ which is ramified at ∞ and all places in Σ_0 and splits elsewhere. Then it is clear:

Let $\prod_{p\in\Sigma_0} \pi_p \times \pi_f^{(\Sigma_0)}$ occur in $H^1_!(S^G_{K_f}, \tilde{\mathcal{M}})$). Then the π_p for $p \in \Sigma_0$ are determined by $pi_f^{(\Sigma_0})$. If one of them is not in the discrete series, then $pi_f^{(\Sigma_0})$ does not occur in $H^\bullet_{\mathrm{genuine}}(\mathcal{S}^{G^*}_{K^*_f}, \mathcal{M})$). If all these π_p are in the discrete series, then we get for any of these π_p the corresponding representations π_p^* of $D(\mathbb{Q})^\times$, and $\prod_{p\in\Sigma_0} \pi_p^* \times \pi_f^{(\Sigma_0)}$ occurs in $H^\bullet_{\mathrm{genuine}}(\mathcal{S}^{G^*}_{K^*_f}, \mathcal{M})$) with multiplicity one.

If in turn $\prod_{p\in\Sigma_0} \pi_p^* \times \pi_f^{(\Sigma_0)}$ occurs in $H^\bullet_{\mathrm{genuine}}(\mathcal{S}^{G^*}_{K^*_f}, \mathcal{M})$, then $\prod_{p\in\Sigma_0} \pi_p \times \pi_f^{(\Sigma_0)}$ occurs in $H^1_!(S^G_{K_f}, \tilde{\mathcal{M}})$). (with multiplicity 2).

We came to our conclusion by looking at the formula

$$\mathrm{tr}(h_f|H^\bullet_!(S^G_{K_f}, \mathcal{M})) + 2\mathrm{tr}(h^*_f|H^\bullet_{\mathrm{genuine}}(\mathcal{S}^{G^*}_{K^*_f}, \mathcal{M})) =$$

$$\sum_{z\in Z(\mathbb{Q})} \mathrm{tr}(z|\mathcal{M})\big(-\tau(G^{(1)}))\frac{1}{12}h_f(z) + \tau(G^{*(1)})\frac{1}{12}\prod_{p\notin U}(1+\frac{1}{p})h^*_f(z)\big).$$

only for pairs (h_p, h^*_p) which have matching orbital integrals and which vanish at central elements.

Now we plug into the formula elements of the form $\prod_{p\in\Sigma_0} h_p^{EP} \times \chi_{K_f^{(\Sigma_0)}}$, $\prod_{p\in\Sigma_0} h_p^{*,EP} \times \chi_{K_f^{(\Sigma_0)}}$. They have also matching orbital integrals and it is clear that the left hand side is zero. (Apply the previous assertion). Then the right hand side is also zero, after a small computation we get the equality of the Tamagawa numbers.

This example is discussed in the book J-L. There the authors work with the Selberg trace formula, which is an analytic trace formula. This makes the entire consideration technically much more complicated, but they also get a much finer result. The topological trace formula avoids the analytic difficulties

I give a second application of the same principle

2.2 Example 2

(Langlands Antwerp).

2.2.1 The modular interpretation

Our group is $\mathrm{Gl}_2/\mathbb{Q}$ we choose an open subgroup $K_f = K_f(N) \subset G(\mathbb{A}_f)$ (see III.1.1.3). We assume $N \geq 3$. Our space $S^G_{K_f}$ is actually set of complex points of a curve

$$\mathcal{S}^G_{K_f} \to \mathrm{Spec}(\mathbb{Z}[\frac{1}{N}])$$

and

$$S^G_{K_f} = \mathcal{S}^G_{K_f}(\mathbb{C}).$$

The curve is the moduli scheme of elliptic curves with N-level structure, to be more precise we have an elliptic curve

$$\begin{array}{c}(\mathcal{E}, e_1, e_2)\\ \downarrow\uparrow\uparrow\\ \mathcal{S}^G_{K_f},\end{array}$$

where the sections e_1, e_2 (the two upwards arrows) form a basis for the group of three division points (See Chapter on moduli). The value $< e_1, e_2 >$ of the Weil pairing generates an extension $\mathcal{O}_N \supset \mathbb{Z}[\frac{1}{N}]$ which is isomorphic to the ring of $\{N\}$ integers in the field of N-th roots of unity.

This gives us a diagram

$$\begin{array}{ccc}\mathcal{S}^G_{K_f} & & \\ \downarrow & \searrow & \\ \mathrm{Spec}(\mathbb{Z}[\frac{1}{N}]) & \leftarrow & \mathrm{Spec}(\mathcal{O}_N).\end{array}$$

The base change $\mathcal{S}^G_{K_f} \times_{\mathbb{Z}[\frac{1}{N}]} \mathrm{Spec}(\mathbb{C})$ decomposes into irreducible components which correspond to the $\phi(N)$ arrows

$$t_\nu : \mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathcal{O}_N),$$

they form a principal homogenous space under the action of $\mathrm{Gal}(\mathcal{O}_N/\mathbb{Z}[\frac{1}{N}])$.

We have seen these components in Chap III,1.1.3.

2.2.2 The ℓ-adic cohomology as Hecke$\times$Galois module

Now we can look at the ℓ-adic cohomology. We fix an algebraic closure $\bar{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$ and put $\mathcal{M}_\ell = \mathcal{M} \otimes \bar{\mathbb{Q}}_\ell$. In this situation we better start from the cohomology with compact supports. To define it we recall that we can compactify $i : \mathcal{S}^G_{K_f}/\mathbb{Q} \to \mathcal{S}^{G,\wedge}_{K_f}/\mathbb{Q}$ by adding a finite scheme $\mathcal{S}^G_{K_f,\infty}$ and extend our sheaf to the compactification by taking $i_!(\mathcal{M})$ or $R^\bullet i_*(\mathcal{M})$. By definition the cohomology with compact supports is the cohomology with coefficients in $i_!(\mathcal{M})$. We get an exact sequence

$$0 \to H^0(\mathcal{S}^G_{K_f} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathcal{M}_\ell) \to H^0(\mathcal{S}^G_{K_f,\infty} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, R^\bullet i_*(\mathcal{M})) \to H^1_c(\mathcal{S}^G_{K_f} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathcal{M}_\ell) \to$$

$$\to H^1(\mathcal{S}^G_{K_f} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathcal{M}_\ell) \to H^1(\mathcal{S}^G_{K_f,\infty} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, R^\bullet i_*(\mathcal{M})) \to\to H^2_c(\mathcal{S}^G_{K_f} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathcal{M}_\ell) \to 0.$$

This is now a sequence of Hecke$\times$ Galois modules, the Galois modules are unramified at all $p \notin \Sigma$. As explained in Chap. II we know, that $H^1_!(\mathcal{S}^G_{K_f} \times_{\mathbb{Q}}$

$\bar{\mathbb{Q}}, \mathcal{M}_\ell)$ -the image of the cohomology with compact support in the cohomology without supports- is semisimple. We have an isotypical decomposition

$$H^1_!(\mathcal{S}^G_{K_f} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathcal{M}_\ell) = \bigoplus_{\pi_f} H^1_!(\mathcal{S}^G_{K_f} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathcal{M}_\ell)(\pi_f)$$

of this module and an isotypical component is of the form

$$H_{\pi_f} \otimes W(\pi_f)$$

where $W(\pi_f)$ is a two dimensional Galois-module. It is is unramified at all $p \not| N$. We want to understand this Galois-module, for instance we want to express the conjugacy class of the Frobenius elements $\Phi_p \in \mathrm{Gl}(W(\pi_f))$ n terms of π_f.

The answer is relatively easy in this case (if we stick to the places p not dividing N) and is given by the Eichler-Shimura relations (See Thm 3 in Chapter II)

To get these relations we compare the topological trace formula to a suitable Lefschetz trace formula. We pick a prime $p \not| N$ and take the reduction $\mod p$. On this reduction

$$\mathcal{S}^G_{K_f} \times_{\mathrm{Spec}([\mathbb{Z}[1/N])} \bar{\mathbb{F}}_p$$

we consider operators $\mathrm{F}_p^m \times h_f^{(p)}$ where F_p^m is the m-th power of the Frobenius at p and $h_f^{(p)}$ is a Hecke operator outside p. Then we try to compute the trace of this operator on the cohomology with compact support

$$\begin{aligned}&\mathrm{tr}(\mathrm{F}_p^m \times h_f^{(p)} | H^1_c(\mathcal{S}^G_{K_f} \times_{\mathbb{F}_p} \bar{\mathbb{F}}_p, \mathcal{M}_\ell) = \\ &\qquad \mathrm{tr}(\mathrm{F}_p^m \times h_f^{(p)} | H^0_{\mathrm{Eis}}) + \sum_{\pi_f^{(p)}} \mathrm{tr}(h_f^{(p)} | \pi_f^{(p)}) \cdot \mathrm{tr}(\mathrm{F}_p^m | H^1_!(\pi_f^{(p)})).\end{aligned}$$

This is a χ-expansion for a Hecke operator multiplied by a power of the Frobenius.

For this trace we have the arithmetic trace formula (Grothendieck-Illusie-Pink). We have to count fixed points and sum up the local contributions (Langlands, Kottwitz). One of the essential points is the theorem of Pink which asserts that for $m >> 0$ the contributions from the fixed points at infinity are zero.

To count the fixed points we start from the modular interpretation and view the points as isomorphism classes of elliptic curves with some level structure.

2.2.3 Elliptic curves over finite fields

We divide the curves into isogeny classes. A geometric point in $\mathcal{S}^G_{K_f}(\bar{\mathbb{F}}_p)$ is an elliptic curve $\mathcal{E}$ with level structure, this level structure is an isomorphism $\psi : \mathcal{E}[N] \xrightarrow{\sim} \mathbb{Z}/N\mathbb{Z}$. The elliptic curve is already defined over some finite field $\mathbb{F}_{p^r}$. Then $\mathrm{F}_{\mathrm{p^r}}$ is an endomorphism of this curve. For all $\ell \neq p$ this endomorphism induces an endomorphism on the $\ell-$ adic cohomology $H^1(\mathcal{E} \times_{\mathbb{F}_{p^r}} \bar{\mathbb{F}}_p, \mathbb{Q}_\ell)$, there it is equal to the endomorphism Φ_p^{-r} which is obtained from the Galois action of $\mathrm{Gal}(\bar{\mathbb{F}}_{p^r}/\mathbb{F}_{p^r})$.

The following can be found in many books, which cover the theory of elliptic curves.

The ring of endomorphisms $\mathrm{End}(\mathcal{E})$ is a finite algebra over $\mathbb{Z}$, the extension $\mathrm{End}_{\mathbb{Q}}(\mathcal{E}) = \mathrm{End}(\mathcal{E}) \otimes \mathbb{Q}$ is a field, which may be non commutative. An element in $\mathrm{End}_{\mathbb{Q}}(\mathcal{E})$ can be written as $\frac{\psi}{m}$ and may also be viewed as a diagram

$$\begin{array}{ccc} \psi_{\mathbb{Q}} : \mathcal{E} & & \mathcal{E}' \\ & \searrow & \downarrow m\mathrm{Id} \\ & & \mathcal{E}' \end{array}$$

with an integer $m > 0$.

On $\mathrm{End}(\mathcal{E})$ we have an involution $\phi \mapsto^t \phi$ which is obtained from the self duality of the elliptic curve (See for instance [book],) We can define the trace of an endomorphism by $\mathrm{tr}(\phi)\mathrm{Id} = \phi +^t \phi$. Then we have $\mathrm{tr}(\phi^t \phi) = \deg(\phi)$ and $\mathrm{tr}(\phi^t\phi) > 0$ for $\phi \neq 0$. We know that the we get an injection

$$\mathrm{End}(\mathcal{E}) \otimes \mathbb{Z}_\ell \hookrightarrow \mathrm{End}(H^1(\mathcal{E} \times_{\mathbb{F}_{p^r}} \bar{\mathbb{F}}_p, \mathbb{Z}_\ell)),$$

and a famous theorem of Tate (Deuring?) asserts that we get an isomporphism

$$\mathrm{End}(\mathcal{E}) \otimes \mathbb{Z}_\ell \xrightarrow{\sim} \mathrm{End}_{\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_{p^r})}((H^1(\mathcal{E} \times_{\mathbb{F}_{p^r}} \bar{\mathbb{F}}_p, \mathbb{Z}_\ell)).$$

The Frobenius satisfies a quadratic equation (the characteristic polynomial)

$$\mathrm{F}_{p^r}^2 - a\mathrm{F}_{p^r} + p^r = 0,$$

where $a = \mathrm{tr}(\mathrm{F}_{p^r})$, the polynomial does not depend on ℓ. The roots of this polynomial generate a quadratic extension $E = \mathbb{Q}(\mathrm{F}_{p^r})$ of $\mathbb{Q}$ or they are in $\mathbb{Q}$. If we take r sufficiently large (in the sense of divisibility) then we see that

$$\mathrm{F}_{p^r} = \begin{cases} \begin{pmatrix} \Pi_{\mathfrak{p}} & 0 \\ 0 & \Pi_{\bar{\mathfrak{p}}} \end{pmatrix} & \text{if } [\mathbb{Q}(\mathrm{F}_{p^r}) : \mathbb{Q}] = 2 \\ \begin{pmatrix} p^{r/2} & 0 \\ 0 & p^{r/2} \end{pmatrix} & \text{if } \mathrm{F}_{p^r} \in \mathbb{Q} \text{ for } r \text{ large} \end{cases},$$

where $\Pi_{\mathfrak{p}}\Pi_{\bar{\mathfrak{p}}} = p^r$, the prime ideal (p) decomposes into $(p) = \mathfrak{p}\bar{\mathfrak{p}}$, $\mathfrak{p} \neq \bar{\mathfrak{p}}$ and $\mathfrak{p}^r = (\Pi_{\mathfrak{p}})$.

We get for the possible types of of fields of rational endomorphisms

A) Imaginary quadratic extensions $E = \mathbb{Q}(\mathrm{F}_{p^r})/\mathbb{Q}$ which split at p.

B) The quaternion algebra $D/\mathbb{Q}$ which is ramified at infinity and p and nowhere else.

The curves having a field of type A) as rational endomorphism are called ordinary. The field extensions is imaginary because the trace $\mathrm{tr} : \mathrm{End}(\mathcal{E}) \to \mathbb{Z}$ satisfies $\mathrm{tr}(\phi^t\phi) > 0$ for all $\phi \neq 0$.

The other curves are called supersingular. We know that D is a field and we know that $D \times \mathbb{Q}_\ell = \mathrm{End}(T_\ell(\mathcal{E} \otimes \mathbb{Q}))$ for all $\ell \neq p$. The quaternion algebra is the only choice.

We have the result of Tate (Deuring?):

Two elliptic curves over $\bar{\mathbb{F}}_p$ are isogenuos, if and only if they have the same field of rational endomorphisms.

2.2.4 p-adic cohomology

We also need some input from p-adic cohomology of elliptic curves over $\bar{\mathbb{F}}_p$.

To any elliptic curve $\mathcal{E}/\bar{\mathbb{F}}_p$ we associate its Dieudonne-module $D_p(\mathcal{E})$. This a free rank 2 module over the Witt ring $W(\bar{\mathbb{F}}_p)$ together with 2 endomorphisms F, V of the abelian group $D_p(\mathcal{E})$, which have the additional properties

i) $\mathrm{FV} = \mathrm{VF} = p\mathrm{Id}$
ii) The endomorphism F is σ linear and the endomorphism V is σ^{-1} linear.

(Here σ is the Frobenius on $W(\bar{\mathbb{F}}_p)$)

Begin: Dieudonne-modules

To get these Dieudonne-module we consider the system of group schemes $\mathcal{E}[p^m]$, these are finite affine group schemes over $\bar{F}_p$ of rank p^{2m}. This means that their affine algebra $\mathcal{E}[p^m]$ is of rank p^{2m}.

A finite abelian group $G/\mathrm{Spec}(\bar{F}_p)$ is called etale if its algebra $A(G)$ is etale, i.e. a direct sum of copies of $\bar{F}_p$. It is called local if $A(G)$ is a local ring. It is not difficult to show that any finite abelian group scheme over $G/\mathrm{Spec}(\bar{F}_p)$ decomposes canonically

$$G = G_{\mathrm{et}} \times G_{\mathrm{loc}}$$

For such a group scheme we can define its Cartier dual

$$G^{\vee} = \mathrm{Hom}(G, G_m).$$

The simplest local group scheme is

$$\mu_{p^m} = \mathrm{Spec}(\bar{\mathbb{F}}_p[T]/(T^{p^m} - 1) \text{ with comultiplication } T \mapsto T \otimes T$$

this is the scheme of p^m-th roots of unity and it is also the kernel $G_m[p^m]$ of the multiplacation by p^m on G_m. It is easy to see that $\mu_{p^m}^{\vee} = \mathbb{Z}/p^m\mathbb{Z}$ and this is an etale group scheme.

We get a decomposition of the category of finite abelian group schemes over $\bar{F}_p$ into three subcategories:

(et,et) Those G which are etale and where also the dual is etale: This are the finite abelian group schemes whose order is prime to p.

(et,loc) This are the G for which one of the pair $G, G^{\vee}$ is local and the other is etale

and finally the most interesting part

(loc,loc) Those G where G and $G^{\vee}$ are local.

If we have a finite abelian group scheme G which is etale and its dual is local, then this is simply a finite abelian p-group, let us denote it by $\bar{G}$. Then we define the Dieudonne-module $D_p(G) = \bar{G} \otimes W(\bar{\mathbb{F}}_p)$. We still have to say how F and V act: We take the identity $\mathrm{Id}_{\bar{G}}$ and F is the σ linear extension of this identity. Then V is simply the σ^{-1} linear extension of $p\mathrm{Id}_{\bar{G}}$. (It is an important fact that we can take for F any σ-linear isomorphism of $D_p(G)$ and we get an isomorphic $W(\bar{F}_p)[\mathrm{F},\mathrm{V}]$ module. (Langs theorem). Now it is clear how to define the Dieudonne-module for group schemes G which are local and whose dual is etale, we simply take a suitable dual of $D_p(G^\vee)$. It remains to define the Dieudonne-module for the group schemes which are of the type (loc,loc).

We consider the ring of Witt vectors

$$W(\bar{\mathbb{F}}_p) = \{(x_0, x_1, \dots)|x_i \in \bar{\mathbb{F}}_p\}$$

on tis ring the two maps F,V are defined by

$$\mathrm{F}(x_0, x_1, \dots) = (x_0^p, x_1^p, \dots), \mathrm{V}(x_0, x_1, \dots) = (0, x_0, x_1, \dots).$$

Then $\mathrm{V}^n W(\bar{\mathbb{F}}_p)$ is an ideal and we define the quotient ring

$$W_n(\bar{\mathbb{F}}_p) = W(\bar{\mathbb{F}}_p)/V^n W(\bar{\mathbb{F}}_p).$$

Using the Verschiebung V we can define an inclusion

$$W_n(\bar{\mathbb{F}}_p) \hookrightarrow W_{n+1}(\bar{\mathbb{F}}_p).$$

If we now view $W_n(\bar{\mathbb{F}}_p)$ as the group of $\bar{\mathbb{F}}_p$-valued points of a unipotent algebraic group $\mathcal{W}_n$, then we get an inductive system of unipotent group schemes

$$\mathcal{W} = \quad \hookrightarrow \mathcal{W}_n \hookrightarrow \mathcal{W}_{n+1} \hookrightarrow .$$

Now we define for a finite abelian group scheme G which is of type (loc,loc)

$$D_p(G) = \mathrm{Hom}(G, \mathcal{W}) = \lim_{\rightarrow} \mathrm{Hom}(G, \mathcal{W}_n)$$

it is not difficult to see that this stabilizes, i.e. $D_p(G) = \mathrm{Hom}(G, \mathcal{W}_n)$ if $n >> 0$.

Now we come back to our elliptic curve $\mathcal{E}$. Then we put

$$D_p(\mathcal{E}) = \lim_{\leftarrow} D_p(\mathcal{E}[p^m])$$

the operators F,V are obtained from the corresponding operators on $\mathcal{W}$. This is a free $W(\bar{\mathbb{F}}_p)$-module of rank 2, it depends funtorially on $\mathcal{E}$, if $\lambda : \mathcal{E} \to \mathcal{E}'$ then we get a homomorphism of Dieudonne-modules $D_p(\lambda) : D_p(\mathcal{E}') \to D_p(\mathcal{E})$. If λ is an isogeny then $D_p(\lambda)$ is an inclusion.

For any elliptic curve $\mathcal{E}/\bar{\mathbb{F}}_p$ we have its Frobenius transform $\mathcal{E}^{(p)}/\bar{\mathbb{F}}_p$ (we raise the coefficients of the defining equations to their p-th power) and we have the isogeny $\mathrm{F} : \mathcal{E} \to \mathcal{E}'$. This gives us an inclusion $D_p(\mathcal{E}') \hookrightarrow D_p(\mathcal{E})$ and clearly

$$D_p(\mathcal{E}^{(p)}) = \mathrm{F}(D_p(\mathcal{E}))$$

.

End: Dieudonne-modules

If we choose a $W(\bar{\mathbb{F}}_p)$ basis (f_1, f_2) of the module $D_p(\mathcal{E})$ we can represent our σ-linear endomorphism F by a matrix $M(\mathrm{F}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ which is defined by

$$\mathrm{F}(f_1) = af_1 + bf_2$$

$$\mathrm{F}(f_2) = cf_1 + df_2$$

If we change this basis by an invertible matrix A then we get the transformation rule

$$M'(\mathrm{F}) = A^{-1}M(\mathrm{F})A^{\sigma}$$

so the conjugation for linear maps is eplaced by σ conjugation.

In our rank 2 situation we have only two isomorphism classes of Dieudonne-modules. For these modules F is represented by the matrices

A)

$$\mathrm{F} = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$$

B)

$$\mathrm{F} = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$$

respectively.

We can define the endomorphism ring of a Dieudonne-module, this is the ring of $W(\bar{\mathbb{F}}_p)$ endomorphisms which commute with F. It is clear that $\mathrm{End}(D_p(\mathcal{E}), \mathrm{F})$ is a $W(\mathbb{F}_p)$-module, and in our situation here

A)

$$\mathrm{End}(D_p(\mathcal{E}), \mathrm{F}) = \mathbb{Z}_p \oplus \mathbb{Z}_p$$

B)

$$\mathrm{End}(D_p(\mathcal{E}), \mathrm{F}) = D(\mathbb{Z}_p)$$

respectively. Here $D(\mathbb{Z}_p)$ is the maximal order of the quaternion algebra $\mathrm{End}(\mathcal{E}) \otimes \mathbb{Q}_p$.

Explanation for the assertion B) (See also ChapII.1.1.2): I said we can choose a basis e_1, e_2 of $D_p(\mathcal{E})$ such that F is represented by the matrix above. We define the $W(\mathbb{F}_p)$-module $D_p(\mathcal{E})_0 = W(\mathbb{F}_p)e_1 \oplus W(\mathbb{F}_p)e_2$. Then the above matrix defines a linear endomorphism F_0 of $D_p(\mathcal{E})$ and $\mathrm{F} = \mathrm{F}_0 \circ \sigma$. We consider linear endomorphisms ϕ of $D_p(\mathcal{E})_0 \otimes W(\bar{\mathbb{F}}_p)$ which commute with F. Then they also commute with $F^2 = p\mathrm{Id} \circ \sigma^2$ and hence with σ^2. But this last assertion says that $\phi \in \mathrm{End}(D_p(\mathcal{E})_0 \otimes W(\mathbb{F}_{p^2}))$.

Now we construct a homomorphism $W(\mathbb{F}_{p^2})) \hookrightarrow \mathrm{End}(D_p(\mathcal{E}), \mathrm{F})$, we send an element $\alpha \in W(\mathbb{F}_{p^2}))$ to the matrix $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{\sigma} \end{pmatrix}$. Furthermore we see that F_0 is in $\mathrm{End}(D_p(\mathcal{E}), \mathrm{F})$. The algebra of endomorphisms contains $W(\mathbb{F}_{p^2}))$ and another element F_0 which normalizes $W(\mathbb{F}_{p^2}))$, more precisely we have

$$\mathrm{F}_0\alpha = \alpha^{\sigma}\mathrm{F}_0.$$

Finally we have $\mathrm{F}_0^2 = p$. Then it is well known that $W(\mathbb{F}_{p^2}))$ and F_0 generate the maximal order in the unique quaternion algebra over $\mathbb{Z}_p$.

2.2.5 The counting of fixed points

We group the elliptic curves according to isogeny classes or what amounts to the same according to their rational fields of endomorphisms.

The isogeny classes are not empty: The classical theory of complex multiplication tells us: For any imaginary quadratic field $F/\mathbb{Q}$ we find elliptic curves $\tilde{\mathcal{E}} \to \mathrm{Spec}(\mathcal{O}_H)$ over the ring of integers $\mathcal{O}_H$ of the Hilbert class field, such that $\tilde{\mathcal{E}}$ has complex multiplication by $\mathcal{O}_F$. Let $\mathfrak{p}$ be a prime which lies over p. Then we can form $\mathcal{E} = \tilde{\mathcal{E}} \times \mathcal{O}_H/\mathfrak{p}$, this is a curve over $\mathbb{F}_{p^r}$ with complex multiplication by $\mathcal{O}_F$. If now the extension $F/\mathbb{Q}$ splits at p then this is an ordinary curve whose field of rational endomorphisms is F. If it does not split we get a supersingular curve.

We pick an elliptic curve $\mathcal{E}/\bar{\mathbb{F}}_p$ with level structure, let us denote the field of its rational endomorphisms by $E/\mathbb{Q}$, so this may also be the quaternion algebra. Let us denote by $H/\mathbb{Q}$ the algebraic group with $H(\mathbb{Q}) = E^\times$. To our curve we have the collection of cohomological data

$$\mathcal{H}^\bullet(\mathcal{E}) = (D_p(\mathcal{E}), \{\dots, H^1(\mathcal{E}, \mathbb{Z}_\ell), \dots\}_{\ell \neq p}, \mathrm{Level}) =$$
$$(D_p(\mathcal{E}), H^1(\mathcal{E}, \hat{\mathbb{Z}}^{(p)}), \psi : H^1(\mathcal{E}, \hat{\mathbb{Z}}^{(p)})/NH^1(\mathcal{E}, \hat{\mathbb{Z}}^{(p)}) \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^2).$$

We want to describe the set Y^H of elliptic curves which are isogenous to $\mathcal{E}$ and we also want to describe the action of the Frobenius Φ_p on this set. Let $\mathcal{E}'$ be a curve and $\lambda : \mathcal{E} \to \mathcal{E}'$ an isogeny. Then this isogeny induces inclusions

$$\lambda_p : D_p(\mathcal{E}') \hookrightarrow D_p(\mathcal{E})$$
$$\lambda^{(p)} : H^1(\mathcal{E}', \hat{\mathbb{Z}}^{(p)}) \hookrightarrow H^1(\mathcal{E}, \hat{\mathbb{Z}}^{(p)})$$

The cokernel of $\lambda^{(p)}$ is of finite index $n^{(p)}$ and coprime to p. The kokernel of λ_p is finite $W(\mathbb{F}_p)$-module of lenght d_p, we say that $\lambda_p(D_p(\mathcal{E}')$ is a cofinite submodule. Then we have

$$\deg(\lambda) = n^{(p)} p^{d_p}$$

In addition we have

$$\psi' : H^1(\mathcal{E}', \hat{\mathbb{Z}}^{(p)})/NH^1(\mathcal{E}', \hat{\mathbb{Z}}^{(p)}) \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^2.$$

It is clear that $\mathcal{E}'$ is determined by these submodules + the datum ψ'. On the other hand we can prescribe submodules

$$X_p \subset D_p(\mathcal{E}) \text{ satisfying } \mathrm{F}(X_p) \subset X_p$$
$$X^{(p)} \subset H^1(\mathcal{E}, \hat{\mathbb{Z}}^{(p)})$$

where X_p is cofinite and $X^{(p)}$ of finite index and we can choose $\psi' : X^{(p)}/NX^{(p)} \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^2$. Then we get a unique $\mathcal{E}'$ with level structure which is isogenous to $\mathcal{E}$ and provides these modules. Finally we observe that we can divide the system of submodules by any non zero integer m, then we get

$$\frac{1}{m} X_p \subset D_p(\mathcal{E}) \otimes B(\bar{\mathbb{F}}_p) \text{ satisfying } \mathrm{F}(\frac{1}{m} X_p) \subset \frac{1}{m} X_p$$
$$\frac{1}{m} X^{(p)} \subset H^1(\mathcal{E}, \hat{\mathbb{Z}}^{(p)}) \otimes \mathbb{Q}$$

and we have

$$\frac{1}{m}X^{(p)} \otimes \mathbb{Z}/N\mathbb{Z} \xrightarrow{\sim} X^{(p)} \otimes \mathbb{Z}/N\mathbb{Z}.$$

These are called submodules up to multiplication by $\mathbb{Q}^\times$, and they also provide elliptic curves with level N structure.

How do we get isogenous curves or how do we get these systems of submodules modulo $\mathbb{Q}^\times$?

At the prime p we consider elements

$$g_p \in \mathrm{Gl}(D_p(\mathcal{E}) \otimes B(\bar{\mathbb{F}}_p))/\mathrm{Gl}(D_p(\mathcal{E})) \text{ for which } \mathrm{F} g_p D_p(\mathcal{E}) \subset g_p D_p(\mathcal{E}).$$

Let us call this set $\mathcal{Y}_p^H$. It is easy to describe:

If we are in the case A) and if we choose a basis as above then

$$\mathcal{Y}_p^H = \{\begin{pmatrix} p^a & 0 \\ 0 & p^b \end{pmatrix}\}$$

and in case B) we get

$$\mathcal{Y}_p^H = \{F^m\}.$$

We observe that $\mathcal{Y}_p^H = H(\mathbb{Q}_p)/H(\mathbb{Z}_p)$, where $H(\mathbb{Z}_p)$ is the maximal compact subgroup.

Outside of p we take elements $g_f^{(p)} \in G(\mathbb{A}^{(p)})/K_f^{(p)}(N)$. Then we get the submodules modulo $\mathbb{Q}^\times$:

$$X_p = g_p D_p(\mathcal{E}), X^{(p)} = g_f^{(p)} H^1(\mathcal{E}, \mathbb{Z}^{(p)})$$

and $g_f^{(p)}$ provides an isomorphism $g_f^{(p)} H^1(\mathcal{E}, \mathbb{Z}^{(p)})/N g_f^{(p)} H^1(\mathcal{E}, \mathbb{Z}^{(p)}) \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^2$.

When do two such elements $(g_p, g_f^{(p)}), (u_p, u_f^{(p)})$ give the same elliptic curve? This is clearly the case if and only if we can find an element $\gamma \in H(\mathbb{Q})$ such that

$$\gamma(g_p, g_f^{(p)}) = (u_p, u_f^{(p)})$$

Hence we come to the conclusion that

$$Y^H = H(\mathbb{Q}) \backslash \mathcal{Y}_p^H \times G(\mathbb{A}^{(p)})/K_f(N).$$

We assume that our Hecke operator $h_f^{(p)}$ is the characteristic function of a double coset $K_f^{(p)} \underline{y}_f^{(p)} K_f^{(p)} \subset \mathrm{Gl}_2(\mathbb{A}^{(p)})$. What does the Hecke operator $\mathrm{F}_p^m \times h_f^{(p)}$ do to an arbitrary curve $\mathcal{E}$ with N-level structure.?

As explained above the p-component of the operator transforms $\mathcal{E}$ into $\mathcal{E}^{(p^m)}$. Outside p we assume that our Hecke operator $h_f^{(p)}$ is given by the characteristic function of a double coset $K_f(N) \backslash y_f^{(p)} K_f(N)$ and we write

$$K_f(N) y_f^{(p)} K_f(N) = \bigcup_{\xi_f^{(p)} \in K_f(N)/(K_f(N) \cap g^{(p)} K_f(N) (g^{(p)})^{-1}} \xi_f^{(p)} y_f^{(p)} K_f(N).$$

We choose a basis $e_1^{(p)}, e_2^{(p)}$ of $H^1(\mathcal{E}, \mathbb{Z}^{(p)})$, we assume that the two basis elements are mapped to $(1,0),(0,1)$ under the map ψ. Now we can apply the elements $\xi_f^{(p)} y_f^{(p)}$ to $H^1(\mathcal{E}, \mathbb{Z}^{(p)})$ and get lattices

$$\{\xi_f^{(p)} y_f^{(p)} H^1(\mathcal{E}, \mathbb{Z}^{(p)})\}_{\xi_f}.$$

Then the operator $F^m \times h_f^{(p)}$ transforms the curve

$$X_p = g_p D_p(\mathcal{E}), X^{(p)} = g_f^{(p)} H^1(\mathcal{E}, \mathbb{Z}^{(p)})$$

into the curves $\mathcal{E}^{(p^m)}(\xi_f^{(p)} y_f^{(p)})$ which are given by the lattices ($\mod \mathbb{Q}^\times$)

$$\mathrm{F}^m X_p = \mathrm{F}^m(g_p D_p(\mathcal{E})), \xi_f^{(p)} y_f^{(p)} X^{(p)} = g_f^{(p)} \xi_f^{(p)} y_f^{(p)} H^1(\mathcal{E}, \mathbb{Z}^{(p)}))$$

What does it mean that we have a fixed point. This means that we can find a $\gamma \in H(\mathbb{Q})$ and a ξ such that $\ker(\gamma) = \ker(\mathcal{E} \to \mathcal{E}^{(p^m)}(\xi))$ and γ induces the identity on $H^1(\mathcal{E}, \hat{\mathbb{Z}}^{(p)})/N H^1(\mathcal{E}, \hat{\mathbb{Z}}^{(p)})$.

This gives a fixed point if and only if we can find an element $\gamma \in \mathrm{End}_{\mathbb{Q}}(\mathcal{E})$ such that

$$\mathrm{F}_p^m g_p D_p(\mathcal{E}) = \gamma g_p D_p(\mathcal{E}) \text{ and } (g_f^{(p)})^{-1} \gamma g_f^{(p)} \in \tilde{K}_f^{(p)}(N) \underline{y}_f^{(p)} \tilde{K}_f^{(p)}(N) \qquad (fix)$$

As in the case of the topological trace formula we may modify $(g_p, g_f^{(p)})$ by an element in the centralizer $Z_\gamma^H(\mathbb{A}_f)$ of γ. We get the set of fixed points

$$Z_\gamma^H(\mathbb{Q}) \backslash Z_\gamma^H(\mathbb{A}_f)(g_p, g_f^{(p)}) K_f^{(p)}(N)/K_f^{(p)}(N).$$

This is equal to

$$Z_\gamma^H(\mathbb{Q}) \backslash Z_\gamma^H(\mathbb{A}_f)/K^Z(g_p, g^{(p)}) = Z_\gamma^H(\mathbb{Q}) \backslash Z_\gamma^H(\mathbb{A}_f)/K_p^Z \times K^Z(g_f^{(p)})$$

where $K^Z(g_p, g^{(p)})$ is the following subgroup $K^Z(g_p, g^{(p)}) = K_p^Z \times K^Z(g_f^{(p)}) \subset Z_\gamma^H(\mathbb{A}_f)$: The group $K_p^Z = Z_\gamma^H(\mathbb{Q}_p) \cap H(\mathbb{Z}_p)$ and this is the maximal compact sugroup in $Z_\gamma^H(\mathbb{Q}_p)$ and

$$K^Z(g_f^{(p)}) = H(\mathbb{A}^{(p)}) \cap g_f^{(p)} K_f^{(p)}(N) (g_f^{(p)})^{-1}.$$

Hence the total contribution of $\gamma, (g_p, g_f^{(p)})$ will be to the trace of our operator will be

$$\mathrm{tr}(\gamma | \tilde{\mathcal{M}}) \#(Z_\gamma^H(\mathbb{Q}) \backslash Z_\gamma^H(\mathbb{A}_f)/K^Z(g_p, g^{(p)})),$$

provided the relation (fix) holds, otherwise it is zero.

Then for a given γ we have to count how many $(g_p, g^{(p)})$ satisfy (fix). Our considerations in the the section 3.1 yield after a small manipulation of measures

$$\sum_{\gamma \in H(\mathbb{Q})/\sim} \mathrm{tr}(\gamma | \mathcal{M}_k) \mathrm{vol}_{dz_f}(Z_\gamma^H(\mathbb{Q}) \backslash Z_\gamma^H(\mathbb{A}_f) \prod_{\ell \neq p} \int_{Z_\gamma^H(\mathbb{Q}_\ell) \backslash G(\mathbb{Q}_\ell)} (t_{\underline{y}_f^{(p)}})_\ell (\bar{g}_\ell^{-1} \gamma \bar{g}_\ell) d\bar{g}_\ell) \chi_p[\gamma, m],$$

where $\chi_p[\gamma, m]$ is the sum over the $g_p \in Z_\gamma^H(\mathbb{Q}_p) \backslash \mathrm{Gl}(D_p(\mathcal{E}) \otimes \mathbb{Q}_p)/\mathrm{Gl}(D_p(\mathcal{E}))$ which satisfy $\mathrm{F} g_p D_p(\mathcal{E}) \subset g_p D_p(\mathcal{E})$ and $\mathrm{F}^m g_p D_p(\mathcal{E}) = \gamma g_p D_p(\mathcal{E})$. This is a so called twisted orbital integral.

Summation over all possible $H/\mathbb{Q}$ yields the contribution over the fixed points in the Grothendieck-Lefschetz fixed point formula

$$\sum_{H/\mathbb{Q}} \sum_{\gamma \in H(\mathbb{Q})/\sim} \mathrm{tr}(\gamma|\mathcal{M}_k)\mathrm{vol}_{dz_f}(Z^H_\gamma(\mathbb{Q})\backslash Z^H_\gamma(\mathbb{A}_f))\times$$
$$\prod_{\ell\neq p} \int_{Z^H_\gamma(\mathbb{Q}_\ell)\backslash G(\mathbb{Q}_\ell)} (t_{\underline{y}_f^{(p)}})_\ell(\bar{g}_\ell^{-1}\gamma\bar{g}_\ell)d\bar{g}_\ell)\chi_p[\gamma,m].$$

2.2.6 The comparison

We now follow the strategy outlined in 1.2. We recall that we defined the element $h_p^{(m)}$ in the unramified Hecke algebra at p by the rule

$$\hat{h}_p^{(m)}(\lambda_p) = \mathrm{tr}(r(\lambda_p)) = \alpha_p^m + \beta_p^m$$

Our aim is to prove that (See 1.2)

$$\mathrm{tr}((h_p^{(m)} \times h_f^{(p)} \mid H_c^\bullet(S^G_{K_f(N)}, \tilde{\mathcal{M}}_k) - 2\mathrm{tr}(\mathrm{F}_p^m \times h_f^{(p)} \mid H_c^\bullet(\mathcal{S}^G_{K_f(N)} \times \bar{\mathbb{F}}_p, \tilde{\mathcal{M}}_{k,\ell}) = 0.$$

We will fail, but we compute the difference anyway.

At first we compare the index sets for the summation. We consider that part of the sum, where $H/\mathbb{Q}$ is the multiplicative group of an imaginary quadratic extension $E/\mathbb{Q}$ which splits at p. (This is case (A)). Our considerations on the Dieudonne-module imply that the element γ must have the prime decompostion $(\gamma)_p = \mathfrak{p}^m$, hence it can not be in $\mathbb{Q}$. The pairs of conjugate (under the Galois group) elements $\gamma, \bar{\gamma}$ are in one to one correspondence with elliptic conjugacy classes in $\mathrm{Gl}_2/\mathbb{Q}$ which generate an extension isomorphic to E. But this also clear that only one of the two elements $\gamma, \bar{\gamma}$ can contribute to the sum. Hence we see that the indices of summation in

$$\sum_{H/\mathbb{Q}, H \text{ splits at} p} \quad \sum_{\gamma \in H(\mathbb{Q}), (\gamma)_p = \mathfrak{p}^m}$$

and

$$\sum_{\gamma \in G(\mathbb{Q})_{\mathrm{ell}}/\sim, \gamma \text{ splits at} p}$$

can be identified.

Basically the same holds if we are in case (B). Then $H(\mathbb{Q}) = D^\times$ and we have a natural identification

$$H(\mathbb{Q})/\sim = \text{elliptic, } p \text{ -elliptic elements in } G(\mathbb{Q})_{\mathrm{ell}}/\sim$$

Hence we see that we have a term by term bijection between the (possibly non zero) terms in the two trace formulae.

Now we have to compare the summands, they are

$$\mathrm{tr}(\gamma|\mathcal{M}_k)\mathrm{vol}_{dz_f}(Z^H_\gamma(\mathbb{Q})\backslash Z^H_\gamma(\mathbb{A}_f))\times$$
$$\Big(\prod_{\ell\neq p} \int_{Z^H_\gamma(\mathbb{Q}_\ell)\backslash G(\mathbb{Q}_\ell)} (t_{\underline{y}_f^{(p)}})_\ell(\bar{g}_\ell^{-1}\gamma\bar{g}_\ell)d\bar{g}_\ell)\Big)\chi_p[\gamma,m]$$

and

$$\mathrm{tr}(\gamma|\mathcal{M}_k)\mathrm{vol}_{\omega^{\mathrm{GB}}}(Z_\gamma(\mathbb{Q})\backslash Z_\gamma(\mathbb{A})/K_\infty^Z)\times$$

$$(\prod_{\ell\neq p}\int_{Z_\gamma(\mathbb{Q}_\ell)\backslash G(\mathbb{Q}_\ell)}(t_{\underline{y}_f^{(p)}})_\ell(\bar{g}_\ell^{-1}\gamma\bar{g}_\ell)d\bar{g}_\ell))\int_{Z_\gamma(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)}h_p^{(m)}(\bar{g}_p^{-1}\gamma\bar{g}_p)\bar{d}g_p$$

where $\mathrm{vol}_{\omega^{\mathrm{GB}}}(Z_\gamma(\mathbb{Q})\backslash Z_\gamma(\mathbb{A})/K_\infty^Z)$ is an abbreviation for the more complicated term in 4.3.

Now we have to discuss carefully the normalization of measures (See 4.3) and the two volume factors in front of the orbital integrals. For regular elements we have

$$\mathrm{vol}_{\omega^{\mathrm{GB}}}(Z_\gamma(\mathbb{Q})\backslash Z_\gamma(\mathbb{A})/K_\infty^Z) = 2\mathrm{vol}_{dz_f}(Z_\gamma(\mathbb{Q})\backslash Z_\gamma(\mathbb{A}_f))$$

where the second factor is the class number divided by the order of the group of roots of unity and where the factor 2 comes from the fact that γ has a fixed point in the lower and in the upper half plane. (This is the $c_\infty(\gamma)$ in 4.3). The other volume factor is

$$\mathrm{vol}_{dz_f}(Z_\gamma^H(\mathbb{Q})\backslash Z_\gamma^H(\mathbb{A}_f)).$$

But since $Z_\gamma/\mathbb{Q} = Z_\gamma^H/\mathbb{Q}$ the two volumes are equal. Hence we see that for regular elements the factors in front of the orbital integrals differ by a factor 2. For central elements we have a similar comparison which follows from the equality of Tamagawa numbers and the computation of Euler characteristics in terms of this Tamagawa measure. The relation between the two factors can by viewed as a "fundamental lemma" at the infinite place (See section 4.1-4.3).

Then we have to prove another "fundamental lemma" at the prime p. This says for regular elements

$$\chi_p[\gamma,m] = \int_{Z_\gamma(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)} h_p^{(m)}(\bar{g}_p^{-1}\gamma\bar{g}_p)\bar{d}g_p$$

This is not so extremely difficult in the present case. The function $\chi_p[\gamma,m]$ is easy to compute. We have the (reduced) norm $N : H(\mathbb{Q}_p)\to\mathbb{Q}_p^\times$ and we can put $\deg_p(\gamma) = \mathrm{ord}_p(N(\gamma))$, i.e. $p^{\deg_p(\gamma)}$ is the p-component of the degree of γ. Since F^m has degree p^m we see easily that

$$\chi_p[\gamma,m] = 0 \text{ if } \deg_p(\gamma)\neq m.$$

Now we have to distinguish two cases, which are of course again our cases A) and B). If $\mathbb{Q}_p(\gamma)$ is split and not central, then we have

$$\chi_p[\gamma,m] = \begin{cases} 1 & \text{if } \gamma = \begin{pmatrix} p^m & 0\\ 0 & 1\end{pmatrix} \quad \bmod (\mathbb{Z}_p^*)^2 \\ 0 & \text{else}\end{cases}$$

and if γ does not split or is central then

$$\chi_p[\gamma,m] = 1 \text{ if and only if } \deg_p(\gamma) = m.$$

We can also compute the orbital integral on the right hand side. It follows from ??? that for p-hyperbolic elements the orbital integral is

$$\int_{Z_\gamma(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)} h_p^{(m)}(\bar{g}_p^{-1}\gamma\bar{g}_p)\bar{d}g_p = \begin{cases} 1 & \text{if } \gamma = \begin{pmatrix} p^m & 0 \\ 0 & 1 \end{pmatrix} \mod (\mathbb{Z}_p^*)^2 \\ 0 & \text{else} \end{cases}.$$

Here the quotient measure bla bla.

To prove the fundamental lemma for regular p -elliptic elements we have to use the Bruhat Tits building.

Then we can conclude the sum over the local contributions from the Grothendieck-Lefschetz formula and the elliptic contributions in the topological trace formula cancel.

Therefore we have proved that for $m >> 0$

$$\mathrm{tr}(h_p^{(m)}\times h_f^{(p)} \mid H_c^\bullet(S^G_{K_f(N)},\tilde{\mathcal{M}}_k) - 2\mathrm{tr}(\mathrm{F}_p^m\times h_f^{(p)} \mid H_c^\bullet(\mathcal{S}^G_{K_f(N)}\times\bar{\mathbb{F}}_p,\tilde{\mathcal{M}}_{k,\ell}) = \mathrm{tr}_\infty(h_p^{(m)}\times h_f^{(p)}).$$

We study the term

$$\mathrm{tr}_\infty(h_p^{(m)}\times h_f^{(p)}) = \mathrm{tr}((h_p^{(m)}\times h_f^{(p)})^B | H^\bullet(\partial S^G_{K_f(N)},\tilde{\mathcal{M}}_k))$$

We also have the tautological formulae

$$\mathrm{tr}(h_p^{(m)}\times h_f^{(p)} \mid H_c^\bullet(S^G_{K_f(N)},\tilde{\mathcal{M}}_k) =$$

$$\mathrm{tr}(h_p^{(m)}\times h_f^{(p)} \mid H_!^\bullet(S^G_{K_f(N)},\tilde{\mathcal{M}}_k) + \mathrm{tr}(h_p^{(m)}\times h_f^{(p)} \mid H_{\mathrm{Eis}}^\bullet(S^G_{K_f(N)},\tilde{\mathcal{M}}_k)$$

and

$$\mathrm{tr}(\mathrm{F}_p^m\times h_f^{(p)} \mid H_c^\bullet(\mathcal{S}^G_{K_f(N)}\times\bar{\mathbb{F}}_p,\tilde{\mathcal{M}}_{k,\ell}) =$$

$$\mathrm{tr}(\mathrm{F}_p^m\times h_f^{(p)} \mid H_!^\bullet(\mathcal{S}^G_{K_f(N)}\times\bar{\mathbb{F}}_p,\tilde{\mathcal{M}}_{k,\ell}) + \mathrm{tr}(\mathrm{F}_p^m\times h_f^{(p)} \mid H_{\mathrm{Eis}}^\bullet(\mathcal{S}^G_{K_f(N)}\times\bar{\mathbb{F}}_p,\tilde{\mathcal{M}}_{k,\ell})$$

At this point I want to make some general remarks. It is the goal of Eisenstein cohomology to describe $H_{\mathrm{Eis}}^\bullet(S^G_{K_f(N)},\tilde{\mathcal{M}}_k)$ as a submodule of $H^\bullet(\partial S^G_{K_f(N)},\tilde{\mathcal{M}}_k)$. We already mentioned that this is easy in our present case, but also some higher dimensional cases have been studied. We have another rather general theorem of Pink, which gives us the structure of the Galois module

$$H^\bullet(\partial S^G_{K_f(N)},\tilde{\mathcal{M}}_k) = H^\bullet(\mathcal{S}^G_{K_f,\infty}\times_{\mathbb{Q}}\bar{\mathbb{Q}}, R^\bullet i_*(\mathcal{M}_\ell))$$

and these two instruments together provide some understanding of $H_{\mathrm{Eis}}^\bullet(S^G_{K_f(N)},\tilde{\mathcal{M}}_k\otimes\mathbb{Q}_\ell)$ as Hecke $\times$ Galois modules.

If we apply this method in our special situation (details will be supplied later) we find

$$\mathrm{tr}(h_p^{(m)}\times h_f^{(p)} \mid H_{\mathrm{Eis}}^\bullet(S^G_{K_f(N)},\tilde{\mathcal{M}}_k) - 2\mathrm{tr}(\mathrm{F}_p^m\times h_f^{(p)} \mid H_{\mathrm{Eis}}^\bullet(\mathcal{S}^G_{K_f(N)}\times\bar{\mathbb{F}}_p,\tilde{\mathcal{M}}_{k,\ell}) = \mathrm{tr}_\infty(h_p^{(m)}\times h_f^{(p)}).$$

Hence we finally get

$$\mathrm{tr}(h_p^{(m)}\times h_f^{(p)} \mid H_!^\bullet(S^G_{K_f(N)},\tilde{\mathcal{M}}_k) - 2\mathrm{tr}(\mathrm{F}_p^m\times h_f^{(p)} \mid H_!^\bullet(\mathcal{S}^G_{K_f(N)}\times\bar{\mathbb{F}}_p,\tilde{\mathcal{M}}_{k,\ell}) = 0.$$

If we now make the isotypical decomposition

$$H^1_!(\mathcal{S}^G_{K_f} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathcal{M}_\ell) = \bigoplus_{\pi_f} H^1_!(\mathcal{S}^G_{K_f} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathcal{M}_\ell)(\pi_f).$$

We write $\pi_f = \pi_p \times \pi_f^{(p)}$ and if $\pi_p = \pi(\alpha_p, \beta_p)$ then we get (Eichler-Shimura):

$$\mathrm{tr}(\Phi_p^{-m} | H^\bullet_!(\mathcal{S}^G_{K_f(N)} \times \bar{\mathbb{F}}_p, \tilde{\mathcal{M}}_{k,\ell})(\pi_f^{(p)})) = (\alpha_p^m + \beta_p^m) m(\pi_f^{(p)})/2.$$

We discussed a proof of this result using the congruence relations earlier and that proof seems to be much easier. But this approach using the (topological) trace formula generalizes to cases where the congruence relations alone are not enough.

2.3 Example 3

(Endoscopy, Langlands-Labesse)

2.3.1 The cohomolgy for Sl_2

In this section we want to apply the trace formula to the cohomology of spaces which are attached to the group $\mathrm{Sl}_2/\mathbb{Q}$. Essentially we will discuss the the previous examples for this group. We will encounter some very interesting and subtle phenomena.

We want to call this group $G^{(1)}/\mathbb{Q}$, the group $\mathrm{Gl}_2/\mathbb{Q}$ is still denoted by $G/\mathbb{Q}$.

As always we choose the full congruence subgroup $K_f^{(1)} = K_f^{(1)}(N) \subset G^{(1)}(\mathbb{A}_f)$, the group $K_\infty^{(1)} = \mathrm{SO}(2)$ and then $X^{(1)} = G^{(1)}(\mathbb{R})/\mathbb{K}_\infty^{(1)} = \mathbb{H}$ is the upper half plane. The first important fact is that our locally symmetric space $S^{G^{(1)}}_{K_f^{(1)}}$ is not the set of complex points of a scheme over $\mathbb{Q}$. To give a modular interpretation of this space we have to pass to the scheme $\mathrm{Spec}(\mathbb{Z}[\frac{1}{N}, \zeta_N])$, where ζ_N is a primitive N-th root of unity. Then we can define the functor which attaches to any scheme $S \to \mathrm{Spec}(\mathbb{Z}[\frac{1}{N}, \zeta_N])$ the set of elliptic curves with N-level structure

$$\begin{array}{c} (\mathcal{E}, e_1, e_2) \\ \downarrow\uparrow\uparrow \\ S \end{array}$$

where $\mathcal{E}$ is an elliptic curve over S, the two upwards arrows are sections e_1, e_2 which provide a basis for the N-division points and where the value of the Weil pairing on the pair e_1, e_2 is ζ_N. Then this functor is represented by a scheme

$$\mathcal{S}^{G^{(1)}}_{K_f^{(1)}} \to \mathrm{Spec}(\mathbb{Z}[\frac{1}{N}, \zeta_N]).$$

Before we can speak of the complex points of this scheme we have to choose an embedding $\mathbb{Z}[\frac{1}{N}, \zeta_N]) \hookrightarrow \mathbb{C}$, i.e. we have to choose a primitive N-th root of unity, let us send ζ_N to $e^{2\pi i/N}$. Then our locally symmetric space

$$S^{G^{(1)}}_{K_f^{(1)}} = \Gamma(N)\backslash\mathbb{H} \xrightarrow{\sim} \mathcal{S}^{G^{(1)}}_{K_f^{(1)}}(\mathbb{C})$$

Again we have the standard representations on the space $\mathcal{M}_k$ of homogenous polynomials in two variables and degree k, we can study the cohomolgy

$$H^\bullet(S^{G^{(1)}}_{K_f^{(1)}}, \tilde{\mathcal{M}}_k) = H^\bullet(\Gamma(N)\backslash\mathbb{H}, \tilde{\mathcal{M}}_k)$$

as a module under the Hecke algebra $\mathcal{H}^{(1)} = \mathcal{C}_c(G^{(1)}(\mathbb{A}_f)//K_f^{(1)})$. IWe can multiply the coefficient system by $\mathbb{Q}_\ell$ then we can consider the ℓ adic cohomology

$$H^1(\mathcal{S}^G_{K_f} \times_{\mathcal{O}_N} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}_{k,\ell})$$

this is a module for

$$\mathcal{H}^{(1)} \otimes \mathrm{Gal}(\mathbb{Q}/\mathbb{Q}(\zeta_N)),$$

and again we want to investigate the structure of this module.

We want to relate this scheme to the corresponding scheme $\mathcal{S}^G_{K_f} \to \mathrm{Spec}(\mathbb{Z}[\frac{1}{N}])$, whichis attached to Gl_2. In 2.2.1 we gave a morphism $\mathcal{S}^G_{K_f} \to \mathrm{Spec}(\mathcal{O}_N)$

This gives us isomorphisms

$$\begin{array}{ccc} t_\nu : \mathcal{S}^{G^{(1)}}_{K_f^{(1)}} & \to & \mathcal{S}^G_{K_f} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbb{Z}[\frac{1}{N}, \zeta_N]) & \xrightarrow{t_\nu} & \mathrm{Spec}(\mathcal{O}_N) \end{array}.$$

We can pick a reference morphism by sending ζ_N to $e^{2\pi i/N}$, then we can represent the t_ν by elements $\underline{t}_\nu \in I_\mathbb{Q}/\mathbb{Q}^\times\mathbb{R}^\times_{>0}\mathfrak{U}_N \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^\times$ (See III, 1.1.3), we even assume that the $\underline{t}_\nu \in \hat{\mathbb{Z}}_f^\times$

On the level of the locally symmetric spaces, these morphisms are obtained from the inclusions

$$\underline{t}_\nu : G^{(1)}(\mathbb{Q})\backslash\mathbb{H} \times G^{(1)}(\mathbb{A}_f)/K_f^{(1)} \subset G(\mathbb{Q})\backslash\tilde{X} \times G(\mathbb{A}_f)/K_f$$

$$\underline{t}_\nu : (z, \underline{g}_f K_f^{(1)}/K_f^{(1)}) \mapsto (t_\infty z, \underline{t}_{\nu,f}\underline{g}_f K_f/K_f),$$

these maps provide isomorphisms of $S^{G^{(1)}}_{K_f^{(1)}}$ with the different connected components of $G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K_f$.

Hence we get an isomorphism of cohomology groups

$$H^\bullet(S^G_{K_f}, \tilde{\mathcal{M}}_k) \xrightarrow{\oplus t_\nu^\bullet} \bigoplus_\nu H^\bullet(S^{G^{(1)}}_{K_f^{(1)}}, \tilde{\mathcal{M}}_k). \qquad iso$$

This is an isomorphism between Hecke modules if we restrict the action of the Hecke-algebra $\mathcal{H}^{(1)}$. If we consider the resulting isomorphisms in ℓ-adic cohomology

$$H^\bullet(\mathcal{S}^G_{K_f} \times \bar{\mathbb{Q}}, \tilde{\mathcal{M}}_{k,\ell}) \xrightarrow{\oplus t_\nu^\bullet} \bigoplus_\nu H^\bullet(\mathcal{S}^{G^{(1)}}_{K_f^{(1)}} \times_{t_\nu} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}_{k,\ell}) \qquad iso_\ell$$

This is now an isomorphism of $\mathcal{H}^{(1)}\otimes\mathrm{Gal}(\mathbb{Q}/\mathbb{Q}(\zeta_N))$ modules and we will use our previous results on the cohomology of $S^{G^{(1)}}_{K_f^{(1)}}$ to understand the Sl_2-case. Here we will combine the classical Clifford theory and the topological trace formula. We use the topological trace formula to make a comparison like in the first example between $G^{(1)}$ and the norm one group of a quaternion.

2.3.2 Clifford theory

We need some information concerning the relationship between the representations of a pair of groups $H \subset G$ where H is a normal subgroup of G and the factor group G/H is a finite abelian group. We consider modules whch are vector spaces over an algebraically closed field of characteristic zero.

We have the functors of restriction res_H^G of representations from G to H and we have induction from representations from H to G. If N is an H module and $g \in G$ then we can define the conjugate module N^g: The underlying abelian group is N but H acts on N^g by

$$h \mapsto \{n \mapsto ghg^{-1}n\}.$$

It is clear that the isomorphism class of N^g only depends on the class of g in G/H.

If we have an irreducible G module M then it is clear that we get an isotypical decomposition of the restriction of M to H:

$$M|H = \bigoplus N_\nu.$$

It is rather obvious that the N_ν are conjugate to each other: Let N be one of the irreducible constituents of $M|H$, then we define $H_N = \{g \in G | N^g \xrightarrow{\sim} N\}$ and

$$M|H = \bigoplus_{g \in G/H_N} (N^g)^m \qquad (*)$$

We may also consider the induced module

$$M' = \mathrm{Ind}_H^G N$$

and ask for the relationship between M and M'.

We have extreme cases:
(A) We have $H_N = H$.
In this case it follows directly from Frobenius reciprocity that

$$M = \mathrm{Ind}_H^G N \text{ is irreducible,}$$

we have

$$M|H = \bigoplus_{g \in G/H} N^g,$$

and for any character $\chi : G/H \to \mathbb{C}^\times$ we have

$$M \xrightarrow{\sim} M \otimes \chi$$

(B) Our irreducible G module M stays irreducible if we restrict to H.
In this case we have for $M|H = N$ that $N_H = G$. Then

$$\mathrm{Ind}_H^G N = \bigoplus_{\chi \in \hat{G}} M \otimes \chi$$

and $M \otimes \chi$ and M are not isomorphic if $\chi \neq 1$.

Now we study the general case. Let M be an irreducible G module and let N be an irreducible constituent of $M|H$. Let $g \in G/H$ be an element $\neq e$ and assume that $N^g \xrightarrow{\sim} N$. (If we do not find such an element then we are in case (A)) Then it is not so difficult to show that we can extend the action of H on N to an action of the group $H' =< H, g >$. Then it is again Frobenius reciprocity which yields that the restriction of M to H' contains an irreducible H' submodule $N \otimes \eta$ with η in the character group of H'/H. What we need that this restriction contains an irreducible H' submodule N' which stays irreducible if we restrict it to H. We proceed and see that we can find a subgroup H_1 and an irreducible constituent N_1 of $M|H_1$ such that

(i) The restriction of N_1 to H_1 is irreducible
(ii) If $g \in G$ and $N_1^g \xrightarrow{\sim} N_1$ then $g \in H_1$.

Then we see that with respect to the pair G, H_1 we are in case (A) and especially we get

$$M = \mathrm{Ind}_{H_1}^G N_1 \text{ and } M \otimes \chi \xrightarrow{\sim} M \text{ for all } \chi \in \widehat{G/H_1}.$$

The restriction

$$M|H_1 = \bigoplus_{g \in G/H_1} N_1^g$$

It now may happen that two different modules N_1, N_1^g become isomorphic if we restrict them to H. This is the case if and only if the multiplicity $m > 1$.

We are in the case (B) if and only if $H_1 = G$. Hence we see that $M|H$ becomes reducible if and only if we can find a $\chi \in \widehat{G/H}, \chi \neq 1$ such that $M \otimes \chi \xrightarrow{\sim} M$.

If in turn we have such a $\chi \neq 1$ we can consider the kernel H_χ and we may consider the restriction $M|H_\chi$. We apply the formula (*) to the pair (G, H_χ) aund get with $H_N \supset H_\chi$

$$M|H_\chi = \bigoplus_{g \in G/H_N} (N^g)^m$$

Since H/H_χ is cyclic we can extend the H_χ-module structure on N to H_N and conclude that M contains the irreducible submodule $\mathrm{Ind}_{H_N}^G N$, from this we get $m = 1$ and $H_N = H_\chi$. The conclusion is that under our conditions

$$M|H_\chi = \bigoplus_{g \in G/H_\chi} N^g,$$

the summands on the right hand side are pairwise non isomorphic.

The nice case is if we have $m = 1$. As I said, in this case the N^g are non isomorphic if we restrict them to H. We see easily that then

$$M \otimes \chi \xrightarrow{\sim} M \text{ if and only if } \chi \in \widehat{G/H_1}$$

this gives us a different description of the group H_1.

With respect to the pair (G,H_1) and for our given module M we are in case (A) , if N_1 is one of the constituents of $M|H_1$, then N_1 with respect to (H_1,H) is of type (B).

The number of irreducible constituents in $M|H$ is equal to the number of characters $\chi \in \widehat{G/H}$ for which $M \otimes \chi \xrightarrow{\sim} M$.

2.3.3 The local restriction from Gl_2 to Sl_2

We want to apply this to the restriction of representations of $\mathrm{Gl}_2(\mathbb{Q}_p)$ to $\mathrm{Sl}_2(\mathbb{Q}_p)$. More precisely we consider irreducible admissible representations π_p (See III 1.3.1) to $\mathrm{Sl}_2(\mathbb{Q}_p)$. This is actually not quite the right thing to do since the index is not finite. We enlarge $\mathrm{Sl}_2(\mathbb{Q}_p)$ by the centre $G_m(\mathbb{Q}_p)$ of $\mathrm{Gl}_2(\mathbb{Q}_p)$ then we get an isomorphism

$$\mathrm{Gl}_2(\mathbb{Q}_p)/G_m(\mathbb{Q}_p)\mathrm{Sl}_2(\mathbb{Q}_p) \xrightarrow{\sim} \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2$$

The restriction of π_p to $G_m(\mathbb{Q}_p)\mathrm{Sl}_2(\mathbb{Q}_p)$ will be a tensor product

$$\mathrm{res}(\pi_p) = \omega_p(\pi_p) \otimes {\pi_p}^{(1)},$$

where $\omega_p(\pi_p)$ is the central character of π_p and ${\pi_p}^{(1)}$ is a representation of $\mathrm{Sl}_2(\mathbb{Q}_p)$ which is of course the object of interest.

We apply Clifford theory to this situation. We a Lemma in Labesse-Langlands which asserts that the irreducible components of ${\pi_p}^{(1)}$ come with multiplicity one. This Lemma uses the local theory of Whittaker models. Hence we get a decomposition

$$\mathrm{res}(\pi_p) = \omega_p(\pi_p) \otimes {\pi_p}^{(1)} = \bigoplus_{g \in G/H_\chi} \omega_p(\pi_p) \otimes ({\tilde{\pi}_p}^{(1)})^g.$$

Here $G_m(\mathbb{Q}_p)\mathrm{Sl}_2(\mathbb{Q}_p) \subset H_\chi \subset \mathrm{Sl}_2(\mathbb{Q}_p)$ is as above and ${\tilde{\pi}_p}^{(1)}$ is irreducible for $G_m(\mathbb{Q}_p)\mathrm{Sl}_2(\mathbb{Q}_p)$. The collection representations $\{({\tilde{\pi}_p}^{(1)})^g\}_{g \in G/H_\chi}$ is the L-packet associated to π_p. We may wonder whether there are non trivial L-packets.

2.3.4 L-packets

The fact that the $\mathbb{Q}_p$-rational elliptic regular conjugacy classes $c \in C_G(\mathbb{Q}_p)$ decompose into 2 conjugacy classes under $G(\mathbb{Q}_p)$ must be reflected in representation theory, since characters ch_π of irreducible representations have to separate these classes. It turns out that there will be specific representations attached to the anisotropic tori which provide this separation. These representations come in so called L-packets consisting of two and sometimes four members, which are somewhat related to each other.

There is a very specific L-packet which I am going to describe. We look at induced representations (Chap III.1.2.1)

$$\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\chi = \{f : G(\mathbb{Q}_p) \to \bar{\mathbb{Q}} | f(bg) = b^{\chi+\rho} f(g)\}.$$

We assume that χ ist unitary and hence the induced representations are so too. The character of these representations is zero on the elliptic regular elements hence, they are totally incapable to separate the above mentioned conjugacy classes in a stable class.

These representations are irreducible with one exception: If

$$\chi = \chi_E : \begin{pmatrix} p^\nu & * \\ 0 & p^{-\nu} \end{pmatrix} \to (-1)^\nu,$$

then

$$\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \chi_E = \pi^+ \oplus \pi^-$$

and the characters ch_{π^+}, ch_{π^-} are non zero on the elliptic elements and the values add up to zero on regular elliptic elements which are conjugate over $\bar{\mathbb{Q}}_p$ but not over $\mathbb{Q}_p$. These are the so called are unstable characters. Here the subscript E indicates that our character χ is also the character attached to the unramified quadratic extension $E/\mathbb{Q}_p$.

We can easily relate this to Clifford theory, we have $G/\mathbb{Q} = \mathrm{Sl}_2/\mathbb{Q} \subset \tilde{G} = \mathrm{Gl}_2/Q$ and we can extend χ_E to a character $\tilde{\chi}_E : \tilde{B}(\mathbb{Q}_p) \to \mathbb{Q}^\times$, in the notation of Chap III, loc.cit. we may choose $\alpha' = i, \eta_p' = -i$, then the central character is trivial. Then the induced representation $\tilde{\pi}_p = \mathrm{Ind}_{\tilde{B}(\mathbb{Q}_p)}^{\tilde{G}(\mathbb{Q}_p)} \tilde{\chi}_E$ of $\tilde{G}(\mathbb{Q}_p)$ is irreducible but we observe that

$$\tilde{\pi}_p \otimes \chi_E \xrightarrow{\sim} \tilde{\pi}_p$$

and hence its restriction to $G(\mathbb{Q}_p)$ becomes reducible.

Of course we can not distinguish between the two representations π^+, π^-. But if we choose an extension of $G/\mathbb{Q}_p$ to a semi-simple group scheme $\mathcal{G}/\mathrm{Spec}(\mathbb{Z}_p)$ then this gives us a maximal compact subgroup $\mathcal{G}(\mathbb{Z}_p)$ and exactly one of the two irreducible representations will have a $\mathcal{G}(\mathbb{Z}_p)$ invariant vector, this component is then distinguished.

We can also see L-packets on the norm one group of the division algebra. It is known that we have a surjective homomorpism from $G^*(\mathbb{Q}_p)$ to the group of elements of norm 1 in $\mathbb{F}_{p^2}^*$.

One checks that two elements $x, x^{-1} = \bar{x}$ are stably conjugate (they are conjugate in $D(\mathbb{Q}_p) = D^*$ (Skolem-Noether), but they are certainly not conjugate in $G^*(\mathbb{Q}_p)$ if they do not map to ± 1 in the group of elements of norm 1. Hence a pair of characters

$$\chi : N_1 \mathbb{F}_{p^2}^* \to \mathbb{C}^\times$$

with $\chi \neq \chi^{-1}$ provides a nice little L-packet. It separates the elements x, x^{-1} if $\chi(x) \neq \pm 1$.

For the group $\mathrm{Sl}_2/\mathbb{Q}_p$ it has been shown by Langlands and Labesse that for any anisotropic $T/\mathbb{Q}_p$ in G or G^* and any unitary character $\varphi_p : T(\mathbb{Q}_p) \to \mathbb{C}^*$ there is a L-packet $\Pi(\varphi_p)$, which is somehow created by φ_p. In the case that $T/\mathbb{Q}_p$ splits over the unramified quadratic extension $E/\mathbb{Q}$ and φ is the trivial character we get again the L-packet constructed from the reducible principal series representation.

For general semi simple groups over p-adic fields the theory of L-packets is still conjectural.

2.3.5 The restriction of the cohomology of Gl_2 to Sl_2.

We recall the situation in 2.3.1. We have the Hecke algebra $\mathcal{H} = \mathcal{C}_c(G(\mathbb{A}_f//K_f)$ action on the cohomology $H^\bullet(\mathcal{S}^G_{K_f} \times \bar{\mathbb{Q}}, \tilde{\mathcal{M}}_{k,\ell})$, on the summands on the right hand side of $(iso_\ell$ we have only the action of $\mathcal{H}^{(1)}$. But we can enlarge this action slightly to an action of an enhaced Hecke algebra $\mathcal{H}^{(1)}_{N,\mathrm{enh}}$. We observe that a Hecke operator $t_{\underline{a}_f} \in \mathcal{H}$ which is defined by the characteristic function of a double coset $K_f\underline{a}_fK_f$ leaves the irreducible components fixed if $det(\underline{a}_f)$ considered as an element in $I_{\mathbb{Q}}$ maps to the identity in $I_{\mathbb{Q}}/\mathbb{Q}^\times(\mathbb{R}^\times_{>0} \times \mathfrak{U}_N) = (\mathbb{Z}/N\mathbb{Z})^\times$. (See Chap III.1.1.3)

Hence we can define the enhanced Hecke algebra $\mathcal{H}^{(1)}_{N,\mathrm{enh}}$ *as the Hecke algebra generated by these Hecke operators. The enhanced Hecke algebra commutes with the action of the Galois group* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\zeta_N)$

Then we can say that the Hecke algebra $\mathcal{H}$ contains the group consisting of the double cosets $K_f\underline{t}_\nu K_f$, which is isomorphic to $(\mathbb{Z}/N\mathbb{Z})^\times$. We observe that the $\underline{t}_\nu$ normalize K_f. We write

$$\mathcal{H} = \mathcal{H}^{(1)}_{N,\mathrm{enh}} \times (\mathbb{Z}/N\mathbb{Z})^\times$$

We can apply Clifford theory to this situation. If we have an irreducible module π_f for $\mathcal{H}$, then we can restrict it to $\mathcal{H}^{(1)}_{N,\mathrm{enh}}$ and as such it will decompose into

$$\pi_f = \pi_f^{(1)} \oplus (\pi_f^{(1)})^{\underline{t}} \oplus ..$$

where $\underline{t} \in (\mathbb{Z}/N\mathbb{Z})^\times/H$ and H is the stabilizer of the isomorphism class of $\pi_f^{(1)}$. For a character $\chi \in (\mathbb{Z}/\hat{N}Z)^\times$ we have $\pi_f \xrightarrow{\sim} \pi_f \otimes \chi$ if and only if χ is trivial on H.

The module π_f occurs with multiplicity two in $H^1_!(\mathcal{S}^G_{K_f} \times \bar{\mathbb{Q}}, \tilde{\mathcal{M}}_{k,\ell})$ and we have

$$H^1_!(\mathcal{S}^G_{K_f} \times \bar{\mathbb{Q}}, \tilde{\mathcal{M}}_{k,\ell})(\pi_f) = \pi_f \otimes W(\pi_f),$$

where $W(\pi_f)$ is a two dimensional Galois module. Now we consider our isomorphism (iso) and (iso_ℓ). If we restrict to π_f, then we have to take the twisted modules $\pi_f \otimes \chi$ into account. We sum over the $\chi \in (\mathbb{Z}/\hat{N\mathbb{Z}})^\times)/((\mathbb{Z}/N\hat{\mathbb{Z}})^\times/H)$ and $\underline{t} \in (\mathbb{Z}/N\mathbb{Z})^\times/H$ and get an isomorphism

$$\bigoplus_\chi H^1_!(\mathcal{S}^G_{K_f} \times \bar{\mathbb{Q}}, \tilde{\mathcal{M}}_{k,\ell})(\pi_f \otimes \chi) \xrightarrow{\oplus t_\nu^\bullet} \bigoplus_\nu \bigoplus_{\underline{t}} H^1_!(\mathcal{S}^{G^{(1)}}_{K_f^{(1)}} \times_{t_\nu} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}_{k,\ell})(\pi_f^{(1)})^{\underline{t}}$$

Then a simple dimension count gives us a formula for the multiplicities

$$m(\pi_f^{(1)}) = m(\pi_f)/[(\mathbb{Z}/N\mathbb{Z})^\times : H]$$

hence we see

If the restriction of π_f *to the enhanced Hecke algebra is irreducible then* $m(\pi_f^{(1)}) = 2$ *and the Galois module* $W^{(1)}(\pi^{(1)})$ *is the restriction of* $W(\pi_f)$ *to* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\zeta_N))$. *If the restriction of* π_f *is reducible, then we have exactly one quadratic character* χ *such that* $\pi_f \xrightarrow{\sim} \pi_f \otimes \chi$. *Then* $W^{(1)}(\pi_f^{(1)})$ *is of dimension*

1 and given by a character $\varphi_\ell : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\zeta_N)) \to \bar{\mathbb{Q}}_\ell^\times$. The quadratic character χ determines a quadratic extension $E/\mathbb{Q}$ and the restriction of the module $W(\pi_f)$ to $\mathrm{Gal}(\bar{\mathbb{Q}}/E)$ is reducible over. It decomposes into two one dimensional summands which are given by two different ℓ-adic characters $\tilde{\varphi}_\ell, \tilde{\varphi}_\ell{}' : \mathrm{Gal}(\bar{\mathbb{Q}}/E) \to \bar{\mathbb{Q}}_\ell^\times$. One of these two characters is an extension of φ_ℓ.

It turns out that this quadratic extension is imaginary quadratic, and the ℓ-adic character is induced by an algebraic Hecke character $\tilde{\varphi}$ on the torus $\tilde{T} = R_{E/\mathbb{Q}}(G_m)$. This will be explained in the next section.

2.3.6 Galois representations attached to algebraic Hecke characters

Let $E/\mathbb{Q}$ be an imaginary quadratic extension and as above $\tilde{T} = R_{E/\mathbb{Q}}(G_m)$. An algebraic Hecke character is a continous homomorphism $\tilde{\varphi} : T(\mathbb{Q})\backslash T(\mathbb{A}) \to \mathbb{C}^\times$ whose restriction φ_∞ to the infinite component $T(\mathbb{R}) = \mathbb{C}^\times$ is of the form

$$\varphi_\infty(z) = z^a \bar{z}^b$$

whith two integers a, b. (The general form of such a restriction is of the form $z \mapsto (z\bar{z})^s(\frac{z}{\bar{z}})^m$ where $s \in \mathbb{C}$ and $m \in \mathbb{Z}$. So the requirement for being algebraic is $s+m, s-m \in \mathbb{Z}$. To such an algebraic Hecke character we can attach an ℓ-adic representation

$$\varphi_\ell : \mathrm{Gal}(\bar{\mathbb{Q}}/E) \to \mathcal{O}_{L,\mathfrak{l}}^\times,$$

here $\mathcal{O}_{L,\mathfrak{l}}$ is the ring of integers in an $\ell-$adic completion of a finite extension of $\mathbb{Q}_\ell$.

To get this representation we observe that the restriction $\varphi_f$to $T(\mathbb{A}_f)$ takes values in a finite extension L of E. Let $\underline{t}_f = (1, \dots, t_\mathfrak{p}, \dots) \in T(\mathbb{A}_f)$. This element defines a divisor, if we raise $\underline{t}_f$ into a suitably high power $\underline{t}_f^N$, then we can assume that this divisor becomes principal. After raising into a still higher power we may assume that we find an $x \in E^\times$, such that $x\underline{t}_f^N$ is a unit at all places and satisfies congruences at the primes, where φ is ramified. Then

$$\varphi_f(\underline{t}_f)^N = \varphi(x\underline{t}_f^N) = \varphi_\infty(x) = x^a\bar{x}^b \in E^\times.$$

Then the assertion follows easily from the finiteness class groups. We choose a prime $\mathfrak{l}$ in L, which lies above ℓ. This defines our ring $\mathcal{O}_{L,\mathfrak{l}}$.

Now we observe that we have to construct representations

$$\varphi_{l,M} : \mathrm{Gal}(E[\ell^M N]/E) \to (\mathcal{O}_{L,\mathfrak{l}}/\mathfrak{l}^{M'})^\times$$

where $E[\ell^M N]$ is the maximal abelian extension, which has some bounded ramification outside the primes above ℓ, which is controlled by the ramifcation of φ and some high ramification over ℓ, which goes to infinity. Then we have by class field theory

$$r : \mathrm{Gal}(E[\ell^M N]/E) \xrightarrow{\sim} E^\times\backslash I_E/\mathbb{C}^\times\mathfrak{U}_E(N\ell^M).$$

Using the theorem of weak approximation we see, that an element in $r(\sigma) \in E^\times\backslash I_E/\mathbb{C}^\times\mathfrak{U}_E(N\ell^M)$ can be represented by an $\underline{x} \in I_E$ which is congruent to one $\mod \ell^M N$. Then we define

$$\varphi_{l,M}(\sigma) = \varphi_f(\underline{x}_f) \in (\mathcal{O}_{L,\mathfrak{l}}/\mathfrak{l}^{M'})^\times.$$

If we take two different representatives $\underline{x}, \underline{x}'$ of such a class, then $\underline{x} = \underline{x}'(z_\infty, \underline{u}_f)\alpha$, with $\underline{u}_f \in \mathfrak{U}_E(N\ell^M)$ and $\alpha \in E^\times$. We conclude that $\alpha \equiv 1 \mod N\ell^M$ and hence $\varphi((\alpha)_f) = \alpha^{-a}\bar{\alpha}^{-b} \equiv 1 \mod \ell^M N$. This shows that $\varphi_{l,M}$ is well defined. If now $M \to \infty$ then $M' \to \infty$ and this gives us the representation.

I want to indicate, that the ℓ-adic characters, which we obtain from algebraic Hecke characters are indeed very special ℓ-adic characters.

The ring $\mathcal{O}_{E,\ell}$ is by definition the ring of integers in $E \otimes \mathbb{Q}_\ell$. Inside the group of units $\mathcal{O}_{E,\ell}^\times$ we have congruence subgroups $\mathcal{O}_{E,\ell}^\times(r)$ of units which are congruent to $1 \mod \ell^r$ for some integer r. These unit groups are $\mathbb{Z}_\ell$ modules and it is not hard to see that they are free of rank 2.

If we pass to the limit then the reciprocty isomorphism of class field theory gives us a homomorphism

$$\mathcal{O}_{E,\ell}^\times(1) \to E^\times \backslash I_E/\mathbb{C}^\times \to \mathrm{Gal}(E[\ell^\infty N]/E)$$

and hence we see that any ℓ-adic character

$$\phi_\ell : \mathrm{Gal}(E[\ell^\infty N]/E) \to \mathcal{O}_{L,\mathfrak{l}}^\times$$

induces a homomorphism

$$\phi' : \mathcal{O}_{E,\ell}^\times(1) \to \mathcal{O}_{L,\mathfrak{l}}^\times.$$

We also have the subgroup $\mathcal{O}_{L,\mathfrak{l}}^\times(1)$ of 1-units and this is also a $\mathbb{Z}_\ell$ module, we can define ϵ^u for $\epsilon \in \mathcal{O}_{L,\mathfrak{l}}^\times(1)$ and $u \in \mathbb{Z}_\ell$. Hence to give such a ϕ' which has values in $\mathcal{O}_{L,\mathfrak{l}}^\times(1)$ we simply have to say where the generators go, they may take any values. But if the ϕ' results from an algebraic Hecke character, the we can find an element $\alpha \in \mathcal{O}_E$ such that $\alpha \equiv 1 \mod \ell N$ and then $\alpha, \bar{\alpha}$ generate a $\mathbb{Z}_\ell$ submodule of finite index in $\mathcal{O}_{E,\ell}^\times(1)$. Then we see that on this submodule the values of ϕ' are given by the algebraic expression $\phi'(\alpha) = \alpha^a\bar{\alpha}^b$.

This means that the Galois modules obtained from algebraic Hecke characters are cristalline.

Now we encounter a problem: For which Hecke characters $\tilde{\varphi} : T(\mathbb{Q})\backslash T(\mathbb{A}) \to \mathbb{C}^\times$ do we find a π_f which occurs in $H^1_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})$ such that the resulting Galois module $W(\pi_f)$ decompses into two one dimensional $\mathrm{Gal}(\bar{\mathbb{Q}}/E)$-modules where one of these Galois modules results from $\tilde{\varphi}$ and the other one from its conjugate.

2.3.7 Functoriality

2.3.8 Some general remarks on conjugacy classes

The first example describes how we can apply the topological trace formula to compare the cohomology of two different groups $G/\mathbb{Q}, G^*/\mathbb{Q}$ which are inner forms of each other as modules under the Hecke algebra. But in more general cases the situation is not so simple because a new phenomenon occurs which is called endoscopy. I want to explain this phenomenon in a special case, which is actually only a slight modification of our example 1.

Before I discuss the example I want to make some general remarks. If $G/\mathbb{Q}$ is a reductive group, then a geometric conjugacy class $C/\mathbb{Q}$ is a Zariski open dense subset in a Zariski closed subvariety $\bar{C}/\mathbb{Q}$, such that its geometric points $C(\bar{\mathbb{Q}})$ form an orbit under conjugation. We will see below that such geometric conjugacy classes are not necessarily closed. It does happen that $C(\mathbb{Q}) = \emptyset$, even if the subvariety is not empty. If $C(\mathbb{Q}) \neq \emptyset$ and if $\gamma \in C(\mathbb{Q})$, then we can identify $C = Z_\gamma \backslash G$, where Z_γ is the centralizer of γ. It also happens that the action of $G(\mathbb{Q})$ on $C(\mathbb{Q})$ is not transitive, we may even have infinitely many orbits. Let us denote by $\mathcal{C}_G(\mathbb{Q})$ the set of geometric conjugacy classes over $\mathbb{Q}$, I do not say that this is the set of $\mathbb{Q}$-valued points of a variety over $\mathbb{Q}$.

If we have wo reductive groups $G/\mathbb{Q}, G^*/\mathbb{Q}$, then we say that these two groups are forms of each other if we can find an isomorphism

$$f : G_{\mathbb{Q}} \times \bar{\mathbb{Q}} \xrightarrow{\sim} G^*_{\mathbb{Q}} \times \bar{\mathbb{Q}},$$

i.e. the become isomorphic over the algebraic closure. We say that they are inner forms of each other if we can find an f such that for any $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ the automorphism $f^{-1} \circ f^\sigma$ is inner, i.e. we find an $g_\sigma \in G(\bar{\mathbb{Q}})/\mathrm{centerof}(G)(\bar{\mathbb{Q}})$ such that $f^{-1} \circ f^\sigma(x) = g_\sigma^{-1} x g_\sigma$ for all $x \in G(\bar{\mathbb{Q}})$(See also Chap II.1.1.2) It is clear

If $G/\mathbb{Q}, G^*/\mathbb{Q}$ are inner forms of each other then such an f induces a bijection

$$f_c : \mathcal{C}_G(\bar{\mathbb{Q}}) \xrightarrow{\sim} \mathcal{C}_{G^*}(\bar{\mathbb{Q}})$$

In our first example we found an inclusion of the set of $G^*(\mathbb{Q})$ conjugacy classes of elliptic elements into the set of $G(\mathbb{Q})$ conjugacy classes of elliptic elements. Hence the index set for the summation in the trace formula on one side was a subset of the index set on the other side. Moreover the rational points in a conjugacy classes always formed an orbit under the action of $G(\mathbb{Q})$ (resp.$G^*(\mathbb{Q})$). This will not be true in general and this fact is responsible for endoscopy.

2.3.9 The specific example

We take for our groups $G/\mathbb{Q}$ (resp. $G^*/\mathbb{Q}$) the group $\mathrm{Sl}_2/\mathbb{Q}$ (resp. the norm 1 group of a quaternion algebra over $\mathbb{Q}$ which ramifies at infinity.)

Now we encounter the phenomenon that the conjugacy classes of (elliptic) elements don't match anymore. We have the two trace maps

$$\mathrm{tr} : G \to \mathbb{A}^1, \mathrm{tr}^* : G^* \to \mathbb{A}^1$$

the fibres over the points $\mathbb{A}^1 \setminus \{2, -2\}$ are the regular semi simple (geometric) conjugacy classes, the fibres over $\{2, -2\}$ contain unipotent elements. These fibres over $\{2, -2\}$ consist of two geometric conjugacy classes, the central elements and the non central (regular) unipotent elements.

The first trace map has a section defined over $\mathbb{Q}$, this is the Steinberg section and defined by

$$\mathrm{St} : a \mapsto \gamma_a = \begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}$$

observe that for $a = \pm 2$ the section has a non central value.

Our two groups become isomorphic over $\bar{\mathbb{Q}}$, they are inner forms of eachother. But over $\mathbb{Q}$ we just learned that tr is surjective on the rational points whereas tr^* is not. We can even say more: over $\mathbb{R}$ the image of tr^* is the intervall $[-2, 2]$.

The elliptic semi simple conjugacy classes are those in the inverse image of this intervall.

2.3.10 The group Sl_2

Now we *stabilize* the topological trace formula for $\mathrm{Sl}_2/\mathbb{Q}$. By this we mean that we want to replace the summation over the index set $G(\mathbb{Q})_{\mathrm{ell}}/\sim$ by a summation over the geometric conjugacy classes over $\mathbb{Q}$.

For any value of $a \in \mathbb{A}^1(\mathbb{Q})_{\mathrm{ell}} = \{a \in \mathbb{Q} | a^2 \leq 4\}$ we consider the rational points in the fibre $\mathrm{tr}^{-1}(a) = C_a$. The group $\mathrm{Sl}_2(\mathbb{Q})$ acts on $C_a(\mathbb{Q})$ and the action is not transitive. Let γ_a^{ss} be the semi simple part of γ_a. In our special case it is equal to γ_a, if $a^2 \neq 2$, and equal to $\pm \mathrm{Id}$ if $a = \pm 2$. Then the centralizer $Z_{\gamma_a^{ss}}$ is a reductive subgroup. It is either the group $G/\mathbb{Q}$ itself (if γ is central) or a maximal torus $T/\mathbb{Q}$ (if γ_a^{ss} is regular, i.e. $a \neq \pm 2$).

We discuss the regular terms, in our special case the others are not interesting. Then $\gamma_a = \gamma_a^{ss}$. For any field extension $\mathbb{Q} \hookrightarrow k$ the orbits of $\mathrm{Sl}_2(k)$ on the semi simple elements in $C_a(k)$ are in one-to-one correspondence to the elements of

$$\mathcal{D}H^1(k, Z_{\gamma_a}) = \{\xi \in H^1(k, Z_{\gamma_a}) | \xi \mapsto 1 \text{ in } H^1(k, G)\}.$$

Since $H^1(\mathbb{Q}, G) = H^1(\mathbb{Q}, \mathrm{Sl}_2) = 0$ we find

$$\mathcal{D}H^1(k, Z_{\gamma_a}) = H^1(k, Z_{\gamma_a})$$

If $a \neq 2, -2$ then $\gamma_a = \gamma$ defines an imaginary quadratic extension $E/\mathbb{Q}$ this defines the torus $R_{E/\mathbb{Q}}(G_m)$ and this contains the norm one torus $T/\mathbb{Q} \subset R_{E/\mathbb{Q}}(G_m)/\mathbb{Q}$. Then $Z_\gamma = T$ and it is well known that

$$H^1(\mathbb{Q}, Z_\gamma) = H^1(\mathbb{Q}, T) = \mathbb{Q}^\times / N_{E/\mathbb{Q}}(E^\times)$$

We look at the retriction to the places and get

$$H^1(\mathbb{Q}, T) \xrightarrow{i} \bigoplus H^1(\mathbb{Q}_v, T) \xrightarrow{\sim} H^1(\mathbb{R}, T) \oplus \bigoplus H^1(\mathbb{Q}_\ell, T)$$

The groups $H^1(\mathbb{Q}_v, T) = \{\pm 1\}$ if $E/\mathbb{Q}$ does not split at v and trivial otherwise. Class field theory implies that i is injective and the image is the subgroup of those elements where the product of the entries is one. Hence we conclude that the map followed the by restriction to the finite primes

$$H^1(\mathbb{Q}, T) \xrightarrow{\sim} \bigoplus H^1(\mathbb{Q}_\ell, T) \qquad (stable)$$

is a bijection.

We rewrite the summation of the elliptic terms in the trace formula

$$\sum_{\gamma \in G(\mathbb{Q})_{\mathrm{ell}}/\sim} = \sum_{\mathbb{A}^1(\mathbb{Q})_{\mathrm{ell}}} \sum_{\gamma \in C_a(\mathbb{Q})/\sim}.$$

Now we observe that the first two factors in any of the summands only depends on the class C_a then this implies that we can pull the inner summation over these two factors and then (*stable*) allows us to interchange the inner summation and the product over the finite primes. Hence we get for the sum over the regular elliptic terms in the stable topological trace formula

$$\sum_{\mathbb{A}^1(\mathbb{Q})_{\mathrm{ell}}} \mathrm{tr}(\gamma_a|M)\mathrm{vol}(Z_{\gamma_a}(\mathbb{Q})\backslash Z_{\gamma_a}(\mathbb{A}_f))\prod_\ell \int_{C_a(\mathbb{Q}_\ell)} h_\ell(x_\ell)dx_\ell.$$

For a prime ℓ where γ_a is ℓ-elliptic the local integral $\int_{C_a(\mathbb{Q}_\ell)} h_\ell(x_\ell)dx_\ell$ .at a prime ℓ consists of two terms. This is not so good if we want to form the infinite product over all primes ℓ But for almost all primes ℓ the function h_ℓ is the characteristic function of the maximal compact subgroup $\mathrm{Sl}_2(\mathbb{Z}_\ell)$ and $\gamma_a \in \mathrm{Sl}_2(\mathbb{Z}_\ell)$. The element a defines a discriminant $\Delta(a) = a^2-4$, if this number is an ℓ adic unit, then the element γ_a is still regular if we reduce it $\mod \ell$, its eigenvalues are different $\mod \ell$. Then we will see-and this is not really difficult -that $\int_{Z_{\gamma_a}(\mathbb{Q}_\ell)\backslash G(\mathbb{Q}_\ell)} h_\ell(\bar{x}_\ell\gamma_a\bar{x}_\ell)d\bar{x}_\ell = 1$. But -we still assume that γ_a does not split at ℓ

$$\int_{C_a(\mathbb{Q}_\ell)} h_\ell(x_\ell)dx_\ell = \int_{Z_{\gamma_a}(\mathbb{Q}_\ell)\backslash G(\mathbb{Q}_\ell)} h_\ell(\bar{x}_\ell^{-1}\gamma_a\bar{x}_\ell)d\bar{x}_\ell + \int_{Z_{\gamma'_a}(\mathbb{Q}_\ell)\backslash G(\mathbb{Q}_\ell)} h_\ell(\bar{x}_\ell^{-1}\gamma'_a\bar{x}_\ell)d\bar{x}_\ell$$

where γ'_a is a representative of the other $G(\mathbb{Q}_\ell)$ conjugacy class in $C_a(\mathbb{Q}_\ell)$.

We will see later that the second sumand is zero under the given assumptions.

We make the following remark: If we have an ℓ elliptic element $x \in G(\mathbb{Q}_\ell)$ then the conjugacy class $C_x(\mathbb{Q}_\ell)$ decomposes into two $G(\mathbb{Q}_\ell)$ conjugacy classes. In general we do not know which class is which, we can not distinguish between the two classes. This does not matter as long as we take the sum of the two orbital integrals. But in the next section we have to take the difference and then this problem becomes more serious.

But if we have choosen the structure of an scheme $\mathcal{G}/\mathrm{Spec}(\mathbb{Z})$ and if $x \in \mathcal{G}(\mathbb{Z}_\ell)$, if the group scheme structure is semi simple at ℓ and if x is still regular after reduction $\mod \ell$, then I just said, that for the other class

$$\{y^{-1}x'y | y \in G(\mathbb{Q}_\ell\} \cap \mathcal{G}(\mathbb{Z}_\ell) = \emptyset$$

Hence we see that the choice of an integral structure gives us a way to make a distinction between the two $G(\mathbb{Q}_\ell)$-conjugacy classes at those primes ℓ where the integral structures are semi-simple and the conjugacy class is regular $\mod \ell$.

We can even do better and we come back to this later.

2.3.11 The group $G^*/\mathbb{Q}$

If we have a (regular) element $\gamma \in G^*(\mathbb{Q})$ and if $Z_\gamma/\mathbb{Q} = T/\mathbb{Q}$ is its centralizer then the situation is different. If $C_\gamma/\mathbb{Q}$ is the conjugacy class of γ then the orbits of $G^*(\mathbb{Q})$ for a principal homogenous space under

$$\mathcal{D}^*H^1(\mathbb{Q},T) = \{\xi \in H^1(\mathbb{Q},T) | \xi \mapsto 1 \text{ in } H^1(\mathbb{Q},G^*)\}.$$

Now we have $H^1(\mathbb{Q},G^*) = H^1(\mathbb{R},G^*) = \{\pm 1\}$ we get

$$\mathcal{D}^*H^1(\mathbb{Q},T) = \{\xi | \xi_\infty = 1\}.$$

This implies that the injective homomorphism

$$\mathcal{D}^*H^1(\mathbb{Q},T) \to \bigoplus H^1(\mathbb{Q}_\ell,T)$$

is not surjective, it has a kokernel of order 2. We introduce the tautological character

$$\kappa_\ell^T : H^1(\mathbb{Q}_\ell,T) \to \{\pm 1\},$$

it is trivial of T splits over $\mathbb{Q}_\ell$ and an isomorphism if not. We put $\kappa^T = \prod_\ell \kappa_\ell^T$.

Now let us choose in any conjugacy class C_a^*, for which $C_a^*(\mathbb{Q}) \neq \emptyset$, an element $\gamma \in C_a^*(\mathbb{Q})$. Then our of regular elliptic terms becomes

$$\sum_{\gamma' \in G(\mathbb{Q})_{\mathrm{ell,reg}}/\sim} = \sum_{\mathbb{A}^1(\mathbb{Q})_{\mathrm{ell,reg}}, C_a^*(\mathbb{Q})\neq\emptyset} \quad \sum_{\gamma' \in C_a(\mathbb{Q})/\sim}$$

We look at the summation over the $\gamma'+\xi$. We can view ξ as a collection of local classes $\underline{\xi} = (\dots,\xi_\ell,\dots)$ where almost all entries are zero and where $\kappa^T(\underline{\xi}) = 1$. Then

$$\sum_{\gamma' \in G(\mathbb{Q})_{\mathrm{ell,reg}}/\sim} =$$
$$\sum_{\mathbb{A}^1(\mathbb{Q})_{\mathrm{ell,reg}}, C_a^*(\mathbb{Q})\neq\emptyset} \mathrm{tr}(\gamma|\,\mathcal{M})\mathrm{vol}(Z_{\gamma_a}(\mathbb{Q})\backslash Z_{\gamma_a}(\mathbb{A}_f))\times$$
$$\sum_{\gamma+\underline{\xi} \in C_a(\mathbb{Q})/\sim, \kappa^T(\underline{\xi})=1} \prod_\ell \int_{Z_{\gamma+\xi_\ell}\backslash G(\mathbb{Q}_\ell)} h_\ell(g_\ell^{-1}(\gamma+\xi_\ell)g_\ell)d\bar{g}_\ell$$

The inner sum can now be written as

$$\sum_{\gamma+\xi \in C_a(\mathbb{Q})/\sim} \frac{(1+\kappa^T(\xi))}{2} \int_{Z_{\gamma+\xi_\ell}\backslash G(\mathbb{Q}_\ell)} h_\ell(g_\ell^{(-1)}(\gamma+\xi_\ell)g_\ell d\bar{g}_\ell.$$

The character κ_ℓ^T induces a function also called $\kappa_\ell^{\gamma_\ell}$:

$$\kappa_\ell : C_a(\mathbb{Q}_\ell) \to \{\pm 1\}$$

which is defined by

$$\gamma + \xi_\ell \mapsto \kappa_\ell^T(\xi_\ell),$$

this depends on our choice of $\gamma \in C_a(\mathbb{Q})$.

We get for the regular elliptic terms in the topological trace formula for $G^*/\mathbb{Q}$ the following expression

$$\frac{1}{2} \sum_{a \in \mathbb{A}^1(\mathbb{Q})_{\mathrm{ell,reg}}} \mathrm{tr}(\gamma|\,\mathcal{M})\mathrm{vol}(Z_{\gamma_a}(\mathbb{Q})\backslash Z_{\gamma_a}(\mathbb{A}_f))\times$$
$$(\prod_\ell \int_{C_a(\mathbb{Q}_\ell)} h_\ell(x_\ell)dx_\ell + \prod_\ell \int_{C_a(\mathbb{Q}_\ell)} \kappa_\ell^{\gamma_\ell}(\xi_\ell)h_\ell(x_\ell)dx_\ell),$$

here we set a summand equal to zero if $C_a(\mathbb{Q}) = \emptyset$. This is justified because then the Hasse principle implies that there is an ℓ for which $C_a(\mathbb{Q}_\ell) = \emptyset$.

The first integral is stable the second one is unstable. We noticed already that for almost all primes ℓ we have $\int_{C_a(\mathbb{Q}_\ell)} h_\ell(x_\ell)dx_\ell = \int_{C_a(\mathbb{Q}_\ell)} \kappa_\ell(\xi_\ell)h_\ell(x_\ell)dx_\ell = 1$, hence the infinite product makes sense.

2.3.12 The fundamental Lemma

The imaginary quadratic extensions $\mathbb{Q} \subset E \subset \bar{\mathbb{Q}} \subset \mathbb{C}$ form a set, for any of them we find a positive squarefree integer $d > 0$ and an element $\alpha \in E$, which satisfies $\alpha^2 = -d$ and viewed as an element in $\mathbb{C}$ we have $\alpha = i\sqrt{d}$, where the square root is positive. For any such extension we choose an embedding $j_E : E \to D$, i.e. an element $j(\alpha) = \alpha'$, provided this is possible.

Then this defines a torus $T_\alpha/\mathbb{Q} \subset G^*/\mathbb{Q}$ whose $\mathbb{Q}$ valued points are

$$T_\alpha(\mathbb{Q}) = \{x + y\alpha' | x^2 + y^2 d = 1\}.$$

We call the choice of an imaginary quadratic extension and such an α' or the choice of j_E an **endoscopic datum**. *Two such data are considered to be equivalent, if $\alpha_1 = \alpha_2$ this means we ignore the actual embedding.*

We can choose representatives of the equivalence classes of endoscopic data, i.e. we choose an α' for any α.

We pick an $a \in \mathbb{Q}$ and assume that $C_a(\mathbb{Q}) \neq \emptyset$. We want to desribe a procedure to choose representatives in $G^*(\mathbb{Q})$ conjugacy classes. The element a defines an imaginary quadratic extension $\mathbb{Q}(\sqrt{a^2-4})$ and the set $C_a(\mathbb{Q})$ has a non empty intersection with exactly one of the $T_\alpha(\mathbb{Q})$ and this intersection consists of two elements which are invers to each other. We modify the choice of the element $\gamma \in C_a(\mathbb{Q})$: Instead of choosing one element in the $G^*(\mathbb{Q})$ conjugacy class $C_a(\mathbb{Q})$ we choose 2 elements, namely the two elements in the intersection with $T_\alpha(\mathbb{Q})$. We get our sum twice

$$\sum_{a \in \mathbb{A}^1(\mathbb{Q})_{\text{ell,reg}}} = \frac{1}{2} \sum_{\alpha} \sum_{\gamma \in T_\alpha(\mathbb{Q}) : \gamma \neq \pm 1} .$$

and look at the sum

$$\frac{1}{4}\text{vol}(T_\alpha(\mathbb{Q}) \backslash T_\alpha(\mathbb{A}_f)) \sum_{\gamma \in T_\alpha(\mathbb{Q}) : \gamma \neq \pm 1} \text{tr}(\gamma | \mathcal{M}) \prod_\ell \int_{C_a(\mathbb{Q}_\ell)} \kappa_\ell^{(\gamma)}(\xi_\ell) h_\ell(x_\ell) dx_\ell,$$

Our local coefficient system is now the module $\mathcal{M} = \mathcal{M}_k$ of homogenous polynomials in two variables of degree 2 with coeficients in $\mathbb{Q}$ (or some finite extension).

We consider an individual term of the sum, we write $\gamma = x + y\alpha'$ then it is given by

$$\frac{(x + y\alpha)^{k+1} - (x - y\alpha)^{k+1}}{(x + y\alpha) - (x - y\alpha)} \cdot \prod_\ell \int_{C_a(\mathbb{Q}_\ell)} \kappa_\ell^{(\gamma)}(\xi_\ell) h_\ell(x_\ell) dx_\ell$$

The denominator is $2y\alpha = 2\alpha \text{sgn}(y)|y|_\infty$, we have $y \in \mathbb{Q}$ and $|y|_\infty = \prod_\ell |y|_\ell^{-1}$. Therefore our last expresion is equal to

$$\frac{(x + y\alpha)^{k+1} - (x - y\alpha)^{k+1}}{2\alpha} \text{sgn}(y) \prod_\ell |y|_\ell \int_{C_a(\mathbb{Q}_\ell)} \kappa_\ell^{(\gamma)}(\xi_\ell) h_\ell(x_\ell) dx_\ell$$

We rewrite the sign factor: It is minus one iff y is not a norm at the infinite place, we know from class field theory that $\mathrm{sgn}(y) = \prod_\ell \varepsilon_\ell(y)$ where $\varepsilon_\ell(y) = -1$ iff y is not in the norm group of $(E \otimes \mathbb{Q}_\ell)^\times$.

For our given torus T_α and a prime ℓ we analyze the function

$$\gamma = x + y\alpha' \to |y|_\ell \varepsilon_\ell(y) \cdot \int_{C_a(\mathbb{Q}_\ell)} \kappa_\ell^{T_\alpha}(\xi_\ell) h_\ell(x_\ell) dx_\ell = {h_\ell}^{T_\alpha}(\gamma)$$

on $T_\alpha(\mathbb{Q}_\ell)$.

At the places $\ell \notin \Sigma$ we have $G^*(\mathbb{Q}_\ell \xrightarrow{\sim} Sl_2(\mathbb{Q}_\ell)$ and $K_\ell = \mathrm{Sl}_2(\mathbb{Z}_\ell)$. Let T_0 be the standard diagonal maximal torus and

$$\mathrm{Hom}_{\mathrm{unv}}(T_0(\mathbb{Q}_\ell), \mathbb{C}^*) = \Lambda.$$

Then we know that $\mathcal{H}_\ell = \mathbb{C}_c(G(\mathbb{Q}_\ell)//K_\ell)$ is commutative and I explained in the beginning that

$$\mathrm{Specmax}(\mathcal{H}_\ell) = \Lambda/W$$

where W is the Weylgroup consisting of two elements. The Hecke-algebra of the torus is $\mathcal{C}_c(T_0(\mathbb{Q}_\ell)/T_o(\mathbb{Z}_\ell)$ and its spectrum is Λ. The natural map $\Lambda \to \Lambda/W$ is induced by the Satake map

$$Sat : \mathcal{H}_\ell = \mathcal{C}_c(\mathrm{Sl}_2(\mathbb{Q}_\ell//\mathrm{Sl}_2(\mathbb{Z}_\ell) \to \mathcal{H}_\ell^{T_0} = \mathcal{C}_c(T_0(\mathbb{Q}_\ell)/T_0(\mathbb{Z}_\ell)).$$

(This should be rewritten.) Inside Λ we have the specific element

$$\chi_E : \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mapsto (-1)^{\mathrm{ord}_p(t)}$$

this is a character of order 2 and it is invariant under W. The role of this character will be explained in 2.5.1.

Now we can state the

Fundamental lemma

a) For all primes the function $\gamma \to {h_\ell}^{T_\alpha}(\gamma)$ is smooth.

b) For $\ell \notin \Sigma$ we get

(i) At the places ℓ where our torus splits we get

$$\hat{{h_\ell}^{T_\alpha}}(\lambda) = \hat{h}_\ell(\lambda)$$

i.e. ${h_\ell}^{T_\alpha}$ is the image of h_ℓ under the Satake homomorphism.

(ii) At the places where the torus is non-split and unramified ${h_\ell}^{T_\alpha}$ is constant and its value is $\hat{h}_\ell(\chi_E)$.

c) If $\ell \notin \Sigma$ and if $T_\alpha/\mathbb{Q}_\ell$ is ramified then ${h_\ell}^{T_\alpha}(\gamma) \equiv 0$.

The assertion b) (i) is the clear from the Satake isomorphism. It is the assertion b) ii) which is more difficult to prove and causes serious headaches in more general cases.

We now suppress the subscript $_\alpha$ at our torus and the elements $x + y\alpha'$ are denoted by γ. On our torus we have a character $\delta : T \times_{\mathbb{Q}} \bar{\mathbb{Q}} \to G_m \times_{\mathbb{Q}} \bar{\mathbb{Q}}$

which is defined by $x + y\alpha' \mapsto x + y\alpha$. The powers δ^n provide one dimensional representations $\mathcal{M}_n$ of $T \times \bar{\mathbb{Q}}$.

If $\ell \notin \Sigma$ and if h_ℓ is the characteristic function of K_ℓ then $\hat{h}_\ell(\lambda) = 1$, hence we can define ${h_f}^T$ to be the product of the local functions, this is now a Hecke operator on the torus. (See remark below)). We rewrite the unstable term in the trace formula coming from our torus as

$$1/4 \cdot \sum_{T/\mathbb{Q}} \mathrm{vol}(T(\mathbb{Q})\backslash T(\mathbb{A}_f)) \sum_{\gamma \in T(\mathbb{Q})\backslash\{\pm 1\}} (\delta^{k+1}(\gamma) - \delta^{k+1}(\gamma^{-1})) \cdot {h_f}^T(\gamma).$$

Let me just mention that it is condition c) that cuts the summation over the tori down to a finite sum.

If we replace γ by γ^{-1} then the sign of the $\kappa_\ell^{(\gamma)}$ orbital integral changes if and only if γ is not $G^*(\mathbb{Q}_\ell)$ conjugate to γ^{-1}, that happens at an even number of places. (Proof: We find an element $u \in D(\mathbb{Q})^\times$, which normalizes $T/\mathbb{Q}$ and induces the non trivial Galois automorphism. Then $u^2 = a \in \mathbb{Q}^\times$ and the reduced norm of u is $-a$. Since our quaternion algebra is definite we have $a < 0$. Hence $-a$ is a norm at the infinite place. Now it is clear that γ is not $G^*(\mathbb{Q}_\ell)$ conjugate to γ^{-1} iff $-a$ is not a norm from $E \otimes \mathbb{Q}_\ell$ and this happens at an even number of places, hence at an even number of finite places.)

But also the $\epsilon_\ell(y)$ changes and this happens at an odd number of finite places, because we have a sign change at the infinite place. Hence ${h_f}^T(\gamma^{-1}) = -{h_f}^T(\gamma)$. Especially we se that ${h_f}^T(\gamma^{-1})(\pm 1) = 0$ and therefore this last sum is

$$1/2 \cdot \sum_{T/\mathbb{Q}} \mathrm{vol}(T(\mathbb{Q})\backslash T(\mathbb{A}_f)) \sum_{\gamma \in T(\mathbb{Q})} \delta^{k+1}(\gamma) \cdot {h_f}^T(\gamma).$$

The unstable term may now be computed backwards. We attach a locally symmetric space to the torus $T/\mathbb{Q}$, namely

$$\mathcal{S}^T_{K_f} = T(\mathbb{Q}) \setminus (T(\mathbb{R})/K^T_\infty) \times T(\mathbb{A}_f)/K^T_f.$$

This topological space is actually a finite set, but it carries the structure of an abelian group. (We suppress a supscript K^T_f and write K_f instead). The one-dimensional module $\mathcal{M}^T = \mathcal{M}^T_{k+1}$ which provides sheaves $\tilde{\mathcal{M}}^T$ on $\mathcal{S}^T_{K_f}$ and now the sum over all $T/\mathbb{Q}$ of the instable parts of the trace is the $\mathcal{O}$-expansion of

$$\frac{1}{2} \sum_T \mathrm{tr}({h_f}^T | H^\bullet(\mathcal{S}^T_{K_f}, \mathcal{M}^T)).$$

i.e. it is the sum of traces of "new" Hecke operators on "new" spaces. Note that this manipulation can not be performed for the stable term, because we can not replace the summation $\sum_{\gamma \in T(\mathbb{Q}), \gamma \neq \pm 1}$ by $\sum_{\gamma \in T(\mathbb{Q})}$.

Remark Notice that we have used the fundamental lemma at this point, because we need to know that we can view ${h_f}^T$ as an element in the Hecke algebra. Hence we must know that for all ℓ the function h^T_ℓ is smooth on $T(\mathbb{Q}_\ell)$ and that for almost all ℓ this function is simply the characteristic function of $T(\mathbb{Z}_\ell)$. I come back to this point when I discuss question D).

The cohomology $H^\bullet(\mathcal{S}^T_{K_f}, \mathcal{M}^T))$ is now a module for the Hecke algebra $\mathcal{C}_c(T(\mathbb{A}_f)//K_f)$, but since $T(\mathbb{A}_f)//K_f$ is an abelian group, this is also a module under this group. We want to decompose this module into irreducibles, this is a spacial case of our general problem. In this particular case this is relatively easy. We need the notion of an *algebraic Hecke character* , this is a homomorphism

$$\varphi : T(\mathbb{Q})\backslash T(\mathbb{A})/K_f \to \mathbb{C}^\times$$

whose restriction to the component at infinity $T(\mathbb{R}) = \{z|z\bar{z} = 1\}$ is algebraic, this means that $\varphi_\infty : z \mapsto z^n$ for some n(Actually in this special situation it is clear that it is algebraic provided it is continous). The number n is called the type of the Hecke character. The Hecke characters of a fixed type on $T(\mathbb{Q})\backslash T(\mathbb{A})/K_f$ form a principal homogenous space under the group of characters of the finite abelian group $T(\mathbb{Q})\backslash T(\mathbb{A})/T(\mathbb{R})K_f$, hence their number is equal to the order of that group. For each such Hecke character φ let φ_f be its restriction to $T(\mathbb{A}_f)$, this retriction determines the component at infinity. So we may say that $\varphi_f : T(\mathbb{A}_f)/K_f \to \mathbb{C}^\times$ is a Hecke character of type n if it is the finite component of a (then unique) Hecke character φ of type n.

Now it is not difficult to write down the decomposition of the cohomology:

$$H^\bullet(\mathcal{S}^T_{K_f}, \mathcal{M}^T)) = \bigoplus_{\varphi_f:\text{type}\varphi=k+1} H^\bullet(\mathcal{S}^T_{K_f}, \mathcal{M}^T))(\varphi_f)$$

and this isotypical constituents come with multiplicity one. (See [Ha-GL2]). We write down the χ-expansion for these traces and find

$$\sum_T \sum_{\varphi_f:\text{type}\varphi_f=k+1} \text{tr}({h_f}^T|\varphi_f)$$

Now we write h_f and φ_f as products of the factors inside Σ and the factors outside Σ, i.e $h_f = h_{f,\Sigma} \times {h_f}^{(\Sigma)}$, $\varphi_f = \varphi_{f,\Sigma} \times \varphi_f^{(\Sigma)}$. Recall that we have choosen a group scheme structure (flat of finite type) $\mathcal{G} \to \text{Spec}(\mathbb{Z})$, which is semi simple outside Σ. We adapted the choice of the tori in such a way that for all $\ell \notin \Sigma$ the maximal compact subgroup $T(\mathbb{Z}_\ell) \subset \mathcal{G}(\mathbb{Z}_\ell)$. Now we attach to $\varphi_f^{(\Sigma)}$ a representation

$$\pi_T^{(\Sigma)}(\varphi_f^{(\Sigma)}) = \bigotimes_{\ell\notin\Sigma} \pi_{T,\ell}(\varphi_\ell)$$

of $G(\mathbb{A}_f^{(\Sigma)})$ which is defined as follows:

$$\pi_{T,\ell}(\varphi_\ell) = \begin{cases} \text{Ind}_{B(\mathbb{Q}_\ell)}^{G(\mathbb{Q}_\ell)}\varphi_\ell & \text{if } E \text{ splits at } \ell \\ \pi^+_{E_\ell} & \text{if } E_\ell = E \otimes \mathbb{Q}_\ell \text{ is unramified non split} \\ \text{the null vector space} & \text{else} \end{cases}$$

The fundamental lemma says that for all $\ell \notin \Sigma$ and for all h_ℓ in the unramified Hecke algebra we have

$$\text{tr}(h_\ell^T|\varphi_\ell) = \text{tr}(h_\ell|\pi_{T,\ell}(\varphi_\ell)).$$

Then

$$\text{tr}(h_f^T|\varphi_f) = \text{tr}(h_{f,\Sigma}^T|\varphi_\Sigma)\text{tr}(h_f^{T^{(\Sigma)}}|\varphi_f^{(\Sigma)}) = \text{tr}(T_{f,\Sigma}|\varphi_\Sigma)\text{tr}({h_f}^{(\Sigma)}|\pi_T(\varphi_f^{(\Sigma)})).$$

So we are finally ready to write down the stabilized trace formula on the cohomology of the quaternion group $G^*/\mathbb{Q}$:

$$\begin{gathered}\mathrm{tr}_{\mathrm{ell}}(h_f|H^\bullet(\mathcal{S}^{G^*},\mathcal{M})) = \textstyle\sum_{\gamma=\pm1}\tau(G^*)h_f(\gamma) + \\ 1/2\cdot\textstyle\sum_{T/\mathbb{Q}}\mathrm{vol}(T(\mathbb{Q})\backslash T(\mathbb{A}_f))\sum_{\gamma\in T(\mathbb{Q})\setminus\{\pm1\}}\mathrm{tr}(\gamma|\mathcal{M})\prod_\ell\int_{C_a(\mathbb{Q}_\ell)}h_\ell(x_\ell)dx_\ell + \\ 1/2\sum_T\sum_{\varphi_f:\mathrm{type}\varphi_f=k+1}\mathrm{tr}(h^T_{f,\Sigma}|\varphi_{f,\Sigma})\mathrm{tr}(h_f{}^{(\Sigma)}|\pi_T(\varphi^{(\Sigma)})).\end{gathered}$$

k We come back to our comparison. We choose a Hecke operator h_f on $G = \mathrm{Sl}_2/\mathbb{Q}$ and another one h_f^* s.t. the local stable orbital integrals match. (This makes sense!!). Since Σ contains at least one finite place we see that the contribution from infinity are zero. The elliptic stable contributions cancel and the difference of traces is

$$\begin{gathered}\mathrm{tr}(h_f|H^\bullet(\mathcal{S}^G,\mathcal{M})) + 2\cdot\mathrm{tr}(h_f^*|H^\bullet(\mathcal{S}^{G^*},\mathcal{M})) = \\ 1/2\sum_T\sum_{\varphi_f:\mathrm{type}\varphi_f=k+1}\mathrm{tr}(h^T_{f,\Sigma}|\varphi_{f,\Sigma})\mathrm{tr}(h_f^{\Sigma)}|\pi_T(\varphi^{(\Sigma)})).\end{gathered}$$

How can we interprete this formula?

2.3.13 Back to the comparison

We assume that the places where our quaternion algebra is ramified are inside Σ. Hence we can identify the parts of the two Hecke algebras which are outside Σ, i.e. we have

$$\mathcal{H}^{(\Sigma)} = \mathcal{H}^{*(\Sigma)}$$

and we can look at $H^1(\mathcal{S}^G,\mathcal{M})) - 2H^0(\mathcal{S}^{G^*},\mathcal{M})$ as a virtual $\mathcal{H}^{(\Sigma)}$-module. This module contains copies of $\pi_T(\varphi^{(\Sigma)})$ and it it not so hard to check that two such $\pi_T(\varphi^{(\Sigma)}), \pi_{T_1}(\varphi_1^{\Sigma})$ cannot be isomorphic unless T, T_1 are conjugate and $\varphi_f = \varphi_{1f}$. We stick to a particular pair T,φ. Then we will find some modules for the full Hecke algebras $\mathcal{H},\mathcal{H}^*$ occurring in one of the two modules upstairs and which are of the form

$$\pi_f = \prod_{p\in\Sigma}\pi_p\times\pi_T(\varphi^{(\Sigma)}) \text{ or } \pi_f^* = \prod_{p\in\Sigma}\pi_p^*\times\pi_T(\varphi^{(\Sigma)})$$

The idea is that their are not to many choices for the components π_p, π_p^* , they have to be taken from the so called L-packets $\Pi(\varphi_p), \Pi(\varphi_p^*)$. If we write $\pi_\Sigma \in \Pi(\varphi_\Sigma)$ then we mean that this is a product of local representations $\pi_p \in \Pi(\varphi_p)$. We denote the multiplicities of $\pi_f = \pi_\Sigma\times\pi_T(\varphi^{(\Sigma)})$ resp. $\pi^*{}_\Sigma\times\pi_T(\varphi^{(\Sigma)})$ by $m(\pi_\Sigma)(\varphi_f)$ resp. $m^*(\pi_\Sigma^*)(\varphi_f)$ and get

$$\sum_{\pi_\Sigma}m(\pi_\Sigma)\mathrm{tr}(h_{f\Sigma}|\pi_\Sigma) - \sum_{\pi_\Sigma^*}m^*(\pi_\Sigma^*)\mathrm{tr}(h^*_{f\Sigma}|\pi^*_\Sigma) = \mathrm{tr}(h_f^{*T}{}_\Sigma|\varphi_{f\Sigma})$$

Now we see that the trace formula can also be used to give us at least an idea of what happens inside an L-packet. If we assume for instance that $h_{f\Sigma} = \prod_{p\in\Sigma}h_p$ contains a factor h_{p_1} for which the **stable orbital integral** vanishes identically. Then we are allowed to choose the corresponding factor in $h_f{}^*{}_\Sigma$ equal to zero. This implies that the second term and the right hand side vanish. So the first term has to be zero. This suggests that the multiplicity $m(\pi_\Sigma)$ should be constant on the L-packet and we should have (under our assumption on this local factor)

$$\sum_{\pi_{p_1}\in\Pi(\varphi_{p_1})} \operatorname{tr}(h_{p_1}|\pi_{p_1}) = 0.$$

This is a so called character relation for the characters of the members of the L-packet. The constancy of the multiplicities is nothing else than the stability of the trace formula for $G/\mathbb{Q}$.

On the other hand it is clear that the multiplicities $m^*(\pi^*_\Sigma)$ cannot be constant on the L-packet. We can try to do the same trick as above. Then we can forget the first term and we find a way out if we believe the following: We can define a sign $\epsilon(\pi_{p_1})$ by

$$\epsilon(\pi^*_{p_1})\operatorname{tr}(h^*_{p_1}|\pi^*_{p_1}) = \frac{1}{|\Pi(\varphi_{p_1})|}\operatorname{tr}({h^*_{p_1}}^T|\varphi_{p_1}),$$

whenever the stable integral for $h^*_{p_1}$ vanishes. Now we can live comfortably with the multiplicity formulae

$$m(\pi_\Sigma) = 1 \text{ and } m^*(\pi^*_\Sigma) = 1/2(1+\epsilon(\pi^*_\Sigma)).$$

2.3.14 Some questions

A)It should not be so difficult to prove a topological twisted trace formula in the rank one case a la Bewersdorff. We could for instance start from a quadratic extension $E/\mathbb{Q}$ and study the two groups $G/\mathbb{Q} = \mathrm{Sl}_2/\mathbb{Q}$ and $G^*/\mathbb{Q} = R_{E/\mathbb{Q}}(\mathrm{Sl}_2/E)$. Now the group G^* admits an automorphism σ so that G becomes the group of fixed points. This automorphism σ extends to the space

$$\mathcal{S}^* = G^*(\mathbb{Q})\backslash X^* \times G^*(\mathbb{A})/K_f$$

(if we are careful enough in choosing K_f.) If we also choose a module $\mathcal{M}^*$ which comes from the group $G/\mathbb{Q}$ then we have Hecke operators

$$h^* \times \sigma : H^\bullet(\mathcal{S}^*, \mathcal{M}^*) \to H^\bullet(\mathcal{S}^*, \mathcal{M}^*)$$

and we would like to have a formula for the trace.

B). After such a formula has been obtained, it would be interesting to make a comparison between the twisted trace formula for G^* and the untwisted trace formula for G.

C). I come back to the discussion of the Eichler-Shimura relations. If we start start from the group $G/\mathbb{Q} = \mathrm{Sl}_2/\mathbb{Q}$ and if we choose a full congruence subgroup $K_f = K_N =$ full congruence subgroup $\mod N$ then this does not correspond to a Shimura variety. The set

$$\mathcal{S}^G_N = G(\mathbb{Q})\backslash X \times G(\mathbb{A})/K_N = \Gamma(N)\backslash H$$

is not the set of complex points of a moduli space over $\mathbb{Q}$ anymore. It classifies elliptic curves together with a basis $< e_1, e_2 >$ of N-division points where the value of the Weil pairing $e_N(e_1,e_2)=\zeta_N$ is a fixed N-th root of unity. Hence the curve is defined over $\mathbb{Q}[\zeta_N]$, let us write $\mathcal{S}_N/\mathbb{Q}[\zeta_N]$ for it. Then the cohomology

$$H^\bullet(\mathcal{S}_N\times_{\mathbb{Q}[\zeta_N]}\bar{\mathbb{Q}},\mathcal{M})$$

will be a Hecke$\times$Gal$(\bar{\mathbb{Q}}/\mathbb{Q}[\zeta_N])$ module. The question is whether the trace formula allows us for instance to decide the following question:

In our discussion above we found the modules $\pi_\Sigma\times\pi_T(\varphi^{(\Sigma)})$ with multiplicity one in the cohomology (now Σ is the set of primes dividing N); we have a copy

$$H^\bullet(\mathcal{S}_N\times_{\mathbb{Q}[\zeta_N]}\bar{\mathbb{Q}},\mathcal{M})(\pi_\Sigma\times\pi_T(\varphi^{(\Sigma)}))$$

in the cohomology. How does the Galois group act on this space? It is quite clear that the action will be given by the λ-adic character $\varphi_\lambda : Gal(\bar{\mathbb{Q}}/\mathbb{Q}[\zeta_N])\to\mathbb{Q}_\lambda^\times$ or its inverse (maybe after a Tate twist). Which one is it. The answer will be given by our $\epsilon(\pi_\Sigma)$. Does the trace formula give it???

D) I want to come back to the remark above. Let us assume that we have some way to may sense out of the equality

$$\mathrm{tr}(h_f|H^\bullet(\mathcal{S}^G,\mathcal{M}))+2\cdot\mathrm{tr}(h_f^*|H^\bullet(\mathcal{S}^{G^*},\mathcal{M}))=$$

$$\sum_T\sum_{\varphi_f:\mathrm{type}\varphi_f=k+1}\mathrm{tr}({h_f}^T|\varphi_f)=\frac{1}{2}\sum_T\mathrm{tr}({h_f}^T|H^\bullet(\mathcal{S}^T,\mathcal{M}^T))$$

We can choose two disjoint finite sets of primes A,B outside Σ and a pair of adapted $h_{f,A},h_{f,A}^*$ Hecke operators such that we have exactly one $\pi_f^{(C)}$ such that $\mathrm{tr}(h_{f,A}|\pi_{f,A})=1$ and that it is zero for all other $(\pi'_f)^{(C)}$. We do the same for the other side with a representation π_f^* and a pair $h_{f,B},h_{f,B}^*$.

Then these $\pi_f^{(C)},(\pi_f^*)^{(C)}$ can be extended to representations π_f,π_f^* in possibly different ways.

Now we put $C=\Sigma\cup A\cup B$ and consider operators

$$h_f=h_{f,\Sigma}\times h_{f,A}\times h_{f,B}\times h_f^{(C)},h_f^*=h_{f,\Sigma}^*\times h_{f,A}^*\times h_{f,B}^*\times {h^*}_f^{(C)}$$

In the next formula we sum over the extensions π_f of $\pi_f^{(C)}$ and π_f^* of $(\pi_f^*)^{(C)}$. We find

$$\sum m(\pi_f)\mathrm{tr}(h_{f,\Sigma}|\pi_\Sigma)\mathrm{tr}(h_{f,B}|\pi_B)\mathrm{tr}(h_f^{(C)}|\pi_f^{(C)})-$$

$$\sum 2m(\pi_f^*)\mathrm{tr}(h_{f,\Sigma}^*|\pi_{f,\Sigma}^*)\mathrm{tr}(h_{f,A}^*|\pi_A^*)\mathrm{tr}({h^*}^{(C)}|{\pi^*}_f^{(C)})=$$

$$\sum_T\sum_{\varphi_f:\mathrm{type}\varphi_f=k+1}\mathrm{tr}({h_f}^T|\varphi_f)$$

Now the only interesting case is that $h_{f,C}^T\neq 0$. Now we look at places ℓ outside C. The representations π_f,π_f^* have Satake parameters α_ℓ,β_ℓ (resp) $\alpha_\ell^*,\beta_\ell^*$. We choose $h^{(m,C)},h^{(*,m,C)}$ to be the product of the identity outside $\{\ell\}$ and such that their ℓ-th component has eigenvalue $\alpha_\ell^m+\beta_\ell^m$ and $(\alpha_\ell^*)^m+(\beta_\ell^*)^m$

respectively. Then they are adapted. Now we write down our formula for these operators: Let us abbreviate the factors in front by H, H^* then

$$H(\alpha_\ell^m+\beta_\ell^m)-2H^*((\alpha_\ell^*)^m+(\beta_\ell^*)^m)=\sum_T \sum_{\varphi_f:\mathrm{type}\varphi_f=k+1} \mathrm{tr}({h_{f,C}}^T|\varphi_{f,C})\mathrm{tr}({h_f}^{(C),T}|\varphi_f^{(C)})$$

The term on the right is of the form

$$\sum_T \sum_{\varphi_f:\mathrm{type}\varphi_f=k+1} G(\varphi)\mathrm{tr}(h_\ell^T|\varphi_\ell)$$

Let us look at an individual summand. If the torus T splits at ℓ then we know from the easy part of the fundamental lemma that

$$\mathrm{tr}(h_\ell^T|\varphi_\ell)=\varphi_\ell(\varpi_\ell^m)+\varphi_\ell((\varpi_\ell')^m),$$

where $\varpi_\ell, \varpi_\ell'$ are the two uniformizing elements.

If ℓ ist inert then $T(\mathbb{Q}_\ell)$ is compact, our character takes value one on it and hence we get

$$\mathrm{tr}(h_\ell^T|\varphi_\ell)=\int_{T(\mathbb{Q}_\ell)} h_\ell^T(t_\ell)dt_\ell$$

where I admit that we do not quite know that the integral makes sense, unless we invoke the difficult part of the fundamental lemma.

I think we are not very far from concluding that there can be only one summand on the right hand side, therefore we have a unique character φ_f such that

$$H(\alpha_\ell^m+\beta_\ell^m)-2H^*((\alpha_\ell^*)^m+(\beta_\ell^*)^m)=\mathrm{tr}({h_{f,C}}^T|\varphi_{f,C})\mathrm{tr}({h_f}^{(C),T}|\varphi_f^{(C)})$$

So we found a formula which is equivalent to the earlier one without using the fundamental lemma, instead we used some weaker and inprecice assumptions concerning our functions h^T. It seems to me plausible that we can derive the fundamental lemma in this case, if we assume the validity of this type of formulae.

2.4 The classical case of the trace formula

Before we discuss the proof of the trace formula in general, I consider a special case. We assume that our group is $G=\mathrm{Gl}_2/\mathbb{Q}$, the level is the standard maximal compact subgroup $K_f=\prod_p \mathrm{Gl}_2(\mathbb{Z}_p)$. In this case we have

$$S_{K_f}^G=G(\mathbb{Q})\backslash G(\mathbb{R})/K_\infty\times G(\mathbb{A}_f)/K_f=\mathrm{Sl}_2(\mathbb{Z})\backslash\mathbb{H}$$

see (Chap.III.1.1.2), we even assume that $N=1$). The symmetric space $G(\mathbb{R})/K_\infty$ has two connected components if we restrict to the subgroup $G^+(\mathbb{R})$ of matrices with positive determinant we get

$$G^+(\mathbb{Q})\backslash G^+(\mathbb{R})/K_\infty\times G(\mathbb{A}_f)/K_f=\mathrm{Sl}_2(\mathbb{Z})\backslash\mathbb{H}.$$

We choose as our coefficient system the module $\mathcal{M}_k[0]=\mathcal{M}_k$ with k even (See Chap. III.1.9). We have the Hecke operator $T_{p^m}^{(1)}$ which is defined by the double coset

$$\underline{t}_{p^m} = K_f(\ldots,1,\ldots\begin{pmatrix} p^m & 0 \\ 0 & 1\end{pmatrix},\ldots,1,\ldots)K_f.$$

We denote the characteristic function of this double coset also by $\underline{t}_{p^m}$. We assume $m>0$.

We want to write down the formula for the trace of t_{p^m} on the cohomology $H^\bullet(S^G_{K_f},\tilde{\mathcal{M}}_k)$. Recall that the definition of the Hecke operator involves the choice of a measure on $G(\mathbb{A}_f)$, we choose the one which gives volume one to K_f.

Remark: This is not the operator which corresponds to the operator T_{p^m} in the classical theory of modular forms, it is its primitive part: We get the classical operator T_{p^m}, if we take the double coset of matrices

$$M_2(p^m) = \{g\in M_2(\mathbb{Z}_p)\mid \det(g)\in p^m\mathbb{Z}_p^\times\}.$$

We recall the computations in Chap.II on page 36. They need some corrections: The operator T_{p^m} in Chap.2 is the $T^{(1)}_{p^m}$ here. The recursion formula on p. 39 has to be corrected.

In Chap. II. 1.2.3 we gave the formula for the Fourier transform of this operator.

Of course we have an easy relation between the to operators

$$T^{(1)}_{p^m} = T_{p^m} - p^k T_{p^{m-2}}$$

I prefer to work with the primitive operator because it has only isolated fixed points.

We compute the elliptic terms in the fixed point formula for $T^{(1)}_{p^m}$: To find fixed points we have to find points (τ,g_f) and an element $\gamma\in G(\mathbb{Q})$ such that

$$\gamma(\tau,g_f)\in(\tau,g_f\underline{t}_{p^m}).$$

This means

a) that γ is conjugate in $G(\mathbb{R})$ to an element of K_∞ , this means that γ is elliptic.

b) For each prime $\ell\neq p$ we have $g_\ell^{-1}\gamma g_l\in G(\mathbb{Z}_\ell)$

c) For the prime p we have $g_p^{-1}\gamma g_p\in Gl(\mathbb{Z}_p)\begin{pmatrix} p^m & 0\\ 0 & 1\end{pmatrix}Gl(\mathbb{Z}_p)$.

From a) we get that the determinant of γ is positive and $\mathrm{tr}(\gamma)^2<4\det(\gamma)$ and b) and c) imply that $\mathrm{tr}(\gamma)=a\in\mathbb{Z}$ and $\det(\gamma)=p^m$. Then it is obvious that γ is not central. This in turn implies that γ has exactly two isolated fixed point in $G(\mathbb{R})/K_\infty$. Then the centralizer $Z_\gamma(\gamma)$ is a maximal torus which splits over the imaginary quadratic extension $\mathbb{Q}(\gamma)=\mathbb{Q}(\sqrt{a^2-4p^m})$, more precisely this is the torus $R_{Q(\gamma)/\mathbb{Q}}$.

Let $\tau_\pm$ be the two fixed points of γ in $G(\mathbb{R})/K_\infty$ then γ provides a set of fixed points which is the image of

$$F(\gamma)=\{(\tau_\pm,g_fK_f/K_f)\in G(\mathbb{R})/K_\infty\times G(\mathbb{A}_f)/K_f\mid g_f^{-1}\gamma g_f\in\underline{t}_p\}$$

in the quotient $S^G_{K_f}$. On this set we have an action of the group $Z_\gamma(\mathbb{Q})\backslash Z_\gamma(\mathbb{A})$ by multiplication from the right. Then the component at infinity $Z_\gamma(\mathbb{R})$ acts trivially. If we define $K_f^Z(g_f)$ as the stabilizer of $g_f K_f/K_f$ in $Z_\gamma(\mathbb{A}_f)$ then we get an injection

$$Z_\gamma(\mathbb{Q})\backslash Z_\gamma(\mathbb{A})/(Z_\gamma(\mathbb{R}) \times K_f^Z(g_f) \hookrightarrow F(\gamma).$$

One has to check that the local contribution of a fixed point in $F(\gamma)$ to the trace is $\mathrm{tr}(\gamma|\mathcal{M}_k)$. Hence we get as the total contribution of the isolated fixed points of $F(\gamma)$:

$$\mathrm{tr}(\gamma|\mathcal{M}_k)\#(Z_\gamma(\mathbb{Q})\backslash Z_\gamma(\mathbb{A}_f)/K_f^Z(g_f)) \sum_{\bar{g}_f \in Z_\gamma(\mathbb{A}_f)\backslash G(\mathbb{A}_f)/K_f} \underline{t}_{p^m}(\bar{g}_f^{-1}\gamma\bar{g}_f)$$

Now we replace K_f by an open subgroup K_f' which is sufficiently small. By this we mean that

For any $g_f \in G(\mathbb{A}_f)$ the intersection $G(\mathbb{Q}) \cap g_f K_f' g_f^{-1}$ is torsion free $\quad (Neat)$

Then we observe that t_{p^m} also defines endomorphisms on the different cohomology groups $H^\bullet_?(?S^G_{K_f'}, \tilde{\mathcal{M}}_k)$ and its traces on these groups are the same as the traces on the corresponding groups with level K_f. (See ???)

So we apply the fixed point formula ([book], Chap IV) to $H^\bullet(S^G_{K_f'}, \tilde{\mathcal{M}}_k)$. The fixed point set $F(\gamma)$ only depends on the $G(\mathbb{Q})$ conjugacy classe of γ and the condition $(Neat)$ implies that different conjugacy classes yield disjoint fixed point sets. Hence we get

$$\mathrm{tr}(\underline{t}_{p^m}|H^\bullet(S_{K_f'}, \tilde{\mathcal{M}}_k) = \mathrm{tr}_\infty(\underline{t}_{p^m}) +$$
$$\sum_{\gamma \in G(\mathbb{Q})_{\mathrm{ell}}/\sim} 2\mathrm{tr}(\gamma|\mathcal{M}_k)\#(Z_\gamma(\mathbb{Q})\backslash Z_\gamma(\mathbb{A}_f)/K_f'^{,Z}(g_f)) \sum_{\bar{g}_f \in Z_\gamma(\mathbb{A}_f)\backslash G(\mathbb{A}_f)/K_f'} \underline{t}_{p^m}(\bar{g}_f^{-1}\gamma\bar{g}_f),$$

the factor 2 comes from the two fixed points on the symmetric space. We want to write the sum on the right as an integral. On $G(\mathbb{A}_f)$ we chose the measure dg_f which gives value one to K_f. In $Z_\gamma(\mathbb{A}_f)$ we have a unique maximal compact subgroup $K^Z_{f,\max}$ we choose the measure dz_f so that $K^Z_{f,\max}$ gets volume one. Then we get a quotient measure $d\bar{g}_f$ on the quotient $Z_\gamma(\mathbb{A}_f)\backslash G(\mathbb{A}_f)$, so that $dg_f = dz_f d\bar{g}_f$.

We have the homomorphism

$$Z_\gamma(\mathbb{Q})\backslash Z_\gamma(\mathbb{A}_f)/K_f'^{,Z}(g_f) \to Z_\gamma(\mathbb{Q})\backslash Z_\gamma(\mathbb{A}_f)/K^Z_{f,\max}$$

Again $(Neat)$ implies that the kernel is $K^Z_{f,\max}/K_f^Z(g_f) = W(\gamma)$, where $W(\gamma)$ is the group of roots of unity in $Z_\gamma(\mathbb{Q}) = \mathbb{Q}(\gamma)^\times$. Therefore we get

$$\#(Z_\gamma(\mathbb{Q})\backslash Z_\gamma(\mathbb{A}_f)/K_f'^{,Z}(g_f)) = \frac{\#(Z_\gamma(\mathbb{Q})\backslash Z_\gamma(\mathbb{A}_f)/K^Z_{f,\max})}{\#W(\gamma)} [K^Z_{f,\max} : K_f'^{,Z}(g_f)]$$

The expression in the numerator of our formula above is the class number $h(\mathbb{Q}(\gamma))$ of $\mathbb{Q}(\gamma)$.

Looking at the definition of the quotient measure we see that the volume of an K'_f orbit $\bar{g}_f K'_f \subset Z_\gamma(\mathbb{A}_f)\backslash G(\mathbb{A}_f))$ is

$$[K^Z_{f,\max} : K'^{,Z}_f(g_f)] = \frac{1}{\mathrm{vol}_{dz_f}(K'^{,Z}_f(g_f))}.$$

Therefore

$$\sum_{\bar{g}_f \in Z_\gamma(\mathbb{A}_f)\backslash G(\mathbb{A}_f)/K_f} \frac{1}{\mathrm{vol}_{dz_f}(K'^{,Z}_f(g_f))} \underline{t}_{p^m}(\bar{g}_f^{-1}\gamma\bar{g}_f) = \int_{Z_\gamma(\mathbb{A}_f)\backslash G(\mathbb{A}_f)} \underline{t}_{p^m}(\bar{g}_f^{-1}\gamma\bar{g}_f) d\bar{g}_f.$$

and our final formula for the elliptic contribution is

$$\sum_{\gamma \in G(\mathbb{Q})_{\mathrm{ell}}/\sim} 2\mathrm{tr}(\gamma|\mathcal{M}_k) \frac{h(\mathbb{Q}(\gamma))}{\#W(\gamma))} \int_{Z_\gamma(\mathbb{A}_f)\backslash G(\mathbb{A}_f)} \underline{t}_{p^m}(\bar{g}_f^{-1}\gamma\bar{g}_f) d\bar{g}_f =$$

$$\sum_{\gamma \in G(\mathbb{Q})_{\mathrm{ell}}/\sim} 2\mathrm{tr}(\gamma|\mathcal{M}_k) \frac{h(\mathbb{Q}(\gamma))}{\#W(\gamma))} \prod_\ell \int_{Z_\gamma(\mathbb{Q}_\ell)\backslash G(\mathbb{Q}_\ell)} \underline{t}_{p^m}(\bar{g}_\ell^{-1}\gamma\bar{g}_\ell) d\bar{g}_\ell$$

Since we assume that $m > 0$ we only have isolated fixed points. Further down we will also discuss the case $m = 0$, where we have non isolated fixed points.

It is now clear that we have to develop some understanding of the orbital integrals, we consider them in the two following section. After that we resume our discussion of the trace formula.

2.5 The orbital integrals.

Let us now make the assumption that K_f is the standard maximal compact subgroup. We want to compute the local orbital integrals.

2.5.1 The case $\ell \neq p$

. We represent the conjugacy classes of elliptic elements by matrices $\gamma = \gamma_a = \begin{pmatrix} a & -1 \\ p^m & 0 \end{pmatrix}$. In this case the local component $(t_{p^m})_\ell$ is simply the characteristic function $\chi_{\mathrm{Gl}_2(\mathbb{Z}_\ell}$ and our local integral is

$$\int_{Z_\gamma(\mathbb{Q}_\ell)\backslash G(\mathbb{Q}_\ell)} \chi_{\mathrm{Gl}_2(\mathbb{Z}_\ell)}(\bar{g}_\ell^{-1}\gamma\bar{g}_\ell) d\bar{g}_\ell = \sum_{\bar{g}_\ell \in Z_\gamma(\mathbb{Q}_\ell)\backslash G(\mathbb{Q}_\ell)/K_\ell} [K^Z_{\ell,\max} : K^Z_\ell(g_\ell)] \chi_{\mathrm{Gl}_2(\mathbb{Z}_\ell)}(\bar{g}_\ell^{-1}\gamma\bar{g}_\ell)$$

Now we observe that the condition $\chi_{\mathrm{Gl}_2(\mathbb{Z}_\ell)}(\bar{g}_\ell^{-1}\gamma\bar{g}_\ell) = 1$ means that for a representative g_ℓ of $\bar{g}_\ell$ the free $\mathbb{Z}_\ell$ module $g_\ell\mathbb{Z}_\ell^2 \subset \mathbb{Q}_\ell^2$ is in fact a $\mathbb{Z}_\ell[\gamma]$ module. Two such $\mathbb{Z}_\ell[\gamma]$ modules $g_\ell\mathbb{Z}_\ell^2, g'_\ell\mathbb{Z}_\ell^2$ are the same if and only if $g_\ell \in g'_\ell\mathrm{Gl}_2(\mathbb{Z}_\ell)$ and the are isomorphic if and only if $g_\ell \in Z_\gamma(\mathbb{Q}_\ell)g'_\ell$. Hence we see that our orbital integral is equal to the sum

$$\sum_{[M_\ell]} \mathrm{vol}_{dz_\ell}(\mathrm{Aut}(M_\ell))^{-1}$$

over the isomorphism classes of $\mathbb{Z}_\ell[\gamma]$ submodules $M \subset \mathbb{Q}_\ell^2$ which are $\mathbb{Z}_\ell$ modules of rank 2.

It is a tedious computation to evaluate these sums.

We have to distinguish the two cases that $\mathbb{Q}_\ell(\gamma_a)$ is a field (*nonsplit*) or a sum of two fields (*split*). We say that γ_a is *maximal* at ℓ if $\mathbb{Z}_\ell[\gamma_a]$ is a maximal order in the case (*nonsplit*) or if the two eigenvalues are different $\mod \ell$ in the case (*split*). For γ maximal at ℓ the value of the orbital integral is equal to 1. If $\ell \neq 2$ then the element γ_a is maximal at ℓ if ℓ^2 does not divide $a^2 - 4p^m$.

In general let $\ell^{m(a,\ell)}$ be the exact power of ℓ dividing $a^2 - 4p^m$ and let $r(a,\ell) = [\frac{m(a,\ell)}{2}]$ the Gauss bracket.

We begin with the case $\ell > 2$. Then it is clear that the extension $\mathbb{Q}_\ell(\sqrt{a^2 - 4p^m})$ is unramified if and only if $m(a,\ell)$ is even (we consider the split extension as unramified). Then we get for the orbital integrals

$$\frac{\ell^{r(a,\ell)+1} + \ell^{r(a,\ell)} - 2}{\ell - 1} \text{ if } \mathbb{Q}_\ell(\sqrt{a^2 - 4p^m}) \text{ is an unramified field extension}$$

$$\ell^{r(a,\ell)} \text{ if } \mathbb{Q}_\ell(\sqrt{a^2 - 4p^m}) \text{ is } \mathbb{Q}_\ell \oplus \mathbb{Q}_\ell$$

and

$$\frac{\ell^{r(a,\ell)+1} - 1}{\ell - 1} \text{ if } \mathbb{Q}_\ell(\sqrt{a^2 - 4p^m}) \text{ is a ramified extension.}$$

For $\ell = 2$ we get

$$2^{r(a,2)+1} + 2^{r(a,2)} - 2 \text{ if } \mathbb{Q}_2(\sqrt{a^2 - 4p^m}) \text{ is an unramified field extension}$$

$$2^{r(a,2)} \text{ if } \mathbb{Q}_2(\sqrt{a^2 - 4p^m}) \text{ is split}$$

and

$$2^{r(a,2)} - 1 \text{ if } \mathbb{Q}_2(\sqrt{a^2 - 4p^m}) \text{ is a ramified field extension.}$$

We see a slight difference in the expression in the ramified case. Furthermore we notice that for an odd value of a the extension $\mathbb{Q}_2(\sqrt{a^2 - 4p^m})$ is unramified, then $r(a,2) = 0$ and in both cases (split or non split). we get the value 1 for the orbital integral.

Finally we observe that the orbital integral at ℓ becomes large if the eigenvalues of γ_a are ℓ- adically close to each other. We also check easily that the extension $\mathbb{Q}_\ell(\sqrt{a^2 - 4p^m})$ can only split if $m(a,\ell)$ is even and then it is also clear that $\ell^{r(a,\ell)} = |\alpha(\gamma_a) - 1)|_\ell$ and in the split case we get our previous results. (germ expansion)

2.5.2 The case $\ell = p$

Recall that t_{p^m} is the characteristic function of $G(\mathbb{Z}_p)\begin{pmatrix} p^m & 0 \\ 0 & 1 \end{pmatrix}G(\mathbb{Z}_p)$. We want to compute

$$O(t_{p^m}, \gamma_a) = \int_{Z_\gamma(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)} t_{p^m}(\bar{g}_p \gamma_a \bar{g}_p^{-1}) d\bar{g}_p$$

We consider the greatest integer t which satisfies

$$t \leq m/2 \text{ and } p^t \mid a$$

and let us put

$$\chi(a,p) = \begin{cases} 1 & \text{if } \mathbb{Q}_p(\sqrt{a^2 - 4p^m}) \text{ splits} \\ 0 & \text{if } \mathbb{Q}_p(\sqrt{a^2 - 4p^m}) \text{ is a ramified extension} \\ -1 & \text{if } \mathbb{Q}_p(\sqrt{a^2 - 4p^m}) \text{ is an unramified field extension} \end{cases}$$

then

$$O(t_{p^m}, \gamma_a) = \begin{cases} 1 & \text{if } t = 0 \\ p^t(1 - \chi(a,p)\frac{1}{p}) & \text{if } t > 0 \end{cases}$$

We do not prove these formulas here, they can be derived from the consideration of the Bruhat-Tits building. (Consider fixed points (later))

2.6 The contribution from the fixed points at infinity.

We want to study the contributions from the fixed points at infinity. Before we do this we have a look at the cohomology of the boundary. our special case her we have

$$H^0(\partial S^G_{K_f}, \tilde{\mathcal{M}}_k) \xrightarrow{\sim} H^1(\partial S^G_{K_f}, \tilde{\mathcal{M}}_k) \xrightarrow{\sim} \mathbb{Q}$$

and the Hecke operator t_{p^m} acts on both spaces by the eigenvalue

$$p^{m(k+1)} + (1 - \frac{1}{p})\Big(p^{(m-1)(k+1)} + p^{(m-2)(k+1)} + \cdots + p^{(k+1)}\Big) + 1$$

We consider this as an ordered sum, the terms are ordered according to their size or as we sometimes say weights. If m is even, then we have a term in the middle. Let us write this sum accordingly as

$$t^+_{p^m} + t^{[1/2]}_{p^m} + t^-_{p^m},$$

where the first term collects the weights above the middle, the second term is the contribution of middle weight and the last term collects the terms of weights below the middle. The contribution of middle weight is of course zero if m is odd. We suppress the dependence on k in the notation.

Now the results of the dissertation of J. Bewersdorff provide a rule to compute the remaining terms coming from the fixed points at infinity. He says that we have to truncate the Hecke operator in the neighborhood of the cusps. This will be explained in detail later. Basically it says that in a neighborhood of the cusps we can decompose into an expanding and a contracting part

$$T^{(1)}_{p^m}{}_{|\infty} = T^{(1,\exp)}_{p^m}{}_{|\infty} + T^{(1,\mathrm{contr})}_{p^m}{}_{|\infty}.$$

(What the expanding and contracting parts are is decided by an observer which sits at infinity, on the boundary of the Borel-Serre compactification.)

Both terms act on the cohomology of the boundary., we are interested in the action of the contracting term. It acts by the eigenvalues

$$t^+_{p^m} + t^{[1/2]}_{p^m} = p^{m(k+1)} + + (1 - \frac{1}{p})(p^{(m-1)(k+1)} + p^{(m-2)(k+1)} + \cdots + p^{(m-\left[\frac{m}{2}\right])(k+1)})$$

on $H^1(\partial S^G_{K_f}, \tilde{\mathcal{M}}_k)$ and by the eigenvalue

$$t^-_{p^m} + t^{[1/2]}_{p^m}$$

on $H^0(\partial S^G_{K_f}, \tilde{\mathcal{M}}_k)$. The middle terms occurs in both expressions..

Now Bewersdorff has shown that

$$\mathrm{tr}_\infty(t_{p^m}) = \mathrm{tr}(T^{(1,\mathrm{contr})}_{p^m} \mid H^\bullet(\partial S^G_{K_f}, \tilde{\mathcal{M}}_k)) = -t^+_{p^m} + t^-_{p^m}.$$

We notice that the " middle" drops out.

Hence we have now the formula

$$\mathrm{tr}(t_{p^m}|H^\bullet(t_{p^m}|H^\bullet(S^G_{K_f}, \tilde{\mathcal{M}}_k) =$$

$$\sum_{\gamma \in G(\mathbb{Q})_{\mathrm{ell}}/\sim} 2\mathrm{tr}(\gamma|\mathcal{M}_k)\frac{h(\mathbb{Q}(\gamma))}{\#W(\gamma))}\prod_\ell \int_{Z_\gamma(\mathbb{Q}_\ell)\backslash G(\mathbb{Q}_\ell)} \underline{t}_{p^m}(\bar{g}_\ell^{-1}\gamma\bar{g}_\ell)d\bar{g}_\ell - t^+_{p^m} + t^-_{p^m}$$

We have the tautological exact sequence

$$0 \to H^\bullet_!(S^G_{K_f}, \tilde{\mathcal{M}}_k) \to H^\bullet(S^G_{K_f}, \tilde{\mathcal{M}}_k) \to H^\bullet_{\mathrm{Eis}}(S^G_{K_f}, \tilde{\mathcal{M}}_k) \to 0$$

Then we know under our assumtions (K_f standard maximal compact subgroup)

$$\text{For } k > 0 \text{ we have } H^\bullet_{\mathrm{Eis}}(S^G_{K_f}, \tilde{\mathcal{M}}_k) = H^1_{\mathrm{Eis}}(S^G_{K_f}, \tilde{\mathcal{M}}_k) = \mathbb{Q}$$

$$\text{For } k = 0 \text{ we have } H^\bullet_{\mathrm{Eis}}(S^G_{K_f}, \tilde{\mathcal{M}}_k) = H^0_{\mathrm{Eis}}(S^G_{K_f}, \tilde{\mathcal{M}}_k) = \mathbb{Q}$$

where t_{p^m} acts by the eigenvalue.

$$p^{m(k+1)} + (1 - \frac{1}{p})\Big(p^{(m-1)(k+1)} + p^{(m-2)(k+1)} + \cdots + p^{(k+1)}\Big) + 1$$

Moving the term coming from the Eisenstein cohomology on the left hand side to the right we get for $k > 0$

$$\mathrm{tr}(\underline{t}_{p^m}|H^\bullet_!(S_{K_f}, \tilde{\mathcal{M}}_k)) = \mathrm{tr}_{\mathrm{ell}}(t_{p^m}) + t^{[1/2]}_{p^m} + 2t^-_{p^m} =$$

$$\mathrm{tr}_{\mathrm{ell}}(t_{p^m}) + \begin{cases} (1-\frac{1}{p})p^{\frac{m}{2}(k+1)} + 2((1-\frac{1}{p})p^{(\frac{m}{2}-1)(k+1)} + \cdots + 2 \\ 2(1-\frac{1}{p})p^{(\left[\frac{m}{2}\right])(k+1)} + \cdots + 2 \end{cases}.$$

The second term collects the trace on the Eisenstein cohomology and the contributions from the fixed points at infinity.

If $k = 0$ we get a similar relation, except that now the Eisenstein cohomology sits in degree zero. This has the effect, that the correcting term will be

$$\mathrm{tr}(\underline{t}_{p^m}|H^\bullet_!(S_{K_f}, \mathbb{Q})) = \mathrm{tr}_{\mathrm{ell}}(t_{p^m}) - 2t^+_{p^m} - t^{[1/2]}_{p^m}.$$

It is easy to see and also follows from the classical theory of modular forms that in our special case the inner cohomology $H^1_!(S^G_{K_f}, \tilde{\mathcal{M}}_k) = 0$ for $k = 0, 2, 4, 6, 8, 12$ hence we get that the trace on the left hand side is zero. This implies the relations

$$\sum_{\{a\in\mathbb{Z}|a^2<4p^m\}} \mathrm{tr}(\gamma_a|\tilde{\mathcal{M}}_k)\frac{h(\sqrt{a^2-4p^m})}{\#W(\gamma_a)}\prod_\ell \int_{Z_\gamma(\mathbb{Q}_\ell)\backslash G(\mathbb{Q}_\ell)} \underline{t}_{p^m}(\bar{g}_\ell^{-1}\gamma_a\bar{g}_\ell)d\bar{g}_\ell =$$

$$\begin{cases} +2t^+_{p^m} - t^{[1/2]}_{p^m} & \text{if } k = 0 \\ -2t^-_{p^m} - t^{[1/2]}_{p^m} & \text{if } k > 0 \end{cases}$$

for the above values of k.

These are the famous class number relations.

Since there are many sources for making stupid mistakes I checked these equalities for $m = 1, 2, 3$ and quite a number of primes p. (See enclosed programs in Mathematica)

2.7 Non isolated fixed points

So far we considered Hecke operators, which have only isolated fixed points, this came from our assumption $m > 0$. The contrasting case is the identity operator, this means $m = 0$. We consider the neat case first. Then we see immedeately that only the element $\gamma = e = \mathrm{Id}$ contributes by a fixed point set, so we have only one elliptic term. But the fixed point is not isolated we have

$$\mathrm{tr}(\mathrm{Id} \mid H^\bullet(S^G_{K_f}, \tilde{\mathcal{M}})) = \mathrm{tr}_{\mathrm{ell}}(e) + \mathrm{tr}_\infty(\mathrm{Id})$$

Now the term on the left hand side is by definition the Euler characteristic of the cohomology groups. It is clear how this Euler characteristic depends on the coefficient system, we have

$$\mathrm{tr}(\mathrm{Id} \mid H^\bullet(S^G_{K_f}, \tilde{\mathcal{M}})) = \chi(H^\bullet(S^G_{K_f}, \tilde{\mathcal{M}})) = \chi(H^\bullet(S^G_{K_f}, \mathbb{Q})) \cdot \dim(\mathcal{M})$$

The Euler characteristic of the space $S^G_{K_f}$ can be computed by the Gauss-Bonnet formula. There is an invariant measure $\omega^{\mathrm{GB}}_\infty$ on $X = G(\mathbb{R})/K_\infty$, which can be computed from the local differential geometric data (the curvature tensor) on X, which has the following property: For any torsion free arithmetic subgroup $\Gamma(g_f) = G(\mathbb{Q}) \cap g_f^{-1}K_f g_f$ we have

$$\int_{\Gamma(g_f)\backslash X} \omega^{\mathrm{GB}}_\infty = \chi(\Gamma(g_f)\backslash X)$$

We go back to the beginning of this chapter, we apply the fixed point formula to the identity on $H^\bullet(S^G_{K_f}, \tilde{\mathcal{M}}_k) = H^\bullet(\mathrm{Sl}_2(\mathbb{Z})\backslash\mathbb{H}, \tilde{\mathcal{M}}_k)$. We easily see, that the only contributions are obtained from the central elements $\pm\mathrm{Id} = \pm e \in \mathrm{Gl}_2(\mathbb{Q})$ and the conjugacy classes containing the elements of finite order 6,3 or 4

$$\begin{pmatrix} \pm 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

(remember that we restricted to the matrices with positive determinant.)

If we now consider the terms coming from the the central elements, then we have to replace the Euler-characteristic by the virtual Euler-characteristic (See 4.2 and 4.2.1. below) $\chi_{\mathrm{orb}}(\mathrm{Sl}_2(\mathbb{Z})\backslash\mathbb{H})$. The value of this number is easy to compute: We pass to the congruence subgroup (Chap III, 1.1.3) $K_f(3)$ and it is not so difficult to see that for a connected component of $S^G_{K_f(3)}$ we have

$$H^0(S^G_{K_f(3)}, \mathbb{Z}) = \mathbb{Z}, H^1(S^G_{K_f(3)}, \mathbb{Z}) = \mathbb{Z}^3,$$

(looking at a component we see easily that it a Riemann sphere with four punctures.) Hence we see that the Euler-characteristic of $S^G_{K_f(3)}$ is -2. If we take both components together then we get -4. The order of $K_f/K_f(3)$ is $2 \cdot 3 \cdot (3+1) \cdot (3-1) = 48$. Hence we get

$$\chi_{\mathrm{orb}}(\mathrm{Sl}_2(\mathbb{Z})\backslash\mathbb{H}) = -\frac{1}{12}.$$

We have two central terms in the trace formula, therefore the contribution of the central terms to the elliptic terms in the fixed point formula will be

$$-\frac{1}{6}\dim \mathcal{M}_k = -\frac{k+1}{6}.$$

The other elliptic terms come from the elements of finite order we get as total contribution

$$-\frac{k+1}{6} + \frac{1}{2}\mathrm{tr}(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} |\mathcal{M}_k) + \frac{2}{3}\mathrm{tr}(\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} |\mathcal{M}_k).$$

(We have the factor 2/3 because we can take $a = \pm 1$.) The trace of the matrices are again easy to compute

$$\mathrm{tr}(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} |\mathcal{M}_k) = \begin{cases} 1 & \text{if } k \equiv 0 \mod 4 \\ -1 & \text{if } k \equiv 2 \mod 4 \end{cases}$$

$$\mathrm{tr}(\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} |\mathcal{M}_k) = \begin{cases} 1 & \text{if } k \equiv 0 \mod 3 \\ 0 & \text{if } k \equiv 2 \mod 3 \\ -1 & \text{if } k \equiv 1 \mod 3 \end{cases}$$

For the values $k = 0, 2, \ldots, 10$ we get for the Euler-characteristic

$$\chi(\mathrm{Sl}_2(\mathbb{Z})\backslash\mathbb{H}, \mathcal{M}_k) = 1, -1, -1, -1, -1, -3$$

3 The proof of the general trace formula (elliptic terms)

In this section we derive the general topological trace formula and especially we will write down the elliptic contributions more carefully and discuss some normalizing of measures at infinity.

We assume that the derived group $G^{(1)}$ of G is simply connected and we make two further assumptions

(A) The center $C/\mathbb{Q}$ is a split torus $C_s/\mathbb{Q}$ times a torus $C^{(1)}/\mathbb{Q}$ which is anisotropic over $\mathbb{R}$.

(B) The semi-simple group $G^{(1)} \times \mathbb{R}$ has a maximal torus $T/\mathbb{R}$ which is anisotropic.

We start from our general situation in Chap. III.1.2. If we have an element $h_f = \prod_p h_p$ in the Hecke algebra $\mathcal{H}(G(\mathbb{A}_f//K_f)$, i.e. an element of level K_f, then it induces an endomorphism in any of the cohomology groups

$$H^\bullet(S^G_{K'_f}, \tilde{\mathcal{M}}), H^\bullet_c(S^G_{K'_f}, \tilde{\mathcal{M}}), H^\bullet(\partial S^G_{K'_f}, \tilde{\mathcal{M}})$$

where $K'_f \subset K_f$. We are interested in the traces of these operators and derive a Lefschetz trace formula for it.

3.1 The change of level

Recall that the definition of a Hecke operator involves the choice of a measure on $G(\mathbb{A}_f)$. Let us take the measure that gives volume one to K_f, but we use the same measure for the Hecke operators acting on the cohomology with respect to K'_f. Then we have for any of these cohomology groups

$$\mathrm{tr}(h_f|H^\bullet_?(?S^G_{K'_f}, \tilde{\mathcal{M}}) = \mathrm{tr}(h_f|H^\bullet_?(?S^G_{K_f}, \tilde{\mathcal{M}}).$$

This formula is not entirely obvious. We do not prove it here, but we give some hints. It requires reduction theory. The essential point is that we can write down a complex, which is obtained from a finite covering of the space by nice open sets, so that the Čech-complex computes the cohomology (See Chap. II ???, and [book],VI). This means that the intersections of the open sets are contractible. We can also build in the Hecke operators. We compare the two Čech complexes which compute the cohomology on the levels K_f and K'_f.

Some more details will be provided later.

We know already that the trace formula we be a sum of contributions coming from the fixed points and a part of this sumis given by the conjugacy classes of elliptic elements in $G(\mathbb{Q})$. The central elements in $G(\mathbb{Q})$ are elliptic by definition and to compute their conribution we have to compute the Euler characteristic of $S^G_{K_f}$. We make these two assumptions above because otherwise this Euler-characteristic become zero. If for instance $G/\mathbb{Q} = \mathrm{Sl}_n/\mathbb{Q}$ and $n > 2$ then (B) does not hold and the Euler-characteric of any associated locally symmetric space is zero.

3.2 The Euler-characteristic of $S^G_{K_f}$

Our assumptions have the effect that our locally symmetric space is a disjoint union of locally symmetric spaces attached to $G^{(1)}/\mathbb{Q}$. To see this we first look at the component at infinity, we have isometric embeddings

$$j_{x_i} : G^{(1)}(\mathbb{R})/K^{(1)}{}_\infty \to G(\mathbb{R})/K_\infty$$

which are given by $g \mapsto gx_i$ and where x_i is any point stabilized by $K^{(1)}{}_\infty$. Since $G^{(1)}$ is simply connected the symmetric space on the left is connected and the space on the right may be not. Then the x_i are specific points in the different connected components. Now we look at the exact sequence

$$1 \to G^{(1)} \to G \to C' \to 1$$

we get an exact sequence for the group of finite adeles

$$1 \to G^{(1)}(\mathbb{A}_f) \to G(\mathbb{A}_f) \to C'(\mathbb{A}_f) \to 1$$

because the Galois cohomology for a simply connected group over a local field is trivial. We choose K_f sufficiently small and put $K_f{}^{(1)} = K_f \cap G^{(1)}(\mathbb{A}_f)$ and $K_f^{C'} = K_f/K_f{}^{(1)} \subset C'(\mathbb{A}_f)$. Now we notice that our locally symmetric space

$$C'(\mathbb{Q})\backslash C'(\mathbb{R})/C_s^0(\mathbb{R}) \times C'(\mathbb{A}_f)/K_f^{C'}$$

attached to C' is a finite abelian group, its order is a generalized ideal class group. The quotient $C'(\mathbb{R})/C_s^0(\mathbb{R})$ is of course the group of connected components $\pi_0(C'(\mathbb{R}))$.

We have the two exact sequences

$$\begin{array}{ccccccccc}
1 & \to & G^{(1)}(\mathbb{R}) & \to & G(\mathbb{R}) & \to & C'(\mathbb{R}) & \to & H^1(\mathbb{R}, G^{(1)}) \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
1 & \to & G^{(1)}(\mathbb{Q}) & \to & G(\mathbb{Q}) & \to & C'(\mathbb{Q}) & \to & H^1(\mathbb{Q}, G^{(1)})
\end{array}$$

The last upwards arrow is an isomorphism because of the Hasse principle.

Therefore we see that our morphism

$$S^G_{K_f} \to S^{C'}_{K_f^{C'}}$$

induces an injective map between the set of connected components

$$\pi_0(S^G_{K_f}) \to S^{C'}_{K_f^{C'}},$$

the image is a subgroup of index $[\pi_0(C'(\mathbb{R})) : \pi_0(G(\mathbb{R}))]$.

We get the desription of $S^G_{K_f}$ as union of connected components. We choose representatives (x_i, c_f) for the elements in the image and we extend the above map to the adeles, i.e. we define

$$j_{x_i,c_f} : S^{(1)}_{K_f^{(1)}(c_f)} \to S^G_{K_f}$$

by

$$(g, g_f) \mapsto (g, g_f)(x_i, c_f)$$

and here of course again we have $K_f^{(1)}(c_f) = c_f K_f^{(1)}(c_f)^{-1}$.

3.3 The comparison of measures

On the Lie algebra $\mathfrak{g}^{(1)}/\mathbb{Q}$ of $G^{(1)}$ we have the Killing-form $B : \mathfrak{g}^{(1)} \times \mathfrak{g}^{(1)} \to \mathbb{Q}$, this a non-degenerate symmetric bilinear form. It also provides an identification of $\mathfrak{g}$ and its dual space $\mathfrak{g}^\vee$. We choose a lattice $\mathfrak{g}_\mathbb{Z}^{(1)} \subset \mathfrak{g}^{(1)}$ which is closed under the Lie-bracket. Then for $U, V \in \mathfrak{g}_\mathbb{Z}^{(1)}$ the value of the Killing-form $B(U,V) = \mathrm{tr}(\mathrm{ad}([U,V]) \in \mathbb{Z}$. We define a dual lattice with respect to the Killing form

$$\mathfrak{g}_\mathbb{Z}^{(1)^\vee} = \{U \in \mathfrak{g}^{(1)} | B(\mathfrak{g}_\mathbb{Z}^{(1)}, U) \in \mathbb{Z}\} \subset \mathfrak{g}^{(1)}$$

Now we have that the highest exterior power $\bigwedge^{\dim G} \mathfrak{g}_\mathbb{Z}^{(1)^\vee} \xrightarrow{\sim} \mathbb{Z}$, we have a canonical generator up to a sign. This is a fifferential form of top degree at the origin and this generator yields an invariant form of top degree $\omega_{G^{(1)}} = \omega_{G^{(1)}}^{\mathfrak{g}_\mathbb{Z}^{(1)}}$ on $G^{(1)}/\mathbb{Q}$ which is unique up to a sign. This form provides by the standard procedure the Tamagawa measure $\omega_{G^{(1)}}^{\mathrm{Tam}}$ on $G^{(1)}(\mathbb{A})$. This Tamagawa measure does not deped on the choice of the lattice.

But if we decompose the Tamagawa measure into its component at infinity and its finite component

$$\omega_{G^{(1)}}^{\mathrm{Tam}} = \omega_{G^{(1)}{}_\infty}^{\mathrm{Tam}} \times \omega_{G^{(1)},f}^{\mathrm{Tam}}$$

then both components depend on the choice of the lattice. In the computations which have to be done in the sequel it is important that we choose good lattices. We want that the index $[\mathfrak{g}^{(1)}{}_\mathbb{Z}^\vee : \mathfrak{g}^{(1)}{}_\mathbb{Z}]$ is as small as possible.

The Euler characteristic of $S_{K_f}^G$ is equal to

$$\chi(S_{K_f}^G) = \chi(S_{K_f^{(1)}}^{G^{(1)}}) \frac{\# S_{K_f^{C'}}^{C'}}{[\pi_0(C'(\mathbb{R}) : \pi_0(G(\mathbb{R}))]}$$

We look at the first factor on the right hand side. It can computed by using the Gauss-Bonnet theorem. It says that we can define a measure $\omega^{\mathrm{GB}} \times \omega_f$ on

$$X \times G(\mathbb{A}_f) = G^{(1)}(\mathbb{R})/K^{(1)}{}_\infty \times G^{(1)}(\mathbb{A}_f)$$

such that

$$\chi(S_{K_f^{(1)}}^{G^{(1)}}) = \int_{G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)} \omega^{\mathrm{GB}} \times \omega_f$$

The measure ω^{GB} is actually invariant under the action of $G^{(1)}(\mathbb{R})$ and ω_f is also invariant and satisfies $\omega_f(K^{(1)}{}_f) = 1$.

We can define an invariant measure

$$\tilde{\omega}^{\mathrm{GB}} = \omega^{\mathrm{GB}} \times dk$$

on $G^{(1)}(\mathbb{R})$ by requiring $\int_{K^{(1)}{}_\infty} dk = 1$. Then

$$\chi(S_{K_f^{(1)}}^{G^{(1)}}) = \int_{G^{(1)}(\mathbb{Q})\backslash G^{(1)}(\mathbb{R}) \times G^{(1)}(\mathbb{A}_f)} \tilde{\omega}^{\mathrm{GB}} \times \omega_f$$

Now the two measures $\omega_\infty^{\mathrm{Tam}}$ and $\tilde{\omega}^{\mathrm{GB}}$ on $G^{(1)}(\mathbb{R})$ differ by a constant factor $c_\infty(\mathfrak{g}^{(1)}{}_\mathbb{Z})$, which depends on the lattice. We can replace the measure $\tilde{\omega}^{\mathrm{GB}} \times \omega_f$ by the Tamagawa measure ω^{Tam} and get

$$\chi(S^{G^{(1)}}_{K^{(1)}_f}) = c_\infty(\mathfrak{g}^{(1)}{}_{\mathbb{Z}})\mathrm{vol}_{\omega^{\mathrm{Tam}}_{G^{(1)},f}}(K^{(1)}{}_f)^{-1}\int_{G^{(1)}(\mathbb{Q})\backslash G^{(1)}(\mathbb{A})}\omega^{\mathrm{Tam}}_{G^{(1)}}$$

The last factor is by definition the Tamagawa number $\tau(G^{(1)})$, it is known to be equal to one, but we keep it as it is and write

$$\chi(S^G_{K_f}) = \frac{c_\infty(\mathfrak{g}^{(1)}{}_{\mathbb{Z}})}{[\pi_0(C'(\mathbb{R}) : \pi_0(G(\mathbb{R}))]}\frac{\# S^{C'}_{K^{C'}_f}}{\mathrm{vol}_{\omega^{\mathrm{Tam}}_{G^{(1)},f}}(K^{(1)}{}_f)}\tau(G^{(1)})$$

The last ratio can be slightly rewritten. On $\pi_0(C'(\mathbb{R}))\times C'(\mathbb{A}_f)$ we define the Tamagawa measure $\omega^{\mathrm{Tam}}_{C',f}$ simply as the measure which gives volume one to the unique maximal compact subgroup $(1)\times C'(\hat{\mathbb{Z}})$ of $\pi_0(C'(\mathbb{R}))\times C'(\mathbb{A}_f)$. Then we define as Tamagawa measure on $G(\mathbb{A}_f)$ the product measure $\omega^{\mathrm{Tam}}_G = \omega^{\mathrm{Tam}}_{G^{(1)}}\times\omega^{\mathrm{Tam}}_{C',f}$ on $G(\mathbb{A}_f)$. We put $C'(\mathbb{Q})^+ = C'(\mathbb{Q})\cap C'(\mathbb{R})^{(0)}$ and

$$h(C') = \mathrm{vol}_{\omega^{\mathrm{Tam}}_{C',f}}(C'(\mathbb{Q})^+\backslash C'(\mathbb{A}_f).$$

and get

$$\chi(S^G_{K_f}) = \frac{c_\infty(\mathfrak{g}^{(1)}{}_{\mathbb{Z}})}{[\pi_0(C'(\mathbb{R}) : \pi_0(G(\mathbb{R}))]}\frac{1}{\mathrm{vol}_{\omega^{\mathrm{Tam}}_{G,f}}(K_f)}\tau(G^{(1)})h(C')$$

In this formula the depence on the choice of K_f is quite clear. If we replace K_f by a subgroup K'_f of finite index, then the Euler characteristic gets multiplied by the index $[K_f : K'_f]$. This allows us to define define the virtual or orbifold Euler characteristic for an arbitrary choice of an open compact subgroup as:

$$\chi_{\mathrm{orb}}(S^G_{K_f}) = \frac{c_\infty(\mathfrak{g}^{(1)}{}_{\mathbb{Z}})}{[\pi_0(C'(\mathbb{R}) : \pi_0(G(\mathbb{R}))]}\frac{1}{\mathrm{vol}_{\omega^{\mathrm{Tam}}_{G,f}}(K_f)}\tau(G^{(1)})h(C').$$

If the group K_f is not small, if for instance we have a non trivial intersection $G(\mathbb{Q})\cap G(\mathbb{A}_f)$ then $\chi_{\mathrm{orb}}(S^G_{K_f})$ is not necessarily equal to the Euler characteristic of the topological space $S^G_{K_f}$. We have seen this in 3.3.

We need a moment of meditation. We see that the Euler-characteristic is a product of four (or three) factors. These factors are the ratio in front, then the inverse of the volume of the maximal compact subgroup K_f with respect to the Tamagawa measure and the product of the last two factors, which in a sense can be viewed as one factor.

The first factor depends of course on the choice of $\mathfrak{g}^{(1)}{}_{\mathbb{Z}}$, if we choose a second lattice $\mathfrak{g}_1{}^{(1)}{}_{\mathbb{Z}}$ then the intersection $\mathfrak{g}_1{}^{(1)}{}_{\mathbb{Z}}\cap\mathfrak{g}^{(1)}{}_{\mathbb{Z}}$ has finite index d (resp.) d_1 in $\mathfrak{g}^{(1)}{}_{\mathbb{Z}}$ (resp.)$\mathfrak{g}_1{}^{(1)}{}_{\mathbb{Z}}$ and it is clear by definition that

$$dc_\infty(\mathfrak{g}^{(1)}{}_{\mathbb{Z}}) = d_1 c_\infty(\mathfrak{g}_1{}^{(1)}{}_{\mathbb{Z}})$$

Essentially this factor depends only on the infinite place and is relatively elementary to compute (See the example below).

The change of the constant will be compensated by a factor occuring in the second factor, in other words

$$\frac{c_\infty(\mathfrak{g}^{(1)}{}_{\mathbb{Z}})}{\mathrm{vol}_{\omega^{\mathrm{Tam}}_{G,f}}(K_f)}$$

is a number which does not depend on the choice of the lattice $\mathfrak{g}^{(1)}{}_{\mathbb{Z}}$. The denominator depends only on the groups $G/\mathbb{Q}_p$ for all places hence it is an infinite product of local contributions at the finite places. These local contributions are relatively easy to compute for all primes outside an exceptional set. (See the example below.)

The most important fact is that the factor in the middle does not depend on any choice of measures. It is a rather deep theorem that for a simply connected group the Tamagawa number

$$\tau(G^{(1)}) = 1.$$

This leaves us with the $h(C')$. This is a generalized ideal class group and which may be difficult to compute.

3.3.1 Bruhat-Tits integral structures

I come back the computation of the ratio $\frac{c_\infty(\mathfrak{g}^{(1)}{}_{\mathbb{Z}})}{\mathrm{vol}_{\omega^{\mathrm{Tam}}_{G,f}}(K_f)}$. Sometimes this may be an extremely unpleasant task. Of course it is clear that the dependence on the variable K_f is simple. If we replace K_f by a subgroup K'_f then $\mathrm{vol}_{\omega^{\mathrm{Tam}}_{G,f}}(K_f) = [K_f : K'_f]\mathrm{vol}_{\omega^{\mathrm{Tam}}_{G,f}}(K'_f)$. Hence it suffices to compute this ratio for a specific choice of K_f. In many cases it is not so difficult to extend the group $G^{(1)}/\mathbb{Q}$ to a smooth group scheme $\mathcal{G}^{(1)}/\mathbb{Z}$ and amoung these extensions are some which are very nice. If for instance $G^{(1)}/\mathbb{Q}$ is split then we may extend $G^{(1)}/\mathbb{Q}$ to a Chevalley scheme over $\mathbb{Z}$ (See Chap. IV and also the next section). If we have such an extension $\mathcal{G}^{(1)}/\mathbb{Z}$ then we can do two things. On one hand we may choose the lattice $\mathfrak{g}^{(1)}{}_{\mathbb{Z}} = \mathrm{Lie}(\mathcal{G}^{(1)}/\mathbb{Z}) \subset \mathfrak{g}^{(1)}$, on the other hand we may define $K^0_f = K_f(\mathcal{G}) = \prod_p \mathcal{G}(\mathbb{Z}_p)$ as open compact subgroup. Then we see that we can attach a number

$$A_\infty(\mathcal{G}^{(1)}) = \frac{c_\infty(\mathrm{Lie}(\mathcal{G}^{(1)}/\mathbb{Z}))}{\mathrm{vol}_{\omega^{\mathrm{Tam}}_{G,f}}(K^0_f)}$$

It is a very general fact that two such extensions $\mathcal{G}^{(1)}/\mathbb{Z}, \mathcal{G}_1{}^{(1)}/\mathbb{Z}$ are equal over some non empty subset $U \subset \mathrm{Spec}(\mathbb{Z})$ and we may choose U so that $\mathcal{G}^{(1)} \times_{\mathrm{Spec}(\mathbb{Z})} \times U = \mathcal{G}^{(1)}/U$ is even semisimple. Then extending this semi-simple scheme into the primes $p \notin U$ essentially amounts to choosing a parahoric subgroup in $G^{(1)}(\mathbb{Q}_p)$ for all these primes. This amounts to choosing a face σ_p in a simplex of the Bruhat-Tits building for all these primes. To this choice we can attach an extension of the semi simple group scheme $\mathcal{G}^{(1)}/U$ to a smooth group scheme $\mathcal{G}^{(1)}/U(\{\sigma_p : p \notin U\})/\mathrm{Spec}(\mathbb{Z})$.

The group $G^{(1)}(\mathbb{Q}_p)$ acts upon the Bruhat-Tits building and

The constant

$$A_\infty(\mathcal{G}^{(1)}/U(\{\sigma_p\}_{p\notin U})$$

only depends on the orbit of $\{\sigma_p\}_{p\notin U}$ *under* $\prod_{p\notin U} G(\mathbb{Q}_p)$

3.3.2 A simple example

We consider the group $G/\mathbb{Q} = \mathrm{Sl}_2/\mathbb{Q}$, the Lie algebra over $\mathbb{Q}$ is generated by the matrices

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

These elements H, E_+, E_- form a basis of the Lie algebra of the Lie algebra of the semi simple group scheme $\mathrm{Sl}_2/\mathbb{Z}$. Hence we want to compute the constant $A_\infty(\mathrm{Sl}_2/\mathbb{Z})$.

We check easily that $B(H,H) = 8$ and $B(E_+, E_-) = 4$ and the other values are zero. Then $\frac{1}{8}H, \frac{1}{4}E_+, \frac{1}{4}E_-$ form a basis of the dual lattice and

$$\frac{1}{2^7} H \wedge E_+ \wedge E_-$$

yields the invariant form $\omega = \omega^{\mathfrak{g}_\mathbb{Z}}$.

Now we consider the Cartan involution Θ on $G(\mathbb{R})$, it is given by $g \mapsto^t g^{-1}$ and on the Lie algebra it is given by $U \mapsto -^tU$. (It is accidental and does not play any role that the Cartan involution is defined over $\mathbb{Q}$). We have $\Theta(H) = -H, \theta(E_+) = -E_-, \Theta(E_-) = -E_+$ and with $Y = E_+ - E_-, V = E_+ + E_-$ we get the Cartan decomposition of the Lie-algebra into the + and - eigenspace

$$\mathfrak{g}_\mathbb{R} = \mathbb{R}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \mathbb{R}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \mathbb{R}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \mathbb{R}Y \oplus \mathbb{R}V \oplus \mathbb{R}H = \mathrm{Lie}(K_\infty) \bigoplus \mathfrak{p}.$$

Our symmetric space $X = \mathbb{H} = G(\mathbb{R})/K_\infty$ has the distinguished point $e = i$, and under the differential of the projection $\pi : G(\mathbb{R}) \to X$ the subspace $\mathfrak{p}$ maps isomorphically to the tangent space at i. The Riemannian metric at the point i is given by $(U, U_1) \mapsto B(U, U_1)$. We have $B(V,V) = B(H,H) = 8$ and these to tangent vectors are orthogonal to each other.

Our volume form with respect to this new basis of $\mathfrak{g}^{(1)}$ will be given by

$$\frac{1}{2^8} Y \wedge V \wedge H$$

The element Y generates the Lie algebra of K_∞. The element $I = \exp(\frac{\pi}{2}Y)$ sends V to H and H to $-V$, it defines the complex structure on $\mathfrak{p}$ and an orientation: The basis V, H is positively oriented. We observe that K_∞ is the group of real points of a maximal torus $T/\mathbb{R}$ which is anisotropic, so condition (B) is satisfied. We have

$$T(\mathbb{R}) = K_\infty = \exp(tY) = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}.$$

The element Y defines a volume form ω_Y on this group and with respect to this volume form we have

$$\int_{K_\infty} \omega_Y = 2^4 \int_0^{2\pi} dt = 2^4\pi.$$

Hence we can write our volume form as

$$\frac{Y}{2^4\pi} \wedge \frac{\pi}{2^4}(V \wedge H)$$

and $\frac{\pi}{2^4}(V \wedge H)$ defines now an invariant volume form on $X = \mathbb{H}$, the factor relating it to the the Gauss-Bonnet form ω^{GB} is our constant $c_\infty(\mathfrak{g}_{\mathbb{Z}})$.

Of course at some point we have to "bite into the sour apple" and to open a book on differential geometry. We must find out how we can compute such a Gauss-Bonnet form. We learn that we have to compute the curvature tensor R of a suitable connection on our Riemannian manifold $\mathbb{H}$. The curvature tensor on a Riemannian manifold M at a point $x \in M$ is an element in

$$R_x \in \mathrm{Hom}_{\mathrm{alt},2}(T_{M,x}, \mathrm{End}_{\mathrm{alt}}(T_{M,x}))$$

hence it attaches to any pair U, V of tangent vectors at x an alternating (with respect to the Riemannian metric) endomorphism $R(U, U_1)$ of the tangent space at x. In our case we have tangent space $\mathfrak{p}$ at i, if we have two elements $U, U_1 \in \mathfrak{p}$, then their Lie bracket $[U, U_1] \in \mathrm{Lie}(K_\infty)$ and bracketing with this element gives an endomorphism of $\mathfrak{p}$. It is in [Ko-No] that this gives us the curvature tensor (See also the next section)

$$R(U, U_1)(V_1) = -2[[U, U_1], V_1]$$

The Gauss-Bonnet form ω^{GB} is now obtained by the following rule: We take a pair of orthonormal vectors V_1, V_2 of equal lenght which form a positively oriented basis of $\mathfrak{p}$. Then

$$R(U, U_1)V_1 = \kappa(U, U_1)V_2,$$

where κ is an alternating 2-form. The value of this form κ on the pair (V, H) is obtained from

$$R(V, H)V = \kappa(V, H)H = 8H,$$

hence we have $\kappa(V, H) = 4$. Then the curvature form is (look up the differential geometry book)

$$\omega^{GB} = (-1)^d \frac{1}{2^{2d}\pi^d}\kappa \text{ where } d = \frac{1}{2}\dim X(\text{ which in this case is } = 1)$$

We get

$$\omega^{GB}(V, H) = -\frac{2}{\pi}$$

On the other hand we have

$$\frac{\pi}{2^4}(V \wedge H)(V, H) = \frac{\pi}{2^4}B(V, V)B(H, H) = 4\pi$$

and we conclude that the value

$$c_\infty(\mathfrak{g}_{\mathbb{Z}}) = -\frac{1}{2\pi^2}$$

I want to say a few words on the second factor. We want to compute the volume of K_f with respect to the Tamagawa measure $\omega^{\mathrm{Tam}}_{G_f}$.

In our special case we can extend our group scheme $\mathrm{Sl}_2/\mathbb{Q}$ to a semi simple group scheme $\mathrm{Sl}_2/\mathbb{Z}$ (See Chap IV, we sometimes abbreviate $\mathrm{Spec}(\mathbb{Z})$ by $\mathbb{Z}$) The choice of such an extension is essentially equivalent to the choice of a standard maximal compact subgroup $\prod_p \mathrm{Sl}_2(\mathbb{Z}_p) = K_f^0 \subset G(\mathbb{A}_f)$. We assume that our open compact subgroup $K_f = \prod_p K_p$ where $K_p \subset K_p^0$ is defined by congruence conditions and for almost all p we have equality. Then we have of course

$$\int_{K_f} \omega_{G_f}^{\mathrm{Tam}} = \frac{1}{[K_f^0 : K_f]} \int_{K_f^0} \omega_{G_f}^{\mathrm{Tam}}$$

Now the finite component of the Tamagawa measure was obtained from the differential form $\omega^{\mathfrak{g}_\mathbb{Z}}$. The group scheme $\mathrm{Sl}_2/\mathbb{Z}$ has a Lie algebra and this is our lattice $\mathfrak{g}_\mathbb{Z}$. Our differential form is an alternating 3-form on $\mathfrak{g}_\mathbb{Z}$ and takes value 1 on the element (H, E_+, E_-). This means that the invariant form ω is a generator of $\Omega^3_{\mathrm{Sl}_2/\mathbb{Z}}(\mathrm{Sl}_2)$.

We can consider the full congruence subgroups $K_p^0(p^\nu)$ of matrices which are congruent to $1 \mod p^\nu$. The exponential map

$$\exp : p^\nu \mathfrak{g}_\mathbb{Z} \otimes \mathbb{Z}_p \to K_p^0(p^\nu)$$

which is defined by

$$p^\nu U \mapsto \mathrm{Id} + p^\nu U + \frac{1}{2!} p^{2\nu} U^2 \cdots = \exp(p^\nu U)$$

provides an isomorphism between $p^\nu \mathfrak{g}_\mathbb{Z} \otimes \mathbb{Z}_p$ and $K_p^0(p^\nu)$. Clearly the volume of $p^\nu \mathfrak{g}_\mathbb{Z} \otimes \mathbb{Z}_p$ with respect to the transported measure $\exp^*(\omega)$ is equal to $p^{-3\nu}$. Therefore we get by an elementary computation

$$\int_{K_p^0} \omega_p^{\mathrm{Tam}} = [K_p^0 : K_p^0(p^\nu)] p^{-3\nu} = p(p^2-1)p^{3\nu-3}p^{-3\nu} = (1 - \frac{1}{p^2})$$

We conclude that

$$\mathrm{vol}_{\omega_{G,f}^{\mathrm{Tam}}}(K_f^0) = \prod_p (1 - \frac{1}{p^2}) = \frac{1}{\zeta(2)}.$$

The formula for the orbifold Euler characteristic becomes

$$\chi_{\mathrm{orb}}(S_{K_f}^G) = -\frac{1}{2\pi^2}[K_f^0 : K_f]\zeta(2) = -\frac{1}{12}[K_f^0 : K_f]$$

This is not the Euler characteristic of the topological space $S_{K_f}^G$ unless K_f is small enough. This smallness condition is certainly fulfilled if $K_f \subset K_f^0(N)$ and $N \geq 3$. Another result of this calculation is

$$A_\infty(\mathrm{Sl}_2/\mathbb{Z}) = -\frac{1}{12}$$

3.3.3 The curvature calculation

To get these constants $c_\infty(\mathfrak{g}^{(1)}{}_\mathbb{Z})$, we may also use an argument which is called "Hirzebruch proportionality principle".

We consider the "compact twin" of our symmetric space $\mathbb{H}$: We start from the observation that the group $\mathrm{Sl}_2(\mathbb{R}) \subset \mathrm{Sl}_2(\mathbb{C})$ and that is the subgroup of elements fixed by conjugation, i.e. fixed by the action of the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. We define a twisted action of the Galois group where the complex conjugation c is replaced by $c \circ \Theta$. This defines a new group $G_c/\mathbb{R}$ whose real points are given by

$$G_c(\mathbb{R}) = \{g \in \mathrm{Sl}_2(\mathbb{C}) | g = (c \circ \Theta)g =^t \bar{g}^{-1}\}$$

and this is the unitary group $\mathrm{SU}(2)$. The group $K_\infty = \mathrm{SU}(1)$ sits in both groups. The "compact twin" is now

$$X_c = G_c(\mathbb{R})/K_\infty = \mathrm{SU}(2)/\mathrm{SU}(1) = S^2.$$

Now we have the distinguished point $\bar{e} \in X_c$ and the tangent space at this point is $\mathfrak{p} \otimes \sqrt{-1} \subset \mathfrak{g} \otimes \mathbb{C}$. We have an isomorphism between the tangent space $\mathfrak{p}$ at $i \in \mathbb{H}$ and the tangent space at $\bar{e} \in X_c$ which is given by $U \mapsto U \otimes i$. We also have a curvature form ω_c^{GB} on the compact twin and under the isomorphism of tangent spaces the curvature form ω^{GB} is mapped to $-\omega_c^{GB}$.

At this point we observe, that ω_c^{GB} can be defined as the unique invariant 2-form on X_c for which

$$\int_{X_c} \omega_c^{GB} = 2$$

it does not depend on any choice of a metric.

On the Lie algebra

$$\mathrm{Lie}(G_c/\mathbb{R}) = \mathfrak{g}_c = \mathbb{R}Y \oplus \mathbb{R}H \oplus \mathbb{R}V$$

we have the Killing form B which is negative definite, the above basis vectors are orthonormal and we have

$$B(Y, Y) = B(i \otimes H, \otimes H) = B(i \otimes V, i \otimes V) = -8$$

this suugests the euclidian metric

$$B_0 = -\frac{1}{8}B$$

on $\mathfrak{g}_c$. With respect to this metric the basis Y, V, H is orthonormal.

We consider the projection

$$G_c(\mathbb{R}) \xrightarrow{p} G_c(\mathbb{R})/K_\infty = X_c.$$

We now have an explicit identification of X_c with the 2-sphere S^2 (with respect to B_0) in $\mathfrak{g}_c$. This is obtained from the adjoint action of $G_c(\mathbb{R}) = SU(2)$ on $\mathfrak{g}_c$: The group $K_\infty = \{\exp(tY)\}$ is the stabilizer of $Y \in S^2$, the projection p can be written as

$$p : g \mapsto \mathrm{Ad}(g)(Y).$$

This yields the isomorphism of tangent spaces

$$D_p : \mathfrak{p} \xrightarrow{\sim} T_{X_c, \bar{e}},$$

which is equivariant with respect to the action of K_∞.
It is given explicitely by

$$i \otimes U \mapsto [i \otimes U, Y] = \mathrm{ad}(i \otimes U)(Y)$$

This tells us that

$$D_p(i \otimes H) = 2i \otimes V, D_p(i \otimes V) = -2i \otimes H$$

so it not an isometry, it is a conformal equivalence and the metric is multiplied by 4, i.e.

$$4B_0(i \otimes U, i \otimes V) = B_0(D_p(i \otimes U), D_p(i \otimes V))$$

We can identify the dual spaces of the two tangent spaces $T_{S^2,Y}$, (resp $\mathfrak{p} = T_{X_c,\bar{e}}$) to the tangent spaces themselves using the respective euclidian metric on them. Then the Gauss-Bonnet form at the point Y will be

$$\omega^{GB}_{S^2} = \frac{1}{2\pi}\, i \otimes V \wedge i \otimes H.$$

This simply expresses the fact that $i \otimes V \wedge i \otimes H$ can be viewed as the standard invariant volume form on the 2-sphere of radius 1 and this gives volume 4π. We can use D_p to transport this form to a form on $\mathfrak{p}$, then we get the Gauss-Bonnet form on $\mathfrak{p}$ and

$$\omega^{GB}_c(i \otimes U, i \otimes V) = \omega^{GB}_{S^2}(D_p(i \otimes U), D_p(i \otimes V)) = \frac{2}{\pi}$$

in other words

$$\omega^{GB}_c = \frac{2}{\pi}\, i \otimes H \wedge i \otimes V$$

Now the form B_0 gives us an invariant Riemannian metric on $\mathfrak{g}_c$ and the volume form defined by that metric is

$$\omega^{B_0} = Y \wedge i \otimes H \wedge i \otimes V$$

Then we get a volume form $\tilde{\omega}^{B_0}$ on X_c such that

$$\int_{G_c(\mathbb{R})} f(g)\omega^{B_0}(dg) = \int_{G_c(\mathbb{R})/K_\infty} \Big(\int_{K_\infty} f(gk)dk\Big)\tilde{\omega}^{B_0}(d\bar{g})$$

where the measure dk is normalized such that $\mathrm{vol}_{dk}(K_\infty) = 1$ for all continous functions f. The element Y defines a 1-form ω_Y on K_∞ and since Y has length 1 we get

$$\int_{K_\infty} 1 = \int_0^{2\pi} dt = 2\pi$$

Hence we can conclude that

$$\frac{1}{4\pi^2}\tilde{\omega}^{B_0} = \omega^{GB}_c$$

At this point I want to make an important remark. The choice of the Killing form and the form B_0 is somewhat arbitrary and not so relevant. What really counts is the conformal class of this form. The form ω_c^{GB} only depends on the conformal structure and can be obtained from the structure of the Lie algebra.

Finally I want to mention that we find a certain discrepancy with the results in [Ko-No], they do not have the factor 2 in front of the definition of the 2-form κ. This will be clarified later.

3.3.4 The computation of $A_\infty(\mathcal{G}^*)$

If $G^*/\mathbb{Q}$ is the algebraic group attached to the quaternion algebra $D/\mathbb{Q}$ which is non split at the infinite place and at the primes in Σ_0. If we want to tackle the problem formulated in the headline of this section, we have to choose an extension of $G^*/\mathbb{Q}$ to a smooth group scheme $\mathcal{G}^*/\mathbb{Z}$. Let $U = \mathrm{Spec}(\mathbb{Z}) \setminus \Sigma_0$. It is possible to extend $G^*/\mathbb{Q}$ to a semi-simple group scheme $\mathcal{G}_U^*/U$, two such extensions are isomorphic. Now we choose for all primes $p \in \Sigma_0$ an extension of $G^* \times_{\mathbb{Q}} \mathbb{Q}_p$ to a smooth group scheme $\mathcal{G}_p^*/\mathbb{Z}_p$. For simplicity I assume that $p \neq 2$. We recall the construction from Chap II.1.1.2. We choose a number $b \in \mathbb{Z}_p^*$ which is not a square, and we take $a = p$. Then $\mathbb{Q}_p(\sqrt{b})$ is unramified, let $O_L \subset L$ be the ring of integers. We know what the group scheme $\mathrm{Gl}_2/\mathbb{Z}_p$ is. This group scheme corresponds to a vertex in the Bruhat-Tits building. This vertex lies in a simplex σ_p of the building, this simplex corresponds to the Iwahori subgroup

$$\mathcal{I}_p = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Gl}_2(\mathbb{Z}_p) | c \equiv 0 \mod p\}$$

We know from the theory of Bruhat-Tits that we can extend $\mathrm{Gl}_2/\mathbb{Q}_p$ to a smooth group scheme $\mathcal{G}_{\sigma_p}/\mathbb{Z}_p$, such that

$$\mathcal{G}_{\sigma_p}(\mathbb{Z}_p) = \mathcal{I}_p.$$

The automorphism $\mathrm{Ad}(\begin{pmatrix} 0 & p \\ 1 & 0 \end{pmatrix})$ extends to an automorphism of $\mathcal{G}_{\sigma_p}/\mathbb{Z}_p$. It defines a 1-cocycle for the etale cohomology

$$H^1(O_L/\mathbb{Z}_p, \mathrm{Ad}(\mathcal{G}_{\sigma_p}/\mathbb{Z}_p))$$

and this cocycle yields a $\mathbb{Z}_p$ form $\mathcal{G}_{\sigma_p}^*/\mathbb{Z}_p$ of $\mathcal{G}_{\sigma_p}/\mathbb{Z}_p$.

These extensions can be glued to $\mathcal{G}_U^*/U$ and provide an extension $\mathcal{G}^*/\mathbb{Z}$ this is a smooth Bruhat-Tits group scheme. Now we can ask for the value of $A_\infty(\mathcal{G}^*)$.

The point is that we have a modified Bruhat-Tits group scheme structure $\mathcal{G}'/\mathbb{Z}$ on $\mathrm{Sl}_2/\mathbb{Q}$. This smooth group scheme a subscheme of $\mathrm{Sl}_2/\mathbb{Z}$. At the points $p \notin U$ we have $\mathcal{G}' \otimes_{\mathbb{Z}} \mathbb{Z}_p = \mathrm{Sl}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_p$, at the primes $p \in U$ we have $\mathcal{G}'(\mathbb{Z}_p) = \mathcal{I}_p$.

Now it is clear that $A_\infty(\mathcal{G}') = -A_\infty(\mathcal{G}^*)2^?$. But clearly $\mathrm{Lie}(\mathcal{G}'/\mathbb{Z})$ has index $\prod_{p \notin U} p$ in $\mathrm{Lie}(\mathrm{Sl}_2)/\mathbb{Z}$ and hence we get

$$c_\infty(\mathrm{Lie}(\mathcal{G}'/\mathbb{Z})) = -\frac{1}{2\pi^2} \prod_{p \notin U} p$$

On the other hand we have $[\prod_p \mathrm{Sl}_2(\mathbb{Z}_p) : \prod_p \mathcal{G}'(\mathbb{Z}_p)] = \prod_{p \notin U}(p+1)$ and hence

$$A_\infty(\mathcal{G}^*) = \frac{1}{12} \prod_{p \notin U} (1 + \frac{1}{p})$$

(This formula should be checked again)

3.3.5 Skip reading

The general Hirzebruch proportionality principle deals with the following situation: We assume that we have a semi simple group $G/\mathbb{R}$ and we also assume that $X = G(\mathbb{R})/K_\infty$ is a hermitian symmetric domain. Assume that we have a torsion free discrete group $\Gamma \subset G(\mathbb{R})$ which is cocompact, i.e. $\Gamma\backslash G(\mathbb{R})$ is compact. Then $\Gamma\backslash X$ is a compact complex manifold.

Parallel to this space we can consider a compact twin $Y = G_c(\mathbb{R})/K_\infty$, where $G_c/\mathbb{R}$ is a compact form of $G/\mathbb{R}$, this quotient is a homogenous projective algebraic variety. Again we can compare the tangent spaces at the points e, e_c in both symmetric spaces. We have the Cartan decompositions

$$\mathfrak{g} \otimes \mathbb{R} = \mathrm{Lie} \oplus \mathfrak{p}, \mathfrak{g}_c \otimes = \mathrm{Lie} \oplus \mathfrak{p} \otimes i,$$

the second summands are identified to the two tangent spaces respectively and this gives us the identification of the tangent spaces. Hence we can also identify their highest exterior powers.

We can construct holomorphic vector bundles $\mathcal{E}_\lambda, \mathcal{E}'_\lambda$ from (irreducible) representations λ of K_∞. We can apply the Hirzebruch-Riemann-Roch theorem to compute the Euler-characteristics

$$\chi(H^\bullet(\Gamma\backslash X, \mathcal{E}_\lambda)) = \sum_\nu (-1)^\nu \dim H^\nu(\Gamma\backslash X, \mathcal{E}_\lambda)$$

$$\chi(H^\bullet(Y, \mathcal{E}'_\lambda)) = \sum_\nu (-1)^\nu \dim H^\nu(Y, \mathcal{E}'_\lambda)$$

These Euler-characteristics can be expressed in terms of Chern numbers obtained from the Chern classes of the bundles and the tangent bundles (which are also obtained from representations of K_∞.)

Now the Chern classes are represented by differential forms $\omega_i \in H^{2i}(\Gamma\backslash X, \mathbb{C})$ and $\omega'_i \in H^{2i}(Y, \mathbb{C})$ and these form are computable in terms of curvature data and λ. A Chern number is an evaluation

$$\int_{\Gamma\backslash X} \omega_{i_1} \wedge \omega_{i_2} \wedge \cdots \wedge \omega_{i_r} = \omega_{i_1} \cdot \omega_{i_2} \cdot \cdots \cdot \omega_{i_r},$$

or

$$\int_Y \omega'_{i_1} \wedge \omega'_{i_2} \wedge \cdots \wedge \omega'_{i_r} = \omega'_{i_1} \cdot \omega'_{i_2} \cdot \cdots \cdot \omega'_{i_r},$$

where of course the degrees add up to $\dim(\Gamma\backslash X) = \dim Y$.

Let us call the expressions $\omega_{i_1} \wedge \omega_{i_2} \wedge \cdots \wedge \omega_{i_r}$ and $\omega'_{i_1} \wedge \omega'_{i_2} \wedge \cdots \wedge \omega'_{i_r}$, they are invariant diffential forms in top degree. Hence they are determined by their value at the distinguished points e, e_c. Hence we can compare them, because we identified the tangent spaces. It follows from the differential geometric procedure how the Chern classes are computed that there is a constant $a(G, G_c)$ such that these Chern forms are proportional, i.e.

$$\omega_{i_1} \wedge \omega_{i_2} \wedge \cdots \wedge \omega_{i_r} = a(G, G_c)\omega'_{i_1} \wedge \omega'_{i_2} \wedge \cdots \wedge \omega'_{i_r}.$$

Now we have the two Riemannian metrics, which also can be compared. They provide volume forms on X, Y and we get for the Chern numbers

$$\omega_{i_1} \cdot \omega_{i_2} \cdot \dots \cdot \omega_{i_r} = \frac{\mathrm{vol}(\Gamma\backslash X)}{\mathrm{vol}(Y)} a(G, G_c) \omega'_{i_1} \cdot \omega'_{i_2} \cdot \dots \cdot \omega'_{i_r}.$$

This implies by the Hirzebruch-Riemann-Roch theorem that

$$\chi(H^\bullet(\Gamma\backslash X, \mathcal{E}_\lambda)) = \frac{\mathrm{vol}(\Gamma\backslash X)}{\mathrm{vol}(Y)} a(G, G_c) \chi(H^\bullet(Y, \mathcal{E}'_\lambda)),$$

and this is the Hirzebruch proportionality principle.

This principle can be extended if we consider indices of certain elliptic operators on $\Gamma\backslash X$ and Y, the the index theorem of Atiyah-Singer gives the same conclusion.

Of course the result in the above example and the following example can be viewed as the determination of such a constant.

3.3.6 A general formula for the Euler characteristic for Chevalley schemes.

We consider a totally real field $F/\mathbb{Q}$, let $\mathcal{O}_F$ and a split semisimple simply connected group scheme $\mathcal{G}_0/\mathcal{O}_F$. It is by definiton obtained by base change from a Chevalley scheme $\mathcal{G}_{00}/\mathbb{Z}$. We look at its generic fiber $G_0 = \mathcal{G}_\mathcal{O} \times F$ and take the base restriction $G = R_{F/\mathbb{Q}}(G_0)$. Inside $G(\mathbb{A}_f)$ we choose as maximal compact subgroup $K_f^0 = \prod_\mathfrak{p} \mathcal{G}_0(\mathcal{O}_\mathfrak{p})$. Here $\mathfrak{p}$ runs over the finite places of F and $\mathcal{O}_\mathfrak{p}$ is the ring of integers in the completion of F at $\mathfrak{p}$. Then

$$\mathcal{S}^G_{K_f^0} = \mathcal{G}_0(\mathcal{O}_F)\backslash G(\mathbb{R})/K_\infty = \mathcal{G}_0(\mathcal{O}_F)\backslash \prod_v \mathcal{G}_{00}(\mathbb{R})/K^0_\infty.$$

A typical example is provided by the group $\mathrm{Sl}_2/\mathcal{O}_F$ and the resulting space is a Hilbert Blumenthal variety.

Now we assume that the real group of real points $\mathcal{G}_{00}(\mathbb{R})$ has a maximal torus $T(\mathbb{R})$ which is compact, we assume that our maximal compact group K^0_∞ contains $T(\mathbb{R})$. Let W_G be the Weyl group of $\mathcal{G}_{00}/\mathbb{Z}$ and W_K the Weyl group of $T(\mathbb{R})$ in K^0_∞, i.e the group $N_{K^0_\infty}(T(\mathbb{R}))/T(\mathbb{R})$. Let $n = [F : \mathbb{Q}]$ and r be the rank of $\mathcal{G}_{00}/\mathbb{Z}$. Let $m_1, \dots, m_r$ be the degrees of the invariant polynomials, then we have a very clean formula for the orbifold Euler characteristic

$$\chi_{\mathrm{orb}}(\mathcal{S}^G_{K_f^0}) = \frac{(\#W_G)^n}{2^{rn}(\#W_K)^n} \prod_{i=1}^{i=r} \zeta_F(1 - m_i)$$

This formula can be interpretet as computation of a Hirzebruch proportionality constant. If we consider the compact twin $Y = G_c(\mathbb{R})/K^0_\infty$ of our symmetric space $X = \mathcal{G}_{00}(\mathbb{R})/K^0_\infty$ then the Euler characteristic of it is

$$\chi(Y) = \frac{\#W_G}{\#W_K}$$

The two Euler characteristics

$$\chi_{\mathrm{orb}}(\mathcal{G}_0(\mathcal{O}_F)\backslash X^n) \text{ and } \chi(Y^n)$$

are given by two corresponding Chern numbers. Their ratio is

$$\frac{1}{2^{rn}}\prod_{i=1}^{i=r}\zeta_F(1-m_i)$$

and this is the Hirzebruch proportionality constant in this case.

I want to point out that this is really a beautiful formula: We want to compute a number and the data entering its definition are a totally real number field $F/\mathbb{Q}$ and a simply connected semi simple group Chevalley scheme. The Chevaley scheme depends only on a Dynkin diagram. This Dynkin diagram has a rank r and produces the numbers $m_1,\dots,m_r$. The number field produces the $\zeta-$ function, to get numbers from this ζ-function, we have to evaluate it. I think there is no simpler way to produce a number from these data than by the above formula.

Now we may have a situation where X and Y carry natural invariant complex structures, this means that X is hermitian symmetric and Y is a generalized flag variety.

Then the Hirzebruch proportionality principle does not work so cleanly because the quotient $\mathcal{S}^G_{K_f^0}$ has singularities and it is any quasi projective. We need to compactify it. Then the numbers

$$\chi(H^\bullet(\Gamma\backslash X,\mathcal{E}_\lambda))=\sum_\nu(-1)^\nu\dim H^\nu(\Gamma\backslash X,\mathcal{E}_\lambda)$$

can not be expressed only in terms of the Chern numbers, we get correction terms from the singularities and the boundary. I think that these correction terms may be of lower order of magnitude (in a certain sense) but they are very difficult to handle.

End skip reading

3.4 The contribution of an elliptic element

Let us assume that we have a Hecke operator h_f which is actually the characteristic function of a double coset

$$h_f=\mathrm{char}_{K_f a_f K_f}.$$

where we still assume that K_f is open compact and sufficiently small and $a_f\in G(\mathbb{A}_f)$

To get fixed points we have to start from an element $\gamma\in G(\mathbb{Q})_{\mathrm{ell}}$. This element has a set of fixed points

$$X^\gamma\subset X=G(\mathbb{R})/K_\infty.$$

The centralizer Z_γ is again a connected reductive group and satisfies our assumptions (A) and (B) above.

Our fixed point set is a finite union of orbits under $Z_\gamma(\mathbb{R})$.

(Example: Look at $\mathrm{Gl}_2(\mathbb{R})/K_\infty=H^+\cup H^-$, then we get for an elliptic regular element γ that $Z_\gamma(\mathbb{R})=\mathbb{C}^\times$. It has an isolated fixed point in H^+ and H^-. Hence we have two orbits.)

We choose $x_i \in X^\gamma$ for each orbit and we get a map

$$Z_\gamma(\mathbb{R})/K_\infty^{Z_\gamma}(x_i) \hookrightarrow X^\gamma$$

where $K_\infty^{Z_\gamma}(x_i)$ is the stabilizer of x_i. We can choose the x_i so that this group does not depend on i. The connected component of the identity $K_\infty^{Z_\gamma} \subset K^{Z_\gamma}(x_i)$ is a group of type $K_\infty^{Z_\gamma}$, this is the kind of subgroup we always select. We put

$$c_\infty(\gamma) = \frac{\# \text{ connected components of } X^\gamma}{[K_\infty^{Z_\gamma}(x_i) : K_\infty^{Z_\gamma}]}.$$

On X^γ we have the Gauss-Bonnet measure $\omega_{X^\gamma}^{GB}$ which provides a measure $\omega_{Z_\gamma}^{GB}$ on $Z_\gamma(\mathbb{R})/C_s^0(\mathbb{R})$ if we normalize the volume of $K_\infty^{Z_\gamma}/C_s^0(\mathbb{R})$ to one.

For $g_f \in G(\mathbb{A}_f)$ we define $K^{Z_\gamma}(g_f)$ to be the stabilizer

$$\{z_f \in Z_\gamma(\mathbb{A}_f) | z_f g_f K_f = g_f K_f\}$$

and since we assumed that K_f is small we get an embedding

$$Z_\gamma(\mathbb{Q}) \setminus \Big(Z_\gamma(\mathbb{R})/K_\infty^{Z_\gamma}(x_i) \times Z_\gamma(\mathbb{A}_f)/K_f^{Z_\gamma}(g_f)\Big) \hookrightarrow S_K^G$$

given by

$$z \mapsto (z_\infty x_i, z_f g_f).$$

This is now a union of connected components of fixed points of the Hecke h_f operator if and only if

$$g_f^{-1}\gamma g_f \in K_f a_f K_f = \text{ supp } (h_f).$$

Let us denote this set of fixed points by

$$S_{K_f(x_i,g_f)}^{Z_\gamma} = Z_\gamma(\mathbb{Q}) \setminus \Big(Z_\gamma(\mathbb{R})/K_\infty^{Z_\gamma}(x_i) \times Z_\gamma(\mathbb{A}_f)/K_f^{Z_\gamma}(g_f)\Big)$$

The contribution of this set of fixed points to the fixed point formula will is given by

$$\mathrm{tr}(\gamma|\mathcal{M})\chi\Big(S_{K_f(x_i,g_f)}^{Z_\gamma}\Big).$$

The Euler characteristic will now be computed by the general rules above. The group ${Z^{(1)}}_\gamma$ is the semisimple part of Z_γ and as above we define the constant $c_\infty({\mathfrak{z}_\gamma}^{(1)})$. Finally we put $C'_\gamma = Z_\gamma/{Z_\gamma}^{(1)}$ and get

$$\chi_{\text{orb}}\Big(S_{K_f(x_i,g_f)}^{Z_\gamma}\Big) = \frac{c_\infty({\mathfrak{z}_\gamma}^{(1)})c_\infty(\gamma)}{[\pi_0(C'_\gamma)(\mathbb{R}) : \pi_0(Z_\gamma(\mathbb{R}))]}\tau({Z_\gamma}^{(1)})h(C'_\gamma)\frac{1}{\mathrm{vol}_{\omega_{G,f}^{\mathrm{Tam}}}(K_f^{Z_\gamma}(g_f))}.$$

Let us abbreviate the factor in front by $C_\infty(\gamma)$.

We have to sum over the conjugacy classes and multiply by the orbital integral. We observe that with our choices of measures

$$\omega_{G,f}^{\mathrm{Tam}} = \omega_{Z_\gamma,f}^{\mathrm{Tam}}\, \omega_{Z_\gamma\backslash G,f}^{\mathrm{Tam}}.$$

If we now take into account that $K_f = \prod K_\ell$ then we get the following expression for the contribution of γ to the trace

$$\mathrm{tr}(\gamma|\mathcal{M})C_\infty(\gamma)\tau({Z_\gamma}^{(1)})h(C'_\gamma)\prod_\ell \frac{1}{\mathrm{vol}(\omega^{\mathrm{Tam}}_{G,\ell}(K_\ell))}\int_{Z_\gamma(\mathbb{Q}_\ell)\backslash G(\mathbb{Q}_\ell)} h_\ell(\bar{g}_\ell^{-1}\gamma\bar{g}_\ell)\omega^{\mathrm{Tam}}_{Z_\gamma\backslash G,f}(d\bar{g}_\ell) =$$

$$\mathrm{tr}(\gamma|\mathcal{M})C_\infty(\gamma)\tau({Z_\gamma}^{(1)})h(C'_\gamma)\frac{1}{\mathrm{vol}(\omega^{\mathrm{Tam}}_{G,f}(K_f))}\prod_\ell \int_{Z_\gamma(\mathbb{Q}_\ell)\backslash G(\mathbb{Q}_\ell)} h_\ell(\bar{g}_\ell^{-1}\gamma\bar{g}_\ell)\omega^{\mathrm{Tam}}_{Z_\gamma\backslash G,f}(d\bar{g}_\ell)$$

Comment It may seem that we have rewritten the contribution of the conjugacy class γ in a much to complicated way. We observe that the whole expression must be a rational number, but we will see that some of the factors contain powers of π which therefore must cancel out. We will see further down, that the factor $C_\infty(\gamma)$ contains powers of π which are in a certain sense of differential geometric origin. (Nobody is surprised to see powers of π in the formula for the volume of a sphere.) But also the denominator $\mathrm{vol}(\omega^{\mathrm{Tam}}_{G,f}(K_f))$ has some powers of π in it which come from the values of the Riemann $\zeta-$ function at even integers. And also the product of orbital integrals will produce powers of π which are also of arithmetic origin. All these powers cancel out.

The reason for complicated way of writing is, that we can understand the variation of the factors in front of the product of orbital integrals if we vary γ in its geometric conjugacy class. If we accept the fact that the Tamagawa number $\tau({Z_\gamma}^{(1)}) = 1$ then it is clear that all factors are constant on the geometric conjugacy class, except the factor $C_\infty(\gamma)$. It will be shown in the next section that this factor can change by a sign, but miraculously this sign change can be "pushed" into the product of local orbital integrals and is absolutely necessary to make them stable.

3.5 The contribution from the cusps

3.5.1 An application

We still assume that . and (B) are true, otherwise the following considerations are uninteresting.

We pick a prime p and a discrete series representation π_p of $G(\mathbb{Q}_p)$. Let K_p be a level for π_p (See ???) we choose a level $K_f^{\{p\}}$ outside of p and put $K_f = K_p \times K_f^{\{p\}}$. Then we consider $H^\bullet(S^G_{K_f}, \tilde{\mathcal{M}})$ as a module for the Hecke algebra $\mathcal{H}_{K_p}$, i.e. we consider only the factor at p. We can ask for the multiplicity $m^q(\pi_p)$ in $H^\bullet(S^G_{K_f}, \tilde{\mathcal{M}})$ and define

$$\chi_{\pi_p}(H^q(S^G_{K_f}, \tilde{\mathcal{M}})) = \sum_q (-1)^q m(\pi_p).$$

This number is of course equal to

$$\frac{1}{\dim(\pi_p^{K_p})}\sum_q (-1)^q \dim H^q(S^G_{K_f}, \tilde{\mathcal{M}})(\pi_p)$$

We choose a suitable function in the Hecke algebra, namely $h = < v_i, g\phi_i > \times\chi_{K_f^{\{p\}}}$ where the first factor is a matrix coefficient of level K_p. We compute the trace

$$\mathrm{tr}(T_h|H^\bullet(S^G_{K_f},\tilde{\mathcal{M}}).$$

Of course we get from the orthogonality relations that this trace is equal to

$$\frac{1}{d(\pi_p)}\chi_{\pi_p}(H^\bullet(S^G_{K_f},\tilde{\mathcal{M}})).$$

On the other hand we can apply the trace formula. We choose $K_f^{\{p\}}$ so small, that the only elliptic term is given by the unit element e, therefore the right hand side is

$$\dim(\mathcal{M})C_\infty(e)\tau(G^{(1)})h(C'_e)\frac{1}{\mathrm{vol}(\omega^{\mathrm{Tam}}_{G,f}(K_f))}+\sum_P(-1)^{d(P)+1}\mathrm{tr}(T_h^P|\partial S^G_{K_f},\tilde{\mathcal{M}})).$$

Now our assumptions (A) and (B) imply that the factor $C_\infty(e)\neq 0$. We see that the elliptic contribution is equal to a non zero constant times the index of K_f in a reference level subgroup $K_f^{(0)}$. On the other hand it is not difficult to see that the truncated operators T_h^P are of the form $h_p^P\times\chi_{P(\mathbb{A}_f^{\{p\}})\cap K_f^{\{p\}}}$ and then it is clear that the growth rate for $\mathrm{tr}(T_h^P|\partial S^G_{K_f},\tilde{\mathcal{M}}))$ is a constant c_P times the index of $P(\mathbb{A}_f^{\{p\}})\cap K_f^{\{p\}}$ in $P(\mathbb{A}_f^{\{p\}})\cap K_f^{0,\{p\}}$.

Hence we can conclude that the contribution from the elliptic element dominates the $\mathcal{O}$ expansion, and we find that for $K_f^{0,\{p\}}$ small enough, the Euler characteristic $\mathrm{tr}(T_h|H^\bullet(S^G_{K_f},\tilde{\mathcal{M}})\neq 0$.

3.5.2 Another application from global to local

This is essentially the adelic version of Chap VI ???

3.6 The stabilization (modulo fundamental lemma)

As we did in the case Sl_2 we want to stabilize the trace formula and this means that we want to rearrange the terms such that the outer sum becomes a sum over the set rational geometric conjugacy classes. Let us also assume that the connected component of the identity of the center of our group G is a split torus, i.e. of the form G_m^s. Then our torus C' is also split and we have a generalized determinant map $\mu: G\xrightarrow{\mu}C'$

We assume that $G^{(1)}$ is simply connected we have the dominant fundamental weights $\omega_1,\dots,\omega_r$ which give us the fundamental $G^{(1)}$ modules $V_{\omega_1},\dots,V_{\omega_r}$. We can extend the action of $G^{(1)}$ on these modules to an action of G. Then we get a generalized trace-determinant map

$$(\mathrm{tr},\mu): G\to\mathbb{A}^r\times C'=\mathbb{A}^G$$

which is defined by

$$(\mathrm{tr},\mu): g\mapsto(\mathrm{tr}|V_{\omega_1},\dots,\mathrm{tr}|V_{\omega_r},\mu(g)).$$

Over an algebraically closed field k two semisimple elements $g_1,g_2\in G(k)$ are conjugate if and only if $(\mathrm{tr},\mu)(g_1)=(\mathrm{tr},\mu)(g_2)$. We define $(\mathbb{A}^r\times C')(\mathbb{Q})_{\mathrm{ell}}$

to be the set of elements for which the resulting semi-simple conjugacy class is elliptic. For any element $a \in \mathbb{A}^r \times C'(\mathbb{Q})$ let C_a be the geometric conjugacy class of elements in the fiber of $(\mathrm{tr}, \mu)^{-1}(a)$. We observe that the trace $\mathrm{tr}(\gamma|\mathcal{M})$ only depends on the geometric conjugacy class determined by γ. The elliptic terms of the topological trace formula are given by a sum

$$\sum_{(\mathbb{A}^r \times C')(\mathbb{Q})_{\mathrm{ell}}} \frac{\mathrm{tr}(\gamma|\mathcal{M})}{\mathrm{vol}(\omega_f^{G,\mathrm{Tam}}(K_f))} \sum_{\gamma \in C_a(\mathbb{Q})/\sim} \chi_\infty(\gamma)\tau({Z_\gamma}^{(1)})h(C'_\gamma)\times$$
$$\prod_\ell \int_{Z_\gamma(\mathbb{Q}_\ell)\backslash G(\mathbb{Q}_\ell)} h_\ell(\bar{g}_\ell^{-1}\gamma\bar{g}_\ell)\omega^{\mathrm{Tam}}_{Z_\gamma\backslash G,f}(d\bar{g}_\ell).$$

The factor $\chi_\infty(\gamma)$ is obtained as a ratio of Gauss-Bonnet and other measures, it is a differential geometric quantity and depends only on $G \times \mathbb{R}$. The two next factors are Tamagawa numbers and class numbers and they depend only the geometric conjugacy class a. We rewrite the expression: We drop the summation over a and forget some of the factors only depending on a

$$\mathrm{tr}(\gamma|\mathcal{M}) \sum_{\gamma \in C_a(\mathbb{Q})/\sim} \chi_\infty(\gamma) \prod_\ell \int_{Z_\gamma(\mathbb{Q}_\ell)\backslash G(\mathbb{Q}_\ell)} h_\ell(\bar{g}_\ell^{-1}\gamma\bar{g}_\ell)\omega^{\mathrm{Tam}}_{Z_\gamma\backslash G,f}(d\bar{g}_\ell)$$

If we choose $\gamma \in C_a(\mathbb{Q})$ then is well known and easy to see that

$$C_a(\mathbb{Q})/\sim = \ker(H^1(\mathbb{Q}, Z_\gamma) \longrightarrow H^1(\mathbb{Q}, G)) = \mathcal{D}H^1(\mathbb{Q}, Z_\gamma).$$

This is quite general and also true for an arbitrary ground field. Later on we will apply it to the groups over local fields.

We put $Z'_\gamma = Z_\gamma \cap G^{(1)}$. (This is of course not necessarily the derived group of Z_γ.) We have the diagram

$$\begin{array}{ccc}
H^1(\mathbb{Q}, Z'_\gamma) & \xrightarrow{j_1} & H^1(\mathbb{Q}, G^{(1)}) \\
\downarrow & & \downarrow \\
H^1(\mathbb{Q}, Z_\gamma) & \xrightarrow{j} & H^1(\mathbb{Q}, G) \\
\downarrow & & \downarrow \\
H^1(\mathbb{Q}, Z_\gamma/Z'_\gamma) & \xrightarrow{\sim} & H^1(\mathbb{Q}, G/G^{(1)})
\end{array}$$

Hence we get

$$\mathcal{D}H^1(\mathbb{Q}, Z_\gamma) = \mathrm{Im}((j_1)^{-1}(\ker(H^1(\mathbb{Q}, G^{(1)})) \to H^1(\mathbb{Q}, G)).$$

The Hasse principle yields $H^1(\mathbb{Q}, G^{(1)}) \xrightarrow{\sim} H^1(\mathbb{R}, G^{(1)})$. We have the surjective map $C'(\mathbb{Q}) \to \pi_0(C'(\mathbb{R}))$ and then we get easily

$$\ker(H^1(\mathbb{Q}, G^{(1)})) \to H^1(\mathbb{Q}, G)) = \ker(H^1(\mathbb{R}, G^{(1)})) \to H^1(\mathbb{R}, G)) = \pi_0(C'(\mathbb{R})/\pi_0(G(\mathbb{R}))$$

hence

$$\mathcal{D}H^1(\mathbb{Q}, Z_\gamma) = \mathrm{Im}(H^1(\mathbb{Q}, Z'_\gamma) \to H^1(\mathbb{Q}, Z_\gamma)) \cap \ker(H^1(\mathbb{Q}, Z_\gamma) \to H^1(\mathbb{R}, G)).$$

3.7 Excursion into Galois-cohomology:

Let us consider a connected reductive group $H/\mathbb{Q}$, let $H^{(1)}$ be its derived subgroup. We assume that the real group $H^{(1)} \times \mathbb{R}$ contains a maximal compact Cartan subgroup.

We have the sequence

$$1 \longrightarrow H^{(1)} \longrightarrow H \longrightarrow H/H^{(1)} \longrightarrow 1$$

and this sequence yields the diagram

$$\begin{array}{ccccc} H^1(\mathbb{Q}, H^{(1)}) & \rightarrow & H^1(\mathbb{Q}, H) & \rightarrow & H^1(\mathbb{Q}, H/H^{(1)}) \\ \downarrow & & \downarrow & & \downarrow \\ H^1(\mathbb{R}, H^{(1)}) & \rightarrow & H^1(\mathbb{R}, H) & \rightarrow & H^1(\mathbb{R}, H/H^{(1)}) \end{array}$$

We define $H^1_\infty(\mathbb{Q}, H)$ to be the subset of classes which are trivial at the infinite place. We have the following Lemma(Borovoi)

Lemma:
We can define a composition

$$H^1_\infty(\mathbb{Q}, H) \times H^1(\mathbb{Q}, H) \longrightarrow H^1(\mathbb{Q}, H).$$

This composition defines the structure of an abelian group on $H^1_\infty(\mathbb{Q}, H)$, and $H^1(\mathbb{Q}, H)$ is a finite union of "cosets" under this operation, the set of cosets maps bijectively to $H^1(\mathbb{R}, H)$. The cosets are principal homogeneous spaces.

To see this we use the old trick of Kneser. Let us pick a class $\xi \in H^1(\mathbb{Q}, H)$, let $Supp(\xi)$ be the finite set of places where ξ is not zero. We find a maximal torus $T'/\mathbb{Q} \subset H^{(1)}$ which is anisotropic at infinity and at all places in $Supp(\xi)$.

This torus lies in a maximal torus $T/\mathbb{Q}$ of $H/\mathbb{Q}$, and we get the diagram

$$\begin{array}{ccccccccc} 1 & \rightarrow & T' & \rightarrow & T & \rightarrow & H/H^{(1)} & \rightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow \wr & & \\ 1 & \rightarrow & H^{(1)} & \rightarrow & H & \rightarrow & H/H^{(1)} & \rightarrow & 1 \end{array}$$

We want to show that our given element ξ lies in the image of $H^1(\mathbb{Q}, T) \to H^1(\mathbb{Q}, H)$ and that we may even find an element $\tilde{\xi}$ in the preimage that maps to zero at the infinite place if ξ is in $H_\infty(\mathbb{Q}, H)$, i.e. $\tilde{\xi} \in H_\infty(\mathbb{Q}, T)$.

The torus T' has the property that it satisfies the Hasse-principle in degree 2, and we have

$$H^2(\mathbb{Q}_v, T') = 0$$

for all places v in the support of $Supp(\xi)$. This implies that the image $\bar{\xi}$ of ξ in $H^1(\mathbb{Q}, H/H^{(1)})$ goes to zero under the boundary map

$$H^1(\mathbb{Q}, H/H^{(1)}) \longrightarrow H^2(\mathbb{Q}, T')$$

hence it lifts to an element $\tilde{\xi}'$ in $H^1(\mathbb{Q}, T)$.

If we send this element to $H^1(\mathbb{Q}, H)$ it will not necessarily be mapped to ξ. We want to look at the difference of the image of $\tilde{\xi}'$ and ξ. Therefore we twist by $\tilde{\xi}$ we get a twisted form of $H/\mathbb{Q}$ which will be denoted by the same letter. In this group $\tilde{\xi}'$ becomes the neutral element and ξ maps to zero in $H^1(\mathbb{Q}, H/H^{(1)})$. We lift it to an element $\eta \in H^1(\mathbb{Q}, H^{(1)})$. This element is now in the image of $H^1(\mathbb{Q}, T^{(1)}) \to H^1(\mathbb{Q}, H^{(1)})$ by the same argument as before. This gives the first part of the assertion.

To prove the second part, we start from the observation that our torus $H/H^{(1)}$ is compact at the infinite place (this is part of our assumption) and hence the map $H^1(\mathbb{R}, H^{(1)}) \to H^1(\mathbb{R}, H)$ is injective and the same holds if we replace $H/\mathbb{Q}$ by the torus. With the same reasoning as before we see, that we have to fulfill the second requirement only if we have already our class $\eta \in H^1(\mathbb{Q}, H^{(1)})$. We introduce the universal cover $\tilde{H}^{(1)}/\mathbb{Q}$, and we consider the exact sequences

$$1 \to \mu \to \tilde{H}^{(1)} \to H^{(1)} \to 1$$

$$1 \to \mu \to \tilde{T}^{(1)} \to T^{(1)} \to 1$$

We lifted η to an element η^T in $H^1(\mathbb{Q}, \tilde{T}^{(1)}$. The restriction to the infinite place of the boundary of η in $H^2(\mathbb{Q}, \mu)$ is zero. Hence the element η^T_∞ lifts to a class $\tilde{\eta}^T_\infty$. A theorem of Tate asserts that this class $\tilde{\eta}^T_\infty$ is the restriction of a class $\tilde{\eta}^T \in H^1(\mathbb{Q}, \tilde{T}^{(1)})$. We modify η^T by subtracting the image of $\tilde{\eta}^T$ from it. Then this modified class is trival at infinity and we see easily that it has the same image in $H^1(\mathbb{Q}, \tilde{H}^{(1)})$.

I suggest to call this method the approximation of the cohomology by the cohomology of tori. It yields of course that any finite number of classes in $H^1(\mathbb{Q}, H)$ can be lifted simultaneously to the cohomology of a torus.

If we have two classes ξ, ξ' and one of them is in $H_\infty(\mathbb{Q}, H)$ then we lift both to the cohomology of a suitable torus and we require the right one lifts to a trivial class at infinity. We add the lifted classes in the torus and define the sum to be the image of the sum.

We introduce the notation

$$H^1(\mathbb{Q}, H \times \mathbb{A}_f) = \prod_p H^1(\mathbb{Q}_p, H).$$

Here the product is restricted, almost all components are trivial. It is clear that our method of approximation yields a structure of an abelian group on $H^1(\mathbb{Q}, H \times \mathbb{A}_f)$.

The same reasoning also gives us that we have a bijection

$$H^1_\infty(\mathbb{Q}, H) \xrightarrow{\sim} H^1_\infty(\mathbb{Q}, H/H^{(1)})$$

3.8 The sign

We look at a class C_a which has a rational point, i.e. $C_a(\mathbb{Q}) \neq \emptyset$ and we choose an element $\gamma \in C_a(\mathbb{Q})$. Now the other $G(\mathbb{Q})$ conjugacy classes in $C_a(\mathbb{Q})$ correspond one-to one to the elements in $\mathcal{D}H^1(\mathbb{Q}, Z_\gamma)$, for an element in $\xi \in \mathcal{D}H^1(\mathbb{Q}, Z_\gamma)$ I introduce the notation $\gamma + \xi$ for the resulting $G(\mathbb{Q})$-conjugacy class, I also choose a representative for this class and denote by the same letter. Then $C_a(\mathbb{Q})$ is the union of orbits, which are equal to $Z_{\gamma+\xi}(\mathbb{Q})\backslash G(\mathbb{Q})$.

We want to investigate the function in the variable ξ

$$\xi \mapsto \frac{c_\infty(\mathfrak{z}_{\gamma+\xi}{}^{(1)})c_\infty(\gamma+\xi)}{[\pi_0(C'_{\gamma+\xi})(\mathbb{R}) : \pi_0(Z_{\gamma+\xi}(\mathbb{R}))]} = \chi_\infty(\gamma+\xi)$$

. This is not entirely easy. We need the comparison between the Gauss-Bonnet measure and the measure induced by the Killing form. To do this we refer to the computation in my paper on the Gauss-Bonnet formula. I claim (??) that this factor is equal to a product of a positive number $W_\infty(\gamma+\xi) = W_\infty(\gamma)$ which does not depend on ξ, and a sign factor. If X^γ is the set of fixed points of γ we define:

$$W_\infty(\gamma)(-1)^{\frac{\dim X^{\gamma+\xi}}{2}} = W_\infty(a)(-1)^{\frac{\dim X^{\gamma+\xi}}{2}},$$

here the last notation indicates that W_∞ is a function of the conjugacy class.

The sign factor $\frac{\dim X^{\gamma+\xi}}{2}$ depends on our initial choice of γ but after that it only depends on the restriction $\xi_\infty = \xi|H^1(\mathbb{R}, Z_\gamma)$.

The group Z_γ is reductive over $\mathbb{Q}$ and quasi-split at almost all finite places. For each prime p let us put

$$d_p(\gamma) = \text{ split rank of } Z_\gamma \times \mathbb{Q}_p - \text{split rank of the quasi-split form of } \mathbb{Z}_\gamma.$$

We have another lemma which is due to Kottwitz:

The sign

$$(-1)^{\frac{\dim X^{\gamma+\xi}}{2} + \sum d_p(\gamma+\xi)} = \epsilon(< \gamma >) = \epsilon(a)$$

does not depend on $\xi \in \mathcal{D}H^1(\mathbb{Q}, Z_\gamma)$.

We put $\epsilon_p(\gamma) = (-1)^{d_p(\gamma)}$ and $\epsilon_\infty(\gamma) = (-1)^{\frac{\dim X^\gamma}{2}}$. If γ is regular, then all these signs are $+1$.

3.8.1 The first step towards stabilization

We make a change of notation, we will have to work with character modules of tori. The elements in there are named by greek letters $\alpha, \beta, \gamma, \dots$ and therefore we change the notation γ_a into x_a and so on.

We come to the process of rearranging the sum in the $\mathcal{O}$-expansion. For any $a \in \mathbb{A}^r(\mathbb{Q})$ we choose a representative $x_a \in C_a(\mathbb{Q})$ (if possible) and then the contribution of this conjugacy class (we forget the factors which depend only on a

$$\sum_{\xi \in \mathcal{D}H^1(\mathbb{Q}, Z_{x_a})} \epsilon_\infty(x_a+\xi)\mathrm{tr}(x_a|\mathcal{M}) \prod_\ell \int_{Z_{x_a+\xi}(\mathbb{Q}_\ell)\backslash G(\mathbb{Q}_\ell)} \epsilon_\ell(x_a+\xi)_\ell h_\ell(\bar{g}_\ell^{-1}(x_a+\xi)\bar{g}_\ell)\omega^{\mathrm{Tam}}_{Z_{\gamma+\xi}\backslash G,f}(d\bar{g}_\ell)$$

The goal of the process of stabilization is to manipulate the sum so that it will again be a finite sum of terms which are products of local integrals. Basically this means that we want to push the inner sum inside the product.

We consider the map

$$\mathcal{D}H^1(\mathbb{Q}, Z_{\gamma_a}) \to H^1(\mathbb{Q}, Z_{\gamma_a} \times \mathbb{A}_f) = \bigoplus_\ell H^1(\mathbb{Q}_\ell, Z_{\gamma_a})$$

and we know that $\mathcal{D}H^1(\mathbb{Q}, Z_{\gamma_a})$ consists of those classes which under restriction to $\mathbb{R}$ go to $\mathcal{D}H^1(\mathbb{R}, Z_{\gamma_a})$. More precisely we can say that

$$\mathcal{D}H^1(\mathbb{Q}, Z_{\gamma_a}) = \bigcup_{\eta \in \mathcal{D}H^1(\mathbb{R}, Z_\gamma)} \eta + H^1_\infty(\mathbb{Q}, Z_{\gamma_a})$$

where we have to choose representatives. Recall that we defined $C'_{x_a} = Z_{\gamma_a}/Z_{\gamma_a}{}^{(1)}$.. We get from 2.3.1 and Kneser's theorem that

$$\begin{array}{ccc} H^1(\mathbb{Q}, Z_{\gamma_a} \times \mathbb{A}_f) & \xrightarrow{\sim} & H^1(\mathbb{Q}, C'_{x_a} \times \mathbb{A}_f) \\ \uparrow & & \uparrow \\ H^1_\infty(\mathbb{Q}, Z_{\gamma_a}) & \xrightarrow{\sim} & H^1_\infty(\mathbb{Q}, C'_{x_a}) \end{array}$$

The module of cocharacters $X_*(C'_{x_a})$ is a free $\mathbb{Z}$-module with an action of a finite quotient $\Gamma = \mathrm{Gal}(K/\mathbb{Q})$ on it. This group Γ contains a complex conjugation $c \in \Gamma$ which acts by multiplication by -1 on $X_*(C'_{x_a})$. Let I_Γ be the augmentation ideal in the group ring of Γ. Then we have the exact sequence resulting from Tate-Nakayama

$$0 \to \underline{|\,|\,|}(C'_{x_a}) \to H^1(\mathbb{Q}, C'_{x_a}) \to H^1(\mathbb{Q}, C'_{x_a} \times \mathbb{A}) \to X_*(C'_{x_a})/I_\Gamma(X_*(C'_{x_a})) \to 0$$

the group on the left is the Tate Shafarewic group and the last arrow on the the right is surjective because $H^2(\mathbb{Q}, C'_{x_a})$ satisfies the Hasse-principle. We decompose

$$H^1(\mathbb{Q}, C'_{x_a} \times \mathbb{A}) = H^1(\mathbb{R}, C'_{x_a}) \times H^1(\mathbb{Q}, C'_{x_a} \times \mathbb{A}_f)$$

and we have a surjection

$$H^1(\mathbb{R}, C'_{x_a}) = X_*(C'_{x_a})/I_{<c>}X_*(C'_{x_a}) = X_*(C'_{x_a})/2X_*(C'_{x_a}) \to X_*(C'_{x_a})/I_\Gamma(X_*(C'_{x_a}))$$

We now define a function $J(x_a, y) : X_*(C'_{x_a})/I_\Gamma(X_*(C'_{x_a})) \to \mathbb{Z}$ by

$$J(x_a, y) = \sum_{\eta \in \mathcal{D}H^1(\mathbb{R}, Z_{x_a}) | \eta \mapsto y} \epsilon_\infty(x_a + \eta)$$

(If our element x_a is regular, then this function depends only on a it counts the number of elements in the preimage of $y \in X_*(C'_{x_a})/I_\Gamma(X_*(C'_{x_a}))$ in $\mathcal{D}H^1(\mathbb{R}, Z_{x_a})$, since $\mathcal{D}H^1(\mathbb{R}, Z_{x_a})$ is not a group this function may be weird.)

We introduce the group of characters

$$\Xi_a = \mathrm{Hom}(X_*(C'_{x_a})/I_\Gamma(X_*(C'_{x_a})), \mu_2)$$

and define the Fourier transform

$$\hat{J}(x_a, \kappa) = \frac{1}{\#\Xi_a} \sum J(x_a, y)\kappa(y)$$

(If we consider for instance the case, where $G \times \mathbb{R}$ is anisotropic, then the function $J(x_a, y) = 1$ for $y = 0$ and it takes value 0 otherwise. This results from the fact that in this case $\mathcal{D}H^1(\mathbb{R}, Z_{x_a})$ is the set consisting of the neutral element. In this case we find that $\hat{J}(x_a, \kappa) = \frac{1}{\#\Xi_a}$.)

The characters κ induce characters on

$$\kappa : H^1(\mathbb{Q}, Z_{\gamma_a} \times \mathbb{A}) = \bigoplus_v H^1(\mathbb{Q}_\ell, Z_{\gamma_a}) \to \mu_2$$

and as such they are product of local characters, i.e. $\kappa = \kappa_\infty \times \kappa_f = \kappa_\infty \times \prod_\ell \kappa_\ell$.

Recall that we have choosen an element $\gamma \in C_a(\mathbb{Q})$ (if possible), we get an isomorphism of affine varieties

$$C_a \times \mathbb{Q}_\ell \xrightarrow{\sim} Z_{\gamma_a} \backslash G \times \mathbb{Q}_\ell$$

The orbits of $G(\mathbb{Q}_\ell)$ on $C_a(\mathbb{Q}_\ell)$ form a torsor under the elements in $H^1(\mathbb{Q}_\ell, C'_{x_a})$. This torsor is has a neutral element given by x_a. An this set of orbits we have the functions ϵ_ℓ and $\kappa_\ell^{\gamma_a}(\gamma_a + \xi_\ell) = \kappa_\ell(\xi_\ell)$. The function $\kappa_\ell^{\gamma_a}$ depends on the choice of γ_a. Then it is clear

$$\sum_{\xi \in \mathcal{D}H^1(\mathbb{Q}, Z_\gamma)} \prod_\ell \int_{Z_{\gamma+\xi}(\mathbb{Q}_\ell) \backslash G(\mathbb{Q}_\ell)} \epsilon_\ell(x_a + \xi)_\ell h_\ell(\bar{g}_\ell^{-1}(x_a + \xi)\bar{g}_\ell) \omega^{\mathrm{Tam}}_{Z_{x_a+\xi} \backslash G, f}(d\bar{g}_\ell) =$$

$$\sum_{\kappa \in \Xi_a} \hat{J}(x_a, \kappa) \prod_\ell \int_{C_a(\mathbb{Q}_\ell)} \kappa(x_\ell)\epsilon_\ell(x)_\ell h_\ell(x_\ell) \omega^{\mathrm{Tam}}_{C_a,\ell}(d\bar{x}_\ell)$$

We get a new expression for the contribution of the class a

$$\mathrm{junk}(a)\epsilon(x_a) \times \sum_{\kappa \in \Xi_a} \hat{J}(x_a, \kappa) \mathrm{tr}(x_a|\mathcal{M}) \prod_\ell \int_{C_a(\mathbb{Q}_\ell)} \kappa_\ell(x_\ell)\epsilon_\ell(x)_\ell h_\ell(x_\ell) \omega^{\mathrm{Tam}}_{C_a,\ell}(d\bar{x}_\ell)$$

Here the term with $\kappa = 1$ is the stable term, the other terms are the instable contributions.

The goal is to rewrite the instable orbital integrals as stable integrals over endoscopic groups and this needs the fundamental lemma.

We have to understand the contributions

$$\hat{J}(x_a, \kappa) \mathrm{tr}(x_a|\mathcal{M}) \prod \int_{C_a(\mathbb{Q}_\ell)} \kappa_\ell^{\gamma_a}(x_\ell)\epsilon_\ell(x_\ell) h_\ell(x_\ell) \omega^{\mathrm{Tam}}_{C_a,\ell}(d\bar{x}_\ell),$$

we manipulate them in the same way as we did in the case Sl_2.

We choose a system $\Delta^+ \subset X^*(T)$ of positive roots, this system is of course not invariant under the action of the Galois group. Let λ be the highest weight of an irreducible module $\mathcal{M}$. Since we assumed that or form is inner, we can realise a certain multiple of it as a representation over $\mathbb{Q}$, but then this means we can ignore this subtletey and assume that the representation itself is defined over $\mathbb{Q}$.

The character formula gives for regular x

$$\mathrm{tr}(\gamma|\mathcal{M}) = \frac{\Sigma \mathrm{sgn}(w) \cdot (\lambda + \rho)(x)}{\prod_{\alpha \in \Delta^+}((\frac{\alpha}{2}(x)) - (\frac{\alpha}{2}(x)^{-1}))}.$$

For non regular γ we have to take a limit. We have to think for a second: We have a normal field extension $\mathbb{Q} \subset L \subset \mathbb{Q}$, which splits the torus, then then numbers $(\lambda+\rho)(wx)$ and $\prod_{\alpha\in\Delta^+}((\frac{\alpha}{2}(x))-(\frac{\alpha}{2}(x)^{-1}))=\rho(x)^{-1}\prod_{\alpha\in\Delta^+}(\alpha(x)-1)$ lie in L. (Our group is simply connected). If we take only partial products in the denominator, then we have to pay attention to the resulting half sum of roots.

The endoscopic group H_κ.

For any root we have the coroot χ_α and we say that α is κ-short if $\kappa(\chi_\alpha)=1$, otherwise it is κ-long. We consider the subgroup W_κ of the Weyl group which is generated by the reflections at κ-short roots. The system of κ-short roots is invariant under the action of W_κ and under the action of the Galois group.

To the choice of positive roots we have a Borel subgroup $B\subset G\times\bar{\mathbb{Q}}$ and Weyl chamber $\mathcal{C}_B\subset(X_*(T)\otimes\mathbb{R}\setminus X_\alpha$, the X_α are the hyperplanes orthogonal to the roots.

We can find a set of roots $\pi_H\in\Delta^+(T)$ which form the system of positive roots of the root system Δ_H consisting of κ short roots. To get this system we look at a chamber $\mathcal{C}_{B^H}\subset(X_*(T)\otimes\mathbb{R}\setminus X_\alpha,\alpha\ \kappa-\text{short}$. I assume that this chamber contains $\mathcal{C}_B$. Then the roots in in π_H are the positve roots in $\Delta^+(T)$ which are orthogonal to the walls of this chamber. We have an action of the Galois group on this system of roots: We take the homomorphism $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ into the Weyl group of $X^*(T)$, an element σ sends the chamber $\mathcal{C}_{B^H}$ into another chamber, and we find an element in W_κ which transports this chamber back. The composition of these two elements induces an automorphism of the chamber $\mathcal{C}_{B^H}$ and hence a permutation of the roots in π_H.

Hence we have a root system with Galois action. We can define a semi simple simply connected quasi split group $H_\kappa^{(1)}$ having this root system. If C_κ is the the connected torus on which the roots in Δ_H vanish, then we can form the product $H_\kappa^{(1)}\times C_\kappa$. The considerations on Galois cohomology show that the roots in Z_γ are κ-short, we find a unique quotient H_κ of this group which contains the centralizer Z_γ.

This group has the following important properties.

(i) I mentioned already that we have

$$Z_\gamma\subset H_\kappa$$

(ii) We have a diagram of spaces of conjugacy classes

$$\begin{array}{ccc} & & \mathbb{A}^G \\ & \nearrow & \\ \mathbb{A}^{Z_\gamma} & & \\ & \searrow & \\ & & \mathbb{A}^{H_\kappa} \end{array}$$

(iii) The image a_κ of γ in $\mathbb{A}^{H_\kappa}$ defines a geometric conjugacy class C_{a_κ} in H_κ. This conjugacy class is κ stable, the value κ on this class is constant =1.

(iv) This last property can be reformulated: For any place the local component κ_v factors over $H^1(\mathbb{Q}_v,H_\kappa)$, i.e. we have a diagram

$$\begin{array}{ccccc} H^1(\mathbb{Q}_v,Z_\gamma) & & \xrightarrow{\kappa_v} & & \mu_2 \\ & \searrow & & \nearrow & \\ & & H^1(\mathbb{Q}_v,H_\kappa) & & \end{array}$$

We have to say what it means that two such data (x_a, Z_{x_a}, κ) and $(x_{a'}, Z_{x_{a'}}, \kappa')$ are equivalent. First of all we have the two tori C_κ and $C_{\kappa'}$. The first requirement is that these tori lie in the same inner conjugacy class, this means that we can find a $g \in G(\bar{\mathbb{Q}})$ which conjugates C_κ into $C_{\kappa'}$ such that the morphism $\mathrm{Ad}(g) : C_\kappa \to C_{\kappa'}$ is defined over $\mathbb{Q}$.

Now we assume that $x_a, x_{a'}$ are regular, so we have two tori $C_\kappa \subset T, C_{\kappa'} \subset T'$. We split the two tori by an extension of the ground field, then we have the two system of roots

$$\pi_{H_\kappa} \subset \Delta_{H_\kappa} \subset X^*(T/C_\kappa) \subset X^*(T)$$
$$\pi_{H'_\kappa} \subset \Delta_{H'_\kappa} \subset X^*(T/C'_\kappa) \subset X^*(T),$$

we have an action of the Galois group on $\pi_{H_\kappa}, \pi_{H'_\kappa}$. Now we call the two data equivalent, if we can find a $\tilde{g} \in G(\bar{Q})$ which conjugates T into $\mathcal{T}'$, where the conjugation restricted to C_κ is equal to $\mathrm{Ad}(g)$ and where it induces an isomorphism between the two systems Galois system of roots $\pi_{H_\kappa}, \pi_{H_{\kappa'}}$. (I hope this is the correct thing to do !)

The discussion is similar if the elements are not regular.

This gives us that the two endoscopic groups H_κ and $H_{\kappa'}$ are isomorphic, but the isomorphism is not canonical, even not up to inner isomorphisms.

We come back to our expression for the trace and write the denominator as a product of two factors and move one of them into the numerator.

We write down an illegal expression

$$\mathrm{tr}(x_a|\mathcal{M}) = \frac{\Sigma_w sgn(w)(w(\lambda+\rho))(\gamma_a)/\prod_{\alpha\in\Delta^+_{\kappa-\mathrm{short}}}(\frac{\alpha}{2}(x_a)-(\frac{\alpha}{2}(x_a)^{-1}))}{\prod_{\alpha\in\Delta^+_{(\kappa\text{-long})}}(\frac{\alpha}{2}(x_a)-(\frac{\alpha}{2}(x_a)^{-1}))}$$

It follows from the definition (???) that for our given γ the denominator is not zero. Our expression is not legal, because neither the numerator nor the denominator make sense. Let us look at the numerator.

We would like to interpret it as a trace of γ on a virtual representation $\mathcal{M}^H = \sum_{w\in W_\kappa\backslash W} sign(w)\mathcal{M}_{(\lambda+\rho))^w-\rho_H}$ it is an "alternating" sum over representations with highest weight $(\lambda+\rho)^w - \rho_H$ where $w \in W^H\backslash W$ and we assume that $(\lambda+\rho)^w - \rho_H$ is in the fundamental chamber for the Borel of H_κ. But of course it can happen that ρ_H is not in the character module. Then we can only view the above alternating sum as a virtual representation of the covering $H_\kappa^{(1)} \times C_\kappa$. In other words the expression in the numerator gives us a trace of a virtual representation, only if x_a is an element in the covering group.

But now we can find a character χ_κ on C_κ such that the tensor product $\mathcal{M}_{(\lambda+\rho))^w-\rho_H} \otimes \chi_\kappa$ becomes an honest representation of H_κ. The trace picks up the factor $\chi_\kappa(x_a)$ and now the numerator makes sense and is

$$\frac{\Sigma_w sgn(w)(w(\lambda+\rho))(\gamma_a)}{\prod_{\alpha\in\Delta^+_{(\kappa-\mathrm{short}}}(\frac{\alpha}{2}(x_a)-(\frac{\alpha}{2}(x_a)^{-1}))}\chi_\kappa(x_a)$$

Accordingly we multiply the denominator by the same factor and the we get for the trace

$$\mathrm{tr}(x_a|\mathcal{M})) = \frac{\frac{\Sigma_w sgn(w)(w(\lambda+\rho))(\gamma_a)}{\prod_{\alpha\in\Delta^+_{(\kappa-\mathrm{short}}}(\frac{\alpha}{2}(x_a)-(\frac{\alpha}{2}(x_a)^{-1}))}\chi_\kappa(x_a)}{\prod_{\alpha\in\Delta^+_{(\kappa\text{-long})}}(\frac{\alpha}{2}(x_a)-(\frac{\alpha}{2}(x_a)^{-1})\chi_\kappa(x_a)} = \frac{\Sigma_{w\in W_\kappa\backslash W} sgn(w)\mathrm{tr}(x_a|\mathcal{M}_{(\lambda+\rho))^w-\rho_H}\otimes\chi_\kappa)}{\prod_{\alpha\in\Delta^+_{(\kappa\text{-long}}}(\frac{\alpha}{2}(x_a)-(\frac{\alpha}{2}(x_a)^{-1})\chi_\kappa(x_a))}$$

Now the numerator and the denominator make perfect sense.

As in the case of Sl_2 we modify our system of representatives. We consider the normalizer $N(Z_{\gamma_a})$ and define the Weyl group $W_Z = N(Z_{\gamma_a})/Z_{\gamma_a}$, this is a finite group scheme over $\mathrm{Spec}(\mathbb{Q})$. This group scheme has rational points $W_Z(\mathbb{Q})$ and for any $s \in W_Z(\mathbb{Q})$ we consider the new representative γ_α^s and we write the same expression the trace. We sum over all $s \in W_Z(\mathbb{Q})$ and our previous expression becomes

$$\frac{1}{\#W_Z(\mathbb{Q})} \sum_{s\in W_Z(\mathbb{Q}),\kappa\in\Xi_a} \hat{J}(x_a^s,\kappa)\mathrm{tr}(x_a^s|\mathcal{M}) \prod \int_{C_a(\mathbb{Q}_\ell)} \kappa_\ell^{\gamma_a^s}(x_\ell)\epsilon_\ell(x_\ell)h_\ell(x_\ell)\omega_{C_a,\ell}^{\mathrm{Tam}}(d\bar{x}_\ell).$$

We abbreviate the orbital integrals by $O(h_\ell, \kappa^{\gamma_a^s}, C_a)$ (so our first choice of the x_a is still visible) and now we we implement our above expression for the trace. We get

$$\frac{1}{\#W_Z(\mathbb{Q})} \sum_{s,\kappa} \frac{\hat{J}(x_a^s,\kappa)\mathrm{tr}(x_a^s|\mathcal{M}^H\otimes\chi_\kappa)}{\prod_{\alpha\in\Delta^+_{(\kappa\text{-long})}}(\frac{\alpha}{2}(x_a^s)-(\frac{\alpha}{2}(x_a^s)^{-1}))\chi_\kappa(x_a)} \prod O(h_\ell,\kappa_\ell^s,C_a)$$

We forget the factor $\hat{J}$ for a second. We are left with

$$\frac{\mathrm{tr}(x_a^s|\mathcal{M}^H\otimes\chi_(x_a))}{\prod_{\alpha\in\Delta^+_{(\kappa\text{-long})}}(\frac{\alpha}{2}(x_a^s)-(\frac{\alpha}{2}(x_a^s))^{-1})\chi_\kappa(x_a)} \prod O(h_\ell,\kappa_\ell^s,C_a).$$

We consider the denominator.

Our aim is to write it as a product of local factors at the finite primes, i.e. to push the denominator into the infinite product. The most subtle part is to write a sign at infinity as a product of signs at the finite places

Let us assume for a moment that we do not need the χ_κ. As it stands the denominator does not make sense, because we extract square roots but

$$\prod_{\alpha\in\Delta^+_{(\kappa\text{-long})}} (\frac{\alpha}{2}(x_a^s)-(\frac{\alpha}{2}(x_a^s))^{-1}) = \frac{\rho(x^s)}{\rho_H(x^s)} \prod_{\alpha\in\Delta^+_{(\kappa\text{-long})}} (\alpha(x_a^s)-1) =$$

$$|\prod_{\alpha\in\Delta^+_{(\kappa\text{-long})}} (\frac{\alpha}{2}(x_a^s)-(\frac{\alpha}{2}(x_a^s))^{-1})|_\infty \eta(x^s,\Delta^+).$$

now the second term in the first line tells us that product makes sense and is a number in the field $L \subset \bar{\mathbb{Q}} \subset \mathbb{C}$, which splits our torus. (Here we use again our assumption on ρ_H).

An element $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$ induces a permutation on the roots-the Galois group is mapped to the Weyl group- and it respects the decomposition into κ-short and κ-long roots. Then is becomes clear that

$$\sigma\big(\prod_{\alpha\in\Delta^+_{(\kappa\text{-long})}} (\frac{\alpha}{2}(x_a^s)-(\frac{\alpha}{2}(x_a^s))^{-1}) = sign_{\kappa-\mathrm{long}}(\sigma)((\frac{\alpha}{2}(x_a^s)-(\frac{\alpha}{2}(x_a^s))^{-1}\big),$$

where the $sign_{\kappa-\mathrm{long}}(\sigma)$ counts the number of times a positive $\kappa-$ long becomes negative.

We conclude that

$$\sigma\Big(\prod_{\alpha\in\Delta^+_{(\kappa\text{-long})}}(\frac{\alpha}{2}(x_a^s)-(\frac{\alpha}{2}(x_a^s))^{-1})\Big)=\Delta_\kappa(x_a^s)\in F_\kappa\subset L$$

where $F_\kappa=\mathbb{Q}$ or a quadratic extension of $\mathbb{Q}$.We write

$$\Delta_\kappa(x_a^s)=|\Delta_\kappa(x_a^s)|_\infty\eta_\infty(x_a^s,\kappa)$$

where $\eta_\infty(x_a^s,\kappa)$ is a fourth root of unity. We know that it is an odd power of i if the number of positive κ-long roots is odd, I claim that this must be the case under our assumption.

Hence we can write

$$\eta_\infty(x_a^s,\kappa)=\eta(\kappa)(-1)^{n(x_a^s,\kappa)}$$

The factor $\eta(\kappa)$ is i or 1 according to our rule above. Now the product formula gives us

$$(|\Delta_\kappa(x_a^s)|_\infty)^{-1}=\prod_\ell|\Delta_\kappa(x_a^s)|_\ell,$$

this numbers (the number on the left and the individual factors on the right) only depend on a and κ. Consequently we change notation and write

$$(|\Delta_\kappa(a)|_\infty)^{-1}=\prod_\ell|\Delta_\kappa(a)|_\ell.$$

We use the product formula to move the absolute value in the denominator into the the product over the infinite primes. We get for our expression

$$\eta(\kappa)^{-1}(-1)^{n(x_a^s,\kappa)}(\mathrm{tr}(x_a^s|\mathcal{M}^H)\prod_\ell|\Delta_\kappa(a)|_\ell\int_{C_a(\mathbb{Q}_\ell)}\kappa_\ell^{\gamma_a^s}(x_\ell)\epsilon_\ell(x_\ell)h_\ell(x_\ell)\omega^{\mathrm{Tam}}_{C_a,\ell}(d\bar{x}_\ell).$$

There is still the sign $(-1)^{n(x_a^s,\kappa)}$ in front which causes us some headaches.

We have an easy case, namely if the number of κ-short roots is odd. In this case we know that $F_\kappa=\mathbb{Q}(\Delta_\kappa(x_a^s))$ is imaginary quadratic.

Now we do what we also did in the case of Sl_2, namely we consider the set of imaginary quadratic extensions $E/\mathbb{Q}, E\subset\bar{\mathbb{Q}}\subset\mathbb{C}$. Any such field contains a unique element α_E whose square is a squarefree integer D_E and which of the form $i\times$ a positive number. For any place v of $\mathbb{Q}$ we consider the norm map

$$N_v:F_{\kappa,v}^\times\to\mathbb{Q}_v^\times$$

the image has index 1 or 2, in any case we have a canonical character δ_v of the kokernel to ±1. We know that for any $x\in\mathbb{Q}^\times$ we have the product relation $\prod_v N_{F_{\kappa,v}/\mathbb{Q}_v}(x)=1$. Hence we can say

$$(-1)^{n(x_a^s,\kappa)}=\prod_\ell\delta_\ell(\Delta_\kappa(x_a^s)/\alpha_{F_\kappa})$$

and now we get for our expression

$$\eta(\kappa)^{-1}(\mathrm{tr}(x_a^s|\mathcal{M}^H)\prod_\ell \delta_\ell(\Delta_\kappa(x_a^s)/\alpha_{F_\kappa})|\Delta_\kappa(a)|_\ell \int_{C_a(\mathbb{Q}_\ell)} \kappa_\ell^{\gamma_a^s}(x_\ell)\epsilon_\ell(x_\ell)h_\ell(x_\ell)\omega^{\mathrm{Tam}}_{C_a,\ell}(d\bar{x}_\ell).$$

The factors $\delta_\ell(\Delta_\kappa(x_a^s)/\alpha_{F_\kappa})|\Delta_\kappa(a)|_\ell$ are the so called transfer factors, the most subtle part in these factors are the signs. I know in principle how I can define them in general. In any case I wlll explain how to do this for rank 2 groups.

3.9 The fundamental lemma

We abbreviate the notation for the transfer factors

$$\delta_\ell(\Delta_\kappa(x_a^s)/\alpha_{F_\kappa})|\Delta_\kappa(a)|_\ell = \tilde{\Delta}_\ell(\kappa, x_a)$$

The fundamental lemma at the finite places deals with the evaluation of the individual terms

$$\tilde{\Delta}_\ell(\kappa, x_a)\int_{C_a(\mathbb{Q}_\ell)} \kappa_\ell^{\gamma_a^s}(x_\ell)\epsilon_\ell(x_\ell)h_\ell(x_\ell)\omega^{\mathrm{Tam}}_{C_a,\ell}(d\bar{x}_\ell),$$

these linear combinations of orbital are called κ_ℓ orbital integrals, they are instable if $\kappa_\ell \neq 1$. We want to express these instable integrals as stable integrals on an endoscopic group. Note that the expression above can be defined purely in local terms at the prime ℓ.

3.9.1 A global consideration

There is of course still something awkward in the process and this is the choice of the representative x_a. We have to find a consistent rule to do this for the different choices of the a simultaneously.

I think to proceed we have to choose the an extension of $G/\mathrm{Spec}(\mathbb{Q})$ to a smooth group scheme $\mathcal{G}/\mathrm{Spec}(\mathbb{Z})$, this group scheme should be semi simple at all place where G splits (or is quasisplit over an unramified extension) and as good as possible at the remaining places. For a given κ we should also find such a group scheme structure for the endoscopic group $\mathcal{H}_\kappa$, which in a certain sense is compatible with the choice of $\mathcal{G}/\mathrm{Spec}(\mathbb{Z})$.

Now the global integral structure gives us local integral structures and we want to formulate rules how to choose representatives $\gamma_{a,\ell}$ in the conjugacy class $C_a(\mathbb{Q}_\ell)$. A representative $\gamma_{a,\ell}$ yields a "neutral" conjugacy class in $C_a(\mathbb{Q}_\ell)$, and what we actually want to find a rule that distinguishes a neutral class.

Let us consider a semi simple elliptic element $\gamma_\ell \in G(\mathbb{Q}_\ell)$. Let $\kappa_\ell : H^1(\mathbb{Q}_\ell, Z_\gamma) \to \mu_2$ be a character. The element γ_ℓ lies in a torus $T(\mathbb{Q}_\ell)$. This torus has a maximal anisotropic subtorus T_c and $T_c(\mathbb{Q}_\ell)$ is compact. We have a unique extension of $T/\mathbb{Q}_\ell$ to a smooth flat group scheme $\mathcal{T}/\mathbb{Z}_\ell$ such that $\mathcal{T}_c(\mathbb{Z}_\ell) = T_c(\mathbb{Q}_\ell)$. Here we assume that the torus splits over a tamely ramified extension.

We say that our γ_ℓ is *white* with respect to the given integral structure if we can conjugate this torus by an element in $G(\mathbb{Q}_\ell)$ such that $\mathcal{T}_c(\mathbb{Z}_\ell) \subset \mathcal{G}(\mathbb{Z}_\ell)$.

Definition:

We say what it means that the torus is $T \times \mathbb{Q}_\ell$ is κ_ℓ unramified:

Assume we have another embedding $\mathcal{T}_c(\mathbb{Z}_\ell) \xrightarrow{i_1} \mathcal{G}(\mathbb{Z}_\ell)$ such that these two embeddings are geometrically conjugate, i.e. there is an $y \in G(\mathbb{Q}_\ell)$ such that $\mathrm{Ad}(y) \circ i_1$ is our first embedding, then the two embeddings differ by an element in $\delta(1, i_1) \in H^1(\mathbb{Q}_\ell, Z(T_c))$. Then our torus is κ_ℓ unramified if $\kappa_\ell(\delta(1, i_1))$ for all possible i_1.

In other words κ_ℓ takes constant values on the *white* elements.

By $\mathrm{char}_\mathcal{G}(\mathbb{Z}_\ell)$ I mean the characteristic function of $\mathcal{G}(\mathbb{Z}_\ell)$ I think that I can prove

If we have an element γ_ℓ which is not κ_ℓ unramified, then the κ_ℓ orbital integral

$$O(char_{\mathcal{G}(\mathbb{Z}_\ell)}, \gamma_\ell, \kappa_\ell) = 0$$

This is in principle the easy case of the fundamental Lemma. It says: If the integral structure does not give a rule to distinguish *white* and *black* elements, then the above κ_ℓ orbital integral over the characteristic function is zero.

Now we assume that we have a semi simple element γ_ℓ, which lies in a conjugacy class $C_a(\mathbb{Q}_\ell) = C_{[\gamma_\ell]}(\mathbb{Q}_\ell)$ and we have a κ_ℓ. We assume that κ_ℓ can always be considered as the ℓ component of some global element $\kappa : X_*(C'_{x_a})/I_\Gamma(X_*(C'_{x_a})) \to \mu_2$ for some $C'_{x_a})$ which is anisotropic over $\mathbb{R}$. Such a κ is defined by its restriction to the infinite place but not by its restriction to ℓ. We choose such a κ as additional datum and use this κ to define H_κ. Then our γ_ℓ also yields a conjugacy class $C^H_{[\gamma_\ell]}$ in the endoscopic group $H = H_\kappa$. Now we formulate:

If γ_ℓ is κ_ℓ unramified and "white" then

$$\tilde{\Delta}_\ell(\kappa, x_a) \int_{C_a(\mathbb{Q}_\ell)} char_{\mathcal{G}(\mathbb{Z}_\ell)}(x_\ell)\kappa_\ell^{\gamma_\ell}(x_\ell)\epsilon_\ell(x_\ell)\omega^{\mathrm{Tam}}_{C_a,\ell}(d\bar{x}_\ell) =$$

$$\int_{C^H_{[\gamma_\ell]}(\mathbb{Q}_\ell)} char_{\mathcal{H}(\mathbb{Z}_\ell)}(x_\ell)\epsilon_\ell(x_\ell)\omega^{\mathrm{Tam}}_{C^H_{[\gamma_\ell]}}(x_\ell)(d\bar{x}_\ell)$$

This is the most difficult assertion of the fundamental lemma at the finite places.

Finally the fundamental lemma makes a general statement:

Let us assume that κ_ℓ, κ are given. We have a linear map which attaches any function h_ℓ in the Hecke algebra, a function h^κ in the Hecke algebra of the endoscopic group such that

$$\tilde{\Delta}_\ell(\kappa, x_a) \int_{C_a(\mathbb{Q}_\ell)} h_\ell(x_\ell)\kappa_\ell^{\gamma_\ell}(x_\ell)\epsilon_\ell(x_\ell)\omega^{\mathrm{Tam}}_{C_a,\ell}(d\bar{x}_\ell) = \int_{C^H_{[\gamma_\ell]}(\mathbb{Q}_\ell)} h^\kappa_\ell(x_\ell)\epsilon^H_\ell(x_\ell)\omega^{\mathrm{Tam}}_{C^H_{[\gamma_\ell]}}(x_\ell)(d\bar{x}_\ell)$$

As I said this is a very general statement, but it is less sharp than our first assertion. Actually we can generalize the first statement to an assertion concerning the unramified Hecke algebra. We assume that ℓ is a place, where our group scheme $\mathcal{G}$ is semi simple and we also assume that κ is unramified at ℓ. Then we get from the Satake isomorphism an inclusion

$$i^\kappa : \mathcal{H}_\ell^{\mathcal{G}} = \mathcal{C}_c(\mathcal{G}(\mathbb{Q}_\ell)//\mathcal{G}(\mathbb{Z}_\ell)) \hookrightarrow \mathcal{H}_\ell^{\mathcal{H}_\kappa} = \mathcal{C}_c(\mathcal{H}_\kappa(\mathbb{Q}_\ell)//\mathcal{H}_\kappa(\mathbb{Z}_\ell))),$$

which is quite explicit such that we have under the assumptions above

$$\tilde{\Delta}_\ell(\kappa, x_a)\int_{C_a(\mathbb{Q}_\ell)} h_\ell(x_\ell)\kappa_\ell^{\gamma_\ell}(x_\ell)\epsilon_\ell(x_\ell)\omega_{C_a,\ell}^{\mathrm{Tam}}(d\bar{x}_\ell) = \int_{C^{H_\kappa}_{[\gamma_\ell]}(\mathbb{Q}_\ell)} i^\kappa(h_\ell(h_\ell))\epsilon_\ell^{H_\kappa}(x_\ell)\omega^{\mathrm{Tam}}_{C^H_{[\gamma_\ell]}}(x_\ell)(d\bar{x}_\ell).$$

Eventually the stabilization of the topological trace formula would lead to a statement of the following kind:

Let us fix a level K_f. We assume that $K_\ell = \mathcal{G}(\mathbb{Z}_\ell)$ for all primes outside our usual set of exceptions Σ. Then we have finitely inner conjugacy classes of endoscopic data tori in $G/\mathbb{Q}$ which are unramified at all places $\ell \notin \Sigma$. This yields a finite number of endoscopic groups H_κ.

Then we have have for a Hecke operator $h_f = \prod_\ell h_\ell$, which is unramified outside Σ and for any of the finitely many κ a Hecke operator $h^\kappa = \prod_{\ell\in\Sigma} h^\kappa \times \prod_{\ell\notin\Sigma} i^\kappa(h_\ell)$ and constants $a(\kappa)$ such that

$$\mathrm{tr}_{\mathrm{ell}}(h_\ell|H^\bullet(S^G_{K_f},\mathcal{M}) = \sum_\kappa a(\kappa)\mathrm{tr}_{\mathrm{ell,stable}}(h_\ell^\kappa|H^\bullet(S^H_{K_f^H},\mathcal{M}_\kappa)$$

3.10 The rank 2 cases

3.10.1 The case A_2

We choose an imaginary quadratic extension $E/\mathbb{Q}$ and we consider a simply connected semi simple group $G/\mathbb{Q}$ of type A_2, which is an outer form and becomes inner over E. (The picture explains the notation) All our extensions of $\mathbb{Q}$ will be considered as subfields of $\bar{Q}$ and $\bar{Q}$ is the field of algebraic numbers in $\mathbb{C}$.

If $T/\mathbb{Q}$ is a maximal torus of $G/\mathbb{Q}$ we get an action of a Galois group $\mathrm{Gal}(K/\mathbb{Q})$ on the character module $X^*(T)$, the field K contains E. This action factors through the group $W \times \{c\}$, where c acts as -1 on $X^*(T)$.

We assume that the image of Galois contains -1 since we want an elliptic torus.

We look for Galois invariant linear maps $X_*(T) \to \mu_2$, this is a Galois fixed element in $\kappa \in X^*(T)/2X^*(T)$. If we find a $\kappa \neq 0$ then the action of Galois is reduced to a 2-Sylow subgroup, hence we can assume that $\kappa = \gamma_1$ and the image of Galois is contained in the subgroup generated by c and perhaps a reflection at α_2. The κ-short roots are $\alpha_2, -\alpha_2$. The torus C_κ is the subtorus on which α_2 vanishes, it is isomorphic to $T^{(1)}_{E/\mathbb{Q}}$, the subtorus of norm one elements in $R_{E/\mathbb{Q}}(G_m)$.

We have only one endoscopic class of non trivial endoscopic data.

As in the case of Sl_2 or more precisely in the case of the norm 1 group of the division algebra we define the notion of an endoscopic datum. We introduce an equivalence relation on the pairs (T, κ) and the endoscopic data are the equivalence classes.

If $\kappa = 1$ the trivial character then (T, κ) is regardless what T is the trivial endoscopic datum $[e]$. If $\kappa \neq 1$ then saw that κ yields a torus C_κ. This torus sits in its centralizer $Z(C_\kappa)$ and we also have $T \subset Z(C_\kappa)$. Now we have the sequence

$$1 \to Z(C_\kappa)^{(1)} \to Z(C_\kappa) \to C_\kappa^{(1)} \to 1$$

(See excursion into Galois cohomology). We get a homomorphism

$$X_*(T)/I_\Gamma \to X_*(C'_\kappa)/I_\Gamma X_*(C'_\kappa) \to 0$$

and clearly κ factorizes over the module on the right. Hence we see that the only information that counts is C_κ but the torus T is not relevant.

Hence in this case we have up to equivalence only two endoscopic data. The non trivial endoscopic datum is simply an embedding of the torus $T^{(1)}_{E/\mathbb{Q}}$ into $G/\mathbb{Q}$ such that over $\bar{Q}$ this an embedding into a maximal torus T' and the image is the subtorus on which a pair of roots is trivial.

On the other hand if such a torus we can find an embedding $T \hookrightarrow H_\kappa$ Notice: We do not necessarily have $H_\kappa \xrightarrow{\sim} Z(C_\kappa)$ This endoscopic group is

$$H_\kappa = H_\kappa^{(1)} \times C_\kappa/\mu_2$$

where $H_\kappa^{(1)} = Sl_2$ and μ_2 the common copy of μ_2. The semi simple component $Z(C_\kappa)^{(1)}$ must not split. One should notice that the formation of the quotients makes (iii) valid and this is also the reason that the embedding of $T \to H_\kappa$ is also "stable" in the sense that any family of local changes within the local geometric conjugacy class can be obtained by a global change. of the embedding (This is (iii), it will be explained better in a revised version)

If we choose as fundamental chamber the cone generated by γ_1, γ_2, then we can choose as fundamental chamber for W_κ the half-space of elements x with $< \alpha_2, x >\geq 0$. Now we can choose representatives $e, w_1 = s_1 = s_{\alpha_1}, w_2 = s_1 s_2$.

Now we choose a highest weight λ and the representation $\mathcal{M}_\lambda$ of $G \times_{\mathbb{Q}} E$. We write the character formula for a regular element $x \in T(\mathbb{Q})$

$$\mathrm{tr}(x|\mathcal{M}) = \frac{\sum_w sign(w)(\lambda+\rho)^w(x)}{\prod((\alpha/2)(x) - \prod(\alpha/2)(x)^{-1}})$$

Here ρ is of course the half sum of positive roots of G. We rewrite the denominator

$$(\lambda+\rho)(\gamma)-(\lambda+\rho)(x)^{-1}-((\lambda+\rho)^{w_1}(x)-(\lambda+\rho)^{w_1}(x)^{-1})+(\lambda+\rho)^{w_2}(x)-(\lambda+\rho)^{w_2}(x)^{-1}$$

We consider the highest weight representations of $H_\kappa^{(1)} \times C_\kappa$ given by the highest weights $\lambda + \rho - \rho_H = \lambda + \rho - \alpha_2/2, (\lambda+\rho)^{w_1} - \rho_H, (\lambda+\rho)^{w_2} - \rho_H$ and let us put

$$\mathcal{M}^H = \mathcal{M}_{\lambda+\rho-\rho_H} - \mathcal{M}_{(\lambda+\rho)^{w_1}-\rho_H} + \mathcal{M}_{(\lambda+\rho)^{w_2}-\rho_H}.$$

For an element $\tilde{x} \in \tilde{T}(\bar{Q})$ ($\tilde{x} \in \tilde{T}(\bar{Q})$ is the inverse image of T in $H_\kappa^{(1)} \times C_\kappa$ we get

$$\mathrm{tr}(\tilde{x}|\mathcal{M}) = \frac{\mathrm{tr}(\tilde{x}|\mathcal{M}^H)}{((\alpha_1/2)(\tilde{x}) - (\alpha_1/2)(\tilde{x})^{-1})((\alpha_1+\alpha_2)/2)(\tilde{x}) - ((\alpha_1+\alpha_2)/2(\tilde{x})^{-1})}.$$

Given $\gamma \in T(\mathbb{Q})$ we have to choose a $\tilde{\gamma}$ to make sense out of the numerator and the denominator.

But now we have $\gamma_1/2 \in X^*(\tilde{T})$ and we may multiply then numerator and the denominator by $\gamma_1/2(\tilde{\gamma})$. Then we can rewrite our formula

$$\mathrm{tr}(\tilde{x}|\mathcal{M}) = \frac{\mathrm{tr}(\tilde{x}|\mathcal{M}^H \otimes (\gamma_1/2))}{((\alpha_1/2)(\tilde{x}) - (\alpha_1/2)(\tilde{x})^{-1})((\alpha_1+\alpha_2)/2)(\tilde{x}) - ((\alpha_1+\alpha_2)/2(\tilde{x})^{-1})(\gamma_1/2)(\tilde{x})}.$$

But now the representation in the numerator is a representation of H_κ and we can now evaluate at $\gamma \in T(\mathbb{Q})$ and get

$$\mathrm{tr}(\gamma|\mathcal{M}) = \frac{\mathrm{tr}(\gamma|\mathcal{M}^H \otimes (\gamma_1/2))}{((\alpha_1/2)(\gamma) - (\alpha_1/2)(\gamma)^{-1})((\alpha_1+\alpha_2)/2)(\gamma) - ((\alpha_1+\alpha_2)/2(\gamma)^{-1})(\gamma_1/2)(\gamma)}.$$

Now we want to write the denominator as a constant, depending only on κ times a product of local factors at the finite primes..

We can find an algebraic Hecke character $\phi : T(\mathbb{Q})\backslash T(\mathbb{A}) \to \mathbb{C}^\times$ which is of type $-\gamma_1/2$, these Hecke characters form a torsor under a certain ideal class group. (Here we make a choice but I think this is reasonable in view of our final goal: The cohomology groups $H^\bullet(S^H, \mathcal{M}^H)$ can be decomposed according to a central character which then will be a Hecke character of the same type (or inverse?).

Then we can write

$$\gamma_1/2(\tilde{\gamma})^{-1} = \prod_\ell \phi_\ell(\tilde{x})$$

and if we abbreviate

$$\Delta^{long}(\tilde{x}) = ((\alpha_1/2)(\gamma) - (\alpha_1/2)(\tilde{\gamma})^{-1})((\alpha_1+\alpha_2)/2)(\tilde{\gamma}) - ((\alpha_1+\alpha_2)/2(\tilde{\gamma})^{-1}),$$

then we get for $\sigma \in \mathrm{Gal}(K/F)$

$$\sigma(\Delta^{long}(\tilde{x})) = \chi(\sigma)\Delta^{long}(\tilde{x})$$

where χ is the non trivial character on the Galois group which is trivial on s_{α_2}.

Hence we see that $\sigma(\Delta^{long}) \in E$. We can write it as

$$\Delta^{long}(\tilde{x})) = i(-1)^{n(\gamma,\kappa)}|\Delta^{long}(\tilde{x}))|_\infty$$

and

$$(-1)^{n(\gamma,\kappa)} = \prod_\ell \delta_\ell(\Delta^{long}(\tilde{x}))/\alpha_E).$$

But now $\delta_\ell((\Delta^{long}(\tilde{x}))/\alpha_E))\phi_\ell(\tilde{x})$ only depends on γ and not on the lifting $\tilde{x}$. Hence we can define local transfer factors

$$\Delta_\ell(x) = \delta_\ell(\Delta^{long}(x))/\alpha_E)\phi_\ell(x)|\Delta^{long}(x))|_\ell$$

and hope that the fundamental lemma is true and says

$$\int_{C_a(\mathbb{Q}_\ell)} \Delta_\ell(x_\ell)\kappa_\ell(x_\ell)\epsilon_\ell(x_\ell)h_\ell(x_\ell)\omega^{\mathrm{Tam}}_{C_a,\ell}(d\bar{x}_\ell) = \int_{C^H_{[\gamma_\ell]}(\mathbb{Q}_\ell)} h^\kappa_\ell(\phi_\ell)(x_\ell)\epsilon^H_\ell(x_\ell)\omega^{\mathrm{Tam}}_{C^H_{[\gamma_\ell]}}(x_\ell)(d\bar{x}_\ell),$$

where $h^\kappa_\ell(\phi_\ell)$ fulfills similar properties as the old h^κ_ℓ.

Now we want to compute the contribution of the conjugacy class a, which allows a $\kappa \neq 1$. Let us now assume that the class a is regular, then $Z = T$. The group $W_T(\mathbb{Q})$ is the centralizer of the action of the Galois group on $X^*(T)$, so it may be equal to the Weyl group if the Galois group is cyclic of order 2, or it is the group generated by the reflection s_{α_2}.

In the first case we have a trivial action of the Galois group on $X^*(T)/2X^*(T)$ and hence three choices of a non trivial κ which are equivalent under the Weyl group. In the second case we have only one choice of a non trivial κ. We pick the term corresponding to the class a in our formula *prestab)* and ignore the factor in front. We have only one κ the summation over $W_Z(\mathbb{Q})$ makes the expression independent of the choice of the representative x_a, once we choose it in $Z(C_\kappa)$. We get

$$\frac{1}{\#W_Z(\mathbb{Q})}\sum_s \frac{\hat{J}(x_a^s)\mathrm{tr}(x_a^s|\mathcal{M}^H\otimes\chi_\kappa)}{\prod_{\alpha\in\Delta^+_{(\kappa\text{-long})}}(\frac{\alpha}{2}(x_a^s)-(\frac{\alpha}{2}(x_a^s))^{-1})\chi_\kappa(x_a)}\prod O(h_\ell,\kappa_\ell^s,C_a) =$$

$$\sum_s \hat{J}(x_a^s,\kappa)\mathrm{tr}([x_a^s]|\mathcal{M}^H\otimes(\gamma_1/2))\prod\int_{C^H_{[x_a^s]}(\mathbb{Q}_\ell)}\phi_\ell(x_\ell)h^\kappa_\ell(x_\ell)\epsilon^H_\ell(x_\ell)\omega^{\mathrm{Tam}}_{C^H_{[x_a^s]}}(x_\ell)(d\bar{x}_\ell)$$

where $[x_a^s]$ is the stable class in H_κ defined by x_a^s. If now look at the classes $b\in C^H(\mathbb{Q})$ which are of the form $[x^s]$ for some s and we denote by $W_b\subset W(\mathbb{Q})$ the set of s so that $[x^s]=b$ then we get

$$\sum_b\sum_{s\in W_b}\hat{J}(x_a^s,\kappa)\mathrm{tr}(b|\mathcal{M}^H\otimes(\gamma_1/2))\prod\int_{C^H_b(\mathbb{Q}_\ell)}\phi_\ell(x_\ell)h^\kappa_\ell(x_\ell)\epsilon^H_\ell(x_\ell)\omega^{\mathrm{Tam}}_{C^H_b}(x_\ell)(d\bar{x}_\ell).$$

Now we need a fundamental Lemma at the infinite place. Recall that for a given class a we have still the factor in front

$$\frac{1}{\mathrm{vol}(\omega_f^{G,\mathrm{Tam}}(K_f))}\tau(Z^{(1)}{}_{x_a})h(C'_{x_a})\epsilon(x_a)W_\infty(x_a).$$

The Tamagawa volume of K_f is just what it is. The first and the second factor depend only on a and they do not change if we consider $Z^{(1)}_{x_a}$ as subgroup of the endoscopic group. I claimed that $W_\infty(x_a)$ only depends on a, but I do not remember why. (3.6.2.) So lets forget it. This factor also can depend on the ambient group which is $G/\mathbb{Q}$, so we write $W^G_\infty(x_a)$ for it. Recall that also the centralizer Z_{x_a} and the group Ξ_a given to us. Then the fundamental Lemma must say

There exists a constant $a(\kappa)$ such that for any class b

$$\frac{1}{\#W_{Z_{x_a}}}\sum_{\kappa\in\hat{\Xi}_a}\sum_{s\in W_b}\hat{J}(x_a^s,\kappa)W^G_\infty(x_a^s)=a(\kappa)W^{H_\kappa}_\infty(([x_a^s]),$$

Then we get eventually that the elliptic κ instable contribution is given by

$$\frac{\#\underline{\sqcup\,\sqcup}(C'_\kappa)\hat{J}(\kappa))}{\mathrm{vol}(\omega_f^{G,\mathrm{Tam}}(K_f))}\sum_{b\in\mathbb{A}^H(\mathbb{Q})}\tau(Z^{(1)}{}_b)h(C'_b)\epsilon(b)W^{H_\kappa}_\infty(b)\times$$
$$\mathrm{tr}(b|\mathcal{M}^H)\prod_\ell\int_{C^H_a(\mathbb{Q}_\ell)}\phi_\ell(x_\ell)\epsilon_\ell(x)_\ell h^\kappa_\ell(\phi_\ell)(x_\ell)\omega^{\mathrm{Tam}}_{C^H_b,\ell}(d\bar{x}_\ell)$$

Hence we come to the final formula :(See 1.1) (The Tate Shafarewic group is trivial, we need to compare two Tamagawa volumes) We have constants $A(\kappa)$ such that

$$\mathrm{tr}_{\mathrm{ell}}(h_f|H^\bullet(S^G,\tilde{\mathcal{M}})) = A(1)\mathrm{tr}_{\mathrm{stable,ell}}(h_f|H^\bullet(S^G,\tilde{\mathcal{M}}))+A(\kappa)\mathrm{tr}_{\mathrm{stable,ell}}(h_f^\kappa(\phi_f)|H^\bullet(S^{H_\kappa},\tilde{\mathcal{M}}^{H_\kappa})).$$

and this does not look so bad afterall. The dependence on ϕ_f seems to be ok.

Such a formula with explicitly computed constants seems to be the final goal of the stabilisation of the topological trace formula.

I have not yet verified the above fundamental Lemma. It must be a special case of the fundamental Lemma in Langlands Shelstadt, which they proved over $\mathbb{R}$. But in the case where $G\times\mathbb{R}$ is anisotropic (and x_a is regular ?) it seems to be very easy.

3.10.2 The case B_2

We have to look for Galois invariant elements $\kappa\in X^*(T)/2X^*(T)$, let Γ is the Galois group acting on $X^*(T)$.

We have two types of endoscopic data. The first type is given by γ_2

We observe that the class of γ_2 is invariant only if Γ fixes the line $\mathbb{Z}\gamma_2$. If this so then γ_2 gives us an endoscopic datum $\kappa=\gamma_2 \mod 2$. Clearly the κ-short roots are $\alpha_1,-\alpha_1$ and C_κ is of dimension 1. We see that it splits over an imaginary quadratic extension $E/\mathbb{Q}$ and $C_\kappa \xrightarrow{\sim} T^{(1)}_{E/\mathbb{Q}}$. The endoscopic group is $C_\kappa\times \mathrm{Sl}_2$.

If we perform the same manipulation with the expression for the trace $\mathrm{tr}(x_a|\mathcal{M})$ then we find four irreducible representations in the numerator and in the denominator we have three factors. We get

$$\mathrm{tr}(x_a|\mathcal{M}) = \frac{\mathrm{tr}(x_a|\mathcal{M}^H)}{(\frac{\alpha_2}{2}(x_a)-\frac{\alpha_2}{2}(x_a)^{-1})(\frac{\alpha_1+\alpha_2}{2}(x_a)-\frac{\alpha_1+\alpha_2}{2}(x_a)^{-1})(\frac{\alpha_1+2\alpha_2}{2}(x_a)-\frac{\alpha_1+2\alpha_2}{2}(x_a)^{-1})}$$

In this case we can say that the square of the denominator is a negative integer and we can proceed as before.

For any imaginary quadratic extension $E/\mathbb{Q}$ we get such an endoscopic datum since the embedding is unique up to inner classes.

Then we have to show that after the fixing of the level we only find finitely many endoscopic classes which are κ unramified outside the set Σ which contains the ramified primes of the level K_f.

The second type is given by γ_1. We observe that $\gamma_1 \mod 2$ is always invariant, hence we see that it provides a non zero κ for any torus $T/\mathbb{Q}$ no matter what the Galois group might be. The κ-short roots are the short roots. If we choose a chamber in $X^*(T)$ -for instance the cone generated by γ_1,γ_2 then we get as the system of posite roots π_H for the endoscopic group the roots $\alpha_1+\alpha_2=\gamma_1,\alpha_2$ and hence we have a consistent way to say which root is which.

We consider the splitting field $E/\mathbb{Q}$ of our torus $T/\mathbb{Q}$. Then it is clear that its Galois group contains the element c which is send to -1 in the Weyl group. It is central and therefore E is a totally imaginary quadratic extension of a totally real field $K/\mathbb{Q}$. This extension $K/\mathbb{Q}$ is totally real and a multiquadratic extension of degree 1,2 or 4.

We have the homomorphism $\Gamma \to W$, this gives us $\Gamma/\{c\} \hookrightarrow W/\{c\}$ and$W/\{c\}$ is the elementary abelian 2 group generated by the images of the reflections $s_{\alpha_1}, s_{\alpha_2}$.

We get an action of the Galois group on the set of positive roots π_H, this action factors over an extension $\mathrm{Gal}(F/\mathbb{Q})$, where $F/\mathbb{Q}$ is either a real quadratic extension or trivial. Hence we find that the endoscopic group is either

$$R_{F/\mathbb{Q}}(\mathrm{Sl}_2/F)/\mu_2 \text{ or } \mathrm{Sl}_2 \times \mathrm{Sl}_2/\mu_2.$$

The endoscopic group is split, i.e. $F = \mathbb{Q}$ if and only if if the Galois Γ lies in W_κ in this case this means that it is contained in the group generated by reflections at the κ-short roots.

The element ρ_H is in the character module, the Weyl character formula becomes

$$\mathrm{tr}(x_a|\mathcal{M}) = \frac{\mathrm{tr}(x_a|\mathcal{M}^H)}{(\frac{\alpha_1}{2}(x_a) - \frac{\alpha_1}{2}(x_a)^{-1})(\frac{\alpha_1+2\alpha_2}{2}(x_a) - \frac{\alpha_1+2\alpha_2}{2}(x_a)^{-1})}.$$

The following considerations are not entirely convincing!

We observe that we are allowed to look at the two individual factors

$$\Delta_1(x_a) = (\frac{\alpha_1}{2}(x_a) - \frac{\alpha_1}{2}(x_a)^{-1}), \Delta_2(x_a) = (\frac{\alpha_1+2\alpha_2}{2}(x_a) - \frac{\alpha_1+2\alpha_2}{2}(x_a)^{-1})$$

in the denominator. Let us now assume that the reflection at the root α_1 is not in the Galois group, this means that $F = \mathbb{Q}$.

The product $\Delta_1(x_a)\Delta_2(x_a) \in Q$, we need to know the sign of this number. If $K = \mathbb{Q}$ then $E/\mathbb{Q}$ is an imaginary quadratic extension and we can write the sign as a product of local signs at the finite places.

If $K \neq \mathbb{Q}$ then the the reflection s_{α_2} is in the Galois group it has a fixed field $E_1/\mathbb{Q}$ which is imaginary quadratic. (There is a second extension, which is the fixed field of cs_{α_2}.)

Then we can again write the sign as a product of local sings using the norm map from E_1 to $\mathbb{Q}$. (Or the other one?)

If $F/\mathbb{Q}$ is quadratic, then $\Delta_1(x_a)\Delta_2(x_a) \in F$.

In this situation we may even have to take into account, that $\mathrm{tr}(x_a|\mathcal{M}^H)$ is only in F ????????

Then we are in the situation, where we know how to define the sign and the transfer factors.