Chapter IV
Semi-simple group schemes and Lie algebras

June 9, 2013

Contents

1 The classical groups and their realization as split semi-simple group schemes over $\mathrm{Spec}(\mathbb{Z})$

We will not discuss the general notion of a semi-simple group scheme over a base S, instead we will discuss the examples of classical groups and explain the main structure theorems in examples.

1.1 The group scheme $\mathrm{Sl}_n/\mathrm{Spec}(\mathbb{Z})$

We consider a free module M of rang n over $\mathrm{Spec}(\mathbb{Z})$. We define the group scheme $\mathrm{Sl}(M)/\mathrm{Spec}(\mathbb{Z})$: for any $\mathbb{Z}$ algebra R we have $\mathrm{Sl}(M)(R) = \mathrm{Sl}(M \otimes_{\mathbb{Z}} R)$.

This is clearly a semi simple group scheme over $\mathrm{Spec}(\mathbb{Z})$ because :

a) The group scheme is smooth over $\mathrm{Spec}(\mathbb{Z})$
b) For any field k -which is of course a $\mathbb{Z}$-algebra we have

$$\mathrm{Sl}(M) \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(k) = \mathrm{Sl}(M \otimes_{\mathbb{Z}} k)/\mathrm{Spec}(k)$$

and for any k this group scheme does not contain a normal subgroup scheme, which is isomorphic to $G_a^r/\mathrm{Spec}(k)$ (hence it is reductive) and its center is a finite group scheme.

If we fix a basis $e_1, e_2, \ldots, e_n$ then we get a split maximal torus $T/\mathrm{Spec}(\mathbb{Z})$ this is the sub group scheme which fixes the lines $\mathbb{Z}e_i$, with respect to this basis we have

$$T(R) = \{ \begin{pmatrix} t_1 & 0 & \ldots & 0 \\ 0 & t_2 & \ldots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & t_n \end{pmatrix} \quad | \quad t_i \in R^\times, \prod t_i = 1\}$$

With respect to this torus $T/\mathrm{Spec}(\mathbb{Z})$ we define root subgroups. This are smooth subgroup schemes $U \subset G$ which are isomorphic to the additive group scheme $G_a/\mathrm{Spec}(\mathbb{Z})$ and which are normalized by T. It is clear that these root subgroups are given by

$$\tau_{ij} : G_a \to \mathrm{Sl}(M)$$

$$\tau_{ij} : x \to \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & \ddots & x & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where the entry x is placed in the i-th row and j-th collumn. Let us denote the image by $U_{\alpha_{ij}}$.

Then we get the relation

$$t\tau_{ij}(x)t^{-1} = \tau_{ij}((t_i/t_j)x)$$

(If I write such a relation then I always mean that $t, x..$ are elements in $T(R), G_a(R)...$ for some unspecified $\mathbb{Z}$ algebra R.)

1.2 The root system

The characters

$$\alpha_{ij} : T \to G_m$$

$$\alpha_{ij} : \begin{pmatrix} t_1 & 0 & \dots & 0 \\ 0 & t_2 & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & t_n \end{pmatrix} \to t_i/t_j$$

are form the set Δ of simple roots in the character module of the torus. We may select a subset of positive roots

$$\Delta^+ = \{\alpha_{ij} \mid i < j\}.$$

Then the torus T and the $U_{\alpha_{ij}}$ with $\alpha_{ij} \subset \Delta^+$ stabilize the flag

$$\mathcal{F} = (0) \subset \mathbb{Z}e_1 \subset \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \subset \dots \subset M.$$

The stabilizer of the flag is a smooth sub group scheme $B/\mathrm{Spec}(\mathbb{Z})$. It is so-but not entirely obvious- that B is a maximal solvable sub group scheme. These maximal subgroup schemes are called Borel subgroups.

It is clear that the morphism

$$T \times \prod_{\alpha_{ij}, i<j} U_{\alpha_{ij}} \to B,$$

which is induced by the multiplication is an isomorphism of schemes.

The set Δ^+ of positive roots contains the subset $\pi \subset \Delta$ of simple roots t_i/t_{i+1}. Every positive root can be written as a sum of simple roots with positive coefficients.

1.3 The flag variety

It is not so difficult to see that the flags form a projective scheme $\mathrm{Gr}/\mathrm{Spec}(\mathbb{Z})$. From this it follows:

For any Dedekind ring A and its quotient field K we have

$$\mathrm{Gr}(K) = \mathrm{Gr}(A).$$

If A is even a discrete valuation ring then we can show easily

The group $\mathrm{Sl}_n(A)$ acts transitively on $\mathrm{Gr}(A)$.

The whole point is, that results of this type are true for arbitrary split semi simple groups $\mathcal{G}/\mathrm{Spec}(\mathbb{Z})$. This is not so easy to explain and also much more difficult to prove. But we have the series of so called classical groups and for those these results are again easy to see. (The main problem in the general approach is that we have to start from an abstract definition of a semi simple group and not from a group which is given to us in a rather explicit way like Sl_n or the classical groups)

1.4 The group scheme $\mathrm{Sp}_g/\mathrm{Spec}(\mathbb{Z})$

Now we choose again a free $\mathbb{Z}$ module M but we assume that we have a non degenerate alternating pairing

$$< , >: M \times M \to \mathbb{Z}$$

where non degenerate means: If $x \in M$ and $< x, M > \subset a\mathbb{Z}$ with some integer $a > 1$, then $x = ay$ with $y \in M$. It is well known and also very easy to prove that M is of even rank $2g$ and that we can find a basis

$$\{e_1, \dots, e_g, f_g, \dots, f_1\}$$

such that $< e_i, f_i >= - < f_i, e_i >= 1$ and all other values of the pairing on basis elements are zero.

The automorphism group scheme of $\mathcal{G} = \mathrm{Aut}((M, < \ . \ >))$ is the symplectic group $\mathrm{Sp}_g/\mathrm{Spec}(\mathbb{Z})$. Again it is easy to find out how a maximal torus must look like. With respect to our basis we can take

$$T = \{ \begin{pmatrix} t_1 & 0 & & & \dots & 0 \\ 0 & \ddots & & & \dots & 0 \\ 0 & 0 & t_g & & & 0 \\ 0 & 0 & 0 & t_g^{-1} & \dots & \\ 0 & & & & \ddots & 0 \\ & 0 & & & & t_1^{-1} \end{pmatrix} \}$$

We can say that the torus is the stabilizer of the ordered collection of rank 2 submodules $\mathbb{Z}e_i, \mathbb{Z}f_i$. We can define a Borel subgroup $B/\mathbb{Z}$ which is the stabilizer of the flag

$$\mathcal{F} = (0) \subset \mathbb{Z}e_1 \subset \dots \subset \mathbb{Z}e_1 \dots \oplus \dots \mathbb{Z}e_g \subset \mathbb{Z}e_1 \dots \oplus \dots \mathbb{Z}e_g \oplus \mathbb{Z}f_g \subset \dots \subset M$$

(A flag starts with isotropic subspaces until we reach half the rank of the module. But then this lower part of the flag determines the upper half, because it is given by the orthogonal complements of the members in the lower half).

We can define the root subgroups (with respect to T)

$$\tau_\alpha : G_a \xrightarrow{\sim} U_\alpha \subset \mathcal{G}$$

which are normalized by T. As before we have the relation

$$t\tau(x)t^{-1} = \tau(\alpha(t)x),$$

where $\alpha \in \Delta \subset X^*(T)$.

Now it is not quite so easy to write down what these root subgroups are, we write down the simple positive roots in the the case $g = 2$: We have the maximal torus

$$T = \{ \begin{pmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 & t_1^{-1} \end{pmatrix} \}$$

and we want to find one-parameter subgroups $U_\alpha \subset \mathcal{G}$ which stabilize the flag.

The one parameter subgroups corresponding to the simple roots are

$$\tau_{\alpha_1} : x \mapsto \{e_1 \mapsto e_1, e_2 \mapsto e_2 + xe_1, f_2 \mapsto f_2, f_1 \mapsto f_1 - xf_2\}$$

$$\tau_{\alpha_2} : y \mapsto \{e_1 \mapsto e_1, e_2 \mapsto e_2, f_2 \mapsto f_2 + ye_2, f_1 \mapsto f_1\}$$

where $\alpha_1(t) = t_1/t_2, \alpha_2(t) = t_2^2$. The unipotent radical is then

$$\{\begin{pmatrix} 1 & x & v & u \\ 0 & 1 & y & -v \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 \end{pmatrix}\}$$

As before it is not so difficult to show that the flags form a smooth projective scheme $\mathcal{X}/\mathrm{Spec}(\mathbb{Z})$ (see also [book], V.2.4.3). Show that for any discrete valuation ring A the group $\mathcal{G}(A)$ acts transitively on $\mathcal{X}(A) = \mathcal{X}(K)$. It is also easy to verify the statements in 1.1.

1.5 The group scheme $\mathrm{SO}(n, n)/\mathrm{Spec}(\mathbb{Z})$

We can play the same game with symmetric forms. Let M together with its basis as above, we replace g by n. But now we take the quadratic form F

$$F : M \to \mathbb{Z}$$

which is defined by

$$F(x_1e_1 \cdots + x_ne_n + y_nf_n + \cdots + y_1f_1) = \sum x_iy_i$$

and all other values of the pairing on basis elements are zero. We define the group scheme of isomorphisms but in addition we require the the determinant is one. Hence

$$\mathrm{SO}(n, n)/\mathrm{Spec}(\mathbb{Z}) = \mathrm{Aut}(M, F, \det = 1).$$

The maximal torus and the flags look pretty much the same as in the previous case. But the set of roots looks different. For $n = 2$ the torus and the unipotent radical are given by

$$T = \{\begin{pmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 & t_1^{-1} \end{pmatrix}\},\ U = \{\begin{pmatrix} 1 & x & y & -xy \\ 0 & 1 & 0 & -y \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 \end{pmatrix}\}.$$

The system of positive roots consists of two roots $\alpha_1(t) = t_1/t_2, \alpha_2(t)t_1t_2$. This is the Dynkin diagram $A_1 \times A_1$ hence there exists a homomorphism (isogeny) between group schemes over $\mathrm{Spec}(\mathbb{Z})$:

$$\mathrm{Sl}_2 \times \mathrm{Sl}_2 \to \mathrm{SO}(2, 2).$$

It is an amusing exercise to write down this isogeny.

Another even more interesting excercise is the computation of the roots for the torus (here $n = 3$)

$$T = \{ \begin{pmatrix} t_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & t_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & t_3^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & t_1^{-1} \end{pmatrix} \}.$$

In this case we have the root subgroups

$$\tau_{\alpha_1} : x \mapsto \begin{pmatrix} 1 & x & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \ \tau_{\alpha_2} : x \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & x & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -x & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\tau_{\alpha_3} : x \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & x & 0 & 0 \\ 0 & 0 & 1 & 0 & -x & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where

$$\alpha_1(t) = t_1/t_2, \ \alpha_1(t) = t_2/t_3, \ \alpha_3(t) = t_2 t_3$$

Use the result of this computation to show that we have an isogeny

$$\mathrm{Sl}_4 \to \mathrm{SO}(3,3).$$

How can we give a linear algebra interpretation of this isogenies.

1.6 The group scheme $\mathrm{SO}(n+1,n)/\mathrm{Spec}(\mathbb{Z})$

Of course we can also consider quadratic forms in an odd number of variables. We take a free $\mathbb{Z}$-module of rank $2n+1$ with a basis

$$\{e_1, \dots, e_n, h, f_n, \dots, f_1\}.$$

On this module we consider the quadratic form

$$F : M \to \mathbb{Z}$$

$$F(\sum x_i e_i + zh + \sum y_i f_i) = \sum x_i y_i + z^2.$$

From this quadratic fom we get the bilinear form

$$B(u,v) = F(u+v) - F(u) - F(v).$$

We have the relation

$$F(u) = 2B(u,u),$$

hence we can reconstruct the quadratic form if we extend $\mathbb{Z}$ to a larger ring where 2 is invertible.

We consider the automorphism scheme

$$\mathcal{G} = \mathrm{SO}(n+1,n)/\mathrm{Spec}(\mathbb{Z}) = \mathrm{Aut}(M, F, det = 1)/\mathrm{Spec}(\mathbb{Z})$$

and I claim that this is indeed a semi simple group scheme over $\mathrm{Spec}(\mathbb{Z})$. To see this I strongly recommend to discuss the case $n = 1$.

We have of course the maximal torus

$$T = \{\begin{pmatrix} t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t^{-1} \end{pmatrix}\}.$$

It is also the stabilizer of the collection of three subspaces $\mathbb{Z}e, \mathbb{Z}h, \mathbb{Z}f$, here we use the determinant condition.

Now one has to discuss the root subgroups with respect to this torus.

From this we can derive that we have an isogeny

$$\mathrm{Sl}_2 \to \mathrm{SO}(2,1)$$

It is also interesting to look at the case $n = 2$. In this case we can compare the root systems of Sp_2 and $\mathrm{SO}(3,2)$ they are isomorphic. Now it is a general theorem in the theory of split semi simple group schemes that the root system determines the group scheme up to isogeny. Hence we we should be able to construct an isogeny between Sp_2 and $\mathrm{SO}(3,2)$. Who can do it?

2 Some facts concerning the system of roots

2.1 Positive roots, simple roots and so on

I do not discuss the general definition of a semi-simple group scheme over $\mathrm{Spec}(\mathbb{Z})$, I hope that the examples above give some idea of what it should be. Our examples are split semi simple group schemes, because they have a maximal split torus, which is split over $\mathrm{Spec}(\mathbb{Z})$.

Of course we also have the notion of a semi simple group scheme over any field k or even over any base scheme. The official definition for G/k to be semi-simple is that the groups scheme is smooth and $G \times_k \bar{k}$ does not have a non trivial connected solvable subgroup which is normal.

Such a semi-simple group scheme has always a maximal torus $T/k \subset G/k$. We can always find a finite separable normal extension E/k such that the extension $T \times_k E$ is a split maximal torus. The semi simple group scheme is split if it has a maximal torus T/k, which is split. This means that we can find a torus for which we can choose $E = k$.

Then we have a finite system Δ of roots in the character module $X^*(T \times E)$ of our maximal torus and for any of these roots $\alpha \in \Delta$ we have a unique sub group scheme $\tau_\alpha : G_a \hookrightarrow G/k$, such that $t\tau_\alpha(x)t^{-1} = \tau_\alpha((\alpha(t))x)$.

We have the action of the Galois group $\mathrm{Gal}(E/k)$ on $X^*(T \times E)$ it acts by permutations on set of roots Δ.

If we choose a Borel subgroup $B \supset T \times E$, then we saw in our examples that it is a semidirect product of the torus T and the unipotent radical U. This unipotent radical contains (and is generated by) the root subgroups U_α where $\alpha \in \Delta_B^+$, these are the positive roots with respect to B. The choice of a Borel subgroup is the same as the choice of a system Δ^+ of positive roots which is closed under addition and satisfies $\Delta^+ \cup -\Delta^+ = \Delta$. In Δ^+ we have the subset π_B of simple roots, these are the roots which can not be written as a sum of two positive roots. Then any positive root can be written as sum $\beta = \sum_{\alpha \in \pi_B} m_\alpha \alpha$ with $m_\alpha \geq 0$. If $\beta \in \Delta^+$ and if β is not simple then we can find a simple root α_1 such that $\beta - \alpha_1 \in \Delta^+$. The system of simple positive roots can be visualised by its Dynkin graph.

We say that G/k is quasi split if we can find a torus T/k which is contained in a Borel subgroup. This is the same as saying that $X^*(T \times E)$ contains a set of simple roots which is invariant under the action of the Galois group.

2.2 The Weyl group

Now I state a general theorem, which is easy to prove in our examples. We take a root α. Then it can be shown that $\mathbb{Q}\alpha \cap \Delta = \{\alpha, -\alpha\} \subset \Delta$. Then we can consider the torus $T^{(\alpha)}$ which is the connected component of the kernel of α. It is a torus of codimension 1.

Now the theorem says that

The root subgroups U_α, resp. $U_{-\alpha}$ are uniquely determined by α (resp. $-\alpha$) and can construct a unique homomorphism

$$\pi_\alpha : \mathrm{Sl}_2 \to \mathcal{G}$$

which sends the root subgroups

$$U_+ = \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\} \xrightarrow{\sim} U_\alpha$$

$$U_- = \{\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\} \xrightarrow{\sim} U_{-\alpha}.$$

The image of the maximal torus T_1 in Sl_2 is a one dimensional torus $T_{(\alpha)} \subset T$.

The element $w_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{Sl}_2(\mathbb{Z})$ maps to an element $w_\alpha \in \mathcal{G}(\mathbb{Z})$. This element lies in the normalizer $N(T)$ of T, and $t \mapsto w_\alpha t w_\alpha^{-1}$ induces the identity on $T^{(\alpha)}$ and $t \mapsto t^{-1}$ on $T_{(\alpha)}$.

These elements w_α generate the group $W = N(T)(\mathbb{Z})/T(\mathbb{Z})$, this is the Weyl group. This Weyl group acts simply transitively on the set of Borel subgroups containing the torus T.

This is also easy to verify in our examples.

The elements w_α induce automorphisms s_α of order two of the character module $X^*(T)$. If course we can define a positive definite quadratic form on $X^*(T)$, let $< \ , \ >$ the associated bilinear form. Then these elements s_α are reflections, we have

$$s_\alpha(\gamma) = \gamma - 2\frac{<\gamma, \alpha>}{<\alpha, \alpha>}\alpha$$

Excercise: Determine the structure of the Weyl group in our examples above and make pictures of the root systems in the cases $\mathrm{Sl}_2, \mathrm{Sl}_3, \mathrm{Sp}_2, \mathrm{SO}(3,2)$.

I forgot to say that the rank of $\mathcal{G}$ is the dimension of the maximal torus.

Excercise Can there be more root systems of rank 2?

2.3 The simply connected and the adjoint group

It we have two split tori T_1, T_2 over $\mathrm{Spec}(\mathbb{Z})$, then we have their character modules $X^*(T_1), X^*(T_2)$ and a canonical isomorphism

$$\mathrm{Hom}(T_1, T_2) = \mathrm{Hom}(X^*(T_2), X^*(T_1))$$
$$\phi \mapsto^t \phi.$$

A morphism $\pi : T_1 \to T_2$ is an isogeny if it is surjective and has a finite kernel. This is the case if and only if ${}^t\phi : X^*(T_2) \to X^*(T_1)$ is injective and the image has finite index. The kernel $\ker(\pi)$ is the a product of group schemes μ_d of d-th roots of unity.

We come back to our split semi simple group scheme. Of course the set Δ generates a sublattice $\mathbb{Z}\Delta = X^*(T_{\mathrm{ad}}) \subset X^*(T)$, this is the root lattice. It is clear that this lattice is of finite index. It is also clear, that these two lattices are equal if the center of $\mathcal{G}/\mathbb{Z}$ is trivial, then our group is an adjoint group, and we write $\mathcal{G}/\mathrm{center} = \mathcal{G}_{\mathrm{ad}}$.

How much bigger can $X^*(T)$ be? There is of course a constraint:

For all $\gamma \in X^*(T)$ the number $2\frac{<\gamma,\alpha>}{<\alpha,\alpha>} \in \mathbb{Z}$.

We choose a Borel subgoup, i.e. a set of positive roots Δ^+ and for $\alpha \in \pi_B$ we define the coroots (or fundamental weights) $\omega_\alpha \in X^*(T) \otimes \mathbb{Q}$ by the rule

$$\frac{2 < \omega_\alpha, \beta >}{< \beta, \beta >} = \delta_{\alpha,\beta}$$

then these span a lattice $X^*(T_{\mathrm{sc}})$. We can construct the simply connected cover $\mathcal{G}_{\mathrm{sc}} \to \mathcal{G}$ whose maximal torus is T_{sc}.

We have an isogeny $T_{\mathrm{sc}} \to T_{\mathrm{ad}}$ the kernel is of multiplicative type and dual to $X^*(T_{\mathrm{sc}})/X^*(T_{\mathrm{ad}})$. For the module of cocharacters we have an inclusion in the opposite direction $X_*(T_{\mathrm{sc}}) \hookrightarrow X_*(T_{\mathrm{ad}})$.

Excercise Consider our classical groups, which one is simply connected, which one is adjoint?

2.4 Highest weight representations

The characters ω_α are called the fundamental dominant weights. They play a role in the theory of representations.

Let us assume that G/k is a split semi simple group scheme over a field k of characteristic zero. We consider rational representations of G/k, this are homomorphisms between algebraic groups over k

$$r : G/k \to \mathrm{Gl}_n(\mathcal{M})/k,$$

here $\mathcal{M}$ is a k-vector space. These representations have an important property: They are semi-simple, i.e. if we have an invariant submodule $\mathcal{M}_1 \subset \mathcal{M}$, then we can find a complement $\mathcal{M}_2$ which is also invariant and of course $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$. Hence any rational representation is a direct sum of irreducibles.

Now we have a fundamental fact:

Let G/k be a split semi simple group over a field k, let $r : G/k \to \mathrm{Gl}(\mathcal{M})$ be an irreducible rational representation. Let $T/k \subset G/k$ be a split maximal torus, let $B/k \supset T/k$ be a Borel subgroup, let $U/k \subset B/k$ be its unipotent radical. Then the space $\mathcal{M}^U$ of invariants is of dimension one, the torus T/k acts on this space by a character $\lambda \in X^*(T)$. This character is a linear combination $\lambda = \sum_{\alpha \in \pi_B} n_\alpha \omega_\alpha$ with $n_\alpha \in \mathbb{Z}, n_\alpha \geq 0$ and it is called the highest weight of the representation. If the group G/k is simply connected, then there exists up to isomorphism exactly one representation $\mathcal{M}_\lambda$ to a given highest weight $\lambda = \sum_{\alpha \in \pi_B} n_\alpha \omega_\alpha$.

2.5 The reductive groups of similitudes

All these classical semi simple group schemes can be embedded in a natural way in a slightly larger reductive group scheme. In the first case we have $\mathrm{Sl}(M) \subset \mathrm{Gl}(M)$. The other groups are defined as stabilizers of a bilinear or a quadratic form. This can be considered as an element in $< \, , \, > \in M^\vee \otimes M^\vee$ and we can also consider the stabilizer of the line $\mathbb{Z} < \, , \, > \subset M^\vee \otimes M^\vee$ and this then will be the group of similitudes.

If we do this for instance for the symplectic group, then we get the group scheme $\mathrm{GSp}_g/\mathrm{Spec}(\mathbb{Z})$. We have a character $\alpha : \mathrm{GSp}_g \to G_m$ which is defined by $< gv, gw >= \alpha(g) < v, w >$, and we have a center $Z \subset \mathrm{GSp}_g$ consisting of the diagonal matrices. If we restrict α to the center then we get the character $t \mapsto t^2$.

2.6 Arbitrary semi-simple group schemes over k.

If we have an arbitrary semi simple group scheme G/k, then we can always find a finite, normal separable extension E/k, such that $G \times_k E$ is split. Let us choose a maximal split torus $T_0/F \subset G \times_k E$ and a Borel subgroup $B_0 \supset T_0$. If now $\sigma \in \mathrm{Gal}(E/k)$, then we can conjugate the pair (T_0, B_0) by this element and get (T_0^σ, B_0^σ). Now it follows from the general theory -and can easily be verified in our examples- that we can find an element $g_\sigma \in G(E)$ such that

$$g_\sigma^{-1} T_0 g_\sigma = T_0^\sigma, g_\sigma^{-1} B_0 g_\sigma = B_0^\sigma$$

and this element is unique up to multiplication by an element $t_\sigma \in T_0(F)$ from the left. Hence the element g_σ provides a unique bijection between the set of simple positive roots $\Pi_{B_0} \subset X^+(T_0)$ and $\Pi_{B_0^\sigma} \subset X^*(T_0^\sigma)$. Hence we see that we may speak of *the system* Π_G of positive roots of any semi simple group scheme. It can be represented by a Dynkin graph. But also the element σ yields a bijection between these two sets of simple roots. Since these two sets are identified we see easily

that any semi simple group G/k comes with a Dynkin graph Π_G together with an action of some Galois group $\mathrm{Gal}(E/k)$ on Π_G.

It is also clear from the general theory that there is a unique split group scheme G_0/k such that $G \times_k E \xrightarrow{\sim} G_0 \times_k E$. One says that two semi simple group schemes $G_1/k, G_2/k$ are k-forms of each other, if they become isomorphic over a suitable finite normal extension. So we saw that the k forms of a given split semi simple group scheme G_0/k come with an action of the Galois group on the system of positive roots. Of course this action is trivial for G_0/k itself. We say that a k-form G_1/k of G_0/k is an inner form, if this Galois action is trivial.

The theory of representation becomes a little bit more complicated. If we have an irreducible rational representation

$$r : G/k \to \mathrm{Gl}(\mathcal{M})$$

then it may become reducible if we extend the ground field. Actually we may pass to an algebraic closure $\bar{k}$ of k, then we get an isotypical decomposition

$$\mathcal{M} \otimes_k \bar{k} = \bigoplus_{\lambda} \mathcal{M}_{\lambda}^{m(\lambda)}.$$

It is easy to see that the weights λ which occur with positive multiplicity form an orbit under the above action of the Galois group $\mathrm{Gal}(\bar{k}/k)$ and the multiplicities $m(\lambda)$ are all the same on this orbit.

These multiplicities are not necessarily equal to one, the determination of the numbers involves Galois cohomology.

In the following we mean by an absolutely irreducible representation of G/k a rational representation

$$r : G \times_k F \to \mathrm{Gl}(\mathcal{M}_F),$$

where F/k is a finite extension and where this representation stays irreducible if we extend the field F/k further.

The field extension is not unique in general. It certainly contains the finite extension F_0/k, which is determined by requiring that its Galois group $\mathrm{Gal}(\bar{k}/F_0)$ fixes the highest weight of $\mathcal{M}_F$. But after that we may have several choices.

2.7 Minimal parabolic subgroups, relative simple roots

If G/k is a semisimple group then we may consider a maximal split torus $S/k \subset G/k$, a theorem of Borel and Tits asserts that all maximal split tori in G/k are conjugate by an element in $G(k)$. Given such a maximal split torus we define $M = Z(S)$ this is the centralizer of S. We can embed S/k into a maximal torus $T/k \subset G/k$ which is then a subtorus of M/k. We choose a normal splitting field E/k for our torus. We consider the root system $\Delta \subset X^*(T \times E)$. It contains the set Δ_M of those roots whose restriction to S becomes trivial. This set is the set of simple roots of the derived group $M^{(1)}$ the semisimple part of M. We can choose a cocharacter $\chi : \mathbb{G}_m \to S/k$ which is sufficiently generic, this means that $< \chi, \alpha >= 0$ if and only if $\alpha \in \Delta_M$. Then we get two more subsets

$$\Delta_U^+ = \{\alpha | < \chi, \alpha >> 0\},\ \Delta_U^- = \{\alpha | < \chi, \alpha >< 0\}$$

and $\Delta = \Delta_M \cup \Delta_U^+ \cup \Delta_U^-$, the sets are invariant under the action of the Galois group $\mathrm{Gal}(E/k)$. We have two minimal parabolic subgroups P^+ and P^- containing M and whose unipotent radicals U^+ resp.U^- contain and are generated by the one parameter subgroups $U_\alpha, \alpha \in \Delta_U^+$ resp. $U_\alpha, \alpha \in \Delta_U^-$. Since our torus S was maximal split it follows that $M^{(1)}$ is anisotropic and therefore we have that P^+ and P^- are minimal parabolic.

We can find a Borel subgroup $B \subset P^+$ which also contains the torus $T \times E$. Then this defines a subset $\Pi_G \subset \Delta^+$. This set is the union of the two sets $\Pi_G = \Pi_M \cap \tilde{\pi}_G$, where $\Pi_M = \Pi_G \cap \Delta_M$ and $\tilde{\pi}_G = \Pi_G \cap \Delta_U^+$.

Our torus T decomposes in $T^{(1)} \cdot C$, where $T^{(1)} = M^{(1)} \cap T$ and where C is the connected component of the identity of the center of M. We get a decomposition of character modules $X^*(T \times E) = X^*_{\mathbb{Q}}(T^{(1)} \times E) \oplus X^*_{\mathbb{Q}}(C \times E)$. Under the restriction from T to C the set $\tilde{\pi}_G$ maps injectively to $\pi_G^* \subset X^*_{\mathbb{Q}}(C \times E)$. The following lemma is easy to prove

Lemma 2.1. *The set π_G^* is invariant under the action of the Galois group $\mathrm{Gal}(E/k)$. Two simple roots have the same restriction to S if and only if they are conjugate under the Galois group.*

The image of π_G^* in $X^*_{\mathbb{Q}}(S)$ is denoted by π_G, it is the set of relative simple roots. Its elements are the orbits of the action of $\mathrm{Gal}(E/k)$ on π_G^*. We denote this set $\pi_G = \{\alpha_1, \ldots, \alpha_r\}$ and identify it to the set $I = \{1, \ldots, r\}$. Here r is the k-rank of G/k. Actually π_G^* is only a supporting actor, it serves to define a permutation action on $\tilde{\pi}_G \subset \Pi_G$ via the above bijection, it disappears from the stage.

If our group G/k is split then our torus $S = T$ is a maximal, then $\Pi_G = \pi_G$. If our group is quasisplit then the set $\Pi_G = \tilde{\pi}_G$,

Attached to the roots $\alpha \in \tilde{\pi}_G$ we have the fundamental dominant weight $\gamma_\alpha \in X^*_{\mathbb{Q}}(T \times E)$. It is clear that the Galois group acts by permutations on the set of these weights. Our α_ν or more consequently $\nu \in \{1, \ldots, r\} = I$ are orbits of elements in $\tilde{\pi}_G$, if $\tilde{\nu}$ is the orbit corresponding to ν then we put

$$\gamma_\nu = \sum_{\alpha \in \tilde{\nu}} \gamma_\alpha \in X^*_{\mathbb{Q}}(T)$$

these characters are in fact defined over $\mathbb{Q}$.

2.7.1 Parabolic subgroups

We want to describe the set of $G(k)-$ conjugacy classes of parabolic subgroups. It is clear that each such conjugacy class contains a unique parabolic subgroup $P \supset P^+$. Hence we only have to describe the parabolic subgroups above P^+.

We get these parabolic subgroups if we choose a subset $J \subset I$ and look at the subtorus $S_J \subset S$ where all $\alpha_\nu, \nu \in J$ are trivial. (Hence we have $S = S_\emptyset$) Then we can look at the centralizer M_J of S_J this is a reductive subgroup and there is a unique parabolic subgroup $P_J \supset P^+$ whose Levi subgroup is M_J. We want to assume $J \neq I$ because we do not consider G itself as a parabolic subgroup.

We get maximal parabolic subgroups if we take $J_\nu = I \setminus \{\nu\}$ in this case we denote the parabolic subgroup by P_ν. The parabolic subgroup P_J can be written as intersection

$$P_J = \bigcap_{\nu \in I \setminus J} P_\nu$$

Finally we consider the character groups $\mathrm{Hom}(P_J, \mathbb{G}_m) = \mathrm{Hom}(M_J, \mathbb{G}_m)$. The central torus C_J of M_J contains S_J. If we divide M_J by its derived subgroup $M_J^{(1)}$ the the quotient $M_J/M_J^{(1)} = C'_J$ is again a torus and the composition yields an isogeny

$$C_J \xrightarrow{j_J} C'_J.$$

The torus C'_J has a maximal anisotropic subtorus and if we divide by this subtorus we get the maximal split quotient

$$P_J \to M_J \to C'_J \to S'_J.$$

The composition gives us an isogeny

$$S_J \to S'_J.$$

Then we clearly get

$$\mathrm{Hom}(P_J, \mathbb{G}_m) = X^*(S'_J) \text{ and } X^*_{\mathbb{Q}}(S'_J) = X^*_{\mathbb{Q}}(S_J)$$

For $\nu \in I \setminus J$ we take the restriction of the relative simple root α_ν to S_J and denote this restriction by $\alpha_\nu^{[J]}$. Then these $\alpha_\nu^{[J]}, \nu \in I \setminus J$ form a basis of $X^*_{\mathbb{Q}}(S_J)$. Using the above identification we may also view these characters in $X^*_{\mathbb{Q}}(S'_J)$ and the provide a basis

$$\{\dots, \alpha_\nu^{[J]}, \dots\}_{\nu \in I \setminus J}$$

of $X^*_{\mathbb{Q}}(S'_J)$.

On the other hand for $\nu \in I \setminus J$ the fundamental dominant weight γ_ν is a character on S'_J and we get a second basis for $X^*_{\mathbb{Q}}(S'_J)$:

$$\{\dots, \gamma_\nu, \dots\}_{\nu \in I \setminus J}.$$

We can write one basis in terms of the other and get

$$\gamma_\nu = \sum_{\nu,\mu} c_{\nu,\mu}^{[J]} \alpha_\mu^{[J]}$$

where the coefficients $c_{\nu,\mu}^{[J]}$ are rational numbers and it is of importance to notice that $c_{\nu,\mu}^{[J]} \geq 0$.

The parabolic subgroups P_J have an unipotent radical U_{P_J}. Let Δ_J^+ be the set of roots occurring in U_{P_J}, this are those roots $\beta \in \Delta$ for which $\tau_\beta : \mathbb{G}_a \hookrightarrow U_{P_J} \times E$. The set of these roots is easy to describe. Recall that we have the projection $\tilde{\pi}_G \to I$ let us denote by $\tilde{J}^\dagger$ the set of those simple roots in $\tilde{\pi}_G$ which map to an element in $J^\dagger = I \setminus J$. Then we write a root $\beta \in \Delta^+$ as

$$\beta = \sum_{\alpha \in \Pi, \alpha \notin \tilde{J}^\dagger} m_{\beta\alpha}\alpha + \sum_{\alpha \in \tilde{J}^\dagger} m_{\beta\alpha}\alpha$$

where $m_{\beta\alpha} \geq 0$, and $\beta \in \Delta_J^+$ if and only if at least one of the coefficients $m_{\beta\alpha}$ in the second sum is not zero.

The characters (half sums of roots)

$$\rho_J = \frac{1}{2} \sum_{\beta \in \Delta_J^+} \beta$$

are defined over k. We apply this to the case $J \setminus \{\nu\}$. Then

$$\rho_{I\setminus\{\nu\}} = \rho_\nu = \sum_{\beta : m_{\beta\alpha} > 0 \text{ for some } \alpha \mapsto \nu} \beta = f_\nu \gamma_\nu$$

and since a simple root $\alpha \in \tilde{\pi}_G$ which maps to ν it follows that the coefficients in the equation above are greater or equal to zero and it also follows that $c_{\nu,\nu}^{[J]} > 0$.

2.8 The relative Weyl group

We have the Weyl group invariant pairing on $X^*(T \times E)$ and this induces

3 Semi-simple Lie algebras over fields of characteristic zero.

We assume that the of k of characteristic zero. We consider Lie-algebras $\mathfrak{g}/k$. A Lie-algebra is called abelian, if the Lie-bracket is identically zero. An ideal $\mathfrak{a} \subset \mathfrak{g}$ is a subspace for which $[\mathfrak{a}, \mathfrak{g}] \subset \mathfrak{a}$. An ideal is always two sided and it is always a subalgebra. We can form the quotient $\mathfrak{g}/\mathfrak{a}$, this is again a Lie-algebra. The commutators $\{[U, V], U, V \in \mathfrak{g}\}$ generate a vector space $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}$, which then itself is an ideal. We can form the quotient algebra $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ it is is abelian. The Lie-algebra $\mathfrak{g}$ is called nilpotent if $[[\dots[\mathfrak{g}, \mathfrak{g}], \mathfrak{g}], \dots, \mathfrak{g}] = 0$, provided the number of brackets is large enough. The Lie-algebra $\mathfrak{g}/k$ is called solvable if the commutator algebra $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.

We say that $\mathfrak{g}/k$ is semi simple, if it does not contain a non zero solvable ideal, we have the famous Killing criterion

The Lie algebra $\mathfrak{g}/k$ is semi simple, if and only if the Killing form

$$\begin{aligned} B : \mathfrak{g} \times \mathfrak{g} &\to k \\ (U, V) &\mapsto \mathrm{tr}(\mathrm{ad}(U), \mathrm{ad}(V)) \end{aligned}$$

is non degenerate

A Cartan -subalgebra $\mathfrak{t} \subset \mathfrak{g}$ is a maximal commutative subalgebra, for which all for all $H \in \mathfrak{t}$ the endomorphism $\mathrm{ad}(H) : \mathfrak{g} \to \mathfrak{g}$ are semi simple. This means that the extensions to endomorphisms of $\mathfrak{g} \otimes \bar{k}$ become diagonalizable .

A Cartan algebra is called split, if all $\mathrm{ad}(H)$ are diagonalizable, i.e. all the eigenvalues of $\mathrm{ad}(H)$ are in k. In this case we can diagonalize the endomorphisms $\mathrm{ad}(H)$ simultaneously and decompose

$$\mathfrak{g} = \mathfrak{t} \bigoplus \oplus_{\alpha \in \Delta} \mathfrak{g}_\alpha,$$

where Δ is a subset of the dual $\mathfrak{t}^\vee$, (the set of roots) where

$$\mathfrak{g}_\alpha = \{U \in \mathfrak{g} | [H, U] = \alpha(H)U\}.$$

Then we have

(i) All elements α are non zero
(ii) If $\alpha \in \Delta$ then $-\alpha \in \Delta$ and if $r \in \mathbb{Q}, r\alpha \in \Delta$ then $r = \pm 1$.
(iii) The spaces $\mathfrak{g}_\alpha$ are of dimension one, we write $\mathfrak{g}_\alpha = ke_\alpha$.
(iv) If $\alpha, \beta \in \Delta, \alpha \neq -\beta$ then

$$[e_\alpha, e_\beta] = \begin{cases} N_{\alpha,\beta} e_{\alpha+\beta} & \textit{if } \alpha + \beta \in \Delta \\ 0 & \textit{else} \end{cases}$$

where $N_{\alpha,\beta} \neq 0$.
(v) $0 \neq [e_\alpha, e_{-\alpha}] \in \mathfrak{t}$

It is not hard to see that the connected component of 1 of the group of automorphisms of a semi simple Lie-algebra is in fact a semi-simple group scheme $G/k = \mathrm{Aut}^{(0)}(\mathfrak{g})$. This is then the adjoint group. If in turn G/k is any semi simple group scheme over k, then its Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ is semi simple. If $T/k \subset G/k$ is a maximal torus, then its Lie algebra $\mathfrak{t} = \mathrm{Lie}(T)$ is a Cartan subalgebra. The torus T/k is split if and only if $\mathfrak{t}$ is split, and the roots $\Delta \subset X^*(T)$ can be identified to the roots $\Delta \subset \mathfrak{t}^\vee$. The Lie-algebra $\mathrm{Lie}(U_\alpha)$ can be identified to $ke_\alpha = \mathfrak{g}_\alpha$. If $B \supset T$ is a Borel subgroup, then its Lie algebra $\mathfrak{b}$ is a maximal solvable sub algebra and we have

$$\mathfrak{b} = \mathfrak{t} \bigoplus \oplus_{\alpha \in \Delta^+} ke_\alpha,$$

where Δ^+ is the system of positive roots corresponding to B/k.

4 The universal enveloping algebra, its centre and the Harish-Chandra Isomorphism

To any Lie-algebra $\mathfrak{g}/k$ we can attach its universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$. This algebra is the quotient of the tensor algebra

$$T(\mathfrak{g}) = k \oplus \mathfrak{g} \oplus \mathfrak{g} \otimes \mathfrak{g} \oplus \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g} \cdots \oplus \mathfrak{g}^{\otimes n} \oplus \ldots \xrightarrow{\pi} \mathfrak{U}(\mathfrak{g})$$

by an ideal I, which is the two sided ideal generated by the tensors $U \otimes V - V \otimes U - [U, V]$, in other words inside $\mathfrak{U}(\mathfrak{g}) = T(\mathfrak{g})/I$ in inside $\mathfrak{U}(\mathfrak{g})$ we have the relation $U \cdot V - V \cdot U = [U, V]$. Hence $\mathfrak{U}(\mathfrak{g})$ is an infinite dimensional (see below) associative k algebra, it is a vector space with an increasing filtration $\mathfrak{U}(\mathfrak{g})_0 \subset \mathfrak{U}(\mathfrak{g})_1 \cdots \subset \mathfrak{U}(\mathfrak{g})_n \subset \ldots$, The subspace $\mathfrak{U}(\mathfrak{g})_n$ is of finite dimension and generated by products $U_1 \cdot U_2 \cdots U_k$ with $k \leq n$.

The symmetric tensor algebra $S(\mathfrak{g})$ is the quotient of the tensor algebra $T(\mathfrak{g})/I_0$ where I_0 is the two side vector space $\mathfrak{g}^\vee$. It is graded by the subspaces $S_n(\mathfrak{g})$ of tensors of degree n and this graduation defines a filtration on $S(\mathfrak{g})$.

We have an inclusion $j : S(\mathfrak{g}) \to T(\mathfrak{g})$: If we represent an element $F \in S(\mathfrak{g})$ by a tensor $U_1 \otimes U_2 \otimes \dots U_n$ then

$$j(F) = \sum_{\sigma \in S_n} U_{\sigma(1)} U_{\sigma(2)} \otimes \dots \otimes U_{\sigma(n)}.$$

From this inclusion we get a linear map

$$\pi \circ j : \ S(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g})$$

and the theorem of Poincare-Birkhoff-Witt asserts that this is an isomorphism of filtered vector spaces. This theorem looks quite obvious, but its proof is non trivial at all. (See)

If the Lie algebra $\mathfrak{g}$ is the Lie algebra of an affine group scheme G/k and if $A(G)/k$ is the affine algebra, then every element $U \in \mathfrak{g}$ defines a derivation $D_U : A(G) \to A(G)$ which is defined by

$$f((e + \epsilon U)g) = f(g) + \epsilon D_U(f)(g),$$

this is a first order differential operator. Then taking products we easily see that we get an isomorphism

$$\mathfrak{U}(\mathfrak{g}) \xrightarrow{\sim} \text{right invariant differential operators on } A(G).$$

In this situation the Poincare-Birkhoff-Witt theorem becomes obvious.

The universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ has a centre $\mathfrak{Z}(\mathfrak{g})$. We study this centre in the case that $\mathfrak{g}/k$ is semi-simple. Since our Lie-algebra will be split after a finite extension L/k and since $\mathfrak{Z}(\mathfrak{g}) \otimes L = \mathfrak{Z}(\mathfrak{g} \otimes L)$ we assume that already $\mathfrak{g}/k$ is split, let $\mathfrak{t}$ a maximal split torus.

The adjoint action of the adjoint group $G/k = \mathrm{Aut}(\mathfrak{g})^{(0)}/k$ on $\mathfrak{g}$ extends to an action on $\mathfrak{U}(\mathfrak{g})$ and clearly we have

$$\mathfrak{Z}(\mathfrak{g}) = \mathfrak{U}(\mathfrak{g})^G.$$

If we consider the sub group of $\mathrm{Aut}(G)$, which fixes $\mathfrak{t}$ (or the maximal torus T), then this is the normalizer $N(T)$ of the torus. If we restrict these automorphisms in $N(T)$ to $\mathfrak{t}$, then the action of T becomes trivial and we get back the operation of the Weyl group $W = W(k) = N(T)(k)/T(k)$. (Here it plays a role that $\mathfrak{t}$ is split.)

Of course we also get an operation on the dual spaces: If $g \in G(k)$ and $\phi \in \mathfrak{g}^\vee = \mathrm{Hom}_k(\mathfrak{g}, k)$ then we write the evaluation of ϕ at an element $U \in \mathfrak{g}$ as $U(\phi)$, i.e we consider $\mathfrak{g}$ as the space of linear forms on $\mathfrak{g}^\vee$. The we write

$$(\mathrm{Ad}(g)(U))(\phi) = U(\phi^g),$$

and then we have the rule $(\phi^{g_1})^{g_2} = \phi^{g_1 g_2}$.

We write as before

$$\mathfrak{g} = \mathfrak{t} \bigoplus \oplus_{\alpha \in \Delta} k e_\alpha,$$

let us choose a basis $H_1, H_2, \dots, H_r$ of $\mathfrak{t}$. we represent an element $D \in \mathfrak{Z}(\mathfrak{g})$ as a symmetric tensor in $T(\mathfrak{g})$. Then it is a sum of pure tensors $aU_1 \otimes U_2 \otimes \dots U_s$ where the U_i are either an H_i or one of the e_α. Then we write

$$D = D_{\mathfrak{t}} + D_e$$

where $D_{\mathfrak{t}} \in S(\mathfrak{t})$ contains pure tensors with no e_α in it and where D_e is a sum of pure tensors with an e_α in it.

Then we have a first assertion (See ???)

The map $D \mapsto D_{\mathfrak{t}}$ provides an isomorphism of vector spaces

$$\mathrm{HC}_0 : \mathfrak{Z}(\mathfrak{g}) \xrightarrow{\sim} S(\mathfrak{t})^W$$

where the term on the right denotes the invariants under the Weyl group

We choose a system Δ^+ of positive roots we write a new decomposition, which depends on this choice. We can write any element $D \in \mathfrak{Z}(\mathfrak{g})$ in the form

$$D = D'_{\mathfrak{t}} + D'$$

where now $D'_{\mathfrak{t}} \subset S(\mathfrak{t})$ and where $D' \in \oplus_{\alpha \in \Delta^+} \mathfrak{U}(\mathfrak{g}) e_\alpha$ (See Knapp, p.220 ff).

Clearly we have for any $D' \in \oplus_{\alpha \in \Delta^+} \mathfrak{U}(\mathfrak{g}) e_\alpha$ and $H \in \mathfrak{t}$ that $D' \cdot H \in \oplus_{\alpha \in \Delta^+} \mathfrak{U}(\mathfrak{g}) e_\alpha$ and hence we see that $D \mapsto D'_{\mathfrak{t}}$ is an algebra homomorphism.

Once we have chosen a Borel subgroup or a system Δ^+ of positive roots then we define

$$\rho_B = \rho_{\Delta^+} = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha,$$

and the choice of this element in turn fixes $B \supset T$. In the following we fix the choice of $B \supset T$ and drop the subscript B at ρ, i.e. $\rho = \rho$.

We define a twisted action of the Weyl group on $S(\mathfrak{t})$. An element in $S(\mathfrak{t})$ is defined by its values on $\mathfrak{t}^\vee$. For an $F \in S(\mathfrak{t})$ we define

$$(w \cdot_\rho F)(\mu) = F(\mu^w - \rho^w + \rho).$$

We define $S(\mathfrak{t})^{W \cdot_\rho}$ to be the ring of invariants under this action. Clearly we have an isomorphism

$$i_\rho : S(\mathfrak{t})^W \xrightarrow{\sim} S(\mathfrak{t})^{W \cdot_\rho},$$

which is the restriction of the translation morphism $T_\rho : F(\ ) \mapsto F(\ +\rho)$ on $S(\mathfrak{t})$.

Now we can write down the Harish-Chandra isomorphism

$$\begin{array}{c} \mathrm{HC}_\rho : \mathfrak{Z}(\mathfrak{g}) \xrightarrow{\sim} S(\mathfrak{t})^{W \cdot_\rho} \\ \mathrm{HC}_\rho = i_\rho \circ \mathrm{HC}_0 \\ \mathrm{HC}_\rho : D \mapsto D'_{\mathfrak{t}} \end{array} .$$

We notice that the Harish-Chandra homomorphism refers to the choice of a system of positive roots or the choice of a Borel-subalgebra $\mathfrak{b} \supset \mathfrak{t}$.

But we saw already something else that refers to the choice of a set of positive roots. If we investigate the different kind of representations of G/k, the finite dimensional rational representations or if $k = \mathbb{R}$ the infinite dimensional representations of $G(\mathbb{R})$ then the first step is always to construct some kind of induced representations from B to G. Here we see the reference to a choice of B.

For instance in section 2.4. we explained the highest weight representations $\mathcal{M}_\lambda$. This is an irreducible representation of G/k, for which $\mathcal{M}_\lambda^U = ke_\lambda$.

The centre $\mathfrak{Z}(\mathfrak{g})$ acts on $\mathcal{M}_\lambda$ by scalars we get a homomorphism

$$\begin{array}{c} \chi_\lambda : \mathfrak{Z}(\mathfrak{g}) \to k \\ Dm = \chi_\lambda(D)m \end{array}.$$

Especially we have $De_\lambda = D'_{\mathfrak{t}}e_\lambda$ and so we end up with the formula

$$\chi_\lambda(D) = \mathrm{HC}_\rho(D)(\lambda).$$

This formula is the essential content of the Harish-Chandra isomorphism.

We have a special element $C \in \mathfrak{Z}(\mathfrak{g})$, this is the Casimir operator. The Killing form B is a symmetric bilinear form on $\mathfrak{g}$, since our Lie algbera is semi simple it is non degenerate it yields an identification $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^\vee$ and hence we can consider it as a symmetric tensor $B^\vee = \sum a_i e_i \otimes e_i$, where the e_i form an orthonormal basis of $\mathfrak{g}/k$. Then we get

$$B(U,V) = \sum_i a_i B(e_i,U)B(e_i,V).$$

We send this symmetric tensor to $\mathfrak{U}(\mathfrak{g})$, the image is the Casimir operator C and since the Killing form is invariant under $\mathrm{Ad}(G)$ we see that it lies in the centre.