Chapter III
Cohomology in the language of Adeles

May 29, 2017

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1 The spaces

1.1 The (generalized) symmetric spaces

Our basic datum is a connected reductive group $G/\mathbb{Q}$. Let $G^{(1)}/\mathbb{Q}$ be its derived group and let $C/\mathbb{Q}$ its centre. Then $G^{(1)}/\mathbb{Q}$ is semi simple and $C/\mathbb{Q}$ is a torus. The multiplication provides a canonical map

$$m : C \times G^{(1)} \to G \tag{1}$$

it is is an isogeny, this means that the kernel $\mu_C = C \cap G^{(1)}$ of this map is a finite group scheme of multiplicative type. A finite group scheme of multiplicative type is simply an abelian group together with an action of the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on it. If we have such an isogeny as in (1) we write $G = C \cdot G^{(1)}$.

Let $S/\mathbb{Q}$ be the maximal $\mathbb{Q}$ -split torus in $C/\mathbb{Q}$. Up to isogeny we have $C = C_1 \cdot S$ where C_1 is the maximal anisotropic subtorus of $C/\mathbb{Q}$. We also introduce the group $G_1 = G^{(1)} \cdot C_1$. We have an exact sequence

$$1 \to G^{(1)} \to G \xrightarrow{d_C} C' \to 1,$$

the quotient C' is a torus and the restricted map $d_C : C \to C'$ is an isogeny.

If $\tilde{G}^{(1)}/\mathbb{Q}$ is the simply connected covering of $G^{(1)}$, then we get an isogeny

$$m_1 : \tilde{G} = \tilde{G}^{(1)} \times C_1 \times S \to G \tag{2}$$

Let $\mathfrak{g}, \mathfrak{g}^{(1)}, \mathfrak{c}, \mathfrak{c}_1, \mathfrak{z}$ be the Lie algebras of $G/\mathbb{Q}, G^{(1)}/\mathbb{Q}, C/\mathbb{Q}, C_1/\mathbb{Q}, S/\mathbb{Q}$, then the differential of m_1 induces an isomorphism

$$D_{m_1} : \mathfrak{g} \to \mathfrak{g}^{(1)} \oplus \mathfrak{c}_1 \oplus \mathfrak{z} \tag{3}$$

On $\mathfrak{g}$ we have the Killing form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{Q}$ be the Killing form, it is defined by the rule

$$(T_1, T_2) \mapsto \mathrm{trace}(\mathrm{ad}(T_1) \circ \mathrm{ad}(T_2)) \tag{4}$$

(See [chap2] 1.2.2) The Killing form is actually a bilinear form on $\mathfrak{g}^{(1)} = \mathfrak{g}/(\mathfrak{c}_1 \oplus \mathfrak{z})$ and the restriction $B : \mathfrak{g}^{(1)} \times \mathfrak{g}^{(1)} \to \mathbb{Q}$ is nondegenerate (see chap2 and chap4).

An automorphism $\Theta : \tilde{G}^{(1)} \times_{\mathbb{Q}} \mathbb{R} \to \tilde{G}^{(1)} \times_{\mathbb{Q}} \mathbb{R}$ is called a Cartan involution if $\Theta^2 = \mathrm{Id}$ and if the bilinear form

$$B_\Theta(T_1, T_2) = B(T_1, \Theta(T_2)) \tag{5}$$

on $\mathfrak{g} \otimes \mathbb{R}$ is negative definite.

If Θ is a Cartan involution then it induces an automorphism -also called Θ- on the Lie algebra $\mathfrak{g}_{\mathbb{R}} = \mathfrak{g} \otimes \mathbb{R}$ and decomposes it into a $+$ and a $-$ eigenspace

$$\mathfrak{g}_{\mathbb{R}} = \mathfrak{k} \oplus \mathfrak{p} \tag{6}$$

and then clearly the $+$ eigenspace $\mathfrak{k}$ is a Lie subalgebra and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. The Killing form is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$. This explains the above assertion on B_Θ.

The topological group of real points $\tilde{G}^{(1)}(\mathbb{R})$ is connected (see ref?). Then we have the classical theorem

Theorem 1.1. *The fixed group $K_\infty^{(1)} = \tilde{G}^{(1)}(\mathbb{R})^\Theta$ is a maximal compact subgroup and it is also connected. The Cartan involutions are conjugate under the action of $\tilde{G}^{(1)}(\mathbb{R})$, and therefore the maximal compact subgroups of $\tilde{G}^{(1)}(\mathbb{R})$ are conjugate.*

We introduce the space $\tilde{X}^{(1)}$ of Cartan involutions on $\tilde{G}^{(1)} \times_{\mathbb{Q}} \mathbb{R}$, it is a homogenous space under the action of $\tilde{G}^{(1)}(\mathbb{R})$ by conjugation and if we choose a Θ or $K_\infty^{(1)}$ then

$$\tilde{X}^{(1)} = \tilde{G}^{(1)}(\mathbb{R})/K_\infty^{(1)} \tag{7}$$

This is the symmetric space attached to $\tilde{G}^{(1)} \times_{\mathbb{Q}} \mathbb{R}$.

Proposition 1.1. *The symmetric space $\tilde{X}^{(1)} = \tilde{G}^{(1)}(\mathbb{R})/K_\infty^{(1)}$ is diffeomorphic to $\mathbb{R}^d$, where $d = \dim \mathfrak{p}$, it carries a Riemannian metric which is $\tilde{G}^{(1)}(\mathbb{R})$ invariant.*

We have to be aware that it may happen that Θ is the identity. Then $\tilde{G}^{(1)}(\mathbb{R}) = K_\infty^{(1)}$ and our symmetric space is a point.

We extend Θ to an involution on $\tilde{G} \times \mathbb{R}$ it will be simply the identity on the other two factors. Then it also induces an involution, again called Θ on $G \times \mathbb{R}$.

We return to our reductive group $G/\mathbb{Q}$. We compare it to $\tilde{G}$ via the homomorphism m_1 in (2). Let K_∞^C be the connected component of the identity of the maximal compact subgroup in $C_1(\mathbb{R})$ and let $Z'(\mathbb{R})^0$ be the connected component of the identity of the group of real points a subtorus $Z' \subset S$. Then we put

$$K_\infty = m_1(K_\infty^{(1)} \times K_\infty^C \times Z'(\mathbb{R})^0)$$

This group K_∞ is connected and if we divide by $Z'(\mathbb{R})^0$ it is compact, more precisely we can say that $K_\infty/Z'(\mathbb{R})^0$ is the connected component of a maximal compact subgroup in $G(\mathbb{R})/Z'(\mathbb{R})^0$. The choice of the subtorus Z' is arbitrary and in a certain sense irrelevant. We could choose $Z' = Z$ then we call K_∞ *saturated* , this choice is very convenient but it certain situations it is better to make a different choice, for instance we may choose $Z' = 1$.

To such a pair (G, K_∞) we attach the *(generalized) symmetric space*

$$X = G(\mathbb{R})/K_\infty.$$

Here are a few comments concerning the structure of this space. (see also Chap II. 1.3) We observe that by construction K_∞ is connected, hence we have that $K_\infty \subset G(\mathbb{R})^0$. So if as usual $\pi_0(G(\mathbb{R}))$ denotes the set of connected components, then we see that

$$\pi_0(X) = \pi_0(G(\mathbb{R})).$$

The connected component of the identity of $\tilde{G}(\mathbb{R})$ maps under m_1 to the connected component of he identity of $G(\mathbb{R})$, i.e.

$$\tilde{G}(\mathbb{R}) = \tilde{G}^{(1)}(\mathbb{R}) \times C_1(\mathbb{R})^0 \times S(\mathbb{R})^0 \to G(\mathbb{R})^0$$

and if we divide by $K_\infty^{(1)} \times K_\infty^C \times Z'(\mathbb{R})^0$, resp. K_∞ we get a diffeomorphism with the connected component corresponding to the identity

$$\tilde{G}^{(1)}(\mathbb{R})/K_\infty^{(1)} \times C_1(\mathbb{R})^0/K_\infty^C \times S(\mathbb{R})^0/Z'(\mathbb{R}) \xrightarrow{\sim} X_1 \subset X.$$

We want to describe the other connected components of X. It is well known that we can find a maximal split torus $\tilde{S}_1 \subset \tilde{G}^{(1)} \times \mathbb{R}$ which is invariant under our given Cartan involution Θ. The homomorphism m_1 maps $\tilde{G}^{(1)}(\mathbb{R}) \to G^{(1)}(\mathbb{R})$. The fixed group $G^{(1)}(\mathbb{R})^\Theta$ is a compact subgroup whose connected component of the identity is the image of $K_\infty^{(1)}$ under m_1. Our torus $\tilde{S}_1$ sits as the first component in the maximal split torus

$$\tilde{S}_2 = \tilde{S}_1 \times C_1^{\text{split}} \times S$$

Then it is clear that Θ induces the involution $t \mapsto t^{-1}$ on $\tilde{S}_1$. Let S_2 be the image of $\tilde{S}_2$ under m_1. We have the following proposition

Proposition 1.2. *a) The group of 2-division points $S_2[2]$ normalizes K_∞.*

b) We have an exact sequence

$$\to \tilde{S}_2[2] \to S_2[2] \xrightarrow{r} \pi_0(G(\mathbb{R})) \to 0$$

c) If K_∞^0 is the image of $K_\infty^{(1)} \times K_\infty^C$ then $K_\infty^0 \cdot S_2[2]$ is a maximal compact subgroup of $G(\mathbb{R})$.

Proof. Rather obvious, the surjectivity of r requires an argument in Galois cohomology. (Details later) $\square$

Now we can write down all the connected components. We choose a system Ξ of representatives for $S_2[2]/\tilde{S}_2[2]$ and for any $\xi \in \Xi$ we get a diffeomorphism

$$\begin{gathered} \tilde{G}^{(1)}(\mathbb{R})/K_\infty^{(1)} \times C_1(\mathbb{R})^0/K_\infty^C \times S(\mathbb{R})^0/Z'(\mathbb{R}) \to X_\xi \subset X \\ g \mapsto g\xi \end{gathered} \tag{8}$$

We may formulate this differently

Proposition 1.3. *The multiplication from the left by $S_2[2]$ on $G(\mathbb{R})$ induces an action of $S_2[2]/\tilde{S}_2[2]$ on X and this action is simple transitive on the set of connected components.*

Let $x_0 = K_\infty \in X$. For any other point $x \in X$ we find an element $g \in X$ which translates x_0 to x. Then the derivative of the translation provides an isomorphism between the tangent spaces

$$D_g : T_{x_0} = \mathfrak{p} \xrightarrow{\sim} T_x.$$

This isomorphism depends of course on the choice of g. (This will play a role in section (4.1)). But we apply this to the highest exterior power and get an isomorphism

$$D_g : \Lambda^d(\mathfrak{p}) \xrightarrow{\sim} \Lambda^d(T_x)$$

which does not depend on the choice of g because the connected group K_∞ acts trivially on $\Lambda^d(\mathfrak{p})$. Hence we can say that we can find a *consistent* orientation on X : We chose a generator in $\Lambda^d(\mathfrak{p})$ the D_g yields a generator in $\Lambda^d(T_x)$.

If our reductive group is an anisotropic torus $T/\mathbb{Q}$, then we have for the connected component of the identity

$$T(\mathbb{R})^{(0)} \xrightarrow{\sim} (\mathbb{R}_{>0}^\times)^a \times (S^1)^b.$$

Then our maximal compact subgroup K_∞^T is simply the product of the circles and

$$X_T = T(\mathbb{R})/K_\infty^T$$

is nothing else than as disjoint union of copies of $\mathbb{R}^a$. The situation is similar for a split torus but then we have the freedom, to divide out the connected component of a subtorus.

As a standard example we can take $G/\mathbb{Q} = \mathrm{Gl}_2/\mathbb{Q}$, then the connected component of the real points of the centre is $\mathbb{R}_{>0}^\times$ and in this case we can take $K_\infty = SO(2) \cdot \mathbb{R}_{>0}^\times \subset \mathrm{Gl}_2(\mathbb{R})$). In this case the symmetric space is the union of an upper and a lower half plane. It we choose for our split torus $S_1/\mathbb{R}$ the standard diagonal torus, then $S_1[2]$ is the group of diagonal matrices with entries ± 1 and this normalizes K_∞.

1.2 The locally symmetric spaces

Let $\mathbb{A}$ be the ring of adeles, we decompose it into its finite and its infinite part: $\mathbb{A} = \mathbb{R} \times \mathbb{A}_f$. We have the group of adeles $G(\mathbb{A}) = G(\mathbb{R}) \times G(\mathbb{A}_f)$. We denote elements in the adele group by underlined letters $\underline{g}, \underline{h} \dots$ and so on. If we decompose an element $\underline{g}$ into its finite and its infinite part then we denote this by $g_\infty \times \underline{g}_f$. Let K_f be a (variable) open compact subgroup of $G(\mathbb{A}_f)$. We always assume that this group is a product of local groups $K_f = \prod_p K_p$.

To get such subgroups we choose an integral structure (explain at some other place) $\mathcal{G}/\mathrm{Spec}(\mathbb{Z})$. Then we know that we have $K_p = \mathcal{G}(\mathbb{Z}_p)$ for almost all p. Furthermore we know that $\mathcal{G} \times \mathrm{Spec}(\mathbb{Z}_p)/\mathrm{Spec}(\mathbb{Z}_p)$ is a reductive group scheme for almost all primes p.

If $\mathcal{G}/\mathrm{Spec}(\mathbb{Z})$ and K_f are given, then we select a finite set Σ of finite primes which contains the primes p where $\mathcal{G}/\mathbb{Z}_p$ is not reductive and those where K_p is not equal to $\mathcal{G}(\mathbb{Z}_p)$. This set Σ will be called the set of *ramified* primes.

The general agreement will be that we use letters $\mathcal{G}, \mathcal{T}, \mathcal{U}, \dots$ for group schemes over the integers, or over $\mathbb{Z}_p$ and then their general fiber will be $G, T, U, \dots$.

Readers who are not so familiar with this language may think of the simple example where $G/\mathbb{Q} = GSp_n/\mathbb{Q}$ is the group of symplectic similitudes on $V = \mathbb{Q}^{2n} = \mathbb{Q}e_1 \oplus \cdots \oplus \mathbb{Q}e_n \oplus \mathbb{Q}f_1 \oplus \cdots \oplus \mathbb{Q}f_n$ with the standard symplectic form which is given by $< e_i, f_i >= 1$ for all i and where all other products zero. The vector space contains the lattice $L = \mathbb{Z}^{2n} = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n \oplus \mathbb{Z}f_1 \oplus \cdots \oplus \mathbb{Z}f_n$. This lattice defines a unique integral structure $\mathcal{G}/\mathbb{Z}$ on $G/\mathbb{Q}$ for which $\mathcal{G}(\mathbb{Z}_p) = \{g \in G(\mathbb{Q}_p) | g(L \otimes \mathbb{Z}_p) = (L \otimes \mathbb{Z}_p)\}$. In this case the group scheme is reductive over $\mathrm{Spec}(\mathbb{Z})$. This integral structure gives us a privileged choice of an open maximal compact subgroup: Within the ring $\mathbb{A}_f$ of finite adeles we have the ring $\hat{\mathbb{Z}} = \lim\limits_{\leftarrow} \mathbb{Z}/m\mathbb{Z}$ of integral finite adeles and we can consider $K_f^0 = \mathcal{G}(\hat{\mathbb{Z}}) = \prod_p \mathcal{G}(\mathbb{Z}_p)$. This is a very specific choice. In this case the set $\Sigma = \emptyset$, we say that $K_f = K_f^0$ is unramified.

Starting from there we can define new subgroups K_f by imposing some congruence conditions at a finite set Σ of primes. These congruence conditions then define congruence subgroups $K_p \subset K_p^0$. This set Σ of places where we impose congruence condition will then be the set of ramified primes.(See the example further down.) Then we define the level subgroup

$$K_f = \prod_{p \in \Sigma} K_p \times \prod_{p \notin \Sigma} \mathcal{G}(\mathbb{Z}_p). \tag{9}$$

The space $(G(\mathbb{R})/K_\infty) \times (G(\mathbb{A}_f)/K_f)$ can be seen as a product of the symmetric space and an infinite discrete set, on this space $G(\mathbb{Q})$ acts properly discontinuously (see below) and the quotients

$$\mathcal{S}^G_{K_f} = G(\mathbb{Q}) \setminus (G(\mathbb{R})/K_\infty \times G(\mathbb{A}_f)/K_f)$$

are the locally symmetric spaces whose topological properties we want to study. We denote by

$$\pi : G(\mathbb{R})/K_\infty \times G(\mathbb{A}_f)/K_f \to \mathcal{S}^G_{K_f} = G(\mathbb{Q}) \setminus (G(\mathbb{R})/K_\infty \times G(\mathbb{A}_f)/K_f)\,,$$

the projection map.

To get an idea of how this space looks like we consider the action of $G(\mathbb{Q})$ on the discrete space $G(\mathbb{A}_f)/K_f$. It follows from classical finiteness results that this quotient is finite, let us pick representatives $\{\underline{g}_f^{(i)}\}_{i=1..m}$. We look at the stabilizer of the coset $\underline{g}_f^{(i)} K_f/K_f$ in $G(\mathbb{Q})$. This stabilizer is obviously equal to $\Gamma^{\underline{g}_f^{(i)}} = G(\mathbb{Q}) \cap \underline{g}_f^{(i)} K_f (\underline{g}_f^{(i)})^{-1}$ which is an arithmetic subgroup of $G(\mathbb{Q})$. This subgroup acts properly discontinuously on X (See Chap. II, 1.6).

Now we call the level subgroup K_f neat, if all the subgroups $\Gamma^{\underline{g}_f^{(i)}}$ are torsion free. It is not hard to see, that for any choice of K_f we can pass to a subgroup of finite index K_f', which is neat. Then we have

1.2.1 *For any subgroup K_f the space $\mathcal{S}^G_{K_f}$ is a finite union of quotient spaces $\Gamma^{\underline{g}_f^{(i)}} \setminus X$ where $X = G(\mathbb{R})/K_\infty$ and the $\Gamma_i = \Gamma^{\underline{g}_f^{(i)}}$ are varying arithmetic congruence subgroups. If K_f is neat, these spaces are locally symmetric spaces. If K_f is not neat then we may pass to a neat subgroup K_f' which is even normal in K_f: We get a covering $\mathcal{S}^G_{K_f'} \to \mathcal{S}^G_{K_f}$ which induces coverings $\Gamma_j'\backslash X \to \Gamma_i\backslash X$, where the Γ_j' are torsion free and normal in Γ_i. So we see that in general the quotients are orbifold locally symmetric spaces. For any point $y \in \mathcal{S}^G_{K_f}$ we can find a neighborhood V_y such that $\pi^{-1}(V_y)$ is the disjoint union of connected components $W_{\underline{x}}, \underline{x} = (x_\infty, \underline{g}_f) \in \pi^{-1}(y)$, and $V_y = \Gamma_{x_\infty}\backslash W_{\underline{g}_f}$, where Γ_{x_∞} is the stabilizer of x_∞ intersected with $\Gamma^{\underline{g}_f}$.*

We will consider the special case where $G/\mathbb{Q}$ is the generic fibre of a split reductive scheme $\mathcal{G}/\mathbb{Z}$. In that case we can choose $K_f = \prod_p \mathcal{G}(\mathbb{Z}_p)$, this is then a maximal compact subgroup in $G(\mathbb{A}_f)$. Then K_f is unramified we will also say that the space $\mathcal{S}^G_{K_f}$ is unramified. If in addition the derived group $G^{(1)}/\mathbb{Q}$ is simply connected, then it is not difficult to see, that $G(\mathbb{Q})$ acts transitively on $G(\mathbb{A}_f)/K_f$ and hence we get

$$\mathcal{S}^G_{K_f} \xrightarrow{\sim} \mathcal{G}(\mathbb{Z})\backslash X.$$

The homomorphism $\mathcal{G}(\mathbb{Z}) \to \pi_0(C'(\mathbb{R}))$ is surjective we can conclude that $\mathcal{G}(\mathbb{Z})$ acts transitively on $\pi_0(X)$ and if Γ_0 is the stabilizer of a connected component X^0 of X then we find

$$\mathcal{S}^G_{K_f} \xrightarrow{\sim} \Gamma_0\backslash X^0$$

especially we see that the quotient is connected. We discuss an example.

We start from the group $G/\mathrm{Spec}(\mathbb{Z}) = \mathrm{Gl}_n/\mathrm{Spec}(\mathbb{Z})$ then we may choose $K_\infty = \mathrm{SO}(n) \times \mathbb{R}^\times_{>0} \subset \mathrm{Gl}_n(\mathbb{R})$. and $X = \mathrm{Gl}_n(\mathbb{R})/K_\infty$ is the disjoint union of two copies of the space X of positive definite symmetric $(n \times n)$ matrices up to homothetie by a positive scalar (or what amounts to the same with determinant one). If we choose K_f as above then we find

$$\mathcal{S}^G_{K_f} = \mathrm{Sl}_n(\mathbb{Z})\backslash X.$$

We have another special case. Let us assume that $G/\mathbb{Q}$ is semi simple and simply connected. The group $G \times \mathbb{R}$ is a product of simple groups over $\mathbb{R}$ and we assume in addition that no no simple factor is compact. Then we have the strong approximation theorem (Kneser and Platonov) which says that for any choice of K_f the map from $G(\mathbb{Q})$ to $G(\mathbb{A}_f)/K_f$ is surjective, i.e. any $\underline{g}_f \in G(\mathbb{A}_f)$ can be written as $\underline{g}_f = a\underline{k}_f, a \in G(\mathbb{Q}), \underline{k}_f \in K_f$. This clearly implies that then

$$\mathcal{S}^G_{K_f} = \Gamma \backslash G(\mathbb{R})/K_\infty \tag{10}$$

where $\Gamma = K_f \cap G(\mathbb{Q})$.

There is a contrasting case, this is the case when $G/\mathbb{Q}$ is still semi simple and simply connected, but where $G(\mathbb{R})$ is compact. In this case our symmetric space X is simply a point $*$ and

$$\mathcal{S}^G_{K_f} = G(\mathbb{Q}) \backslash (* \times G(\mathbb{A}_f)/K_f).$$

In this case the topological space is just a discrete set of points. So it looks as if this is an entirely uninteresting and trivial case, but this is not so. To determine the finite set and the stabilizers is a highly non trivial task. Later we will construct sheaves and discuss the action of the Hecke algebra on the cohomology of these sheaves. Then it turns out that this case is as difficult as the case where $\Gamma\backslash X$ becomes an honest space.

In the choice of our group K_∞ a subtorus $Z' \subset S$ enters. The choice of this subtorus has very little influence on the structure of our locally symmetric space $\mathcal{S}^G_{K_f}$. Remember that the isogeny m in (1) induces an isogeny $C \to C'$ and this isogeny yields an isogeny from S to the maximal split subtorus $S' \subset C'$. This homomorphism induces an isomorphism $S(\mathbb{R})^0 \to S'(\mathbb{R})^0$. If $G_1(\mathbb{R})$ is the inverse image of the the group of 2-division points $S'[2]$ then we get from this isomorphism that $G(\mathbb{R}) = G_1(\mathbb{R}) \times S(\mathbb{R})^0$. If we now consider the two spaces $\mathcal{S}^G_{K_f}$ and $(\mathcal{S}^G_{K_f})^\dagger$, the first one defined with an arbitrary torus Z' the second one with $Z' = S$ then the arguments above imply that

$$\mathcal{S}^G_{K_f} = (\mathcal{S}^G_{K_f})^\dagger \times (S(\mathbb{R})^0/Z'(\mathbb{R})^0) \tag{11}$$

the second factor on the right hand side is isomorphic to $\mathbb{R}^b$ and since we are interested in the cohomology group of this space, it is irrelevant.

In certain situations we encounter cases where it is natural to choose a subgroup K_∞ which is slightly larger and not connected. If this is the case we denote the connected component $K_\infty^{(1)}$ and we get two locally symmetric spaces and a finite map

$$G(\mathbb{Q}) \setminus \left(G(\mathbb{R})/K_\infty^{(1)} \times G(\mathbb{A}_f)/K_f\right) \to G(\mathbb{Q}) \setminus (G(\mathbb{R})/K_\infty \times G(\mathbb{A}_f)/K_f) \tag{12}$$

This map is a covering if K_f is neat and the space on the right is a quotient of the space on the left by an action of the finite elementary abelian [2]-group $K_\infty/K_\infty^{(1)}$.

In accordance with the terminology in number theory we call the space $\mathcal{S}^G_{K_f}$ *narrow* if $K_\infty^{(1)} = K_\infty$ and in general we call the space on the left the *narrow cover* of $G(\mathbb{R})/K_\infty \times G(\mathbb{A}_f)/K_f$.

1.3 The group of connected components, the structure of $\pi_0(\mathcal{S}^G_{K_f})$.

If we keep our assumptions that $G/\mathbb{Q}$ is split and $G^{(1)}/\mathbb{Q}$ simply connected. Then it is straightforward to show that under our assumptions we have a bijection

$$\pi_0(\mathcal{S}^G_{K_f}) \xrightarrow{\sim} \pi_0(\mathcal{S}^{C'}_{K^{C'}_\infty \times K^{C'}_f}) \tag{13}$$

We have seen in the previous section that we can choose a consistent orientation on $X = G(\mathbb{R})/K_\infty$ provided K_∞ is narrow. Then it clear this induces also a consistent orientation on $\mathcal{S}^G_{K_f}$.

1.4 The Borel-Serre compactification

In general the space $\mathcal{S}^G_{K_f}$ is not compact. Recall that in the definition of this quotient the choice of a subtorus $Z'/\mathbb{Q}$ of $S/\mathbb{Q}$ enters. This If $Z' \neq S$ then the quotient will never be compact. But this kind of non compactness is "uninteresting". In the following we assume that $Z' = S$.

In this case we have the criterion of Borel - Harish-Chandra which says

The quotient space $\mathcal{S}^G_{K_f}$ is compact if and only if the group $G/\mathbb{Q}$ has no proper parabolic subgroup over $\mathbb{Q}$.

If we have a non trivial parabolic subgroup $P/\mathbb{Q}$ then we add a boundary part $\partial_P \mathcal{S}^G_{K_f}$ to $\mathcal{S}^G_{K_f}$ it will depend only the $G(\mathbb{Q})$-conjugacy class of P. We will describe this boundary piece later. We define the Borel-Serre boundary

$$\partial(\mathcal{S}^G_{K_f}) = \bigcup_P \partial_P \mathcal{S}^G_{K_f},$$

where P runs over the set of $G(\mathbb{Q})$ conjugacy classes of parabolic subgroups. We will put a topology on this space and if $Q \subset P$ then $\partial_Q \mathcal{S}^G_{K_f}$ will be in the closure of $\partial_P \mathcal{S}^G_{K_f}$. Then

$$\mathcal{S}^{\bar{G}}_{K_f} = \mathcal{S}^G_{K_f} \cup \partial(\mathcal{S}^G_{K_f})$$

will be a compact Hausdorff-space.

We describe the construction of this compactification in more detail. In chap4.pdf 2.7.1 we studied the group $\mathrm{Hom}(P, \mathbb{G}_m)$ and have seen that

$$\mathrm{Hom}(P, \mathbb{G}_m) \otimes \mathbb{Q} = \mathrm{Hom}(S_P, \mathbb{G}_m) \otimes \mathbb{Q}.$$

For any character $\gamma \in \mathrm{Hom}(P, \mathbb{G}_m)$ we get a homomorphism $\gamma_A : P(\mathbb{A}) \to \mathbb{G}_m(\mathbb{A}) = I_\mathbb{Q}$, the group of ideles. We have the idele norm $| \ | : \underline{x} \mapsto |\underline{x}|$ from the idele group to $\mathbb{R}^\times_{>0}$ and then we get by composing

$$|\gamma| : P(\mathbb{A}) \to \mathbb{R}^\times_{>0}.$$

It is obvious that we can extend this definition to characters $\gamma \in \mathrm{Hom}(P, \mathbb{G}_m) \otimes \mathbb{Q}$, for such a γ we find a positive non zero integer m such that $m\gamma \in \mathrm{Hom}(P, \mathbb{G}_m)$ and then we define

$$|\gamma| = (|m\gamma|)^{\frac{1}{m}}$$

Later we will even extend this to a homomorphism $\mathrm{Hom}(P, \mathbb{G}_m) \otimes \mathbb{C} \to \mathrm{Hom}(P(\mathbb{A}), \mathbb{C}^\times)$ by the rule $\boxed{\text{XtimesC}}$

$$\gamma \otimes z \mapsto |\gamma|^z \tag{14}$$

If we have a parabolic subgroup $P/\mathbb{Q}$ and a point $(x, \underline{g}_f) \in X \times G(\mathbb{A}_f)/K_f$ then we attach to it a (strictly positive) number

$$p(P, (x, \underline{g}_f)) = \mathrm{vol}_{d_x u}(U(\mathbb{Q}) \cap \underline{g}_f K_f \underline{g}_f^{-1} \backslash U(\mathbb{R})). \tag{15}$$

This needs explanation. The group $U(\mathbb{Q}) \cap \underline{g}_f K_f \underline{g}_f^{-1} = \Gamma_{U, \underline{g}_f}$ is a cocompact discrete lattice in $U(\mathbb{R})$, we can describe it as the group of elements $\gamma \in U(\mathbb{Q})$ which fix $\underline{g}_f K_f$, so it can be viewed as a lattice of integral elements where integrality is determined by $\underline{g}_f$. The point x defines a positive definite bilinear form B_{Θ_x} on the Lie algebra $\mathfrak{g} \otimes \mathbb{R}$, and this bilinear form can be restricted to the Lie-algebra $\mathfrak{u}_P \otimes \mathbb{R}$ and this provides a volume form $d_x u$ on $U(\mathbb{R})$ the above number is the volume of the nilmanifold $\Gamma_{U, \underline{g}_f} \backslash U(\mathbb{R})$ with respect to this measure.

If we are in the special case that $G = \mathrm{Sl}_2/\mathbb{Q}$ and $K_f = \mathrm{Sl}_2(\hat{\mathbb{Z}})$ then a parabolic subgroup P is a point $r = \frac{p}{q} \in \mathbb{P}^1(\mathbb{Q})$ (or ∞) and then $p(P, (z, 1))$ is small if z lies in a small Farey circle, i.e. it is close to r.

These numbers have some obvious properties

a) They are invariant under conjugation by an element $a \in G(\mathbb{Q})$, this means we have

$$p(a^{-1} P a, (x, \underline{g}_f)) = p(P, a(x, \underline{g}_f))$$

b) If $\underline{p} \in P(\mathbb{A})$ then we have

$$p(P, \underline{p}(x, \underline{g}_f)) = p(P, (x, \underline{g}_f)) |\rho_P|^2$$

The $G(\mathbb{Q})$ conjugacy classes of parabolic are in one to one correspondence with the subsets π' of the set relative simple roots π_G :The minimal parabolic corresponds to the empty set, the non proper parabolic subgroup $G/\mathbb{Q}$ corresponds to π_G itself. In general π' is the set of relative simple roots of the semi simple part of the reductive quotient of the parabolic subgroup. For a parabolic subgroup P' corresponding to π' we put $d(P') = \#(\pi_G \setminus \pi')$. For any $i \in \pi_G \setminus \pi'$ we have a fundamental character

$$\gamma_i : P \to \mathbb{G}_m.$$

We have the Borel-Serre compactification

$$i : \mathcal{S}^G_{K_f} \to \bar{\mathcal{S}}^G_{K_f}$$

The compactification is a manifold with corners, the boundary is stratified

$$\partial(\bar{\mathcal{S}}^G_{K_f}) = \bigcup_P \partial_P(\bar{\mathcal{S}}^G_{K_f})$$

where P runs over the $G(\mathbb{Q})$ conjugacy classes of parabolic subgroups. If $P \subset Q$ then the stratum $\partial_Q(\bar{\mathcal{S}}^G_{K_f}) \subset \overline{\partial_P(\bar{\mathcal{S}}^G_{K_f})}$.

Locally at a point $x \in \partial_P(\bar{\mathcal{S}}^G_{K_f})$ we find neighborhoods of x in $\bar{\mathcal{S}}^G_{K_f}$ which are of the form

$$U_x = W_x \times \{\dots, u_i, \dots\}_{i \in \pi_G \setminus \pi';\, 0 \le u_i < \epsilon} \tag{16}$$

where W_x is a neighborhood of x in the orbifold $\partial_P(\bar{\mathcal{S}}^G_{K_f})$. The intersection $\overset{\circ}{U}_x = U_x \cap \mathcal{S}^G_{K_f}$ consists of those elements where all the $u_i > 0$.

1.5 The easiest but very important example

If we take for instance $\mathcal{G}/\mathbb{Z} = \mathrm{Gl}_2/\mathbb{Z}$ and if we pick an integer N then we can define the congruence subgroup $K_f(N) = \prod_p K_p(N) \subset \mathcal{G}(\hat{\mathbb{Z}})$. It is defined by the condition that at all primes p dividing N the subgroup

$$K_p(N) = \{\gamma \in \mathcal{G}(\hat{\mathbb{Z}}) | \gamma \equiv \mathrm{Id} \mod p^{n_p}\}$$

where of course p^{n_p} is the exact power of p dividing N. At the other primes we take the full group of integral points. For the discussion of the example we put $K_f(N) = K_f$.

If we consider the action of $G(\mathbb{Q})$ on $G(\mathbb{A}_f)/K_f$ then the determinant gives us a map

$$\mathrm{Gl}_2(\mathbb{Q}) \backslash \mathrm{Gl}_2(\mathbb{A}_f)/K_f \to \mathbb{G}_m(\mathbb{A}_f)/\mathbb{Q}^* \mathfrak{U}_N$$

where $\mathfrak{U}_N$ is the group of unit ideles in $I_{\mathbb{Q},f} = \mathbb{G}_m(\mathbb{A}_f)$ which satisfy $u_p \equiv 1 \mod p^{n_p}$. This map is a bijection as one can easily see from strong approximation in Sl_2, and the right hand side is equal to $(\mathbb{Z}/N\mathbb{Z})^*/\{\pm 1\}$. At the infinite place we have that our symmetric space has two connected components, we have

$$X = \mathrm{Gl}_2(\mathbb{R})/SO(2) = \mathbb{C} \setminus \mathbb{R} = \mathbb{H}_+ \cup \mathbb{H}_-$$

where $\mathbb{H}_\pm$ are the upper and lower half plane, respectively. We have a complex structure on X which is invariant under the action of $\mathrm{Gl}_2(\mathbb{R})$. The connected components of this quotient correspond (one to one)to the elements in

$$\mathbb{G}_m(\mathbb{A})/\mathbb{G}_m(\mathbb{Q})(\mathbb{G}_m(\mathbb{R})^0 \times \mathfrak{U}_N) = I_{\mathbb{Q}}/\mathbb{Q}^* \mathbb{R}^*_{>0} \mathfrak{U}_N = (\mathbb{Z}/N\mathbb{Z})^*.$$

We put $\Gamma(N) = G(\mathbb{Q}) \cap K_f$ and then the components are

$$\Gamma(N) \backslash \begin{pmatrix} \underline{t}_\infty & 0 \\ 0 & 1 \end{pmatrix} H_+ \times \begin{pmatrix} \underline{t}_f & 0 \\ 0 & 1 \end{pmatrix} K_f/K_f$$

where $\underline{t}$ runs through a set of representatives of $I_{\mathbb{Q}}/\mathbb{Q}^* \mathbb{R}^*_{>0} \mathfrak{U}_N = (\mathbb{Z}/N\mathbb{Z})^*$.

These connected components are Riemann surfaces which are not compact. They can be compactified by adding a finite number of points, the so called *cusps*. These are in one to one correspondence with the orbits of $\Gamma(N)$ on $\mathbb{P}^1(\mathbb{Q})$ (see reduction theory).

(Compare to Borel-Serre)

2 The sheaves, their cohomology and the action of the Hecke algebra

2.1 Basic data and simple properties

Let $\mathcal{M}$ be a finite dimensional $\mathbb{Q}$-vector space, let

$$r : G/\mathbb{Q} \to \mathrm{Gl}(\mathcal{M})$$

a rational representation. This representation r provides a sheaf $\tilde{\mathcal{M}}$ on $\mathcal{S}^G_{K_f}$ whose sections on an open subset $V \subset \mathcal{S}^G_{K_f}$ are given by

$$\tilde{\mathcal{M}}(V) = \{s : \pi^{-1}(V) \to \mathcal{M} | s \text{ locally constant and } s(\gamma v) = r(\gamma)s(v), \gamma \in G(\mathbb{Q})\}.$$

We call this the *right module description* of $\tilde{\mathcal{M}}$.

We can describe the stalk of the sheaf in a point $y \in \mathcal{S}^G_{K_f}$, we choose a point $\underline{x} = (x_\infty, \underline{g}_f)$ in $\pi^{-1}(y)$ and we choose a neighborhood V_y as in 1.2.1. Then we can evaluate an element $s \in \tilde{\mathcal{M}}(V_y)$ at $\underline{x}$ and this must be an element in $\mathcal{M}^{\Gamma_{x_\infty}}$, this means we get an isomorphism

$$e_{\underline{x}} : \tilde{\mathcal{M}}_y \xrightarrow{\sim} \mathcal{M}^{\Gamma_{x_\infty}}.$$

By definition we have $e_{\gamma\underline{x}} = \gamma e_{\underline{x}}$.

In our previous example such a representation r is of the following form: We take the homogeneous polynomials $P(X,Y)$ of degree n in two variables and with coefficients in $\mathbb{Q}$. This is a $\mathbb{Q}$-vector space of dimension $n+1$, we choose another integer m and now we define an action of $\mathrm{Gl}_2/\mathbb{Q}$ on this vector space

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} P(X,Y) = P(aX+cY, bX+dY)\det(\begin{pmatrix} a & b \\ c & d \end{pmatrix})^m.$$

This Gl_2 module will be called $\mathcal{M}_n[m]$ and it yields sheaves $\tilde{\mathcal{M}}_n[m]$ on our space $\mathcal{S}^G_{K_f}$.

It is sometimes reasonable to start from an absolutely irreducible representation and therefore we consider representations defined after a base change $r : G \times_{\mathbb{Q}} F \to \mathrm{Gl}(\mathcal{M})$ where $\mathcal{M}$ is a finite dimensional F vector space and the action is absolutely irreducible. Since $G(\mathbb{Q})$ acts on $\mathcal{M}$ we can define a sheaf $\tilde{\mathcal{M}}$ of F vector spaces.

If our group is a torus $T/\mathbb{Q}$, then we can find a finite normal extension $E/\mathbb{Q}$ such that $T \times_{\mathbb{Q}} E$ is split and then we denote by

$$X^*(T) = \mathrm{Hom}(T \times E, \mathbb{G}_m) \text{ resp } X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T \times_{\mathbb{Q}} E) \tag{17}$$

the character (resp.) cocharacter module of $T/\mathbb{Q}$. Both modules come with an action of the Galois group $\mathrm{Gal}(E/\mathbb{Q})$. In this case an absolutely irreducible representation is simply a character $\gamma \in X^*(T)$ and we denote by $E[\gamma]$ a one dimensional E-vector space on which $T/\mathbb{Q}$ acts by γ. Then $\tilde{E[\gamma]}$ is a sheaf of F-vector spaces on $S^T_{K^T_f}$.

2.1.1 Integral coefficient systems

We assume again that we have a rational representation of our group $G/\mathbb{Q}$, the following considerations easily generalize to the case of an arbitrary number field as base field. We want to define a subsheaf $\tilde{\mathcal{M}}_{\mathbb{Z}} \subset \tilde{\mathcal{M}}$. To do this we embed the field $\mathbb{Q} \hookrightarrow \mathbb{A}_f$ and we consider the resulting sheaf of $\mathbb{A}_f$-modules $\tilde{\mathcal{M}} \otimes \mathbb{A}_f$. We consider the diagram

$$\begin{tikzcd}
 & G(\mathbb{R})/K_\infty \times (G(\mathbb{A}_f)/K_f) \arrow[dr, "\pi"] & \\
G(\mathbb{R})/K_\infty \times G(\mathbb{A}_f) \arrow[ur, "\pi'"] \arrow[rr, "\Pi"] \arrow[dr, "\Pi_1"'] & & \mathcal{S}^G_{K_f} \\
 & G(\mathbb{Q})\backslash G(\mathbb{R})/K_\infty \times G(\mathbb{A}_f) \arrow[ur, "\Pi_2"'] &
\end{tikzcd} \tag{18}$$

this means that the division by the action by K_f on the right and by $G(\mathbb{Q})$ on the left (this gives Π) is divided into two steps: In the lower diagram the projection Π_1 is division by the action of $G(\mathbb{Q})$ and then Π_2 gives the division by the action of K_f on the right.

The sheaf $\tilde{\mathcal{M}} \otimes_{\mathbb{Q}} \mathbb{A}_f$ can be rewritten. For any open subset $V \subset \mathcal{S}^G_{K_f}$ we consider $W = \Pi^{-1}(V)$ and by definition

$$\tilde{\mathcal{M}} \otimes_{\mathbb{Q}} \mathbb{A}_f(V) = \{s : \Pi^{-1}(W) \to \mathcal{M} \otimes_{\mathbb{Q}} \mathbb{A}_f | s(\gamma(x_\infty, \underline{g}_f \underline{k}_f)) = \gamma(s(x_\infty, \underline{g}_f)),$$

where these sections s are locally constant in the variable x_∞. For any $s \in \mathcal{M} \otimes \mathbb{A}_f(V)$ we define a map $\tilde{s} : W \to \mathcal{M} \otimes \mathbb{A}_f$ by the formula

$$\tilde{s}(x_\infty, \underline{g}_f) = \underline{g}_f^{-1} s(x_\infty, \underline{g}_f \underline{K}_f),$$

this makes sense because $\mathcal{M} \otimes \mathbb{A}_f$ is a $G(\mathbb{A}_f)-$ module. For $\gamma \in G(\mathbb{Q})$ we have $\tilde{s}(\gamma(x_\infty, \underline{g}_f)) = \tilde{s}((x_\infty, \underline{g}_f))$ hence we can view $\tilde{s}$ as a map

$$\tilde{s} : G(\mathbb{Q})\backslash G(\mathbb{R})/K_\infty \times G(\mathbb{A}_f) \to \mathcal{M} \otimes_{\mathbb{Q}} \mathbb{A}_f.$$

We consider the projection

$$\Pi_2 : G(\mathbb{Q})\backslash G(\mathbb{R})/K_\infty \times G(\mathbb{A}_f) \to G(\mathbb{Q})\backslash G(\mathbb{R})/K_\infty \times G(\mathbb{A}_f)/K_f = \mathcal{S}^G_{K_f}$$

and then it becomes clear that $\tilde{\mathcal{M}} \otimes \mathbb{A}_f$ can be described as

$$\widetilde{\mathcal{M} \otimes \mathbb{A}_f}(V) = \{\tilde{s} : (\Pi_1^{-1}(V) \to \mathcal{M} \otimes_{\mathbb{Q}} \mathbb{A}_f |$$
$$\tilde{s} \text{ locally constant in } x_\infty \text{ and } \tilde{s}((x_\infty, \underline{g}_f \underline{k}_f)) = \underline{k}_f^{-1} \tilde{s}((x_\infty, \underline{g}_f))\}.$$

Hence we have identified the sheaf $\tilde{\mathcal{M}} \otimes_{\mathbb{Q}} \mathbb{A}_f$ which is defined in terms of the action of $G(\mathbb{Q})$ on $\mathcal{M}$ to the sheaf $\widetilde{\mathcal{M} \otimes_{\mathbb{Q}}} \mathbb{A}_f$ which is defined in terms of the action of K_f on $\mathcal{M} \otimes_{\mathbb{Q}} \mathbb{A}_f$.

Now we assume that our group scheme $G/\mathbb{Q}$ is the generic fiber of a flat group scheme $\mathcal{G}/\mathrm{Spec}(\mathbb{Z})$ (See 1.2). We choose our maximal compact subgroup $K_f = \prod_p K_p$ such that $K_p \subset \mathcal{G}(\mathbb{Z}_p)$ and with equality for all primes outside a finite set Σ. We can extend the vector space $\mathcal{M}$ to a free $\mathbb{Z}$ module $\tilde{\mathcal{M}}_{\mathbb{Z}}$ of the same rank which provides a representation $\mathcal{G}/\mathrm{Spec}(\mathbb{Z}) \to \mathrm{Gl}(\mathcal{M}_{\mathbb{Z}})$.

As usual $\hat{\mathbb{Z}}$ will be the ring of integral adeles. Then it is clear that $\mathcal{M}_{\mathbb{Z}} \otimes \hat{\mathbb{Z}} \subset \mathcal{M} \otimes_{\mathbb{Q}} \mathbb{A}_f$ is invariant under K_f and hence we can define the sub sheaf

$$\widetilde{\mathcal{M}_{\mathbb{Z}} \otimes} \hat{\mathbb{Z}} \subset \widetilde{\mathcal{M} \otimes_{\mathbb{Q}}} \mathbb{A}_f,$$

this is the sheave where the sections $\tilde{s}$ have values in $\mathcal{M}_{\mathbb{Z}} \otimes \hat{\mathbb{Z}}$. We put

$$\tilde{\mathcal{M}}_{\mathbb{Z}} = \widetilde{\mathcal{M}_{\mathbb{Z}} \otimes} \hat{\mathbb{Z}} \cap \tilde{\mathcal{M}},$$

of course it depends on our choice of $\mathcal{M}_{\mathbb{Z}} \subset \mathcal{M}$. We get two exact sequences of sheaves

$$\begin{array}{ccccccc} 0 & \to & \tilde{\mathcal{M}}_{\mathbb{Z}} & \to & \tilde{\mathcal{M}} & \to & \widetilde{\mathcal{M} \otimes (\mathbb{Q}}/\mathbb{Z}) \to 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & \widetilde{\mathcal{M} \otimes} \hat{\mathbb{Z}} & \to & \widetilde{\mathcal{M} \otimes_{\mathbb{Q}}} \mathbb{A}_f & \to & \widetilde{\mathcal{M} \otimes} (\mathbb{A}_f/\hat{\mathbb{Z}}) \to 0 \end{array}$$

The far most vertical arrow to the right is an isomorphism, the inclusions $\mathbb{Z} \hookrightarrow \hat{\mathbb{Z}}$ and $\mathbb{Q} \hookrightarrow \mathbb{A}_f$ are flat. Writing down the resulting long exact sequences provides a diagram

$$\begin{array}{ccccc} \to & H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathbb{Z}}) & \xrightarrow{j_{\mathbb{Q}}} & H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}) & \to \\ & \downarrow i_{\mathbb{Z}} & & \downarrow i_{\mathbb{Q}} & . \\ \to & H^\bullet(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M} \otimes} \hat{\mathbb{Z}}) & \xrightarrow{j_{\mathbb{A}}} & H^\bullet(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M} \otimes_{\mathbb{Q}}} \mathbb{A}_f) & \to \end{array}$$

The above remarks imply that the vertical arrows are injective, the horizontal arrows in the middle have the same kernel and kokernel. This implies

Proposition 2.1. *The integral cohomology*

$$H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathbb{Z}})$$

consists of those elements in $H^\bullet(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M} \otimes} \hat{\mathbb{Z}})$ *which under* $j_{\mathbb{A}}$ *go to an element in the image under* $i_{\mathbb{Q}}$ *or in brief*

$$H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathbb{Z}}) = j_{\mathbb{A}}^{-1}(\mathrm{im}(i_{\mathbb{Q}}))$$

This generalizes to the case where we have a representation $r : G \times F \to \mathrm{Gl}(\mathcal{M})$ where $\mathcal{M}$ is a vector space over F. If our group scheme is an extension of a flat group scheme $\mathcal{G}/\mathrm{Spec}(\mathcal{O}_F)$ then can find a lattice $\mathcal{M}_{\mathcal{O}_F}$ which yields a representation of $\mathcal{G} \to \mathrm{Gl}(\mathcal{M}_{\mathcal{O}_F})$. Then we can define the sheaf $\tilde{\mathcal{M}}_{\mathcal{O}_F}$ and define the cohomology groups

$$H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathcal{O}_F})$$

2.1.2 Sheaves with support conditions

We can extend the sheaves to the Borel-Serre compactification. We have the inclusion

$$i : \mathcal{S}^G_{K_f} \to \bar{\mathcal{S}}^G_{K_f}$$

and we can extend the sheaf by the direct image functor $i_*(\tilde{\mathcal{M}})$. It follows easily from the description of the neighborhood of a point in the boundary (see 16) that $R^q i_*(\mathcal{M}) = 0$ for $q = 0$ and hence we get that the restriction map

$$H^\bullet(\bar{\mathcal{S}}^G_{K_f}, i_*(\tilde{\mathcal{M}})) \to H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})$$

is an isomorphism.

We may also extend the sheaf by zero (See [Vol I], 4.7.1), this yields the sheaf $i_!(\tilde{\mathcal{M}})$ whose stalk at $x \in \mathcal{S}^G_{K_f}$ is equal to $\tilde{\mathcal{M}}_x$ and whose stalk ist zero in points $x \in \partial\mathcal{S}^G_{K_f}$. Then we have by definition

$$H^\bullet_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}) = H^\bullet(\bar{\mathcal{S}}^G_{K_f}, i_!(\tilde{\mathcal{M}}))$$

this is the cohomology with compact supports.

We are interested in the *integral* cohomology modules $H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\mathbb{Z}), H^\bullet_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\mathbb{Z})$. We introduced the boundary $\partial\mathcal{S}^G_{K_f}$ of the Borel-Serre compactification then we have a first general theorem, which is due to Raghunathan.

Theorem 2.1. *(i) The cohomology groups $H^i(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\mathbb{Z}), H^i(\partial\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\mathbb{Z})$ and $H^i_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\mathbb{Z})$ are finitely generated.*

(ii) We have the well known **fundamental long exact sequence** *in cohomology*

$$\longrightarrow H^{i-1}(\partial\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\mathbb{Z}) \longrightarrow H^i_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\mathbb{Z}) \longrightarrow H^i(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\mathbb{Z}) \xrightarrow{r} H^i(\partial\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\mathbb{Z}) \longrightarrow .$$

We introduce the notation $H_?(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\mathbb{Z})$ meaning that for ? = blank we take the cohomology without support, for ? = c we take the cohomology with compact support and for ? = ∂ we take cohomology of the boundary of the Borel-Serre compactification. Later on we will also allow ? =! this denotes the inner cohomology. The above proposition holds for all choices of ?.

Let $\Sigma = \{P_1, \dots, P_s\}$ be a finite set of parabolic subgroups, we assume that none of them is a subgroup of another parabolic subgroup in this set. The union of the closures of the strata

$$\bigcup_i \bigcup_{Q \subset P_i} \partial_Q(\mathcal{S}^G_{K_f}) = \partial_\Sigma(\mathcal{S}^G_{K_f})$$

is closed .

$$j_\Sigma : \mathcal{S}^G_{K_f} \to \bar{\mathcal{S}}^G_{K_f} \setminus \partial_\Sigma(\bar{\mathcal{S}}^G_{K_f}), j^\Sigma : \bar{\mathcal{S}}^G_{K_f} \setminus \partial_\Sigma(\bar{\mathcal{S}}^G_{K_f}) \to \bar{\mathcal{S}}^G_{K_f}.$$

The inclusion $i : \mathcal{S}^G_{K_f} \to \bar{\mathcal{S}}^G_{K_f}$ is the composition $i = j^\Sigma \circ j_\Sigma$ we define the intermediate extension

$$i_{\Sigma,*,!}(\tilde{\mathcal{M}}) = j^\Sigma_! \circ j_{\Sigma,*}(\tilde{\mathcal{M}}). \tag{19}$$

For these sheaves with intermediate support conditions we can also formulate assertion like in the above theorem. Later we will discuss an increasing filtration

$$W_0 H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}) = H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}) \subset W_1 H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}) \subset \dots \tag{20}$$

on the cohomology, the bottom of this filtration will be the inner cohomology,

2.1.3 Functorial properties

The groups have some functorial properties if we vary the level subgroup K_f. If we pass to a smaller open subgroup $K'_f \subset K_f$ then we get a surjective map

$$\pi_{K_f, K'_f} : \mathcal{S}^G_{K'_f} \to \mathcal{S}^G_{K_f},$$

whose fibers are finite. This induces maps between cohomology groups

$$\pi_{K'_f, K_f}{}^\bullet_? : H^\bullet_?(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\mathbb{Z}) \to H^\bullet_?(\mathcal{S}^G_{K'_f}, \tilde{\mathcal{M}}_\mathbb{Z}),$$

for $? = c$ we exploit the fact that the fibers are finite.

We construct homomorphisms in the opposite direction. We exploit the finiteness a second time and find that the direct image functor $(\pi_{K'_f, K_f})_*$ is exact and hence

$$H^\bullet_?(\mathcal{S}^G_{K'_f}, \tilde{\mathcal{M}}_\mathbb{Z}) = H^\bullet_?(\mathcal{S}^G_{K_f}, (\pi_{K'_f, K_f})_*(\tilde{\mathcal{M}}_\mathbb{Z})).$$

We define a trace homomorphism $(\pi_{K'_f, K_f})_*(\tilde{\mathcal{M}}_\mathbb{Z}) \to \tilde{\mathcal{M}}_\mathbb{Z}$: A section $s \in (\pi_{K'_f, K_f})_*(\tilde{\mathcal{M}}_\mathbb{Z})(V)$ is a map $\tilde{s} : \Pi^{-1}(V) \to \tilde{\mathcal{M}}_\lambda \otimes \hat{\mathbb{Z}}$ such that

$$\tilde{s}(\gamma(x_\infty, \underline{g}_f \underline{k}'_f)) = (k'_f)^{-1} \tilde{s}((x_\infty, \underline{g}_f)) \text{ for all } k'_f \in K'_f.$$

This is a section of $\tilde{\mathcal{M}}_\mathbb{Z}$ if and only if the corresponding section s takes values in $\mathcal{M}$. Then we define

$$\mathrm{tr}(\tilde{s})(x_\infty, \underline{g}_f) = \sum_{\underline{\xi}_f \in K_f/K'_f} \underline{\xi}_f^{-1} \tilde{s}(x_\infty, \underline{g}_f)$$

and this now satisfies

$$\mathrm{tr}(\tilde{s})(\gamma(x_\infty, \underline{g}_f \underline{k}_f)) = k_f^{-1} \tilde{s}((x_\infty, \underline{g}_f)) \text{ for all } k_f \in K_f.$$

and since the corresponding section $\mathrm{tr}(s)$ takes values in $\mathcal{M}$ we see that $\mathrm{tr}(\tilde{s}) \in \tilde{\mathcal{M}}_\mathbb{Z}(V)$.

Remark: It may happen that this trace map is not the optimal choice, it can be the integral multiple of another homomorphism between these two sheaves. This happens the intersection $C(\mathbb{Q}) \cap K_f$ is non trivial.

Then the homomorphism between the sheaves induces

$$H^\bullet_?(\mathcal{S}^G_{K'_f}, \tilde{\mathcal{M}}_\mathbb{Z}) = H^\bullet_?(\mathcal{S}^G_{K_f}, (\pi_{K'_f, K_f})_*(\tilde{\mathcal{M}}_\mathbb{Z})) \xrightarrow{\pi_{K'_f, K_f}} {}^\bullet H^\bullet_?(\mathcal{S}^G_{K_f}, (\tilde{\mathcal{M}}_\mathbb{Z})).$$

Later on our maps between the spaces will be denoted $\pi, \pi_1, \dots$ and the notation simplifies accordingly.

2.2 The rational cohomology groups and the Hecke-algebra

In this section we assume that our coefficient systems are obtained from rational representations of a reductive group scheme $G/\mathbb{Q}$ hence they are $\mathbb{Q}$ vector spaces. We discuss some further properties of the *rational* cohomology groups

$$H^i_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathbb{Q}}), H^i(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathbb{Q}}) \dots .$$

These cohomology groups are finite dimensional $\mathbb{Q}$-vector spaces and we have the same exact fundamental sequence. We can also pass to the direct limit

$$H^i_?(\mathcal{S}^G, \tilde{\mathcal{M}}) = \lim_{K_f} H^i_?(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}).$$

Proposition 2.2. *On these limits we have an action of the group $\pi_0(G(\mathbb{R})) \times G(\mathbb{A}_f)$. We recover the cohomology with fixed level K_f by taking the invariants under this action, i.e. we have*

$$H^i_?(\mathcal{S}^G, \tilde{\mathcal{M}})^{K_f} = H^i_?(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})$$

To define this action we represent an element in $\pi_0(G(\mathbb{R}))$ by an element k_∞ in the in the normalizer of K_∞ in $G(\mathbb{R})$. An element $\underline{x} = (k_\infty, \underline{x}_f) \in G(\mathbb{R}) \times G(\mathbb{A}_f))$ defines by multiplication from the right an isomorphism of spaces

$$m_{\underline{x}} : G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K_f \xrightarrow{\sim} G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/\underline{x}_f^{-1} K_f \underline{x}_f.$$

It is clear from the definition that $m_{\underline{x}}$ yields a bijection between the fibers $\pi^{-1}(\underline{\bar{g}}), \underline{\bar{g}} \in G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K_f$ and $\pi^{-1}(m_{\underline{x}})(\underline{\bar{g}})$ and since the sheaf is described in terms of the left action by $G(\mathbb{Q})$ we get $m_{\underline{x},*}(\tilde{\mathcal{M}}) = \tilde{\mathcal{M}}$. Passing to the limit gives us the action. The second assertion is obvious, but here we need that our coefficients are $\mathbb{Q}$ vector spaces, we need to take averages.

We introduce the Hecke algebra, it acts on the cohomology with a fixed level. It consists of the compactly supported $\mathbb{Q}-$valued functions $h : G(\mathbb{A}_f) \to \mathbb{Q}$ which are biinvariant under the action of K_f and is denoted by $\mathcal{H} = \mathcal{H}_{K_f} = \mathcal{C}_c(G(\mathbb{A}_f)//K_f, \mathbb{Q})$. An element $h \in \mathcal{H}$ is simply a finite linear combination of characteristic functions $h = \sum c_{\underline{a}_f} \chi_{K_f \underline{a}_f K_f}$ with rational coefficients $c_{\underline{a}_f}$. The algebra structure is given by convolution with respect to the Haar measure on $G(\mathbb{A}_f)$ which gives volume 1 to K_f. This convolution is given by

$$h_1 * h_2(\underline{g}_f) = \int_{G(\mathbb{A}_f)} h_1(\underline{x}_f) h_2(\underline{x}_f{}^{-1} \underline{g}_f) d\underline{x}_f.$$

With this choice of the measure it is clear that the characteristic function of K_f is the identity element of this algebra.

The action of the group $G(\mathbb{A}_f)$ induces an action of $\mathcal{H}_{K_f}$ on the cohomology with fixed level $H^i_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}), H^i(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}), \cdots$: For an element $v \in H^i_?(\mathcal{S}^G, \tilde{\mathcal{M}})$ we define

$$T_h(v) = \int_{G(\mathbb{A}_f)} h(\underline{x}_f) \underline{x}_f v d\underline{x}_f,$$

where the measure is still the one that gives volume 1 to K_f. Clearly we have $T_{h_1 * h_2} = T_{h_1} T_{h_2}$.

(Actually the integral is a finite sum: We find an open subgroup $K'_f \subset K_f$ such that v is fixed by K'_f and then it is clear that

$$T_h(v) = \int_{G(\mathbb{A}_f)} h(\underline{x}_f)\underline{x}_f v d\underline{x}_f = \imath[K_f : K'_f] \sum_{\underline{a}_f} \sum_{\underline{\xi}_f \in G(\mathbb{A}_f)/K'_f} c_{\underline{a}_f} \chi_{K_f \underline{a}_f K_f)}(\underline{\xi}_f)\underline{\xi}_f v.$$

This makes it clear why we need rational coefficients .)

It is clear that $T_h(v) \in H^i_?(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})$ and hence T_h gives us an endomorphism of $H^i_?(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})$. We will show later that we also get endomorphisms on the cohomology of the boundary and therefore $\mathcal{H}$ also acts on the long exact sequence (Seq) .

If our function h is the characteristic function of a double coset $K_f \underline{x}_f K_f$ then we change notation and write $T_h = \mathbf{ch}(\underline{x}_f)$. We give another definition of the Hecke operator $\mathbf{ch}(\underline{x}_f)$ in terms of sheaf cohomology. We imitate the construction of the Hecke operators in Chap.II 2.2. We put $K_f^{(\underline{x}_f)} = K_f \cap \underline{x}_f K_f \underline{x}_f^{-1}$ and consider the diagram

$$\begin{array}{ccccc} S^G_{K_f^{(\underline{x}_f)}} & & \xrightarrow{m_{\underline{x}_f}} & & S^G_{K_f^{(\underline{x}_f^{-1})}} \\ & \searrow \pi_1 & & \swarrow \pi_2 & \\ & & \mathcal{S}^G_{K_f} & & \end{array} \qquad Hop1$$

where $m_{\underline{x}_f}$ is induced by the multiplication by $\underline{x}_f$ from the right. This yields in cohomology

$$H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}) \xrightarrow{\pi_1^\bullet} H^\bullet(S^G_{K_f^{(\underline{x}_f)}}, \tilde{\mathcal{M}}) \xrightarrow{m_{\underline{x}_f,*}} H^\bullet(S^G_{K_f^{(\underline{x}_f^{-1})}}, m_{\underline{x}_f,*}(\tilde{\mathcal{M}})) \qquad (Hop2).$$

Since we described the sheaf by the action of $G(\mathbb{Q})$ and the map $m_{\underline{x}_f}$ by multiplication from the right we have $m_{\underline{x}_f,*}(\tilde{\mathcal{M}}) = \tilde{\mathcal{M}}$, this yields an isomorphism $i_{\underline{x}_f}$. Since π_2 is finite we have the trace homomorphism

$$\pi_{2,\bullet} : H^\bullet(S^G_{K_f^{(\underline{x}_f)^{-1}}}, \tilde{\mathcal{M}}) \to H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})$$

and the composition is our Hecke operator

$$\pi_{2,\bullet} \circ i_{\underline{x}_f} \circ m_{\underline{x}_f,*} \circ \pi_1^\bullet = \mathbf{ch}(\underline{x}_f) : H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}) \to H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}).$$

This is simpler than the construction Chap.II 2.2. because we do not need the intermediate homomorphism u_α. But we we do not get Hecke operators on the integral cohomology.

2.3 The integral cohomology as a module under the Hecke algebra

We resume the discussion of the integral Hecke algebra acting on $H^\bullet_?(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\mathbb{Z})$ from Chapter II. Inside the Hecke algebra we may also look at the sub algebra of $\mathbb{Z}$ -valued functions. This is in principle the algebra which is generated by the characteristic functions $\mathbf{ch}(\underline{x}_f)$ of double cosets $K_f\underline{x}_fK_f$. These characteristic functions act by convolution on the cohomology $H^\bullet(\mathcal{S}^G_{K_f},\mathcal{M})$ but this does not induce an action on the integral cohomology. Our next aim is to define a fractional ideal $\mathfrak{n}(\underline{x}_f)\subset\mathbb{Q}$ or more generally $\mathfrak{n}(\underline{x}_f)\subset F$ such that for any $a\in\mathfrak{n}(\underline{x}_f)$ we can define an endomorphism

$$a\cdot\mathbf{ch}(\underline{x}_f): H^\bullet(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda)\to H^\bullet(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda\ )$$

and if we send this to the rational cohomology then on $H^\bullet(\mathcal{S}^G_{K_f},\mathcal{M})$ this will be the convolution endomorphism induced by $\mathbf{ch}(\underline{x}_f)$ multiplied by a.

This ideal will depend on $\underline{x}_f$ and on λ and further down we compute it in special cases.

(iv) These endomorphisms $a\cdot\mathbf{ch}(\underline{x}_f)$ generate an algebra $\mathcal{H}^{(\lambda)}_{\mathbb{Z}}$ acting on the integral cohomology and the arrows in our sequence above commute with this action.

v) Moreover, we have an action of $\pi_0(G(\mathbb{R}))$ on the above sequence and this action also commutes with the action of the Hecke algebra. Hence we know that our above sequence is long exact sequence of $\pi_0(G(\mathbb{R}))\times\mathcal{H}^{(\lambda)\cdot}_{\mathbb{Z}}$

We come to the definition of the ideal.

If we are in the special case that our group has strong approximation then we have

$$\Gamma\backslash X \xrightarrow{\sim} G(\mathbb{Q})\backslash X\times G(\mathbb{A}_f)/K_f$$

(See (10)). We pick an element $\alpha\in G(\mathbb{Q})$. In Chap. II , 2.2 we defined the Hecke operator $T(\alpha,u_\alpha)$ where $u_\alpha:\mathcal{M}^{(\alpha)}\to\mathcal{M}$ is the canonical choice. Let us denote the image of α in $G(\mathbb{A}_f)$ by $\underline{\alpha}_f$. We just attached a Hecke operator to the double coset $K_f\underline{\alpha}_f.K_f$. We have the diagram of spaces

$$\begin{tikzcd}
\Gamma(\alpha)\backslash X \arrow[r] \arrow[d, "l(\alpha^{-1})"] & G(\mathbb{Q})\backslash G(\mathbb{R})/K_\infty\times G(\mathbb{A}_f)/K_f^{\underline{\alpha}_f} \arrow[d, "r(\underline{\alpha}_f)"] \\
\Gamma(\alpha^{-1})\backslash X \arrow[r] & G(\mathbb{Q})\backslash G(\mathbb{R})/K_\infty\times G(\mathbb{A}_f)/K_f^{\underline{\alpha}_f^{-1}}
\end{tikzcd} \tag{21}$$

Here the horizontal arrows are the isomorphisms provided by strong approximation, the arrow $l(\alpha^{-1})$ is the isomorphism induced by left multiplication by α^{-1} and $r(\underline{\alpha}_f)$ by right multiplication by $\underline{\alpha}_f$. These two maps enter in the definition of the Hecke operators $T(\alpha^{-1},u_{\alpha^{-1}})$ and $\mathbf{ch}(\underline{\alpha}_f)$ and a straightforward inspection of the sheaves yields

$$\mathbf{ch}(\underline{\alpha}_f)=T(\alpha^{-1},u_{\alpha^{-1}}).$$

Hence we can conclude that under this assumption our newly defined Hecke operators coincide with the Hecke operators defined in Chap. II. This also tells us what we have to do if we want to define Hecke operators on integral cohomology.

To define the action of the Hecke algebra on the integral cohomology without the assumption of simple connectedness we have to translate their definition into the right module description. Then our sheaf $\widetilde{\mathcal{M}\otimes\mathbb{A}_f}$ is described by the action of K_f on $\mathcal{M}\otimes\mathbb{A}_f$ and this allows us to define the sub sheaf $\mathcal{M}_\mathbb{Z}\otimes\hat{\mathbb{Z}}$. We look at the same diagram. But now the sheaf $m_{\underline{x}_f,*}(\widetilde{\mathcal{M}\otimes\mathbb{A}_f})$ is the sheaf described by the the $K_f^{(\underline{x}_f)^{-1}}$ module $(\mathcal{M}\otimes\mathbb{A}_f)^{(\underline{x}_f)}$. This module is $\mathcal{M}\otimes\mathbb{A}_f$ as abelian group, but $\underline{g}_f\in K_f^{(\underline{x}_f)^{-1}}$ acts by $\underline{m}_f\mapsto \underline{x}_f\underline{g}\underline{x}_f^{-1}\underline{m}_f$. The map $\underline{m}_f\to\underline{x}_f\underline{m}_f$ induces an isomorphism $[\underline{x}_f]$ between the two $K_f^{(\underline{x}_f)^{-1}}$ modules $(\mathcal{M}\otimes\mathbb{A}_f)^{(\underline{x}_f)}$ and $(\mathcal{M}\otimes\mathbb{A}_f)$. We now consider the diagram $Hop1$. and replace in the sequence of maps the homomorphism $i_{\underline{x}_f}$ by the map $[\underline{x}_f^\bullet]$ induced by the isomorphism $[\underline{x}_f]$ between the sheaves. Then we can proceed as before and get an operator

$$p_{1,*}\circ[\underline{x}_f]^\bullet\circ m_{\underline{x}_f,*}\circ p_2^*=\mathbf{ch}(\underline{x}_f).$$

It is straightforward to check that this operator is an extension $\pi_{2,\bullet}\circ i_{\underline{x}_f}\circ m_{\underline{x}_f,*}\circ\pi_1^\bullet$ to $H^\bullet(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}\otimes\mathbb{A}_f)$.

Our right module sheaf contains the submodule sheaf $\tilde{\mathcal{M}}_\lambda\otimes\hat{\mathbb{Z}}$, we can write the same diagram but now it can happen that $[\underline{x}_f]$ does not map $\mathcal{M}_\mathbb{Z}\otimes\hat{\mathbb{Z}}$ into itself. This forces us to make the following definition

$$\mathfrak{n}(\underline{x}_f)=\{a\in\mathbb{Q}|\ [a\underline{x}_f]:\mathcal{M}_\mathbb{Z}\otimes\hat{\mathbb{Z}}\subset\mathcal{M}_\mathbb{Z}\otimes\hat{\mathbb{Z}}\}$$

Then we can again go back to our above diagram and it becomes clear that we can define Hecke operators

$$a\cdot\mathbf{ch}(\underline{x}_f):H^\bullet(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\mathbb{Z})\to H^\bullet(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\mathbb{Z})\text{ for all }a\in\mathfrak{n}(\underline{x}_f).$$

2.3.1 The case of a split group

We want to discuss this in the special case that $\mathcal{G}/\mathrm{Spec}(\mathbb{Z})$ is split reductive, we assume that the derived group $\mathcal{G}^{(1)}/\mathrm{Spec}(\mathbb{Z})$ is simply connected, we assume that the center $\mathcal{C}/\mathrm{Spec}(\mathbb{Z})$ is a (split)-torus and that $\mathcal{C}\cap\mathcal{G}^{(1)}$ is equal to the center $Z^{(1)}$ of $\mathcal{G}^{(1)}$. This center is a finite multiplicative group scheme (See 1.1).

Accordingly we get decompositions up to isogeny of the character and cocharacter modules of the torus

$$X^*(\mathcal{T})\hookrightarrow X^*(\mathcal{T}^{(1)})\oplus X^*(\mathcal{C})\ X_*(\mathcal{T}^{(1)})\oplus X_*(\mathcal{C})\hookrightarrow X_*(\mathcal{T})\tag{22}$$

they become isomorphisms after taking the tensor product by $\mathbb{Q}$. We numerate the simple positive roots $I=\{1,2,\dots,r\}=\{\alpha_1,\alpha_2,\dots,\alpha_r\}\subset X^*(\mathcal{T})$ and we define dominant fundamental weights $\gamma_i\in X^*(\mathcal{T})_\mathbb{Q}$ which restricted to $\mathcal{T}^{(1)}$ are the usual fundamental dominant weights and restricted to $\mathcal{C}$ are trivial. Then a dominant weight can be written as

$$\lambda=\sum_{i\in I}a_i\gamma_i+\delta=\lambda^{(1)}+\delta,\tag{23}$$

where $\delta \in X^*(\mathcal{C})$ and we must have the congruence condition

$$(\lambda^{(1)} + \delta)|Z^{(1)} = 1 \tag{24}$$

We can construct a highest weight module $\mathcal{M}_{\lambda,\mathbb{Z}}$. We pick a prime p, we assume that is unramified (with respect to K_f), this means that $K_p = \mathcal{G}(\mathbb{Z}_p)$. Any element $t_p \in T(\mathbb{Q}_p)$ defines a double coset $K_p t_p K_p$. Of course only the image of t_p in $T(\mathbb{Q}_p)/\mathcal{T}(\mathbb{Z}_p)$ matters and

$$T(\mathbb{Q}_p)/\mathcal{T}(\mathbb{Z}_p) = X_*(T)$$

we find $\chi \in X_*(T)$ such that $\chi(p) = t_p$. We take a χ in the positive chamber, i.e. we assume $<\chi, \alpha> \geq 0$ for all α. We can produce the element

$$\underline{\chi}_p = (1\ldots, 1, \ldots, \chi(p), 1\ldots, 1, \ldots) \in T(\mathbb{A}_f)$$

and the Hecke operator

$$\mathbf{ch}(\underline{\chi}_p) : H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}} \otimes \mathbb{Q}) \to H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}} \otimes \mathbb{Q})$$

We have to look at the ideal of those integers a for which

$$a\, \mathbf{ch}(\underline{\chi}_p)(\mathcal{M}_{\lambda,\mathbb{Z}} \otimes \mathbb{Z}_p) \subset (\mathcal{M}_{\lambda,\mathbb{Z}} \otimes \mathbb{Z}_p).$$

This is easy: We have the decomposition into weight spaces

$$\mathcal{M}_{\lambda,\mathbb{Z}} = \oplus_\mu \mathcal{M}_{\lambda,\mathbb{Z}}(\mu)$$

and on a weight space the torus element $\mathbf{ch}(\underline{\chi}_p)$ acts by

$$\mathbf{ch}(\underline{\chi}_p) x_\mu = p^{<\chi,\mu>} x_\mu.$$

We get the smallest exponent if we choose for μ, the lowest weight vector. We denote by w_0 the longest element in the Weyl group, which sends all the positive roots into negative roots. The the element $-w_0$ induces an involution $i \to i'$ on the set of simple roots. Then

$$\mu = w_0(\lambda) = -\sum a_{i'}\gamma_i + \delta. \tag{25}$$

We say that our weight is (essentially) *self dual* if we have $a_i = a_{i'}$. If our weight is self dual then $\mu = -\lambda^{(1)} + \delta$

Hence we see that our ideal is the principal ideal is given by

$$(p^{-<\chi, w_0\lambda^{(1)}> - <\chi,\delta>}) \text{ or if } \lambda \text{ self dual } (p^{<\chi,\lambda^{(1)}> - <\chi,\delta>}) \tag{26}$$

and therefore, we have the Hecke operator

$$T^{\mathrm{coh},\lambda}_{p,\chi} = p^{-<\chi, w_0\lambda^{(1)}> - <\chi,\delta>} \cdot \mathbf{ch}(\underline{\chi}_p) : H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,\mathbb{Z}}) \to H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,\mathbb{Z}}) \tag{27}$$

The number $-<\chi, w_0\lambda^{(1)}>$ is the relevant contribution in the exponent (let us call this the semi-simple term), the second term $-<\chi,\delta>$ is a correction term (the abelian contribution) and it takes care of the central character. We come back to this in section 3.1.4.

2.4 Excursion: Finite dimensional $\mathcal{H}-$modules and representations.

We fix a level $K_f = \prod_p K_p$ and drop it in the notations. It follows from the theorem 2.1 that we have a finite Jordan-Hölder series on our cohomology groups such that the subquotients are irreducible Hecke-modules. If we take the tensor product with a suitable finite extension $F/\mathbb{Q}$ then we can refine the Jordan-Hölder series such that the quotients become absolutely irreducible modules for the Hecke algebra, we say a few words concerning the absolutely irreducible Hecke-modules.

We have a decomposition

$$\mathcal{H} = \bigotimes_p{}' \mathcal{H}_p = \bigotimes_p{}' \mathcal{C}_c(G(\mathbb{Q}_p)//K_p).$$

As the notation indicates we take the tensor product over all finite primes. This tensor product has to be taken in a restricted sense: for an element of the form $h_f = \otimes h_p$ the local factor h_p is equal to the identity element for almost all primes p (this is the characteristic function of K_p). All other elements are finite linear combinations of elements of the form above. We have the obvious embedding $\mathcal{H}_p \hookrightarrow \mathcal{H}$ we simply send $h_p \mapsto 1 \otimes \cdots \otimes h_p \otimes 1 \ldots$. The subalgebras $\mathcal{H}_p$ commute with each other.

We are interested in categories of modules for the Hecke algebra, which will be finite dimensional $\mathbb{Q}-$ vector spaces V together with a homomorphism $\mathcal{H} \to \mathrm{End}_{\mathbb{Q}}(V)$. If Let us call this category $\mathbf{Mod}_{\mathcal{H}}$. For any extension $L/\mathbb{Q}$ we may consider the extension $\mathcal{H}_L = \mathcal{H} \otimes L$ and the resulting category $\mathbf{Mod}_{\mathcal{H}_L}$. If we have an extension $L \hookrightarrow K$ the tensor product yields a functor $\mathbf{Mod}_{\mathcal{H}_L} \to \mathbf{Mod}_{\mathcal{H}_K}$.

We briefly recall the theory of modules over a finite dimensional $\mathbb{Q}$-algebra $\mathcal{A}$ more precisely for any extension $L/\mathbb{Q}$ we consider the category $\mathbf{Mod}_{\mathcal{A}_L}$ of finite dimensional L-vector spaces V together with a homomorphism $\mathcal{A}_L \to \mathrm{End}_L(V)$.

We say that a finite dimensional $\mathcal{A}_L$ module V irreducible, if V does not contain a non trivial $\mathcal{A}_L$ submodule. We say that V is absolutely irreducible if $V \otimes \bar{L}$ is irreducible. We say that V is indecomposable if it can not be written as the direct sum of two non zero submodules.

We call such an algebra $\mathcal{A}$ semi-simple if it does not contain a non trivial two sided ideal $\mathcal{N}$ consisting of nilpotent elements. It is well known that this is equivalent to the semi simplicity of the category $\mathbf{Mod}_{\mathcal{A}}$, this means that for any $\mathcal{A}$-module V (finite dimensional over $\mathbb{Q}$) and any submodule $W \subset V$ we can find a $\mathcal{A}$ submodule W' such that $V = W \oplus W'$. It is also well known that $\mathcal{A}$ is semi simple if it has a faithful semi-simple (finite dimensional) module $V \in \mathbf{Ob}(\mathbf{Mod}_{\mathcal{A}})$, where faithful means that $\mathcal{A} \to \mathrm{End}_{\mathbb{Q}}(V)$ is injective and semi simple means of course that any $\mathcal{A}$-submodule $W \subset V$ admits a complement.

It follows from a simple Galois-theoretic argument, that $\mathcal{A}$ is semi simple if and only if $\mathcal{A} \otimes_{\mathbb{Q}} L$ is semi simple for any extension $L/\mathbb{Q}$.

If we have two modules V_1, V_2 in $\mathbf{Mod}_{\mathcal{A}_L}$ and these modules become isomorphic after some extension $L \hookrightarrow K$, then they are already isomorphic over L. The isomorphism classes of irreducible modules for $\mathcal{A}_L$ form a set which is called $\mathrm{Spec}(\mathcal{A}_L)$. It is a standard fact from the theory of semi simple algebras that this spectrum can be identified to the set of two sided maximal ideals.

We also know that we can write the identity element as a sum of commuting idempotents

$$1 = \sum_{\phi \in \mathrm{Spec}(\mathcal{A}_L)} e_\phi; e_\phi^2 = e_\phi; e_\phi e_\psi = 0 \text{ for } \phi \neq \psi.$$

Then $\mathcal{A}_L e_\psi$ is simple, i.e. has no non trivial two sided ideal. The maximal ideal corresponding to ϕ is $\oplus_{\psi:\psi\neq\phi}\mathcal{A}e_\psi$. We have the decomposition

$$\mathcal{A}_L = \sum_{\phi \in \mathrm{Spec}(\mathcal{A}_L)} \mathcal{A}_L e_\phi \tag{28}$$

Our algebra $\mathcal{A}_L$ has a center $\mathfrak{Z}_L$, which is a commutative algebra over L and since it does not have nilpotent elements it is a direct sum of fields. The idempotents $e_\phi \in \mathfrak{Z}_L$ and clearly

$$\mathfrak{Z}_L = \oplus_{\phi \in \mathrm{Spec}(\mathcal{A}_L)} \mathfrak{Z} e_\phi$$

where $\mathfrak{Z}_L e_\phi$ is a field. Hence we get an identification $\mathrm{Spec}(\mathcal{A}_L) = \mathrm{Spec}(\mathfrak{Z}_L)$.

We conclude that a semi-simple algebra $\mathcal{A}_L$ whose center $\mathfrak{Z}_L$ is a field is actually simple and then the structure theorem of Wedderburn implies

$$\mathcal{A}_L \xrightarrow{\sim} M_n(\mathcal{D})$$

where the right hand side is a matrix algebra of a central division algebra $\mathcal{D}/\mathfrak{Z}_L$. This algebra has only one irreducible non zero module: It acts by multiplication from the left on itself, any non zero minimal left ideal yields an irreducible module. These modules (minimal left ideals) are isomorphic to the ideal given by $\mathfrak{c}_i$ where $\mathfrak{c}_i$ consists of those matrices which have zero entries outside the i-th column. In this case $\mathrm{Spec}(\mathcal{A}_L) = (0)$ is the zero ideal. The unique irreducible module is not absolutely irreducible if $\mathcal{D} \neq \mathfrak{Z}_L$ We may choose an extension K/L which splits the division algebra, then $\mathcal{A}_F = M_{nd}(K)$ where $[\mathcal{D} : L] = d^2$. If this is the case we call the algebra $\mathcal{A}_K$ absolutely simple. The spectrum does not change.

This tells us that in general the set of isomorphism classes of irreducible $\mathcal{A}_L$ is canonically isomorphic to $\mathrm{Spec}(\mathcal{A}_L)$ for any irreducible $\mathcal{A}_L$ module Y_ϕ we have exactly one ϕ such that $e_\phi Y = Y$, and for all $\psi \neq \phi$ $e_\psi Y = 0$. One the other hand our construction above yields exactly one module irreducible module Y_ϕ for a given ϕ. For any $\mathcal{A}_L$ -module X we get the isotypical decomposition

$$X = \sum_{\phi \in \mathrm{Spec}(\mathcal{A})} e_\phi X,$$

The isotypical component where the isotypical component $e_\phi X = Y_\phi^{m(X,\phi)}$, and where $m(X,\phi)$ is the multiplicity of this component. If we extend our ground field further $Y_\phi \otimes_L K$ may become reducible, but if our extension $L/\mathbb{Q}$ is large enough then Y_ϕ will be absolutely irreducible.

Let us start from a semi simple algebra $\mathcal{A}/\mathbb{Q}$. Then its center $\mathfrak{Z}$ is a direct sum of fields, $\mathfrak{Z} = \oplus \mathfrak{Z}_i$. We say that a finite extension $F/\mathbb{Q}$ is a *splitting field* for

$\mathcal{A}$ if it is normal and if any summand $\mathfrak{Z}_i$ can be embedded into F. Then we get

$$\mathcal{A}_F = \mathcal{A} \otimes_{\mathbb{Q}} F = \bigoplus_{\iota \in \mathrm{Hom}(\mathfrak{Z}, F)} \mathcal{A} \otimes_{\mathfrak{Z},\iota} F$$

Clearly the center $\mathcal{A} \otimes_{\mathfrak{Z},\iota} F = F$ and hence we see that this decomposition is the same as the above decomposition (28), we get

Proposition 2.3. *If $F/\mathbb{Q}$ is a splitting field of $\mathcal{A}/\mathbb{Q}$ the we get an action of the Galois group on* $\mathrm{Spec}(\mathcal{A}_F)$. *The orbits of this actions are in one to one correspondence with the elements in* $\mathrm{Spec}(\mathcal{A})$ *in this is the set of summands of the decomposition of $\mathfrak{Z}_{\mathbb{Q}}$ into a direct sum of fields.*

A summand $\mathcal{A}e_\phi F$ has only one non zero irreducible module (up to isomorphism). This module Y_ϕ is not necessarily absolutely irreducible because $\mathcal{A}e_\phi \xrightarrow{\sim} M_n(\mathcal{D})$ where $\mathcal{D}/F$ may be non trivial (we have a non trivial Schur multiplier).

We say that $\mathcal{A}/\mathbb{Q}$ has trivial Schur multiplier if for all $\phi \in \mathrm{Spec}(\mathcal{A})$ the division algebra $\mathcal{D}$ is trivial, i.e. equal to the center.

We apply these general principles to our Hecke -algebra and its action on the cohomology $H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$. We define the ideal $I^!_{K_f}$ to be the kernel of this action, then $\mathcal{H}/I^!_{K_f} = \mathcal{A}$ is a finite dimensional algebra. It is known- and will be proved later - that $H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})$ is a semi simple module and hence we see that $\mathcal{A}$ is semi simple. Then we define the scheme

$$\mathrm{Coh}(H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)) = \mathrm{Spec}(\mathcal{A}).$$

We will denote the set of geometric points of this scheme, or more simple minded the set of isomorphism classes occurring in this cohomology, by $\mathrm{Coh}_!(G, K_f, \lambda)$.

More generally we may consider the set of isomorphism classes of absolutely irreducible Hecke modules occurring in the Jordan-Hlder filtration of any of our cohomology modules $H^\bullet_?(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda))$ and denote this set by $\mathrm{Coh}_?(G, K_f, \lambda)$. Since we have a fixed level K_f they are all defined over a suitable finite extension $F/\mathbb{Q}$.

2.4.1 A central subalgebra

We still consider the action of $\mathcal{H}/I^!_{K_f} = \mathcal{A}$ on $H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}) = \oplus_q H^q_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})$. For all p outside the finite set Σ we have $K_p = \mathcal{G}(\mathbb{Z}_p)$. In this case the algebra $\mathcal{H}_p$ is finitely generated, integral and commutative. We say that $\mathcal{H}_p$ is *unramified* if $K_p = \mathcal{G}(\mathbb{Z}_p)$. For an unramified Hecke-algebra $\mathcal{H}_p$ its maximal spectrum $\mathrm{Hom}_{\mathrm{alg}}(\mathcal{H}_p, \mathbb{C})$,- i.e. the set of isomorphism classes of absolutely irreducible modules over $\mathbb{C}$-,is described by a theorem of Satake which we will recall in the next section.

The subalgebra

$$\mathcal{H}^{(\Sigma)} = \bigotimes_{p \notin \Sigma} \mathcal{H}_p$$

is commutative and its image in $\mathcal{H}/I^!_{K_f}$ lies in the center and hence also in the center of $\mathcal{A}$. Hence we can conclude that for a splitting field F for $\mathcal{A}$ and any irreducible module Y_ϕ for $\mathcal{A}_F$ the restriction of the action to $\mathcal{H}^{(\Sigma)}$ is given by a homomorphism

$$\phi^{(\Sigma)} : \mathcal{H}^{(\Sigma)} \to F.$$

Hence the module Y_ϕ is determined by the action of $\mathcal{H}_\Sigma = \prod_{p\in\Sigma} \mathcal{H}_p$ in $\mathcal{A}_F$. If we assume that Y_ϕ is absolutely irreducible, then it follows from a standard argument that $Y_\phi \xrightarrow{\sim} \otimes_{p\in\Sigma} Y_{\phi_p}$ where Y_{ϕ_p} is an absolutely irreducible $\mathcal{H}_p$-module. For $p \notin \Sigma$ let V_{ϕ_p} be the one dimensional F vector space F with canonical basis element $1 \in F$ and an $\mathcal{H}_p$ action given by the homomorphism $\phi_p : \mathcal{H}_p \to F$. Then we get an isomorphism

$$Y_\phi \xrightarrow{\sim} \bigotimes_p{}' Y_{\phi_p}, \qquad (Fl)$$

where we take the restricted tensor product in the usual sense, i.e. at almost all primes the factor in a tensor is equal to 1. Under our assumptions the homomorphism

$$\mathcal{H}_p \to \mathrm{End}_F(Y_{\phi_p})$$

is surjective.

We get a map from the isomorphism classes of irreducible modules $[Y_\phi]$ for $\mathcal{A}_F$ to $\phi^\sigma \in \mathrm{Hom}(\mathcal{H}^{(\Sigma)}, F)$. We say that $\mathcal{H}^{(\Sigma)}$ acts *distinctively* on $H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}\otimes F)$ if this map is injective, i.e. the isomorphism type $[Y_\phi]$ is determined by its restriction to $\mathcal{H}^{(\Sigma)}$.

On the cohomology $H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})$ we still have the action of the group $\pi_0(G(\mathbb{R}))$, this action commutes with the action of the Hecke algebra. (See (2.5.4) This is an elementary abelian 2- group and we may decompose further according to characters $\epsilon : \pi_0(G(\mathbb{R})) \to \{\pm 1\}$.

We say that the $\mathcal{H}$ module $H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})$ has *strong multiplicity one* (with respect to Σ) if $\mathcal{H}^{(\Sigma)}$ acts distinctivly and for any splitting field F and any $\phi^\Sigma : \mathcal{H}^{(\Sigma)} \to F$ we can find a degree q and an ϵ such that

$$H^q_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})(\epsilon) \otimes_{\mathcal{H}^{(\Sigma)},\phi^{(\Sigma)}} F$$

is an absolutely irreducible $\mathcal{H}-$ module.

If this is so then the homomorphism

$$\mathcal{H}_\Sigma \to \mathrm{End}_F(H^q_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})(\epsilon) \otimes_{\mathcal{H}^{(\Sigma)},\phi^{(\Sigma)}} F)$$

is surjective and the Hecke module $H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})$ has trivial Schur multiplier.

2.4.2 Representations and Hecke modules

For $p \in \Sigma$ the category of finite dimensional modules is complicated, since the Hecke algebra will not be commutative in general.

Let F be a field of characteristic zero, let V be an F-vector space. An admissible representation of the group $G(\mathbb{Q}_p)$ is an action of $G(\mathbb{Q}_p)$ on V which has the following two properties

(i) For any open compact subgroup $K_p \subset G(\mathbb{Q}_p)$ the space V^{K_p} of K_p invariant vectors is finite dimensional.

(ii) For any vector $v \in V$ we can find an open compact subgroup K_p so that $v \in V^{K_p}$ in other words $V = \lim_{K_p} V^{K_p}$.

Then is is clear that the vector spaces V^{K_p} are modules for the Hecke algebra $\mathcal{H}_{K_p}$. An admissible $G(\mathbb{Q}_p)$ -module V is irreducible if it does not contain an invariant proper submodule. Given such an irreducible module $V \neq (0)$, we can find a K_p such that $V^{K_p} \neq (0)$. We claim that then V^{K_p} is an irreducible $\mathcal{H}_{K_p}$-module. To see this we take the identity element e_{K_p} in our Hecke algebra, it induces a projector on V and a decomposition

$$V = V^{K_p} \oplus V' = e_{K_p} V \oplus (1 - e_{K_p})V.$$

Let assume we have a proper $\mathcal{H}_{K_p}$-invariant submodule $W \subset V^{K_p}$ Now we convince ourselves that the $G(\mathbb{Q}_p)$-invariant subspace $\tilde{W}$ generated by the elements gw is a proper subspace. We compute the integral

$$\int_{K_p} kgwdk = \int_{K_p \times K_p} k_1 g k_2 w dk_2 dk_1.$$

The first integral gives us the projection to V^{K_p}, the second integral is the Hecke operator, hence the result is in W. We conclude that $e_{K_p}\tilde{W} \subset W$ and tis shows that $(0) \neq \tilde{W} \neq V$.

Now it is not hard to see, that the assignment

$$V \to V^{K_p}$$

from irreducible admissble $G(\mathbb{Q}_p)$-modules with $V^{K_p} \neq (0)$ to finite dimensional irreducible $\mathcal{H}_{K_p}$-modules induces an bijection between the isomorphism classes of the respective types of modules. If we start from V^{K_p} we can reconstruct V by an appropriate form of induction.

2.4.3 The dual module

Let us assume that V is a finite dimensional F-vector space with an action of the Hecke algebra $\mathcal{H}$ (we fix the level). We have an involution on the Hecke algebra which is defined by

$${}^t h(\underline{x}_f) = h(\underline{x}_f^{-1})$$

a simple calculation shows that ${}^t h_1 * {}^t h_2 = {}^t (h_2 * h_1)$.

This allows us to introduce a Hecke-module structure on $V^\vee = \mathrm{Hom}_F(V, F)$ we for $\phi \in V^\vee$we simply put

$$T_h(\phi)(v) = \phi(T_{{}^t h}(v))$$

for all $v \in V$.

2.4.4 Unitary and essentially unitary representations

Here it seems to be a good moment to recall the notion of unitary Hecke modules and unitary representations. In this book we make the convention that a character is a continuous homomorphism from a topological group $H \to \mathbb{C}^\times$, we do not require that its values have absolute value one. If this is the case we call the character unitary. Our ground field will now be $F = \mathbb{C}$, let V be a $\mathbb{C}$ vector space. We pick a prime p. We call a representation $\rho : G(\mathbb{Q}_p) \to \mathrm{Gl}(V)$ unitary if there is given a positive definite hermitian scalar product $< , > V \times V \to \mathbb{C}$ which is invariant under the action of $G(\mathbb{Q}_p)$.

If our representation is irreducible then it has a central character $\zeta_\rho : C(\mathbb{Q}_p) \to \mathbb{C}^\times$. In this case the scalar product is unique up to a scalar. A necessary condition for the existence of such a scalar product is that $|\zeta_\rho| = 1$, in other words ζ_ρ is unitary.

If this is not the case then our representation may still be *essentially unitary*: We have a unique homomorphism $|\zeta_\rho^*| : C'(\mathbb{Q}_p) \to \mathbb{R}_{>0}^\times$ whose restriction to $C(\mathbb{Q}_p)$ under d_C (see 1.1) is equal to $|\zeta_\rho|$. Then we may form the twisted representation $\rho^* = \rho \otimes |\zeta_\rho^*|^{-1}$. Then the central character of ρ^* is unitary. We say that σ is called essentially unitary if ρ^* is unitary.

If our representation is not irreducible we still can define the notion of being essential unitary. This means that there exists a homomorphism $|\zeta_\rho^*| : C'(\mathbb{Q}_p) \to \mathbb{R}_{>0}^\times$, such that the twisted representation $\rho^* = \rho \otimes |\zeta_\rho^*|^{-1}$ is unitary.

The same notions apply to modules for the Hecke algebra. A (finite dimensional) $\mathbb{C}$ vector space V with an action $\pi_p : \mathcal{H}_p \to \mathrm{End}(V)$ is called unitary, if there is given a positive definite scalar product $< , >: V \times V \to \mathbb{C}$ such that

$$< T_h(v), w >=< v, (T_{{}^t h}(w)). \tag{29}$$

Recall that we always assume that our functions $h \in \mathcal{H}_p$ take values in $\mathbb{Q}$, hence we do not need a complex conjugation bar in the expression on the right.

The restriction of π_p to $C(\mathbb{Q}_p)$ in induces a homomorphism $\zeta_{\pi_p} : C(\mathbb{Q}_p) \to \mathbb{C}^\times$. We call π_p isobaric if this action of the center is semi simple - and therefore a direct sum of characters $\zeta_{\pi_p} = \sum \zeta_{\pi_p}^\nu$ - and if all these characters have the same absolute values $|\zeta_{\pi_p}^\nu| = |\zeta_{\pi_p}|$. This means that we can find $|\zeta_{\pi_p}^*|$ as above. Then we call π_p essentially unitary if the Hecke module $\pi_p^* = \pi_p \otimes |\zeta_{\pi_p}^*|^{-1}$ is unitary.

These boring considerations will be needed later, we will see that for an irreducible coefficient system $\mathcal{M}$ the $H_!^\bullet(\mathcal{S}_{K_f}^G, \tilde{\mathcal{M}}) \otimes \mathbb{C}$ is essentially unitary (see 4.2.1).

2.4.5 Satake's theorem

In the formulation of this theorem I will use the language of group schemes, the reader not so familiar with this language may think of Gl_n or the group of symplectic similitudes GSp_n. Since we assumed that for $p \notin \Sigma$ the integral structure $\mathcal{G}/\mathrm{Spec}(\mathbb{Z}_p)$ is reductive it is also quasisplit. We can find a Borel subgroup $\mathcal{B}/\mathrm{Spec}(\mathbb{Z}_p) \subset \mathcal{G}/\mathrm{Spec}(\mathbb{Z}_p)$ and a maximal torus $\mathcal{T}/\mathrm{Spec}(\mathbb{Z}_p) \subset \mathcal{B}/\mathrm{Spec}(\mathbb{Z}_p)$. Then our torus $\mathcal{T}/\mathrm{Spec}(\mathbb{Z}_p)$ splits over an unramified extension $E_p/\mathbb{Q}_p$ and the Galois group $\mathrm{Gal}(E_p/\mathbb{Q}_p)$ acts on the character module $X^*(\mathcal{T} \times E_p) = \mathrm{Hom}(\mathcal{T} \times E_p, \mathbb{G}_m)$. Let $\{\alpha_1, \alpha_2, \dots, \alpha_r\} \subset X^*(\mathcal{T} \times E_p)$ be the set of positive simple roots, it is invariant under the action of the Galois group. Let $W(\mathbb{Z}_p)$ be the centralizer of the Galois action in the absolute Weyl group W. We introduce the module of unramified characters on the torus this is

$$\mathrm{Hom}_{\mathrm{unram}}(\mathcal{T}(\mathbb{Q}_p), \mathbb{C}^\times) = \mathrm{Hom}(\mathcal{T}(\mathbb{Q}_p)/\mathcal{T}(\mathbb{Z}_p), \mathbb{C}^\times) = \Lambda(\mathcal{T}).$$

We also view $\omega_p \in \Lambda(\mathcal{T})$ as a character $\omega_p : B(\mathbb{Q}_p) \to \mathbb{C}^\times,\ \lambda \mapsto \lambda(b) = b^{\omega_p}$. The group of characters $\mathrm{Hom}(\mathcal{T}, \mathbb{G}_m) = X^*(T)^{\mathrm{Gal}(E_p/\mathbb{Q}_p)}$ is a subgroup of $\Lambda(T)$: An element $\gamma \in X^*(\mathcal{T})^{\mathrm{Gal}(E_p/\mathbb{Q}_p)}$ defines a homomorphism $\mathcal{T}(\mathbb{Q}_p) \to \mathbb{Q}_p^\times$ and this gives us the following element $x \mapsto |\gamma(x)|_p \in \ \Lambda(\mathcal{T})$ which we denote by $|\gamma|$. We can even do this for elements $\gamma \otimes \frac{1}{n} \in X^*(T) \otimes \mathbb{Q}$, then $\gamma \otimes \frac{1}{n}(x) = |\gamma(x)|_p^{1/n} \in \mathbb{R}_{>0}^\times$. Our open compact subgroup $K_p = \mathcal{G}(\mathbb{Z}_p)$. Since we have the Iwasawa decomposition $G(\mathbb{Q}_p) = B(\mathbb{Q}_p)\mathcal{G}(\mathbb{Z}_p) = B(\mathbb{Q}_p)K_p$ we can attach to any $\omega_p \in \Lambda(\mathcal{T})$ a *spherical function*

$$\phi_{\omega_p}(g) = \phi_{\omega_p}(b_p k_p) = (\omega_p + |\rho|_p)(b_p) \tag{30}$$

here $\rho \in \Lambda(\mathcal{T}) \otimes \mathbb{Q}$ is the half sum of positive roots. This spherical function is of course an eigenfunction for $\mathcal{H}_p$ under convolution, i.e. for $h_p \in \mathcal{H}_p$

$$\int_{G(\mathbb{Q}_p)} \phi_{\omega_p}(gx) h_p(x) dx = \hat{h}_p(\omega_p)\phi_{\omega_p}(g) \tag{31}$$

and $\mathfrak{s}(\omega_p) : h_p \mapsto \hat{h}_p(\omega_p)$ is an algebra homomorphism from $\mathcal{H}_p$ to $\mathbb{C}$. Of course the measure dx gives volume 1 to $\mathcal{G}(\mathbb{Z}_p) = K_p$.

The theorem of Satake asserts:

Theorem 2.2. *The group $W(\mathbb{Z}_p)$ acts on $\Lambda(\mathcal{T})$, we have $\mathfrak{s}(w\omega_p)) = \mathfrak{s}(\omega_p)$ and*

$$\Lambda(\mathcal{T})/W(\mathbb{Q}_p) \xrightarrow{\mathfrak{s}} \mathrm{Hom}_{\mathrm{alg}}(\mathcal{H}_p, \mathbb{C})$$

is an isomophism.

We will write irreducible modules in this case as $\pi_p = \pi_p(\omega_p)$ and $\omega_p \in \Lambda(\mathcal{T})/W(\mathbb{Q}_p)$ is the so called *Satake parameter* of π_p.

The Hecke algebra is generated by the characteristic functions of double cosets $K_p t_p K_p$ where $t_p \in T(\mathbb{Q}_p)$ and where for all simple roots $\alpha \in \pi$ we have $|\alpha(t_p)|_p \leq 1$, i.e. $t_p \in T_+(\mathbb{Q}_p)$. Then the evaluation in (31) comes down to the computation the integrals

$$\int_{K_p t_p K_p} \phi_{\omega_p}(gx) dx = \hat{t}_p(\omega_p)\phi_{\omega_p}(g) \tag{32}$$

We discuss this evaluation in (3.1.3)

2.4.6 Spherical representations

Now we assume that Let $F' \subset \mathbb{C}$ be a finite extension of $\mathbb{Q}$ and let V/F be a vector space. We choose $K_p = \mathcal{G}(\mathbb{Z}_p)$, i.e. p is unramified. An admissible representation

$$\tilde{\pi}_p : G(\mathbb{Q}_p) \to \mathrm{Gl}(V)$$

is called *spherical* if $V^{K_p} \neq 0$, and this space is a module for the Hecke algebra. If the representation is absolutely irreducible, then it is well known that $\dim_{F'} V^{K_p} = 1$, this is a one dimensional module for $\mathcal{H}_{K_p}$, i.e. a homomorphism $\pi_p : \mathcal{H}_{K_p} \to F'$. Let $\omega_p \in \Lambda(\mathcal{T})$ the corresponding Satake parameter, it is well defined modulo the action of the group $W(\mathbb{Q}_p)$. We consider the field F' which is generated by the values $\hat{t}_p(\omega_p)$. Then the one dimensional F' vector space

$$H_{\pi_p} = F'\phi_{\omega_p} \tag{33}$$

will be our standard model for the isomorphism type π_p.

The representation $\tilde{\pi}_p$ can be realized as a submodule J_{π_p} of the induced representation

$$H_{\tilde{\pi}_p} = \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} F'\phi_{\omega_p} = \{f : G(\mathbb{Q}_p) \to F' | f(bg) = \omega_p(b)|\rho|_p(b)f(g)\}$$

where f satisfies the (obvious) condition that there exists a finite index subgroup $K_p' \subset K_p$ such that f is invariant under right translations by elements $k' \in K_p'$. In general this module $H_{\tilde{\pi}_p}$ will be irreducible and then $J_{\pi_p} = H_{\tilde{\pi}_p}$.

If $\tilde{\pi}_p^\vee$ is the spherical representation attached to the Satake parameter ω_p^{-1} then we have a pairing

$$\begin{aligned} H_{\tilde{\pi}_p} \times H_{\tilde{\pi}_p^\vee} &\to \mathbb{C} \\ f_1 \times f_2 &\mapsto \textstyle\int_{K_p} f_1(k_p) f_2(k_p) dk_p \end{aligned} \tag{34}$$

This tells us that the dual module to $H_{\pi_p} = H_{\tilde{\pi}_p}^{K_p}$ has the Satake parameter ω_p^{-1}. The representations $H_{\tilde{\pi}_p}$ are called the representations of the unramified principal series.

We may consider the case that ω_p is a unitary character, this means that $\omega_p : \mathcal{T}(\mathbb{Q}_p)/\mathcal{T}(\mathbb{Z}_p) \to \mathbb{S}^1$. Then we have $\omega_p^{-1}(t) = \overline{\omega_p}(t)$ and our above pairing defines a positive definite hermitian scalar product

$$< \, , \, >: H_{\tilde{\pi}_p} \times H_{\tilde{\pi}_p} \to \mathbb{C} \tag{35}$$

which is given by

$$< f_1, f_2 > = \int_{K_p} f_1(k_p)\overline{f_2(k_p)} dk_p \tag{36}$$

If we allow for $f \in H_{\tilde{\pi}_p}$ all the functions whose restriction to K_p lies in $L^2(K_p)$ then $H_{\tilde{\pi}_p}$ becomes a Hilbert space and the representation of $G(\mathbb{Q}_p)$ on $H_{\tilde{\pi}_p}$ is a unitary representation.

These representations are called the unitary principal series representations. It is not the case that these representations are the only unramified principal series representations which carry an invariant positive definite scalar product. (See [Sat]).

In the following section we discuss the classical case.

2.4.7 The case Gl_2.

In the case of Gl_2 the maximal torus is given by

$$T(\mathbb{Q}_p)=\left\{\begin{pmatrix} t_1 & 0\\ 0 & t_2\end{pmatrix}\right\}.$$

It is contained in the two Borel subgroups $B/\mathbb{Q}$ of upper and $B_-/\mathbb{Q}$ of lower triangular matrices. Let $U/\mathbb{Q}$ be the unipotent radical of B.

If we represent an element $\bar{\omega}_p \in \Lambda(\mathcal{T})/W$ by $\omega_p : T(\mathbb{Q}_p)/T(\mathbb{Z}_p) \to \mathbb{C}^\times$ then we get two numbers

$$\begin{array}{c} \omega_p(\begin{pmatrix} p & 0\\ 0 & 1\end{pmatrix}) = \alpha'_p \\ \omega_p(\begin{pmatrix} 1 & 0\\ 0 & p\end{pmatrix}) = \beta'_p \end{array}.$$

The local algebra is generated by two operators $T_p, T(p,p)$ for which

$$\begin{array}{c} \mathfrak{s}(\bar{\omega}_p)(T_p) = p^{1/2}(\alpha'_p+\beta'_p) = \alpha_p+\beta_p \\ \mathfrak{s}(\bar{\omega}_p)(T(p,p)) = p\alpha'_p\beta'_p = \alpha_p\beta_p \end{array}.$$

These two Hecke operators are -up to a normalizing factor - defined as the characteristic functions of the double cosets

$$\mathrm{Gl}_2(\mathbb{Z}_p)\begin{pmatrix} p & 0\\ 0 & 1\end{pmatrix}\mathrm{Gl}_2(\mathbb{Z}_p) \text{ and } \mathrm{Gl}_2(\mathbb{Z}_p)\begin{pmatrix} p & 0\\ 0 & p\end{pmatrix}\mathrm{Gl}_2(\mathbb{Z}_p).$$

The to numbers $\alpha_p+\beta_p, \alpha_p\beta_p$ determine ω_p. They are also called the Satake parameters.

It is not difficult to prove Satakes theorem for $\mathrm{Gl}_2/\mathbb{Q}_p$. We write $\mathrm{Gl}_2(\mathbb{Z}_p) = K_p$. It is the theorem for elementary divisors that all the double cosets $K_p\backslash G(\mathbb{Q}_p)/K_p$ are of the form

$$K_p\begin{pmatrix} p^a & 0\\ 0 & p^b\end{pmatrix}K_p \text{ with } a \geq b.$$

If we want to understand the function $h \mapsto \hat{h}(\lambda)$ it clearly suffices to compute its value on the characteristic function t_{p^m} of the double coset

$$K_p\begin{pmatrix} p^m & 0\\ 0 & 1\end{pmatrix}K_p$$

To do this we have to evaluate the integral

$$\int_{G(\mathbb{Q}_p)} \phi_\lambda(x)t_{p^m}(x)dx = \hat{t_{p^m}}(\lambda).$$

We abbreviate $y_p = \begin{pmatrix} p^m & 0 \\ 0 & 1 \end{pmatrix}$ and write our double coset as a union of right K_p cosets, i.e.

$$K_p y_p K_p = \bigcup_{\xi \in K_p / K_p \cap y_p K_p y_p^{-1}} \xi y_p K_p.$$

The volume of such a coset is one hence we get

$$\int_{G(\mathbb{Q}_p)} \phi_\lambda(x) t_{p^m}(x) dx = \sum_{\xi} \phi_\lambda(\xi y_p)$$

The group

$$K_p \cap y_p K_p y_p^{-1} = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_p | b \equiv 0 \mod p^m \},$$

this is the group of points $B_-(\mathbb{Z}/p^m)$ of lower triangular matrices. Hence the coset space

$$\mathrm{Gl}_2(\mathbb{Z}/p^m)/B_-(\mathbb{Z}/p^m) = K_p / K_p \cap y_p K_p y_p^{-1} = \mathbb{P}^1(\mathbb{Z}/p^m).$$

The points in $\mathbb{P}^1(\mathbb{Z}/p^m)$ are arrays $\begin{pmatrix} a \\ b \end{pmatrix}$, $a, b \in \mathbb{Z}/p^m, a$ or $b \in (\mathbb{Z}/p^m\mathbb{Z})^\times$, modulo $(\mathbb{Z}/p^m)^\times$. Then K_p acts by multiplication from the left on this coset space and $K_p \cap y_p K_p y_p^{-1}$ is the stablizer of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. We still have an action of $B(\mathbb{Z}/p^m)$ from the left on $\mathbb{P}^1(\mathbb{Z}/p^m)$ and the orbits under this action from the left can be represented by

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ p^\nu \end{pmatrix} \text{ for } \nu = 1, \dots m$$

On these orbits the function $\xi \mapsto \phi_\lambda(\xi y_p)$ is constant. We can take the representatives

$$\xi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ -1 & p^\nu \end{pmatrix}$$

and get the values

$$\phi_\lambda(y_p) = p^{-m} \alpha_p^m$$

$$\phi_\lambda(\begin{pmatrix} 0 & 1 \\ -p^m & p^\nu \end{pmatrix}) = \phi_\lambda(\begin{pmatrix} p^{m-\nu} & * \\ 0 & p^\nu \end{pmatrix} k_p) = \alpha_p^{m-\nu} \beta_p^\nu p^{\nu-m}.$$

The length of these orbits is $p^m, \{p^{m-\nu}(1-\frac{1}{p})\}_{\nu=1,\dots,m-1}, 1$, and we get

$$\hat{t_{p^m}}(\lambda) = \alpha_p^m + \beta_p^m + (1-\frac{1}{p}) \sum_{\nu=1}^{m-1} \alpha_p^{m-\nu} \beta_p^\nu.$$

This formula clearly proves the theorem of Satake in this special case.

2.4.8 A very specific case

We consider the case

2.5 Back to cohomology

2.5.1 The case of a torus and the central character

We consider the case that our group $G/\mathbb{Q}$ is a torus $T/\mathbb{Q}$. This case is already discussed in [Ha-Gl2]. Our torus splits over a finite extension $F/\mathbb{Q}$ and our absolutely irreducible representation is simply a character $\gamma : T \times_{\mathbb{Q}} F \to \mathbb{G}_m$, it defines a one dimensional $T \times_{\mathbb{Q}} F-$ module $F[\gamma]$. Here $F[\gamma]$ is simply the one dimensional vector space F over F with $T \times_{\mathbb{Q}} F$ acting by the character γ.

We recall the notion of an algebraic Hecke character of type γ. We choose an embedding $\iota : F \hookrightarrow \bar{\mathbb{Q}}$ then γ induces a homomorphisms $T(\mathbb{C}) \to \mathbb{C}^\times$. The restriction of this homomorphism to $T(\mathbb{R})$ is called $\gamma_\infty : T(\mathbb{R}) \to \mathbb{C}^\times$.

A continuos homomorphism

$$\phi = \phi_\infty \times \Pi_p \phi_p = \phi_\infty \times \phi_f \ : T(\mathbb{A})/T(\mathbb{Q}) \to \mathbb{C}^\times$$

is called an *algebraic Hecke character of type* γ if the restrictions to the connected component of the identity satisfy

$$\phi_\infty|_{T^{(0)}(\mathbb{R})} = \gamma_\infty^{-1}|_{T^{(0)}(\mathbb{R})}.$$

The finite part $\phi_f : T(\mathbb{A}_f) \to \bar{\mathbb{Q}}^\times$ is trivial on some open compact subgroup $K_f^T \subset T(\mathbb{A}_f)$. We also say that a homomorphism $\phi_1 : T(\mathbb{A}_f)/K_f^T \to \bar{\mathbb{Q}}^\times$ is an algebraic Hecke-character, if it is the finite part of an algebraic Hecke character, which is then uniquely defined.

In [Ha-Gl2], 2.5.5 we explain that the cohomology vanishes (for any choice of K_f^C) if γ is not the type of an algebraic Hecke character. In this case we give the complete description of the cohomology in [Ha-Gl2], 2.6: If we choose $Z' = Z$ (see 1.1) then

$$H^0(S^C_{K_f^C}, F[\gamma] \otimes_{F,\iota} \otimes \bar{\mathbb{Q}}) = \bigoplus_{\phi_f : C(\mathbb{A}_f)/K_f^C) \to \bar{\mathbb{Q}}^\times : \mathrm{type}(\phi_f) = \gamma} \bar{\mathbb{Q}} \phi_f. \tag{37}$$

The property of γ to be the type of an algebraic Hecke character does not depend on the choice of ι. If we fix the level then it is easy to see that the values of the characters ϕ_f lie in a finite extension F_1 of $\iota(F)$ so we may replace in our formula above the algebraic closure $\mathbb{Q}$ by F_1.

If we return to our group $G/\mathbb{Q}$ and if we start from an absolutely irreducible representation $G \times_{\mathbb{Q}} F \to \mathrm{Gl}(\mathcal{M})$ then its restriction to the center $C/\mathbb{Q}$ is a character $\zeta_{\mathcal{M}}$. Our remark above implies that this character must be the type of an algebraic Hecke character if we want the cohomology $H^\bullet_?(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})$ to be non trivial. (Look at a suitable spectral sequence).

In any case we can consider the sub algebra $C_{K_f} \subset \mathcal{H}_{K_f}$ generated by central double cosets $K_f \underline{z}_f K_f = K_f \underline{z}_f$. with $\underline{z}_f \in C(\mathbb{A}_f)$ This provides an action of the group $C(\mathbb{A}_f)/K_f^C$ on the cohomology $H^\bullet_?(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})$. Then the following proposition is obvious

Proposition 2.4. *Let H_{π_f} be an absolutely irreducible subquotient in the Jordan Hölder series in any of our cohomology groups. Then $C(\mathbb{A}_f)/K_f^C$ acts by a character ζ_{π_f} on H_{π_f} and ζ_{π_f} is an algebraic Hecke character of type $\zeta_{\mathcal{M}}$.*

Note that $\zeta_{\mathcal{M}}$ is the restriction of the abelian component δ in $\lambda = \lambda^{(1)} + \delta$ to the center.

2.5.2 The cohomology in degree zero

Let us start from an absolutely irreducible representation $r : G \times F \to \mathrm{Gl}(\mathcal{M})$, we want to understand $H^0(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})$: To do this we have to understand the connected components of the space and the spaces of invariants in $\tilde{\mathcal{M}}$ under the discrete subgroups $\Gamma^{\underline{g}_f}$ in 1.2.1. We assume that the groups $\Gamma^{\underline{g}_f} \cap G^{(1)}(\mathbb{Q})$ are Zariski dense in $G^{(1)}$. Then it is clear that we can have non trivial cohomology in degree zero if $\mathcal{M}$ is one dimensional and $G^{(1)}$ acts trivially. Hence $\mathcal{M}$ is given by a character $\delta : C' \times F \to \mathbb{G}_m \times F$.

To simplify the situation we assume that the assumptions in (1.3) are fulfilled and we have a bijection

$$\pi_0(\mathcal{S}^G_{K_f}) \xrightarrow{\sim} \pi_0(\mathcal{S}^{C'}_{K^{C'}_\infty \times K^{C'}_f}) \tag{38}$$

where $K^{C'}_\infty$ and $K^{C'}_f$ are the images of the chosen compact subgroups respectively. With these data we define $\mathcal{S}^{C'}_{K^{C'}_f}$ and we can view $\mathcal{M}$ as a sheaf on $\mathcal{S}^{C'}_{K^{C'}_f}$, in our previous notation it is the sheaf $\tilde{F}[\delta]$.

Then we get for an absolutely irreducible $G \times F$ module $\mathcal{M}$ -and under the assumption that the $\Gamma^{\underline{g}_f} \cap G^{(1)}(\mathbb{Q})$ are Zariski dense in $G^{(1)}$- that (See 2.5.1)

$$H^0(\mathcal{S}^G_{K_f}, \mathcal{M} \otimes F_1) = \begin{cases} 0 & \text{if } \dim(\mathcal{M}) > 1 \\ \bigoplus_{\phi_f : \mathrm{type}(\phi_f)=\delta} F_1\phi & \text{if } \mathcal{M} = F[\delta] \end{cases} \tag{39}$$

The density assumption is fulfilled if $G^{(1)}/\mathbb{Q}$ is quasisplit. We also observe that we have the isogeny $d_C : C \to C'$ (See (1.1). Then it is clear that the composition $d_C \circ \delta$ is the character $\zeta_{\mathcal{M}}$ in section 2.5.1.

2.5.3 The Manin-Drinfeld principle

For a moment we assume that our coefficient systems are rational vector spaces. This means that we start from a rational (preferably absolutely irreducible) representation $\rho : G \times_{\mathbb{Q}} F_0 \to \mathrm{Gl}(\mathcal{M})$ where $\mathcal{M}$ is a finite dimensional F_0 vector spaces. We have an action of $\mathcal{H}_{F_0}$ on our cohomology groups and we defined the spectra $\mathrm{Coh}(H^\bullet_?(\mathcal{S}^G_{K_f}, \mathcal{M}))$ which now will be a finite scheme over F_0. We will show show that the modules $H^i_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_L)$ are semi simple and if we pass to a splitting field F/F_0 we can decompose

$$H^i_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})(\Pi_f) \otimes F = \bigoplus_{\pi_f} H^i_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})(\pi_f) = \bigoplus_{\pi_f} e_{\pi_f} H^i_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}) \tag{40}$$

Here we changed our notation slightly, we replaced the ϕ by π_f. The isomorphism types π_f are not necessarily absolutely irreducible, but if we extend our field further then they decompose in a direct sum of modules of exactly one isomorphism type. We call the above decomposition the isotypical decomposition and under our assumption on F the summands are absolutely isotypical.

We say that for a cohomology group $H^i(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_F)$ (resp. $H^\bullet_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_F)$ satisfies the *Manin-Drinfeld principle*, if $\mathrm{Coh}(H^i_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_F) \cap \mathrm{Coh}(H^i(\partial(\mathcal{S}^G_{K_f}), \tilde{\mathcal{M}}_F) = \emptyset$ (resp $\mathrm{Coh}(H^i_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_F) \cap \mathrm{Coh}(H^{i-1}(\partial(\mathcal{S}^G_{K_f}), \tilde{\mathcal{M}}_F) = \emptyset$.

We have defined $\mathrm{Coh}(X)(=\mathrm{Spec}(\mathcal{H}/I(X)))$ for any Hecke-module X and if X is a submodule of another Hecke module Y then we say that X satisfies the Manin-Drinfeld principle with respect to Y if $\mathrm{Coh}(X)\cap\mathrm{Coh}(Y/X)=\emptyset$.

If the Manin-Drinfeld principle is valid we get canonical decompositions

$$H^i(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F)= \ \mathrm{Im}(H^i(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F)\longrightarrow H^i(\partial(\mathcal{S}^G_{K_f}),\tilde{\mathcal{M}}_F))\oplus H^i_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F), \tag{41}$$

$$H^i_c(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F)= \ \mathrm{Im}(H^{i-1}(\partial\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F)\longrightarrow H^i_c(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F))\oplus H^i_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F),$$

which is invariant under the action of the Hecke algebra and no irreducible representation $\overline{\pi}_\infty\times\pi_f$ which occurs in $H^i_!(\mathcal{S}^G_{K_f},\mathcal{M}_F)$ can occur as a sub quotient in $\mathrm{Im}(H^{i-1}(\partial\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F)\to H^i_c(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F))$.

In the second case we will call the above image of the boundary cohomology the Eisenstein subspace or compactly supported Eisenstein cohomology and denote it by

$$\mathrm{Im}(H^{i-1}(\partial\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F)\longrightarrow H^i_c(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}))=H^i_{c,\mathrm{Eis}}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}).$$

In the first case we can consider the module $H^i_{\mathrm{Eis}}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F)\subset\ \mathrm{Im}(H^i(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F)\longrightarrow H^i(\partial(\mathcal{S}^G_{K_f}),\tilde{\mathcal{M}}_F))$ as a submodule in $H^i(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F)$ and this submodule is called the Eisenstein cohomology. Under the assumption of the Manin-Drinfeld principle we have a canonical section $s:H^i_{\mathrm{Eis}}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F)\to H^i\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F)$.

If we know the Manin-Drinfeld principle we can ask new questions. We return to the the integral cohomology $H^i_?(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathcal{O}_F})$ and map it into the rational cohomology then the image is called $H^\bullet_?(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}})_{\ \mathrm{int}}\subset H^\bullet_?(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F)$ this is also the module which we get if we divide $H^\bullet_?(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathcal{O}_F})))$ by the torsion. (This may be not true for ? =!)

Our decompositions above do not induce decomposition on the groups $H^\bullet_?(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}})_{\ \mathrm{int}}$. Whenever we have a decomposition $H^\bullet_?(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F)=X\oplus Y$ we can take the intersections $X_{\mathrm{int}}\cap H^\bullet_?(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}})_{\ \mathrm{int}}$ and the same for Y and get a decomposition *up to isogeny*

$$X_{\mathrm{int}}\oplus Y_{\mathrm{int}}\subset H^\bullet_?(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}})_{\ \mathrm{int}},$$

where up to isogeny means that the left hand side is of finite index in the right hand side.

For instance the Manin-Drinfeld decomposition above yields (if it exists) a decomposition up to isogeny

$$H^i_{c,\mathrm{Eis}}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}})_{\ \mathrm{int}}\oplus H^i_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}})_{\ \mathrm{int}}\subset H^i_c(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}})_{\ \mathrm{int}},$$

it is canonical but the direct sum is only of finite index in the right hand side module. The primes dividing the order of the index are called *Eisenstein primes.*

These Eisenstein primes have been studied in the case $G=\mathrm{Gl}_2/\mathbb{Q}$ but they also seem to play a role in more general situation. The general philosophy is that they are related to the arithmetic of special values of L-functions. (See [Ha-Cong])

The same applies to the decomposition of $H^i_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}})_{\text{ int}}$ in isotypical summands. We put

$$H^i_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}})(\pi_f)\cap H^i_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathcal{O}_F})_{\text{ int}} = H^i_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathcal{O}_F})_{\text{ int}}(\pi_f).$$

Then we get an decomposition up to isogeny

$$\bigoplus_{\pi_f} H^i_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathcal{O}_F})_{\text{ int}}(\pi_f)\subset H^i_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathcal{O}_F})_{\text{ int}}. \tag{42}$$

It is a very interesting question to learn something about the the structure of the quotient of the right hand side by the left hand side. The structure of this quotient should be related to the arithmetic of special values of L-functions. (See [Hi]).

2.5.4 The action of $\pi_0(G(\mathbb{R}))$

We have seen that we can choose a maximal torus $T/\mathbb{Q}$ such that $T(\mathbb{R})[2]$ normalizes K_∞. We know that $T(\mathbb{R})[2]\to\pi_0(G(\mathbb{R}))$ is surjective and that $T(\mathbb{R})[2]\cap G^{(1)}(\mathbb{R})\subset K_\infty$. This allows us to define an action of $\pi_0(G(\mathbb{R}))$ on the various cohomology groups and this action commutes with the action of the Hecke-algebra. Therefore we can decompose any isotypical subspace in a cohomology group into eigenspaces under this action

$$H^\bullet_?(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F)(\pi_f)=\bigoplus_{\epsilon_\infty} H^\bullet_?(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F)(\pi_f\times\epsilon_\infty) \tag{43}$$

and for the integral lattices we get a decomposition up to isogeny

$$\bigoplus_{\pi_f\times\epsilon_\infty} H^i_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathcal{O}_F})_{\text{ int}}(\pi_f\times\epsilon_\infty)\subset H^i_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathcal{O}_F})_{\text{ int}} \tag{44}$$

2.6 Some questions and and some simple facts

Of course we can be more modest and we may only ask for the dimension of the cohomology groups $H^i(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}})$, this question will be discussed later in Chapter V and can be answered in some simple cases.

If we are a little bit more modest we can ask for the Euler characteristic

$$\chi(H^\bullet(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}})=\sum_i(-1)^i\dim(H^i(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}))$$

This question can be answered in a certain sense. If the subgroup K_f is neat (See 1.1.2.1) , then $\mathcal{S}^G_{K_f}$ is a disjoint union of locally symmetric spaces. On these spaces exists a differential form of highest degree, which is obtained from differential geometric data, this is the Gauss-Bonnet form ω^{GB}. Then the Gauss-Bonnet theorem yields that

$$\chi(H^\bullet(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}})=\dim(\mathcal{M}))\int_{\mathcal{S}^G_{K_f}}\omega^{GB}.$$

This will be discussed in more detail in Chap V. This implies of course, that for a covering $\mathcal{S}^G_{K'_f} \to \mathcal{S}^G_{K_f}$, where $K'_f \subset K_f$ and both groups are neat,we get

$$\chi(\mathcal{S}^G_{K'_f}, \tilde{\mathcal{M}}) = \chi(H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})[K'_f : K_f],$$

a fact which also follows easily from topological considerations.

This leads us-following C.T.C. Wall- to introduce the orbifold Euler characteristic for a not necessarily neat K_f by

$$\chi_{\mathrm{orb}}(H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})) = \frac{1}{[K'_f : K_f]}\chi(\mathcal{S}^G_{K'_f}, \tilde{\mathcal{M}})$$

where $K'_f \subset K_f$ is a neat subgroup of finite index. The orbifold Euler characteristic may differ from the Euler characteristic $\chi(H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})$ by a sum of contributions coming from the set of fixed points of the Γ_i on X (See 1.1.2.1).

This is perhaps the right moment, to discuss another minor technical point. When we discuss the action of the Hecke algebra $\mathcal{H}_{K_f} = \mathcal{C}_c(G(\mathbb{A}_f)//K_f, \mathbb{Q})$ on $H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})$ then we chose the same K_f for the space and for the Hecke algebra. We also normalized the measure on the group so that it gave volume 1 to K_f. But we have of course an inclusion of Hecke algebras $\mathcal{H}_{K_f} \subset \mathcal{H}_{K'_f}$. Therefore $\mathcal{H}_{K_f}$ also acts on $H^\bullet(\mathcal{S}^G_{K'_f}, \tilde{\mathcal{M}})$. This contains $H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})$ but then the inclusion is not compatible with the action of the Hecke algebra. We therefore choose a measure independently of the level, if we are in a situation where we vary the level. In such a case a measure provided by an invariant form ω_G on G (See 2.1.3) is a good choice. If we now define the action of the Hecke operators by means of this measure. With this choice of a measure the inclusion $\mathcal{H}_{K_f} \subset \mathcal{H}_{K'_f}$ is compatible with the inclusion of the cohomology groups.

Then we see the the new Hecke operator $T_h^{(\omega_G)}$, and the old one are related by the formula

$$T_h = \frac{1}{\mathrm{vol}_{|\omega_G|}(K_f)} T_h^{(\omega_G)}$$

The reader might raise the question, why we work with fixed levels and why we do not pass to the limit. The reason is that for some questions we need to work with the integral cohomology, and this does not behave so well under change of level.

2.6.1 Homology

We may also define homology groups $H_i(\mathcal{S}^G_{K_f}, \underline{\mathcal{M}}_\lambda)$ and $H_i(\mathcal{S}^G_{K_f}, \partial\mathcal{S}^G_{K_f}, \underline{\mathcal{M}}_\lambda)$, here $\mathcal{M}_\lambda$ is a "cosheaf". The "costalk" $\underline{\mathcal{M}}_{\mathbb{Z},x}$ is obtained as follows: We consider $\pi^{-1}(x)$ and

$$\bigoplus_{\underline{y} = y \times \underline{g}_f K_f/K_f} \underline{g}_f \mathcal{M}_\lambda,$$

and the action of $G(\mathbb{Q})$ on this direct sum. Then $\underline{\mathcal{M}}_{\lambda,x}$ is the module of coinvariants. If we pick a point $\underline{y} = y \times \underline{g}_f K_f/K_f$, which maps to $x \in \mathcal{S}^G_{K_f}$ then we get an isomorphism

$$\underline{\mathcal{M}}_{\lambda,x} \simeq (g_f \mathcal{M}_\lambda)_{\Gamma_y^{(\underline{g}_f)}}.$$

We define the chain complex

$$C_i(\mathcal{S}^G_{K_f}, \underline{\mathcal{M}}_\lambda)$$

and the above homology groups are given by the homology of this complex.

If we assume that $\mathcal{S}^G_{K_f}$ is oriented (ref. to prop 1.3) then we know (Chap. II 2. 1. 5) that we have isomorphisms which are compatible with the fundamental exact sequence

$$\begin{array}{ccc}
\downarrow & & \downarrow \\
H^{i-1}(\partial\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) & \xrightarrow{\sim} & H_{d-i}(\partial\mathcal{S}^G_{K_f}, \underline{\mathcal{M}}_\lambda) \\
\downarrow & & \downarrow \\
H^i_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) & \xrightarrow{\sim} & H_{d-i}(\mathcal{S}^G_{K_f}, \underline{\mathcal{M}}_\lambda) \\
\downarrow & & \downarrow \\
H^i(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) & \xrightarrow{\sim} & H_{d-i}(\mathcal{S}^G_{K_f}, \partial\mathcal{S}^G_{K_f}, \underline{\mathcal{M}}_\lambda) \\
\downarrow & & \downarrow \\
H^i(\partial\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) & \xrightarrow{\sim} & H_{d-i-1}(\partial\mathcal{S}^G_{K_f}, \underline{\mathcal{M}}_\lambda) \\
\downarrow & & \downarrow
\end{array}$$

2.6.2 Poincaré duality

We assume that $\mathcal{S}^G_{K_f}$ is connected. If we denote the dual representation by $\mathcal{M}^\vee_\lambda = \mathcal{M}_{w_0(\lambda)}$ (we choose a suitable lattice lattice $\mathcal{M}^\vee_\mathbb{Z}$then we have the canonical homomorphism $\mathcal{M}_\lambda \otimes \mathcal{M}^\vee_\lambda \to \mathbb{Z}$ and the standard pairing between the homology and the cohomology groups yields pairings

$$\begin{array}{ccccc}
H^i_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) \times H_i(\mathcal{S}^G_{K_f}, \partial\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}^\vee_\lambda) & \to & H^0(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \tilde{\mathcal{M}}^\vee_\lambda) & \to & H^0(\mathcal{S}^G_{K_f}, \mathbb{Z}) \\
\downarrow \qquad\qquad \uparrow & & \downarrow & & \downarrow \\
H^i(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) \times H_i(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda^\vee}) & \to & H^0(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \tilde{\mathcal{M}}^\vee_\lambda) & \to & H^0(\mathcal{S}^G_{K_f}, \mathbb{Z})
\end{array}$$

This pairing is of course compatible with the isomorphism between homology and cohomology and then the pairing becomes the cup product. We get the diagram

$$\begin{array}{ccccc}
H^i_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) \times H^{d-i}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}^\vee_\lambda) & \to & H^d_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \tilde{\mathcal{M}}^\vee_\lambda) & \to & H^d_c(\mathcal{S}^G_{K_f}, \mathbb{Z}) \\
\downarrow \qquad\qquad \uparrow & & \downarrow & & \downarrow \\
H^i(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) \times H^{d-i}_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda^\vee}) & \to & H^d_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \tilde{\mathcal{M}}^\vee_\lambda) & \to & H^d_c(\mathcal{S}^G_{K_f}, \mathbb{Z})
\end{array}$$

We know that the manifold with corners $\partial\mathcal{S}^G_{K_f}$ "smoothable" it can be approximated by a $\mathcal{C}-$ manifold and therefore we also have a pairing $< , >_\partial$ on the cohomology of the boundary. This pairing is consistent with the fundamental long exact sequence (Thm. 2.1). We write this sequence twice but the second time in the opposite direction and the pairing $< , >$ in vertical direction:

$$\begin{array}{cccccc}
\to & H^p(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) & \xrightarrow{r} & H^p(\partial\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) & \xrightarrow{\delta} & \\
& \times & & \times & & \\
\leftarrow & H^{d-p}_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda^\vee}) & \xleftarrow{\delta} & H^{d-p-1}(\partial\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda^\vee}) & \leftarrow & \\
& \downarrow < , > & & \downarrow < , >_\partial & & \\
& H^d_c(\mathcal{S}^G_{K_f}, \mathbb{Z}) & \xleftarrow{\delta_d} & H^{d-1}_c(\partial\mathcal{S}^G_{K_f}, \mathbb{Z}) & &
\end{array} \tag{45}$$

then we have: For classes $\xi \in H^p(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda), \eta \in H^{d-p-1}(\partial\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda^\vee})$ we have the equality

$$< \xi, \delta(\eta) >= \delta_d(< r(\xi), \eta >_\partial) \tag{46}$$

2.6.3 Non degeneration of the pairing

The spaces $\mathcal{S}^G_{K_f}$ and $\partial\mathcal{S}^G_{K_f}$ are not connected in general. Let us assume that we have a consistent orientation on $\mathcal{S}^G_{K_f}$. Then each connected component M of $\mathcal{S}^G_{K_f}$ is an oriented manifold which is natural embedded into its compactification $\bar{M}$. It is obvious that the cohomology groups are the direct sums of the cohomology groups of the connected components and that we may restrict the pairing to the components

$$H^p(M, \tilde{\mathcal{M}}_\lambda) \times H^{d-p}_c(M, \tilde{\mathcal{M}}_{\lambda^\vee}) \to H^d_c(M, \mathbb{Z}) = \mathbb{Z}. \tag{47}$$

We recall the results which are explained in Vol. I 4.8.4. The fundamental group $\pi_1(M)$ is an arithmetic subgroup $\Gamma_M \subset G(\mathbb{Q})$ and $\mathcal{M}_\lambda, \mathcal{M}_{\lambda^\vee}$ are Γ_M modules. For any commutative ring with identity $\mathbb{Z} \to R$ the Γ_M modules $\mathcal{M}_\lambda \otimes R, \mathcal{M}_{\lambda^\vee} \otimes R$ provide local systems $\widetilde{\mathcal{M}_\lambda \otimes R}, \widetilde{\mathcal{M}_{\lambda^\vee} \otimes R}$, and we have the extension of the cup product pairing

$$H^p(M, \widetilde{\mathcal{M}_\lambda \otimes R}) \times H^{d-p}_c(M, \widetilde{\mathcal{M}_{\lambda^\vee} \otimes R}) \to H^d_c(M, R) = R$$

Proposition 2.5. *If $R = k$ is a field then the pairing is non degenerate. .*

If R is a Dedekind ring then the pairing then the cohomology may contain some torsion submodules and

$$H^p(M, \widetilde{\mathcal{M}_\lambda \otimes R})/\mathrm{Tors} \times H^{d-p}_c(M, \widetilde{\mathcal{M}_{\lambda^\vee} \otimes R})/\mathrm{Tors} \to H^d_c(M, R) = R$$

is non degenerate.

(See Vol. I 4.8.9)

We want to discuss the consequences of this result for the cohomology of $H^\bullet_?(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$. Before we do this we want to recall some simple facts concerning the representations of the algebraic group $G/\mathbb{Q}$. We consider two highest weights $\lambda, \lambda_1 \in X^*(T \times F)$ which are dual modulo the center. By this we mean that we have (See 22)

$$\lambda = \lambda^{(1)} + \delta, \lambda_1 = -w_0(\lambda^{(1)}) + \delta_1 \tag{48}$$

Then $\delta + \delta_1$ is a character on $X^*(C' \times F)$ and yields a one dimensional module

$H^i_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) \times H^{d-i}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}^\vee_\lambda) \to H^d_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \tilde{\mathcal{M}}^\vee_\lambda) \to H^d_c(\mathcal{S}^G_{K_f}, \mathbb{Z})$ for $G \times F$, of course the action of $G^{(1)}$ on this module is trivial. Then we get a G invariant non trivial pairing

$$\mathcal{M}_{\lambda,F} \times \mathcal{M}_{\lambda_1,F} \to \mathcal{N}_{\lambda\circ\lambda_1}$$

which induces a pairing

$$H^i_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,F}) \times H^{d-i}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda_1,F}) \to H^d_c(\mathcal{S}^G_{K_f}, \mathcal{N}_{\lambda\circ\lambda_1}),$$

this only a slight generalization of the previous pairing.

Now we recall that (under certain assumptions) we have the inclusion $\pi_0(\mathcal{S}^G_{K_f}) \hookrightarrow \pi_0(\mathcal{S}^{C'}_{K^{C'}_\infty \times K^{C'}_f})$ and then we get

$$H^d_c(\mathcal{S}^G_{K_f}, \mathcal{N}_{\lambda\circ\lambda_1}) \subset H^0(\mathcal{S}^{C'}_{K^{C'}_\infty \times K^{C'}_f}, \mathcal{N}_{\lambda\circ\lambda_1}) = \bigoplus_{\chi':\text{type}(\chi')=\lambda\circ\lambda_1} F\chi'$$

The character χ' has a restriction to $C(\mathbb{A})$ let us call this restriction χ.

The group $C(\mathbb{A}_f)$ acts on the cohomology groups and this action has an open kernel K^C_f. Hence we can decompose the cohomology groups on the left hand side according to characters

$$H^i_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,F}) = \bigoplus_{\zeta_f:\text{type}(\zeta_f)=\delta} H^i_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,F})(\zeta_f) \tag{49}$$

$$H^{d-i}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda_1,F}) = \bigoplus_{\zeta_{1,f}:\text{type}(\zeta_{1,f})=\delta_1} H^{d-i}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda_1,F})(\zeta_{1,f}). \tag{50}$$

With these notations we get another formulation of Poincaré duality.

Proposition 2.6. *If we have three algebraic Hecke characters $\zeta_f, \zeta_{1,f}, \chi'_f$ of the correct type and if we have the relation $\zeta_f \cdot \zeta_{1,f} = \chi_f$ then the cup product induces a non degenerate pairing*

$$H^i_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,F})(\zeta_f) \times H^{d-i}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda_1,F})(\zeta_{1,f}) \to F\chi'$$

This is an obvious consequence of our considerations above. Fixing the central characters has the advantage that the target space of the pairing becomes one dimensional over F, The field F should contain the values of the characters.

We return to the diagram (45) and consider the images $\text{Im}(r^q)(\zeta_f) = \text{Im}(H^q_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,F})(\zeta_f) \to H^{d-q-1}_c(\partial\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}^\vee_{\lambda,F})(\zeta_f)$ and $\text{Im}(r^{\vee,d-q-1})$. Then the following proposition is an obvious consequence of the non degeneration of the pairing and (46)

Proposition 2.7. *The images Im$(r^p(\zeta_f))$ and Im$(r^{\vee,d-p-1})(\zeta_{1,f})$ are mutual orthogonal complements of each other with respect to $<\,,\,>_\partial$.*

The pairing in proposition 2.6 induces a non degenerate pairing

$$H^i_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,F})(\zeta_f) \times H^{d-i}_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda_1,F})(\zeta_{1,f}) \to F\chi'.$$

Proof. Let $\eta \in H^{d-p-1}(\zeta_{1,f})$ Then we know from the exactness of the sequence that $\eta \in \text{Im}(r^{\vee,d-p-1})(\zeta_{1,f}) \iff \delta(\eta) = 0 \iff <\delta(\eta), \xi>= 0$ for all $\xi \in H^p(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)(\zeta_f) \iff <\eta, r(\xi)>= 0$ for all $\xi \in H^p(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)(\zeta_f) \iff <\eta, \xi'>_\partial= 0$ for all $\xi' \in \text{Im}(r^q)(\zeta_f)$.

The second assertion is rather obvious. If we have $\xi \in H^p_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)(\zeta_f), \xi_1 \in H^{d-p}_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda^\vee})(\zeta_f)$ then we can lift either of these classes - say ξ_1- to a class $\tilde{\xi_1} \in H^p_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)(\zeta_f)$ and then $<\xi_1, \xi_2>=<\tilde{\xi_1}, \xi_2>$. It is clear that the result does not depend on the choice of class which we lift. It is also obvious that the pairing is non degenrate. □

Of course we also have a version of proposition 2.7 for the integral cohomology. Since we fixed the level we have only a finite number of possible central characters $\zeta_f, \zeta_{1,f}$ of the required type. The values of these characters evaluated on $C(\mathbb{A}_f)$ lie in a finite extension $F/\mathbb{Q}$ and of of course they are integral. If we now invert a few small primes and pass to a quotient ring $R = \mathcal{O}_F[1/N]$ then we get the decomposition (49) but with coefficient systems which are R-modules:

$$H^i_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,R}) = \bigoplus_{\zeta_f:\text{type}(\zeta_f)=\delta} H^i_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,R})(\zeta_f) \tag{51}$$

$$H^{d-i}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda_1,R}) = \bigoplus_{\zeta_{1,f}:\text{type}(\zeta_{1,f})=\delta_1} H^{d-i}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda_1,R})(\zeta_{1,f}) \tag{52}$$

Then it becomes clear that we get an integral version of proposition 2.6 where replace the F-vector space coefficient systems $\tilde{\mathcal{M}}_{\lambda,F}$ by R -module coefficient systems. We get a non degenerate pairing

$$H^i_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,R})(\zeta_f)/\text{Tors} \times H^{d-i}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda_1,R})(\zeta_{1,f})/\text{Tors} \to R\chi' \tag{53}$$

We can also get an integral version of proposition 2.7. To formulate it we need a little bit of commutative algebra. Our ring R is a Dedekind ring and all our cohomology groups are finitely generated R modules. If we divide any finitely generated R-module by the subgroups of torsion elements then the result is a projective R-module and it is locally free for Zariski topology.

An element $\xi \in H^i_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,R})(\zeta_f)/\text{Tors}$ is called *primitive* if the submodule $R\xi$ is locally for the Zariski topology a direct summand or what amounts to the same if $H^i_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,R})(\zeta_f)/\text{Tors}/R\xi$ is torsion free. The assertion that the above pairing is non degenerate f means:

For any primitive element $\xi \in H^i_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,R})(\zeta_f)/\text{Tors}$ *we find an element* $\xi_1 \in H^{d-i}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda_1,R})(\zeta_{1,f})/\text{Tors}$ *such the value of the pairing* $< \xi, \xi_1 >= 1$

We can formulate an integral version of proposition 2.7 we have the same notations as above but now our coefficient system is $\tilde{\mathcal{M}}_{\lambda,R}$.

Proposition 2.8. *Assume that* $H^\bullet(\partial\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,R})$ *and* $H^\bullet(\partial\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,R})$ *are torsion free. Then the images* $Im(r^p(\zeta_f))$ *and* $Im(r^{\vee,d-p-1})(\zeta_{1,f})$ *are mutual orthogonal complements of each other with respect to* $< , >_\partial$ *.*

The pairing in proposition 2.6 induces a non degenerate pairing

$$H^i_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,R})(\zeta_f)/\text{Tors} \times H^{d-i}_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda_1,R})(\zeta_{1,f})/\text{Tors} \to R\chi'.$$

2.6.4 Inner Congruences

We choose a highest weight $\lambda = \lambda^{(1)} + d\delta$ and the dual weight $\lambda^\vee = -w_0(\lambda) - d\delta$. Let us also fix a central character ζ_f whose type is equal to the restriction of $d\delta$ to the central torus C.

We look at the pairing in prop. 2.7 where we assume in addition that $\zeta_{1,f} = \zeta_f^{-1}$ and we take the action of the Hecke algebra into account, i.e we look at the decomposition into eigenspaces (see(40). Then we get a non degenerate pairing between isotypical subspaces

$$H^i_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,F})(\pi_f) \times H^{d-i}_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda^\vee,F})(\pi_f^\vee) \to F$$

where we assume that the central characters of the summands are ζ_f, ζ_f^{-1}.

If we try to extend this to the integral cohomology. In this case the above decomposition yields decompositions up to isogeny

$$\begin{array}{r} H^i_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,R})/\mathrm{Tors} \supset \bigoplus_{\pi_f} H^i_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,R})/\mathrm{Tors}(\pi_f) \\ H^{d-i}_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda^\vee,R})/\mathrm{Tors} \supset \bigoplus_{\pi_f} H^{d-i}_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda^\vee,R})/\mathrm{Tors}(\pi_f^\vee) \end{array} \tag{54}$$

where we should fix the central characters as above. We choose a pair $\pi_f, \pi_f^\vee$. Then our non degenerate pairing from the above proposition induces a pairing

$$H^i_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,R})/\mathrm{Tors}(\pi_f) \times H^{d-i}_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda^\vee,R})/\mathrm{Tors}(\pi_f^\vee) \to R \tag{55}$$

and this pairing is non degenerate if and only if both modules are direct summands in the above decomposition up to isogeny.

But it may happen that the values of the pairing generate a proper ideal $\Delta(\pi_f) \subset R$, and in this case the above submodules will not be direct summands and this implies that we will have congruences between the Hecke-module π_f and some other module in the decomposition up to isogeny. This yields the *inner congruences*.

The ideal $\Delta(\pi_f)$ should be expressed in terms of L-values, in the classical case this has been done by Hida [Hi].

3 The fundamental question

Let Σ be a finite set. Of course any product $V = \otimes H_{\pi_p}$ of finite dimensional absolutely irreducible modules for the $\mathcal{H}_p$, for which $\mathcal{H}_p$ is spherical for all $p \notin \Sigma$ gives us an absolutely irreducible module for the Hecke algebra.

We may ask: Can we formulate non tautological conditions for the irreducible representation V or for the collection $\{\pi_p\}_{p:prime}$, which are necessary or (and) sufficient for the occurrence of $\otimes'_p \pi_p$ in the cohomology

This question can be formulated in the more general framework of the theory automorphic forms, but in this book we only consider "cohomological" (or certain limits of those) automorphic forms. This restricted question is difficult enough. A speculative answer is outlined in the following section

3.1 The Langlands philosophy

Let us start from a product $V = \otimes H_{\pi_p}$. For the primes outside the finite set Σ the module H_{π_p} is determined by its Satake parameter ω_p.

3.1.1 The dual group

There is another way of looking at these Satake parameters ω_p. We explain this in the case that $\mathcal{G}/\mathbb{Z}_p$ is a split reductive group. We choose a maximal split torus $\mathcal{T}$ over $\mathbb{Z}$ and a Borel subgroup $\mathcal{B}/\mathbb{Z}$. For any commutative ring with identity ring R we have a canonical isomorphism $X_*(\mathcal{T}) \otimes R^\times \xrightarrow{\sim} \mathcal{T}(R)$, which is given by $\chi \otimes a \mapsto \chi(a)$. Then $\mathcal{T}(\mathbb{Q}_p)/\mathcal{T}(\mathbb{Z}_p) = X_*(\mathcal{T}) \otimes \mathbb{Q}_p^\times/\mathbb{Z}_p^\times = X_*(\mathcal{T})$. We apply this to the maximal split torus $\mathcal{T}/\mathbb{Z}_p \subset \mathcal{G}/\mathbb{Z}_p$. Then $\Lambda(\mathcal{T}) = \mathrm{Hom}(X_*(\mathcal{T}), \mathbb{C}) = X^*(\mathcal{T}) \otimes \mathbb{C}^\times = T^\vee(\mathbb{C})$ where $T^\vee$ is the torus over $\mathbb{Q}$ whose cocharacter module is $X^*(\mathcal{T})$. This torus over $\mathbb{Q}$ is called the dual torus. There is a canonical construction of a dual group ${}^LG/\mathbb{C}$, this is a reductive group with maximal torus $T^\vee$ such that the Weyl group of $T^\vee$ in this dual group is equal to the Weyl group of $\mathcal{T} \subset \mathcal{G}$ (See also (3.1.7)). This dual torus sits in a Borel subgroup ${}^LB \subset^L G$. Recall that we have a canonical pairing

$$<,>: X_*(\mathcal{T}) \times X^*(\mathcal{T}) \to \mathbb{Z},\ \gamma \circ \chi(x) \mapsto x^{<\chi,\gamma>}. \tag{56}$$

The positive simple roots in $X^*(T^\vee)$ in the dual group ${}^LG/\mathbb{C}$ are the cocharacters $\alpha_i^\vee \in X_*(\mathcal{T}^{(1)})$ defined by

$$< \alpha_i^\vee, \gamma_j >= \delta_{i,j}.$$

Hence we can interpret $\omega_p \in \Lambda(T) = X^*(\mathcal{T}) \otimes \mathbb{C}^\times = T^\vee(\mathbb{C})$ as a semi simple conjugacy class in ${}^LG(\mathbb{C})$. Remember that ω_p is only determined by the local component π_p up to an element in the Weyl group, hence we only get a conjugacy class.

We assume that $\mathcal{G}/\mathbb{Z}$ is a split reductive group scheme. Then the dual group ${}^LG$ is also split over $\mathbb{Z}$ and the absolutely irreducible highest weight modules $\mathcal{M}_\lambda$ for $\mathcal{G}/\mathbb{Z}$ and the highest weight module attached to χ are defined over $\mathbb{Q}$. Let $\pi_f \in \mathrm{Coh}_!(G, K_f, \lambda)$ be absolutely irreducible and defined over a finite extension $E/\mathbb{Q}$. Hence we see that our absolutely irreducible π_f provides a collection of conjugacy classes $\{\omega(\pi_p) = \omega_p\}_{p \notin \Sigma}$ in the dual group ${}^LG(E)$.

A rather vague but also very bold formulation of the general Langlands philosophy predicts:

The isotypical components under the action of the Hecke algebra, namely the $H^i_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})(\pi_f)$, should correspond to a collection $\{\mathbb{M}(\pi_f, r_\chi)\}_{r_\chi}$ of motives (with coefficients in E). The correspondence should be defined via the equality of certain automorphic and motivic L-functions.

This formulation is definitely somewhat cryptic, we will try to make it a little bit more precise in the following sections.

Such a motive could in principle be a "direct summand" the $H^i(X)$ of a smooth projective scheme $X/\mathbb{Q}$, which in a certain sense is cut out by a projector. In some cases, where the space $\mathcal{S}^G_{K_f}$ "is a Shimura variety", these motives have been constructed, we will discuss this issue in Chap. V.

3.1.2 The cyclotomic case

We consider the special case that $G = \mathbb{G}_m/\mathbb{Q}$ and our coefficient system $\mathbb{Q}(n)$ is given by the character $[n] : x \mapsto x^n$. We fix a level K_f and we consider our isotypical decomposition over $\mathbb{Q}$

$$H^0(\mathcal{S}^G_{K_f}, \mathbb{Q}(n)) = \bigoplus_\Phi \mathbb{Q}(\Phi_f).$$

In this case $\mathbb{Q}(\Phi_f)$ is a field, and the action of the group is simply an irreducible action of the group of finite ideles $G(\mathbb{A}_f) = I_{\mathbb{Q},f}$ on the $\mathbb{Q}$-vector space $\mathbb{Q}(\Phi_f)$. If we extend our field to $\bar{\mathbb{Q}}$ we get a decomposition

$$H^0(\mathcal{S}^G_{K_f}, \bar{\mathbb{Q}}(n))) = \bigoplus_{\chi:type(\chi)=[n]} \bar{\mathbb{Q}}(\chi),$$

and the collection of conjugate characters χ are in one to one correspondence with the Φ_f. We can attach two different kinds of L-functions to our isotypical component Φ_f namely an automorphic L-function and a motivic L-function.

Actually we get a collection of such L-functions which are labelled by the embeddings $\iota : \mathbb{Q}(\Phi) \to \bar{\mathbb{Q}} \subset \mathbb{C}$. Such an embedding yields an algebraic Hecke character

$$\chi_f^{(\iota)} = \iota \circ \Phi_f : G(\mathbb{A}_f) = I_{\mathbb{Q},f} \to \bar{\mathbb{Q}}^\times$$

and

$$\chi^{(\iota)} = \iota \circ \Phi : G(\mathbb{Q})\backslash G(\mathbb{A}) = \mathbb{Q}^\times \backslash I_{\mathbb{Q}} \to \mathbb{C}^\times$$

and to any of these Hecke characters we attach the (the automorphic L-function) namely

$$L(\chi^{(\iota)}, s) = \prod_p (1 - \chi^{(\iota)}(p)p^{-s})^{-1}$$

where $\chi\ ^{(\iota)}(p) = \chi\ ^{(\iota)}(1, 1\dots, p, \dots)$ and it is zero of the character is ramified.

Now we can attach a motive $\mathbb{M}(\Phi)$ to our isotypical component. To do this we assume first that $\mathbb{Q}(\Phi) = \mathbb{Q}$, then we have only one embedding. Then we have $\chi(\underline{x}) = \alpha^n(\underline{x}) = \mid \underline{x} \mid^n$ for some integer n. This is an algebraic Hecke character of type $[-n] : x \mapsto x^{-n}$. Then we attach the motive $\mathbb{Z}(-n)$ to this Hecke character. At this moment we do not need to know what a motive is, the only thing we need to know that it provides a compatible system of ℓ -adic representations of the Galois group: For any prime ℓ we define a module To this motive we attach a motivic L function using the compatible system of ℓ-adic representations. For a prime ℓ and a prime $p \neq \ell$ we have the local Euler factor

$$L_p(\mathbb{Z}(-n), s) = \frac{1}{\det(1 - F_p^{-1}|\mathbb{Z}_\ell(-n)p^{-s})} = \frac{1}{1 - p^n p^{-s}},$$

where F_p is the Frobenius at p. The ℓ-adic representation is unramified outside ℓ and the Frobenius F_p corresponds to p under the reciprocity map r. Hence we see that the Frobenius F_p acts by the multiplication by $\alpha^n(p) = \mid p \mid_p^n = p^{-n}$ on $\mathbb{Z}_\ell(-n)$. In the general case we start from the representation $\Phi_f : I_{\mathbb{Q},f} \to \mathbb{Q}(\Phi_f)^\times$, it is unramified outside a finite set Σ of primes. The reciprocity map from class field theory provides a homomorphism $r : I_{\mathbb{Q},f} \to \mathrm{Gal}_\Sigma(\bar{\mathbb{Q}}/\mathbb{Q})_{abelian}$, this is the maximal abelian quotient of the Galois group which is unramified outside Σ, the image of the reciprocity map is dense. If we fix a prime ℓ then we get an ℓ-adic representation

$$\rho(\Phi) : \mathrm{Gal}_\Sigma(\bar{\mathbb{Q}}/\mathbb{Q})_{abelian} \to (\mathbb{Q}(\Phi_f) \otimes \mathbb{Q}_\ell)^\times,$$

which is determined by the rule $\rho(\Phi)(F_p) = \Phi_f(p)$. If we now choose an embedding $\iota : \mathbb{Q}(\Phi_f) \to \bar{\mathbb{Q}}$ and an extension $\mathfrak{l}$ of ℓ to a place of $\bar{\mathbb{Q}}$ and we get a one dimensional $\mathfrak{l}$ adic representation

$$\rho(\iota \circ \Phi) : \mathrm{Gal}_\Sigma(\bar{\mathbb{Q}}/\mathbb{Q})_{abelian} \to \bar{\mathbb{Q}}_{\mathfrak{l}}^\times,$$

from which we get a motivic L-function $(\mathbb{M}(\Phi)\circ\iota, s)$, whose local factor at p is

$$L_p(\mathbb{M}(\Phi)^{(\iota)}, s) = \frac{1}{1-\rho(\iota\circ\Phi)(F_p)^{-1}p^{-s}}$$

These are the collections of ℓ-adic rpresentations of our motives $\mathbb{M}(\Phi)$. Then the relation between the automorphic and the ℓ -adic L functions is:

The collection of automorphic L-functions attached to Φ is equal to the collection of motivic L-functions attached to $\mathbb{M}(\Phi^{-1})$.

We will sometimes modify the notation slightly. If χ is an algebraic Hecke character then this datum corresponds to a pair (Φ,ι) and hence we can attach to it a character $\chi_{\mathfrak{l}} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \bar{\mathbb{Q}}_{\mathfrak{l}}$ and then we get the equality of local L-factors

$$L_p(\chi, s) = \frac{1}{1-\chi(p)p^{-s}} = \frac{1}{1-\chi_{\mathfrak{l}}^{-1}(F_p)^{-1}p^{-s}}$$

(Nochmal ein wenig besser schreiben!!!!!!!!!!!!!!)

3.1.3 The L-functions

Let us choose a cocharacter $\chi : G_m \to T$, we assume that it is in the positive chamber, i.e. we have $<\chi, \alpha_i>\geq 0$ for all positive simple roots . It yields an element $\chi(p)\in T(\mathbb{Q}_p)$. For $\omega_p\in\Lambda(T)$ we put

$$S_{\chi,\omega_p} = p^{<\chi,\rho>}\sum_{w\in W/W_\chi}\omega_p(w(\chi(p))$$

then we get a formula

$$\int_{\mathbf{ch}(\chi(p))}\phi_{\omega_p}(xg)dg = (S_{\chi,\omega_p}+\sum_{\chi'<\chi}a(\chi,\chi')S_{\chi',\omega_p})\phi_{\omega_p}(x) \tag{57}$$

where the χ' are in the positive chamber, $\chi'<\chi$ means that $\chi-\chi'=\sum n_i\chi_i, n_i\geq 0$ and the coefficients $a(\chi,\chi')\in\mathbb{Z}$. The expression on the right hand side is invariant under W and hence only depends on ω_p modulo W. (**Give reference!**)

The number $<\chi,\rho>$ is a half integer, hence $p^{<\chi,\rho>}$ may not lie in a fixed number field if p varies. But for those χ' which may occur in the summation we have $<\chi-\chi',\rho>\in\mathbb{Z}$.

We consider an unramified prime. The theorem of Satake yields that we can define a Hecke operator $S_\chi\in\mathcal{H}_p$ such that $S_\chi*\phi_{\omega_p}=S_{\chi,\omega_p}\phi_{\omega_p}$ and the formula (57) tells us that we get another recursion

$$S_\chi = \mathbf{ch}(\chi)+\sum_{\chi'<\chi}b(\chi,\chi')\mathbf{ch}(\chi') \tag{58}$$

where again $b(\chi,\chi')\in\mathbb{Z}$.

Since we assume that our absolutely irreducible module $V_{\pi_f}, \pi_f=\otimes'\pi_p$ occurs in $\mathrm{Coh}(G, K_f,\lambda)$, the Hecke module is a vector space over a finite extension

$F/\mathbb{Q}$. We can conclude that the eigenvalue of the convolution operator $\mathbf{ch}(\chi)$ is in F and it follows that

$$S_{\chi,\omega_p} \in F$$

for any cocharacter χ.

Since we can replace χ by $n\chi$ for any integer $n \geq 1$ it follows that the numbers $w(\chi(p))$ lie in a finite extension of F and the polynomial

$$\prod_{w \in W/W_\chi} (X \cdot \mathrm{Id} - p^{<\chi,\rho>} w(\chi(p))) \in F[X].$$

Our cocharacter $\chi \in X_*(T)$ can also be interpreted as a character in $X^*(T^\vee)$, i.e it is a character on the dual torus. Since we assumed it to be in the positive chamber we can view χ as the highest weight of an irreducible representation $r_\chi :^L G \to \mathrm{Gl}(\mathcal{E}_\chi)$. (Since we assume that G is split the dual group is also split over $\mathbb{Q}$ and hence r_χ is defined over $\mathbb{Q}$.) The eigenvalues of the endomorphism $r_\chi(\omega_p)$ are of the form $\omega_p(w(\chi'(p))$ where $\chi' \leq \chi$ and this implies that the polynomial

$$\det(X \cdot \mathrm{Id} - p^{<\chi,\rho>} r_\chi(\omega_p)|\mathcal{E}_\chi) \in F[X].$$

We attach a local Euler factor to the data $\pi_p, \omega_p = \omega(\pi_p), \chi$:

$$L_p^{\mathrm{rat}}(\pi_f, r_\chi, s) = \frac{1}{\det(\mathrm{Id} - p^{<\chi,\rho>} r_\chi(\omega_p) p^{-s}|\mathcal{E}_\chi)} \tag{59}$$

which is a formal power series in the variable p^{-s} with coefficients in F. We define

$$L^{\mathrm{rat}}(\pi_f, r_\chi, s) = \prod_{p \in \Sigma} L_p(\pi_f, r_\chi, s)(\prod_{p \notin \Sigma} \frac{1}{\det(\mathrm{Id} - p^{<\chi,\rho>} r_\chi(\omega_p) p^{-s}|\mathcal{E}_\chi)}), \tag{60}$$

at the moment we do not say anything about the Euler factors at the bad primes.

At this moment $L^{\mathrm{rat}}(\pi_f, r_\chi, s)$ is a a product of formal power series in infinitely many variables p^{-s} which in some sense encodes the collection of eigenvalues of the different Hecke eigenvalues.

We want to relate this L -function to some other $L-$ functions which are defined in the theory of automorphic forms.

To define the automorphic L -function we start from an absolutely irreducible Hecke -module V_{π_f} over $\mathbb{C}$, its isomorphism type is still denoted by π_f. This π_f will be the first argument (in our notation) in the automorphic L-function. It has a central character ζ_{π_f} and we assume that this central character is the finite component of a character $\zeta_\pi : C(\mathbb{Q})\backslash C(\mathbb{A}) \to \mathbb{C}^\times$. (In the back of our mind of π_f to be the finite component of an automorphic form π, then this assumption is automatically fulfilled. But for the definition of the L-functions we do not need this.)

Then we define the unitary (automorphic) L-function: Here we require that the central character ζ_{π_f} of π_f is unitary and put

$$L^{\mathrm{unit}}(\pi_f, r_\chi, s) = \prod_{p \in \Sigma} L_p(\pi_f, r_\chi, s)(\prod_{p \notin \Sigma} \frac{1}{\det(\mathrm{Id} - r_\chi(\omega_p) p^{-s}|\mathcal{E}_\chi)}) \tag{61}$$

If the central character is not unitary we define the automorphic L-function essentially by the same formula:

$$L^{\mathrm{aut}}(\pi_f, r_\chi, s) = \prod_{p\in\Sigma} L_p(\pi_f, r_\chi, s)(\prod_{p\notin\Sigma} \frac{1}{\det(\mathrm{Id} - r_\chi(\omega_p)p^{-s}|\mathcal{E}_\chi)}) \tag{62}$$

This $L-$ function is related to an unitary $L-$ function by a shift in the variable s. The isogeny d_C induces a homomorphism $d' : C(\mathbb{Q})\backslash C(\mathbb{A}) \to C'(\mathbb{Q})\backslash C'(\mathbb{A})$ and it is well known that this map has a compact kernel. We compose ζ_π with the the norm $|\ | : \mathbb{C}^\times \to \mathbb{R}^\times_{>0}$, this composition is trivial on the kernel of d'. Therefore we find a homomorphism $|\zeta_\pi|^* : C'(\mathbb{A}_f) \to \mathbb{R}^\times_{>0}$ which satisfies $|\ | \circ \zeta_\pi = |\zeta_\pi|^* \circ d'$. We look at the finite components of these characters and put as in (2.4.4)

$$\pi_f^* = \pi_f \otimes (|\zeta_\pi|^*)^{-1}. \tag{63}$$

This module has a unitary central character. It is easy to see how the Satake parameter changes under the twisting. We have the homomorphism $T(\mathbb{A}) \to C'(\mathbb{A})$ and therefore $(|\zeta_\pi|^*)^{-1}$ induces also a homomorphism from $T(\mathbb{A}_f)$ to $\mathbb{R}^\times_{>0}$. Then it is clear that we get for the Satake parameters the equality

$$\omega(\pi_p \otimes (|\zeta_\pi|^*_p)^{-1}) = \omega(\pi_p)(|\zeta_\pi|^*_p)^{-1} \tag{64}$$

Let us assume that π_f occurs as an isotypical subspace in some $H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{C})$, where $\lambda = \lambda^{(1)} + \delta$. The element δ is an element in $X^*(C') \otimes \mathbb{Q}$. To an element $\eta \in X^*(C') \otimes \mathbb{R}$ we have attached an element $|\eta|$ and since ζ_{π_f} is of type $\delta \circ d_C$ we have

$$(|\zeta_\pi|^*)^{-1} = |\delta|.$$

We also have the cocharacter $\chi : \mathbb{G}_m \to T$ then it is clear that the composition $(|\zeta_\pi|^*)^{-1} \circ \chi$ induces a homomorphism $\mathbb{G}_m(\mathbb{Q})\backslash\mathbb{G}_m(\mathbb{A}) \to \mathbb{R}^\times_{>0}$ which is of the form

$$((|\zeta_\pi|^*)^{-1} \circ \chi)_{\mathbb{A}} : \underline{x} \mapsto |\underline{x}|^{<\chi,\delta>}. \tag{65}$$

Then we have

$$L^{\mathrm{unit}}(\pi_f^*, r_\chi, s) = L^{\mathrm{aut}}(\pi_f, r_\chi, s+ < \chi, \delta >) \tag{66}$$

We now assume that π_f^* is the finite part of a cuspidal unitary representation (See 4.2), then the functions $L^{\mathrm{unit}}(\pi_f^*, r_\chi, s)$ are studied in the theory of automorphic forms. The Euler factors are now meromorphic functions in the variable $s \in \mathbb{C}$. Since π_f^* is unitary it follows that the Satake parameters satisfy some bounds and this implies that the infinite product converges if $\Re(s) >> 0$. If for all $p \notin \Sigma$ the representation π_p^* is in the unitary principal series, i.e $|\omega^*_{i,p}| = 1$ then it follows from standard arguments that the infinite product over $p \notin \Sigma$ converges for $\Re(s) > 1$.

It is a conjecture (proved in some cases) that $L^{\mathrm{unit}}(\pi_f, r_\chi, s)$ has analytic continuation into the entire complex plane and that there is a functional equation relating $L^{\mathrm{unit}}(\pi_f, r_\chi, s)$ and $L^{\mathrm{unit}}(\pi_f^\vee, r_\chi, 1-s)$.

But of course any theorem proved for the L-functions $L^{\mathrm{unit}}(\pi_f^*, r_\chi, s)$ translates into a theorem for the automorphic L functions $L^{\mathrm{aut}}(\pi_f, r_\chi, s)$.

Given a automorphic representation π which occurs in the cuspidal spectrum then we may twist it by any character $\xi : C'(\mathbb{Q})\backslash C'(\mathbb{A}) \to \mathbb{R}^\times_{>0}$, this group of characters is equal to $X^*(C') \otimes \mathbb{R}$. We get a principal homogenous space (a torsor) of automorphic representations $\{\pi \otimes \xi\}_{\xi \in \Xi}$.

For the Euler factors $p \notin \Sigma$ we have

$$\frac{1}{\det(\mathrm{Id} - r_\chi((\omega_p)(\pi_p \otimes \xi_p))p^{-s}|\mathcal{E}_\chi)}) = \frac{1}{\det(\mathrm{Id} - r_\chi((\omega_p)(\pi_p\ ))p^{-<\chi,\xi>-s}|\mathcal{E}_\chi)}) \tag{67}$$

and hence we get for our automorphic L-function

$$L^{\mathrm{aut}}(\pi_f \otimes \xi_f, r_\chi, s) = L^{\mathrm{aut}}(\pi_f, r_\chi, s+ < \chi, \xi >) \tag{68}$$

The representation π^* is then the unique cuspidal (in the above sense) representation in this principal homogeneous spaces $\{\pi \otimes \xi\}_{\xi \in \Xi}$, i.e. it is the unique representation which has a unitary central character. In other words π^*_f provides a trivialization of the torsor. Then we define for any $\pi \otimes \xi$

$$L^{\mathrm{unit}}(\pi_f \otimes \xi_f, r_\chi, s) = L^{\mathrm{unit}}(\pi^*_f, r_\chi, s) \tag{69}$$

the unitary L-function is constant on the torsor, i.e. invariant under twisting.

We compare the automorphic $L-$ function to the rational $L-$ function. We start from an absolutely irreducible module π_f which occurs in $\mathrm{Coh}_!(G, K_f, \lambda)$ and which is defined over some finite extension $F/\mathbb{Q}$. As usual we write $\lambda = \lambda^{(1)} + \delta$, (See(22)). We know that the central character ζ_{π_f} is an algebraic Hecke character of type δ. Our Hecke module π_f is an absolutely irreducible module over F. If we want to compare its L functions to automorphic L-functions we need to choose an embedding $\iota : F \hookrightarrow \mathbb{C}$ and consider the module $V_{\pi_f} \otimes_{F,\iota} \mathbb{C} = V_{\iota\circ\pi_f}$. The we will see in section 4.2 that $\iota \circ \pi_f$ is the finite part of an automorphic representation occurring in the discrete (or the cuspidal) spectrum. Hence we have defined $\tilde{L}^{\mathrm{aut}}(\iota \circ \pi_f, r_\chi, s))$. We can also consider the "extension" of the rational L-function

$$\iota\circ L^{\mathrm{rat}}(\pi_f, r_\chi, s) = \prod_{p\in\Sigma} \iota\circ L_p^{\mathrm{rat}}(\pi_f, r_\chi, s) \prod_{p\notin\Sigma} \frac{1}{\det(\mathrm{Id} - \iota(p^{<\chi,\rho>} r_\chi(\omega_p(\pi_p)))p^{-s}|\mathcal{E}_\chi)}$$

Then it is clear that

$$\iota \circ L^{\mathrm{rat}}(\pi_f, r_\chi, s) = L^{\mathrm{aut}}(\iota \circ \pi_f, r_\chi, s- < \chi, \rho >). \tag{70}$$

The central character of $\iota \circ \pi_f$ is of type δ, it follows from (22) that some non zero multiple $r\delta \in X^*(T)$. Then we put $< \chi, \delta >= \frac{1}{r} < \chi, r\delta >$, this is a rational number. Then we get

$$\iota \circ L^{\mathrm{rat}}(\pi_f, r_\chi, s) = L^{\mathrm{aut}}(\iota \circ \pi_f, r_\chi, s- < \chi, \delta >) \tag{71}$$

We still have another L function which is attached to a Hecke module π_f which occurs in the cohomology, this is the cohomological L function. Let us decompose the representation $\mathcal{E}_\lambda$ into weight spaces

$$\mathcal{E}_\chi = \bigoplus_\nu \mathcal{E}_{\chi,\nu} = \bigoplus_{\nu\in X_{*,+}(T)} \bigoplus_{w\in W/W_\nu} \mathcal{E}_{\chi,w(\nu)}$$

then we get with $m(\nu,\chi) = \dim(\mathcal{E}_{\chi,w(\nu)})$. Such a weight vector space is zero unless we have $\nu < \chi$.

$$\det(\mathrm{Id} - r_\chi(\omega_p)p^{-s}|\mathcal{E}_\chi) = \prod_{\nu\in X_{*,+}(T)} \prod_{w\in W/W_\nu} (1-\omega_p(w(\nu))p^{-s})^{m(\nu,\chi)}$$

For a given ν we expand the inner product

$$\prod_{w\in W/W_\nu} (1-\omega_p(w(\nu))p^{-s}) = 1 - (\sum_{w\in W/W_\nu} \omega_p(w(\nu)))p^{-s}\dots.$$

Now we recall that

$$p^{<\chi,\lambda^{(1)}>-<\chi,\delta>}\mathbf{ch}(\chi) = S_\chi^{(\lambda)}$$

is an operator on the integral cohomology (See (27)). Then our recursion formula (58) implies that

$$p^{<\chi,\lambda^{(1)}>-<\chi,\delta>}S_{\chi'}$$

is an operator on the integral cohomology, we simply have to observe that $<\chi,\lambda^{(1)}> \geq <\chi',\lambda^{(1)}>$. From this it follows directly that for $\nu \in X_{*,+}(T)$ which occurs as a weight in r_χ we have

$$p^{<\chi,\lambda^{(1)}+\rho>-<\chi,\delta>} \sum_{w\in W/W_\nu} \omega_p(w(\nu)) \in \mathcal{O}_F$$

because $<\chi,\lambda^{(1)}> \; > \; <\nu,\lambda^{(1)}>$. Then the right hand side in the above formula can be written

$$1 - p^{<\chi,\lambda^{(1)}+\rho>-<\chi,\delta>}(\sum_{w\in W/W_\nu} \omega_p(w(\nu)))p^{-s-<\chi,\lambda^{(1)}+\rho>+<\chi,\delta>}\dots$$

We introduce the new variable $s' = s + <\chi,\lambda^{(1)}+\rho> - <\chi,\delta>$ and put

$$c(\chi,\lambda) = <\chi,\lambda^{(1)}+\rho> - <\chi,\delta> \tag{72}$$

$$\prod_{w\in W/W_\nu} (1-p^{c(\chi,\lambda)}\omega_p(w(\nu))p^{-s'}) = 1 - p^{c(\chi,\lambda)}(\sum_{w\in W/W_\nu} \omega_p(w(\nu)))p^{-s'}\dots \tag{73}$$

Hence we define the cohomological local Euler factor at p

$$L_p^{\mathrm{coh}}(\pi_f, r_\chi, s) = \frac{1}{\det(\mathrm{Id} - p^{c(\chi,\lambda)}r_\chi(\omega_p)p^{-s})}. \tag{74}$$

(It seems to be reasonable and very adequate to define for any highest weight λ the modified weight $\tilde{\lambda} = \lambda + \rho$.)

We look at this local Euler factor from a slightly different point of view. Our π_f is an absolutely irreducible module which occurs in the cohomology $H^\bullet_?(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda \otimes F)$, where $F/\mathbb{Q}$ is an abstract (normal) finite extension of $\mathbb{Q}$. For an unramified prime p the local factor is simply a homomorphism $\pi_p : \mathcal{H}_p \to E$. The previous computations show that the denominator is equal to a polynomial in the "variable" p^{-s} and with coefficients in $\mathcal{O}_F$, i.e.

$$\det(\mathrm{Id} - p^{c(\chi,\lambda)} r_\chi(\omega_p) p^{-s}) = 1 - A_1(p,\lambda,\chi)(\pi_p)p^{-s} + A_2(p,\lambda,\chi)(\pi_p)p^{-2s} \cdots \in \mathcal{O}_F[p^{-s}] \tag{75}$$

where the $A_i(p,\lambda,\chi)$ are certain explicitly computable elements in $\mathcal{H}^{(\lambda)}_{\mathbb{Z}}$. (We showed this only for $A_1(p,\lambda,\chi)$ but the same kind of reasoning gives it for the other $A_i(p,\lambda,\chi)$.) In the expression of the right hand side the Satake parameter does not enter.

The cohomological L function is defined as

$$L^{\mathrm{coh}}(\pi_f, r_\chi, s) = \prod_{p\in\Sigma} L^{\mathrm{coh}}_p(\pi_p, r_\chi, s) \prod_{p\notin\Sigma} \frac{1}{1 - A_1(p,\lambda,\chi)(\pi_p)p^{-s} + A_2(p,\lambda,\chi)(\pi_p)p^{-2s}\dots}. \tag{76}$$

Again we do not discuss the factors at the primes in Σ.

In the definition of the automorphic L function the Satake parameter is an element in ${}^LG(\mathbb{C})$ or in other words $\omega_p(\nu) \in \mathbb{C}^\times$ and $L^{\mathrm{aut}}_p(\pi_f, r_\chi, s)$ is an honest analytic function in the complex variable s for $\Re(s) >> 0$.

If we want to compare the cohomological L-function to the automorphic L -function we have to pick an element $\iota \in I(F,\mathbb{C})$, then $\iota \circ \pi_f$ is an absolutely irreducible Hecke module over $\mathbb{C}$. To $\iota\circ\pi_p$ belongs a Satake parameter ω_p and then

$$\det(\mathrm{Id} - r_\chi(\omega_p)p^{-s+c(\chi,\lambda)}) = 1 - \iota(A_1(p,\lambda,\chi))(\pi_p))p^{-s} + \iota(A_2(p,\lambda,\chi))(\pi_p)p^{-2s}\dots$$

and this tells us that we have

$$L^{\mathrm{coh}}(\iota\circ\pi_f, r_\chi, s) = L^{\mathrm{aut}}(\iota\circ\pi_f, r_\chi, s - c(\chi,\lambda)) \tag{77}$$

3.1.4 Invariance under twisting

We remember that we introduced the quotient $\mathcal{C}' = \mathcal{T}/\mathcal{T}^{(1)}$ and the isogeny $d_C : \mathcal{C} \to \mathcal{C}'$. (See 1.1). The map d_C in 1.1 induces a map from our locally symmetric space

$$\mathcal{S}^G_{K_f} \xrightarrow{d_{\mathcal{C}'}} \mathcal{S}^{\mathcal{C}'}_{K^{\mathcal{C}'}_\infty \times K^{\mathcal{C}'}_f}$$

We assume that K_∞ is connected and then $K^{\mathcal{C}'}_\infty$ is also connected.

We can modify our system of coefficients if we replace λ by $\lambda + \delta_1$ with $\delta_1 \in X^*(\mathcal{C}')$. Then δ_1 provides a local coefficient system $\mathbb{Z}[\delta_1]$ on $\mathcal{S}^{\mathcal{C}'}_{K^{\mathcal{C}'}_\infty \times K^{\mathcal{C}'}_f}$ and since $K^{\mathcal{C}'}_\infty$ is connected we get a canonical class

$$e_{\delta_1} \in H^0(\mathcal{S}^{\mathcal{C}'}_{K^{\mathcal{C}'}_\infty \times K^{\mathcal{C}'}_f}, \mathbb{Z}[\delta_1])$$

which generates the rank one submodule of type $|\delta_f|^{-1}$ in the decomposition (37). We pull this back by $d'_{\mathcal{C}}$ and we get a class in

$$e_{\delta_1} \in H^0(\mathcal{S}^G_{K_f}, \mathbb{Z}[\delta_1]) \tag{78}$$

(see section (2.5.2)). We have the isomorphism $\mathcal{M}_{\lambda,\mathbb{Z}} \otimes \mathbb{Z}[\delta_1] \xrightarrow{\sim} \mathcal{M}_{\lambda+\delta_1,\mathbb{Z}}$ and then the cup product with e_{δ_1} yields an isomorphism

$$H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,\mathbb{Z}}) \cup e_{\delta_1} \xrightarrow{\sim} H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda+\delta_1,\mathbb{Z}}) \tag{79}$$

This isomorphism is compatible with the action of the integral Hecke algebra provided we choose the right identification

$$\mathcal{H}^{(\lambda)}_{\mathbb{Z}} \to \mathcal{H}^{(\lambda+\delta_1)}_{\mathbb{Z}}$$

which is given by $a \cdot \mathbf{ch}(\underline{x}_f) \mapsto p^{<\mathbf{ch}(\underline{x}_f),\delta_1>} a \cdot \mathbf{ch}(\underline{x}_f)$.

If we extend the coefficients to F then this cup product yields an isomorphism

$$H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,F})(\pi_f) \xrightarrow{\sim} H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda+\delta_1,F})(\pi_f \otimes |\delta_{1,f}|^{-1}) \tag{80}$$

Then our cohomological L-function has the property

$$L^{\mathrm{coh}}(\pi_f \otimes |\delta_{1,f}|^{-1}, r_\chi, s) = L^{\mathrm{coh}}(\pi_f, r_\chi, s) \tag{81}$$

This invariance under twists is of course also a consequence of the definition in terms of the automorphic L-function.

We may interpret this differently. Our λ is a sum of a semi-simple component $\lambda^{(1)}$ plus an abelian part δ We can use the isomorphisms in (80) to define a vector space

$$H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda^{(1)}+,F})\{\pi_f\}, \tag{82}$$

this vector space has a distinguished isomorphism to any of the $H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda+\delta_1,F})(\pi_f \otimes |\delta_{1,f}|^{-1})$, we could say that it the direct limit of all these spaces. By $\{\sigma_f\}$ we understand the array

$$\{\sigma_f\} = \{\dots, \pi_f \otimes |\delta_{1,f}|^{-1}, \}_{\delta_1 \in X^*(\mathcal{C}')}.$$

Using (81) we have now defined $L^{\mathrm{coh}}(\{\pi_f\}, r_\chi, s)$

For any pair $\chi \in X_*(T), \lambda \in X^*(T)$, where χ is in the positive chamber and λ a dominant weight we define the weight

$$\mathbf{w}(\chi, \lambda) =< \chi, \lambda^{(1)} + \rho > . \tag{83}$$

Here we observe that χ provides a highest weight representation $r = r_\chi$ of ${}^L G$ and λ a highest weight representation of G so we could also write

$$\mathbf{w}(\chi, \lambda) = \mathbf{w}(r_\chi, \mathcal{M}_\lambda) = \mathbf{w}(r, \mathcal{M}). \tag{84}$$

This means that we may consider the weight as a number attached to a pair of irreducible rational representations of ${}^L G$ and G. It also depends only on the semi simple part of λ.

3.1.5 A different look

We could look at the previous discussion from another point of view. Given our coefficient system $\mathcal{M}_\lambda$ where $\lambda = \lambda^{(1)} + \delta$ and an absolutely irreducible module $\pi_f \in \mathrm{Coh}_!(G, \lambda, K_f)$. As explained above we get $X^*(C')$ torsor $(\lambda + \delta', \pi_f \otimes |\delta'_f|)$ of such objects. If we choose a $\iota : F \hookrightarrow \mathbb{C}$ then we can think of $\iota \circ \pi_f$ as the finite part of an automorphic representation π. Then we get a second torsor for the above group $\Xi = X^*(C') \otimes \mathbb{R}$. The inclusion $X^*(C') \hookrightarrow \Xi$ yields an interpolation of the first torsor into the second one. To any element $\pi \otimes \xi$ we defined the automorphic L function $L^{\mathrm{aut}}(\iota \circ \pi_f \otimes \xi_f, r_\chi, s)$. Now the unitary and the cohomological L-function are defined as the automorphic L function of a specific point in the torsor, i.e. a specific trivialization.

To define the unitary L function we choose the specific point for which the central character is unitary, for the cohomological L -function we choose the "optimal" point $\pi_f \otimes |\delta'_f|$ for which we have

$$L_p^{\mathrm{coh}}(\pi_f \otimes |\delta'_f|, r_\chi, s)^{-1} \in \mathcal{O}_F[p^{-s}]. \tag{85}$$

If we are investigating analytic questions concerning automorphic forms the unitary L is the right object, but if we want to capture the integral structure of the cohomology we prefer to work with the cohomological L function.

3.1.6 The motives

We consider an isotypical submodule $H_!^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda;F})(\pi_f)$ in the inner cohomology. The Langlands philosophy predicts the existence of a collection of pure motives over $\mathbb{Q}$ with coefficients in F.

$$\{\mathbb{M}(\pi_f, r_\chi)\}_{r_\chi}$$

which has certain properties. We will not be absolutely precise in the following but we list certain properties this motive should have. We should assume that π_f is not some kind of exceptional Hecke module (for instance it should not be endoscopic), and I can not give a precise definition what that means. We will make it more precise later when we discuss the case that our group is Gl_n.

This motive should be invariant under twists, i.e. we want that

$$\mathbb{M}(\pi_f \otimes |\delta_f|, r_\chi) = \mathbb{M}(\pi_f, r_\chi)$$

First of all this motive has a Betti-realization $\mathbb{M}(\pi_f, r_\chi)_B$, which is simply an F vector space of dimension $\dim(r_\chi)$. Such a motive has a de-Rham realization $\mathbb{M}(\pi_f, r_\chi)_{dRh}$, this is another F-vector space of the same dimension. It has a descending filtration

$$\mathbb{M}(\pi_f, r_\chi)_{dRh} = F^0(\mathbb{M}(\pi_f, r_\chi)_{de-Rh}) \supset F^1(\mathbb{M}(\pi_f, r_\chi)_{de-Rh}) \supset \dots$$

$$\dots \supset F^{\mathbf{w}}(F^0(\mathbb{M}(\pi_f, r_\chi)_{dRh}) \supset F^{\mathbf{w}+1}(F^0(\mathbb{M}(\pi_f, r_\chi)_{dRh}) = 0.$$

The number $\mathbf{w} = \mathbf{w}(\pi_f, \chi)$ is the weight of the motive it is equal to $\mathbf{w}(\chi, \lambda)$.

Furthermore we have a comparison isomorphism

$$I_{B-dRh} : \mathbb{M}(\pi_f, r_\chi)_B \otimes \mathbb{C} \xrightarrow{\sim} \mathbb{M}(\pi_f, r_\chi)_{dRh} \otimes \mathbb{C},$$

this yields periods and these periods should be related to π_f, this is rather mysterious.

For any prime ℓ and any prime $\mathfrak{l}|\ell$ in F we get a Galois representation

$$\rho(\pi_f,\chi) : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL(\mathbb{M}(\pi_f, r_\chi)_B \otimes F_{\mathfrak{l}})$$

which is unramified outside $\Sigma \cup \{l\}$ and for any such prime we have

$$\det(\mathrm{Id} - \rho(\pi_f,\chi)(\Phi_p^{-1})p^{-s}, \mathbb{M}(\pi_f, r_\chi)_B \otimes F_{\mathfrak{l}}) = L_p^{\mathrm{coh}}(\pi_f, r_\chi, s)^{-1},$$

or in other words we expect that the semi-simple conjugacy classes

$$\rho(\pi_f,\chi)(\Phi_p^{-1}) \sim p^{c(\chi,\lambda)} r_\chi(\omega_p) \tag{86}$$

and hence we want

$$L^{\mathrm{coh}}(\pi_f, r_\chi, s) = L(\mathbb{M}(\pi_f, r_\chi), s)$$

The existence of these hypothetical motives has a lot of consequences. Once we have established such a relation

$$L^{\mathrm{coh}}(\pi_f, r_\chi, s) = L(\mathbb{M}(\pi_f, r_\chi), s)$$

then we can exploit this in both directions. We have a certain chance to prove the conjectural analytic properties and the conjectural functional equation for the L-function of the motive $\mathbb{M}(\pi_f, r_\chi)$, provided we can prove this for $L^{\mathrm{coh}}(\pi_f, r_\chi, s)$. On the automorphic side we know many cases in which we can prove these properties of the L-function using the theory of automorphic forms.

In the other direction we have Deligne's theorem concerning the absolute values of the Frobenius. This implies Ramanujan (more details later)

We seem to be very far away from proving these conjectures, but there are many instances where some parts of this program have been established and there are also some very interesting cases where this correspondence has been verified experimentally.

3.1.7 The case $G = \mathrm{Gl}_n$

3.1.8 Notations for the dual group ${}^LG$

We want to verify formula (57) in the special case $G = \mathrm{Gl}_n/\mathbb{Z}$. In this case t we have the cocharacters χ_i which send t to the diagonal matrix $t \mapsto \mathrm{diag}(t, \dots, t, 1 \dots, 1)$ where t is placed to the first i dots. They satisfy $<\chi_i, \alpha_j> = \delta_{i,j}$ for $1 \le i \le n, 1 \le j \le n-1$. They are uniquely determined by this condition modulo the cocharacter χ_n which identifies $\mathbb{G}_m$ with the center. For $1 \le \nu \le n-1$ the cocharacter χ_i determines a maximal parabolic subgroup $P_i \supset T$ whose roots $\Delta_{P_i} = \{\alpha| <\chi_i, \alpha> \ge 0\}$. The parabolic subgroup P_i^- will be the opposite parabolic subgroup.

Let $\eta_i : \mathbb{G}_m \to T$ be the cocharacter which sends t to t on the $i-$ th spot on the diagonal and to 1 at all others. If we identify the module of cocharacters with the character group of the dual torus $T^\vee \subset^L G = \mathrm{Gl}_n$ then the differences $\eta_i - \eta_j$ will be the roots, the simple roots are $\eta_i - \eta_{i+1}$ and the fundamental dominant weights are the semi simple components $(\sum_{i=1}^{i} \eta_i)^{(1)}$.

3.1.9 Formulas for the Hecke operators

We consider the homomorphism $r : K_p = \mathrm{Gl}_n(\mathbb{Z}_p) \to \mathrm{Gl}_n(\mathbb{F}_p)$ then we check easily that the intersection $K_p \cap \chi_i(p) K_p \chi_i(p)^{-1} = K_p^{(\chi_i(p))}$ is the inverse image of the parabolic subgroup $P_i^-(\mathbb{F}_p)$ under r.

We want to evaluate the integral

$$\int_{K_p \chi_i(p) K_p} \phi_{\omega_p}(x) dx$$

We write choose representatives ξ for the cosets of $K_p/K_p^{(\chi_i(p))}$ and write $K_p = \cup_\xi \xi K_p^{(\chi_i(p))}$. We observe that ϕ_{ω_p} is constant on the cosets $\xi K_p^{(\chi_i(p))}$. Hence we see that

$$\int_{K_p \chi_i(p) K_p} \phi_{\omega_p}(x) dx = \sum_\xi \phi_{\omega_p}(\xi \chi_i(p))) \tag{87}$$

The Bruhat decomposition gives us a nice system of representatives for $K_p/K_p^{(\chi_i(p))} = \mathrm{Gl}_n(\mathbb{F}_p)/P_i^-(\mathbb{F}_p)$. Let W_{M_i} be the Weyl group of the standard Levi subgroup $M_i = P_i \cap P_i^-$ and we choose a system of representatives W^{P_i} for W/W_{M_i} Then we get a disjoint decomposition

$$\mathrm{Gl}_n(\mathbb{F}_p) = \bigcup_{w \in W^{P_i}} U_B(\mathbb{F}_p) w P_i^-(\mathbb{F}_p),$$

here U_B is the unipotent radical of the standard Borel subgroup. The function ϕ_{ω_p} is constant on the double cosets. If we write a representative in the form $\xi = uw$ then the factor w is determined by ξ but the factor u is not. This factor is only unique up to multiplication from the right by a factor $u \in U_B^{(w,-)}(\mathbb{F}_p) = U_B(\mathbb{F}_p) \cap w P_i^- w^{-1}(\mathbb{F}_p)$. Hence we may choose our u in the subgroup

$$U_B^{(w,+)}(\mathbb{F}_p) = \prod_{\alpha \in \Delta^+ | <\chi_i, w^{-1}\alpha> > 0} U_\alpha(\mathbb{F}_p) \tag{88}$$

and our sum in (87) becomes

$$\sum_{w \in W^{P_i}} \sum_{u \in U_B^{(w,+)}(\mathbb{F}_p)} \phi_{\omega_p}(uw\chi_i(p))) = \sum_{w \in W^{P_i}} p^{l(w)} \phi_{\omega_p}(w \chi_i(p) w^{-1})) \tag{89}$$

where $l(w)$ is the cardinality of the set $\{\alpha \in \Delta^+ | < \chi_i, w^{-1}\alpha >> 0\}$. We recall the definition of the spherical function and get for our integral

$$\sum_{w \in W/W_{M_i}} p^{l(w)} \omega_p(w\chi_i(p)w^{-1})) |\rho|_p(w\chi_i(p)w^{-1})) = \sum_{w \in W/W_{M_i}} p^{l(w) - <\chi_i, w^{-1}\rho>} \omega_p((w\chi_i)(p)) \tag{90}$$

Now one checks easily that $p^{l(w) - <\chi_i, w^{-1}\rho>} = p^{<\chi_i, \rho>}$ and hence we get the desired formula

$$\int_{K_p \chi_i(p) K_p} \phi_{\omega_p}(x) dx = p^{<\chi_i, \rho>} \sum_{w \in W/W_{M_i}} \omega_p((w\chi_i)(p)) \tag{91}$$

This is the formula (57) for the group Gl_n and the special choice of the cocharacters $\chi = \chi_i$. The only cocharacter $\chi' < \chi_i$ is the trivial cocharacter, in our situation its contribution to (57) is zero.

Let us have a brief look at an arbitrary reductive (split or may be only quasisplit) group $G/\mathbb{Q}$, let us assume that the center is a connected torus $C/\mathbb{Q}$. We choose a maximal torus $T/\mathbb{Q}$ which is contained in a Borel subgroup $B/\mathbb{Q}$. We have the homomorphism to the adjoint group $G \to G_{\mathrm{ad}}$ it maps T to $T_{\mathrm{ad}} = T/C$. Again we may also define the fundamental cocharacters $\chi_i : \mathbb{G}_m \to T$ which satisfy $< \chi_i, \alpha_j >= \delta_{i,j}$. They are only well defined modulo cocharacters $\chi : \mathbb{G}_m \to C$ but this does not matter so much. Our above method to compute the eigenvalue of $\mathbf{ch}(\chi_i)$ still works if the cocharacter χ_i is "minuscule" which means that $< \chi_i, \alpha_j > \in \{-1, 0, 1\}$. In this case the formula (91) is still valid, again there is no contribution from the trivial character.

We return to $G = \mathrm{Gl}_n$ and to our speculations about motives. We choose a weight module $\mathcal{M}_\lambda$ where $\lambda = \sum_i a_i \gamma_i + d\delta$, where the γ_i are the fundamental weights and δ is the determinant. The a_i are integers and we have the consistency condition $\sum i a_i \equiv nd \mod n$. Let us pick an isotypical submodule $H^\bullet(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda \otimes F)(\pi_f)$. In section 2.3 we define the Hecke operators

$$T^{\mathrm{coh},\lambda}_\chi : H^\bullet_?(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda) \to H^\bullet_?(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda)$$

and these endomorphisms induce endomorphisms

$$T^{\mathrm{coh},\lambda}_\chi : H^\bullet_{?,\,\mathrm{int}}(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda \otimes F)(\pi_f) \to H^\bullet_{?,\,\mathrm{int}}(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda \otimes F)(\pi_f)$$

Let $\pi_f = \otimes \pi_p$ be an irreducible Hecke module and at an unramified place p let ω_p be the Satake parameter. Our Satake parameter is determined by the n-tuple of numbers

$$\omega_p(\eta_i(p)) = \omega_{i,p} \text{ for } i = 1, \dots, n$$

The cocharacter $\chi_n : \mathbb{G}_m \to T$ identifies $\mathbb{G}_m$ with the center of Gl_n. Our Hecke-module π_f has a central character and this provides a Hecke character

$$\pi_f \circ \chi_n : \mathbb{G}_m(\mathbb{A}_f) = I_{\mathbb{Q},f} \to F^\times$$

The restriction of $\mathcal{M}_\lambda$ to $\mathbb{G}_m$ is the character $\omega_\lambda : t \mapsto t^{nd}$ and the type of $\pi_f \circ \chi_n$ is of course ω_λ.

Our cocharacters χ_i define representations of the dual group which is again Gl_n and in fact χ_1 yields the tautological representation $r_1 : \mathrm{Gl}_n \xrightarrow{\sim} \mathrm{Gl}(V)$. Then χ_i yields the representation $r_i = \Lambda^i(r_1) : \mathrm{Gl}_n \to \mathrm{Gl}(\Lambda^i(V))$. For any subset $I \subset \{1, 2, \dots, n\}$ we define

$$\omega_{I,p} = \prod_{i \in I} \omega_{i,p}$$

and then our formula (91) in combination with the formula (27) in section 2.3 and the observation that $< \chi_i, \delta >= i$ yields

$$T^{\mathrm{coh},\lambda}_{\chi_i}(\pi_p) = p^{<\chi_i, \lambda^{(1)} + \rho > - id} \sum_{I : \# I = i} \omega_{I,p} \tag{92}$$

and by the same token we get for the cohomological L-function

$$L^{\mathrm{coh}}(\pi_f, r_\nu, s) = \prod_{p\in S} L_p^{\mathrm{coh}}(\pi_f, r_i, s) \prod_{p\notin S} \Big(\prod_{I:\#I=i} \frac{1}{(1 - p^{<\chi_i,\lambda^{(1)}+\rho>-id}\omega_{I,p}p^{-s})} \Big) \tag{93}$$

Here we see in a very transparent way the independence of the twist: If we modify λ to $\lambda + r\delta$ then we have to modify π_f to $\pi_f \otimes |\delta_f|^{-r}$. This means that the $\omega_{I,p}$ get multiplied by p^{ir} and the modifications cancel out.

We assume that $\pi_f \in \mathrm{Coh}(H_!^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda))$, then we will see in section 4.2 that π_f is essentially unitary. The central character of $\mathcal{M}_\lambda$ is $x \mapsto x^{nd}$ and hence we get that $\pi_f^* = \pi_f \otimes |\delta_f|^d$ is unitary. Then the Satake parameter of π_f^* is given by

$$\omega^*_{i,p} = \omega_{i,p}p^{-d} \text{ for } i = 1, \dots, n \tag{94}$$

where the factor $p^{-d} = |p|_p^d$ and we observe that these numbers are also invariant under twists by a power of $|\delta_f|$.

Since the operators $T^{\mathrm{coh},\lambda}_{\chi_i}$ operate on the integral cohomology it follows that the numbers $T^{\mathrm{coh},\lambda}_{\chi_i}(\pi_f)$ are algebraic integers. We easily check that for all $i \le n$

$$i(< \chi_1, \lambda^{(1)} + \rho > -d) \ge < \chi_i, \lambda^{(1)} + \rho > -id$$

and this implies that the numbers

$$\sum_{I:\#I=i} \prod_{\nu\in I} p^{<\chi_1,\lambda^{(1)}+\rho>-d}\omega_{\nu,p}$$

are algebraic integers and hence we can conclude

The numbers

$$\tilde{\omega}_{i,p} = p^{<\chi_1,\lambda^{(1)}+\rho>-d}\omega_{i,p} = p^{<\chi_1,\lambda^{(1)}+\rho>}\omega^*_{i,p} \tag{95}$$

are algebraic integers

Observe that these numbers are invariant under twists by a power of $|\delta_f|$.

We want t make few remarks about the relationship between the automorphic and the cohomological L-functions, especially we comment the shift in the variable s.

For the automorphic L -function we assume that we are over $\mathbb{C}$, we have chosen an embedding $\iota : F \hookrightarrow \mathbb{C}$. If our isotypical Hecke module π_f is cuspidal (see Thm. 4.2) then the considerations around this theorem show that π_f is essentially unitary. The center $C = \mathbb{G}_m$, the quotient $C' = \mathbb{G}_m$ and the isogeny $d_C : x \mapsto x^n$.

We come back to the Langlands philosophy. It predicts that for our a "cuspidal" π_f and the cocharacter χ_1 we should be able to attach a motive $\mathbb{M}(\pi_f, r_1) = \mathbb{M}(\pi_f, \chi_1)$ with coefficients in F. This motive provides a compatible system of $\mathfrak{l}$- adic Galois representations

$$\rho_{\mathfrak{l}}(\pi_f, \chi_1) : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gl}_n(F_{\mathfrak{l}}) = \mathrm{Gl}(\mathbb{M}(\pi_f, \chi_1)_{\text{ét},\mathfrak{l}}) \tag{96}$$

which are unramified outside $\{l\} \cup S$ and for $p \notin S \cup \{l\}$ we should have

$$\det(\mathrm{Id} - \rho_{\mathfrak{l}}(\pi_f, \chi_1)(\Phi_p^{-1})p^{-s}) = \prod_i (1 - p^{<\chi_1, \lambda^{(1)}+\rho>-d}\omega_{i,p}p^{-s}) \tag{97}$$

and this means that up to the local factors at the bad primes we should have

$$L^{\mathrm{mot}}(\mathbb{M}(\pi_f, \chi_1), s) = L^{\mathrm{coh}}(\pi_f, \chi_1, s) \tag{98}$$

The existence of the compatible system of Galois representation has been shown by Harris - Kai-Wen Lan -Taylor and Thorne and by P. Scholze.

Once we have the motive for the cocharacter χ_1 we easily get it the other χ_i we simply have to look at the exterior powers $\Lambda^i(\mathbb{M}(\pi_f, \chi_1))$.

Now we see that that numbers $\tilde{\omega}_{\nu,p}$ can be interpreted as the eigenvalues of the Frobenius on $\mathbb{M}_{\text{ét},\mathfrak{l}}(\pi_f, \chi_1)$. Under the assumption that π_f is "cuspidal" we expect that the motive $\mathbb{M}(\pi_f, \chi_1)$ is pure of weight $\mathbf{w}(\chi_1, \lambda)$ we get

$$|\tilde{\omega}_{\nu,p}| = p^{\frac{\mathbf{w}(\chi_1,\lambda)}{2}}$$

and this is the Ramanujan conjecture. We will explain in the section on analytic aspects, that for cuspidal π_f the Ramanujan conjecture says that for any embedding $\iota : F \hookrightarrow \mathbb{C}$ we have

$$|\iota \circ \omega^*_{\nu,p}| = 1$$

This suggests that we call the array $\tilde{\omega}_p = \{\tilde{\omega}_{1,p}, \ldots, \tilde{\omega}_{n,p}\}$ the *motivic* Satake parameter (with respect to the tautological representation r_1 .) Of course it can always be defined, independently of the existence of the motive.

We will see in the next section that the inner cohomology is trivial unless our highest weight is essentially self dual, this means that $\lambda^{(1)} = -w_0(\lambda^{(1)})$. Let us assume that this is the case. If $r_1^\vee$ is the dual of the tautological representation then the eigenvalues of $r_1^\vee(\omega_p)$ are by

$$r_1^\vee(\omega_p) = \{\omega_{1,p}^{-1}, \ldots, \omega_{n,p}^{-1}\}.$$

The highest weight of $r_1^\vee$ is the cocharacter $-\eta_n = \sum_{i=1}^{n-1} \eta_i - \det$ (This has to be read in $X^*(T^\vee)$) Then

$$c(-\eta_n, \lambda) =< \chi_1, -w_0(\lambda^{(1)}) > +d$$

and under our assumption that λ is essentially self dual we know

$$< \chi_1, -w_0(\lambda^{(1)}) >=< \chi_1, \lambda^{(1)} >= \frac{\mathbf{w}(\chi_1, \lambda)}{2}.$$

This implies that the motivic Satake parameters with respect to the dual representation $r_1^\vee$ are the numbers

$$\{p^{<\chi_1,\lambda^{(1)}>+d\delta}\omega_{1,p}^{-1}, \ldots, p^{<\chi_1,\lambda^{(1)}>+d\delta}\omega_{n,p}^{-1}\} \tag{99}$$

In the following section on Poincaré duality we will see that for any isotypical module $H^i_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,F})(\pi_f)$ the dual module $\pi_f^\vee$ appears in $H^{d-i}_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda^\vee,F})$. Then we get an equality of local Euler factors

$$L^{\mathrm{coh}}(\pi_p, r_1^\vee, s) = L^{\mathrm{coh}}(\pi_p^\vee, r_1, s) \tag{100}$$

The concept of motives allows us to define the the dual motive. If our motive has weight $\mathbf{w}(M)$ then Poincaré duality suggests that we define the motive

$$\mathbb{M}^\vee = \mathrm{Hom}(\mathbb{M}, \mathbb{Z}(-\mathbf{w}(M)) \tag{101}$$

The $\mathfrak{l}$ adic realization as Galoismodule gives us

$$\mathbb{M}^\vee_{\text{ét},\mathfrak{l}} = \mathrm{Hom}(\mathbb{M}_{\text{ét},\mathfrak{l}}, \mathbb{Z}_{\mathfrak{l}}(-\mathbf{w}(M))$$

If $\{\alpha_1, \dots, \alpha_m\}$ are the eigenvalues of Φ_p^{-1} on $\mathbb{M}_{\text{ét},\mathfrak{l}}$ then $\{\alpha_1^{-1} p^{\mathbf{w}(M)}, \dots, \alpha_m^{-1} p^{\mathbf{w}(M)}\}$ are the eigenvalues of Φ_p^{-1} on $\mathbb{M}^\vee_{\text{ét},\mathfrak{l}}$.

Therefore we can say: If we find a motive $\mathbb{M}(\pi_f, \chi_1)$ for π_f the we also find the motive for $\pi_f^\vee$ and we have

$$\mathbb{M}(\pi_f^\vee, \chi_1) = \mathbb{M}(\pi_f, \chi_1)^\vee$$

4 Analytic methods

4.1 The representation theoretic de-Rham complex

4.1.1 Rational representations

We start from a reductive group $G/\mathbb{Q}$ for simplicity we assume that the semi simple component $G^{(1)}/\mathbb{Q}$ is quasisplit. There is a unique finite normal extension $F/\mathbb{Q}, F \subset \mathbb{C}$ such that $G^{(1)} \times_{\mathbb{Q}} F$ becomes split, if $T^{(1)}/\mathbb{Q}$ is a maximal torus which is contained in a Borel subgroup $B/\mathbb{Q}$ then the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on $X^*(T^{(1)} \times_{\mathbb{Q}} F)$ and by permutations on the set of positive roots $\pi_G \subset X^*(T^{(1)} \times_{\mathbb{Q}} F)$ corresponding to $B/\mathbb{Q}$. This action factors over the quotient $\mathrm{Gal}(F/\mathbb{Q})$. Then it also acts on the set of highest weights. Since our group is quasi split we find for any highest weight an absolutely irreducible $G \times_{\mathbb{Q}} F$-module $\mathcal{M}_\lambda$.

$$r : G \times_{\mathbb{Q}} K \to \mathrm{Gl}(\mathcal{M}_\lambda)$$

whose highest weight is λ. Since we assumed that $\mathbb{Q} \subset F \subset \bar{\mathbb{Q}} \subset \mathbb{C}$ we get the extension

$$r_{\mathbb{C}} : (G \times_{\mathbb{Q}} K) \times_K \mathbb{C} \to \mathrm{Gl}(\mathcal{M}_\lambda \otimes_F \mathbb{C}).$$

Given such an absolutely irreducible rational representation, we can construct two new representations. At first we can form the dual $\mathcal{M}^\vee_{\lambda,\mathbb{C}} = \mathrm{Hom}_{\mathbb{C}}(\mathcal{M}_\lambda, \mathbb{C})$ and the complex conjugate $\overline{\mathcal{M}}_{\mathbb{C}}$ of our module $\mathcal{M}_\lambda$. On the dual module we have the contragredient representation $r^\vee$, which is defined by $\phi(r_{\mathbb{C}}(g)(v)) = r^\vee_{\mathbb{C}}(g^{-1})(\phi)(v)$.

To get the rational representation on the conjugate module $\bar{\mathcal{M}} \otimes_F \mathbb{C}$, we recall its definition: As abelian groups we have $\mathcal{M} \otimes_F \mathbb{C} = \bar{\mathcal{M}} \otimes_F \mathbb{C}$ but the action of the scalars is conjugated, we write this as $z \cdot_c m = \bar{z} m$. Then the identity gives us an identification

$$\mathrm{End}_{\mathbb{C}}(\mathcal{M} \otimes_F \mathbb{C}) = \mathrm{End}_{\mathbb{C}}(\bar{\mathcal{M}}_\lambda \otimes_F \mathbb{C}).$$

Now we define an action $\bar{r}_{\mathbb{C}}$ on $\bar{\mathcal{M}}_\lambda \otimes_F \mathbb{C}$: For $g \in G(\mathbb{C})$ we put

$$\bar{r}_{\mathbb{C}}(g)m = r_{\mathbb{C}}(g) \cdot_c m.$$

This defines an action of the abstract group $G(\mathbb{C})$, but this is in fact obtained from a rational representation. Therefore $\mathcal{M}_{\mathbb{C}}^\vee$ and $\overline{\mathcal{M}}_C$ both are given by a highest weight.

The highest weight of $\mathcal{M}_\lambda^\vee$ is $-w_0(\lambda)$. Herew_0 is the unique element $w_0 \in W$, which sends the system of positive roots Δ^+ into the system $\Delta^- = -\Delta^+$.

The highest weight of $\bar{\mathcal{M}}_\lambda \otimes_F \mathbb{C}$ is $c(\lambda)$ where $c \in \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \subset \mathrm{Gal}(F/\mathbb{Q})$ is the complex conjugation acting on $X^*(T \times_{\mathbb{Q}} F)$. So we may say: $\overline{\mathcal{M}_{\lambda C}} = \mathcal{M}_{\bar{\lambda}}$.

We will call the module $\mathcal{M}_\lambda$- *conjugate-autodual* or simply *c-autodual* if

$$c(\lambda) = -w_0(\lambda) \tag{102}$$

In the following few sections (until 4.3.5 we will always assume that our local system (resp. the corresponding representation) are local systems in $\mathbb{C}$-vector spaces (resp. $\mathbb{C}$-vector spaces $\tilde{\mathcal{M}}_\lambda$). Therefore we will suppress the factor $\otimes\mathbb{C}$.

4.1.2 Harish-Chandra modules and $(\mathfrak{g}, K_\infty)$-cohomology.

Now we consider the group of real points $G(\mathbb{R})$, it has the Lie algebra $\mathfrak{g}$, inside this Lie algebra we have the Lie algebra $\mathfrak{k}$ of the group K_∞. We have the notion of a $(\mathfrak{g}, K_\infty)$ module: This is a $\mathbb{C}$-vector space V together with an action of $\mathfrak{g}$ and an action of the group K_∞. We have certain assumptions of consistency:

i) The action of K_∞ is differentiable, this means it induces an action of $\mathfrak{k}$, the derivative of the group action.

ii) The action of $\mathfrak{g}$ restricted to $\mathfrak{k}$ is the derivative of the action of K_∞.

iii) For $k \in K_\infty, X \in \mathfrak{g}$ and $v \in V$ we have

$$(\mathrm{Ad}(k)X)v = k(X(k^{-1}v)).$$

Inside V we have have the subspace of K_∞ finite vectors, a vector v is called K_∞ finite if the $\mathbb{C}$- subspace generated by all translates kv is finite dimensional, i.e. v lies in a finite dimensional K_∞ invariant subspace. The K_∞ finite vectors form a subspace $V^{(K_\infty)}$ and it is obvious that $V^{(K_\infty)}$ is invariant under the action of $\mathfrak{g}$, hence it is a $(\mathfrak{g}, K_\infty)$ sub module of V. We call a $(\mathfrak{g}, K_\infty)$ module a Harish-Chandra module if $V = V^{(K_\infty)}$.

For such a $(\mathfrak{g}, K_\infty)$-module we can write down a complex

$$\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), V) = \{0 \to V \to \mathrm{Hom}_{K_\infty}(\Lambda^1(\mathfrak{g}/\mathfrak{k}), V) \to \mathrm{Hom}_{K_\infty}(\Lambda^2(\mathfrak{g}/\mathfrak{k}), V) \to \dots\}$$

where the differential is given by

$$d\omega(X_0, X_1, \dots, X_p) = \sum_{i=0}^{p}(-1)^i X_i \omega(X_0, \dots, \hat{X}_i, \dots, X_p) +$$

$$\sum_{0\le i<j\le p}(-1)^{i+j}\omega([X_i,X_j],X_0,\dots,\hat{X}_i,,\dots,\hat{X}_j,\dots,X_p).$$

A few comments are in order. We have inclusions

$$\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}),V)\subset \mathrm{Hom}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}),V)\subset \mathrm{Hom}(\Lambda^\bullet(\mathfrak{g}),V).$$

The above differential defines the structure of a complex for the rightmost term, we have to verify that the leftmost term is a subcomplex, this is not so difficult.

We define the $(\mathfrak{g},K_\infty)$ cohomology as the cohomology of this complex, i.e.

$$H^\bullet(\mathfrak{g},K_\infty,V)=H^\bullet(\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}),V)).$$

It is clear that the map

$$H^\bullet(\mathfrak{g},K_\infty,V^{(K_\infty)})\to H^\bullet(\mathfrak{g},K_\infty,V)$$

is an isomorphism.

If we have two $(\mathfrak{g},K_\infty)$ modules V_1,V_2 and form the algebraic tensor product $W=V_1\otimes V_2$ the we have a natural structure of a $(\mathfrak{g},K_\infty)$ -module on W : The group K_∞ acts via the diagonal and $U\in\mathfrak{g}$ acts by the Leibniz-rule $U(v_1\otimes v_2)=Uv_1\otimes v_2+v_1\otimes Uv_2$. If both modules are Harish-Chandra modules, then the tensor product is also a Harish-Chandra module.

Of course any finite dimensional rational representation of the algebraic group also yields a Harish-Chandra module.

For us the $(\mathfrak{g},K_\infty)$ module $\mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f)$,- this is the space of functions which are $\mathcal{C}_\infty$ in the variable g_∞- is one of the most important $(\mathfrak{g},K_\infty)$ -modules. We may also consider the limit over smaller and smaller levels K_f we get the space $\mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A}))$, which consists of those functions on $G(\mathbb{A})$, which are left invariant under $G(\mathbb{Q})$, right invariant under a suitably small open subgroup $K_f\subset G(\mathbb{A}_f)$ and which are $\mathcal{C}_\infty$ in the variable g_∞. On these functions the group $G(\mathbb{A})$ acts by translations from the right, since our functions are $\mathcal{C}_\infty$ we also get an action of the Lie algebra $\mathfrak{g}$. Hence this is also a $(\mathfrak{g},K_\infty)\times G(\mathbb{A}_f)$-module.

If we fix the level see that $\mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f))$ is a $(\mathfrak{g},K_\infty)\times\mathcal{H}_{K_f}$, the Hecke algebra acts by convolution. We choose a highest weight module $\mathcal{M}_\lambda$ and apply the previous considerations to the Harish-Chandra module

$$V=\mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f)\otimes\mathcal{M}_\lambda.$$

Notice that we can evaluate an element $f\in\mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f)\otimes\mathcal{M}_\lambda$ in a point $\underline{g}=(g_\infty,\underline{g}_f)$ and the result $f(\underline{g})\in\mathcal{M}_\lambda$. The Hecke algebra acts via convolution on the first factor.

Let us assume that our compact subgroup $K_f\subset G(\mathbb{A}_f)$ is neat, i.e. for any $\underline{g}=(g_\infty,\underline{g}_f)\in G(\mathbb{A})$ we have $\underline{g}^{-1}(K_\infty\times K_f)\underline{g}\cap G(\mathbb{Q})=\{e\}$. In this case we know that $\tilde{\mathcal{M}}$ is a local system and we can form the de-Rham complex $\Omega^\bullet(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}_\lambda})$.

We have an action of the Hecke algebra on this complex and we have the following fundamental fact:

Proposition 4.1. *We have a canonical isomorphism of complexes*

$$\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f) \otimes \mathcal{M}_\lambda) \xrightarrow{\sim} \Omega^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda),$$

this isomorphism is compatible with the action of the Hecke algebra on both sides

This is rather clear. We have the projection map

$$q : G(\mathbb{R}) \times G(\mathbb{A}_f) \to G(\mathbb{R})/K_\infty \times G(\mathbb{A}_f)/K_f = X \times G(\mathbb{A}_f)/K_f$$

let $x_0 \in X \times G(\mathbb{A}_f)/K_f$ be the image of the identity $e \in G(\mathbb{R})$. The differential $D_q(e)$ maps the Lie algebra $\mathfrak{g} =$ tangent space of $G(\mathbb{R})$ at e to the tangent space T_{X,x_0} at $x_0 \times e_f$. This provides the identification $T_{X,x_0} \xrightarrow{\sim} \mathfrak{g}/\mathfrak{k}$.

An element $\omega \in \mathrm{Hom}_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}), \mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f) \otimes \mathcal{M}_\lambda)$ can be evaluated on a p-tuple $(X_0, X_1, \dots, X_{p-1})$ and the result

$$\omega(X_0, X_1, \dots, X_{p-1}) \in \mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f) \otimes \mathcal{M}_\lambda.$$

We want to produce an element $\tilde{\omega}$ in the de-Rham complex $\Omega^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$. Pick a point $x \times \underline{g}_f \in X \times G(\mathbb{A}_f)/K_f$, we find an element $(g_\infty, \underline{g}_f) \in G(\mathbb{R}) \times G(\mathbb{A}_f)$ such that $g_\infty x_0 = x$. Our still to be defined form $\tilde{\omega}$ can be evaluated at a p-tuple $(Y_0, \dots, Y_{p-1})$ of tangent vectors in $x \times \underline{g}_f$ and the result has to be an element in $\tilde{\mathcal{M}}_{\mathbb{C},x}$. We find a p-tuple $(X_0, X_1, \dots, X_{p-1})$ of tangent vectors at x_0 which are mapped to $(Y_0, \dots, Y_{p-1})$ under the differential D_g of the left translation by g. We put

$$\tilde{\omega}(Y_0, \dots, Y_{p-1})(x \times \underline{g}_f) = g_\infty^{-1}\omega(X_0, \dots, X_{p-1})(g_\infty, \underline{g}_f).$$

At this point I leave it as an exercise to the reader that this gives the isomorphism we want. We recall that the de-Rham complex (Reference Book Vol. !) computes the cohomology and therefore we can rewrite the de-Rham isomorphism $\boxed{\text{BodeRh}}$

$$H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) \xrightarrow{\sim} H^\bullet(\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f) \otimes \mathcal{M}_\lambda) \quad (103)$$

From now on the complex $\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f) \otimes \mathcal{M}_\lambda)$ will also be called the de-Rham complex.

By the same token we can compute the cohomology with compact supports $\boxed{\text{BodeRhcs}}$

$$H_c^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) \xrightarrow{\sim} H^\bullet(\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathcal{C}_{c,\infty}(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f) \otimes \mathcal{M}_\lambda) \quad (104)$$

where $\mathcal{C}_{c,\infty}(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f)$ are the $\mathcal{C}_\infty$ function with compact support. These isomorphisms are also valid if we drop the assumption that K_f is neat.

The Poincaré duality on the cohomology is induced by the pairing on the de-Rham complexes:

Proposition 4.2. *If* $\omega_1 \in \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f) \otimes \tilde{\mathcal{M}})$ *is a closed form and* $\omega_2 \in \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathcal{C}_{\infty,c}(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f) \otimes \tilde{\mathcal{M}}^\vee)$ *a closed form with compact support in complementary degree then the value of the cup* *product pairing of the classes* $[\omega_1] \in H^p(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda), [\omega_2] \in H^{d-p}_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}_\lambda}^\vee)$ *is given by*

$$< [\omega_1] \cup [\omega_2] >= \int_{\mathcal{S}^G_{K_f}} < \omega_1 \wedge \omega_2 >$$

(Reference Book Vol. !)

4.1.3 Input from representation theory of real reductive groups.

Let us consider an arbitrary irreducible $(\mathfrak{g}, K_\infty)$- module V. We also assume that for any $\vartheta \in \hat{K_\infty}$ the multiplicity of ϑ in V is finite (we say that V is admissible). Then we can extend the action of the Lie-algebra $\mathfrak{g}$ to an action of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ on V and we can restrict this action to an action of the centre $\mathfrak{Z}(\mathfrak{g})$. The structure of this centre is well known by a theorem of Harish-Chandra, it is a polynomial algebra in $r = \mathrm{rank}(G)$ variables, here the rank is the absolute rank, i.e. the dimension of a maximal torus in $G/\mathbb{Q}$. (See Chap. 4 sect. 4)

Clearly this centre respects the decomposition into K_∞ types, since these K_∞ types come with finite multiplicity we can apply the standard argument, which proves the Lemma of Schur. Hence $\mathfrak{Z}(\mathfrak{g})$ has to act on V by scalars, we get a homomorphism $\chi_V : \mathfrak{Z}(\mathfrak{g}) \to \mathbb{C}$, which is defined by

$$zv = \chi_V(z)v.$$

This homomorphism is called the central character of V.

A fundamental theorem of Harish-Chandra asserts that for a given central character there exist only finitely many isomorphism classes of irreducible, admissible $(\mathfrak{g}, K_\infty)$-modules with this central character.

Of course for any rational finite dimensional representation $r : G/\mathbb{Q} \to \mathrm{Gl}(\mathcal{M}_\lambda)$ we can consider $\mathcal{M}_\lambda \otimes \mathbb{C}$ as $(\mathfrak{g}, K_\infty)$-module. If $\mathcal{M}_\lambda$ is absolutely irreducible with highest weight λ (See chap. IV) then it also has a central character $\chi_{\mathcal{M}} = \chi_\lambda$.

Wigner's lemma: *Let V be an irreducible, admissible $(\mathfrak{g}, K_\infty)$-module, let $\mathcal{M} = \mathcal{M}_\lambda$, a finite dimensional, absolutely irreducible rational representation. Then $H^\bullet(\mathfrak{g}, K_\infty, V \otimes \mathcal{M}_\mathbb{C}) = 0$ unless we have*

$$\chi_V(z) = \chi_{\mathcal{M}^\vee}(z) = \chi_{\mathcal{M}_\lambda{}^\vee}(z) \textit{ for all } z \in \mathfrak{Z}(\mathfrak{g})$$

Since we also know that the number of isomorphism classes of irreducible, admissible $(\mathfrak{g}, K_\infty)$-modules with a given central character is finite, we can conclude that for a given absolutely irreducible rational module $\mathcal{M}_\lambda$ the number of isomorphism classes of irreducible, admissible $(\mathfrak{g}, K_\infty)$-modules V with $H^\bullet(\mathfrak{g}, K_\infty, V \otimes \mathcal{M}_\mathbb{C}) \neq 0$ is finite.

The proof of Wigner's lemma is very elegant. We have $\mathcal{M} \otimes V = \mathcal{M}^\vee \otimes V$ and hence we have $H^0(\mathfrak{g}, K_\infty, \mathcal{M} \otimes V) = \mathrm{Hom}(\mathcal{M}^\vee, V)^{(\mathfrak{g}, K_\infty)} = \mathrm{Hom}_{\mathfrak{g}, K_\infty}(\mathcal{M}^\vee, V)$. In [B-W] , Chap.I 2.4 it is shown, that the category of $\mathfrak{g}, K_\infty$ -modules has enough injective and projective elements (See [B-W], I. 2.5) . If I is an injective

$\mathfrak{g}, K_\infty$-module then $\mathcal{M} \otimes I$ is also injective because for any $\mathfrak{g}, K_\infty$-module A we have $\mathrm{Hom}(A, \mathcal{M} \otimes I) = \mathrm{Hom}(\mathcal{M}^\vee, I)$. Hence an injective resolution $0 \to V \to I^0 \to I^1 \dots$ yields an injective resolution $0 \to \mathcal{M} \to \mathcal{M} \otimes I^0 \to \mathcal{M} \otimes I^1 \dots$ and from this we get

$$H^q(\mathfrak{g}, K_\infty, \mathcal{M} \otimes V) = \mathrm{Ext}^q_{\mathfrak{g}, K_\infty}(\mathcal{M}^\vee, V).$$

Any $z \in \mathfrak{Z}(\mathfrak{g})$ induces an endomorphism of $\mathcal{M}_\lambda$ and V. Since $\mathrm{Ext}^\bullet$ is functorial in both variables, we see that z induces endomorphisms z_1 (via the action on $\mathcal{M}_\lambda$) and z_2 (via the action on V) on $\mathrm{Ext}^q_{\mathfrak{g}, K_\infty}(\mathcal{M}^\vee, V)$. We show that $z_1 = z_2$. This is clear by definition for $\mathrm{Ext}^0_{\mathfrak{g}, K_\infty}(\mathcal{M}^\vee, V) = \mathrm{Hom}_{\mathfrak{g}, K_\infty}(\mathcal{M}^\vee, V)$: For $z \in \mathfrak{Z}(\mathfrak{g})$ and $\phi \in \mathrm{Hom}_{\mathfrak{g}, K_\infty}(\mathcal{M}^\vee, V), m \in \mathcal{M}_\lambda$ we have $z_1\phi(m) = \phi(zm) = z_2(\phi(m))$. To prove it for an arbitrary q we use devissage and induction. We embed V into an injective $\mathfrak{g}, K_\infty$ module I and get an exact sequence

$$0 \to V \to I \to I/V \to 0$$

and from this and $\mathrm{Ext}^q_{\mathfrak{g}, K_\infty}(\mathcal{M}_\lambda, I)$ for $q > 0$ we get

$$\mathrm{Ext}^{q-1}(\mathfrak{g}, K_\infty, \mathcal{M}_\lambda, I/V) = \mathrm{Ext}^q(\mathfrak{g}, K_\infty, \mathcal{M}_\lambda, V) \text{ for } q > 0.$$

Now by induction we know $z_1 = z_2$ on the left hand side, so it also holds on the right hand side.

If now $\chi_V \neq \chi_{\mathcal{M}^\vee}$ then we can find a $z \in \mathfrak{Z}(\mathfrak{g})$ such that $\chi_{\mathcal{M}^\vee}(z) = 0, \chi_V(z) = 1$. This implies that $z_1 = 0$ and $z_2 = 1$ on all $\mathrm{Ext}^q(\mathfrak{g}, K_\infty(\mathcal{M}_\lambda, V)$. Since we know that $z_1 = z_2$ we see that the identity on $\mathrm{Ext}^q(\mathfrak{g}, K_\infty(\mathcal{M}_\lambda, V)$ is equal to zero and this implies the assertion.

On the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ we have an antiautomorphism $u \mapsto^t u$ which is induced by the antiautomorphism $X \mapsto -X$ on the Lie algebra $\mathfrak{g}$. If V is an admissible $(\mathfrak{g}, K_\infty)$-module, then we can form the dual module $V^\vee$ and if we denote the pairing between $V, V^\vee$ by $< \, , \, >_V$ then

$$< Uv, \phi >_V = < v, ^t U\phi >_V \text{ for all } U \in \mathfrak{U}(\mathfrak{g}), v \in V, \phi \in V^\vee.$$

If V is irreducible, then it has a central character and we get

$$\chi_{V^\vee}(z) = \chi_V(^t z).$$

This applies to finite dimensional and infinite dimensional $(\mathfrak{g}, K_\infty)$-modules.

4.1.4 Representation theoretic Hodge-theory.

We consider irreducible unitary representations $G(\mathbb{R}) \to U(H)$. We know from the work of Harish-Chandra:

1) If we fix an isomorphism class ϑ irreducible representations of K_∞ then the isotypical subspace $\dim_{\mathbb{C}} H(\vartheta) \leq \dim(\vartheta)^2$, i.e. ϑ occurs at most with multiplicity $\dim(\vartheta)$.

2) The direct sum $\sum_{\vartheta \subset \hat{K_\infty}} H(\vartheta) = H^{(K)} \subset H$ is dense in H and it is an admissible irreducible Harish-Chandra -module.

We call an irreducible $(\mathfrak{g}, K_\infty)$-module unitary, if it is isomorphic to such an $H^{(K)}$.

For a given $G/\mathbb{R}$ and any rational irreducible module $\mathcal{M}_\lambda$ Vogan and Zuckerman give a finite list of certain irreducible, admissible $(\mathfrak{g}, K_\infty)-$ modules $A_\mathfrak{q}(\lambda)$, for which $H^\bullet(\mathfrak{g}, K_\infty, A_\mathfrak{q}(\lambda) \otimes \mathcal{M}_\lambda) \neq 0$ they compute these cohomology group. This list contains all unitary, irreducible $(\mathfrak{g}, K_\infty)-$modules, which have non trivial cohomology with coefficients in $\mathcal{M}_\lambda$.

For the following we refer to [B-W] Chap. II , §1-2 . We want to apply the methods of Hodge-theory to compute the cohomology groups $H^\bullet(\mathfrak{g}, K_\infty, V \otimes \mathcal{M}_\lambda)$ for an unitary $(\mathfrak{g}, K_\infty)$-module V. This means have a positive definite scalar product $< , >_V$ on V, for which the action of K_∞ is unitary and for $U \in \mathfrak{g}$ and $v_1, v_2 \in V$ we have $< Uv_1, v_2 >_V + < v_1, Uv_2 >_V = 0$.

In the next step we introduce for all p a hermitian form on $\mathrm{Hom}_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}), V \otimes \mathcal{M}_\lambda)$. To do this we construct a hermitian form on $\mathcal{M}_\lambda$.

(The following considerations are only true modulo the centre). We consider the Lie algebra and its complexification $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \otimes \mathbb{C}$. On this complex vector space we have the complex conjugation $^- : U \mapsto \bar{U}$. We rediscover $\mathfrak{g}$ as the set of fixed points under $^-$. We also have the Cartan involution Θ which is the involution which has $\mathfrak{k}$ as its fixed point set. Then we get the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \text{ where } \mathfrak{p} \text{ is the -1 eigenspace of } \Theta.$$

The Killing form is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$, we have for the Lie bracket $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. We consider the invariants under $^- \circ \Theta$, this is the Lie algebra $\mathfrak{g}_c = \mathfrak{k} \oplus \sqrt{-1} \otimes \mathfrak{p}$. On this real Lie algebra the Killing form is negative definite and $\mathfrak{g}_c$ is the Lie algebra of an algebraic group $G_c/\mathbb{R}$ whose base extension $G_c \otimes_\mathbb{R} \mathbb{C} \xrightarrow{\sim} G \otimes_\mathbb{R} \mathbb{C}$ and whose group $G_c(\mathbb{R})$ of real points is compact (this is the so called compact form of G). We still have the representation $G_c/\mathbb{R} \to \mathrm{Gl}(\mathcal{M}_\lambda)$ which is irreducible and hence we find a hermitian form $< , >_\lambda$ on $\mathcal{M}_\lambda$, which is invariant under $G_c(\mathbb{R})$ and which is unique up to a scalar.

This form satisfies the equations

$$< Um_1, m_2 >_\mathcal{M} + < m_1, Um_2 >_\lambda = 0 \text{ for all } m_1, m_2 \in \mathcal{M}_\lambda, U \in \mathfrak{k}$$

this is the invariance under K_∞ and

$$< Um_1, m_2 >_\mathcal{M} = < m_1, Um_2 >_\lambda \text{ for all } m_1, m_2 \in \mathcal{M}_\lambda, U \in \mathfrak{p}$$

this is the invariance under $\sqrt{-1} \otimes \mathfrak{p}$.

Now we define a hermitian metric on $V \otimes \mathcal{M}_\lambda$, we simply take the tensor product $< , >_V \otimes < , >_\lambda = < , >_{V \otimes \lambda}$. Finally we define the (hermitian) scalar product on $\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), V \otimes \mathcal{M}_\lambda)$. We choose and orthonormal (with respect to the Killing form) basis $E_1, E_2, \ldots, E_d$ on $\mathfrak{p}$, we identify $\mathfrak{g}/\mathfrak{k} \xrightarrow{\sim} \mathfrak{p}$. Then a form $\omega \in \mathrm{Hom}_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}), V \otimes \mathcal{M}_\lambda)$ is given by its values $\omega(E_I) \in V \otimes \mathcal{M}_\lambda$, where $I = \{i_1, i_2, \ldots, i_p\}$ runs through the ordered subsets of $\{1, 2, \ldots, d\}$ with p elements. For $\omega_1, \omega_2 \in \mathrm{Hom}_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}), V \otimes \mathcal{M}_\lambda)$ we put

$$< \omega_1, \omega_2 > = \sum_{I, |I|=p} < \omega_1(E_I), \omega_2(E_I) >_{V \otimes \lambda} \tag{105}$$

Now we can define an adjoint operator

$$\delta : \mathrm{Hom}_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}), V \otimes \mathcal{M}_\lambda) \to \mathrm{Hom}_{K_\infty}(\Lambda^{p-1}(\mathfrak{g}/\mathfrak{k}), V \otimes \mathcal{M}_\lambda), \qquad (106)$$

which can be defined by a straightforward calculation. We simply write a formula for δ: For an element E_i we define $E_i^*(v \otimes m) = -E_i v \otimes m + v \otimes E_i m$. Then we can define δ by the following formula:

We have to evaluate $\delta(\omega)$ on $E_J = (E_{i_1}, \dots, E_{i_{p-1}})$ where $J = \{i_1, \dots, i_{p-1}\}$. We put

$$\delta(\omega)(E_J) = \sum_{i \notin J} (-1)^{p(i, J \cup \{i\})} E_i^* \omega_{J \cup \{i\}},$$

where $p(i, J \cup \{i\})$ denotes the position of i in the ordered set $J \cup \{i\}$. With this definition we get for a pair of forms $\omega_1 \in \mathrm{Hom}_{K_\infty}(\Lambda^{p-1}(\mathfrak{g}/\mathfrak{k}), V \otimes \mathcal{M}_\lambda)$ and $\omega_2 \in \mathrm{Hom}_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}), V \otimes \mathcal{M}_\lambda)$ (See [B-W], II, prop. 2.3)

$$< d\omega_1, \omega_2 > = < \omega_1, \delta\omega_2 > \qquad (107)$$

We define the Laplacian $\Delta = \delta d + d\delta$. Then we have ([B-W] , II ,Thm.2.5)

$$< \Delta\omega, \omega > \geq 0 \text{ and we have equality if and only if } d\omega = 0, \delta\omega = 0 \qquad (108)$$

Inside $\mathfrak{Z}(\mathfrak{g})$ we have the the Casimir operator C (See Chap. 4). An element $z \in \mathfrak{Z}(\mathfrak{g})$ acts on $V \otimes \mathcal{M}_\lambda$ by $z \otimes \mathrm{Id}$ via the action on the first factor and by the scalar $\chi_\lambda(z)$ via the action on the second factor. Then we have

Kuga's lemma : *The action of the Casimir operator and the Laplace operator on* $\mathrm{Hom}_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}), V \otimes \mathcal{M}_\lambda)$ *are related by the identity*

$$\Delta = C \otimes \mathrm{Id} - \chi_\lambda(C).$$

If the $\mathfrak{g}, K_\infty$ *module is irreducible, then* Δ *acts by multiplication by the scalar* $\chi_V(C) - \chi_\lambda(C)$

This has the following consequence

If V *is an irreducible unitary* $\mathfrak{g}, K_\infty$*- module and if* $\mathcal{M}_\lambda$ *is an irreducible representation with highest weight* λ *then*

$$H^\bullet(\mathfrak{g}, K_\infty, V \otimes \mathcal{M}_\mathbb{C}) = \begin{cases} 0 & \text{if } \chi_V(C) - \chi_\lambda(C) \neq 0 \\ \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), V \otimes \mathcal{M}_\lambda) & \text{if } \chi_V(C) - \chi_\lambda(C) = 0 \end{cases}.$$

This only applies for unitary $\mathfrak{g}, K_\infty$-modules, but for these it is much stronger: It says that under the assumption $\chi_V(C) = \chi_\lambda(C)$ we have $\chi_V = \chi_\lambda$ (we only have to test the Casimir operator) and it says that all the differentials in the complex are zero.

4.2 Input from the theory of automorphic forms

We apply this to the spaces of square integrable functions on $G(\mathbb{Q})\backslash G(\mathbb{A})/K_f$. Because of the presence of a non trivial center, we have to consider functions which transform in a certain way under the action of the center. We may assume that coefficient system $\mathcal{M}_\lambda$ has a central character and this central character defines a character ζ_λ on the maximal $\mathbb{Q}$-split torus $S \subset C$. This character can be evaluated on the connected component of the identity of the real valued points and induces a (continuous) homomorphism $\zeta_\infty : S^0(\mathbb{R}) \to \mathbb{R}^\times_{>0}$. Then we define

$$\mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f, \zeta_\infty^{-1}) \tag{109}$$

to be the subspace of those $\mathcal{C}_\infty$ functions which satisfy $f(z_\infty \underline{g}) = \zeta_\infty^{-1}(z_\infty) f(\underline{g})$ for all $z_\infty \in S^0(\mathbb{R}), \underline{\in} G(\mathbb{A})$. The isogeny $d_C : C \to C'$ (see 1.1) induces an isomorphism $S^0(\mathbb{R}) \xrightarrow{\sim} S'^{,0}(\mathbb{R})$, where S' is the maximal $\mathbb{Q}$ split torus in C'. Therefore we get a character $\zeta'_\infty : S'^{,0}(\mathbb{R}) \to \mathbb{R}^\times_{>0}$ and this is also a character $\zeta'_\infty : G(\mathbb{R}) \to \mathbb{R}^\times_{>0}$ and its restriction to $S^0(\mathbb{R})$ is ζ_∞. If now $f \in \mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f, \zeta_\infty^{-1})$ then

$$f(\underline{g})\zeta'_\infty(\underline{g}) \in \mathcal{C}_\infty(G(\mathbb{Q})S^0(\mathbb{R})\backslash G(\mathbb{A})/K_f) \tag{110}$$

We say that $f \in \mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f, \zeta_\infty^{-1})$ is square integrable if

$$\int_{(G(\mathbb{Q})S^0(\mathbb{R})\backslash G(\mathbb{A})/K_f)} |f(\underline{g})\zeta'_\infty(\underline{g})|^2 d\underline{g} < \infty \tag{111}$$

and this allows us to define the Hilbert space $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f, \zeta_\infty^{-1})$. Since the space $(G(\mathbb{Q})S^0(\mathbb{R})\backslash G(\mathbb{A})/K_f)$ has finite volume we know that

$$\zeta'_\infty \in L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f, \zeta_\infty^{-1}).$$

The group $G(\mathbb{R})$ acts on $\mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f, \zeta_\infty^{-1})$ by right translations and hence we get by differentiating an action of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ on it. We define by $\mathcal{C}^{(2)}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f, \zeta_\infty^{-1})$ the subspace of functions f for which Uf is square integrable for all $U \in \mathfrak{U}(\mathfrak{g})$.

This allows us to define a sub complex of the de-Rham complex $\boxed{\text{Ltwo}}$

$$\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathcal{C}^{(2)}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f, \zeta_\infty^{-1}) \otimes \mathcal{M}_\lambda). \tag{112}$$

We will not work with this complex because its cohomology may show some bad behavior. (See remark below).

We do something less sophisticated, we simply define $H^\bullet_{(2)}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) \subset H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$ to be the image of the cohomology of the complex (112) in the cohomology. Hence $H^\bullet_{(2)}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$ is the space of cohomology classes which can be represented by square integrable forms.

Remark: Some authors also define L^2 de-Rham complexes, using the above complex (112) and then they take suitable completions to get complexes of Hilbert spaces. These complexes also give cohomology groups which run under the name of L^2-cohomology. These L^2-cohomology groups are related but not necessarily equal to our $H^\bullet_{(2)}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$. They can be infinite dimensional.

The Hilbert space $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f, \zeta_\infty^{-1})$ is a module for $G(\mathbb{R}) \times \mathcal{H}_{K_f}$ the group $G(\mathbb{R})$ acts by unitary transformations and the algebra $\mathcal{H}_{K_f}$ is selfadjoint.

Let us assume that $H = H_{\pi_\infty \times \pi_f}$ is an irreducible unitary module for $G(\mathbb{R}) \times \mathcal{H} = \bigotimes_p' \mathcal{H}_p$ and assume that we have an inclusion of this $G(\mathbb{R}) \times \mathcal{H}$-module

$$j : H \hookrightarrow L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f, \zeta_\infty^{-1}).$$

It follows from the finiteness results in 4.1.4 that induces an inclusion into the space of square integrable $\mathcal{C}_\infty$ functions

$$H^{(K_\infty)} \hookrightarrow \mathcal{C}_\infty^{(2)}(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f, \zeta_\infty^{-1})^{(K_\infty)}.$$

We consider the $(\mathfrak{g}, K_\infty)-$ cohomology of this module with coefficients in our irreducible module $\mathcal{M}_\lambda$, we assume $\chi_V(C) = \chi_\lambda(C)$. We have $H^\bullet(\mathfrak{g}, K_\infty, H \otimes \mathcal{M}_\lambda) = \mathrm{Hom}_{K_\infty}(\mathfrak{g}, K_\infty, H^{(K_\infty)} \otimes \mathcal{M}_\lambda)$ and get

$$H^\bullet(\mathfrak{g}, K_\infty, H^{(K_\infty)} \otimes \mathcal{M}_\mathbb{C}) \xrightarrow{j^\bullet} H^\bullet(\mathfrak{g}, K_\infty, \mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f, \zeta_\infty^{-1})^{(K_\infty)} \otimes \mathcal{M}_\lambda).$$

This suggests that we try to "decompose" $\mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f, \zeta_\infty^{-1})^{(K_\infty)}$ into irreducibles and then investigate the contributions of the irreducible summands to the cohomology. Essentially we follow the strategy of [Bo-Ga] and [Bo-Ca] but instead of working with complexes of Hilbert spaces we work with complexes of $\mathcal{C}_\infty$ forms and modify the arguments accordingly.

It has been shown by Langlands, that we have a decomposition into a discrete and a continous spectrum

$$L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f) = L^2_{\mathrm{disc}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f) \oplus L^2_{\mathrm{cont}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f),$$

where $L^2_{\mathrm{disc}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f)$ is the closure of the sum of all irreducible closed subspaces occuring in $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f)$ and where $L^2_{\mathrm{cont}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f)$ is the complement.

The discrete spectrum $L^2_{\mathrm{disc}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f)$ contains as a subspace the *cuspidal spectrum* $L^2_{\mathrm{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f)$:

A function $f \in L^2(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f)$ is called a *cusp form* if for all proper parabolic subgroups $P/\mathbb{Q} \subset G/\mathbb{Q}$, with unipotent radical $U_P/\mathbb{Q}$ the integral

$$\mathcal{F}^P(f)(g) = \int_{U_P(\mathbb{Q})\backslash U_P(\mathbb{A})} f(\underline{u}\underline{g})d\underline{u} = 0,$$

this means that the integral is defined for almost all $\underline{g}$ and zero for almost all $\underline{g}$. The function $\mathcal{F}^P(f)(\underline{g})$, which is an almost everywhere defined function on $P(\mathbb{Q})\backslash G(\mathbb{A})/K_f$ is called the constant Fourier coefficient of f along $P/\mathbb{Q}$. The cuspidal spectrum the the intersection of all the kernels of the $\mathcal{F}^P$.

If our group is anisotropic, then it does not have any proper parabolic subgroup and in this case we have $L^2_{\mathrm{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f) = L^2_{\mathrm{disc}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f) = L^2(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f)$.

For any unitary $G(\mathbb{R}) \times \mathcal{H}$- module $H_\pi = H_{\pi_\infty} \otimes H_{\pi_f}$ we put $W_{\pi,\mathrm{cusp}} = \mathrm{Hom}_{G(\mathbb{R})\times\mathcal{H}}(H_\pi, L^2_{\mathrm{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f))$. We can ignore the $\mathcal{H}$-module structure and define

$$W_{\pi_\infty,\mathrm{cusp}} = \mathrm{Hom}_{G(\mathbb{R})}(H_{\pi_\infty}) \otimes H_{\pi_f}, L^2_{\mathrm{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f)).$$

It has been shown by Gelfand-Graev and Langlands that

$$m_{\text{cusp}}(\pi_\infty) = \sum_{\pi_f} \dim(W_{\pi,\text{cusp}}) < \infty.$$

We get a decomposition into isotypical subspaces

$$L^2_{\text{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f) = \overline{\bigoplus_{\pi_\infty\otimes\pi_f} (L^2_{\text{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f)(\pi_\infty \times \pi_f)},$$

where $(L^2_{\text{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f)(\pi_\infty \times \pi_f)$ is the image of $W_{\pi,\text{cusp}} \otimes H_\pi$ in $L^2_{\text{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f)$.

The cuspidal spectrum has a complement in the discrete spectrum, this is the *residual spectrum* $L^2_{\text{res}}((G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f)$. It is called residual spectrum, because the irreducible subspaces contained in it are obtained by residues of Eisenstein classes.

Again we define $W_{\pi,\text{res}} = \text{Hom}_{G(\mathbb{R})\times\mathcal{H}}(H_\pi, L^2_{\text{res}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f))$, (resp. $W_{\pi_\infty,\text{res}} = \text{Hom}_{G(\mathbb{R})}(H_{\pi_\infty}, L^2_{\text{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f))$, and it is a deep theorem of Langlands that $m_{\text{res}}(\pi_\infty) = \dim(W_{\pi_\infty,\text{res}}) < \infty$. Hence we get a decomposition

$$L^2_{\text{res}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f) = \overline{\bigoplus_{\pi_\infty\otimes\pi_f} (L^2_{\text{res}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f)(\pi_\infty \times \pi_f)}.$$

If our group $G/\mathbb{Q}$ is isotropic, then the one dimensional space of constants is in the residual (discrete) spectrum but not in the cuspidal spectrum.

Langlands has given a description of the continuos spectrum using the theory of Eisenstein series, we have a decomposition $\boxed{\text{decomp-cont}}$

$$L^2_{\text{cont}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f) = \overline{\bigoplus_{\Sigma} \tilde{H}^+_P(\pi_\Sigma)}, \tag{113}$$

we briefly explain this decomposition following [Bo-Ga]. The Σ are so called cuspidal data, this are pairs (P, π_Σ) where P is a proper parabolic subgroup and π_Σ is a representation of $M(\mathbb{A}) = P(\mathbb{A})/U(\mathbb{A})$ occurring in the discrete spectrum $L^2_{\text{cusp}}(M(\mathbb{Q})\backslash M(\mathbb{A}))$.

Let $M^{(1)}/\mathbb{Q}$ be the semi simple part of M and recall that C/Q was the center of $G/\mathbb{Q}$. We consider the character module $Y^*(P) = \text{Hom}(C \cdot M^{(1)}, \mathbb{G}_m)$. The elements $Y^*(P)\otimes\mathbb{C}$ provide homomorphisms $\gamma\otimes z : M(\mathbb{A})/C(\mathbb{A})M^{(1)}(\mathbb{A}) \to \mathbb{C}^\times$. (See (14)). The module $Y^*(P) \otimes \mathbb{Q}$ comes with a canonical basis which is given by the dominant fundamental weights γ_μ which are trivial on $M^{(1)}$. We define

$$\Lambda_\Sigma = Y^*(P) \otimes i\mathbb{R} = \{\sum_\mu \gamma_\mu \otimes it_\mu | t_\mu \in \mathbb{R}\}$$

this is a group of unitary characters. For $\sigma \in \Lambda_\Sigma$ we define the unitarily induced representation

$$\begin{gathered}\text{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \pi_\Sigma \otimes (\sigma + \rho_P) = I_P^G \pi_\Sigma \otimes \sigma \\ \{f : G(\mathbb{A}) \to L^2_{\text{res}}(M(\mathbb{Q})\backslash M(\mathbb{A}))(\pi_\Sigma) | f(\underline{pg}) = (\sigma + |\rho_P|)(\underline{p})\pi_\Sigma(\underline{p})f(\underline{g})\}\end{gathered} \tag{114}$$

where of course $\underline{p} \in P(\mathbb{A}), \underline{g} \in G(\mathbb{A})$ and $\rho_P \in Y^*(P) \otimes \mathbb{Q}$ is the half sum of the roots in the unipotent radical of P. This gives us a unitary representation of $G(\mathbb{A})$. Let d_Σ be the Lebesgue measure on Λ_Σ then we can form the direct integral unitary representations

$$H_P(\pi_\Sigma) = \int_{\Lambda_\Sigma} I_P^G \pi_\Sigma \otimes \sigma \, d_\Sigma \sigma \tag{115}$$

The theory of Eisenstein series gives us a homomorphism of $G(\mathbb{R}) \times \mathcal{H}$ -modules

$$\mathrm{Eis}_P(\pi_\Sigma) : H_P(\pi_\Sigma) \to L^2_{\mathrm{cont}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f). \tag{116}$$

Let us put

$$\Lambda_\Sigma^+ = \{\sum_\mu \gamma_\mu \otimes it_\mu | t_\mu \geq 0\}$$

then the restriction

$$\mathrm{Eis}_P(\pi_\Sigma) : H_P^+(\pi_\Sigma) = \int_{\Lambda_\Sigma^+} I_P^G \pi_\Sigma \otimes \sigma \, d_\Sigma \sigma \to L^2_{\mathrm{cont}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f). \tag{117}$$

is an isometric embedding. The image will be denoted by $\tilde{H}_P^+(\pi_\Sigma)$ these spaces are the elementary subspaces in [B-G]. Two such elementary subspaces $\tilde{H}_P^+(\pi_\Sigma), \tilde{H}_{P_1}^+(\pi_{\Sigma_1})$ are either orthogonal to each other or they are equal. We get the above decomposition if we sum over a suitable set of representatives of cuspidal data.

Now we are ready to discuss the contribution of the continuous spectrum to the cohomology. If we have a closed square integrable form

$$\omega \in \mathrm{Hom}_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}), \mathcal{C}^2_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f) \otimes \mathcal{M}_\lambda),$$

then we can decompose it

$$\omega = \omega_{\mathrm{res}} + \omega_{\mathrm{cont}},$$

both summands are $\mathcal{C}^2_\infty$ and closed.

Proposition 4.3. *The cohomology class* $[\omega_{\mathrm{cont}}]$ *is trivial.*

Proof. This now the standard argument in Hodge theory, but this time we apply it to a continuous spectrum instead of a discrete one. We follow Borel-Casselman and prove their Lemma 5.5 (see [B-C]) in our context"

We may assume that ω_∞ lies in one of the summands, i.e. $\omega_{\mathrm{cont}} = \mathrm{Eis}(\int_{\Lambda_\Sigma} \omega^\vee(\sigma) d_\Sigma \sigma)$ where $\omega^\vee(\sigma) \in \mathrm{Hom}_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}), I_P^G \pi_\Sigma \otimes \sigma \otimes \mathcal{M}_\lambda))$ is the Fourier transform of ω_∞ in the L^2., (theorem of Plancherel). As it stands the expression $\int_{\Lambda_\Sigma} \omega^\vee(\sigma) d_\Sigma \sigma)$ does not make sense because the integrand is in L^2 and not necessarily in L^1. If we choose a symmetric positive definite quadratic form $h(\sigma) = \sum_{\nu,\mu} b_{\nu,\mu} t_\nu t_\mu$ and a positive real number τ then the function

$$h_\tau(\sigma) = (1 + \tau h(\sigma)^m)^{-1} \in L^2(\Lambda_\Sigma)$$

and then $\omega^\vee(\sigma) h_\tau(\sigma)$ is in L^1 and by definition

$$\lim_{\tau \to 0} \int_{\Lambda_\Sigma} \omega^\vee(\sigma) h_\tau(\sigma) d_\Sigma \sigma) = \int_{\Lambda_\Sigma} \omega^\vee(\sigma) d_\Sigma \sigma \tag{118}$$

where the convergence is in the L^2 sense. Since $\omega_\infty \in \mathrm{Hom}_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}), I_P^G\pi_\Sigma \otimes \sigma \otimes \mathcal{M}_\lambda)$ we get get that $\omega^\vee(\sigma)$ has the following property

For any polynomial $P(\sigma) = \sum a_{\underline{\mu}} t^{\underline{\mu}}$ in the variables t_μ and with real coefficients the section $\boxed{\text{diffmult}}$

$$\omega^\vee(\sigma)P(\sigma) \text{ is square integrable} \tag{119}$$

this follows from the well known rules that differentiating a function provides multiplication by the variables for the Fourier transform.

The Lemma of Kuga implies

$$\Delta(\omega^\vee(\sigma)) = (\chi_\sigma(C) - \chi_\lambda(C))\omega^\vee(\sigma)$$

and if $\sigma = \sum \gamma_\mu \otimes it\_\mu$ the eigenvalue is

$$\chi_\sigma(C) - \chi_\lambda(C) = \sum a_{\nu,\mu} t_\nu t_\mu + \sum b_\mu t_\mu + c_{\pi_\Sigma} - c_\lambda. \tag{120}$$

where c_{π_Σ} is the eigenvalue of the Casimir operator of $M^{(1)}$ on π_Σ If the $t_\mu \in \mathbb{R}$ then this expression is always ≤ 0 especially we see that the quadratic form on the right hand side is negative definite. This implies that for $\sigma \in \Lambda_F$ the expression $\chi_\sigma(C) - \chi_\lambda(C)$ assumes a finite number of maximal values all of them ≤ 0 and hence

$$V_\Sigma = \{\sigma | \chi_\sigma(C) - \chi_\lambda(C) = 0\} \tag{121}$$

is a finite set of point. This set has measure zero, since we assumed that P was a proper parabolic subgroup. The of σ for which $H^\bullet(\mathfrak{g}, K_\infty, H_{\Lambda_\Sigma}(\sigma) \otimes \mathcal{M}_\mathbb{C}) \neq 0$ is finite. We choose a $\mathcal{C}_\infty$ function $h_\Sigma(\sigma)$ which is positive, which takes value 1 in a small neighborhood of V_Σ, which takes values ≤ 1 in a slightly larger neighborhood and which is zero outside this second neighborhood. Then we write

$$\omega_\infty = \mathrm{Eis}(\int_{\Lambda_\Sigma^+} h_\Sigma(\sigma)\omega^\vee(\sigma) d_\Sigma\sigma) + \mathrm{Eis}(\int_{\Lambda_\Sigma^+} (1 - h_\Sigma(\sigma))\omega^\vee(\sigma) d_\Sigma\sigma)$$

We have $d\omega^\vee(\sigma) = 0$ and hence we get

$$\Delta((1 - h_\Sigma(\sigma))\omega^\vee(\sigma) = d\Big((\chi_\sigma(C) - \chi_\lambda(C))(1 - h_{\Sigma(\sigma)})\delta\omega^\vee(\sigma)\Big)$$

and this implies that

$$\mathrm{Eis}(\int_{\Lambda_\Sigma^+}(1-h_\Sigma(\sigma))\omega^\vee(\sigma)d_\Sigma\sigma) = d\mathrm{Eis}(\int_{\Lambda_\Sigma^+}(1-h_\Sigma(\sigma))(\chi_\sigma(C)-\chi_\lambda(C))^{-1}\delta\omega^\vee(\sigma)d_\Sigma\sigma)$$

It is clear that the integrand in the second term- $\int_{\Lambda_\Sigma^+}(1 - h_\Sigma(\sigma))(\chi_\sigma(C) - \chi_\lambda(C))^{-1}\delta\omega^\vee(\sigma)$ still satisfies (119) and then our well known rules above imply that $\psi = \mathrm{Eis}(\int_{\Lambda_\Sigma^+}(1 - h_\Sigma(\sigma))(\chi_\sigma(C) - \chi_\lambda(C))^{-1}\delta\omega^\vee(\sigma)d_\Sigma\sigma)$ is $\mathcal{C}^2_\infty$. Therefore the second term in our above formula is a boundary.

$$\omega_{\text{cont}} = \int_{\Lambda_\Sigma} h_\Sigma(\sigma)\omega(\sigma) d_\Sigma \sigma + d\psi.$$

This is true for any choice of h_Σ. Hence the scalar product $< \omega - d\psi, \omega - d\psi >$ can be made arbitrarily small. Then we claim that the cohomology class $[\omega] \in H^\bullet(\mathrm{Hom}_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}), \mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f) \otimes \mathcal{M}_\lambda)$ must be zero. This needs a tiny final step.

We invoke Poincaré duality: A cohomology class in $[\omega] \in H^p(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$ is zero if and only the value of the pairing with any class $[\omega_2] \in H^{d-p}_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}_\lambda}^\vee)$ is zero. But the (absolute) value $[\omega] \cup [\omega_2]$ of the cup product can be given by an integral (See Prop.4.2). Therefore it can be estimated by the norm $< \omega - d\psi, \omega - d\psi >$ (Cauchy-Schwarz inequality) and hence must be zero. $\square$

As usual we denote by $\widehat{G(\mathbb{R})}$ the unitary spectrum, for us it is simply the set of unitary irreducible representations of $G(\mathbb{R})$. Given $\tilde{\mathcal{M}}_\lambda$, we define

$$\mathrm{Coh}(\lambda) = \{\pi_\infty \in \widehat{G(\mathbb{R})} | \ H^\bullet(\mathfrak{g}, K_\infty, H_{\pi_\infty} \otimes \tilde{\mathcal{M}}_\lambda) \neq 0\}.$$

The theorem of Harish-Chandra says that this set is finite.

Let

$$H_{\mathrm{Coh}(\lambda)} = \bigoplus_{\pi:\pi_\infty \in \mathrm{Coh}(\lambda)} L^2_{\mathrm{disc}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f)(\pi_\infty \times \pi_f),$$

the theorem of Gelfand-Graev and Langlands assert that this is a finite sum of irreducible modules. This space decomposes again into $H^{\mathrm{cusp}}_{\mathrm{Coh}(\lambda)} \oplus H^{\mathrm{res}}_{\mathrm{Coh}(\lambda)}$

Then we get

Theorem (Borel, Garland, Matsushima-Murakami)

a)The map

$$H^\bullet(\mathfrak{g}, K_\infty, H^{(K_\infty)}_{\mathrm{Coh}(\lambda)} \otimes \mathcal{M}_\lambda) = \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), H^{(K_\infty)}_{\mathrm{Coh}(\lambda)} \otimes \mathcal{M}_\lambda) \to H^\bullet_{(2)}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$$

surjective. Especially the image contains $H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$.

b) (Borel) The homomorphism

$$H^\bullet(\mathfrak{g}, K_\infty, H^{(\mathrm{cusp},\mathrm{K}_\infty)}_{\mathrm{Coh}(\lambda)} \otimes \mathcal{M}_\lambda) \to H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$$

is injective.

[Bo-Ga] Prop.5.6, they do not consider the above space $H^\bullet_{(2)}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$ we added an $\epsilon > 0$ to this proposition by claiming that this space is the image.

In general the homomorphism

$$H^\bullet(\mathfrak{g}, K_\infty, H^{\mathrm{res}}_{\mathrm{res}(\lambda)}, K_\infty) \otimes \mathcal{M}_\lambda) \to H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$$

is not injective. We come to this issue in the next section.

If we denote by $H^\bullet_{\text{cusp}}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda)$ the image of the homomorphism in b), then we get a filtration of the cohomology by four subspaces

$$H^\bullet_{\text{cusp}}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda) \subset H^\bullet_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda) \subset H^\bullet_{(2)}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda) \subset H^\bullet(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda). \quad (122)$$

We want to point out that our space $H^\bullet_{(2)}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda)$ is not the space denoted by the same symbol in the paper [Bo-Ca]. They define L^2 cohomology as the complex of square integrable forms, i.e. ω and $d\omega$ have to be square integrable. But then a closed form ω which is in L^2 gives the trivial class in their cohomology if we can write $\omega = d\psi$ where ψ must also be square integrable. In our definition we do not have that restriction on ψ.

4.2.1 A formula for the Poincaré duality pairing

We assume that $-w_0(\lambda) = c(\lambda)$. We have the positive definite hermitian scalar product on $\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), H^{(K_\infty)}_{\mathrm{Coh}(\lambda)} \otimes \mathcal{M}_\lambda)$ (See(105)). On the other hand we have the Poincaré duality pairing

$$H^i_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda)(\omega_f) \times H^{d-i}_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\lambda^\vee})(\omega_{1,f}) \to \mathbb{C} \quad (123)$$

where $\omega_f \cdot \omega_{1,f} = 1$. To relate these two products we recall the Hodge $*$-operator. (See for instance Vol. I. 4.11) This operator yields an isomorphism

$$\begin{array}{l} * : \mathrm{Hom}_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}), \mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f) \otimes \mathcal{M}_\lambda) \xrightarrow{\sim} \\ \quad \mathrm{Hom}_{K_\infty}(\Lambda^{d-p}(\mathfrak{g}/\mathfrak{k}), \mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f) \otimes \mathcal{M}_{c\lambda}) \end{array} \quad (124)$$

We can use the $*$ operator to define the adjoint $\delta = (-1)^{d(p+1)+1} * d *$ and hence the Laplacian Δ (See (106). Especially the $*$ operator yields an identification between the $\mathcal{C}_\infty$-functions and the $\mathcal{C}_\infty$ differential forms in top degree.

We consider two differential forms

$$\omega_1,\omega_2 \in \mathrm{Hom}_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}), \mathcal{C}^2_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f) \otimes \mathcal{M}_\lambda)$$

which are square integrable, then we defined the scalar product (See(105) $<\omega_1,\omega_2>$ of these two forms. By definition this scalar product is an integral over a function

$$<\omega_1,\omega_2> = \int_{\mathcal{S}^G_{K_f}} \{\omega_1,\omega_2\}.$$

If we have two closed forms $\omega_1 \in \mathrm{Hom}_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}), \mathcal{C}^2_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f) \otimes \mathcal{M}_\lambda), \omega_2 \in \mathrm{Hom}_{K_\infty}(\Lambda^{d-p}(\mathfrak{g}/\mathfrak{k}), \mathcal{C}^2_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f) \otimes \mathcal{M}_{\lambda^\vee})$ and if one of these forms has compact support -say ω_2-then they define cohomology classes $[\omega_1] \in H^p(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda), [\omega_2] \in H^{d-p}_c(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\lambda^\vee})$ and the cup product $[\omega_1 \cup [\omega_2]$ is defined and given by an integral (See proposition 4.2) over a form in top degree. Now we check easily - and this is the way how the $*$ operator is designed that for $\omega_1,\omega_2 \in \mathrm{Hom}_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}), \mathcal{C}^2_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f) \otimes \mathcal{M}_\lambda)$ the integrand

$$\{\omega_1,\omega_2\} = <\omega_1 \wedge *\omega_2> .$$

Now we can formulate the

Proposition 4.4. *If* $\omega_1, \omega_2 \in \mathrm{Hom}_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}), H^{(K_\infty)}_{\mathrm{Coh}(\lambda)} \otimes \mathcal{M}_\lambda)$ *and if both classes* $[\omega_1], [*\omega_2]$ *are inner classes, i.e. can be represented by compactly supported forms then*

$$< \omega_1, \omega_2 >= [\omega_1] \cup [*\omega_2]$$

Proof. Postponed We exploit the fact that we can construct a real valued $h : \mathcal{S}^G_{K_f} \to \mathbb{R}_{>0}$ □

This proposition is of course a consequences of Hodge theory if the quotient $\mathcal{S}^G_{K_f}$ is compact, but if this is not the case, then the assertion is delicate. In fact we have the standard example which shows that we need the assumption that both classes $[\omega_1], [*\omega_2]$ are inner. If take $\omega_1 = \omega_2$ to be the form in degree zero given by the constant function 1. Then the left hand side is non zero but the class $*1$ is the volume form which is trivial if $\mathcal{S}^G_{K_f}$ is not compact, and therefore the right hand side is not zero.

The proposition has the following nice corollary

Corollary 4.1. *If* $\omega \in \mathrm{Hom}_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}), H^{(K_\infty)}_{\mathrm{Coh}(\lambda)} \otimes \mathcal{M}_\lambda)$ *is non zero and if the restrictions of* ω *and* $*\,\omega$ *to the boundary are zero then* $[\omega] \neq 0$

Now we remember that in the previous sections we made the convention (See end of (4.1.1)) that our coefficient systems $\mathcal{M}_\lambda$ are $\mathbb{C}$ vector spaces. We now revoke this convention and recall that the coefficient systems $\mathcal{M}_\lambda$ should be replaced by $\mathcal{M}_\lambda \otimes_F \mathbb{C}$. Then in the above list (122) of four subspaces in the cohomology the second and the fourth subspace have a natural structure of F-vector spaces and they have a combinatorial definition, whereas the first and third subspace need some input from analysis in their definition. In other words if we replace $\mathcal{M}_\lambda$ in (122) by $\mathcal{M}_\lambda \otimes_f \mathbb{C}$ then the second and the fourth space can be written as

$$H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) \otimes_F \mathbb{C} \subset H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) \otimes_F \mathbb{C}$$

We believe that also the third space has a combinatorial definition, for this we need the weighted cohomology groups: *Weighted cohomology* ; G. Harder; R. MacPherson; M. Goresky Inventiones mathematicae (1994).

4.3 Consequences.

4.3.1 Vanishing theorems

If V is unitary and irreducible, then we have that $\bar{V} \xrightarrow{\sim} V^\vee$ and this implies for the central character

$$\overline{\chi_V(z)} = \chi_{V^\vee}(z) \text{ for all} z \in \mathfrak{Z}(\mathfrak{g}).$$

Combining this with Wigner's lemma we can conclude

If V *is an irreducible unitary* $(\mathfrak{g}, K_\infty)$*-module,* $\mathcal{M}_\lambda$ *is an irreducible rational representation, and if*

$$H^\bullet(\mathfrak{g}, K_\infty, V \otimes \mathcal{M}_\lambda) \neq 0$$

then $\chi_{\mathcal{M}_\lambda^\vee}(z) = \chi_{\mathcal{M}_\lambda}({}^t z) = \chi_{\bar{\mathcal{M}}_\lambda}(z)$

In other words: For an unitary irreducible $(\mathfrak{g}, K_\infty)$-module V the cohomology with coefficients in an irreducible rational representation $\mathcal{M}$ vanishes, unless we have $\mathcal{M}_\lambda^\vee \xrightarrow{\sim} \bar{\mathcal{M}}_\lambda$, or in terms of highest weights unless $-w_0(\lambda) = c(\lambda)$. (See 3.1.1)

If we combine this with the considerations following Wigner's lemma we get

Corollary *If $\mathcal{M}$ is an absolutely irreducible rational representation and if $\mathcal{M}_\lambda^\vee$ is not isomorphic to $\bar{\mathcal{M}}_\lambda$ then*

$$H^\bullet_{(2)}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) = 0.$$

Hence also

$$H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) = 0.$$

We will discuss examples for this in section 4.3.2

4.3.2 The group $G/\mathbb{Q} = \mathrm{Sl}_2/\mathbb{Q}$

Let us consider the group $G/\mathbb{Q} = \mathrm{Sl}_2/\mathbb{Q}$. We have tautological representation $\mathrm{Sl}_2 \hookrightarrow \mathrm{Gl}(\mathbb{Q}^2) = \mathrm{Gl}(V)$ and we get all irreducible representations of we take the symmetric powers $\mathcal{M}_n = \mathrm{Sym}^n(V)$ of V. (See 2, these are the $\mathcal{M}_n[m]$ restricted to Sl_2, then the m drops out.)

In this case the Vogan-Zuckerman list is very short. It is discussed in [Slzwei] for the groups $\mathrm{Sl}_2(\mathbb{R})$ and $\mathrm{Sl}_2(\mathbb{C})$, where both groups are considered as real Lie-groups.

In the case $\mathrm{Sl}_2(\mathbb{R})$ we have the trivial module $\mathbb{C}$ and for any integer $k \geq 2$ we have two irreducible unitarizable $(\mathfrak{g}, K_\infty)$-modules $\mathcal{D}_k^\pm$ (the discrete series representations) (See [Slzwei], 4.1.5). These are the only $(\mathfrak{g}, K_\infty)$-modules which have non trivial cohomology with coefficients in a rational representation. If we now pick one of our rational representation $\mathcal{M}_n$, then the non vanishing cohomology groups are

$$H^q(\mathfrak{g}, K_\infty, \mathcal{M}_n \otimes \mathbb{C}) = \mathbb{C} \text{ for } l = 0, q = 0, 2$$
$$H^q(\mathfrak{g}, K_\infty, \mathcal{D}_k^\pm \otimes \mathcal{M}_n \otimes \mathbb{C}) = \mathbb{C} \text{ for } l = k-2, q = 1$$

The trivial $(\mathfrak{g}, K_\infty)$-module $\mathbb{C}$ occurs with multiplicity one in $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f)$ hence we get for the trivial coefficient system a contribution

$$H^\bullet(\mathfrak{g}, K_\infty, \mathbb{C}\otimes\mathcal{M}_n\otimes\mathbb{C}) = H^0(\mathfrak{g}, K_\infty, \mathbb{C})\oplus H^2(\mathfrak{g}, K_\infty, \mathbb{C}) = \mathbb{C}\oplus\mathbb{C} \to H^\bullet_{(2)}(\mathcal{S}^G_{K_f}, \mathbb{C}).$$

This map is injective in degree 0 and zero in degree 2.

For the modules $\mathcal{D}_k^\pm$ we have to determine the multiplicities $m^\pm(k)$ of these modules in the discrete spectrum of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f)$. A simple argument using complex conjugation tells us $m^+(k) = m^-(k)$ Now we have the fundamental observation made by Gelfand and Graev, which links representation theory to automorphic forms:

We have an isomorphism

$$\mathrm{Hom}_{\mathfrak{g},K_\infty}(\mathcal{D}_k^+, L^2_{\mathrm{disc}}(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f) \xrightarrow{\sim} S_k(G(\mathbb{Q})\backslash \mathbb{H} \times G(\mathbb{A}_f)/K_f) =$$

space of holomorphic cusp forms of weight k and level K_f

This is also explained in [Slzwei] on the pages following 23. We explain how we get starting from a holomorphic cusp form f of weight k an inclusion

$$\Phi_f : \mathcal{D}_k^+ \hookrightarrow L^2_{\mathrm{disc}}(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f)$$

and that this map $f \mapsto \Phi_f$ establishes the above isomorphim. This gives us the famous Eichler-Shimura isomorphism

$$S_k(G(\mathbb{Q})\backslash \mathbb{H} \times G(\mathbb{A}_f)/K_f) \oplus \overline{S_k(G(\mathbb{Q})\backslash \mathbb{H} \times G(\mathbb{A}_f)/K_f)} \xrightarrow{\sim} H^1_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{k-2}).$$

4.3.3 The group $G/\mathbb{Q} = R_{F/\mathbb{Q}}(\mathrm{Sl}_2/F)$.

For any finite extension $F/\mathbb{Q}$ we may consider the base restriction $G/\mathbb{Q} = R_{F/\mathbb{Q}}(Sl_2/F)$. (See Chap-II. 1.1.1). Here we want to consider the special case the $F/\mathbb{Q}$ is imaginary quadratic. In this case we have $G \otimes \mathbb{C} = \mathrm{Sl}_2 \times \mathrm{Sl}_2/\mathbb{C}$ the factors correspond to the two embeddings of F into $\mathbb{C}$. The rational irreducible representations are tensor products of irreducible representations of the two factors $\mathcal{M}_\lambda = \mathcal{M}_{k_1} \otimes \mathcal{M}_{k_2}$ where again $\mathcal{M}_k = \mathrm{Sym}^k(\mathbb{C}^2)$. These representations are defined over F.

In this case we discuss the Vogan-Zuckerman list in [Slzwei], here we want to discuss a particular aspect. We observe that

$$\mathcal{M}_\lambda^\vee \xrightarrow{\sim} \mathcal{M}_{k_1} \otimes \mathcal{M}_{k_2}, \bar{\mathcal{M}}_\lambda = \mathcal{M}_{k_2} \otimes \mathcal{M}_{k_1}$$

and hence our corollary above yields for any choice of K_f

$$H^\bullet_!(\mathcal{S}^G_{K_f}, \mathcal{M}) = 0 \text{ if } k_1 \neq k_2.$$

In Chapter II we discuss the special examples in low dimensions. We take $F = \mathbb{Q}[i]$ and $\Gamma = \mathrm{Sl}_2[\mathbb{Z}[i]]$ this amounts to taking the standard maximal compact subgroup $K_f = \mathrm{Sl}_2[\hat{\mathcal{O}}_F]$. If now for instance $k_1 > 0$ and $k_2 = 0$, then we get $H^\bullet_!(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda) = 0$. Hence we have by definition $H^\bullet_!(\mathcal{S}^G_{K_f}, \mathcal{M}) = H^\bullet_{\mathrm{Eis}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})$ and we have complete control over the Eisenstein- cohomology in this case. Hence we know the cohomology in this case if we apply the analytic methods.

On the other hand in Chapter II we have written an explicit complex of finite dimensional vector spaces, which computes the cohomology. It is not clear to me how we can read off this complex the structure of the cohomology groups.

We get another example where this phenomenon happens, if we consider the group $\mathrm{Sl}_n/\mathbb{Q}$ if $n > 2$. In Chap. IV 1.2 we described the simple roots $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$, accordingly we have the fundamental highest weights $\omega_1, \ldots, \omega_{n-1}$. The element w_0 (See 4.1.1) has the effect of reversing the order of the weights. Hence we see that for $\lambda = \sum n_i \omega_i$ we have

$$H^\bullet_!(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda) = 0$$

unless we have $-w_0(\lambda) = \lambda$ and this means $n_i = n_{n-1-i}$.

4.3.4 The algebraic K-theory of number fields

I briefly recall the definition of the K-groups of an algebraic number field $F/\mathbb{Q}$. We consider the group $\mathrm{Gl}_n(\mathrm{O}_F)$, it has a classifying space BG_n. We can pass to the limit $\lim\limits_{n\to\infty} \mathrm{Gl}_n(\mathrm{O}_F) = \mathrm{Gl}(\mathrm{O}_F) = G$ and let BG its classifying space. Quillen invented a procedure to modify this space to another space BG^+, whose fundamental group is now abelian, but which has the same homology and cohomology as BG. Then he defines the algebraic K-groups as

$$K_i(\mathrm{O}_F) = \pi_i(\mathrm{BG}^+).$$

The space is an H-space, this means that we have a multiplication $m : \mathrm{BG}^+ \times \mathrm{BG}^+ \to \mathrm{BG}^+$ which has a two sided identity element. Then we get a homomorphism $m^\bullet : H^\bullet(\mathrm{BG}^+, \mathbb{Z}) \to H^\bullet(\mathrm{BG}^+ \times \mathrm{BG}^+, \mathbb{Z})$ and if we tensorize by $\mathbb{Q}$ and apply the Künneth-formula then we get the structure of a Hopf algebra on the Cohomology

$$m^\bullet : H^\bullet(\mathrm{BG}^+, \mathbb{Q}) \to H^\bullet(\mathrm{BG}^+, \mathbb{Q}) \otimes H^\bullet(\mathrm{BG}^+, \mathbb{Q})$$

Then a theorem of Milnor asserts that the rational homotopy groups

$$\pi_i(\mathrm{BG}^+) \otimes \mathbb{Q} = \ \mathrm{prim}(H^i(\mathrm{BG}, \mathbb{Q}),$$

where prim are the primitive elements, i.e. those elements $x \in H^i(\mathrm{BG}, \mathbb{Q})$ for which

I sketch a second application. We discuss the group $G = R_{F/\mathbb{Q}}(\mathrm{Gl}_n/F)$, where $F/\mathbb{Q}$ is an algebraic number field. the coefficient system $\tilde{\mathcal{M}}_\lambda = \mathbb{C}$ is trivial. In this case Borel, Garland and Hsiang have shown hat in low degrees $q \leq n/4$

$$H^q(\mathcal{S}^G_{K_f}, \mathbb{C}) = H^q_{(2)}\mathcal{S}^G_{K_f}, \mathbb{C}).$$

On the other hand it follows from the Vogan-Zuckerman classification, that the only irreducible unitary $(\mathfrak{g}, K_\infty)$ modules V, for which $H^q(\mathfrak{g}, K_\infty, V) \neq 0$ and $q \leq n/4$ are one dimensional.

Hence we see that in low degrees

$$H^q(\mathfrak{g}, K_\infty, \mathbb{C}) \to H^q(\mathcal{S}^G_{K_f}, \mathbb{C})$$

is an isomorphism (Injectivity requires some additional reasoning.)

On the other hand we have $H^q(\mathfrak{g}, K_\infty, \mathbb{C}) = \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathbb{C})$ and obviously this last complex is isomorphic to the complex $\Omega^\bullet(X)^{G(\mathbb{R})}$ of $G(\mathbb{R})$-invariant forms on the symmetric space $G(\mathbb{R})/K_\infty$. Our field has different embeddings $\tau : F \hookrightarrow \mathbb{C}$, the real embeddings factor through $\mathbb{R}$, they form the set S^{real}_∞ and the pairs of may conjugate embeddings into $\mathbb{C}$ form the set S^{comp}_∞ Then

$$X = \prod_{v \in S^{\mathrm{real}}_\infty} \mathrm{Sl}_n(\mathbb{R})/SO(n) \times \prod_{S^{\mathrm{comp}}_\infty} \mathrm{Sl}_n(\mathbb{C})/SU(n).$$

Now the complex $\Omega^\bullet(X)^{G(\mathbb{R})}$ of invariant differential forms (all differentials are zero) does not change if we replace the group

$$G(\mathbb{R}) = \prod_{v \in S^{\mathrm{real}}_\infty} \mathrm{Sl}_n(\mathbb{R}) \times \prod_{S^{\mathrm{comp}}_\infty} \mathrm{Sl}_n(\mathbb{C})$$

by its compact form $G_c(\mathbb{R})$ and then we get the complex of invariant forms on the compact twin of our symmetric space

$$X_c = \prod_{v \in S_\infty^{\mathrm{real}}} SU_n(\mathbb{R})/SO(n) \times \prod_{S_\infty^{\mathrm{comp}}} (SU(n) \times SU(n))/SU(n),$$

but then

$$\Omega(X_c)^{G_c(\mathbb{R})} = H^\bullet(X_c, \mathbb{C}).$$

The cohomology of the topological spaces like the one on the right hand side has been computed by Borel in the early days of his career.

If we let n tend to infinity, we can consider the limit of these cohomology groups, then the limit becomes a Hopf algebra and we can consider the primitive elements

4.3.5 The semi-simplicity of the inner cohomology

Now we assume again that our representation $\tilde{\mathcal{M}}_\lambda$ is defined over some number field F we consider it as a subfield of $\mathbb{C}$. In other word we have a representation $r : G \times F \to \mathrm{Gl}(\mathcal{M}_\lambda)$. We have defined $H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})$, this is a finite dimensional F-vector space and Theorem 2 in Chapter II asserts that this is a semi simple module under the Hecke algebra. This is now an easy consequence of our results above.

The module $H_1 \subset L^2_{\mathrm{disc}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f)$ can also be decomposed into a finite direct sum of irreducible $G(\mathbb{R}) \times \mathcal{H}_{K_f}$ modules

$$H_1 = \bigoplus_{\pi_\infty \otimes \pi_f \in \hat{H}_1} (H_{\pi_\infty} \otimes H_{\pi_f})^{m_1(\pi_\infty \times \pi_f)},$$

this module is clearly semi-simple. Of course it is not a $(\mathfrak{g}, K_\infty)$-module, but we can restrict to the K_∞-finite vectors and get

$$H^\bullet(\mathfrak{g}, K_\infty, H_1^{(K_\infty)} \otimes \mathcal{M}_\lambda \otimes \mathbb{C}) = \bigoplus_{\pi_\infty \otimes \pi_f \in \hat{H}_1} \left(\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), H_{\pi_\infty} \otimes \mathcal{M}_\mathbb{C}) \otimes H_{\pi_f}\right)^{m_1(\pi_\infty \times \pi_f)}$$

This is a decomposition of the left hand side into irreducible $\mathcal{H}_{K_f}$ modules. Now we have the surjective map

$$H^\bullet(\mathfrak{g}, K_\infty, H_1^{(K_\infty)} \otimes \mathcal{M}_\lambda \otimes \mathbb{C}) \to H^\bullet_{(2)}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{C})$$

hence it follows that $H^\bullet_{(2)}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{C}))$ is a semi simple $\mathcal{H}_{K_f}$ module and hence also $H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$ is a semi simple $\mathcal{H}_{K_f}$ module.

At this point we encounter an interesting problem. We have the three subspaces (See end of 3.2)

$$H^\bullet_{\mathrm{cusp}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{C}) \subset H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) \otimes \mathbb{C} \subset H^\bullet_{(2)}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{C}) \subset H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) \otimes \mathbb{C},$$

note the positions of the tensor symbol $\otimes$. The first and the third space are only defined after we tensorize the coefficient system by $\mathbb{C}$, whereas the second and the fourth cohomology groups by definition F vector spaces tensorized by $\mathbb{C}$.

Now the question is whether the first and the third space also have a natural F -vector space structure. Of course we get a positive answer, if the Manin-Drinfeld principle holds. All the vector spaces are of course modules under the Hecke algebra and we and we can look at their spectra

$$\begin{array}{cc} \Sigma(H^\bullet_{\text{cusp}}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda\otimes\mathbb{C}))=\Sigma_{\text{cusp}} & \Sigma(H^\bullet_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda\otimes\mathbb{C}))=\Sigma_! \\ \Sigma(H^\bullet_{(2)}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda\otimes\mathbb{C}))=\Sigma_{(2)} & \Sigma(H^\bullet(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda\otimes\mathbb{C}))=\Sigma \end{array}.$$

If now for instance $\Sigma_{\text{cusp}}\cap(\Sigma_!\setminus\Sigma_{\text{cusp}}=\emptyset$ then we can define $H^\bullet_{\text{cusp}}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda)\subset H^\bullet_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda)$ as the subspace which is the sum of the isotypical components in Σ_{cusp}.

If this is the case we say that the cuspidal cohomology is *intrinsically definable* and we get a canonical decomposition

$$H^\bullet_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda)=H^\bullet_{\text{cusp}}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda)\oplus H^\bullet_{!,\text{noncusp}}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda).$$

The classical Manin-Drinfeld principle refers to the two spectra $\Sigma_!\subset\Sigma$, if it is true in this case we get a decomposition

$$H^\bullet(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda)=H^\bullet_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda)\oplus H^\bullet_{\text{Eis}}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda)$$

the canonical complement is called the Eisenstein cohomology. (See Chap. II 2.2.3 and Chap III 5.)

4.4 Franke's Theorem

: ....

5 Modular symbols

5.1 The general pattern

We start from the following data. Let $H/\mathbb{Q}$ be a (reductive) subgroup of our group $G/\mathbb{Q}$. Let $K_\infty^{H,(1)}$ be the connected component of the identity of a maximal compact subgroup of $H(\mathbb{R})$ we put $X^H=H(\mathbb{R})/K_\infty^{H,(1)}$. We have the spaces

$$\mathcal{S}^G_{K_f}=G(\mathbb{Q})\backslash X\times G(\mathbb{A}_f)/K_f,\mathcal{S}^H_{K_f^H}=H(\mathbb{Q})\backslash X^H\times H(\mathbb{A}_f)/K_f.$$

From the inclusion $i:H\to G$ we get maps between these locally symmetric spaces

$$j(x,\underline{g}_f):\mathcal{S}^H_{K_f^H}\to\mathcal{S}^G_{K_f}$$

which depend on the choice of "pin points" $(x,\underline{g}_f)\in X\times G(\mathbb{A}_f)$. These pin points have to be chosen with some care:

a) The point $x\in X$ can be viewed as a Cartan involution Θ_x on $G(\mathbb{R})$ and Θ_x should fix $H(\mathbb{R})$. Hence it is also a Cartan involution on H and we require that it is the identity on our chosen $K_\infty^{H,(1)}$. Let us denote this subset of X by $X^{(H,K_\infty^{H,(1)})}$. Let N be the subgroup of the normalizer of $H/\mathbb{Q}$ which also normalizes $K_\infty^{H,(1)}$. Then $N(\mathbb{R})$ acts on $X^{(H,K_\infty^{H,(1)})}$. I think that this action is transitive and the orbits under the group $N(\mathbb{R})^{(1)}$ are the connected components.

b)The element $\underline{g}_f$ has to satisfy a similar condition:

$$K_f^H \underline{g}_f K_f = \underline{g}_f K_f$$

(Recall that we always have make careful choices of the level if we deal with integral cohomology.)

Choosing $(x, \underline{g}_f)$ we get a map

$$j(x, \underline{g}_f) : H(\mathbb{Q})\backslash H(\mathbb{R})/K_\infty^H \times H(\mathbb{A}_f)/K_f^H \longrightarrow \mathcal{S}_{K_f}^G$$

which is defined by

$$(h_\infty, \underline{h}_f) \mapsto (h_\infty x, \underline{h}_f \underline{g}_f).$$

Now we assume that we have coefficient systems $\tilde{\mathcal{M}}_\mathcal{O}, \mathcal{O}_\mu$ coming from representations of $\rho : \mathcal{G}/\mathbb{Z} \to \mathrm{Gl}(\mathcal{M}_\mathcal{O})$ resp. a one dimensional representation $\mu : \mathcal{H}/\mathbb{Z} \to \mathbb{G}_m$. We assume that we also have a homomorphism from the restriction of ρ to $\mathcal{H}/\mathbb{Z}$ to μ, i.e

$$r_{\lambda,\mu} : \mathcal{M}_\mathcal{O} \to \mathcal{O}_\mu$$

which invariant under the action of $\mathcal{H}$. This induces a homomorphism of sheaves

$$r_{\lambda,\mu}^* : j(x, \underline{g}_f)^*(\tilde{\mathcal{M}}_\mathcal{O}) \to \tilde{\mathcal{O}}_\mu. \tag{125}$$

Then these data provide a homomorphism for the cohomology groups

$$j(x, \underline{g}_f)^\bullet : H^\bullet(\mathcal{S}_{K_f}^G, \tilde{\mathcal{M}}) \to H^\bullet(\mathcal{S}_{K_f^H}^H, \mathcal{O}_\mu)$$

We are interested in this homomorphism in degree $d_H = \dim \mathcal{S}_{K_f^H}^H$.

In this degree we know the compactly supported cohomology of $\mathcal{S}_{K_f^H}^H$

$$H_c^{d_H}(\mathcal{S}_{K_f^H}^H, \mathcal{O}_\mu) = H^{d_H}(\mathcal{S}_{K_f^H}^H, i_!(\tilde{\mathcal{O}})_\mu) = \bigoplus_\chi H^{d_H}(\mathcal{S}_{K_f^H}^H, i_!(\tilde{\mathcal{O}})_\mu)[\chi_f]$$

where we sum over characters $\tilde{\chi}_f$ of type μ. on $\pi_0(H(\mathbb{R})) \times H(\mathbb{A}_f)$ (See (2.5.2)) The eigenspaces are projective $\mathcal{O}$- modules of rank one let us assume that they are free and that we have chosen generators c_χ. We will call such generators modular symbols.

We see that the homomorphism $j(x, \underline{g}_f)^\bullet$ is not yet good enough it has the wrong target, if we want to evaluate cohomology classes on the fundamental cycles of $H^{d_H}(\mathcal{S}_{K_f^H}^H, i_!(\tilde{\mathcal{O}})_\mu)$. We need to modify the source.

We study the extension of $j(x, \underline{g}_f)$ to the compactification

$$\bar{j}(x, \underline{g}_f) : \bar{\mathcal{S}}_{K_f^H}^H \to \bar{\mathcal{S}}_{K_f}^G$$

We recall the construction of sheaves with intermediate support conditions (2.1.2.Let us assume that we can find a Σ such that the image of $\partial(\bar{\mathcal{S}}^H_{K_f^H})$ factors through $\partial_\Sigma(\bar{\mathcal{S}}^G_{K_f})$. Then our homomorphism r yields a homomorphism between sheaves (see (19))

$$r^!_{\lambda,\mu} : \bar{j}(x, \underline{g}_f)^*(i_{\Sigma,*,!}(\tilde{\mathcal{M}})) \to i_!(\tilde{\mathcal{O}}_\mu). \tag{126}$$

and hence we get a homomorphism in cohomology

$$\bar{j}((x, \underline{g}_f), r_{\lambda,\mu})^{d_H} : H^{d_H}(\mathcal{S}^G_{K_f}, i_{\Sigma,*,!}(\tilde{\mathcal{M}})) \to H^{d_H}(\mathcal{S}^H_{K_f^H}, i_!(\tilde{\mathcal{O}}_\mu)) \tag{127}$$

If we change x inside a connected component of $X^{(H,K_\infty^{H,(1)})}$ then $\bar{j}((x, \underline{g}_f), r_{\lambda,\mu})^{d_H}$ does not change, and hence we can view x as a discrete variable.

We still have the variable $\underline{g}_f$. This has to satisfy the above condition b), it has to respect the level and we have to fix the level because we want to get integral cohomology groups. If we tensorize our coefficient systems with F (the quotient field of $\mathcal{O}$) then we can consider the limit

$$\lim_{K_f} H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_F) = H^\bullet(\mathcal{S}^G, \tilde{\mathcal{M}}_F),$$

and this limit is now a $\pi_0(G(\mathbb{R})) \times G(\mathbb{A}_f)$ module. Doing this also with $\mathcal{S}^H_{K_f^H}$ we can forget the constraint on $\underline{g}_f$ and we get an intertwining operator

$$\bar{j}((x, \underline{g}_f), r_{\lambda,\mu})^{d_H} : H^{d_H}(\mathcal{S}^G, i_{\Sigma,*,!}(\tilde{\mathcal{M}}))_{\bar{\mathbb{Q}}}) \to H^{d_H}(\mathcal{S}^H, i_!(\bar{\mathbb{Q}}_\mu)) = \bigoplus_\chi \bar{\mathbb{Q}}[\tilde{\chi}_f] \tag{128}$$

where the direct sum on the right hand side is now infinite, we sum over all characters of type μ.

Assume that we have chosen a basis element $c_\chi \in H^{d_H}(\mathcal{S}^H, i_!(\bar{\mathbb{Q}}_\mu))[\chi]$ (a modular symbol) for all χ. For a class $\xi \in H^{d_H}(\mathcal{S}^G, i_{\Sigma,*,!}(\tilde{\mathcal{M}}))_{\bar{\mathbb{Q}}})$ we get

$$\bar{j}((x, \underline{g}_f), r_{\lambda,\mu})^{d_H}(\xi) = \sum_\chi F_\chi(\xi, (x, \underline{g}_f))c_\chi \tag{129}$$

The cohomology $H^{d_H}(\mathcal{S}^G, i_{\Sigma,*,!}(\tilde{\mathcal{M}}))_{\bar{\mathbb{Q}}})$ is a $\pi_0(G(\mathbb{R})) \times G(\mathbb{A}_f)$-module.

Lemma 5.1. *We get get an intertwining operator between $\pi_0(G(\mathbb{R})) \times G(\mathbb{A}_f)$-modules*

$$J_{c_\chi}(r_{\lambda,\mu}) : H^{d_H}(\mathcal{S}^G, i_{\Sigma,*,!}(\tilde{\mathcal{M}}))) \to \mathrm{Ind}_{\pi_0(H(\mathbb{R}))\times H(\mathbb{A}_f)}^{\pi_0(G(\mathbb{R}))\times G(\mathbb{A}_f)} \tilde{\chi}_f^{-1}$$

The question arises to compute this operator. Of course it is not so clear what this means. First of all we have the problem that we do not know the left hand side. Recall that the left hand side still sits in an exact sequence

$$0 \to H^{d_H-1}(\partial_\pi(\mathcal{S}^G), \tilde{\mathcal{M}}_{\bar{\mathbb{Q}}}) \to H^{d_H}(\mathcal{S}^G, i_{\Sigma,*,!}(\tilde{\mathcal{M}}))_{\bar{\mathbb{Q}}}) \to H^{d_H}_!(\mathcal{S}^G, \tilde{\mathcal{M}})_{\bar{\mathbb{Q}}}) \to 0.$$

We try to produce absolutely irreducible submodules

$$H^{d_H}(\mathcal{S}^G, i_{\Sigma,*,!}(\tilde{\mathcal{M}}))_F)(\pi_f) \subset H^{d_H}(\mathcal{S}^G, i_{\Sigma,*,!}(\tilde{\mathcal{M}}))_F$$

and restrict the intertwining operator to this submodule. Then we may be lucky and the space of $\pi_0(G(\mathbb{R})) \times G(\mathbb{A}_f)$ homomorphisms of this submodule into $\mathrm{Ind}_{\pi_0(H(\mathbb{R}))\times H(\mathbb{A}_f)}^{\pi_0(G(\mathbb{R}))\times G(\mathbb{A}_f)} \tilde{\chi}_f^{-1}$ is one dimensional and contains some kind of canonical generator . In this case the intertwining operator is essentially given by a number

1) We may, of course, consider first the boundary map

$$H^{d_H-1}(\partial\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathcal{O}}) \longrightarrow H_c^{d_H}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathcal{O}}),$$

and restrict the map J_{c_χ} to its image.

If we want to understand this restriction – perhaps we should simply denote it by ∂J_{c_χ} – then we have to look at the image of c_χ under the boundary map

$$\begin{array}{ccccc} \partial & : & H_{d_H}(\mathcal{S}^H_{K_f^H}, \partial\mathcal{S}^H_{K_f^H}, \tilde{\mathcal{M}}_{\mathcal{O}}) & \longrightarrow & H_{d_H-1}(\partial\mathcal{S}^H_{K_f^H}, \tilde{\mathcal{M}}_{\mathcal{O}}) \\ & & & & \Big\downarrow j(x,\underline{g}_f) \\ & & & & H_{d_H-1}(\partial\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathcal{O}}). \end{array}$$

I think this restriction is not so interesting, since we are basically dealing with a smaller group.

In certain cases it happens that

$$j(x,\underline{g}_f)(\partial c_\chi) = 0 \qquad (M_1)$$

If this condition is satisfied, then we know that J_{c_χ} factorizes over

$$J_{c_\chi} : H_!^{d_H}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathcal{O}}) \longrightarrow \mathrm{Ind}_{\tilde{H}(\mathbb{A}_f)}^{\tilde{G}(\mathbb{A}_f)} \ \tilde{\chi}_f^{-1}.$$

If this is the case we are somewhat better off, because cohomology classes in $H_!^{d_H}(\mathcal{S}^G_{K_f}, \mathcal{M}_{\mathbb{C}})$ can be constructed and described using automorphic forms (Θ-series or Fourier expansions (See 2.2.2).) Moreover we know that after tensorization with the quotient field F of $\mathcal{O}$ the inner cohomology becomes semi simple and we can restrict J_{c_χ} to isotypical submodules. (See next section)

Of course we are always in this special case it the group $H/\mathbb{Q}$ is anisotropic, because in this case $j(x,\underline{g}_f) \in H_c^{d-d_H}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_F) = H_{d_H}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathcal{O}})$.

In this case we may even pair $j(x,\underline{g}_f)$ with elements in $H^{d_H}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathcal{O}})$

2) Another condition that may be satisfied is the Manin-Drinfeld principle, i.e. we have an isotypical decomposition

$$H^{d_H}_{\mathrm{Eis}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_F) \oplus H_!^{d_H}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_F). \qquad (M_2)$$

Then we may restrict J_{c_χ} to the second summand. We get

$$J_{c_\chi,!} : H_!^{d_H}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_F) \longrightarrow \mathrm{Ind}_{\tilde{H}(\mathbb{A}_f)}^{\tilde{G}(\mathbb{A}_f)} \ \tilde{\chi}_f.$$

3)

5.1.1 Model spaces

I want to introduce some abstract concept of the production of cohomology classes and the evaluation of these intertwining operators on these classes. To do this we introduce model spaces.

We assume that we have a family of local smooth and admissible representations $\{X_{\pi_v}\}$ where v runs over all places. For almost all finite places p the representation $\{X_{\pi_p}\}$ should be an unramified irreducible principal series representation. We assume that X_{π_∞} is an irreducible Harish-Chandra module with non trivial cohomology $H^\bullet(\mathfrak{g}, K_\infty, X_{\pi_\infty} \otimes \mathcal{M}_\mathbb{C}) \neq 0$. Furthermore we assume that we have an intertwining operator of $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$-modules

$$\Phi : X_{\pi_\infty} \otimes \bigotimes_p X_{\pi_p} \longrightarrow \mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})).$$

This induces of course an intertwining operator

$$\begin{aligned} H^\bullet(\mathfrak{g}, K_\infty, X_{\pi_\infty} \otimes \mathcal{M}_\mathbb{C}) \otimes \textstyle\bigotimes_p X_{\pi_p} \quad &\xrightarrow{\Phi^\bullet} \quad H^\bullet(\mathfrak{g}, K_\infty, \mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})) \otimes \mathcal{M}_\mathbb{C}) \\ &= \quad H^\bullet(\mathcal{S}^G, \tilde{\mathcal{M}}_\mathbb{C}) \end{aligned}$$

We introduce a subspace of $\mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A}))$. We consider the subspace of functions of moderate growth and inside this space we consider the space of functions which are cuspidal along the strata $\partial_P(\mathcal{S}^G)$ for the parabolic subgroups $P \in \Sigma$, i.e. which satisfy

$$\int_{U_P(\mathbb{Q})\backslash U_P(\mathbb{A})} f(\underline{u}g) d\underline{u} \equiv 0$$

for these parabolic subgroups. Let us call this subspace $\mathcal{C}_\infty^{(\Sigma)}(G(\mathbb{Q})\backslash G(\mathbb{A}))$. We assume that our intertwining operator factors through the subspace of Σ cuspidal functions

$$\Phi : X_{\pi_\infty} \otimes \bigotimes_p X_{\pi_p} \longrightarrow \mathcal{C}_\infty^{(\Sigma)}(G(\mathbb{Q})\backslash G(\mathbb{A})) \tag{130}$$

and we assume in addition that we have multiplicity one, this means that Φ is unique up to scalar.

We have an action of $\pi_0(G(\mathbb{R}))$ on $H^\bullet(\mathfrak{g}, K_\infty, X_{\pi_\infty} \otimes \mathcal{M}_\mathbb{C})$ let $\epsilon : \pi_0(G(\mathbb{R})) \to \{\pm 1\}$ be a character and let ω_ϵ be a differential form representing an eigenclass $[\omega_\epsilon]$. In [Ha-Gl2] we explain how a Hecke character χ_f extends to a character $\tilde{\chi}_f : \pi_0(H(\mathbb{R}))H(\mathbb{A}_f) \to \{\pm 1\}$. We have the homomorphism $\pi_0(H(\mathbb{R})) \to \pi_0(G(\mathbb{R}))$ and we require that $\chi_\infty = \epsilon_\infty$

We get a diagram

$$\begin{array}{ccc} H^\bullet(\mathfrak{g}, K_\infty, X_{\pi_\infty} \otimes \mathcal{M}_\mathbb{C})(\epsilon_\infty) \otimes \otimes_p X_{\pi_p} & & \\ \downarrow \Phi^{d_H} & & \\ H^{d_H}(\mathfrak{g}, K_\infty, \mathcal{C}_\infty^{(\Sigma)}(G(\mathbb{Q})\backslash G(\mathbb{A})) \otimes \mathcal{M}_\mathbb{C}) & \xrightarrow{\Phi^{d_H,\Sigma}} & H^{d_H}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) \otimes \mathbb{C} \\ & & \uparrow i_\Sigma^{d_H} \otimes \mathbb{C} \\ \mathrm{Ind}_{\pi_0(H(\mathbb{R}))\times H(\mathbb{A}_f)}^{\pi_0(G(\mathbb{R}))\times G(\mathbb{A}_f)} \tilde{\chi}_f^{-1} \otimes \mathbb{C} & \xleftarrow{J_{c_\chi}} & H^{d_H}(\mathcal{S}^G, i_{\Sigma,*,!}(\tilde{\mathcal{M}})) \otimes \mathbb{C} \end{array}$$

Proposition 5.1. *The image of Φ^{d_H} is contained in the image of $i_\Sigma^{d_H} \otimes \mathbb{C}$*

Proof. Careful analysis using reduction theory $\square$

We now make the further assumption that the Manin-Drinfeld principle is valid for the image $H^{d_H}_{\Sigma,!}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$ of $i_\Sigma^{d_H}$, this means that we have unique $G(\mathbb{A}_f)$-invariant section

$$s_\Sigma^{d_H} : H^{d_H}_{\Sigma,!}(\mathcal{S}^G, \tilde{\mathcal{M}}) \to H^{d_H}(\mathcal{S}^G, i_{\Sigma,*,!}(\tilde{\mathcal{M}}))) \tag{131}$$

Then we get an arrow

$$H^{d_H}(\mathfrak{g}, K_\infty, \mathcal{C}_\infty^{(\Sigma)}(G(\mathbb{Q})\backslash G(\mathbb{A})) \otimes \mathcal{M}_\mathbb{C}) \to \mathrm{Ind}_{\pi_0(H(\mathbb{R}))\times H(\mathbb{A}_f)}^{\pi_0(G(\mathbb{R}))\times G(\mathbb{A}_f)} \tilde{\chi}_f^{-1} \otimes \mathbb{C}$$

which should be placed into the middle of the above diagram. The cohomology on the left hand side can by computed by the de-Rham complex.

Theorem 5.1. *This arrow is given by the integral*

$$J_{c_\chi}(\xi)((x,\underline{g}_f), r_{\lambda,\mu})([\omega]) = \int_{\mathcal{S}^H_{K^H_f}} r_{\lambda,\mu}(j^*(x,\underline{g}_f)(\omega))$$

We can take the composition

$$\Phi^{d_H} : H^{d_H}(\mathfrak{g}, K_\infty, \pi_\infty \otimes \mathcal{M}_\mathbb{C})(\epsilon_\infty) \otimes \bigotimes_p X_{\pi_p} \longrightarrow H^{d_H}_!(\mathcal{S}^G, \tilde{\mathcal{M}}_\mathbb{C}) \xrightarrow{J_{c_\chi,!}} \mathrm{Ind}_{\tilde{H}(\mathbb{A}_f)}^{\tilde{G}(\mathbb{A}_f)} \tilde{\chi}_f^{-1}$$

Let us pick a form in the ε-eigenspace

$$\omega_\varepsilon \in \mathrm{Hom}_{K_\infty}(\Lambda^{d_H}(\mathfrak{g}/\mathfrak{k}), \pi_\infty \otimes \tilde{\mathcal{M}}_\mathbb{C})$$

and let us assume that the restriction of ε to $\pi_0(H(\mathbb{R}))$ is the infinity component of $\tilde{\chi}$. Then we get a new intertwining operator

$$J_{c_{\chi,!}}(\omega_\varepsilon) : \bigotimes_p X_{\pi_p} \longrightarrow \mathrm{Ind}_{H(\mathbb{A}_f)}^{G(\mathbb{A}_f)} \chi_f^{-1}$$

which is defined by

$$J_{c_{\chi,!}}(\omega_\varepsilon)(\psi_f) = J_{c_{\chi,!}} \circ \Phi^{d_H}(\omega_\varepsilon \otimes \psi_f)].$$

Again we have the problem to compute this operator. The situation has changed. The source and the target of $J_{c_{\chi,!}} \circ \Phi^{d_H}$ are restricted tensor products of local representations. A necessary condition for $J_{c_{\chi,!}} \circ \Phi^{d_H} \neq 0$ is that for all primes p the vector space

$$\mathrm{Hom}_{G(\mathbb{Q}_p)}(X_{\pi_p}, \mathrm{Ind}_{H(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \chi_p^{-1}) \neq 0. \tag{I_p}$$

Therefore we assume that this condition is fulfilled. There are cases where the above condition is not always true, see for instance the Hilbert modular surfaces [H-L-R].

If the local condition (I_p) is satisfied for all primes p, then we have interesting special cases where

$$\dim \mathrm{Hom}_{G(\mathbb{Q}_p)}(X_{\pi_p}, \mathrm{Ind}_{H(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\chi_p^{-1}) = 1 \qquad (I_{pp})$$

Let us assume that the representations X_{π_p} are somehow given to us as very concrete representations and (I_{pp}) is true for all primes p. Then it may be possible to select at each prime p a natural generator

$$I^{\mathrm{loc}}_{\chi_p} \in \mathrm{Hom}_{G(\mathbb{Q}_p)}(X_{\pi_p}, \mathrm{Ind}_{H(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\chi_p^{-1}).$$

(This will be discussed in our examples.) We can define

$$I^{\mathrm{loc}}_{\chi_f} = \bigotimes_p I^{\mathrm{loc}}_{\chi_p} \in \mathrm{Hom}_{G(\mathbb{A}_f)}(\bigotimes_p X_{\pi_p}, \mathrm{Ind}_{H(\mathbb{A}_f)}^{G(\mathbb{Q}_p)}\chi_f^{-1})$$

and now we can formulate the following question:

The operator $J_{c_{\chi,!}}(\omega_\epsilon)$ is a multiple of the product of local operators, the problem arises to compute the proportionality factor in

$$J_{c_{\chi,!}}(\omega_\epsilon) = \mathcal{L}(\pi_f,\chi)\cdot I^{\mathrm{loc}}_{\chi_f}.$$

The general idea is that this proportionaly factor is related to a special value of an L-function attached to $\bigotimes_v \pi_v$.

5.2 Rationality and integrality results

We assume that we have fixed a finite level. We assume that the Manin-Drinfeld principle (131) is valid we get a decomposition up to isogeny

$$H^{d_H-1}(\partial(\mathcal{S}^G_{K_f}),\tilde{\mathcal{M}}) \oplus H^{d_H}_{\Sigma,!}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}) \subset H^{d_H}(\mathcal{S}^G_{K_f}, i_{\Sigma,*,!}(\tilde{\mathcal{M}}))). \qquad (132)$$

An absolutely irreducible isotypical submodule $H^{d_H}_{\Sigma,!}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}})_F(\pi_f) \subset H^{d_H}_{\Sigma,!}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}})_F$ can also be viewed as a submodule in $H^{d_H}(\mathcal{S}^G_{K_f}, i_{\Sigma,*,!}(\tilde{\mathcal{M}})_F))$.

We intersect $H^{d_H}_{\Sigma,!}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}})_F(\pi_f)$ with the integral cohomology $H^{d_H}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathcal{O}_F})$ and get the submodule $H^{d_H}_{\Sigma,!}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathcal{O}_F})_{\mathrm{int}}(\pi_f) \subset H^{d_H}_{\Sigma,!}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}})_{\mathcal{O}_F})_{\mathrm{int}}$. The same procedure gives us a submodule

$$H^{d_H}(\mathcal{S}^G_{K_f}, i_{\Sigma,*,!}(\tilde{\mathcal{M}}_{\mathcal{O}_F}))_{\mathrm{int}}(\pi_f) \subset H^{d_H}(\mathcal{S}^G_{K_f}, i_{\Sigma,*,!}(\tilde{\mathcal{M}}_{\mathcal{O}_F}))_{\mathrm{int}} \qquad (133)$$

The map

$$r_{\Sigma,!}: H^{d_H}(\mathcal{S}^G_{K_f}, i_{\Sigma,*,!}(\tilde{\mathcal{M}}))_{\mathcal{O}_F})_{\mathrm{int}}(\pi_f) \to H^{d_H}_{\Sigma,!}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathcal{O}_F})_{\mathrm{int}}(\pi_f) \qquad (134)$$

becomes an isomorphism if we tensorize it by F and hence the image of this map is a submodule of finite index. We define

$$\Delta(\pi_f) = [H^{d_H}_{\Sigma,!}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathcal{O}_F})_{\text{ int}}(\pi_f) : r_{\Sigma,!}(H^{d_H}(\mathcal{S}^G_{K_f}, i_{\Sigma,*,!}(\tilde{\mathcal{M}})_{\mathcal{O}_F})_{\text{ int}}(\pi_f))] \tag{135}$$

We return to our model space and assume that we have multiplicity one (130). Our isotypical subspace in (133) is defined over the field F. We now assume that all the local components X_{π_p} are defined over F, i.e. the local representations are defined over F. Then we get for any embedding $\sigma : F \to \mathbb{C}$ an isomorphism

$$\Phi^H_\sigma(\omega_\epsilon) : (\bigotimes_p X_{\pi_p}) \otimes_\sigma \mathbb{C}) \to H^{d_H}(\mathcal{S}^G, i_{\Sigma,*,!}(\tilde{\mathcal{M}}_F))(\pi_f \times \epsilon_\infty) \otimes_\sigma \mathbb{C} \tag{136}$$

these are isomorphisms over $\mathbb{C}$ between absolutely irreducible $G(\mathbb{A}_f)$ modules which are defined over F. Hence we can find numbers (the periods) $\Omega(\pi_f \times \epsilon, \sigma) \in \mathbb{C}^\times$ such that

$$\frac{\Phi^H_\sigma(\omega_\epsilon)}{\Omega(\pi_f \times \epsilon, \sigma)} : \bigotimes_p X_{\pi_p} \xrightarrow{\sim} H^{d_H}(\mathcal{S}^G, i_{\Sigma,*,!}(\tilde{\mathcal{M}}_F))(\pi_f \times \epsilon_\infty) \tag{137}$$

is an isomorphism over F. We can choose these periods consistent with the action of the Galois group and then it becomes clear that these period arrays are unique up to an element in $F^\times$.

We may also assume that after fixing a level we have an integral structure on our model space, i.e we have lattices $X^{K_p}_{\pi_p,\mathcal{O}_F}$ which are modules under the Hecke algebra. If we invert some primes and pass to $\mathcal{O}_F[\frac{1}{N}]$ then we can arrange our periods in such a way that

$$\frac{\Phi^H_\sigma(\omega_\epsilon)}{\Omega(\pi_f \times \epsilon, \sigma)} : (\bigotimes_p X^{K_p}_{\pi_p,\mathcal{O}_F} \otimes \mathcal{O}_F[\frac{1}{N}]) \xrightarrow{\sim} H^{d_H}(\mathcal{S}^G_{K_f}, i_{\Sigma,*,!}(\tilde{\mathcal{M}}_{\mathcal{O}_F}))_{\text{ int}}(\pi_f \times \epsilon_\infty) \otimes \mathcal{O}_F[\frac{1}{N}]) \tag{138}$$

This pins down the periods up to an element in $\mathcal{O}_F[\frac{1}{N}]^\times$.

We get a formula

$$j((x, \underline{g}_f), r_{\lambda,\mu})(\Phi^{d_H}(\frac{[\omega_\epsilon]}{\Omega(\pi_f, \omega_\epsilon)} \times \psi_f)) = \frac{\mathcal{L}(\pi \otimes \chi, \mu)}{\Omega(\pi_f, \omega_\epsilon)} I^{\text{loc}}_{\chi_f}(\psi_f)(\underline{g}_f) c_\chi \tag{139}$$

By definition of the expression $\Phi^{d_H}(\frac{[\omega_\epsilon]}{\Omega(\pi_f,\omega_\epsilon)} \times \psi_f)$the left hand side is rational if $\psi_f \in \bigotimes_p X_{\pi_p,F}$ and we get a rationality statement for the value of the L-function provided we know that $I^{\text{loc}}_{\chi_f}(\psi_f)(\underline{g}_f)$ is non zero and in F.

We have to choose $\psi_f \in \bigotimes_p X^{K_p}_{\mathcal{O}_F[\frac{1}{N}]}$, and we choose $\underline{g}_f$ such that $K^H_f \underline{g}_f K_f = \underline{g}_f K_f)$). The first choice provides an integral cohomology class in $H^{d_H}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathcal{O}_F[\frac{1}{N}]})(\pi_f)$. But this class is not necessarily the image of an integral class under $r_{\Sigma,!}$ this will be the case if we multiply it with $\Delta(\pi_f)$. Once we have done this we get that

$$j((x,\underline{g}_f), r_{\lambda,\mu})(\Phi^{d_H}(\frac{[\omega_\epsilon]}{\Omega(\pi_f,\omega_\epsilon)} \times \Delta(\pi_f)\psi_f)) = \Delta(\pi_f)\frac{\mathcal{L}(\pi\otimes\chi,\mu)}{\Omega(\pi_f,\omega_\epsilon)} I^{\mathrm{loc}}_{\chi_f}(\psi_f)(\underline{g}_f)c_\chi \tag{140}$$

is a number in $\mathcal{O}_F[\frac{1}{N}]$.

Then we have to optimize the choice of $\underline{g}_f$, this means that we have to keep the numerator of $I^{\mathrm{loc}}_{\chi_f}(\psi_f)(\underline{g}_f)$ small. Then we get an integrality result for the L-value.

We discuss this in the next example.

5.3 The special case Gl_2

We consider the special case $G = Gl_2/\mathbb{Q}$. In this case we have very nice model spaces, namely the Whittaker model, our map Φ is given by the Fourier expansion and the theory of the Kirillow-model gives us a canonical choice for the local intertwining operators. Let $\mathcal{M}_n$ be the $\mathbb{Q}$-vector space of homogeneous polynomials $P(X,Y)$ of degree n and with coefficients in $\mathbb{Q}$. An element $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts by $(\gamma P)(X,Y) = P(aX+cY, bX+dY)$. Sometimes we twist this action by a power of the determinant $\det(\gamma)^r$, then the module is denoted by $\mathcal{M}_n[r]$. From now on $\mathcal{M}$ will be one of the modules $\mathcal{M}_n[r]$, i.e. our highest weight will be the pair $\lambda = (n,r)$. The subgroup which provides the modular symbols will be our standard maximal torus T and the $r_{\lambda,\mu}$ will be the projections to $X^{n-\mu}Y^\mu$.

We assume that a K_f is been chosen. Let us assume that we selected a K_f stable lattice $\tilde{\mathcal{M}}_\mathbb{Z}$ and we consider the exact sequence of modules under the Hecke algebra

$$\to H^0(\partial\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\mathbb{Z}) \to H^1_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\mathbb{Z}) \to H^1(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\mathbb{Z}) \to H^1(\partial\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\mathbb{Z}).$$

We can tensorize our sequence by $\mathbb{Q}$, and then in this case the Manin-Drinfeld principle is valid

$$H^1_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}) = H^1_{\mathrm{Eis}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})) \oplus H^1_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}).$$

The first summand can be described in terms of induced representations

$$H^0(\partial\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}) \otimes \overline{\mathbb{Q}} = \bigoplus_{\chi:\ \mathrm{type}(\chi)=\lambda} \left(\mathrm{Ind}_{\tilde{B}(\mathbb{A})}^{\tilde{G}(\mathbb{A})}\tilde{\chi}\right)^{K_f}$$

where λ is the highest weight of our module, where χ runs over the Hecke characters with some restriction conditions dictated by K_f, and where $\tilde{\chi}$ is the character on $\pi_0(T(\mathbb{R})) \times T(\mathbb{A}_f)$ attached to it (see $[GL_2]$, ...).

The module $H^1_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})$ is semisimple, if we tensorize by $\overline{\mathbb{Q}}$, then we get an isotypical decomposition

$$H^1_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\overline{\mathbb{Q}}}) = \bigoplus_{\pi_f} H^1(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\overline{\mathbb{Q}}})(\pi_f)$$

where π_f is an isomorphism class of a (finite dimensional) $\overline{\mathbb{Q}}$-vector space with an irreducible action of $\mathcal{H}$ on it. Since we fixed the level we have only finitely many of them. The Galois action on $\overline{\mathbb{Q}}$ induces a permutation of the π_f, if $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, then we can define the isomorphism class π_f^σ. It is clear that we have a finite extension $\mathbb{Q}(\pi_f) \subset \overline{\mathbb{Q}}$ such that $\pi_f^\sigma = \pi_f$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\pi_f))$. The field $\mathbb{Q}(\pi_f)$ is the field of definition of the representation π_f.

For almost all primes p we have $K_p = Gl_2(\mathbb{Z}_p)$ and the local Hecke algebra $\mathcal{H}(G(\mathbb{Q}_p)//K_p) = \mathbb{Q}[T_p, Z_p, Z_p^{-1}]$ and π_p is simply determined by the eigenvalues ω_p, ω_p of T_p and $T_{p,p}$. on the one dimensional vector space of K_p invariant vectors. Then $\mathbb{Q}(\pi_p) = \mathbb{Q}[\omega_p, \omega_p]$.

5.3.1 Input from the theory of automorphic forms 2

The theory of automorphic forms for Gl_2 provides the following extra informations:

(i) The multiplicity of $H^1(\mathcal{S}^G_{K_f}, \mathcal{M}_{\overline{\mathbb{Q}}})(\pi_f)$ is two. (Multiplicity one.)

(ii) If we know the numbers $\omega_p(\pi_f)$, $\omega_p(\pi_f)$ for almost all unramified prime, then π_f is uniquely determined. (Strong multiplicity one.)

(iii) On $H^1(\mathcal{S}^G_{K_f}, \mathcal{M}_{\overline{\mathbb{Q}}})(\pi_f)$ we have an action of $\pi_0(G_\infty)$. This group is the quotient of

$$T(\mathbb{R}) \cap K_\infty \xrightarrow{\sim} \pi_0(T_\infty) \xrightarrow{\sim} \left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \right\},$$

by the subgroup generated by $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

Under the action of $\pi_0(G_\infty)$ an eigenspace decomposes into two pieces

$$H^1_!(\mathcal{S}^G_{K_f}, \mathcal{M}_{\overline{\mathbb{Q}}})(\pi_f) = \bigoplus_{\varepsilon :: \pi_0(G_\infty) \to \{\pm 1\}} H^1_!(\mathcal{S}^G_{K_f}, \mathcal{M}_{\overline{\mathbb{Q}}})(\varepsilon, \pi_f).$$

Both pieces have multiplicity equal to one.

Of course we can find a finite extension $F/\mathbb{Q}$ such that we have this decomposition already over F. If we also invoke the Manin-Drinfeld decomposition, we find

$$H^1_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_F) = H^1_{\mathrm{Eis}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_F) \oplus \bigoplus_{\pi_f, \varepsilon} H^1_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_F)(\varepsilon, \pi_f).$$

Now we consider the ring $\mathcal{O}_F \subset F$. For any cohomology group we define the image

$$\mathrm{Im}(H^\bullet_?(?, \tilde{\mathcal{M}}_{\mathcal{O}_F}) \longrightarrow H^\bullet_?(?, \tilde{\mathcal{M}}_F)) =: H^\bullet_?(?, \tilde{\mathcal{M}}_{\mathcal{O}_F})_{\mathrm{int}}$$

it is also simply this cohomology divided by the torsion. Then we get a decomposition up to finite quotient isogeny

$$H^1_{\mathrm{Eis}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathcal{O}_F})_{\mathrm{int}} \oplus H^1_!(\mathcal{S}^G_K, \tilde{\mathcal{M}}_{\mathcal{O}_F})_{\mathrm{int}}$$

Then the submodules

$$H^1_{!,\varepsilon}(\mathcal{S}^G_{K_f}, \mathcal{M}_{\mathcal{O}})(\pi_f)_{\mathrm{int}}$$

are the isotypical summands in the cohomology $H^1_!(\mathcal{S}^G_{K_f}, \mathcal{M}_{\mathcal{O}})_{\text{int}}$.

We may also define isotypical quotients. They are obtained if we divide $H^1_!(\mathcal{S}^G_{K_f}, \mathcal{M}_{\mathcal{O}})_{\text{int}}$ by the complementary summand to $H^1_!(\mathcal{S}^G_{K_f}, \mathcal{M}_{\mathcal{O}})_{\text{int}}$, and we denote these quotients by

$$H^1_!(\mathcal{S}^G_{K_f}, \mathcal{M}_{\mathcal{O}})[\varepsilon, \pi_f]_{\text{int}}.$$

We have a natural inclusion

$$H^1_!(\mathcal{S}^G_{K_f}, \mathcal{M}_{\mathcal{O}})(\varepsilon, \pi_f)_{\text{int}} \longrightarrow H^1_!(\mathcal{S}^G_{K_f}, \mathcal{M}_{\mathcal{O}})[\varepsilon, \pi_f]_{\text{int}},$$

and the quotient is a finite module.

5.3.2 The Whittaker model

We assume that π_f is a representation which occurs in the decomposition of $H^1_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_F)$. Let π_∞ be the discrete series representation which has nontrivial cohomology with coefficients in $\mathcal{M}_{\mathbb{C}}$. Now we choose an additive character $\tau : \mathbb{A}_{\mathbb{Q}}/\mathbb{Q} \to S^1$. It may be the best to choose the standard character which is trivial on $\hat{\mathbb{Z}} \subset \mathbb{A}_f$ and at infinity is $x \mapsto e^{2\pi i x}$.

Our representation $\pi_\infty \otimes \pi_f$ (which is known as a module of $\mathbb{C}$-vector spaces) has a unique Whittaker model

$$\mathcal{W}(\pi_\infty \otimes \pi_f, \tau)_{\mathbb{C}}.$$

This is the unique subspace in

$$\mathcal{W}(\tau)_{\mathbb{C}} = \left\{ f : G(\mathbb{A})/K_f \to \mathbb{C} \mid f\left(\begin{pmatrix} 1 & \underline{u} \\ 0 & 1 \end{pmatrix} \underline{g}\right) = \tau(\underline{u}) f(\underline{g}) \right\},$$

which is invariant unter $GL_2(\mathbb{R}) \times \mathcal{H}$ and isomorphic to $\pi_\infty \otimes \pi_f$. The Fourier expansion provides an inclusion

$$\mathcal{W}(\pi_\infty \otimes \pi_f, \tau) \xrightarrow{\mathcal{F}} \mathcal{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A}))$$

$$\mathcal{F}(f)(\underline{g}) = \sum_{t \in \mathbb{Q}^\times} f\left(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \underline{g}\right),$$

where $\mathcal{A}_0$ means the space of cusp forms. This gives us an isomorphism

$$H^1(\mathfrak{g}, K_\infty, \mathcal{W}(\pi_\infty, \tau) \otimes \mathcal{M}_{\mathbb{C}}) \otimes \mathcal{W}(\pi_f, \tau) \xrightarrow{\sim} H^1_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathbb{C}})(\pi_f).$$

We have

$$\begin{aligned} H^1(\mathfrak{g}, K_\infty, \mathcal{W}(\pi_\infty, \tau) \otimes \mathcal{M}_{\mathbb{C}}) &= \mathrm{Hom}_{K_\infty}(\Lambda^1(\mathfrak{g}/\mathfrak{k}), \mathcal{W}(\pi_\infty, \tau) \otimes \tilde{\mathcal{M}}_{\mathbb{C}}) \\ &= \mathbb{C}\,\omega_n + \mathbb{C}\,\omega_{-n} \end{aligned}$$

where I will pin down these two generators later. We assume that $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \omega_n = \omega_{-n}$. Then

$$\begin{aligned} \omega_+ &= \tfrac{1}{2}(\omega_n + \omega_{-n}) \\ \omega_- &= \tfrac{1}{2}(\omega_n - \omega_{-n}) \end{aligned}$$

form generators of the spaces

$$\mathrm{Hom}_{K_\infty}(\Lambda^1(\mathfrak{g}/\mathfrak{k}), \mathcal{W}(\pi_\infty,\tau)\otimes\tilde{\mathcal{M}}_{\mathbb{C}})_{\pm}.$$

Now our general procedure outlined in 2.1.1 provides intertwining operators

$$\mathcal{F}_1^1(\omega_\varepsilon)\ :\ \bigotimes_p \mathcal{W}(\pi_p,\tau) \to H^1(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathbb{C}})_\varepsilon(\varepsilon,\pi_f) \tag{141}$$

5.3.3 The integral model for $\mathcal{W}(\pi_p,\tau)$.

Our representation π_p has a field of definition $\mathbb{Q}(\pi_p)$ which is a finite extension of $\mathbb{Q}$. To get this field of definition we look at the space of $\overline{\mathbb{Q}}$-valued functions

$$\mathcal{W}_{\overline{\mathbb{Q}}}(\tau)=\left\{ f: G(\mathbb{Q}_p)\to\overline{\mathbb{Q}} \mid f\left(\begin{pmatrix}1 & u_p\\ 0 & 1\end{pmatrix} g_r\right)=\tau(u_p)f(g_p)\right\}.$$

On this space I defined an action of the Galois group ([Ha-Mod]) as follows.

$$f^\sigma(g)=\left(f\left(\begin{pmatrix}t_\sigma^{-1} & 0\\ 0 & 1\end{pmatrix} g\right)\right)^\sigma,$$

and $\mathbb{Q}(\pi_p)$ is the number field for which $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\pi_p))$ is the stabilizer of $\mathcal{W}(\pi_p,\tau)$.

The space $\mathcal{W}_{\overline{\mathbb{Q}}}(\pi_p,\tau)$is finite dimensional over $\overline{\mathbb{Q}}$, and the space of functions which are invariant under $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\pi_p))$ is a $\mathbb{Q}(\pi_p)$ vector space $\mathcal{W}(\pi_p,\tau)$ on which $\mathcal{H}(G(\mathbb{Q}_p)//K_p)$ acts absolutely irreducible. We have $\mathcal{W}(\pi_p,\tau)\otimes_{\mathbb{Q}(\pi_p)}\overline{\mathbb{Q}}=\mathcal{W}_{\overline{\mathbb{Q}}}(\pi_p,\tau)$.

Of course $\mathbb{Q}(\pi_p)\subset\mathbb{Q}(\pi_f)$, and we define a subring $\mathcal{O}(\pi_f)\subset\mathbb{Q}(\pi_f)$. This is the ring of integers in $\mathbb{Q}(\pi_f)$ but we invert the primes which occur in the conductor of π_f, i.e. all the primes where π_p is ramified. Let us denote the product of these primes by N.

We have the action of $\mathcal{H}^{coh}_{\mathbb{Z}}$ (See 1.2.1.(ii)) on the cohomolgy and hence we get an action of the algebra $\mathcal{H}(G(\mathbb{Q}_p)//K_p)_{\mathbb{Z}}$ on $\mathcal{W}(\pi_p,\tau)$ and this gives us a finitely generated $\mathrm{O}(\pi_p)$- module of endomorphisms. Hence we can find invariant lattices $\mathcal{W}(\pi_p,\tau)_{\mathrm{O}(\pi_p)}$. If we invert a few more primes then we can achieve that two such choices just differ by an element $a\in\mathrm{O}(\pi_p)$. We assume that such a choice of lattices has been made at all primes p. If we are in the unramified case then we will make a very particular choice later. We put $\mathcal{W}_{\mathcal{O}(\pi_f)}(\pi_f,\tau)=\bigotimes_p\mathcal{W}_{\mathcal{O}(\pi_f)}(\pi_p,\tau)$ (See 2.2.7).

If we take an element $\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ then it conjugates the representation π_p into π_p^σ and we get a map

$$\begin{array}{ccc}\mathcal{W}(\pi_p,\tau) & \xrightarrow{\tilde\sigma} & \mathcal{W}(\pi_p^\sigma,\tau)\\ \\ f & \mapsto & f^\sigma\end{array}$$

This map is a semilinear isomorphism.

5.3.4 The periods

Now we have constructed the intertwining operator

$$\mathcal{F}_1^{(1)}(\omega_\varepsilon):\bigotimes_p \mathcal{W}_{\mathcal{O}(\pi_f)}(\pi_p,\tau)\otimes\mathbb{C}\longrightarrow H^1(\mathcal{S}^G,\mathcal{M}_{\mathcal{O}})(\varepsilon,\pi_f)\otimes\mathbb{C},$$

and we can define a complex number $\Omega_\varepsilon(\pi_f)$ such that

$$\Omega_\varepsilon(\pi_f)^{-1}\cdot\mathcal{F}_1^{(1)}(\omega_\varepsilon):\bigotimes_p \mathcal{W}_{\mathcal{O}(\pi_f)}(\pi_f,\tau)\xrightarrow{\sim} H^1(\mathcal{S}^G,\mathcal{M}_{\mathcal{O}})(\varepsilon,\pi_f) \tag{142}$$

provided $\mathcal{O}(\pi_f)$ has class number one. Then this number is called a period and it is unique up to an element in $\mathcal{O}(\pi_f)^\times$. We may also look at the conjugates of $\ldots\pi_f^\sigma\ldots$ of π_f. We can choose these periods consistently (see [Ha-Mod]) and hence we even get a period vector

$$\Omega_\varepsilon(\Pi_f)^{-1}=(\ldots\Omega_\varepsilon(\pi_f^\sigma)^{-1}\ldots)_{\sigma:\mathbb{Q}(\pi_f)\to\mathbb{C}}.$$

5.3.5 The modular symbols for Gl_2

We start from $GL_2/\mathbb{Q}$ and a coefficient system $\mathcal{M}_n[r]$. Now we consider the modular symbols arising from the subgroup

$$H=T=\left\{\begin{pmatrix} t_1 & 0\\ 0 & t_2\end{pmatrix}\right\}$$

Our module $\mathcal{M}_n[r]_{\mathbb{Z}}=\bigoplus_{\nu=0}^{n}\mathbb{Z}X^\nu Y^{n-\nu}$ decomposes into eigenspaces $\mathbb{Z}X^\nu Y^{n-\nu}$. Hence we get

$$H^0(\mathcal{S}^T_{K_f^T},\tilde{\mathcal{M}}_{\mathcal{O}})=\bigoplus_{\nu=0}^{n}\bigoplus_{\chi:\text{type}(\chi)=\gamma_\nu}\mathcal{O}c_\chi,$$

and since the Manin-Drinfeld principle is valid we get a canonical decomposition

$$H^1_c(\mathcal{S}^G,\tilde{\mathcal{M}}_{\mathbb{Q}})=H^1_{\mathrm{Eis}}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathbb{Q}})\oplus H^1_!(\mathcal{S}^G,\tilde{\mathcal{M}}_{\mathbb{Q}}),$$

and this means that we have a canonical section

$$H^1_!(\mathcal{S}^G,\tilde{\mathcal{M}}_{\mathbb{Q}})\longrightarrow H^1_c(\mathcal{S}^G,\tilde{\mathcal{M}}_{\mathbb{Q}}),$$

and hence we can define the intertwining operator

$$J_{c_{\chi,!}}:H^1_!(\mathcal{S}^G,\tilde{\mathcal{M}}_F^\vee)\longrightarrow \mathrm{Ind}_{H(\mathbb{A}_f)}^{G(\mathbb{A}_f)}\ \chi_f^{-1}.$$

Let us assume that we have an isotopical component $H^1_!(\mathcal{S}^G,\tilde{\mathcal{M}}^\vee_{\mathbb{Q}(\pi_f)})(\pi_f)$, then we can consider the composition

$$J_{c_{\chi,!}}\circ\Omega_\varepsilon(\pi_f)^{-1}\mathcal{F}_1^{(1)}(\omega_\varepsilon):\bigotimes_p\ \mathcal{W}_{\mathcal{O}(\pi_f)}(\pi_p,\tau)\longrightarrow \mathrm{Ind}_{\tilde{H}(A_f)}^{\tilde{G}(\mathbb{A}_f)}\ \tilde{\chi}_f^{-1}.$$

5.3.6 The local intertwining operators

We need to investigate the space of intertwining operators

$$\mathrm{Hom}_{G(\mathbb{Q}_p)}(\mathcal{W}(\pi_p,\tau_p)\ ,\ \mathrm{Ind}_{T(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\chi_p^{-1}).$$

Of course we need to assume that the central character $\omega(\pi_p)$ is equal to the character χ_p restricted to the centre. We introduce the subtorus

$$T_1(\mathbb{Q}_p)=\left\{\begin{pmatrix} t & 0\\ 0 & 1\end{pmatrix}\right\}$$

of $T(\mathbb{Q}_p)$ and we restrict χ_p to this subgroup and call this restriction $\chi_p^{(1)}$. For $t\in\mathbb{Q}_p^\times$ we denote by $h(t)$ the matrix $h(t)=\begin{pmatrix} t & 0\\ 0 & 1\end{pmatrix}$.

Now it is easy to write down an intertwining operator, namely

$$I_p(f)(g)=\int_{T_1(\mathbb{Q}_p)} f(h(t)g)\chi_p^{(1)}(h(t))d^\times t,$$

where of course $d^\times t$ is an invariant measure on $T_1(\mathbb{Q}_p)$. Of course we have to discuss the convergence of this integral.

Before doing that we convince ourselves that this is the only intertwing operator operator up to a scalar factor, the condition (I_{pp}) is valid. If we apply Frobenius reciprocity we see that

$$\mathrm{Hom}_{G(\mathbb{Q}_p)}(\mathcal{W}(\pi_p,\tau_p)\ ,\ \mathrm{Ind}_{T(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\chi_p^{-1})=\mathrm{Hom}_{T(\mathbb{Q}_p)}(\mathcal{W}(\pi_p,\tau_p)\ ,\ \chi_p^{-1})$$

The restriction of the functions in $\mathcal{W}(\pi_p,\tau_p)$ to $T_1(\mathbb{Q}_p)$ is injective (See [Go]) and the image of the restriction map is called the*Kirillov model* $\mathcal{K}(\pi_p,\tau_p)$. On this Kirillov model the torus $T_1(\mathbb{Q}_p)$ acts by translation. It is known that the Kirillov model contains the space $\mathcal{C}_c(\mathbb{Q}_p^\times)$ of Schwartz functions, this are the locally constant functions with compact support on $\mathbb{Q}_p^\times$. This space of Schwartz functions has at most codimension 2 and it is of course invariant under $T_1(\mathbb{Q}_p)$. Hence it is clear that the restriction of our intertwning operator to the space of Schwartz functions is (up to a scalar factor) given by the integral. If our representation is supercuspidal then $\mathcal{K}(\pi_p,\tau_p)=\mathcal{C}_c(\mathbb{Q}_p^\times)$ and we we get existence and uniqueness up to a scalar of the intertwining operator very easily. In the general case we have to show that it extends and for this we have to invoke the theory of local L-functions. If we introduce a parameter $s\in\mathbb{C}$, then the integral

$$\int_{T_1(\mathbb{Q}_p)} f(h(t)g)\chi^{(1)}(h(t))\cdot|t|^{s-1}d^\times t$$

is convergent for $\Re(s)>>0$ and can be analytically continued to a meromorphic function in the entire plane with at most two poles (see [J-L], [Go]). In [J-L] the authors attach a local L-function $L(\pi_p\otimes\chi_p^{(1)},s)$ to $\pi_p\otimes\chi_p^{(1)}$ which has exactly poles for those values of s where the integral does not converge and then

$$I^{loc}(\pi_p,\chi_p^{-1},s)f(g)=L(\pi_p\otimes\chi_p^{(1)},s)^{-1}\int_{T_1(\mathbb{Q}_p)} f(h(t)g)\chi^{(1)}(h(t))\cdot|t|^{s-1}d^\times t$$

provides an intertwining operator

$$I^{loc}(\pi_p, \chi_p^{-1}, s) : \mathcal{W}(\pi_p, \tau_p)_{\mathbb{C}} \to \mathrm{Ind}_{T(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \chi_p^{-1} |\ |^{1-s}$$

which is everywhere holomorphic and non zero. If we evaluate at $s = 1$ we get a generator

$$I^{loc}(\pi_p, \chi_p^{-1}) : \mathcal{W}(\pi_p, \tau_p)_{\mathbb{C}} \to \mathrm{Ind}_{T(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \chi_p^{-1}.$$

The arithmetic properties of this operator will be discussed in the next section.

In defining the local L-function we have to be a little bit careful, we will give a precise formula further expression for the unramified case further down. Our local L-factor will differ by a shift by 1/2 in the variable s from the L-factor in [J-L] etc. Will will come back to this point later.

5.3.7 The unramified case

To see what is going on we consider the special case that $\pi_p = \pi_p(\lambda_p)$ is an unramified principal series representation. This means that

$$\lambda_p : \begin{pmatrix} t_1 & * \\ 0 & t_2 \end{pmatrix} \longrightarrow \lambda_{p,1}(t_1) \cdot \lambda_{p,2}(t_2)$$

is an unramified character and $\pi_p(\omega_p)$ is the representation obtained by unitary induction from ω_p, i.e. we consider the space of functions

$$\mathrm{Ind}_{\mathrm{un}}(\lambda_p) = \left\{ f : G(\mathbb{Q}_p) \to \mathbb{C} \mid f\left(\begin{pmatrix} t_1 & * \\ 0 & t_2 \end{pmatrix} g \right) = \lambda_{p,1}(t_1)\lambda_{p,2}(t_2) \cdot \left| \frac{t_1}{t_2} \right|_p^{\frac{1}{2}} f(g) \right\},$$

where the functions are locally constant. In this case it is not difficult to compute the intertwining operator to the Whittaker model

$$R_p : \mathrm{Ind}_{\mathrm{un}}(\lambda_p) \longrightarrow \mathcal{W}(\pi_p(\lambda_p), \tau_p),$$

it is given by

$$R_p(f)(g) = \int_{U(\mathbb{Q}_p)} f(wug)\overline{\tau_p(u)} du,$$

where $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Again we have a problem of convergence. To solve this we simply compute the integral. Let us also assume that the additive character τ_p is trivial on

$$\mathbb{Z}_p = U(\mathbb{Z}_p) = \left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \;\middle|\; u \in \mathbb{Z}_p \right\},$$

and nontrivial on $\frac{1}{p}\mathbb{Z}_p$. We know that $f(wug)$ becomes constant in the variable u if $u \in p^m\mathbb{Z}$ with m large. Hence we have to compute

$$\sum_{\nu=1}^{\infty} \int_{p^{-\nu+m}\mathbb{Z}_p \setminus p^{-\nu+1+m}\mathbb{Z}_p} f(wug)\overline{\tau_p(u)} du,$$

and for convergence we have to discuss what happens if $\nu \to \infty$. We write $u = p^{-n}\varepsilon$ with $n >> 0$ and $\varepsilon \in \mathbb{Z}_p^\times$. Then $wu = wuw^{-1}w$ and

$$wuw^{-1} = \begin{pmatrix} 1 & 0 \\ -p^{-n}\varepsilon & 1 \end{pmatrix} = \begin{pmatrix} p^n\varepsilon^{-1} & -1 \\ 0 & p^{-n}\varepsilon \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & p^n\varepsilon^{-1} \end{pmatrix}.$$

Then

$$f(wug) \quad = \quad f\left(\begin{pmatrix} p^n\varepsilon^{-1} & -1 \\ 0 & p^{-n}\varepsilon \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & p^n\varepsilon^{-1} \end{pmatrix} wg\right)$$

$$\lambda_{p,1}(p)^n\lambda_{p,2}(p)^{-n}p^{-n}f\left(\begin{pmatrix} 0 & -1 \\ 1 & p^n\varepsilon^{-1} \end{pmatrix} wg\right),$$

and $f\left(\begin{pmatrix} 0 & -1 \\ 1 & p^n\varepsilon^{-1} \end{pmatrix} wg\right) = f(g)$ if $n >> 0$, especially it will not depend on ε. This means that for $n >> 0$

$$\int_{p^{-n}\mathbb{Z}_p\setminus p^{1-n}\mathbb{Z}_p} f(wug)\overline{\tau(u)}du = \text{const}\int_{p^{-n}\mathbb{Z}_p\setminus p^{+1-n}\mathbb{Z}_p} \overline{\tau(u)} = 0,$$

and hence our integral is actually a finite sum.

Let us consider the special case where $f = f_{\lambda_p} \in Iun(\lambda)$ is the spherical function which takes the value 1 at the identity. This means that for $g = b \cdot k$ with $k \in Gl_2(\mathbb{Z}_p)$

$$f_{\lambda_p}\left(\begin{pmatrix} t_1 & * \\ 0 & t_2 \end{pmatrix} k\right) = \lambda_{p,1}(t_1)\lambda_{p,2}(t_2)\cdot\left|\frac{t_1}{t_2}\right|^{\frac{1}{2}},$$

and we keep our assumption on τ_p. Then our computation yields

$$\begin{aligned} R_p(f_{\lambda_p})(e) \quad &= \quad \textstyle\int_{U(\mathbb{Q}_p)} f_{\lambda_p}(wu) = \\ & \textstyle\int_{U(\mathbb{Z}_p)} f_{\lambda_p}(wu)\overline{\tau(u)}du + \displaystyle\sum_{\nu=1}^{\infty}\int_{p^{-\nu}\mathbb{Z}_p\setminus p^{-\nu+1}\mathbb{Z}_p} f_{\lambda_p}(wu)\overline{\tau(u)}du = \\ & 1 + \textstyle\int_{p^{-1}\mathbb{Z}_p\setminus\mathbb{Z}_p} f_{\lambda_p}(wu)\overline{\tau(u)}du = 1 - \frac{\lambda_{p,1}(p)}{\lambda_{p,2}(p)}p^{-1}, \end{aligned}$$

because all the terms with $\nu \geq 2$ vanish since $\tau_p \mid \frac{1}{p}\mathbb{Z}_p \neq 1$.

The same kind of computation gives us also the value

$$R_p(f_{\lambda_p})\begin{pmatrix} p^k & 0 \\ 0 & 1 \end{pmatrix}.$$

It is zero for $k < 0$ and for $k \geq 0$ we get

$$p^{-\frac{k}{2}}\left(\lambda_{p,2}(p)^k + \left(1-\tfrac{1}{p}\right)\lambda_{p,2}(p)^{k-1}\lambda_{p,1}(p)\ldots + \left(1-\tfrac{1}{p}\right)\lambda_{p,1}(p)^k - \tfrac{\lambda_{p,1}(p)^{k+1}}{\lambda_{p,2}(p)}p^{-1}\right) =$$

$$p^{-\frac{k}{2}}\left(\lambda_{p,2}(p)^k + \lambda_{p,2}(p)^{k-1}\lambda_{p,1}(p) + \ldots + \lambda_{p,1}(p)^k\right)\left(1 - \tfrac{\lambda_{p,1}}{\lambda_{p,2}}(p)p^{-1}\right).$$

We put

$$\frac{1}{1-\frac{\lambda_{p,1}}{\lambda_{p,2}}(p)p^{-1}} \quad R_p(f_{\lambda_p}) = \Psi_{\lambda_p}.$$

(If $\frac{\lambda_{p,1}}{\lambda_{p,2}}(p)p^{-1} = 1$ then the induced representation is not irreducible.) This means that Ψ_{λ_p} is the spherical Whittaker function which has value 1 at the identity element.

Now we can discuss the integral Whittaker model at an unramified place p. In this case we assume that $K_p = Gl_2(\mathbb{Z}_p)$ and we put $\mathcal{W}(\pi_p)_{\mathrm{O}(\pi_p)} = \mathrm{O}(\pi_p)\Psi_{\lambda_p}$, the module is of rank one.

We return to our intertwining operator from the Whittaker model to the induced representation $\mathrm{Ind}_{H(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \chi_p^{-1}$. We assume that $\chi_p^{(1)}$ is also unramified, we normalize $d^\times t(\mathbb{Z}_p^\times) = 1$. We want to compute the value of the local intertwining operator on Ψ_{λ_p}. Then

$$\begin{aligned}
&\int_{\mathbb{Q}_p^\times} \Psi_{\lambda_p}\left(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\right) \chi_p^{(1)}(t)|t|^{s-1}d^\times t = \\
&\sum_{k=0}^{\infty} \Psi_{\lambda_p}\left(\begin{pmatrix} p^k & 0 \\ 0 & 1 \end{pmatrix}\right) \chi_p^{(1)}(p)^k p^{k(1-s)} = \\
&\sum_{k=0}^{\infty} p^{\frac{k}{2}} \left(\lambda_{p,2}(p)^k + \lambda_{p,2}(p)^{k-1}\lambda_{p,1}(p) + \ldots + \lambda_{p,1}(p)^k\right) \chi_p^{(1)}(p)^k p^{-ks} = \\
&\frac{1}{\left(1 - p^{\frac{1}{2}}\lambda_{p,2}(p)\chi_p^{(1)}(p)p^{-s}\right)\left(1 - p^{\frac{1}{2}}\lambda_{p,1}(p)\chi_p^{(1)}(p)p^{-s}\right)}
\end{aligned}$$

Now we work with the module $\mathcal{M}_n$, i.e. we do not make a twist by the determinant. If we look at the definition of the Hecke operators on the integral cohomology (See [Heck]) then we notice that in this case we do not need a modification of the operators $T_p, T_{p,p}$ to get them acting on the integral cohomology. We conclude that the numbers

$$p^{1/2}\lambda_{p,1}(p) = \alpha_p, p^{1/2}\lambda_{p,2}(p) = \beta_p$$

are algebraic integers. Since the central character is of type $x \mapsto x^n$ we conclude $\alpha_p\beta_p$ has absolute value p^{n+1} and of course the Weil conjectures imply $|\alpha_p| = |\beta_p| = p^{(n+1)/2}$. The numbers $\alpha_p + \beta_p, \alpha_p\beta_p$ generate the field $\mathbb{Q}(\pi_p)$ and the number $L(\pi \otimes \chi^{(1)}, 1) \in \mathbb{Q}(\pi_p, \chi^{(1)})$. From this we conclude that the local intertwining operator $I^{loc}(\pi_p, \chi_p^{-1})$ is defined over $\mathbb{Q}(\pi_p, \chi^{(1)})$ we get

$$I^{loc}(\pi_p, \chi_p^{-1}) : \mathcal{W}(\pi_p, \tau_p)_{\mathbb{Q}(\pi_p,\chi^{(1)})} \to (\mathrm{Ind}_{T(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\chi_p^{-1})_{\mathbb{Q}(\pi_p,\chi^{(1)})}$$

In fact it transforms the spherical function Ψ_{λ_p} into the spherical function in the induced module which also takes value one at the identity element.

A similar consideration shows that also at the finitely many remaining places we can define a local intertwining operator $I^{loc}(\pi_p, \chi_p^{-1})$ over $\mathbb{Q}(\pi_p, \chi^{(1)})$. Here we have to look up the table for the local L factors in [Go]. We define the so called local intertwing operator as restricted tensor product

$$I^{loc}(\pi_f, \chi_f^{-1}) = \bigotimes_p I^{loc}(\pi_p, \chi_p^{-1})$$

These local operators are almost compatible with the action of the Galois action. We observe for $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ we have the transformation rule $L(\pi_p \otimes \chi_p^{(1)}, 1)^\sigma = L(\pi_p^\sigma \otimes (\chi_p^{(1)})^\sigma, 1)^\sigma$. But the integral is not quite compatible with the action of the Galois group. We have the following commutative diagram: For $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

$$\begin{array}{rccc}
I^{loc}(\pi_f, \chi_f^{-1}) : & \mathcal{W}(\pi_f, \tau) & \longrightarrow & \mathrm{Ind}_{H(\mathbb{A}_f)}^{G(\mathbb{A}_f)} \chi_f^{-1} \\
 & \downarrow \sigma & & \downarrow \sigma \\
I^{loc}(\pi_f, \chi_f^{-1})\chi_f^{(1)}(\underline{t}_\sigma) : & \mathcal{W}(\pi_f^\sigma, \tau) & \longrightarrow & \mathrm{Ind}_{H(\mathbb{A}_f)}^{G(\mathbb{A}_f)} (\chi_f^\sigma)^{-1}
\end{array} .$$

We discuss the local case where π_p is unramified and $\chi_p^{(1)}$ is ramified and its conductor is $f_p > 0$ Let $T_1(\mathbb{Z}_p)(p^{f_p}) \subset T_1(\mathbb{Z}_p)$ be the subgroup of units $\equiv 1 \mod p^{f_p}$, then character $\chi_p^{(1)}$ is trivial on the on this subgroup but not on $T_1(\mathbb{Z}_p)(p^{f_p-1})$ We normalize $d^\times t_p$ to give $T_1(\mathbb{Z}_p)(p^{f_p})$ the volume one. Again an intertwining operator is given by the integral

$$f \mapsto \int_{T_1(\mathbb{Q}_p)} f(h(t_p)g)\chi^{(1)}(h(t_p)) \cdot d^\times t_p = I_{\chi_p} f(e)$$

We have to optimize our choices (See 5.2). For our function f we have to take the spherical Whittaker function Ψ_{λ_p}. For g_p we choose an element

$$g_p = (\begin{pmatrix} 1 & \frac{1}{p^n} \\ 1 & 1 \end{pmatrix}).$$

We want $T_1(\mathbb{Z}_p)(p^{f_p})g_pK_p = g_pK_p$ a simple calculation says that this is the case if and only if

$$\begin{pmatrix} 1 & -\frac{1}{p^n} \\ 1 & 1 \end{pmatrix} h(t_p) \begin{pmatrix} 1 & \frac{1}{p^n} \\ 1 & 1 \end{pmatrix} h(t_p) \in K_p$$

and this says

$$\begin{pmatrix} 1 & (t_p-1)\frac{1}{p^n} \\ 1 & 1 \end{pmatrix} \in K_p.$$

Since $t_p \equiv 1 \mod p^{f_p}$ we see that this is the case if and only if $n \leq f_p$. Let us choose such an n, i.e. a g_p.

To compute the intertwining operator we have to evaluate at e (Frobenius reciprocity) and we observe

$$I_{\chi_p}(\Psi_{\lambda_p})(\begin{pmatrix} 1 & \frac{1}{p^n} \\ 1 & 1 \end{pmatrix}) = I_{\chi_p}(\begin{pmatrix} 1 & \frac{1}{p^n} \\ 1 & 1 \end{pmatrix} \Psi_{\lambda_p})(e)$$

By definition this operator is given

$$I_{\chi_p}(\begin{pmatrix} 1 & \frac{1}{p^n} \\ 1 & 1 \end{pmatrix} \Psi_{\lambda_p})(e) = \int_{T_1(\mathbb{Q}_p)} \Psi_{\lambda_p}(\begin{pmatrix} 1 & \frac{t_p}{p^n} \\ 1 & 1 \end{pmatrix} h(t_p))\chi^{(1)}(h(t_p)) \cdot d^\times t_p$$

Since Ψ_{λ_p} is in the Whittaker model the last integral becomes

$$\int_{T_1(\mathbb{Q}_p)} \tau_p(\frac{t_p}{p^n})\Psi_{\lambda_p}(h(t_p))\chi^{(1)}(h(t_p)) \cdot d^\times t_p$$

The value $\Psi_{\lambda_p}(h(t_p))$ depends only on $\operatorname{ord}_p(t_p) = \nu_p$ and hence our integral becomes

$$\sum_{\nu_p=0}^{\infty} \Psi_{\lambda_p}(\begin{pmatrix} p^{\nu_p} & 0 \\ 0 & 1 \end{pmatrix})\chi^{(1)}(p^{\nu_p}) \int_{T^{(1)}(\mathbb{Z}_p)} \tau_p(p^{\nu_p-n}\epsilon)\chi^{(1)}(\epsilon)d^\times\epsilon$$

The integral is a Gauss sum, it vanishes unless $\nu_p - n \leq -f_p$, since we have $n \leq f_p$ and $\nu_p \geq 0$, the only non zero term is $\nu_p = 0, n = f_p$.

Hence we see that the local contribution at a prime p where π_p is unramified and $\chi^{(1)}$ is ramified is given by the Gauss sum $G(\chi^{(1)}, \tau_p)$. Hence we get for a

π_f which is globally unramified and a character χ and for the above choice of $\underline{g}_p$ and $\Psi_{\pi_f} = \otimes \Psi_{\lambda_p}$

$$j((x,\underline{g}_f), r_{\lambda,\mu})(\mathcal{F}(\frac{[\omega_\epsilon]}{\Omega(\pi_f,\nu_\epsilon)} \times \Psi_{\pi_f})) = \frac{L(\pi_f\otimes\chi,\mu)}{\Omega(\pi_f,\omega_\epsilon)} \prod_p G(\chi_p^{(1)},\tau_p)c_\chi \quad (143)$$

5.3.8 Fixing the period

The actual of computation the period may be a highly non trivial. Actually this may even not be so important. But it is indeed of interest to compute the factorization of the L-values, this means we have to compute the numbers

$$\mathrm{ord}_{\mathfrak{p}}(\frac{L(\pi_f\otimes\chi,\mu)}{\Omega(\pi_f,\omega_\epsilon)}) \quad (144)$$

for as many $\mathfrak{p}\subset\mathcal{O}_F$ as possible.

Of course we have problems to fix the period if the class number of $\mathcal{O}_F$ is not one, but this does not matter for the above question, we have to fix a prime p and then we have to choose a good period locally at p. This means we solve the problem alluded to in (138) only locally at p.

We discuss this problem in a very special case where our group $G=\mathrm{Gl}_2$, the maximal compact subgroup $K_f=\prod_p \mathrm{Gl}_2(\mathbb{Z}_p)$ and our coefficient system $\mathcal{M}$ is the module of homogenous polynomials $P(X,Y)$ of degree n and coefficients in $\mathbb{Z}$. Hence the Hecke algebra $\mathcal{H}_{K_f}=\otimes'_p\mathcal{H}_{K_p}$ is unramified at all primes p it is commutative. Our isotypical component π_f defines an ideal $\mathcal{I}(\Pi_f)\subset\mathcal{H}_{K_f}$ and the quotient $\mathcal{H}_{K_f}/\mathcal{I}(\Pi_f)$ is an order in the field $\mathbb{Q}(\mathcal{I}(\Pi_f))=\mathcal{H}_{K_f}/\mathcal{I}(\Pi_f)\otimes\mathbb{Q}$, which is finite extension of $\mathbb{Q}$. (I replaced π_f by Π_f because the ideal does not change if we conjugate π_f the ideal $\mathcal{I}(\Pi_f)$ is associated to the Galois orbit of π_f. I prefer to view $\mathbb{Q}(\Pi_f)$ as an abstract extension of $\mathbb{Q}$.) This ideal $\mathcal{I}(\Pi_f)$ defines a submodule $H^1_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}})_{\,\mathrm{int}}(\mathcal{I}(\Pi_f))=\mathrm{Ann}(\mathcal{I}(\Pi_f))$, this is the submodule annihilated by $\mathcal{I}(\Pi_f)$.

We can think of π_f as simply being a modular cusp form f of weight $k=n+2$. To get our isotypical module $H^1_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathcal{O}_F})_{\,\mathrm{int}}$ we have to find a homomorphism $\sigma:\mathcal{H}_{K_f}/\mathcal{I}(\Pi_f)\to\mathcal{O}_F$ and then

$$H^1_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathcal{O}_F})_{\,\mathrm{int}}(\pi_f)=H^1_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}})_{\,\mathrm{int}}(\mathcal{I}(\Pi_f))\otimes_{\mathcal{H}_{K_f},\sigma}\mathcal{O}_F \quad (145)$$

We have the action of complex conjugation, i.e. of $\pi_0(G(\mathbb{R}))$, on the cohomology $H^1_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}})_{\,\mathrm{int}}(\mathcal{I}(\Pi_f))$ we get the decomposition (up to an isogeny of degree 2^m)

$$H^1_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}})_{\,\mathrm{int}}(\mathcal{I}(\Pi_f))\supset H^1_{!,+}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}})_{\,\mathrm{int}}(\mathcal{I}(\Pi_f))\oplus H^1_{!,-}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}})_{\,\mathrm{int}}(\mathcal{I}(\Pi_f)) \quad (146)$$

and after taking the tensor product by $\mathbb{Q}$ both summands become one dimensional vector spaces over $\mathbb{Q}(\mathcal{I}(\Pi_f))$. But it is by no means clear that the integral modules are isomorphic.

This becomes a little bit better if tensor by $\mathcal{O}_F$ then then we have again

$$H^1_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathcal{O}_F})_{\text{ int}}(\pi_f) \supset H^1_{!(,+}\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathcal{O}_F})_{\text{ int}}(\pi_f) \oplus H^1_{!(,-}\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathcal{O}_F})_{\text{ int}}(\pi_f) \tag{147}$$

and now the two summands are are $\mathcal{O}_F$ modules of rank one and get their structure as Hecke-modules from the homomorphism σ. (In a sense $\pi_f = (\Pi_f, \sigma)$) But still they are not necessarily isomorphic. If we want to define the periods we need class number one. But instead of defining a period we define a local periods. If we tensor the semilocal ring $\mathcal{O}_{F,p} = \mathcal{O}_F \otimes \mathbb{Z}_{(p)}$ then the class number problem disappears we can choose a period such that we get an isomorphism

$$\Omega^{(p)}_{\pm}(\pi_f)^{-1} \cdot \mathcal{F}^{(1)}_1(\omega_\pm) : \bigotimes_p \mathcal{W}_{\mathcal{O}_{F,p}(\pi_f)}(\pi_f,\tau) \xrightarrow{\sim} H^1_{!,\pm}(\mathcal{S}^G_{K_f},\mathcal{M}_{\mathcal{O}_{F,p}})(\pi_f) \tag{148}$$

Recall that we viewed π_f as a modular form f of weight k we change the notation for the periods slightly and denote them by $\Omega^{(p)}_{\pm}(f)$. Our character χ will now be unramified which implies that it is uniquely determined by its type μ. We put $\nu = \mu + 1$ then we get for $\nu = 1, 2, \dots, k-1$ the following integrality statement

$$\Delta(f)\frac{L(f,\nu)}{\Omega_\pm(f)} \in \mathcal{O}_{F,p} \tag{149}$$

But we can still do a little bit better. Recall that we have to evaluate our integral cohomology class on a modular symbol c_μ. This modular symbol is a relative cycle from 0 to $i\infty$ (just along the imaginary axis) loaded by an element $e_\mu = X^\mu Y^{n-\mu}$, we denote it by $[0, i\infty] \times e_\nu$. The index μ runs from zero to n. This is a relative cylce and defines a class in $H_1(\mathcal{S}^G_{K_f}, \partial(\mathcal{S}^G_{K_f}), \tilde{\mathcal{M}})$. We have the boundary operator

$$\partial : H_1(\mathcal{S}^G_{K_f}, \partial(\mathcal{S}^G_{K_f}), \tilde{\mathcal{M}}) \to H_0(\partial(\mathcal{S}^G_{K_f}), \tilde{\mathcal{M}}). \tag{150}$$

We represent the boundary by the circle at $i\infty$ then it is clear that

$$\partial(e_\mu) = e_\mu - we_\mu \tag{151}$$

and we see that $\partial(e_\mu)$ is a torsion class if $\mu \neq 0, n$. Not only that it is a torsion class it is annihilated by a power of the Hecke-operator T^n_p. This implies that $T^n_p([0,i\infty]\times e_\mu)$ can be lifted to a homology class in $\tilde{E_\mu} \in H_1(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})$. But then it is clear that the evaluation of our generator $\xi_\pm$ in $H^1_{!,\pm}(\mathcal{S}^G_{K_f}, \mathcal{M}_{\mathcal{O}_{F,p}})(\pi_f)$ on this lifted cycle gives an integral value. Since $\xi_\pm$ is an eigenvalue for the Hecke operator we get for $\mu = 1, .., n-1$ and $\nu = \mu + 1$

$$<\xi_\pm, \tilde{E_\mu}> = \pi_f(T_p)^n <\xi, e_\mu> = \pi_f(T_p)^n \frac{L(f,\nu)}{\Omega_{\epsilon(\nu)}(f)} \in \mathcal{O}_{F,p} \tag{152}$$

This means that we do not need the factor $\Delta(f)$ in front.

We choose a prime $\mathfrak{p}$ in $\mathcal{O}_F$ lying above p. Let us now assume that $\pi_f(T_p)$ is a unit, i.e. f is ordinary at $\mathfrak{p}$ then we can conclude that

$$\frac{L(f,\nu)}{\Omega^{(p)}_\pm(f)} \in \mathcal{O}_{F,\mathfrak{p}}$$

and consequently

$$\mathrm{ord}_\mathfrak{p}(\frac{L(f,\nu)}{\Omega^{(p)}_\pm(f)}) \geq 0 \text{ for all } 2 \leq \nu \leq k-2 \tag{153}$$

We also know what we should expect at the argument $\nu = k-1$. In this case $\partial(e_n)$ is not a torsion element, but we know that for all primes ℓ the element $(\ell^{k-1}+1-\pi_f(T_\ell))\partial(e_n)$ is annihilated by a power of T_p. If $b_\mathfrak{p}(f)$ is the minimum of the numbers $\mathrm{ord}_\mathfrak{p}(\ell^{k-1}+1-\pi_f(T_\ell))$ then we can conclude that

$$\mathrm{ord}_\mathfrak{p}(\frac{L(f,\mu)}{\Omega^{(p)}_\pm(f)}) + b_\mathfrak{p}(f) \geq 0 \text{ for } \mu = 1, k-1 \tag{154}$$

Hence we can say (still a little bit conjecturally and using Poincare'-duality and the fact that the modular symbols c_μ generate the relative homology. (H. Gebertz, Diploma Thesis Bonn .)

If $\mathfrak{p}$ is ordinary then the numbers $\Omega^{(p)}_\pm(f)$ are the right periods at $\mathfrak{p}$ if and only if one of the non negative numbers in the $+$ or $-$ part of the lists (153),(154)

$$\mathcal{L}_{f,\mathfrak{p}} = \{\ ord_\mathfrak{p}(\frac{L(f,k-1)}{\Omega^{(p)}_-(f)}) + b_\mathfrak{p}(f),\ ord_\mathfrak{p}(\frac{L(f,k-2)}{\Omega^{(p)}_+(f)}),\ldots,\ ord_\mathfrak{p}(\frac{L(f,\nu)}{\Omega^{(p)}_\pm(f)}),\ldots\}$$

is zero.

This discussion is interesting in view of the conjectures on congruences in [Ha-Cong]. In this note we make conjectures about some congruences between Siegel and elliptic modular forms, these congruences are congruences modulo a "large" prime and I do not really say what a large prime should be. Already in [Ha-Cong] I address the issue that we have to choose the right period, but there the choice is rather ad hoc.

Now we have a better recipe. The heuristic argument for the existence of the congruences only works if the prime is ordinary for the modular form f. But in this case we have now a much more precise rule to compute the period. For an ordinary prime $\mathfrak{p}$ we should expect a congruence if for one of the members in the above lists we find a strictly positive value. Here we should still be a little bit more careful, my heuristic argument predicts congruences if $\mathfrak{p}$ occurs in the denominator of a ratio

$$\mathrm{ord}_\mathfrak{p}(\frac{\mathcal{L}_{f,\mathfrak{p}}(\nu)}{\mathcal{L}_{f,\mathfrak{p}}(\nu+1)}) < 0\ , \nu = k-2, k-3, \ldots, k/2+1$$

so we should pay attention to possible cancellations.

Checking the list of the list of the modular forms of weight 12,16,18,20,22,26 we find that the only cases of ordinary primes for which we expect congruences are indeed the cases $k = 22, \ell = 41$ and $k = 26, \ell = 29, 43, 97$ and they are already in [Ha-Cong]. Here is no cancellation.

It will be very interesting to check the case of the two dimensional space of cusp forms of weight 24. In this case the field $F = \mathbb{Q}(\sqrt{144169})$. Again we find very few instances of ordinary candidates, these are the primes dividing 73, 179 and the congruences have been checked.

But apart from these two cases we have the two divisors of 13, they occur rather frequently in our list $\mathcal{L}_{f,\mathfrak{p}}$ and it seems to be interesting to see what happens.

The modular form f of weight 24 has an expansion with coefficients in $\mathbb{Q}(\omega)$ where $\omega^2 = 144169$, we write the first few terms

$$\begin{aligned} f(q) = q + 12(45-\omega)q^2 + 36(4715+16\cdot\omega)q^3 + 32(395729-405\cdot\omega)q^4 + \\ 1410(25911+128\cdot\omega)q^5 \cdots + 658(3325311035-23131008\cdot\omega)q^{13}\dots \end{aligned} \tag{155}$$

and this provides the two modular forms $f^{(+)}$ (resp. $f^{(-)}$) with real coefficients which we get if we send ω to the positive root $\sqrt{144169}$ (resp. negative root).

We have the periods $\Omega_\pm(f^{(+)}), \Omega_\pm(f^{(-)})$ and we know that

$$\frac{L(f^{(+)},\nu)}{\Omega_{\epsilon(\nu)}(f^{(+)})}, \frac{L(f^{(-)},\nu)}{\Omega_{\epsilon(\nu)}(f^{(-)})} \in \mathbb{Q}(\sqrt{144169}) \tag{156}$$

Looking at the norms of these numbers we find some factors of 13. The prime 13 decomposes in $\mathbb{Z}[\omega]$ and we see that the two prime factors above thirteen are given by the homomorphism $\phi_5 : \omega \mapsto 5 \mod 13$. and $\phi_8 : \omega \mapsto 8 \mod 13$ We check that $f^{(+)}$ is ordinary at ϕ_8 but not at ϕ_5. But if we look at the prime factor decomposition of $\frac{L(f^{(+)},\nu)}{\Omega_{\epsilon(\nu)}(f^{(+)})}$ then we see that ϕ_5 occurs non trivially but ϕ_8 does not. Hence we do not expect the existence of a Siegel modular form and a congruence modulo ϕ_5 because ϕ_5 is not ordinary for $f^{(+)}$. The prime ϕ_8 is ordinary for $f^{(+)}$ but this prime does not occur in the L-values.

5.3.9 Anton's Congruence

The issue to fix the period becomes even more delicate once we allow ramification. Let us consider the case of the congruence subgroup $\Gamma_0(p)$, this means that our open compact subgroup will be $K_{0,f}(p) = \prod_{q:q\neq p} \mathrm{Gl}_2(\mathbb{Z}_q) \times K_0(p)$. Again we can determine the periods locally at a prime ℓ by evaluating period integrals against certain modular symbols. The point is that we have more modular symbols, because we allow ramification. To get control over these modular symbols we consider the representation $\mathrm{Ind}_{K_{0,f}(p)}^{K_f}\mathbb{1}$, i.e. the induced from the trivial representation of $K_{0,f}(p)$ to the maximal compact subgroup K_f. This representation can be viewed as a representation of $\mathrm{Gl}_2(\mathbb{F}_p)$, it is of dimension $p+1$ and it has the Steinberg-module St_p of dimension p. Then we can consider the cohomology $H^1(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_n \otimes \mathrm{St}_p)$, and new forms f for $\Gamma_0(p)$ correspond to eigenclasses in $H^1_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_n \otimes \mathrm{St}_p)$.

We can construct modular symbols with coefficients in $\tilde{\mathcal{M}}_n \otimes \mathrm{St}_p$. The standard torus $T(\mathbb{F}_p)$ acts on St_p and under this action we get a decomposition into eigenspaces (we invert the divisors of $p(p-1)$ let $R=\mathbb{Z}[\frac{1}{p(p-1)}]$)

$$\mathrm{St}_p\otimes R=\bigoplus_{\chi:\mathbb{F}_p^\times\to\mu_{p-1}} Re_\chi \tag{157}$$

(The trivial character occurs two times)

Hence we can define modular symbols $e_\mu\otimes e_\chi$ where e_μ is as above. Then we get integrality for the values

$$\frac{L(f\otimes\chi,\mu)}{\Omega_{\epsilon(\mu,\chi)}(f)}G(\chi,\tau) \tag{158}$$

Since we inverted p the Gaussian sum does not play any role. We assume that the modular symbols $e_\mu\otimes e_\chi$ generate the relative homology $H_1(\mathcal{S}^G_{K_f},\partial(\mathcal{S}^G_{K_f}),\tilde{\mathcal{M}}_n\otimes \mathrm{St}_p\otimes R)$. Hence we can fix the periods locally at a prime ℓ which does not divide $p(p-1)$ and which is ordinary for f. We compute the L-values and then we must have

$$\mathrm{ord}_{\mathfrak{l}}(\frac{L(f\otimes\chi,\mu)}{\Omega_{\epsilon(\mu,\chi)}(f)})\geq 0 \tag{159}$$

and for both signs $\epsilon(\mu,\chi)$ at least one of these numbers has to be zero. Here $\mathfrak{l}$ runs over the divisors of ℓ in $\mathcal{O}_F[\zeta_{p-1}]$.

We want to consider the special case of modular forms of weight 4 for $\Gamma_0(p)$. In this case we have only three critical values $L(f\otimes\chi,\mu)$ for $\mu=1,2,3$.

We are interested in this case because we want to understand the conjectures in [Ha-Cong] also in the case of a non regular coefficient system, especially we want to look at the case of the trivial coefficient system, i.e. the case where the representation is one dimensional. Then we find modular forms of weight four in the boundary cohomology and this forces us to allow ramification. But we want to keep ithe ramification as small as possible.

We start from the group $G=\mathrm{GSp}_2/\mathbb{Z}$, we choose as level subgroup the group $K_f=K^G_{f,p}=\prod_{q:q\neq p}G(\mathbb{Z}_q)\times K_0(p)$, where $K_0(p)$ is the group of $\mathbb{Z}_p$ valued points of the unique non special maximal parahoric subgroup scheme $\mathcal{P}_{\gamma_1}$. (Here γ_1 is the fundamental weight attached to the short root viewed as a cocharacter, we have $<\gamma_1,\alpha_1>=1,<\gamma_1,\alpha_2>=0.$). This choice $K^G_{f,p}$ defines an arithmetic subgroup $\Gamma_p\subset\mathrm{GSp}_2(\mathbb{Q})$ which is called the paramodular group.

We consider the homomorphism

$$H^3(\mathcal{S}^G_{K_f},R)\xrightarrow{r}H^3(\partial(\mathcal{S}^G_{K_f}),R) \tag{160}$$

The right hand side contains a contribution coming from the cuspidal cohomology of the stratum of the Siegel parabolic subgroup, this is the contribution $H^1_!(\mathcal{S}^M_{K^M_f},H^2(\mathfrak{u}_P,R))$. The point is that now that $K^M_f=K_{0,f}(p)=\prod_{q:q\neq p}\mathrm{Gl}_2(\mathbb{Z}_q)\times K_0(p)$, which we introduced above. The M-module $H^2(\mathfrak{u}_P,R)$ is the standard three dimensional representation. Hence this cohomology is described by the space of modular forms of weight 4 for the group $\Gamma_0(p)$.

Any modular (new) form f of weight 4 for $\Gamma_0(p)$, yields a contribution

$$H^1_!(\mathcal{S}^M_{K^M_f},H^2(\mathfrak{u}_P,R))[f]$$

of rank one over $R \otimes \mathcal{O}_F$. Let us consider the inverse image $H^3(\mathcal{S}^G_{K_f}, R)[f] = r^{-1}(H^1(\mathcal{S}^M_{K_f^M}, H^2(\mathfrak{u}_P, R)[f]))$. We consider the restriction

$$H^3(\mathcal{S}^G_{K_f}, R)[f] \xrightarrow{r_f} H^1(\mathcal{S}^M_{K_f^M}, H^2(\mathfrak{u}_P, R)[f] \tag{161}$$

We invoke results from Eisenstein cohomology. Schwermer has shown: This restriction map is surjective if and only if we have $L(f,2) = 0$ otherwise we encounter a pole of an Eisenstein class.

I also discuss an analogous situation in the appendix of [Ha-Eis]. There I assume that we have no ramification, but I discuss non trivial non regular coefficient systems. A rather speculative computation using the comparison between the Lefschetz and the topological trace formula suggests that in this case

r_f has a non trivial kernel $H^3_!(\mathcal{S}^G_{K_f}, R)[f]$ if and only if the sign of the functional equation for $L(f,s)$ is minus one.

Let us believe that the same is true in this case (and if we do not believe in the trace formula we could also try to explain this kernel as a Gritsenko lift) and we get the exact sequence

$$0 \to H^3_!(\mathcal{S}^G_{K_f}, R)[f] \to H^3(\mathcal{S}^G_{K_f}, R)[f] \xrightarrow{r_f} H^1(\mathcal{S}^M_{K_f^M}, H^2(\mathfrak{u}_P, R)[f], \tag{162}$$

where $H^3_!(\mathcal{S}^G_{K_f}, R)[f]$ is the Scholl motive attached to f. This yields an extension class of motives

$$\mathcal{X}(f) \in \mathrm{Ext}^1(R(-2), H^3_!(\mathcal{S}^G_{K_f}, R)[f]). \tag{163}$$

Tony Scholl suggests to attach a number to such an extension. More precisely he suggests to construct a suitable biextension, this can be done by the Anderson construction introducing an auxiliary prime p_0.) and then this number should be essentially

$$\frac{\frac{L'(f,2)}{\Omega_+(f)}}{\frac{L(f,3)}{\Omega_-(f)}} \tag{164}$$

Under this assumption the denominator $\frac{L(f,3)}{\Omega_-(f)}$ becomes interesting. Since we fixed the period, we can ask whether ordinary primes $\mathfrak{l}$ dividing this number yield denominators of Eisenstein classes and hence congruences. Such a congruence has been detected by Anton Mellit in the case $p = 61$ and $\ell = 43$. Checking the tables of W. Stein we find that for $p = 61$ the cohomology $H^1_!(\mathcal{S}^M_{K_f^M}, H^2(\mathfrak{u}_P, R))$ is of rank 2×15 and decomposes into a 12-dimensional and a 18 dimensional piece (over $\mathbb{Q}$). The 6 dimensional piece corresponds to a modular cusp form f of weight 4 for $\Gamma_0(61)$ its coefficients lie in a field of degree 6 over $\mathbb{Q}$. The sign in the functional equation is -1 and we should look for the prime decomposition of the number

$$\frac{L(f,3)}{\Omega_-(f)} \tag{165}$$

over $\ell = 43$. We know that there is a Siegel modular form for Γ_{61} which is not a Gritsenko lift and satisfies the congruence (Poor-Yuen).The question is whether a divisor $\mathfrak{l}|\ell$ occurs in the value above. But then it becomes clear that we to obey strict rules to fix the period.

We may also check some other primes p and compute the ratios in (165) and look whether they are divisible by interesting primes and whether these primes yield congruences for non Gritsenko lifts.

5.4 The L-functions

Again I have to say a few words concerning L-functions.

To get the automorphic L-functions at the unramified places we have to introduce the dual group $G^\vee(\mathbb{C})$ (this is $\mathrm{Gl}_2(\mathbb{C})$ in this case) and a finite dimensional representation r of this group. The definition of the dual group is designed in such a way that the Satake parameter ω_p of an unramified representation at p can be interpreted as a semi simple conjugacy class in $G^\vee(\mathbb{C})$ (see [La]). Therefore we can form the expression

$$L(\pi_p, r, s) = \det(\mathrm{Id} - r(\omega_p)p^{-s})^{-1}$$

and then the global L function $L(\pi, r, s)$ is defined as the product over all these unramified L -factors times a product over suitable L-factors at the finite primes. If we do this for our automorphic forms on Gl_2 and if $r = r_1$ is the tautological representation of $Gl_2(\mathbb{C})$ then we get the local L-factors

$$L(\pi_p, r_1, s) = \frac{1}{(1-\lambda_{p,2}(p)p^{-s})\,(1-\lambda_{p,1}(p)p^{-s})}$$

and we see that it differs by a shift by 1/2 from our previous definition. Our earlier L -function was the motivic L-function, its definition does not require the additional datum r. Our automorphic form π defines a motive $\mathbb{M}(\pi)$. This motive has the disadvantage that it does not occur in the cohomology of a variety, it occurs only after we apply a Tate twist to it. The central character $\omega(\pi)$ has type $x \mapsto x^n$ and defines a Tate motive. The automorphic form $\pi \otimes \omega(\pi)^{-1} = \pi^\vee$ occurs in the cohomology

$$H^1(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}[-n]) \supset H^1(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}[-n])(\pi \otimes \omega(\pi)^{-1}) = H^1(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}[-n])(\pi^\vee)$$

where $\mathcal{M}_n[-n]$ is obtained by twisting the original module by the $-n$-th power of the determinant. (See [Ha-Eis], III). This motive occurs in the cohomology of a quasiprojective scheme (See also [Scholl]) Now we adopt the point of view that π_f is a pair (Π_f, ι) (See 1.2.6) and then $\mathbb{M}(\pi)$ defines a system of $\mathfrak{l}$-adic representations $\rho(\pi)_\mathfrak{l}$ which are also labelled by the $\iota : \mathbb{Q}(\pi_f) \to \bar{\mathbb{Q}}$. Then it is Delignes theorem that for unramified primes

$$L(\pi_p, r_1, s - \frac{1}{2}) = L_p((\mathbb{M}(\pi^\vee), s) = \det(\mathrm{Id} - \rho(F_p)_p^{-1} | \mathbb{M}(\pi^\vee)_\mathfrak{l}\, p^{-s})$$

for a suitable choice of $\ell \neq p$.

5.4.1 Weights and Hodge numbers

We may of course look at the motives $\mathbb{M}(\pi)$ which are attached to an eigenspace in $H^1_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}[-k])(\pi)$ in other words we twisted the natural module $\mathcal{M}_n$ by the $-k$-th power of the determinant. Again we get an $\mathfrak{l}$-adic representation $\rho_\mathfrak{l}$ and the Weil conjectures imply that the eigenvalues of the inverse Frobenius $\rho_\mathfrak{l}(F_p^{-1})$ all have the same absolute value $p^{\frac{2k-n+1}{2}}$. The number $2k-n+1$ is usually called the weight $w(\rho_\mathfrak{l})$ of the Galois representation or also the weight $w(\mathbb{M}(\pi))$ of the motive $\mathbb{M}(\pi)$.

The central character $\omega(\pi)$ of π has a type and if we make the natural identification of G_m with the centre then the type of $\omega(\pi)$ is an integer $\mathrm{type}(\omega(\pi)) \in \mathbb{Z}$ and the formula for the weight is

$$w(\mathbb{M}(\pi)) = -\mathrm{type}(\omega(\pi)) + 1.$$

This weight plays a role if we want to get a first understanding of the analytic properties of the motivic L-functions. Its abcizza of convergence is the line $\Re(s) = w(\mathbb{M}(\pi)) + 1$.

The special case $k = n$ is special, because in this case our motive occurs in the cohomolgy of a variety. The eigenvalues of the Frobenius are algebraic integers and the non zero Hodge numbers are $h^{n+1,0}$ and $h^{0,n+1}$. If k is arbitrary then the centre acts on $\mathcal{M}_n[-k]$ by the character $t(k) = n-2k$ and the non zero Hodge numbers will be $h^{1+\frac{n-t(k)}{2}, -\frac{n+t(k)}{2}}$. We notice that for an isotypic component $H^1_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}[-k])(\pi)$ the number $t(k)$ is the type of the central character $\omega(\pi)$.

5.5 The special values of L-functions

We now observe that the local L factors $L(\mathbb{M}(\pi^\vee \otimes (\chi^{(1)})^{-1}), s)$ which we introduced in 2.2.6 are actually the local L-factrs of the motivic L-function, i.e.

$$L(\mathbb{M}(\pi^\vee \otimes (\chi^{(1)})^{-1}), s) = L(\mathbb{M}(\pi^\vee \otimes (\chi^{(1)})^{-1}), s)$$

Theorem 5.2. *With these notations we can give a formula for the composition*

$$J_{c_{\chi,!}} \circ \Omega_\varepsilon(\pi_f)^{-1} \cdot \mathcal{F}_1^{(1)}(\omega_\varepsilon) = \frac{L(\mathbb{M}(\pi^\vee \otimes (\chi^{(1)})^{-1}), 1)}{\Omega_\varepsilon(\pi_f)} \cdot I^{loc}(\pi_f, \chi_f^{-1})$$

5.5.1 Applications

We evaluate this formula at elements $\psi_f \in \mathcal{W}(\pi_f, \tau)_{\mathrm{O}(\pi_f,\chi)}$ and an element $\underline{g}_f \in G(\mathbb{A}_f)$. We get $\Omega_\varepsilon(\pi_f)^{-1} \cdot \mathcal{F}_1^{(1)}(\omega_\varepsilon)(\psi_f) = \tilde{\psi}_f \in H^1_{!,\varepsilon}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})_{\mathrm{O}(\pi_f,\chi)}$ and

$$J_{c_{\chi,!}}(\psi_f)(\underline{g}_f) = \frac{L(\mathbb{M}(\pi^\vee \otimes (\chi^{(1)})^{-1}), 1)}{\Omega_\varepsilon(\pi_f)} \cdot I^{loc}(\pi_f, \chi_f^{-1})(\psi_f)(\underline{g}_f))$$

We have seen that $J_{c_{\chi,!}}(\psi_f)(\underline{g}_f)d(\underline{g}_f)$ (Lemma 2.2) is an integer and it is obvious that $d(\underline{g}_f) = \prod_p d(g_p)$. If we choose for ψ_f an element which is also a product $\psi_f(\underline{g}_f) = \prod_p \psi_p(g_p)$ then we get

$$J_{c_{\chi,!}}(\psi_f)(\underline{g}_f) \prod_p d(g_p) = \frac{L(\mathbb{M}(\pi^\vee \otimes (\chi^{(1)})^{-1}), 1)}{\Omega_\varepsilon(\pi_f)} \cdot \prod_p I_p^{loc}(\pi_p, \chi_p^{-1})(\psi_p)(g_p)d(g_p))$$

The factors in the products over all primes are equal to one at almost all places. Then we have to optimize the choices of ψ_p and g_p. First of all we can choose these data such that all local factors are different from zero. Then we conclude that we have an invariance under Galois for the L-values

$$(\frac{L(\mathbb{M}(\pi^\vee \otimes (\chi^{(1)})^{-1}), 1)}{\Omega_\varepsilon(\pi_f)})^\sigma = \chi^{(1)}(\underline{t}_\sigma)\frac{L(\mathbb{M}((\pi^\vee \otimes (\chi^{(1)})^{-1})^\sigma, 1)}{\Omega_\varepsilon(\pi_f^\sigma)}$$

We may observe that the characters $\chi^{(1)}$ can be written as product of a Dirichlet character and a power of the Tate character, i.e. $\chi^{(1)} = \phi{\cdot}\alpha^{-\nu}$ where $\nu = 0, \dots n$. Now we can write

$$\mathbb{M}(\pi^\vee \otimes (\chi^{(1)})^{-1}) = \mathbb{M}(\pi^\vee \otimes \phi^{-1}) \otimes \mathbb{Z}(\nu)$$

and

$$L(\mathbb{M}(\pi^\vee \otimes (\chi^{(1)})^{-1}), 1) = L(\mathbb{M}(\pi^\vee \otimes \phi^{-1}), 1+\nu)$$

and the arguments $1+\nu$ are exactly the critical arguments for the motive $\mathbb{M}(\pi^\vee \otimes \phi^{-1})$ in the sense of Deligne.

Of course we are now able to prove also some integrality results, it is clear that the left hand side is integral, more precisely it is an element in $\mathrm{O}(\pi_f, \chi_f)$. Now we have to work with local representations to find out under which conditions we can force the product of local factors to be a unit or at least to bound the primes dividing it. Hence we have a tool to show that

$$\frac{L(\mathbb{M}(\pi^\vee \otimes (\chi^{(1)})^{-1}), 1)}{\Omega_\varepsilon(\pi_f)} \in \mathrm{O}(\pi_f, \chi_f)$$

at least if we invert a few more primes.

5.5.2 The arithmetic interpretation

It is clear that we have some control of the primes that have to be inverted. I call them *small* primes. The main reason why I am interested in the integrality statement for these special values is, that I want to understand what it means if a *large* prime divides these values.

I strongly believe that the large primes dividing these L-values are related to the denominators of Eisenstein classes for the cohomology of the symplectic group, what this means will be explained in 5.6 and we also refer to the notes [kolloquium.pdf]. In the following section I want to give some idea how such a relationship between the arithmetic properties of the L-values and the integral structure of the cohomology as a Hecke-module should look like.

6 Eisenstein cohomology

Our starting point is a smooth group scheme $\mathcal{G}/\mathrm{Spec}(\mathbb{Z})$ whose generic fiber $G = \mathcal{G} \times_{\mathbb{Z}} \mathbb{Q}$ is reductive and quasisplit. We assume the group scheme is reductive over the largest possible open subset of $\mathrm{Spec}(\mathbb{Z})$ and at the remaining places it is given by a maximal parahoric group scheme structure. If G is split, then we assume that $\mathcal{G}$ is split. We define $K_f = \mathcal{G}(\hat{\mathbb{Z}}) = \prod_p \mathcal{G}(\mathbb{Z}_p) \subset G(\mathbb{A}_f)$

We choose a Borel subgroup $B/\mathbb{Q}$ and a torus $T/\mathbb{Q} \subset B/\mathbb{Q}$. We assume that $T(\mathbb{A}_f) \cap K_f = T/\hat{\mathbb{Z}})$ is maximal compact in $T(\mathbb{A}_f)$. Let $\lambda \in X^*(T)$ be a highest weight, let $\mathcal{M}_\lambda$ be a highest weight module attached to this weight. It is a $\mathbb{Z}$-module, the module $\mathcal{M}_\lambda \otimes \mathbb{Q}$ is a highest weight module for the group $G/\mathbb{Q}$. We consider

6.1 The Borel-Serre compactification

We consider our space

$$\mathcal{S}^G_{K_f} = G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K_f$$

and its Borel-Serre compactification

$$i : \mathcal{S}^G_{K_f} \to \bar{\mathcal{S}}^G_{K_f}.$$

Our highest weight module $\mathcal{M}_\lambda$ provides a sheaf $\tilde{\mathcal{M}}_\lambda$ on these spaces.

We have an isomorphism

$$H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) \xrightarrow{\sim} H^\bullet(\mathcal{S}^{\bar{G}}_{K_f}, \tilde{\mathcal{M}}_\lambda)$$

for any coefficient system $\tilde{\mathcal{M}}_\lambda$ coming from a rational representation $\mathcal{M}$ of $G(\mathbb{Q})$. The boundary $\partial\bar{\mathcal{S}}_K$ is a manifold with corners. It is stratified by submanifolds

$$\partial\bar{\mathcal{S}}_K = \bigcup_P \partial_P \mathcal{S}^G_{K_f},$$

where P runs over the $G(\mathbb{Q})$ conjugacy classes of proper parabolic subgroups defined over $\mathbb{Q}$. We identify the set of conjugacy classes of parabolic subgroups with the set of representatives given by the parabolic subgroups that contain our standard Borel subgroup $B/\mathbb{Q}$. Then we have

$$H^\bullet(\partial_P \mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) = H^\bullet(P(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K_f, \tilde{\mathcal{M}}_\lambda)$$

We have a finite coset decomposition

$$G(\mathbb{A}_f) = \bigcup_{\xi_f} P(\mathbb{A}_f)\xi_f K_f,$$

for any $\underline{\xi_f}$ put $K^P_f(\xi_f) = P(\mathbb{A})_f \cap \xi_f K_f \xi_f^{-1}$. Then we have

$$P(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K_f = \bigcup_{\xi_f} P(\mathbb{Q})\backslash X \times P(\mathbb{A}_f)/K^P_f(\xi_f)\xi_f,$$

If $R_u(P) \subset P$ is the unipotent radical, then

$$M = P/R_u(P)$$

is a reductive group. For any open compact subgroup $K_f \subset G(\mathbb{A}_f)$(resp. for $K_\infty \subset G_\infty$) we define $K^M_f(\xi_f) \subset M(\mathbb{A}_f)$(resp. $K^M_\infty \subset M_\infty$) to be the image of $K^P(\xi_f)$ in $M(\mathbb{A}_f)$ (resp. M_∞). We put

$$\mathcal{S}^M_{K_f(\xi_f)} = M(\mathbb{Q})\backslash M(\mathbb{A})/K^M_\infty K^M_f(\xi_f)$$

and get a fibration

$$\pi_P : P(\mathbb{Q})\backslash X \times P(\mathbb{A}_f)/K_f^P(\xi_f) \to M(\mathbb{Q})\backslash M(\mathbb{A})/M(\mathbb{Q})\backslash K_\infty^M \times K_f^M(\xi_f)$$

where the fibers are of the form $\Gamma_U\backslash R_u(P)(\mathbb{R})$ and where $\Gamma_U \subset U(\mathbb{Z})$ is of finite index and defined by some congruence condition dictated by $K_f^P(\xi_f)$. The Lie-algebra $\mathfrak{u}$ of $R_u(P)$ is a free $\mathbb{Z}$-module and it is clear that we have an integral version of the van -Est theorem which says:

If $R = \mathbb{Z}[\frac{1}{N}]$ where a suitable set of primes has been inverted then

$$H^\bullet(\Gamma_U\backslash R_u(P)(\mathbb{R}), \tilde{\mathcal{M}}_R) \xrightarrow{\sim} H^\bullet(\mathfrak{u}, \tilde{\mathcal{M}}_R).$$

More precisely we know that the local coefficient system $R^\bullet\pi_{P*}(\tilde{\mathcal{M}})$ is obtained from the rational representation of M on $H^\bullet(\mathfrak{u}, \mathcal{M})$.

Hence we get

$$H^\bullet(\partial_P\mathcal{S}, \tilde{\mathcal{M}}_R) = \bigcup_{\xi_f} H^\bullet(\mathcal{S}^M_{K_f^M(\xi_f)}, \widetilde{H^\bullet(\mathfrak{u}, \mathcal{M})}_R),$$

and

$$H^\bullet(\mathfrak{u}, \mathcal{M}_R) = \bigoplus_{w\in W^P} H^{l(w)}(\mathfrak{u}, \mathcal{M}_R)(w\cdot\lambda),$$

where W^P is the set of Kostant representatives of W/W^M and where $w\cdot\lambda = (\lambda+\rho)^w - \rho$ and ρ is the half sum of positive roots.

The primes which we have to be inverted should be those which are smaller than the coefficients of the dominant weights in the highest weight of $\mathcal{M}$. But at this point we may have to enlarge the set of small primes.

We conclude

The cohomology of the boundary strata $\partial_P\mathcal{S}^G_{K_f}$ with coefficients in $\mathcal{M}$ can be computed in terms of the cohomology of the reductive quotient, where we have coefficients in the cohomology of the Lie algebra of the unipotent radical with coefficients in $\mathcal{M}$

In the following considerations we sometimes suppress the subscripts $K_f, K^M_{K_f^M}$ and so on. Then we mean that the considerations are valid for a fixed level or that we have taken the limit over the K_f. (See the remarks below concerning induction)

6.1.1 The two spectral sequences

The covering of the boundary by the strata $\partial_P\mathcal{S}$ provides a spectral sequence, which converges to te cohomology of the boundary. We can introduce the simplex Δ of types of parabolic subgroups, the vertices correspond to the maximal ones and the full simplex corresponds to the minimal parabolic. To any type of a parabolic P let $d(P)$ its rank, we make the convention that $d(P)-1$ is equal to the dimension of the corresponding face in the simplex. Let $M = M_P = P/R_u(P)$ be the reductive quotient (the Levi quotient). If $Z_M/\mathbb{Q}$ is the connected component of the identity of the center of $M/\mathbb{Q}$ then

$d(P)$ is also the dimension of the maximal split subtorus of $Z_M/\mathbb{Q}$ minus the dimension of the maximal split subtorus of $Z_G/\mathbb{Q}$. The covering yields a spectral sequence whose $E_1^{\bullet,\bullet}$ term together with the differentials of our spectral sequence is given by

$$0 \to E_1^{0,q} = \bigoplus_{P,d(P)=1} H^q(\partial_P \mathcal{S}, \mathcal{M}) \xrightarrow{d_1^{0,q}} \cdots \to \bigoplus_{P,d(P)=p+1} H^q(\partial_P \mathcal{S}, \mathcal{M}) \xrightarrow{d_1^{p,q}} \tag{166}$$

where the boundary map $d_1^{p,q}$ is obtained from the restriction maps (See [Gln]). There is also a homological spectral sequence which converges to the cohomology of the boundary. It can be written as a spectral sequence for the cohomology with compact supports. Let d be the dimension of $\mathcal{S}$ then we have a complex

$$\to \bigoplus_{P,d(P)=p+1} H_c^{d-1-p-q-1}(\partial_P \mathcal{S}, \mathcal{M}) \xrightarrow{\delta_1} \bigoplus_{P,d(P)=p} H_c^{d-1-p-q}(\partial_P \mathcal{S}, \mathcal{M}) \to \tag{167}$$

and therefore the $E^1_{\bullet,\bullet}$ term is

$$E^1_{p,q} = \bigoplus_{P,d(P)=p} H_c^{d-1-p-q}(\partial_P \mathcal{S}, \mathcal{M})$$

the (higher) differential go from (p,q) to $(p-r, q+1-r)$.

6.1.2 Induction

The description of the cohomology of a boundary stratum is a little bit clumsy, since we are working with the coset decomposition. The reason is that we are working on a fixed level, if we consider cohomology with integral coefficients. If we have rational coefficients then we can pass to the limit. Then

$$H^\bullet(\partial_P \mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}) = \lim_{K_f} H^\bullet(P(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K_f, \tilde{\mathcal{M}}) =$$

$$\mathrm{Ind}_{\pi_0(M(\mathbb{R})\times P(\mathbb{A}_f)}^{\pi_0(G(\mathbb{R})\times G(\mathbb{A}_f)} \lim_{K_f^M} H^\bullet(\mathcal{S}^M_{K_f^M}, \widetilde{H^\bullet(\mathfrak{u}, \mathcal{M})}) = \mathrm{Ind}_{\pi_0(M(\mathbb{R}))\times P(\mathbb{A}_f)}^{\pi_0(G(\mathbb{R}))\times G(\mathbb{A}_f)} H^\bullet(\mathcal{S}^M, \widetilde{H^\bullet(\mathfrak{u}, \mathcal{M})}),$$

where the induction is ordinary group theoretic induction. We should keep in our mind that the $\pi_0(M(\mathbb{R})) \times P(\mathbb{A}_f)$ -modules are in fact $\pi_0(M(\mathbb{R})) \times M(\mathbb{A}_f)$-modules. We need some simplification in the notation and we will write for any such $\pi_0(M(\mathbb{R})) \times M(\mathbb{A}_f)$ module H

$$\mathrm{Ind}_{\pi_0(M(\mathbb{R}))\times P(\mathbb{A}_f)}^{\pi_0(G(\mathbb{R}))\times G(\mathbb{A}_f)} H = I_M^G H$$

We will use the same notation for an induction from the torus T to M.

Under certain conditions we also have the notion of induction for Hecke - modules and we can work with integral coefficient systems. This will be discussed at another occasion.

But I want to mention that in the case that K_f is a hyperspecial maximal compact subgroup (in the cases where we are dealing with a split semi-simple group scheme over Spec($\mathbb{Z}$) we can take $K_f = \prod \mathcal{G}(\mathbb{Z}_p)$ (see 1.1)) then $G(\mathbb{Q}_p) = P(\mathbb{Z}_p)K_p = B(\mathbb{Z}_p)K_p$ the group theoretic induction followed by taking K_f invariants gives back the original module. In this case we do not have to induce!

Of course we have to understand the coefficient systems $H^\bullet(\mathfrak{u}, \mathcal{M})$, for this we need the theorem of Kostant which will be discussed in the next section.

6.1.3 A review of Kostants theorem

At this point we can make the assumption that our group $G/\mathbb{Q}$ is quasisplit, we also assume that $G^{(1)}/\mathbb{Q}$ is simply connected. Then we may assume that $\mathcal{M}_\mathbb{Z}$ is irreducible and of highest weight λ. Let $B/\mathbb{Q}$ be a Borel subgroup, we choose a torus $T/\mathbb{Q} \subset B/\mathbb{Q}$. Let $X^*(T) = \mathrm{Hom}(T \times_\mathbb{Q} \bar{\mathbb{Q}}, \mathbb{G}_m \times_\mathbb{Q} \bar{\mathbb{Q}}$ be the character module, it comes with an action of a finite Galois group $\mathrm{Gal}(F/\mathbb{Q})$, here F is the smallest sub field of $\bar{\mathbb{Q}}$ over which $G/\mathbb{Q}$ splits. Let $T^{(1)}/\mathbb{Q} \subset T/\mathbb{Q}$ the maximal torus in $G^{(1)}/\mathbb{Q}$, then $X^*(T^{(1)})$ contains the set Δ of roots, the subset Δ^+ of positive roots (with respect to B.) The set of simple roots is identified to a finite index set $I = \{1, 2, \dots, r\}$, i.e we write the set of simple roots as $\pi = \{\alpha_1, \dots, \alpha_i, \dots, \alpha_r\} \subset \Delta^+$. We assume that the numeration is somehow adapted the Dynkin diagram. The finite Galois group $\mathrm{Gal}(F/\mathbb{Q})$ acts on I and π by permutations. Attached to the simple roots we have the dominant fundamental weights $\{, \dots, \gamma_i, \dots, \gamma_j, \dots\}$ they are related to the simple roots by the rule

$$2\frac{< \gamma_i, \beta_j >}{< \beta_j, \beta_j >} = \delta_{i,j}.$$

The dominant fundamental weights form a basis of $X^*(T^{(1)})$.

Our maximal torus $T/\mathbb{Q}$ is up to isogeny the product of $T^{(1)}$ and the central torus $C/\mathbb{Q}$, i.e. $T = T^{(1)} \cdot C$ and the restriction of characters yields an injection

$$j : X^*(T) \to X^*(T^{(1)}) \oplus X^*(C),$$

this becomes an isomorphism if we tensorize by the rationals

$$X^*_\mathbb{Q}(T) = X^*(T) \otimes \mathbb{Q} \xrightarrow{\sim} X^*_\mathbb{Q}(T^{(1)}) \oplus X^*_\mathbb{Q}(C).$$

This isomorphism gives us canonical lifts of elements in $X^*(T^{(1)})$ or $X^*(C)$ to elements in $X^*_\mathbb{Q}(T)$ which will be denoted by the same letter. Especially the fundamental weights $\gamma_1 \dots, \gamma_i, \dots$ are elements in $X^*_\mathbb{Q}(T)$.

Let $\lambda \in X^*(T)$ be a dominant weight, our decomposition allows us to write it as

$$\lambda = \sum_{i \in I} a_i \gamma_i + \delta = \lambda^{(1)} + \delta$$

we have $a_i \in \mathbb{Z}, a_i \geq 0$ and $\delta \in X^*(C)$. To such a dominant weight λ we have an absolutely irreducible $G \times F$ -module $\mathcal{M}_\lambda$.

We consider maximal parabolic subgroups $P/\mathbb{Q} \supset B/\mathbb{Q}$. These parabolic subgroups are given by the choice of a $\mathrm{Gal}(F/\mathbb{Q})$ orbit $\tilde{i} = J \subset I$ Such an orbit yields a character $\gamma_J = \sum_{i\in J} \gamma_i$ The parabolic subgroup $P/\mathbb{Q}$ provided by this datum is determined by its root system $\Delta^P = \{\beta \in \Delta| < \beta, \gamma_J >\geq 0\}$. The choice of the maximal torus $T \subset P$ also provides a Levi subgroup $M \subset P$ but actually it is better to consider M as the quotient P/U_P.

The set of simple roots of $M^{(1)}$ is the subset $\pi_M = \{\dots, \alpha_i, \dots\}_{i\in I_M}$, where of course $I_M = I\backslash J$. We also consider the group $G^{(1)}\cap M = M_1$. It is a reductive group, it has $T^{(1)}$ as its maximal torus. We apply our previous considerations to this group M_1. It has a non trivial central torus $C_1/\mathbb{Q}$. This torus has a simple description, we pick a root $\alpha_i, i \in J$, we know that J is an orbit under $\mathrm{Gal}(F/\mathbb{Q})$. We have the subfield $F_{\alpha_i} \subset F$ such that $\mathrm{Gal}(F/F_{\alpha_i})$ is the stabilizer of α_i. Then it is clear that

$$C_1 \xrightarrow{\sim} R_{F_{\alpha_i}/\mathbb{Q}}(\mathbb{G}_m/F_{\alpha_i}),$$

up to isogeny it is a product of an anisotropic torus $C_1^{(1)}/\mathbb{Q}$ and a copy of $\mathbb{G}_m$. The character module $X^*_{\mathbb{Q}}(C_1)$ is a direct sum

$$X^*_{\mathbb{Q}}(C_1) = X^*_{\mathbb{Q}}(C_1^{(1)}) \oplus \mathbb{Q}\gamma_J. \tag{168}$$

Here $X^*_{\mathbb{Q}}(C_1^{(1)}) = \{\gamma \in X^*_{\mathbb{Q}}(C_1) \mid < \gamma, \sum_{i\in J} \alpha_i >= 0\}$. The half sum of positive roots in the unipotent radical is

$$\rho_U = f_P \gamma_J \tag{169}$$

where $2f_P > 0$ is an integer.

We also have the semi simple part $T^{(1,M)} \subset M^{(1)}$ and again we get the orthogonal decomposition

$$X^*_{\mathbb{Q}}(T^{(1)}) = X^*_{\mathbb{Q}}(T^{(1,M)}) \oplus X^*_{\mathbb{Q}}(C_1) = \bigoplus_{i\in I_M} \mathbb{Q}\alpha_i \oplus \bigoplus_{i\in J} \mathbb{Q}\gamma_i = \bigoplus_{i\in I_M} \mathbb{Q}\gamma_i^M \oplus \bigoplus_{i\in J} \mathbb{Q}\gamma_i.$$

Here we have to observe that the $\gamma_i^M, i \in I_M$ are the dominant fundamental weights for the group $M^{(1)}$, they are the orthogonal projections of the γ_i to the first summand in the above decomposition. We have a relation

$$\gamma_j = \gamma_j^M + \sum_{i\in\tilde{i}} c(j,i)\gamma_i, \text{ for } j \in I_M$$

and we have $c(j,i) \geq 0$ for all $i \in J$.

Let W be absolute Weylgroup and subgroup $W_M \subset W$ the Weyl group of M. For the quotient $W_M\backslash W$ we have a canonical system of representatives

$$W^P = \{w \in W \mid w^{-1}(\pi_M) \subset \Delta^+\}.$$

To any $w \in W$ we define $w \cdot \lambda = w(\lambda+\rho) - \rho$ where ρ us the half sum of positive roots. If we do this with an element $w \in W^P$ then $\mu = w \cdot \lambda$ is a highest weight for $M^{(1)}$ and $w \cdot \lambda$ defines us a module for M. Then Kostants theorem says

$$H^\bullet(\mathfrak{u}_P, \mathcal{M}_\lambda) = \bigoplus_{w\in W^P} H^{\ell(w)}(\mathfrak{u}_P, \mathcal{M})(w \cdot \lambda),$$

the summands on the right hand side are the irreducible modules attached to $w \cdot \lambda$, they sit in degree

$$l(w) = \#\{\alpha \in \Delta^+ | w^{-1}\alpha \in \Delta^-\} \tag{170}$$

Each isomorphism class occurs only once.

We write

$$\begin{aligned} w \cdot \lambda = \quad & \underbrace{\mu^{(1,M)} + \delta_1} \quad & +\delta \\ & \in X^*_{\mathbb{Q}}(T^{(1,M)}) \oplus X^*_{\mathbb{Q}}(C_1) \quad & \oplus X^*(C) \end{aligned} \tag{171}$$

We decompose δ_1 and define the numbers $a(w, \lambda)$ (see (168))

$$\delta_1 = \delta_1' + a(w, \lambda)\gamma_J.$$

Then we get

$$w(\lambda + \rho) - \rho = \mu^{(1,M)} + a(w, \lambda)\gamma_J \tag{172}$$

We also consider the extended Weyl group $\tilde{W}$, this is the group of automorphisms of the root system. Let $w_0 \in W$ be the element sending all positive roots into negative ones. We have an automorphism $\Theta_- \in \tilde{W}$ inducing $t \mapsto t^{-1}$ on the torus. Let $\Theta = w_0 \circ \Theta_-$. This element induces a permutation on the set π of positive roots, which may be the identity and induces -1 on the determinant. Then

$$\Theta\lambda = \sum_{i \in I} a_{\Theta i}\gamma_i - \delta$$

is a dominant weight and the resulting highest weight module is dual module to $\mathcal{M}_\lambda$. Therefore we get a non degenerate pairing

$$H^\bullet(\mathfrak{u}_P, \mathcal{M}_\lambda) \times H^\bullet(\mathfrak{u}_P, \mathcal{M}_{\Theta\lambda}) \to H^{d_{U_P}}(\mathfrak{u}_P, F) = F(-2\rho_U),$$

which respects the decomposition, i.e. we get a bijection $w \mapsto w'$ such that $l(w) + l(w') = d_{U_P}$ and such

$$H^{l(w)}(\mathfrak{u}_P, \mathcal{M}_\lambda)(w \cdot \lambda) \times H^{l(w')}(\mathfrak{u}_P, \mathcal{M}_{\Theta\lambda})(w' \cdot \Theta\lambda) \to H^{d_{U_P}}(\mathfrak{u}_P, F) \tag{173}$$

is non degenerate. We conclude

$$a(w, \lambda) + a(w', \Theta\lambda) = -2f_P. \tag{174}$$

We say that $w \cdot \lambda$ is in the positive chamber if

$$a(w, \lambda) \leq -f_P \tag{175}$$

The element Θ conjugates the parabolic subgroup P into the parabolic subgroup Q, which may be equal to P or not. If $P = Q$ resp. $P \neq Q$ then we say that P is (resp. not) conjugate to its opposite parabolic. If Θ_- is in the Weyl group then all parabolic subgroups are conjugate to their opposite. In this case we have $\Theta = 1$.

Conjugating by the element Θ provides an identification $\theta_{P,Q} : W^P \xrightarrow{\sim} W^{\mathbb{Q}}$. We have two specific Kostant representatives, namely the identity $e \in W^P$ and the element $w_P \in W^P$, this is the element which sends all the roots in U_P to negative roots (the longest element). Its length $l(w_P)$ is equal to the dimension $d_P = \dim(U_P)$.

Any element in $w \in W^P$ can be written as product of reflections

$$w = s_{\alpha_{i_1}} \dots s_{\alpha_{i_\nu}} \tag{176}$$

where $\nu = l(w)$ and the first factor $\alpha_{i_1} \in J$. We always can complement this product to a product giving the longest element

$$s_{\alpha_{i_1}} \dots s_{\alpha_{i_\nu}} s_{\alpha_{i_{\nu+1}}} \dots s_{\alpha_{i_{d_P}}} = w s_{\alpha_{i_{\nu+1}}} \dots s_{\alpha_{i_{d_P}}} = w_P, \tag{177}$$

The inverse of the element $s_{\alpha_{i_{\nu+1}}} \dots s_{\alpha_{i_{d_P}}}$ is

$$w' = s_{\alpha_{i_{d_P}}} \dots s_{\alpha_{i_{\nu+1}}} \in W^Q$$

This defines a second bijection $i_{P,Q} : W^P \xrightarrow{\sim} W^Q$ which is defined by the relation

$$w = w_P \cdot i_{P,Q}(w) = w_P \cdot w', \;\; l(w) + l(w') = d_P \tag{178}$$

The composition $\theta_{P,Q}^{-1} \circ I_{P,Q} : W^P \to W^P$ is the bijection provided by duality.

The element w_P conjugates the Levi subgroup M of P into the Levi subgroup of $Q = w_P P w_P^{-1}$. The element $\tilde{w}_P = \Theta w_P$ conjugates the parabolic subgroup P into its opposite (which is conjugate to Q) and induces an automorphism on the subgroup M which is a common Levi-subgroup of P and its opposite.

If we choose $w = e$ then

$$\sum_{i \in I} a_i \gamma_i + \delta = \sum_{i \in I_M} a_i \gamma_i^M + \sum_{j \in J} (\sum_{i \in I_M} a_i c(i,j) + n_j) \gamma_j + \delta.$$

Since J is the orbit of an element $i \in I$ we see that $< \gamma_j, \alpha_j >$ is independent of j and hence we get easily

$$\sum_{j \in J} (\sum_{i \in I_M} a_i c(i,j) + n_j) \gamma_j = \frac{1}{\# J} (\sum_{j \in J} (\sum_{i \in I_M} a_i c(i,j) + n_j)) \gamma_J + \delta'$$

and hence

$$a(e, \lambda) = \frac{1}{\# J} (\sum_{j \in J} (\sum_{i \in I_M} a_i c(i,j) + a_j))$$

If we choose Θ_P then as an M-module $\mathcal{M}_{\Theta_P \cdot \lambda}$ is dual to $\mathcal{M}_{\Theta\lambda}(-2 f_J \gamma_J)$. We write $\Theta\lambda + \rho = \sum_{i \in I} a_{\Theta i} \gamma_i - \delta$ and then

$$w_P (\sum_{i \in I} a_i \gamma_i + \delta) = \sum_{i \in I_M} n_{\Theta i} \gamma_i^M - \sum_{j \in J} (\sum_{\Theta i \in I_M} a_{\Theta i} c(\Theta i, \Theta j) + a_{\Theta j}) \gamma_j - 2 f_J \gamma_J - \delta.$$

and especially we find

$$a(w_P,\lambda)=-(\frac{1}{\#J}(\sum_{j\in J}(\sum_{i\in I_M}a_{\Theta i}c(\Theta i,\Theta j)+a_{\Theta j}))+2f_J)\gamma_J$$

In general we have the inequalities

$$a(\Theta_P,\lambda)\le a(w,\lambda)\le a(e,\lambda).$$

We can write our relation (172) slightly differently. We can move the half sum of positive roots to the right and split into $\rho=\rho^M+f_P\gamma_J$. We put $\tilde{\mu}^{(1)}=\mu^{(1,M)}+\rho^M$ and then we write

$$w(\lambda+\rho)=\tilde{\mu}^{(1)}+(a(w,\lambda)+f_P)\gamma_J=\tilde{\mu}^{(1)}+b(w,\lambda)\gamma_J \tag{179}$$

and of course now we have

$$b(w,\lambda)+b(w',\Theta\lambda)=0. \tag{180}$$

6.1.4 The inverse problem

Later we will encounter the following problem. Our data are as above and we start from a highest weight for M, we write

$$\mu=\mu^{(1)}+\delta_1+a\gamma_J+\delta=\sum_{i\in I_M}n_{\Theta i}\gamma_i^M+\delta_1+a\gamma_J+\delta.$$

We ask whether we can find a λ such that we can solve the equation ($Kost$). More precisely: We give ourselves only the semi simple component $\mu^{(1)}$ of μ and we ask for the solutions

$$w(\lambda+\rho)=\tilde{\mu}^{(1)}+\dots$$

where $w\in W^P$ and λ dominant, i.e. we only care for the semi simple component.

Let us consider the case where $J=\{i_0\}$, i.e. it is just one simple root. Then the term δ_1 disappears and our equation becomes

$$w(\lambda+\rho)=\tilde{\mu}^{(1)}+b\gamma_{i_0}+\delta,$$

of course the δ is irrelevant, but we want to know the range of the values $b=b(\lambda,w)$ when $\tilde{\mu}^{(1)}$ is fixed, but λ,w vary. Of course it may be empty. Let us fix a w and let us assume we have solved $w(\lambda+\rho)=\tilde{\mu}^{(1)}+\dots$. Then it is clear that the other solutions are of the form $\lambda+\rho+\nu$ where $w\nu\in\mathbb{Q}\gamma_{i_0}$. These ν are of the form $\nu=c\nu_0$ with $c\in\mathbb{Z}$. We write $\nu_0=\sum_{i\in I}b_i\gamma_i$ and it is easy to see that there must be some $b_i>0$ and some $b_j<0$. This implies that $\lambda+c\nu_0$ is dominant if and only if $c\in[M,N]$, an interval with integers as boundary point. This of course implies that -still for a given w - the values $b=b(\lambda,w)$ also have to lie in a fixed finite interval

$$b=b(w,\lambda)\in[b_{\min}(w,\tilde{\mu}^{(1)}),a_{\max}(w,\tilde{\mu}^{(1)}]=I(w,\tilde{\mu}^{(1)}).$$

This will be of importance because these intervals will be related to intervals of critical values of L-functions.

6.2 The goal of Eisenstein cohomology

The goal of the Eisenstein cohomology is to provide an understanding of the restriction map r in theorem (2.1). More precisely we assume that we understand (can describe) the cohomology $H^\bullet(\partial\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda)$ then we want to understand the image $H^\bullet_{\mathrm{Eis}}(\partial\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda)$ in terms of this description. Under certain conditions we will construct a section $\mathrm{Eis}: H^i_{\mathrm{Eis}}(\partial\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\lambda,\mathbb{C}})\to H^i(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\lambda,\mathbb{C}})$. It is clear from the previous considerations that understanding of $H^\bullet(\partial\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda)$ requires understanding cohomology of $H^\bullet(\mathcal{S}^M_{K^M_f},\widetilde{H^\bullet(\mathfrak{u},\mathcal{M})})$ and we have to compute the differentials in the spectral sequence. These differentials will depend on the Eisenstein cohomology of $H^\bullet(\mathcal{S}^M_{K^M_f},\widetilde{H^\bullet(\mathfrak{u},\mathcal{M})})$. Under certain conditions the spectral sequence degenerates at E_2 level and I do not know whether this is true in general. In a certain sense it would be much more interesting if this is not the case.

We consider certain submodules in the cohomology of the Borel-Serre compactification for which we can construct a section as above. We start from a maximal parabolic subgroup $P/\mathbb{Q}$, let $M/\mathbb{Q}$ be its reductive quotient. We define

$$H^\bullet_!(\partial_P\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda)=\bigoplus_{w\in W^P}H^{\bullet-l(w)}_!(\mathcal{S}^M_{K^M_f},H^{l(w)}(\mathfrak{u}_P,\tilde{\mathcal{M}})(w\cdot\lambda))\subset H^\bullet(\partial_P\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda) \tag{181}$$

We will abbreviate $H^{l(w)}(\mathfrak{u}_P,\tilde{\mathcal{M}})(w\cdot\lambda)=\tilde{\mathcal{M}}(w\cdot\lambda)$ where always keep in mind that the element $w\in W^P$ knows what the actual parabolic subgroup is and that $\tilde{\mathcal{M}}(w\cdot\lambda)$ sits in degree $l(w)$.

By definition the inner cohomology is the image of the cohomology with compact supports. This implies that the submodule

$$\bigoplus_{P:d(P)=1}H^q_!(\partial_P\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda)\subset\bigoplus_{P:d(P)=1}H^q(\partial_P\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda)=E_1^{0,q}$$

is annihilated by all differentials $d_\nu^{0,q}$ and hence we get an inclusion

$$i_P:\oplus_{w\in W^P}I^G_PH^{\bullet-l(w)}_!(\mathcal{S}^M_{K^M_f},\mathcal{M}(w\cdot\lambda))\to H^\bullet(\partial\mathcal{S}^G_{K_f},\mathcal{M}_\lambda) \tag{182}$$

Taking the direct sum over the maximal parabolic subgroups yields a submodule

$$H^\bullet_!(\partial\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda)\hookrightarrow H^\bullet(\partial\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda) \tag{183}$$

The Hecke algebra acts on these two modules. Let us assume that this submodule when tensorized by $\mathbb{Q}$ is isotypical in $H^\bullet_!(\partial\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda\otimes\mathbb{Q})$. Then we get a decomposition

$$H^\bullet_!(\partial\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda\otimes\mathbb{Q})\oplus H^\bullet_{\mathrm{non!}}(\partial\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda\otimes\mathbb{Q})=H^\bullet(\partial\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda\otimes\mathbb{Q}). \tag{184}$$

We formulated the goal of the Eisenstein cohomology, we described an isotypical subspace and we know can ask: What is the intersection of $H^\bullet_{\mathrm{Eis}}(\partial\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda\otimes\mathbb{Q})$ with this subspace, or what amounts to the same, what is $H^\bullet_{!,\mathrm{Eis}}(\partial\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda\otimes\mathbb{Q})$.

The element Θ induces an involution on the set of parabolic subgroups containing B (= set of $G(\mathbb{Q})$ conjugacy classes of parabolic subgroups) two parabolic subgroups $P, Q \supset B$ are called associate if $\Theta P = Q$. We can decompose the cohomology $H_!^\bullet(\partial\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{Q})$ into summands attached to the classes of associated parabolic subgroups

$$H_!^\bullet(\partial\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{Q}) = \bigoplus_{P:P=\Theta P} H_!^\bullet(\partial_P\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) \oplus \bigoplus_{[P,Q]} H_!^\bullet(\partial_P\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) \oplus H_!^\bullet(\partial_Q\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) \tag{185}$$

where in the second sum $Q = \Theta P$. Each summand is a sum over the elements of W^P and then we can decompose under the action of the Hecke algebra. We choose a sufficiently large extension $F/\mathbb{Q}$ and in the case $P = \Theta P$ we get

$$H_!^\bullet(\partial_P\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes F) = \bigoplus_{w\in W^P} \bigoplus_{\sigma_f} H_!^{\bullet - l(w)}(\mathcal{S}^M_{K^M_f}, \tilde{\mathcal{M}}(w\cdot\lambda) \otimes F)(\sigma_f) \tag{186}$$

In the case $P \neq \Theta P = Q$ we group the contributions from the two parabolic subgroups together. To any $w \in W^P$ we have the element $i_{P,Q}(w) = w' \in W^Q$. We also group the terms corresponding to w and w' together. To any σ_f which occurs in $H_!^{\bullet-l(w)}(\mathcal{S}^M_{K^M_f}, H^{l(w)}(\mathfrak{u}_P, \tilde{\mathcal{M}})(w\cdot\lambda)\otimes F)$ we find a $\sigma'_f = \sigma_f^{w_P}|\gamma_{\Theta j}|_f^{2f_Q}$, which occurs in the second summand.

The decomposition into isotypical pieces becomes

$$\bigoplus_{\sigma_f}\Big(H_!^{\bullet-l(w)}(\mathcal{S}^M_{K^M_f}, \tilde{\mathcal{M}}(w\cdot\lambda)\otimes F)(\sigma_f) \oplus H_!^{\bullet-l(w')}(\mathcal{S}^{M'}_{K^{M'}_f}, \tilde{\mathcal{M}}(w'\cdot\lambda)\otimes F)(\sigma'_f)\Big) \tag{187}$$

We can define the second step in the filtration (20) as the inverse image of $H_!^\bullet(\partial\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$ under the restriction r.

6.2.1 Induction and the local intertwining operator at finite places

Our modules σ_f are modules for the Hecke algebras $\mathcal{H}^M_{K^M_f} = \otimes_p \mathcal{H}^M_{K^M_p}$. Therefore we can write them as tensor product $\sigma_f = \otimes_p \sigma_p$. We consider a prime p where σ_f is unramified then we get can give a standard model for this isomorphism class. The module H_{σ_p} is the rank one $\mathcal{O}_F$ -module $\mathcal{O}_F$, i.e. it comes with a distinguished generator 1. The Hecke algebra acts by a homomorphism (See 2.3)

$$h(\sigma_p) : \mathcal{H}^{(M,w\cdot\lambda)}_{K^M_p,\mathbb{Z}} \to \mathcal{O}_F \tag{188}$$

and gives us the Hecke-module structure on H_{σ_p}. We can induce H_{σ_p} to a $\mathcal{H}^G_{K^G_p}$ module. This is actually the same $\mathcal{O}_F$ module but now with an action of the algebra $\mathcal{H}^{(G,\lambda)}_{K^G_p,\mathbb{Z}}$. We simply observe that we have an inclusion $\mathcal{H}^{(G,\lambda)}_{K^G_p,\mathbb{Z}} \hookrightarrow \mathcal{H}^{(M,w'\cdot\lambda)}_{K^M_p,\mathbb{Z}}$ and induction simply means restriction.

It follows easily from the description of the description of the spherical (unramified) Hecke modules via their Satake-parameters that the induced modules H_{σ_p} and $H_{\sigma'_p}$ are isomorphic as $\mathcal{H}^{(G,\lambda)}_{K^G_p,\mathbb{Z}}$ -modules and hence we get that after induction the two summands in (187) become isomorphic. We choose a local intertwining operator

$$T_p^{\mathrm{loc}} : H_{\sigma_p} \to H_{\sigma'_p} \tag{189}$$

simply the identity.

We postpone the discussion of a local intertwining operator at ramified places.

6.3 The Eisenstein intertwining operator

We start from an irreducible unitary module $H_{\sigma_\infty} \times H_{\sigma_f} = H_\sigma$ and assume that we have an inclusion $\Phi : H_\sigma \hookrightarrow L^2_{\mathrm{cusp}}(M(\mathbb{Q})\backslash M(\mathbb{A}))$. We assume that σ_f occurs in the cohomology $H^\bullet_{\mathrm{cusp}}(\mathcal{S}^M_{K_f^M}, \tilde{\mathcal{M}}(w\cdot\lambda)_\mathbb{C})$ and we assume that $w\cdot\lambda$ is in the positive chamber. We consider Φ as an element of $W(\sigma)$ and for the moment we identify H_σ to its image under Φ. We stick to our assumption that σ occurs with multiplicity one in the cuspidal spectrum.

Then we we can consider the induced module, recall that this is the space of functions

$$\{f : G(\mathbb{A}) \to H_\sigma | f(\underline{pg}) = \bar{\underline{p}} f(\underline{g})\} \tag{Ind}$$

where $\bar{\underline{p}}$ is the image of $\underline{p}$ in $M(\mathbb{A})$. We can define the subspace $H_\sigma^{(\infty)}$ consisting of those f which satisfy some suitable smoothness conditions and then we can define a submodule $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} H_\sigma^{(\infty)}$ where the $f(g) \in H_\sigma^{(\infty)}$ and the f themselves also satisfy some smoothness conditions.

We embed this space into the space $\mathcal{A}(P(\mathbb{Q})\backslash G(\mathbb{A}))$ by sending

$$f \mapsto \{g \mapsto f(g)(e_M)\},$$

here $\mathcal{A}$ denotes some space of automorphic forms. This an embedding of $G(\mathbb{A})$-modules or an embedding of Hecke modules if we fix a level.

We have the character $\gamma_P : M \to G_m$, for any complex number z this yields a homomorphism $|\gamma_P|^z : M(\mathbb{A}) \to \mathbb{R}^\times$ which is given by $|\gamma_P| : \underline{m} \mapsto |\gamma_P(\underline{m})|^z$. As usual we denote by $\mathbb{C}(|\gamma_P|^z)$ the one dimensional $\mathbb{C}$ vector space on which $M(\mathbb{A})$ acts by the character $|\gamma_P|^z$. Then we may twist the representation H_σ by this character and put $H_\sigma \otimes |\gamma_P|^z = H \otimes \mathbb{C}(|\gamma_P|^z)$. An element $\underline{g} \in G(\mathbb{A})$ can be written as $\underline{g} = \underline{pk}, \underline{p} \in P(\mathbb{A}), \underline{k} \in K_f^0$ where $K_f^0 \supset K_f$ is a suitable maximal compact subgroup and now we define $h(\underline{g}) = |\gamma_P|(\underline{p})$.

Eisenstein summation yields embeddings

$$\mathrm{Eis} : \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} H_\sigma^{(\infty)} \otimes |\gamma_P|^z \to \mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A})), \tag{190}$$

where

$$\mathrm{Eis}(f)(\underline{g}) = \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} f(\gamma \underline{g})(e_M) h(\gamma \underline{g})^z,$$

it is well known that this is locally uniformly convergent provided $\Re(z) >> 0$ and it has meromorphic continuation into the entire z plane (See [Ha-Ch]).

We assumed that H_σ is in the cuspidal spectrum. We get important information concerning these Eisenstein series, if we compute their constant Fourier coefficient with respect to parabolic subgroups: For any parabolic subgroup $P_1/\mathbb{Q} \subset G/\mathbb{Q}$ with unipotent radical $U_1 \subset P_1$ we define (See [Ha-Ch], 4)

$$\mathcal{F}^{P_1}(\mathrm{Eis}(f))(\underline{g}) = \int_{U_1(\mathbb{Q})\backslash U_1(\mathbb{A})} \mathrm{Eis}(f)(\underline{u}\underline{g})(e_M)d\underline{u}.$$

This essentially only depends on the $G(\mathbb{Q})$-conjugacy class of $P_1/\mathbb{Q}$. It it also in [Ha-Ch] , 4 that this constant term is zero unless P_1 is maximal and the conjugacy class of P_1 is equal to the conjugacy class of $P/\mathbb{Q}$ or the conjugacy class of $Q/\mathbb{Q}$. (which may or may not be equal to the conjugacy class of $P/\mathbb{Q}$.)

These constant Fourier coefficients have been computed by Langlands, we have to distinguish the two cases:

a) The parabolic subgroup $P/\mathbb{Q}$ is conjugate to an opposite parabolic $Q/\mathbb{Q}$.

In this case we have a Kostant representative $w^P \in W^P$ which conjugates $Q/\mathbb{Q}$ into $P/\mathbb{Q}$ and it induces an automorphism of $M/\mathbb{Q}$. We get a twisted representation $w^P(\sigma)$ of $M(\mathbb{A})$. In the computation of the the constant term we have to exploit that σ is cuspidal and we get two terms:

$$\begin{gathered}\mathcal{F}^P \circ \mathrm{Eis}: \ \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} H_\sigma \otimes |\gamma_P|^z \to \\ \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} H_\sigma \otimes |\gamma_P|^z \oplus \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} H_{w^P(\sigma)} \otimes |\gamma_Q|^{2f_P - z} \subset \mathcal{A}(P(\mathbb{Q})\backslash G(\mathbb{A})).\end{gathered} \tag{191}$$

We can describe the image. It is well known, that we can define a holomorphic family

$$T^{\mathrm{loc}}(z) : \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} H_\sigma \otimes |\gamma_P|^z \to \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} H_{\sigma^{w^P}} \otimes |\gamma_Q|^{2f_P - z}$$

which is defined in a neighborhood of $z = 0$ and which is nowhere zero. This local intertwining operator is unique up to a nowhere vanishing holomorphic function $h(z)$. It is the tensor product over all places $T^{\mathrm{loc}}(z) = \otimes_v T_v^{\mathrm{loc}}(z)$. For the unramified finite places the local operator is constant, i.e. does not depend on z and is equal to T_p^{loc} in section (6.2.1) and $T^{\mathrm{loc}}(0) = \otimes_p T_p^{\mathrm{loc}}$. At the remaining factors there is a certain arbitrariness for the choice of the local operator and some fine tuning is appropriate.

We also assume that we have chosen nice model spaces $H_{\sigma_\infty}, H_{\sigma'_\infty}$, and an intertwining operator

$$T_\infty^{\mathrm{loc}} : H_{\sigma_\infty} \to H_{\sigma'_\infty} \tag{192}$$

which is normalized by the requirement that it induces the "identity" on a certain fixed K_∞^M type.

Then we get the classical formula of Langlands for the constant term: For $f \in \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} H_\sigma \otimes |\gamma_P|^z$ we get

$$\mathcal{F}^P \circ \mathrm{Eis}(f) = f + C(\sigma, z) T^{\mathrm{loc}}(z)(f), \tag{193}$$

where $C(\sigma, \lambda, z)$ is a product of local factors $C(\sigma_v, z)$ and where $C(\sigma_v, z)$ is a function in z which is holomorphic for $\Re(z) \geq 0$ (here we need that $w \cdot \lambda$ is in the positive chamber.) This function compares our local intertwining operator to an intertwining operator which is defined by the integral.

The computation of this factor is carried out in H. Kims paper in [C-K-M], chap. 6. He expresses the factor in terms of the automorphic L function attached to σ_f. To formulate the result of this computation we have to recall the notion of the dual group (3.1). Inside the dual group ${}^LG$ we have the dual group ${}^LM$ which acts by conjugation on the Lie algebra $\mathfrak{u}_P^\vee$. The set of roots $\Delta^+_{U_P^\vee}$ is a set of cocharacters of $T/\mathbb{Q}$, a coroot $\alpha^\vee \in \Delta^+_{U_P}$ defines a one-dimensional root subgroup $\mathfrak{u}^\vee_{P,\alpha^\vee}$. The ${}^LM$ -module $\mathfrak{u}_P^\vee$ decomposes into submodules. We recall that the maximal parabolic subgroup $P/\mathbb{Q}$ was obtained from the choice of a Galois-orbit $\tilde{i} \subset I$ (6.1.3) and any

$$\alpha^\vee \in \Delta^+_{U_P^\vee},\ \chi = a(\alpha^\vee, \tilde{i})\chi_{\tilde{i}} + \sum_{j \notin \tilde{i}} m_{\tilde{i},j}\chi_j. \tag{194}$$

Here the coefficients are integers ≥ 0 and $a(\alpha^\vee, \tilde{i}) > 0$. For a given integer $a > 0$ we define

$$\mathfrak{u}_P^\vee[a] = \bigoplus_{\alpha^\vee : a(\alpha^\vee, \tilde{i}) = a} \mathfrak{u}^\vee_{P,\alpha^\vee} \tag{195}$$

it is rather obvious that $\mathfrak{u}_P^\vee[a]$ is an invariant submodule under the action of M and actually it is even irreducible. Let us denote the representation of $M/\mathbb{Q}$ on $\mathfrak{u}_P^\vee[a]$ by $r_a^{\mathfrak{u}_P^\vee}$. In the following η_a will be the highest weight of $r_a^{\mathfrak{u}_P^\vee}$.

With these notations we get the following formula for the local factor at p (See[Kim])

$$C_p(\sigma, z) = \prod_{a=1}^{r} \frac{L^{\text{aut}}(\sigma_p, r_a^{\mathfrak{u}_P^\vee}, a(z - f_P))}{L^{\text{aut}}(\sigma_p, r_a^{\mathfrak{u}_P^\vee}, a(z - f_P) + 1)} T_p^{\text{loc}}(z)(f) \tag{196}$$

We do not discuss the ramified finite places, from now on we assume that σ_f is unramified. Then we get

$$C(\sigma, z) = C(\sigma_\infty, z) \prod_p C_p(\sigma_p, z) = C(\sigma_\infty, z) \prod_{a=1}^{r} \frac{L^{\text{aut}}(\sigma_f, r_a^{\mathfrak{u}_P^\vee}, a(z - f_P))}{L^{\text{aut}}(\sigma_f, r_a^{\mathfrak{u}_P^\vee}, a(z - f_P) + 1)}$$

The local factor at infinity depends on the choice of T_∞^{loc}, in 1.2.4 we gave some rules how to fix it, if it is not zero on cohomology.

b) The opposite group $Q/\mathbb{Q}$ is not conjugate to $P/\mathbb{Q}$, then we have to compute two Fourier coefficients namely $\mathcal{F}^P$ and $\mathcal{F}^Q$ in this case we get

$$\mathcal{F} : \text{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} H_\sigma \otimes |\gamma_P|^z \stackrel{\mathcal{F}^P \oplus \mathcal{F}^Q}{\longrightarrow}$$

$$\text{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} H_\sigma \otimes |\gamma_P|^z \oplus \text{Ind}_{Q(\mathbb{A})}^{G(\mathbb{A})} H_\sigma \otimes |\gamma_Q|^{2f_P - z} \subset \mathcal{A}(P(\mathbb{Q})\backslash G(\mathbb{A})) \oplus \mathcal{A}(Q(\mathbb{Q})\backslash G(\mathbb{A})).$$

and again we get

$$\mathcal{F} \circ \text{Eis}(f) = f + C(\sigma_\infty, z) \prod_a \frac{L^{\text{aut}}(\sigma_f, r_a^{\mathfrak{u}_P^\vee}, a(z - f_P))}{L^{\text{aut}}(\sigma_f, r_a^{\mathfrak{u}_P^\vee}, a(z - f_P) + 1)} T^{\text{loc}}(z)(f), \tag{197}$$

where now $T^{\mathrm{loc}}(z)$ is a product of local intertwining operators

$$T_v^{\mathrm{loc}} : \mathrm{Ind}_{P(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)} H_{\sigma_v} \otimes |\gamma_P|^z \to \mathrm{Ind}_{Q(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)} H_{\sigma_v^{w^P}} \otimes (2f_P - z).$$

It is also due to Langlands that the Eisenstein intertwining operator is holomorphic at $z = 0$ if the factor in front of the second term is holomorphic at $z = 0$. Up to here σ can be any representation occurring in the cuspidal spectrum of M.

Now we assume that we have a coefficient system $\mathcal{M} = \mathcal{M}_\lambda$ and a $w \in W^P$ such that our σ_f occurs in $H_!^{\bullet - l(w)}(\mathcal{S}^M_{K_f^M}, \tilde{\mathcal{M}}(w \cdot \lambda) \otimes F)$. Then we find a $(\mathfrak{m}, K_\infty^M)-$ module H_{σ_∞} such that $H^\bullet(\mathfrak{m}, K_\infty^M, H_{\sigma_\infty} \otimes \mathcal{M}(w \cdot \lambda)) \neq 0$. We also find an embedding

$$\Phi_\iota : H_{\sigma_\infty} \otimes H_{\sigma_f} \otimes_{F,\iota} \mathbb{C} \hookrightarrow L^2_{\mathrm{cusp}}(M(\mathbb{Q}) \backslash M(\mathbb{A})) \tag{198}$$

Let us assume that $w \cdot \lambda$ or equivalently σ_f are in the positive chamber. In case a) we have holomorphicity at $z = 0$ if the weight λ is regular (See [Schw]) and in case b) the Eisenstein series is always holomorphic at $z = 0$. In this section that we assume that the Eisenstein series is holomorphic at $z = 0$ and hence we can evaluate at $z = 0$ in (344) and get an intertwining operator

$$\mathrm{Eis} \circ \Phi_\iota : \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} H_\sigma \to \mathcal{A}(G(\mathbb{Q}) \backslash G(\mathbb{A})). \tag{199}$$

We get a homomorphism on the de-Rham complexes

$$\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} H_\sigma \otimes_{F,\iota} \mathbb{C} \otimes \mathcal{M}_\lambda) \to \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathcal{A}(G(\mathbb{Q}) \backslash G(\mathbb{A})) \otimes \tilde{\mathcal{M}}_\lambda) \tag{200}$$

We introduce the abbreviation $H_{\iota \circ \sigma_f} = H_{\sigma_f} \otimes_{F,\iota} \mathbb{C}$ and decompose $H_{\iota \circ \sigma} = H_{\sigma_\infty} \otimes H_{\iota \circ \sigma_f}$. We compose (200) with the constant term and get

$$\mathcal{F} \circ \mathrm{Eis}^\bullet : \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})} H_{\sigma_\infty} \otimes \mathcal{M}_\lambda) \otimes H_{\iota \circ \sigma_f} \to$$
$$\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})} H_{\sigma_\infty} \otimes \mathcal{M}_\lambda) \otimes H_{\iota \circ \sigma_f}) \oplus \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathrm{Ind}_{Q(\mathbb{R})}^{G(\mathbb{R})} H_{\sigma'_\infty} \otimes \mathcal{M}_\lambda) \otimes H_{\iota \circ \sigma'_f}) \tag{201}$$

where $P = Q$ in case a).

We choose an $\omega \in \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})} \otimes \mathcal{M}_\lambda)$ and consider classes $\omega \otimes \psi_f$ and map them by the Eisenstein intertwining operator to the cohomology (or the de-Rham complex) on $\mathcal{S}^G_{K_f}$. Then the restriction of of the Eisenstein cohomology to the boundary is given by the classes

$$\Phi_\iota(\omega \otimes \psi_f + \frac{1}{\Omega(\sigma_f)} C(\sigma_\infty, \lambda) C(\sigma_f, \lambda) T_\infty^{\mathrm{loc}}(\omega) \otimes T_f^{\mathrm{loc}}(\psi_f)) \tag{202}$$

Here the factor $C(\sigma_f, \lambda)$ can be expressed in terms of the cohomological L-function. Translating the formula (196) yields (see 179)

$$C(\sigma_f, \lambda) = \prod_a \frac{L^{\mathrm{coh}}(\sigma_f, r_a^{\mathfrak{u}_P^\vee}, < \eta_a, \tilde{\mu}^{(1)} > - b(w, \lambda) < \eta_a, \gamma_P >)}{L^{\mathrm{coh}}(\sigma_f, r_a^{\mathfrak{u}_P^\vee}, < \eta_a, \tilde{\mu}^{(1)} > - b(w, \lambda) < \eta_a, \gamma_P > + 1)} \tag{203}$$

We may complete the cohomological L-function by the correct factor at infinity and replace the ratio of L-values by the corresponding ratio of values for the completed $L-$ function. By definition we have $<\eta_a,\gamma_P>=a$ and then our formula for the second term in (202) becomes

$$\frac{1}{\Omega(\sigma_f)}\prod_a \frac{\Lambda^{\mathrm{coh}}(\sigma_f, r_a^{\mathfrak{u}_P^\vee}, <\eta_a,\tilde{\mu}^{(1)}>-ab(w,\lambda))}{\Lambda^{\mathrm{coh}}(\sigma_f, r_a^{\mathfrak{u}_P^\vee}, <\eta_a,\tilde{\mu}^{(1)}>-ab(w,\lambda)+1)} C^*(\sigma_\infty,\lambda)T_\infty^{\mathrm{loc}}(\omega)\otimes T_f^{\mathrm{loc}}(\psi_f)) \tag{204}$$

This formula needs some comments. The factor $C^*(\sigma_\infty)T_\infty^{\mathrm{loc}}$ is a representation theoretic contribution it is not easy to understand. Experience shows that becomes very simple at the end. In SecOps.pdf we discuss the special case of the symplectic group.

The number $\Omega(\sigma_f)$ is a period, it will be discussed later.

We see that the constant term is the sum of two terms. The first term reproduces the original class from which we started. We assumed that w or $w\cdot\lambda$ it is in the positive chamber (see(175)). The second term is some kind of scattering term which is the image of the first term under an intertwining operator. In case a) the restriction of the second term gives a class in the same stratum, in case b) the restriction of the second term gives a class in a second stratum.

At this point I formulate a general principle

Under certain circumstances the second term is of fundamental arithmetic interest, it contains relevant arithmetic information.

To exploit this information we have to understand several aspects of the behavior of this second term in the constant term. We have to recall that is obtained as the evaluation of a meromorphic function $C(\sigma_f,\lambda,z)$ at $z=0$, i.e. we have to know whether it has pole at $z=0$ or not. We also have to understand the contribution $C(\sigma_\infty,\lambda)T_\infty^{\mathrm{loc}}$, and we have to understand the arithmetic nature of this term, it is a product and some of the factors yield an algebraic number and the rest will have a motivic interpretation. This is explained further down and in [Mix-Mot-2013.pdf].

We give some more detailed indications how such arithmetic applications may look like. We assume that $w\cdot\lambda$ is in the positive chamber and $l(w)\geq l(w')$. Let us also assume that the Eisenstein intertwining operator is holomorphic at $z=0$. Then we have to look at

$$T_\infty^{\mathrm{loc},\bullet}:\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}),\mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}H_{\sigma_\infty}\otimes\mathcal{M}_\lambda)\to\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}),\mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}H_{\sigma'_\infty}\otimes\mathcal{M}_\lambda) \tag{205}$$

The two complexes can be described by the Delorme isomorphism

$$\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}),\mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}H_{\sigma_\infty}\otimes\mathcal{M}_\lambda)\xrightarrow{\sim}\bigoplus_{w\in W^P}\mathrm{Hom}_{K_\infty^M}(\Lambda^{\bullet-l(w)}(\mathfrak{m}_{\mathbb{C}}^{(1)}/\mathfrak{k}^M)),H_{\sigma_\infty}\otimes\mathcal{M}(w\cdot\lambda)) \tag{206}$$

Our intertwining operator respects this decomposition and we get

$$T_\infty^{\mathrm{loc},\bullet}(w) : \mathrm{Hom}_{K_\infty^M}(\Lambda^{\bullet - l(w)}(\mathfrak{m}_{\mathbb{C}}^{(1)}/\mathfrak{k}^M)), H_{\sigma_\infty} \otimes \mathcal{M}(w \cdot \lambda)) \to$$
$$\mathrm{Hom}_{K_\infty^M}(\Lambda^{\bullet - l(w')}(\mathfrak{m}_{\mathbb{C}}^{(1)}/\mathfrak{k}^M)), H_{\sigma'_\infty} \otimes \mathcal{M}(w' \cdot \lambda))$$

Now we know that for regular representations $\mathcal{M}_\lambda$ the cohomology $H^\nu(\mathfrak{m}, K_\infty^M, H_{\sigma_\infty} \otimes \mathcal{M}(w \cdot \lambda))$ is non zero only for ν in a very narrow interval around the middle degree (See [Vo-Zu], Thm. 5.5). If the difference $|l(w) - l(w')|$ is greater than the length of this interval, then the following condition is fulfilled

In any degree $T_\infty^{\mathrm{loc},\bullet}(w)$ induces zero on the cohomology. (Tzero)

In this cases (and under the assumption that the Eisenstein series is holomorphic at $z = 0$) the Eisenstein intertwining operator gives us a section for the Hecke-modules

$$\mathrm{Eis}_{\mathbb{C}} : H^{q-l(w)})(\mathcal{S}^M_{K_f^M}, \mathcal{M}(w \cdot \lambda) \otimes \mathbb{C})(\sigma_f) \to H^q(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{C}) \tag{207}$$

6.4 The special case Gl_n

Our group is Gl_n/Q and we choose a parabolic subgroup P containing the standard Borel subgroup and with reductive quotient $M = \mathrm{Gl}_{n_1} \times \mathrm{Gl}_{n_2} \times \cdots \times \mathrm{Gl}_{n_r}$. We want to construct Eisenstein cohomology classes in $H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,\mathbb{C}})$ starting from cuspidal classes in $H^\bullet(\partial_P \mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,\mathbb{C}})$. For an element $w \in W^P$ we write

$$w(\lambda + \rho) = \underline{\mu}^{(1)} - b_1(w,\lambda)\gamma_{n_1} - b_2(w,\lambda)\gamma_{n_1+n_2} + \cdots - b_r(w,\lambda)\gamma_{n_1+\cdots+n_{r-1}} + d\delta. \tag{208}$$

It is the sum of the semi simple part (with respect to M)

$$\underline{\mu}^{(1)} = (b_1\gamma_1^M + \cdots + b_{n_1-1}\gamma^M_{n_1-1}) + (b_{n_1+1}\gamma^M_{n_1+1} + \ldots b_{n_1+n_2-1}\gamma^M_{n_1+n_2-1}) + \ldots \tag{209}$$
$$= \mu_1^{(1)} + \cdots + \mu_r^{(1)} \tag{210}$$

and the abelian part $\underline{\mu}^{\mathrm{ab}}$.

We assume that $b_i(w,\lambda) \geq 0$ i.e. $w(\lambda + \rho)$ is in the negative chamber and we also assume that the $\mu_i^{(1)}$ are self dual, this is a condition on λ, w. We decompose the strongly inner cohomology

$$H^\bullet_{\mathrm{cusp}}(\partial_P \mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) = \bigoplus_{w \in W^P} \bigoplus_{\underline{\sigma}_f} \mathrm{Ind}_P^G H^{\bullet - l(w)}_{\mathrm{cusp}}(\mathcal{S}^M_{K_f^M}, \tilde{\mathcal{M}}_{w\cdot\lambda})(\underline{\sigma}_f) \tag{211}$$

The Künneth-theorem implies that $\underline{\sigma}_f = \sigma_{1,f} \otimes \sigma_{2,f} \otimes \cdots \otimes \sigma_{r,f}$. At an unramified place p then this module has a Satake parameter

$$\omega_p(\sigma_f) = \{\omega_{1,p}, \ldots, \omega_{n_1,p}, \omega_{n_1+1,p}, \ldots, \omega_{n_1+n_2,p}, \ldots\}$$

where the first n_1 entries are the Satake parameters of $\sigma_{1,f}$ and so on.

We choose an $\iota : E \to \mathbb{C}$. We take an irreducible submodule $H_{\underline{\sigma}_f}$ then we find an irreducible $(\mathfrak{g}, K_\infty^M)$-module $H_{\underline{\sigma}_\infty}$ and an embedding

$$\Phi : H_{\underline{\sigma}_\infty} \otimes H_{\underline{\sigma}_f} \otimes_{E,\iota} \mathbb{C} = H_{\underline{\sigma}} \hookrightarrow \mathcal{C}_{\text{cusp}}(M(\mathbb{Q})\backslash M(\mathbb{A})) \tag{212}$$

For $\underline{z} = (z_1, z_2, \ldots, z_{r-1}), z_i \in \mathbb{C}$ we define the character

$$|\gamma_P|^{\underline{z}} = |\gamma_{n_1}|^{z_1} |\gamma_{n_1+n_2}|^{z_2} \ldots |\gamma_{n_1+n_2+\cdots+n_{r-1}}|^{z_{r-1}} : M(\mathbb{A}) \to \mathbb{C}^\times$$

By the usual summation process we get an Eisenstein intertwining operator

$$\mathrm{Eis}(\underline{\sigma}, \underline{z}) : I_P^G H_{\underline{\sigma}} \otimes |\gamma_P|^{\underline{z}} \to \mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A})) \tag{213}$$

the series is locally uniformly converging in a region where all $\Re(z_i) >> 0$ and hence the Eisenstein intertwining operator is holomorphic in this region. We know that it admits a meromorphic extension into the entire $\mathbb{C}^{r-1}$.

We want to evaluate at $\underline{z} = 0$ this is possible if $\mathrm{Eis}(\underline{\sigma}, \underline{z})$ is holomorphic at $\underline{z} = 0$, we have to find out what happens at $\underline{z} =$ we have to consider the constant term (constant Fourier coefficient) of $\mathrm{Eis}(\underline{\sigma}, \underline{z})$ along parabolic subgroups P_1. (See [H-C]) These constant Fourier coefficients a given by integrals

$$\mathcal{F}^{P_1} : f(\underline{g}) \mapsto \int_{U_{P_1}(\mathbb{Q})\backslash U_{P_1}(\mathbb{A})} f(\underline{u}\underline{g}) d\underline{u}. \tag{214}$$

It suffices to compute these constant terms only for parabolic subgroups containing our given maximal torus. It is shown in [H-C] that the constant term evaluated at $\mathrm{Eis}(\underline{\sigma}, \underline{z})(f)$ is zero unless P and P_1 are associate, this means that the Levi subgroups M and M_1 are isomorphic. (For this we need the cuspidality condition (See [H-C],)(But then we can find an element in the Weyl group which conjugates M into M_1 and hence we may assume that P and P_1 both contain our given Levi subgroup M. Of course now P_1 may not contain the standard Borel subgroup.)

We may also assume that $n_1 = n_2 = \cdots = n_{j_1} < n_{j_1+1} = \cdots = n_{j_1+j_2} < \cdots < n_{j_1+\ldots j_{s-1}+1} = \cdots = n_{j_1+\cdots+j_s} = n_r$, Then it is easy to see that the number of conjugacy classes of parabolic subgroups which contain M is equal to $r!/j_1!j_2!...j_s!$.

We compute $\mathcal{F}^{P_1} \circ \mathrm{Eis}(\underline{\sigma}, \underline{z})(f)$ following [H-C], . By definition (adelic variables in $U(\mathbb{A}), P(\mathbb{A}), ...$are underlined)

$$\mathcal{F}^{P_1} \circ \mathrm{Eis}(\underline{\sigma}, \underline{z})(f)(\underline{g}) = \int_{U_{P_1}(\mathbb{Q})\backslash U_{P_1}(\mathbb{A})} \sum_{a \in P(\mathbb{Q})\backslash G(Q)} f_{\underline{z}}(a\underline{u}\underline{g}) d\underline{u} \tag{215}$$

Let W_M be the Weyl group of M, the Bruhat decomposition yields $G(\mathbb{Q}) = \bigcup_{w \in W} P(\mathbb{Q})\backslash w P_1(\mathbb{Q})$, put $P_1^{(w)}(\mathbb{Q}) = w^{-1}P(\mathbb{Q})w \cap P_1(\mathbb{Q})$ then our expression becomes (we pull the summation over W to the front)

$$\mathcal{F}^{P_1} \circ \mathrm{Eis}(\underline{\sigma}, \underline{z})(f)(\underline{g}) = \sum_{W_{M_1}\backslash W^{M,M_1}/W_M} \int_{U_{P_1}(\mathbb{Q})\backslash U_{P_1}(\mathbb{A})} \sum_{b \in P_1^{(w)}(\mathbb{Q})\backslash P_1(\mathbb{Q})} f_{\underline{z}}(wb\underline{u}\underline{g}) d\underline{u} \tag{216}$$

where W_M is the Weyl group of M. If now for a given w the intersection of algebraic groups $w^{-1}U_1 w \cap M = V$ has dimension > 0, then this intersection is the unipotent radical of a proper parabolic subgroup of M. Since σ is cuspidal the integral over $V(\mathbb{Q})\backslash V(\mathbb{A})$ is zero, therefore this w contributes by zero. Hence we can restrict our summation over those $w \in W$ which satisfy $wMw^{-1} = M_1$. let us call this set W^{M,M_1}. But then

$$P_1^{(w)}(\mathbb{Q})\backslash P_1(\mathbb{Q}) = w^{-1}U_P(\mathbb{Q})w \cap U_{P_1}(\mathbb{Q})\backslash U_{P_1}(\mathbb{Q})$$

and the above expression becomes

$$\mathcal{F}^{P_1} \circ \mathrm{Eis}(\underline{\sigma}, \underline{z})(f)(\underline{g}) = \textstyle\sum_{W_M\backslash W^{M,M_1}/W_M} \int_{U_{P_1}(\mathbb{Q})\backslash U_{P_1}(\mathbb{A})} \sum_{v \in U_{P_1}^{(w)}(\mathbb{Q})\backslash U_{P_1}(\mathbb{Q})} f_{\underline{z}}(wv\underline{u}\underline{g})d\underline{u} =$$

$$\textstyle\sum_{W_M\backslash W^M/W^{M,M_1}} \int_{(w^{-1}U_P w \cap U_{P_1}\backslash U_{P_1})(\mathbb{A})} f_{\underline{z}}(w\underline{u}\underline{g})d\underline{u} \tag{217}$$

Our parabolic subgroup P contains the standard Borel subgroup, let U_P^- be the unipotent radical of the opposite group. In the argument of $f_{\underline{z}}$ we conjugate by w, then $U_P \cap wU_{P_1}w^{-1} \backslash wU_{P_1}w^{-1} = wU_{P_1}w^{-1} \cap U_P^- = U_{P,P_1}^{-,w}$.

$$\mathcal{F}^{P_1} \circ \mathrm{Eis}(\underline{\sigma}, \underline{z})(f)(\underline{g}) = \sum_{W_{M_1}\backslash W^{M,M_1}/W_M} \int_{U_{P,P_1}^{-,w}(\mathbb{A})} f_{\underline{z}}(\underline{u}w\underline{g})d\underline{u} \tag{218}$$

We pick a w, the group M acts by the adjoint action on $w^{-1}U_{P,P_1}^{-,w}w$ and hence by a character $\delta_{P.P_1}^{(w)}$ on the highest exterior power of the Lie-algebra of this group. Then this operator sends

$$\mathcal{F}^{P_1,w} \circ \mathrm{Eis}(\underline{\sigma}, \underline{z}) : I_P^G H_{\underline{\sigma}} \otimes |\gamma_P|^{\underline{z}} \to I_{P_1}^G H_{\underline{\sigma}^{w^{-1}}} \otimes (|\gamma_P|^{\underline{z}})^{w^{-1}} |\delta_{P.P_1}^{(w)}| \tag{219}$$

The integral is a product of local integrals over all places, we may assume that $f_{\underline{z}} = f_{\infty,\underline{z}} \prod_{p:prime} f_{p,\underline{z}}$. and then

$$\int_{U_{P,P_1}^{-,w}(\mathbb{A})} f_{\underline{z}}(\underline{u}w\underline{g})d\underline{u} = \int_{U_{P,P_1}^{-,w}(\mathbb{R})} f_{\infty,\underline{z}}(u_\infty w g_\infty) \prod_p \int_{U_{P,P_1}^{-,w}(\mathbb{Q}_p)} f_{p,\underline{z}}(u_p w g_p) \tag{220}$$

and here the local integrals yield intertwining operators

$$T_v^{P,P_1,w}(\sigma_v, \underline{z}) : I_P^G H_{\underline{\sigma}_v} \otimes |\gamma_P|_v^{\underline{z}} \to I_{P_1}^G H_{\underline{\sigma}_v^{w^{-1}}} \otimes |\gamma_P|_v^{w^{-1}\underline{z}} \otimes |\delta_{P.P_1}^{(w)}|_v \tag{221}$$

Proposition 6.1. *We can find local intertwining operators*

$$T_v^{P,P_1,w,\mathrm{loc}}(\sigma_v, \underline{z}) : I_P^G H_{\underline{\sigma}_v} \otimes |\gamma_P|_v^{\underline{z}} \to I_{P_1}^G H_{\underline{\sigma}_v^{w^{-1}}} \otimes |\gamma_P|_v^{w^{-1}\underline{z}} \otimes |\delta_{P.P_1}^{(w)}|_v \tag{222}$$

which have the following properties

a) They are holomorphic and nowhere zero in $\Re z_i \geq 0$ (we are still assuming that $\underline{\mu}$ is in the negative chamber.)

b) They have a certain rationality property (For the case of finite places see [Ha-Ra] 7.3.2.1, for the infinite places [Ha-HC]) *c) At the unramified primes $v = p$ they map the spherical vector to the spherical vector.*

and finally we have

$$\mathcal{F}^{P_1,w} \circ \mathrm{Eis}(\underline{\sigma}, \underline{z}) = C(w, P, P_1, \underline{\sigma}, \underline{z})\, T_\infty^{P,P_1,w,\mathrm{loc}}(\sigma_\infty, \underline{z}) \otimes \bigotimes_{p:\mathrm{primes}}{}' \; T_p^{P,P_1,w,\mathrm{loc}}(\sigma_p, \underline{z}) \tag{223}$$

where $C(w, P, P_1, \underline{\sigma}, \underline{z})$ is a meromorphic function in the variable $\underline{z}$. Therefore these functions $C(w, P, P_1, \underline{\sigma}, \underline{z})$ decide whether $\mathrm{Eis}(\underline{\sigma}, \underline{z})$ *is holomorphic at $\underline{z} = 0$, the poles of* $\mathrm{Eis}(\underline{\sigma}, \underline{z})$ *at $\underline{z}$ are the poles of the $C(w, P, P_1, \underline{\sigma}, \underline{z})$.*

We compute these factors $C(w, P, P_1, \underline{\sigma}, \underline{z})$. By definition the group $U_{P,P_1}^{-,w}$ is a subgroup of U_P^- and as such it it easy to describe. Recall that our our group M is $\mathrm{Gl}_{n_1} \times \cdots \times \mathrm{Gl}_{n_r}$ and this corresponds to a decomposition of $\mathbb{Q}^n = X_1 \oplus X_2 \oplus \cdots \oplus X_r$ into subspaces and for any two indices $1 \le i < j \le r$ we define $G_{i,j}$ to be the subgroup $\mathrm{Gl}(X_i \oplus X_j)$ acting trivially on all other summands. For all pairs i, j we define the cocharacters $\chi_{i,j} : \mathbb{G}_m \to T$ where $\chi_{i,j}(t)$ is the diagonal matrix having t as entry at place i, and t^{-1} at place j and 1 everywhere else. We define $\mathbf{w}_{i,\mathrm{j}} := < \chi_{i,j}, \underline{\mu}^{(1)} >$.

The intersection $G_{i,j} \cap U_{P,P_1}^{-,w}$ is either trivial or it is the full left lower block unipotent group $U_{i,i+1}^-$

This tells us that the above integral can be written as iterated integral over subgroups of the form $U_{\nu,\mu}(\mathbb{A})$. To be more precise: If $U_{P,P_1}^{-,w} \neq 1$ then we find an index i such that $U_{i,i+1}$ is not trivial. In a first step we compute the local integral $\int_{U_{i,i+1}(\mathbb{Q}_p)} f_{p,\underline{z}}^{(0)}(u_p w g_p) du_p$ at finite places where our representation $\underline{\sigma}_p$ unramified. We are basically in the situation, that our parabolic subgroup is maximal. The group $P' = P \cap G_{i,i+1}$ contains the standard Borel subgroup, $P_1' = P_1 \cap G_{ii,i+1}$ is the opposite and $w = e$. Then

$$C_p(e, P', P_1', \underline{\sigma}, \underline{z}) = \frac{L^{\mathrm{coh}}(\sigma_{i,p} \times \sigma_{i+1,p}^\vee, \frac{\mathbf{w}_{i,i+1}}{2} + b_i(w,\lambda) + < \chi_{i,i+1}, \underline{z} > -1)}{L^{\mathrm{coh}}(\sigma_{i,p} \times \sigma_{i+1,p}^\vee, \frac{\mathbf{w}_{i,i+1}}{2} + b_i(w,\lambda) + < \chi_{i,i+1}, \underline{z} >)} \tag{224}$$

A standard argument (See Langlands, Kim, Shahidi) tells us that we can reduce the computation of the iterated integral to situations like the one above and then we get at unramified places

$$C_p(w, P, P_1, \underline{\sigma}, \underline{z}) = \prod_{i,j} \frac{L^{\mathrm{coh}}(\sigma_{i,p} \times \sigma_{j,p}^\vee, \frac{\mathbf{w}_{i,j}}{2} + b_{i,j}(w,\lambda) + < \chi_{i,j}, \underline{z} > -1)}{L^{\mathrm{coh}}(\sigma_{i,p} \times \sigma_{j,p}^\vee, \frac{\mathbf{w}_{i,j}}{2} + b_{i,j}(w,\lambda) + < \chi_{i,j}, \underline{z} >} \tag{225}$$

Here the indices i, j run over those indices for which $U_{i,j} \subset U_{P,P_1}^{-,w}$, and $b_{i,j}(w,\lambda) = < \chi_{i,j}, \underline{\mu}^{\mathrm{ab}} >$.

Now we define $C_v(w, P, P_1, \underline{\sigma}, \underline{z})$ for all places v by the above expression, where we express the the cohomological L factor by the automorphic Rankin-Selberg L factor with the shift in the variable s. We go back to equation (223) and define

$$C(w,P,P_1,\underline{\sigma},\underline{z}) = \prod_v C_v(w,P,P_1,\underline{\sigma},\underline{z}). \tag{226}$$

We from the above proposition (6.1) that the factors $C(w,P,P_1,\underline{\sigma},\underline{z})$ determine the analytic behavior of $Eis(\underline{\sigma},\underline{z})$ at $\underline{z}=0$. We have to exploit the analytic properties of the Rankin-Selberg L-functions. Here we have to use Shahidi's theorem which yields -(always remember that $\underline{\mu}$ is in the negative chamber-)

$$L^{\mathrm{coh}}(\sigma_{i,p}\times\sigma_{j,p}^{\vee},\frac{\mathbf{w}_{i,j}}{2}+b_{i,j}(w,\lambda)+<\chi_{i,j},\underline{z}>-1) \tag{227}$$

is holomorphic at $\underline{z}=0$ unless we are in the following special case:

a) In the product in formula (225) we have factors $(i,i+1)$ where $n_i = n_{i+1}, \mu_i^{(1)} = \mu_{i+1}^{(1)}$ and $b_i(w,\lambda)=1$.

b) The pair $\sigma_i\times\sigma_{i+1}$ is a segment, this means that $\sigma_i\otimes\det_i=\sigma_{i+1}$

If these two conditions are fulfilled then $C(w,P,P_1,\underline{\sigma},\underline{z})$ has first order pole along $z_i=0$.

The denominator is always holomorphic and never zero at $\underline{z}=0$. (This is a deep theorem: it is the prime number theorem for Rankin-Selberg L-functions.)

6.4.1 Resume and questions

We see that we get an abundant supply of cohomology classes: Starting from any parabolic P and an isotypical subspace $\mathrm{Ind}_P^G H^{\bullet-l(w)}_{\mathrm{cusp}}(\mathcal{S}^M_{K_f^M},\tilde{\mathcal{M}}_{w\cdot\lambda})(\underline{\sigma}_f)$ we get the Eisenstein intertwining operator (See equation (213)). We analyze what happens at $\underline{z}=0$. If it is holomorphic we get a Hecke invariant homomorphism

$$\mathrm{Eis}^\bullet(0): H^\bullet(\mathfrak{g},K_\infty,\mathrm{Ind}_P^G\sigma_\infty\otimes\tilde{\mathcal{M}})\otimes\mathrm{Ind}_P^G H_{\underline{\sigma}_f}\to H^\bullet(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathbb{C}}) \tag{228}$$

We can restrict these cohomology classes to the boundary and even to boundary strata $\partial_Q(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}})$ where Q runs over the parabolic subgroups associate to P, or more generally those parabolic subgroups which contain an associate to P. This means that the class "spreads out" over different boundary strata These restrictions to these other strata are given by certain linear maps which are product of "local intertwining operators" times certain special values of L functions.

In certain cases this "spreading out" is highly non trivial. We have to clarify some local issues. First of all we have to find out whether the local intertwining operators are non zero and have certain rationality properties. Especially we have to show that these local operators at the infinite places induce non zero maps between the cohomology groups of certain induced Harish-Chandra modules. And we have to show that these maps on the level of cohomology have rationality properties. ([Ha-HC] , [Ha-Ra], 7.3,)

If these local issues are settled then we can argue: The image of the cohomology $H^\bullet(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}})$ in the cohomology of the boundary is defined over $\mathbb{Q}$ (or some number field depending on our data). Since the $L-$ values enter in the description of this image we get rationality statements for special values of L-functions.

This has been exploited in some cases ([Ha-Gl2], [Ha-Gln], [Ha-Mum]) and the so far most general result in this direction is in [Ha-Ra] (See previous section).

But in case we have a pole we may also produce cohomology classes by taking residues, again starting from one boundary stratum. The restriction of these classes to the boundary will spread out over other strata in the boundary and we may play the same game. In this case the non vanishing issue of intertwining operators on cohomological level comes up again and will be discussed in the following section. (See Thm. 6.1)

We also will encounter situation where a pole along a plane $z_i = 0$ (or may be even several such planes) "fights" with a zero along some other planes containing zero. Then this influences the structure of the cohomology. But how? This question has been discussed in [Ha-Gln]. Is the order of vanishing along this zero visible in the structure of the cohomology? Or is it visible in the structure of the cohomology of the boundary, or in the spectral sequence?

6.5 Residual classes

We have seen that our Eisenstein classes may be singular at $\underline{z} = 0$. In this section we look at the extremal case that $\mathrm{Eis}(\sigma, \underline{z})$ has simple poles along the lines $z_i =< \chi_{n_i,n_i+1}, \underline{z} >= 0$, In this case we call these Eisenstein classes residual.

It follows from the work of Moeglin-Waldspurger [M-W] that this can only happen under some very special conditions.

We start from a factorization $n = uv$ we look the parabolic subgroup $P_{u,v}$ which contains the standard Borel subgroup and has reductive quotient $\mathrm{Gl}_u \times \mathrm{Gl}_u \times \cdots \times \mathrm{Gl}_u$. The standard maximal torus is a product $T = \prod_{i=1}^{i=v} T_i$ and accordingly we have $X^*(T) = \bigoplus_{i=1}^{i=v} X^*(T_i)$. We have an obvious identification $T_i = \mathbb{G}_m^u$.

We choose a highest weight $\lambda = \sum a_i\gamma_i + d\delta$, we assume that it is self dual, i.e. $a_i = a_{n-i}$. We have a restriction on the character $\underline{\mu} = w\cdot\lambda = w(\lambda+\rho_N)-\rho_N$, we must have

$$\begin{aligned} w(\lambda + \rho_N) - \rho_N = b_1\gamma_1^M + b_2\gamma_2^M + \cdots + b_{u-1}\gamma_{u-1}^M - (u+1)\gamma_u \\ +b_1\gamma_{1+u}^M + b_2\gamma_{2+u}^M + \cdots + b_{u-1}\gamma_{2u-1}^M - (u+1)\gamma_{2u} + \ldots \\ \ldots \\ b_1\gamma_{(v-1)u+1}^M + b_2\gamma_2^M + \cdots + b_{u-1}\gamma_{vu-1}^M + d\gamma_{uv} \end{aligned} \tag{229}$$

where $\gamma_{uv} = \delta = \det$. The highest weight is a sum $\underline{\mu} = \sum \mu_i$ where

$$\mu_i = \mu^{(1)} - d_i\mathrm{det}_i \text{ and } d_i - d_{i+1} = -1. \tag{230}$$

where the semi simple component $\mu^{(1)} = b_1\gamma_1^M + b_2\gamma_2^M + \cdots + b_{u-1}\gamma_{u-1}^M = b_1\gamma_{1+u}^M + b_2\gamma_{2+u}^M + \cdots + b_{u-1}\gamma_{2u-1}^M \ldots$ is "always the same". We notice that of course we have the self duality condition $b_i = b_{u-i}$. Furthermore we have $\sum d_i = -d$.

We define

$$\mathbb{D}_\mu = \bigotimes_{i=1}^{i=v} \mathbb{D}_{\mu_i} \tag{231}$$

and start from our isotypical $H^\bullet_{\mathrm{cusp}}(\mathcal{S}^M_{K^M_f}, \mathbb{D}_\mu \otimes \mathcal{M}_{w\cdot\lambda})(\sigma_f)$. The Künneth formula yields that we can write $\sigma_f = \sigma_{1,f} \times \sigma_{2,f} \times \cdots \times \sigma_{v,f}$ where all the $\sigma_{i,f}$ occur in the cuspidal cohomology of Gl_u, hence they may be compared. The relation (230) allows us to require that $\sigma_{i+1,f} = \sigma_{i,f} \otimes |\delta|$. If this is satisfied we say that σ_f is a segment. We assume $v \neq 1$ and hence $P \neq G$.

We know that under the assumption that σ_f is a segment (and only under this assumption) the factor $C(\sigma, w_P, \underline{z})$ has a simple poles along the lines $z_i = 0$, and this is the only term in (**??**) having these poles. The operator $T^{\mathrm{loc}}(\sigma, \underline{s})$ is a product of local operators at all places

$$T^{\mathrm{loc}}(\sigma, \underline{z}) = T^{\mathrm{loc}}_\infty(\sigma_\infty, \underline{s}) \times \prod_p T^{\mathrm{loc}}_p(\sigma_p, \underline{z}),$$

and the local factors are holomorphic as long as $\Re(z_i) \geq 0$. We take the residue at $\underline{z} = 0$ i.e. we evaluate

$$(\prod z_i)\mathcal{F}^P \circ \mathrm{Eis}(\sigma \otimes \underline{s})|_{\underline{z}=0} = (\prod z_i) C(\sigma, w_P, \underline{z})|_{\underline{z}=0} T^{\mathrm{loc}}(\sigma, w_P, \underline{0})(f) \tag{232}$$

This tells us that the residue of the Eisenstein class gives us an intertwining operator

$$\mathrm{Res}_{\underline{z}=0}\mathrm{Eis}(\sigma \otimes \underline{z}) : {}^{\mathrm{a}}\mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})} \mathbb{D}_\mu \otimes V_{\sigma_f} \to L^2_{\mathrm{disc}}(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f, \omega^{-1}_{\mathcal{M}_\lambda}|_{S(\mathbb{R})^0}) \tag{233}$$

The image $J_{\sigma_\infty} \otimes J_{\sigma_f}$ is an irreducible module (this is a Langlands quotient) and via the constant Fourier coefficient it injects into ${}^{\mathrm{a}}\mathrm{Ind}_{P(\mathbb{A}))}^{G(\mathbb{A})} \mathbb{D}_{\mu'} \otimes V_{\sigma_f}$. At the infinite place we get a diagram

$$\begin{array}{ccc} \mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})} \mathbb{D}_\mu & \stackrel{T^{(\mathrm{loc})}(D_\mu)}{\longrightarrow} & J_{\sigma_\infty} \\ & & \downarrow \\ & & \mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})} \mathbb{D}_{\mu'} \end{array} \tag{234}$$

It is a - not completely trivial - exercise to write down the solutions for the system of equations (229). We start from a highest weight of a special form

$$\lambda = a_1\gamma_u + a_2\gamma_{2u} + \cdots + a_{v-1}\gamma_{(v-1)u} + d\delta \tag{235}$$

which in addition is essentially self dual, i.e. $a_i = a_{v-i}$ the number d is uninteresting and only serves to satisfy the parity condition.

We choose a specific Kostant representative $w'_{u,v} \in W^P$ whose τ- it is the permutation in the letters $1, 2, \ldots, n$ given by the following rule: write $\nu =$

$i+(j-1)v$ with $1 \le i \le u$ then $w'_{u,v}(\nu) = j+(i-1)v$. Then we compute $w'_{u,v}(\lambda+\rho_N)-\rho_N \in X^*(T\times E)$ and we get

$$\begin{gathered}(w'_{u,v}(\lambda+\rho_N)-\rho_N) = \\ (a_1+v-1)\gamma_1^M + (a_2+v-1)\gamma_2^M + (a_{u-1}+v-1)\gamma_{u-1}^M \\ (a_1+v-1)\gamma_{1+u}^M + (a_2+v-1)\gamma_{2+u}^M + (a_{u-1}+v-1)\gamma_{u-1+u}^M \\ \vdots \\ (a_1+v-1)\gamma_{1+(v-1)u}^M + (a_2+v-1)\gamma_{2+(v-1)u}^M + \dots + (a_{u-1}+v-1)\gamma_{u-1+(v-1)u}^M)+ \\ -(u-1)(\gamma_u+\gamma_{2u}+\dots+\gamma_{(v-1)u}) + d\delta\end{gathered} \tag{236}$$

The length of this Kostant representative is

$$l(w'_{u,v}) = n(u-1)(v-1)/4.$$

Let w_P be the longest Kostant representative which sends all the roots in U_P to negative roots. Then we define the (reflected) Kostant representative $w_{u,v} = w_P w'_{u,v}$. We get

$$\begin{aligned}w_{u,v}(\lambda+\rho)-\rho = \mu = (a_1+v-1)(\gamma_1^M + \gamma_{1+u}^M + \dots + \gamma_{1+(v-1)u}^M)+ \\ (a_2+v-1)(\gamma_2^M + \gamma_{2+u}^M + \dots + \gamma_{2+(v-1)u}^M)+ \\ \vdots \\ (a_{u-1}+v-1)(\gamma_{u-1}^M + \gamma_{u-1+u}^M + \dots + \gamma_{u-1+(v-1)u}^M)+ \\ -(u+1)(\gamma_u+\gamma_{2u}+\dots+\gamma_{(v-1)u}) + d\delta.\end{aligned} \tag{237}$$

Hence we see that we the semi simple component stays the same and the abelian parts differ by $2(\gamma_u+\gamma_{2u}+\dots+\gamma_{(v-1)u})$) We see that we can solve (229) provided $b_i \ge v-1$.

6.5.1 The identification $J_{\sigma_\infty} \xrightarrow{\sim} A_{\mathfrak{q}}(\lambda)$)

Of course we expect

$$H^\bullet(\mathfrak{g}, K_\infty, J_{\sigma_\infty} \otimes \mathcal{M}_\lambda) \ne 0. \tag{238}$$

In the paper [Vo-Zu] the authors give a list of irreducible $(\mathfrak{g}, K_\infty)$ modules $A_{\mathfrak{q}}(\lambda)$ which have non trivial cohomology $H^\bullet(\mathfrak{g}, K_\infty, A_{\mathfrak{q}}(\lambda)\otimes\mathcal{M}_\lambda) \ne 0$. This list contains all unitary modules having this property. On the other hand we know that any such unitary $A_{\mathfrak{q}}(\lambda)$ can be written as a Langlands quotient. In the paper of Vogan and Zuckerman it is explained how we can get a given unitary $A_{\mathfrak{q}}(\lambda)$ as Langlands quotient, basically this means we construct a diagram of the form (234) but where now we have $A_{\mathfrak{q}}(\lambda)$ in the upper right corner instead of J_{σ_∞}. In the following section we describe a specific $A_{\mathfrak{q}}(\lambda)$ and write it as a Langlands quotient (i.e. we find its Langlands parameters) this means we determine the upper left and lower right entries and then check that these entries are the ones in diagram (234). From this we will derive the following

The map

$$H^\bullet(\mathfrak{g}, K_\infty, J_{\sigma_\infty} \otimes \mathcal{M}_\lambda) \otimes J_{\sigma_f} \to H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) \tag{239}$$

is non zero in degree $l(w'_{u,v}) = n(u-1)(v-1)/4$.

See Theorem (6.1)

6.6 Detour: $(\mathfrak{g}, K_\infty)-$ modules with cohomology for $G = \mathrm{Gl}_n$

I want to fix some notations and conventions.

Let $T/\mathbb{Q}$ be the maximal torus in $\mathrm{Gl}_n/\mathbb{Q}$, let $T^{(1)} = Sl_n \cap T$. We put $r = n-1$. We have the standard basis for the character-module $X^*(T)$:

$$e_i : T \to G_m, t \mapsto t_i.$$

The positive (resp. simple roots) roots are $\alpha_{i,j} = e_i - e_j, i < j$, (resp. $\alpha_i = e_i - e_{i+1}$.) We have the determinant $\delta = \sum_1^n e_i$.

The fundamental weights are elements in $X^*(T) \otimes \mathbb{Q}$, they are defined by

$$\gamma_i = \sum_{\nu=1}^{i} e_\nu - \frac{i}{n}\delta,$$

these γ_i are the fundamental weights if we restrict to Sl_n, the image of γ_i under the restriction map lies in $X^*(T^{(1)})$.

From now on my natural basis for $X^*(T) \otimes \mathbb{Q}$ will be

$$\{\gamma_1, \dots, \gamma_i, \dots, \gamma_r, \delta\}.$$

This basis respects the decomposition of T into $T^{(1)} \cdot G_m$, the first factor is its component in Sl_n and the second one is the central torus.

We also have the cocharacters $\chi_i \in X_*(T^{(1)})$ which are given by

$$\chi_i : t \mapsto \begin{pmatrix} 1 & 0 & 0 & & \dots & 0 \\ 0 & \ddots & \dots & \dots & \dots & \dots \\ 0 & 0 & t & 0 & \dots & 0 \\ 0 & \dots & 0 & t^{-1} & \dots & 0 \\ \dots & \dots & \dots & & \ddots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

and the central cocharacter

$$\zeta : t \mapsto \begin{pmatrix} t & 0 & 0 \dots & 0 \\ 0 & t & \dots & \dots \\ 0 & 0 & \ddots & \dots \\ 0 & \dots & 0 & \end{pmatrix}$$

We have the standard pairing $(\chi, \gamma) \mapsto <\chi, \gamma>$ between cocharacters and characters which is defined by $\gamma \circ \chi = \{t \mapsto t^{<\chi,\gamma>}\}$. We have the relations

$$<\chi_j, \gamma_i> = \delta_{ij}, \; <\chi_i, \alpha_i> = 2$$

the character δ is trivial on the χ_i and $\delta \circ \zeta = \{t \mapsto t^n\}$. It is clear that an element $\gamma = \sum_i a_i \gamma_i + d\delta \in X^*(T)$ if and only if the $a_i, nd \in \mathbb{Z}$ and we have the congruence

$$\sum i a_i \equiv nd \mod n.$$

We identify the center of Gl_n with G_m via the cocharacter ζ, the character module of G_m is $\mathbb{Z}$. Hence the central character ω_λ is an integer and we find

$$\omega_\lambda = nd.$$

Actually this central character should be considered as an element in $\mathbb{Z} \mod n$ because we can replace d by $r+d$ and then the central character changes by a multiply of n. If $\lambda \in X^+(T^{(1)})$ is a dominant weight then we write it as

$$\lambda = \sum a_i \gamma_i$$

then we have $a_i \geq 0$.

6.6.1 The tempered representation at infinity

We consider the group $\mathrm{Gl}_n/\mathbb{R}$, we choose a essentially selfdual highest weight $\lambda = \sum_1^{n-1} a_i\gamma_i + d\delta($ i.e. $a_i = a_{n-i})$. The a_i are integers and d is a half integer which satisfies the parity condition

$$d \in \mathbb{Z} \text{ if } n \text{ is odd }, \frac{n}{2} a_{\frac{n}{2}} \equiv nd \mod n \text{ if } n \text{ is even}$$

We want to recall the construction of a specific $(\mathfrak{g}, K_\infty)$ -module $\mathbb{D}_\lambda$ such that

$$H^\bullet(\mathfrak{g}, K_\infty, \mathbb{D}_\lambda \otimes \mathcal{M}_\lambda) \neq 0$$

and we will also determine the structure of this cohomology. This module is the only tempered Harish-Chandra module which has non trivial cohomology with coefficients in $\mathcal{M}_\lambda$. The center $\mathbb{G}_m$ of Gl_n acts on the module $\mathcal{M}_\lambda$ by the character $\omega_\lambda : x \mapsto x^{nd}$. Since we want no zero cohomology the center $S(\mathbb{R})$ of $\mathrm{Gl}_n(\mathbb{R})$ acts by the central character $(\omega_\lambda)_{\mathbb{R}}^{-1}$ on $\mathbb{D}_\lambda$. The module $\mathbb{D}_\lambda$ will be essentially unitary with respect to that character.

We construct our representation $\mathbb{D}_\lambda$ by inducing from discrete series representations. We consider the parabolic subgroup ${}^\circ P$ whose simple root system is described by the diagram

$$\circ - \times - \circ - \times - \cdots - \circ(-\times) \tag{240}$$

i.e. the set of simple roots $I_{{}^\circ M}$ of the semi simple part of the Levi quotient ${}^\circ M$ is consists of those which have an odd index. Let m be the largest odd integer less or equal to $n-1$ then α_m is the last root in the system of simple roots in $I_{{}^\circ M}$. Of course $m = n-1$ if n is even and $m = n-2$ else.

The reductive quotient is equal to $\mathrm{Gl}_2 \times \mathrm{Gl}_2 \times \ldots \mathrm{Gl}_2(\times \mathbb{G}_m)$, where the last factor occurs if n is odd. This product decomposition of ${}^\circ M$ induces a product decomposition of the standard maximal torus $T = \prod_{i:i\mathrm{odd}} T_i(\times \mathbb{G}_m)$ and for the character module we get

$$X^*(T) = \bigoplus_{i:i\mathrm{odd}} X^*(T_i)(\oplus X^*(\mathbb{G}_m)) \tag{241}$$

The semi simple reductive quotient ${}^\circ M^{(1)}(\mathbb{R})$ is $A_1 \times A_1 \times \cdots \times A_1$, the number of factors is

$$ {}^\circ r = (m+1)/2 = \begin{cases} \frac{n}{2} & \text{if n is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases} $$

We also introduce the number

$$ \epsilon(n) = \begin{cases} 0 & \text{if } n \text{ even} \\ 1 & \text{if } n \text{ odd} \end{cases} \tag{242} $$

We have a very specific Kostant representative $w_{\mathrm{un}} \in W^{{}^\circ P}$. The inverse of this permutation it is given by

$$ w_{\mathrm{un}}^{-1} = \{1 \mapsto 1, 2 \mapsto n, 3 \mapsto 2, 4 \mapsto n-1 \dots \}. $$

The length of this element is equal to 1/2 the number of roots in the unipotent radical of ${}^\circ P$, i.e.

$$ l(w_{\mathrm{un}}) = \begin{cases} \frac{1}{4} n(n-2) & \text{if } n \text{ is even} \\ \\ \frac{1}{4}(n-1)^2 & \text{if } n \text{ is odd} \end{cases} \tag{243} $$

We compute

$$ w_{\mathrm{un}}(\lambda+\rho) - \rho = \sum_{i:i \text{ odd}} b_i \gamma_i^{{}^\circ M^{(1)}} + d\delta = \sum_{i:i \text{ odd}} b_i \frac{\alpha_i}{2} + d\delta = \tilde{\mu}^{(1)} + d\delta. \tag{244} $$

(The subscript $_{\mathrm{un}}$ refers to unitary, it refers also to the length being half the dimension of the unipotent radical. Here we have to observe that $w \cdot \lambda$ is an element in $X^*(T)$ but the individual summands may only lie in $X^*(T) \otimes \mathbb{Q} = X^*_{\mathbb{Q}}(T)$. Any element $\gamma \in X^*(T)$ also defines a quasicharacter $\gamma_{\mathbb{R}} : T(\mathbb{R}) \to \mathbb{R}^\times$ (by definition). But an element $\gamma \in X^*_{\mathbb{Q}}(T)$ only defines a quasicharacter $|\gamma|_{\mathbb{R}} : T(\mathbb{R}) \to \mathbb{R}^\times_{>0}$ which is defined by $|\gamma|_{\mathbb{R}}(x) = |m\gamma(x)|^{1/m}$.)

To compute the coefficients b_j we use the pairing (See56) and observe that $< \chi_i, \gamma_j >= \delta_{i,j}$. Then

$$ b_j =< \chi_j, w_{\mathrm{un}}(\lambda+\rho) - \rho >=< w_{\mathrm{un}}^{-1}\chi_j, \lambda+\rho > - < \chi_j, \rho > . \tag{245} $$

Now the choice of w_{un} becomes clear. It is designed in such a way that

$$ w_{\mathrm{un}}^{-1}\chi_1(t) = \begin{pmatrix} t & 0 & 0 & & \dots & 0 \\ 0 & \ddots & \dots & \dots & \dots & \dots \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & & \ddots & 0 \\ 0 & 0 & 0 & \dots & 0 & t^{-1} \end{pmatrix}, w_{\mathrm{un}}^{-1}\chi_3(t) = \begin{pmatrix} 1 & 0 & 0 & & \dots & 0 \\ 0 & t & \dots & \dots & \dots & \dots \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & & t^{-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix} $$

and for the general odd index j we have $w_{\mathrm{un}}^{-1}\chi_j(t) = h_{(j+1)/2}$ where for all $1 \le \nu \le n/2$ we denote by $h_\nu(t)$ the diagonal matrix which has a 1 at all entries different from $\nu, n+1-\nu$ and which has entry t at ν and t^{-1} at $n+1-\nu$. Then $h_\nu = \{t \mapsto h_\nu(t)\}$ is a cocharacter. It is clear that

$$\gamma_i(h_\nu(t)) = \begin{cases} t & \text{if } \nu \le i \le n-\nu \\ 1 & \text{else} \end{cases}$$

This yields for $j = 1, \dots, {}^\circ r$

$$b_{2j-1} = \sum_\nu (a_\nu + 1) < h_j, \gamma_\nu > - < \chi_j, \rho >= (\sum_{j \le \nu \le n-j} (a_\nu + 1)) - 1.$$

We should keep in mind that we assume $a_\nu = a_{n-\nu}$. Then we can rewrite the expressions for the b_ν :

$$b_{2j-1} = \begin{cases} 2a_j + 2a_{j+1} + \cdots + 2a_{\frac{n}{2}-1} + a_{\frac{n}{2}} + n - 2j & \text{if } n \text{ is even} \\ 2a_j + 2a_{j+1} + \cdots + 2a_{\frac{n-1}{2}} + n - 2j & \text{if } n \text{ is odd} \end{cases} \tag{246}$$

The b_{2j+1} will be called the *cuspidal parameters* and we summarize

The b_{2j-1} have the same parity, this parity is odd if n is odd. If n is even then b_{2j-1} has parity of $a_{\frac{n}{2}}$. We have $b_1 > b_3 > \cdots > b_m > 0$. They only depend on the semi simple part $\lambda^{(1)}$.

By Kostants theorem

$$w_{\text{un}} \cdot \lambda = w_{\text{un}}(\lambda + \rho) - \rho$$

is the highest weight of an irreducible representation of ${}^\circ M$. This irreducible representation occurs with multiplicity one in $H^{l(w_{\text{un}})}(\mathfrak{u}_{{}^\circ P}, \mathcal{M}_\lambda)$.

The highest weight of this representation is

$$w_{\text{un}} \cdot \lambda = w_{\text{un}}(\lambda + \rho) - \rho = \sum_{i:i \text{ odd}} b_i \gamma_i^{{}^\circ M^{(1)}} + d\delta - (2\gamma_2 + 2\gamma_4 + \cdots + 2\gamma_{m-1} + \frac{3}{2}\gamma_{m+1}) \tag{247}$$

Digression: *Discrete series representations of* $\mathrm{Gl}_2(\mathbb{R})$*, some conventions*

We consider the group $\mathrm{Gl}_2/\mathrm{Spec}(\mathbb{Z})$, the standard torus T and the standard Borel subgroup B. We have $X^*(T) = \{\gamma = a\gamma_1 + d\delta | a \in \mathbb{Z}, d \in \frac{1}{2}\mathbb{Z}; a + 2d \equiv 0 \mod 2\}$ where

$$\gamma(\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}) = t_1^{\frac{a}{2}+d} t_2^{-\frac{a}{2}+d} = (\frac{t_1}{t_2})^{\frac{a}{2}} (t_1 t_2)^d$$

(Note that the exponents in the expression in the middle term are integers)

A dominant weight $\lambda = a\gamma_1 + d\delta$ is a character where $a \ge 0$. These dominant weights parameterize the finite dimensional representations of $\mathrm{Gl}_2/\mathbb{Q}$. The dual representation is given by $\lambda^\vee = a\gamma_1 - d\delta$. But these highest weights also parameterize the discrete series representations of $\mathrm{Gl}_2(\mathbb{R})$, (or better the discrete series Harish-Chandra modules). The highest weight λ defines a line bundle $\mathcal{L}_{-a\gamma+d\delta}$ on $B\backslash G$ and

$$\mathcal{M}_\lambda = H^0(B\backslash G, \mathcal{L}_{-a\gamma+d\delta})$$

Then we get an embedding and a resulting exact sequence

$$0 \to \mathcal{M}_\lambda \to I_B^G((-a\gamma_1 + d\delta)_\mathbb{R}) \to \mathcal{D}_{\lambda^\vee} \to 0$$

and $\mathcal{D}_{\lambda^\vee}$ is the discrete series representation attached to $\lambda^\vee$. (Note the subscript $_\mathbb{R}$ can not be pulled inside the bracket!).

A basic argument in representation theory yields a pairing

$$I_B^G((-a\gamma_1 - d\delta)_\mathbb{R}) \times I_B^G(((a+2)\gamma_1 + d\delta)_\mathbb{R}) \to \mathbb{R}$$

(here observe that $2\gamma_1 = 2\rho \in X^*(T)$).

From this we get another exact sequence which gives the more familiar definition of the discrete series representation

$$0 \to \mathcal{D}_\lambda \to I_B^G(((a+2)\gamma_1 + d\delta)_\mathbb{R}) \to \mathcal{M}_\lambda \to 0. \tag{248}$$

The module $\mathbb{D}_\lambda$ is also a module for the group $K_\infty = \mathrm{SO}(2)$ and it is well known that it decomposes into K_∞ types

$$D_\lambda = \cdots \oplus \mathbb{C}\psi_\nu \ldots \mathbb{C}\psi_{-a-4} \oplus \mathbb{C}\psi_{-a-2} \oplus \mathbb{C}\psi_{+a+2} \oplus \mathbb{C}\psi_{a+4} \cdots \tag{249}$$

(End of digression)

We return to our formula (247). The group

$$^\circ M = \prod_{i:i\text{odd}} M_i \times (\mathbb{G}_m)$$

where $M_i = \mathrm{Gl}_2$. If T_i is the maximal torus in the i-th factor, then the highest weight is $\gamma_i^{^\circ M^{(1)}}$ and let δ_i be the determinant on that factor. The indices i run over the odd numbers $1, 3, \ldots, m$. If n is odd then let $\delta_n : T \to \mathbb{G}_m$ be the character given by the last entry. Then we have for the determinant

$$\delta = \delta_1 + \delta_3 + \cdots + \delta_m + \begin{cases} 0 \\ \delta_n \end{cases} \tag{250}$$

We want to write the character $2\gamma_2 + 2\gamma_4 + \cdots + 2\gamma_{m-1} + \frac{3}{2}\gamma_{m+1}$ in terms of the δ_i. We recall that

$$\begin{gathered} \gamma_2 = \delta_1 - \tfrac{2}{n}\delta \\ \gamma_4 = \delta_1 + \delta_3 - \tfrac{4}{n}\delta \\ \vdots \\ \gamma_{m-1} = \delta_1 + \delta_3 \cdots + \delta_{m-2} - \tfrac{m-1}{n}\delta \\ \text{and if } n \text{ is odd} \\ \gamma_{m+1} = \delta_1 + \delta_3 \cdots + \delta_m - \tfrac{m+1}{n}\delta \end{gathered} \tag{251}$$

Then the summation over the δ-terms on the right hand side yields

$$-\frac{1}{n}(4 + 8 + \cdots + 2(m-1) - \begin{cases} 0 \\ \frac{3}{2}(m+1) \end{cases}) = -[\frac{n-1}{2}] \tag{252}$$

and if we take our formula (250) into account and also count the number of times a δ_i occurs in the summation we get

$$2\gamma_2+2\gamma_4+\cdots+2\gamma_{m-1}+\frac{3}{2}\gamma_{m+1}=\begin{cases}(\frac{n}{2}-1)\delta_1+(\frac{n}{2}-3)\delta_3+\cdots+(-\frac{n}{2}+1)\delta_{m-2} & n\equiv 0 \mod 2\\ \frac{n-2}{2}\delta_1+\cdots+\frac{-n+4}{2}\delta_m-\frac{n-1}{2}\delta_n & \text{else}\end{cases} \tag{253}$$

Let us denote the coefficient of δ_i in the expressions on the right hand side by $c(i,n.)$ We recall that we still have the summand $d\delta$ in our formula (**??**. Then

$$\underline{\mu}=w_{\mathrm{un}}\cdot\lambda=\sum_{i:i \text{ odd}} b_i\gamma_i^{\circ M^{(1)}}+(c(i,n)+d)\delta_i+\begin{cases}d\delta\\ (-\frac{n-1}{2}+d)\delta_n\end{cases} \tag{254}$$

We claim that the individual summands are in the character modules $X^*(T_i)$ (resp. $X^*(\mathbb{G}_m)$). This means that

$$b_i\gamma_i^{\circ M^{(1)}}+(c(i,n)+d)\delta_i\in X^*(T_i),\ -\frac{n-1}{2}+d\in\mathbb{Z}. \tag{255}$$

We have to verify the parity conditions. If n is odd the the parity condition for λ says that $d\in\mathbb{Z}$. On the other hand we know that in this case the b_i are odd and since the $c(i,n)$ are also odd the parity condition is satisfied for the individual summands.

If n is even then the parity condition for for λ says that $\frac{n}{2}a_{\frac{n}{2}}\equiv nd \mod n$. We know that the b_i all have the same parity: $b_i\equiv a_{\frac{n}{2}} \mod 2$. Hence need that $a_{\frac{n}{2}}\equiv 2d \mod 2$, but this is the parity condition for λ.

For any of the characters μ_i we have the induced representations $I_{B_i}^{\circ M_i}(\mu_i+2\rho_i)$ the discrete series representation $\mathcal{D}_{\mu_i}$ and the exact sequence

$$0\to\mathcal{D}_{\mu_i}\to I_{B_i}^{\circ M_i}(\mu_i+2\rho_i)\to\mathcal{M}_{\mu_i}\to 0. \tag{256}$$

The tensor product

$$\mathcal{D}_\mu=\bigotimes_{i:i\mathrm{odd}}\mathcal{D}_{\mu_i}\otimes\mathbb{C}(-\frac{n-1}{2}+d) \tag{257}$$

is a module for ${}^\circ M$.

Here we have to work with $K_\infty^{\circ M}=K_\infty\cap^\circ M$. This compact group is not necessarily connected, its connected component of the identity is

$$K_\infty^{\circ M}\cap^\circ M^{(1)}(\mathbb{R})=\mathrm{SO}(2)\times\mathrm{SO}(2)\times\cdots\times\mathrm{SO}(2)=K_\infty^{\circ M,(1)}.$$

An easy computation shows

$$K_\infty^{\circ M}=\begin{cases}S(\mathrm{O}(2)\times\mathrm{O}(2)\times\cdots\times\mathrm{O}(2)) & \text{if } n \text{ is even}\\ \mathrm{O}(2)\times\mathrm{O}(2)\times\cdots\times\mathrm{O}(2) & \text{if } n \text{ is odd}\end{cases}, \tag{258}$$

since $K_\infty\subset\mathrm{Sl}_n(\mathbb{R})$ we have the determinant condition in the even case, in the odd case we have the $\{\pm1\}$ in the last factor and this relaxes the determinant condition.

Under the action of $K_\infty^{\circ M,(1)}$ we get a decomposition

$$\mathcal{D}_{\underline{\mu}} = \bigoplus_{\underline{\varepsilon}} \bigotimes_{i=1}^{\circ r} \Big(\bigoplus_{\nu_i=0}^{\infty} \mathbb{C}\psi_{\varepsilon_i(b_i+2+2\nu_i)} \Big) \tag{259}$$

occur with multiplicity one. Here $\underline{\varepsilon} = (\dots, \varepsilon_i, \dots)$ is an array of signs ± 1.

The induced representation (algebraic induction)

$$\mathrm{Ind}_{{}^\circ P(\mathbb{R})}^{G(\mathbb{R})} \mathcal{D}_{\underline{\mu}} = \mathbb{D}_\lambda \tag{260}$$

is an irreducible essentially unitary $(\mathfrak{g}, K_\infty)$ -module, this is the module we wanted to construct. (To be more precise: We first construct the induced representation of $G(\mathbb{R})$ where $G(\mathbb{R})$ is acting on vectors space V_∞ consisting of a suitable class of functions from $G(\mathbb{R})$ with values in $\mathcal{D}_{\underline{\mu}}$ and then we take the K_∞ finite vectors in V_∞.) The restriction of this module to $K_\infty^{(1)}$ s given by

$$\mathrm{Ind}_{K_\infty^{\circ M^{(1)}}}^{K_\infty^{(1)}} \mathcal{D}_{\underline{\mu}} = \bigoplus_{\underline{\varepsilon}} \bigotimes_{i=1}^{\circ r} \Big(\bigoplus_{\nu_i=0}^{\infty} \mathrm{Ind}_{K_\infty^{\circ M^{(1)}}}^{K_\infty^{(1)}} \mathbb{C}\psi_{\varepsilon_i(b_i+2+2\nu_i)} \Big) \tag{261}$$

(The last induced module is defined in terms of the theory of algebraic groups. We consider $K_\infty^{(1)}$ as the group of real points of an algebraic group, namely the connected group of the identity of the fixed points under the Cartan involution Θ. Then $K_\infty^{\circ M^{(1)}}$ is the group of real points of a maximal torus. Then

$$\mathrm{Ind}_{K_\infty^{\circ M^{(1)}}}^{K_\infty^{(1)}} \mathbb{C}\psi_{\varepsilon_i(b_i+2+2\nu_i)} = \\ \{f | f \text{ regular function } f(tk) = \textstyle\prod_j e_i(t)^{\varepsilon_i(b_i+2+2\nu_i)} f(k), \text{ for all } t \in K_\infty^{\circ M^{(1)}}, k \in K_\infty\} \tag{262}$$

)

We compute the cohomology of this module

$$\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}),\ D_\lambda \otimes \mathcal{M}_\lambda) = H^\bullet(\mathfrak{g}, K_\infty,\ D_\lambda \otimes \mathcal{M}_\lambda),$$

i.e. the differentials in the complex on the left hand side are all zero. (Reference to 4.1.4)

We apply Delorme to compute this cohomology. We can decompose ${}^\circ\mathfrak{m} ={}^\circ \mathfrak{m}^{(1)} \oplus \mathfrak{a}$ then ${}^\circ\mathfrak{k} \subset{}^\circ \mathfrak{m}^{(1)}$ and

$$\begin{aligned}\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}),\ D_\lambda \otimes \mathcal{M}_\lambda) = \mathrm{Hom}_{K_\infty^{\circ M}}(\Lambda^\bullet({}^\circ\mathfrak{m}/{}^\circ\mathfrak{k}), \mathcal{D}_{\tilde{\mu}} \otimes \mathcal{M}_{w_{\mathrm{un}}\cdot\lambda}) = \\ \mathrm{Hom}_{K_\infty^{\circ M}}(\Lambda^\bullet({}^\circ\mathfrak{m}^{(1)}/{}^\circ\mathfrak{k}), \mathcal{D}_{\tilde{\mu}} \otimes \mathcal{M}_{w_{\mathrm{un}}\cdot\lambda}) \otimes \Lambda^\bullet(\mathfrak{a}).\end{aligned} \tag{263}$$

If we replace $K_\infty^{\circ M}$ on the right hand side by its connected component of the identity then we have an obvious decomposition

$$\mathrm{Hom}_{K_\infty^{\circ M,(1)}}(\Lambda^\bullet({}^\circ\mathfrak{m}^{(1)}/{}^\circ\mathfrak{k}), \mathcal{D}_\mu \otimes \mathcal{M}_{w_{\mathrm{un}}\cdot\lambda}) = \bigotimes_{i:i \text{ odd}} \mathrm{Hom}_{K_\infty^{i,\circ M,(1)}}(\Lambda^\bullet({}^\circ\mathfrak{m}^{(i,1)}/{}^\circ\mathfrak{k}^i), \mathcal{D}_{b_i} \otimes \mathcal{M}_{b_i}) \tag{264}$$

the factors on the right hand side are of rank two: We have $K_\infty^{i,\circ M,(1)} = \mathrm{SO}(2)$ and under the adjoint action of $K_\infty^{i,\circ M,(1)}$ the module $\mathfrak{m}^{(i,1)}/{}^\circ\mathfrak{k}^i \otimes \mathbb{C}$ decomposes

$$\mathfrak{m}^{(i,1)}/{}^\circ\mathfrak{k}^i \otimes \mathbb{C} = \mathbb{C}P_{i,+}^\vee \oplus \mathbb{C}P_{i,-}^\vee$$

(See [Sltwo.pdf]) Then the two summands are generated by the tensors

$$\omega_{i,+} = P_{i,+}^\vee \otimes \psi_{b_i+2} \otimes m_{-b_i}, \bar{\omega}_{i,-} = P_{i,-}^\vee \otimes \psi_{-b_-2} \otimes m_{b_i} \tag{265}$$

where $m_{\pm(b_i)}$ is a highest (resp.) lowest weight vector for $K_\infty^{i,\circ M}$ acting on $\mathcal{M}_{w_{\mathrm{un}}\cdot\lambda}$. On the tensor product on the right we have an action of the maximal compact subgroup $\mathrm{O}(2)\times\mathrm{O}(2)\times\cdots\times\mathrm{O}(2)$ and under this action it decomposes into eigenspaces of dimension one. These eigenspaces are given by the product of sign characters $\underline{\epsilon} = (\epsilon_1, \epsilon_2, \dots)$.

Then it becomes clear that $\mathrm{Hom}_{K_\infty^{\circ M}}(\Lambda^\bullet({}^\circ\mathfrak{m}^{(1)}/{}^\circ\mathfrak{k}), \mathcal{D}_{\underline{\mu}} \otimes \mathcal{M}_{w_{\mathrm{un}}\cdot\lambda})$ is of rank one if n is odd and for n even it decomposes into two eigenspaces for the action of the group $\mathrm{O}(2)\times\mathrm{O}(2)\times\cdots\times\mathrm{O}(2)/S(\mathrm{O}(2)\times\mathrm{O}(2)\times\cdots\times\mathrm{O}(2)) = \{\pm 1\}$

$$\mathrm{Hom}_{K_\infty^{\circ M}}(\Lambda^\bullet({}^\circ\mathfrak{m}^{(1)}/{}^\circ\mathfrak{k}), \mathcal{D}_{\underline{\mu}} \otimes \mathcal{M}_{w_{\mathrm{un}}\cdot\lambda}) =$$

$$\mathrm{Hom}_{K_\infty^{\circ M}}(\Lambda^\bullet({}^\circ\mathfrak{m}^{(1)}/{}^\circ\mathfrak{k}), \mathcal{D}_{\underline{\mu}} \otimes \mathcal{M}_{w_{\mathrm{un}}\cdot\lambda}))_+ \oplus \mathrm{Hom}_{K_\infty^{\circ M}}(\Lambda^\bullet({}^\circ\mathfrak{m}^{(1)}/{}^\circ\mathfrak{k}), \mathcal{D}_{\underline{\mu}} \otimes \mathcal{M}_{w_{\mathrm{un}}\cdot\lambda}))_-$$

We have to recall that $\mathcal{M}_{\lambda^{\mathrm{un}}_{\circ M}} = H^{l(w_{\mathrm{un}})}(\mathfrak{u}_{\circ P}, \mathcal{M}_\lambda)$ is a cohomology group in degree $l(w_{\mathrm{un}})$. The classes in the factors of the last tensor product lie in degree 1, hence the multiply up to classes in degree ${}^\circ r$. This means that

$$H^q(\mathfrak{g}, K_\infty, \mathbb{D}_\lambda \otimes \mathcal{M}_\lambda) \neq 0 \text{ exactly for } q \in [l(w_{\mathrm{un}}) +^\circ r, l(w_{\mathrm{un}}) + n] \tag{266}$$

in the minimal degree ${}^\circ r$ it is of rank 2 or 1 depending on the parity of n.

6.6.2 The lowest K_∞ type in $\mathbb{D}_\lambda$

The maximal compact subgroup K_∞ is the fixed group of the standard Cartan-involution $\Theta : g \mapsto {}^t g^{-1}$. The subgroup ${}^\circ M$ is fixed under Θ and the subgroup $\mathrm{SO}(2)\times\mathrm{SO}(2)\times\cdots\times\mathrm{SO}(2) = K_\infty^{\circ M,(1)} = T_1^c$ is a maximal torus in K_∞. It is the stabilizer of a direct sum decompositions of $\mathbb{R}^n$ into two dimensional oriented planes V_i plus a line $\mathbb{R}z$ if n is odd, we write

$$\mathbb{R}^n = \bigoplus V_i \oplus (\mathbb{R}z) \tag{267}$$

The Cartan involution is the identity on our torus $T_1^c/\mathbb{R}$. This torus can be supplemented to a $\Theta-$ stable maximal torus by multiplying it by the torus $T_{1,\mathrm{split}}$ which is the product of the diagonal tori acting on the V_i in (267) times another copy of $\mathbb{G}_m$ acting on $\mathbb{R}z$ (if necessary). So we get a maximal torus $T_1 = T_1^c \cdot T_{1,\mathrm{split}}$. Obviously T_1 is the centralizer of T_1^c and the centralizer of $T_{1,\mathrm{split}}$ is the group ${}^\circ M$.

If we base change to $\mathbb{C}$ then T_1^c splits. We identify

$$\mathrm{SO}(2) \xrightarrow{\sim} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \tag{268}$$

and then the character group $X^*(T_1^c \times \mathbb{C}) = \oplus \mathbb{Z} e_\nu$ where on the ν-th component $e_\nu : \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mapsto a+bi = a+b\sqrt{-1}$. Then this choice provides a Borel subgroup $B_c \supset T_1^c \times \mathbb{C}$, for which the simple roots $\alpha_1^c, \alpha_2^c, \dots, \alpha_{{}^\circ r}^c$ are

$$\begin{cases} e_1 - e_2, e_2 - e_3, \dots, e_{{}^\circ r-1} - e_{{}^\circ r}, e_{{}^\circ r-1} + e_{{}^\circ r} & \text{for } n \text{ even} \\ e_1 - e_2, e_2 - e_3, \dots, e_{{}^\circ r} & \text{if } n \text{ is odd} \end{cases}$$

(See [Bou]). For n even we get the fundamental dominant weights

$$\gamma_\nu^c = \begin{cases} e_1 + e_2 + \cdots + e_\nu, & \text{if } \nu <^\circ r-1 \\ \frac{1}{2}(e_1 + e_2 + \cdots + e_{{}^\circ r-1} - e_{{}^\circ r}) & \text{if } \nu =^\circ r-1 \\ \frac{1}{2}(e_1 + e_2 + \cdots + e_{{}^\circ r-1} + e_{{}^\circ r}) & \text{if } \nu =^\circ r \end{cases} \tag{269}$$

and for n odd we get

$$\gamma_\nu^c = \begin{cases} e_1 + e_2 + \cdots + e_\nu, & \text{if } \nu <^\circ r \\ \frac{1}{2}(e_1 + e_2 + \cdots + e_{{}^\circ r}) & \text{last weight} \end{cases} \tag{270}$$

An easy calculation shows

$$\sum_{i=1}^{{}^\circ r} g_i e_i = \begin{cases} (g_1 - g_2)\gamma_1^c + (g_2 - g_3)\gamma_2^c + \cdots + (g_{{}^\circ r-1} - g_{{}^\circ r})\gamma_{{}^\circ r-1}^c + (g_{{}^\circ r-1} + g_{{}^\circ r})\gamma_{{}^\circ r}^c & n \text{ even} \\ (g_1 - g_2)\gamma_1^c + (g_2 - g_3)\gamma_2^c + \cdots + (g_{{}^\circ r-1} - g_{{}^\circ r})\gamma_{{}^\circ r-1} + 2g_{{}^\circ r}\gamma_{{}^\circ r}^c & n \text{ odd} \end{cases} \tag{271}$$

The character $\sum_{i=1}^{{}^\circ r} g_i e_i$ is dominant (with respect to B_c) if

$$\begin{cases} g_1 \geq g_2 \geq \dots g_{{}^\circ r-1} \geq \pm g_{{}^\circ r} & \text{if } n \text{ is even} \\ g_1 \geq g_2 \geq \cdots \geq g_{{}^\circ r-1} \geq g_{{}^\circ r} \geq 0 \end{cases} \tag{272}$$

Under the action of $K_\infty^{(1)}$ the $(\mathfrak{g}, K_\infty^{(1)})$- module $\mathbb{D}_\lambda$ decomposes into a direct sum

$$\mathbb{D}_\lambda = \bigoplus_{\mu^c} \mathbb{D}_\lambda(\Theta_{\mu^c}) \tag{273}$$

where $\mu^c \in X^*(T^c \times \mathbb{C})$ is a highest weight, Θ_{μ^c} is the resulting irreducible K_∞-module and $\mathbb{D}_\lambda(\Theta_{\mu^c})$ is the isotypical component.

We introduce the highest weight (see (246))

$$\mu_0^c(\lambda) = (b_1 + 2)e_1 + (b_3 + 2)e_2 + \cdots + (b_{2^\circ r-1} + 2)e_{{}^\circ r} \tag{274}$$

and and in terms of our dominant weight λ this is

$$\mu_0^c(\lambda) = \begin{cases} 2(a_1 + 1)\gamma_1^c + \cdots + 2(a_{{}^\circ r-1} + 1)\gamma_{{}^\circ r-1}^c + 2(a_{{}^\circ r-1} + a_{{}^\circ r} + 3)\gamma_{{}^\circ r}^c & \text{if } n \text{ is even} \\ 2(a_1 + 1)\gamma_1^c + \cdots + 2(a_{{}^\circ r} + 3)\gamma_{{}^\circ r}^c & \text{if } n \text{ is odd} \end{cases} \tag{275}$$

For $\lambda = 0$ we get an expression (not depending on the parity of n)

$$\mu_0^c(0) = 2\gamma_1^c + \cdots + 2\gamma_{{}^\circ r-1}^c + 6\gamma_{{}^\circ r}^c \tag{276}$$

In the case that n is even the group K_∞ contains the element θ which maps $e_i \to e_i$ for $i \leq^\circ r-1$ and $e_{\circ r} \to -e_{\circ r}$ or what amounts to the same exchanges $\gamma^c_{\circ r-1}$ and $\gamma^c_{\circ r}$ and fixes the other fundamental dominant weights. Then

$$\bar{\mu}^c_0(\lambda) := \vartheta(\mu^c_0(\lambda)) = 2\gamma^c_1 + \cdots + 6\gamma^c_{\circ r-1} + 2\gamma^c_{\circ r} + \vartheta(\lambda^c) \tag{277}$$

Proposition 6.2. *If n is odd then the $K^{(1)}_\infty$- type $\Theta_{\mu^c_0(\lambda)}$ occurs in $\mathbb{D}_\lambda$ with multiplicity one. All other occurring $K^{(1)}_\infty$ types are "larger", i.e. their highest weight satisfies $\mu^c = \mu^c_0(\lambda) + \sum n_i\alpha^c_i$ with $n_i \geq 0$. We have*

$$H^\bullet(\mathfrak{g}, K_\infty, \mathbb{D}_\lambda \otimes \mathcal{M}_\lambda) = \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \Theta_{\mu^c_0(\lambda)} \otimes \mathcal{M}_\lambda)$$

If n is even then the $(\mathfrak{g}, K^{(1)}_\infty)$ module $\mathbb{D}_\lambda$ decomposes into two irreducible sub modules

$$\mathbb{D}_\lambda = \mathbb{D}^+_\lambda \oplus \mathbb{D}^-_\lambda.$$

The $K^{(1)}_\infty$ types $\Theta_{\mu^c_0(\lambda)}$ resp. $\Theta_{\bar{\mu}^c_0(\lambda)}$ occur with multiplicity one (resp. zero) in $\mathbb{D}^+_\lambda$(resp. $\mathbb{D}^-_\lambda$). They are the lowest $K^{(1)}_\infty$ types respectively. We have

$$H^\bullet(\mathfrak{g}, K^{(1)}_\infty, \mathbb{D}_\lambda \otimes \mathcal{M}_\lambda) = H^\bullet(\mathfrak{g}, K^{(1)}_\infty, \mathbb{D}^+_\lambda \otimes \mathcal{M}_\lambda) \oplus H^\bullet(\mathfrak{g}, K^{(1)}_\infty, \mathbb{D}^-_\lambda \otimes \mathcal{M}_\lambda) =$$

$$\mathrm{Hom}_{K^{(1)}_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \Theta_{\mu^c_0(\lambda)} \otimes \mathcal{M}_\lambda) \oplus \mathrm{Hom}_{K^{(1)}_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \Theta_{\bar{\mu}^c_0(\lambda)} \otimes \mathcal{M}_\lambda)$$

Proof. For two fundamental weights we write $\mu^c \geq \mu^c_1$ if μ^c is "larger" than μ^c_1 in the above sense. We start from (261) and consider a single summand $\mathrm{Ind}_{K^\circ_\infty M^{(1)}}^{K^{(1)}_\infty} \mathbb{C}\psi_{\varepsilon_i(b_i+2+2\nu_i)}$. This induced module decomposes into isotypical modules

$$\mathrm{Ind}_{K^\circ_\infty M^{(1)}}^{K^{(1)}_\infty} \mathbb{C}\psi_{\varepsilon_i(b_i+2+2\nu_i)} = \bigoplus_{\mu^c} \mathrm{Ind}_{K^\circ_\infty M^{(1)}}^{K^{(1)}_\infty} \mathbb{C}\psi_{\varepsilon_i(b_i+2+2\nu_i)}(\Theta_{\mu^c}) \tag{278}$$

where μ^c runs over the set of dominant weights, where Θ_{μ^c} is the irreducible module of highest weight μ^c and where $\mathrm{Ind}_{K^\circ_\infty M^{(1)}}^{K^{(1)}_\infty} \mathbb{C}\psi_{\varepsilon_i(b_i+2+2\nu_i)}(\Theta_{\mu^c})$ is the isotypical component. If we pick any dominant weight μ^c then Frobenius reciprocity yields that

$$\begin{gathered}\Theta_{\mu^c} \text{ occurs in } \mathrm{Ind}_{K^\circ_\infty M^{(1)}}^{K^{(1)}_\infty} \mathbb{C}\psi_{\varepsilon_i(b_i+2+2\nu_i)} \text{ with multiplicity } k \iff \\ t \mapsto \textstyle\prod_j e_i(t)^{\varepsilon_i(b_i+2+2\nu_i)} \text{ occurs in } \Theta_{\mu^c} \text{ with multiplicity } k\end{gathered} \tag{279}$$

and if $k > 0$ this implies $\mu^c \geq t \mapsto \prod_j e_i(t)^{\varepsilon_i(b_i+2+2\nu_i)}(t)$. It it easy to see that we get minimal $K^{(1)}_\infty$ types only if all $\nu_i = 0$. But

$$t \mapsto \prod_j e_i(t)^{\varepsilon_i(b_i+2)} \text{ is dominant } \iff \begin{cases} \varepsilon = (1,1,\dots,1,\pm 1) \text{ if } n \text{ even} \\ \varepsilon = (1,1,\dots,1,1) \text{ if } n \text{ odd} \end{cases} \tag{280}$$

and in the n even case these two characters are exactly $\mu^c_0(\lambda)$ and $\bar{\mu}^c_0(\lambda)$ and in the n odd case this character is $\mu^c_0(\lambda)$.

□

6.6.3 The unitary modules with cohomology, cohomological induction.

We start from an essentially self dual highest weight λ and the attached highest weight module $\mathcal{M}_\lambda$. In their paper [Vo-Zu] Vogan and Zuckerman construct a finite family of $(\mathfrak{g}, K_\infty)$ modules denoted by $A_\mathfrak{q}(\lambda)$ which have non trivial cohomology with coefficients in $\mathcal{M}_\lambda$., i.e.

$$H^\bullet(\mathfrak{g}, K_\infty, A_\mathfrak{q}(\lambda) \otimes \mathcal{M}_\lambda) \neq 0$$

They also show that all unitary irreducible $(\mathfrak{g}, K_\infty)$ -modules with non trivial cohomology in with coefficients in $\mathcal{M}_\lambda$. are of this form. We briefly recall their construction and translate it into our language and our way of thinking about these issues.

We introduce the torus $\mathbb{S}^1/\mathbb{R}$ whose group of real points is the unit circle in $\mathbb{C}^\times$ and we chose once for all an isomorphism

$$i_0 : \mathbb{S}^1 \times_\mathbb{R} \mathbb{C} \xrightarrow{\sim} \mathbb{G}_m/\mathbb{C} \tag{281}$$

We consider the free $\mathbb{Z}$ module

$$\mathrm{Hom}_\mathbb{R}(\mathbb{S}^1, T_1^c) = \mathrm{Hom}_\mathbb{R}(\mathbb{S}^1, T_1) = X_*(T_1^c \times_\mathbb{R} \mathbb{C})$$

where of course the last identification depends on the choice of i_0. We have the standard pairing $< \ , \ > : X_*(T_1 \times_\mathbb{R} \mathbb{C}) \times X^*(T_1 \times_\mathbb{R} \mathbb{C}) \to \mathbb{Z}$.

The first ingredient in the construction of an $A_\mathfrak{q}(\lambda)$ is the choice of a cocharacter $\chi : \mathbb{S}^1 \to T_c$ (defined over $\mathbb{R}$). From this cocharacter we get the centralizer Z_χ , this is a reductive subgroup whose set of roots is

$$\Delta_\chi = \{\alpha \in \Delta \subset X^*(T_1 \times_\mathbb{R} \mathbb{C}) | \ < \chi, \alpha >= 0\}.$$

We can also define

$$\Delta_\chi^+ = \{\alpha | < \chi, \alpha > > \ 0\},$$

this set depends on the choice of i_0 (see (281)). This provides a parabolic subgroup $P_\chi \subset G \times_\mathbb{R} \mathbb{C}$ whose system of roots is $\Delta_\chi \cup \Delta_\chi^+$. Clearly $\Theta(P_\chi) = P_\chi$ hence P_χ is the Θ-stable parabolic subgroup attached to the datum χ. This parabolic subgroup is only defined over $\mathbb{C}$, if we intersect it with its conjugate $\bar{P}_\chi$ then we get the centralizer Z_χ of χ. We relate this to the notations in [Vo-Zu]: the $\mathfrak{q}$ in $A_\mathfrak{q}(\lambda)$ is the Lie-algebra of P_χ, the group Z_χ is the L. Let $\mathfrak{u}_\chi$ be the Lie algebra of U_χ. The datum χ determines the $\mathfrak{q}$ in $A_\mathfrak{q}(\lambda)$. We could introduce the notation $A_\mathfrak{q}(\lambda) = A_\chi(\lambda)$. Since T_1 is the centralizer of T_c we can find a generic cocharacter χ_{gen} such that $P_{\chi_{\mathrm{gen}}} = B_c$ our chosen Borel subgroup in ${}^\circ M$.

To a highest weight λ which is trivial on the semi-simple part $Z_\chi^{(1)}$ Vogan-Zuckerman attach an irreducible unitary $(\mathfrak{g}, K_\infty)$ module $A_\mathfrak{q}(\lambda)$ such that

$$H^\bullet(\mathfrak{g}, K_\infty, A_\mathfrak{q}(\lambda) \otimes \mathcal{M}_\lambda) \neq 0.$$

Vogan and Zuckerman show (based on results of many others) that all the unitary irreducible $(\mathfrak{g}, K_\infty)$ modules with non trivial cohomology in $\mathcal{M}_\lambda$ are isomorphic to an $A_\mathfrak{q}(\lambda)$.

Furthermore they give a description of the K_∞ types occurring in $A_{\mathfrak{q}}(\lambda)$ especially they show that $A_{\mathfrak{q}}(\lambda)$ contains a lowest K_∞ type. This lowest K_∞-type is given by a dominant weight which obtained by the following rule:

We consider the action of the group K_∞ on the unipotent radical U_χ and on the Lie algebra $\mathfrak{u}_\chi$ and the restriction of this action to T_1^c. The torus T_1 also acts on $\mathfrak{u}_\chi$ and under this action we get a decomposition into one dimensional eigenspaces

$$\mathfrak{u}_\chi = \bigoplus_{\alpha\in\Delta_\chi^+} \mathfrak{u}_\alpha$$

let us choose generators X_α in these eigenspaces. We observe that the roots $\alpha, \Theta\alpha \in \Delta^+$ induce the same root α_c on T_1^c. The vector $V_{\alpha_c} = X_\alpha - \Theta X_\alpha \in \mathfrak{u}_\chi$ is a non zero eigenvector for T_1^c and

$$\mathfrak{u}_\chi \cap (\mathfrak{p}\otimes\mathbb{C}) = \bigoplus_{(\alpha,\Theta\alpha)\in\Delta_\chi^+} \mathbb{C}V_{\alpha_c}$$

the sum runs over the unordered pairs. Then

$$\mu_c(\chi,\lambda) = \sum_{(\alpha,\Theta\alpha)\in\Delta_\chi^+} \alpha_c + \lambda_c \tag{282}$$

is a highest weight of a representation $\Theta_{\mu_c(\chi,\lambda)}$ of $K_\infty^{(1)}$ and this is the lowest $K_\infty^{(1)}$ type in $A_{\mathfrak{q}}(\lambda)$. We get

$$H^\bullet(\mathfrak{g}, K_\infty^{(1)}, A_{\mathfrak{q}}(\lambda)\otimes\mathcal{M}_\lambda) = \mathrm{Hom}_{K_\infty^{(1)}}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), A_{\mathfrak{q}}(\lambda)\otimes\mathcal{M}_\lambda) = \mathrm{Hom}_{K_\infty^{(1)}}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \Theta_{\mu_c(\chi,\lambda)}\otimes\mathcal{M}_\lambda) \tag{283}$$

The module is determined by these properties:

1) It has non trivial cohomology with coefficients in $\mathcal{M}_\lambda$

2) It has $\mu_c(\chi,\lambda)$ as highest weight of a minimal K_∞ type. (See Thm. 5. 3 in [Vo-Zu].)

Recall that our aim at this moment is to identify the module J_{σ_∞} to an $A_{\mathfrak{q}}(\lambda)$, and to achieve this goal we exhibit a list of very specific $A_{\mathfrak{q}}(\lambda)$'s.

6.6.4 Comparison of two tori

We need to compute $\mu_c(\chi,\lambda)$ and to achieve this goal the author of this book modifies the Cartan involution in order to do the computation in a split group. Our standard torus T is contained in the standard Borel subgroup B of upper triangular matrices. Let w_0 be an element in the normalizer of T which conjugates B into its opposite Borel subgroup. If we replace our Cartan involution by $\Theta_1 = w_0\Theta$ then Θ_1 fixes T and the Borel subgroup B. This is not a Cartan involution, but it is easily seen that it is conjugate to Θ over $\mathrm{Gl}_n(\mathbb{C})$. and

$$\Theta_1 : \begin{pmatrix} t_1 & 0 & 0 & \dots & \\ 0 & t_2 & \dots & \dots & \\ 0 & 0 & \ddots & 0 & \dots \\ 0 & & \dots & t_{n-1} & \\ 0 & & & & t_n \end{pmatrix} \mapsto \begin{pmatrix} t_n^{-1} & 0 & 0 & \dots & \\ 0 & t_{n-1}^{-1} & \dots & \dots & \\ 0 & 0 & \ddots & 0 & \dots \\ 0 & & \dots & t_2^{-1} & \\ 0 & & & & t_1^{-1} \end{pmatrix} \tag{284}$$

We can decompose T up to isogeny into a torus T_c on which Θ_1 acts by the identity and a torus T_{split} where it acts by $x \mapsto x^{-1}$:

$$T_c = \{\begin{pmatrix} t_1 & 0 & 0 & \dots & & \\ 0 & t_2 & \dots & \dots & & \\ 0 & 0 & \ddots & 0 & \dots \\ 0 & & \dots & t_2^{-1} & \\ 0 & & & & t_1^{-1} \end{pmatrix}\} \text{ resp. } T_{\mathrm{split}} = \{\begin{pmatrix} t_1 & 0 & 0 & \dots & & \\ 0 & t_2 & \dots & \dots & & \\ 0 & 0 & \ddots & 0 & \dots \\ 0 & & \dots & t_2 & \\ 0 & & & & t_1 \end{pmatrix}\}$$

It is clear that a suitable permutation matrix conjugates $T_{1,\mathrm{split}}$ into T_{split}. This permutation matrix maps the centralizer of $T_{1,\mathrm{split}}$ (which is ${}^\circ M$) to the centralizer ${}^\circ M'$ of T_{split} and the anisotropic torus T_1^c to an anisotropic torus $T_1^{c\prime}$ in ${}^\circ M'$. Then we can find an element $m \in {}^\circ M'(\mathbb{C})$ which conjugates $T_1^{c\prime} \times \mathbb{C}$ into T_c.

The composition of these conjugations provides an identification of the character modules $X^*(T_1 \times \mathbb{C}) = X^*(T)$ which respects the product decompositions and hence we get

$$X^*(T_1^c \times \mathbb{C}) = X^*(T_c). \tag{285}$$

We choose our conjugating element m such that the $e_i \in X^*(T_1^c \times \mathbb{C})$ are mapped to the element $t \mapsto t_i$ (for $i = 1$ to ${}^\circ r$).

Inside $X^*(T)$ we have the dominant fundamental weights $\gamma_1, \dots, \gamma_{n-1}$, let $\bar{\gamma}_i$ be the restriction of γ_i to T_1^c then we have $\bar{\gamma}_i = \bar{\gamma}_{n-i}$. We can interpret the $\bar{\gamma}_i$ also as elements in $X^*(T_1 \times \mathbb{C}) \otimes \mathbb{Q}$ we require that the restriction of $\bar{\gamma}_i$ to $T_{1,\mathrm{split}}$ is trivial. Then we can write

$$\bar{\gamma}_i = \begin{cases} \frac{1}{2}(\gamma_i + \gamma_{n-i}) & \text{if } i \neq \frac{n}{2} \\ \gamma_i & \text{else} \end{cases} \tag{286}$$

We can relate the dominant weights γ_i^c and the $\bar{\gamma}_i$: If n is even then

$$\gamma_\nu^c = \bar{\gamma_\nu} \text{ for } 1 \leq \nu <^\circ r-1,\ \gamma_{{}^\circ r-1}^c = \bar{\gamma}_{{}^\circ r-1} - \frac{1}{2}\bar{\gamma}_{{}^\circ r},\ \gamma_{{}^\circ r}^c = \frac{1}{2}\bar{\gamma}_{{}^\circ r} \tag{287}$$

For n odd we get

$$\gamma_\nu^c = \bar{\gamma_\nu} \text{ for } 1 \leq \nu <^\circ r,\ \gamma_{{}^\circ r}^c = \frac{1}{2}\bar{\gamma}_{{}^\circ r}$$

The Borel subgroup B is invariant under Θ_1, the root subgroup $U_{i,j}; 1 \leq i < l \leq n$ is conjugated into $U_{n+1-j,n+1-i}$. Inside the unipotent radical we have the half diagonal of spots $({}^\circ r, {}^\circ r + 1 + 2\epsilon(n)), \dots (2, n-1), (1, n)$ The involution is a reflection along this half diagonal and the spots on the left of the half diagonal form a system of representatives for $\sim \Theta_1$. Of course we have a corresponding Borel subgroup $B_1 \supset T_1 \times \mathbb{C}$ of $G \times \mathbb{C}$.

Proposition 6.3. *Under the above identification the restrictions of the* $\gamma_{2i-1}^{{}^\circ M}$ *to* T_c *are equal to the* e_i *in* $X^*(T_1^c \times \mathbb{C})$.

We want to compute $\mu_c(\chi, \lambda)$. By definition this is an element in $X^*(T_c \times \mathbb{C})$ using the identification in (6.6.4) we carry out this computation in $X^*(T_c)$. A cocharacter $\chi : \mathbb{G}_m \to T_c$ is of the form

$$\chi : t \mapsto \begin{pmatrix} t^{m_1} & 0 & 0 & \dots & & \\ 0 & t^{m_2} & \dots & \dots & & \\ 0 & 0 & \ddots & 0 & \dots & \\ 0 & & \dots & t^{-m_2} & & \\ 0 & & & & t^{-m_1} & \end{pmatrix}$$

since we want $P_\chi \supset B_1$ we require $m_1 \geq m_2 \geq m_{\circ r} \geq 0$. (If n is odd then there is an $m_{\circ r+1} = 0$). Let us start with the regular case, this means that all $\geq$ signs are actually strict, i.e. $>$ signs. Then it an easy computation that

$$\mu_c(\chi_{\mathrm{reg}}, \lambda) = \begin{cases} ne_1 + (n-2)e_2 + \dots + 2e_{\circ r} + \lambda_c & \text{if } n \text{ is even} \\ ne_1 + (n-2)e_2 + \dots + 3e_{\circ r} + \lambda_c & \text{if } n \text{ is odd} \end{cases} \tag{288}$$

The set $\Delta^+_{\chi_{\mathrm{reg}}}$ is the set of roots of B modulo the conjugation Θ_1. Hence we see that

$$\mu_c(\chi_{\mathrm{reg}}, \lambda) = \mu_0^c(\lambda).$$

The interesting contribution is in fact $\mu_c(\chi_{\mathrm{reg}}, , 0)$ and this is the number μ_0^c in (276) We can express $\mu_c(\chi_{\mathrm{reg}}, 0)$ in terms of the fundamental weights γ_i (or the $\bar{\gamma}_i$) we use the formulas (287). We get

$$\mu_c(\chi_{\mathrm{reg}}, 0) = 2\bar{\gamma}_1 + 2\bar{\gamma}_2 + \dots + 2\bar{\gamma}_{\circ r-1} + \begin{cases} 2\bar{\gamma}_{\circ r} & n \equiv 0 \mod 2 \\ 6\bar{\gamma}_{\circ r} & n \equiv 1 \mod 2 \end{cases} \tag{289}$$

If χ is not regular then the relevant information extracted from χ is the list

$$t_\chi = (t_1, t_2, \dots, t_s; t_0)$$

(the type of χ) where the t_i are the length of the intervals where the $m_i > 0$ are constant, i.e. $m_1 = m_2 = \dots = m_{t_1} > m_{t_1+1} = \dots = m_{t_1+t_2} > \dots$. The number t_0 is the length of the interval where $m_i = 0$. The Θ stable parabolic P_χ subgroup only depends on t_χ. The types t_χ have to satisfy the (only) constraint

$$2\sum t_\nu + t_0 = n \tag{290}$$

The regular case corresponds to the list $(1, 1, \dots, 1; 0 \text{ or } 1)$. In the general case we get a decorated Dynkin diagram where the crossed out roots are those where the m_i jump.

$$- \times - \circ - \circ - \circ - \times - \times - \circ - \dots - \circ - \times \dots \times - \circ - \dots - \circ$$

This decorated diagram is symmetric under the reflection $i \mapsto n - i$. We look at the connected component of $\circ$-s. These components come in pairs unless the component is invariant under the reflection, i.e. it is central. The non central pairs

$$\pi_{\chi,\nu} = \times - \underset{\alpha_{i_\nu}}{\circ} - \dots - \underset{\alpha_{j_\nu}}{\circ} - \times - \dots \times - \underset{\alpha_{n-j_\nu}}{\circ} - \dots - \underset{\alpha_{n-i_\nu}}{\circ} \tag{291}$$

are labelled by the indices ν for which $t_\nu > 1$, and are of length $t_\nu - 1 = j_\nu - i_\nu + 1$. (The meaning of the indices i_ν, j_ν is explained in the diagram). The central connected component is of length $t_0 - 1$, of course it may be empty. We write it as

$$\pi_{\chi,0} = \begin{array}{ccccc} \times - & \circ & - \cdots - & \circ & - \times \\ & \alpha_{i_0} & & \alpha_{j_0} & \end{array} \tag{292}$$

where of course $i_0 = n - j_0$. Let π_χ be the union of these connected components. Let Δ_ν^+ be the set of positive roots which are sums of roots in π_ν.

To compute $\mu_c(\chi, 0)$ we have to subtract from $\mu_c(\chi_{\text{reg}}, 0)$ the sum of roots in Δ_ν^+ with $j_\nu <^\circ r$ and the sum of roots in $\Delta_0^+/\{\Theta_1\}$.

A simple calculation shows that for $\nu > 0$

$$2\rho^{(\nu)} = \sum_{i=i_\nu}^{i=j_\nu} \gamma_i + \gamma_{n-i} - (t_\nu - 1)(\gamma_{i_\nu - 1} + \gamma_{i_\mu + 1}) \tag{293}$$

where we put $\gamma_{-1} = \gamma_n = 0$. This means that subtracting $2\rho^{(\nu)}$ from the sum which yields $\mu_c(\chi_{\text{reg}}, 0)$ has the effect that the sum $\sum_{i=i_\nu}^{i=j_\nu} \gamma_i + \gamma_{n-i} = 2\sum \bar{\gamma}_i$ cancels out and we have to add $(t_\nu - 1)(\gamma_{i_\nu - 1} + \gamma_{i_\mu + 1})$. Observe that $i_{\nu-1}, j_{\mu+1} \notin \pi_\chi$. We still have to subtract the contribution from the central component Δ_0^+. We have to sum the roots in $\Delta_0^+/\{\Theta_1\}$ this means that we take half the sum of all roots and add half the sum of the symmetric roots. This yields

$$2\rho^{(0)} = \frac{1}{2}((j_0 - i_0 + 1)\alpha_{i_0} + \cdots + (j_0 - i_0 + 1)\alpha_{j_0}) + \frac{1}{2}(\alpha_{i_0} + \cdots + \cdots + \alpha_{j_0}) =$$

$$((j_0 - i_0 + 2)\bar{\alpha}_{i_0} + \cdots + (\ldots)\bar{\alpha}_{\circ r}$$

we see again that the sum $\sum_{i=i_0}^{n-i_0} \bar{\gamma}_i$ drops out and we have to add a term $t_0(\gamma_{i_0 - 1} + \gamma_{i_0 + 1})$.

Hence we get: Let π_χ^c be the union of the π_ν^c and π_0^c. Then

$$\mu_c(\chi, 0) = \sum_{i \notin \pi_\chi^c} (2 + c_i(\chi, 0))\gamma_i^c$$

where

$$c_i(\chi, 0) = \begin{cases} (t_{\nu^-} - 1) + (t_{\nu^+} - 1) & \text{if } \nu \neq 0 \\ (t_{\nu^-} - 1) + t_{\nu^+} & \text{if } \nu = 0 \end{cases} \tag{294}$$

and where $t_{\nu^-} - 1$ is the length of connected component directly to the left of $i_\nu - 1$ and $t_{\nu^+} - 1$ is the length of the component directly to the right of $i_\nu - 1$.

If we have chosen a highest weight $\lambda = \sum a_i \gamma_i$ then we require $a_i = a_{n+1-i} \geq 0$ and we must have $a_i = 0$ for all $i \in \pi_\chi$. Then

$$\mu_c(\chi, \lambda) = \sum_{i \notin \pi_\chi} (2 + c_i(\chi, 0) + 2a_i)\gamma_i^c.$$

For us a special case is of interest. We decompose $n = uv$ and take $\chi_{u,v} = \chi$ of type $t_\chi = (v, v, \ldots, v)$. Hence the reductive quotient of the Θ stable parabolic subgroup is $M^\vee = \mathrm{Gl}_v \times \mathrm{Gl}_v \times \cdots \times \mathrm{Gl}_v$, the number of factors is u. In this case we get

$$\begin{array}{cccccccccccc} \circ & - & \circ & \cdots - & \circ - & \times & - \circ - & \circ - \cdots - & \circ - \times & & - \circ \dots \\ \alpha_1 & & \alpha_2 & & \alpha_{v-1} & \alpha_v & \alpha_{v+1} & & \alpha_{2v-1} & \alpha_{2v} & \end{array} \tag{295}$$

so the indices outside π_χ are the multiples of v. Let us denote by $\mathfrak{q}$ the Lie-algebra of $P_{\chi_{u,v}}$.

$$\mu_c(\chi_{u,v}, \lambda) = \sum_{\nu : \nu v \leq \frac{u}{2}} (2 + 2(v-1) + e(\nu))\gamma^c_{\nu v} + \lambda_c \tag{296}$$

where $e(\nu) = 0$ except in the case that ${}^\circ r \in [\nu v, (\nu+1)v]$ and then it is equal to 1.

6.6.5 The $A_{\mathfrak{q}_{u,v}}(\lambda)$ as Langlands quotients

Let $n = uv$ and $\mathfrak{q} = \mathfrak{q}_{u,v}$ as above. The parabolic is $P_{\chi_{u,v}}$ To realize $A_{\mathfrak{q}_{u,v}}(\lambda)$ as Langlands quotient we apply the procedure described in [Vo-Zu], p.82-83. We have to find a parabolic subgroup $P \subset \mathrm{Gl}_n/\mathbb{R}$ and a tempered representation σ_∞ of $M = P/U$ such that

a) our λ is a character on P,

b) the module ${}^{\mathrm{a}}\mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}\sigma_\infty$ has the right infinitesimal character,

c) the module $\mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}\sigma_\infty$ restricted to K_∞ contains $\mu_c(\chi_{u,v}, \lambda_c)$ as minimal K_∞ type.

To get our parabolic subgroup we choose a cocharater $\eta_{u,v} : \mathbb{G}_m \to T$, this cocharacter is defined as

$$t \mapsto \eta_{u,v}(t) = \begin{pmatrix} t^v & 0 & 0 & & & \dots & \\ 0 & t^{v-1} & \dots & \dots & & & \\ 0 & 0 & \ddots & 0 & \dots & & \\ 0 & & \dots & t^1 & & & \\ 0 & & & & t^v & & \\ 0 & & & & & t^{v-1} & \\ 0 & & & & & & \ddots \end{pmatrix} \tag{297}$$

i.e. we have u copies of the diagonal matrix $\mathrm{diag}(t^v, t^{v-1}, \dots, t)$ on the diagonal.

This cocharacter $\eta = \eta_{u,v}(t)$ yields a parabolic subgroup P_η which contains the torus and has as roots $\Delta_\eta = \{\alpha | < \eta, \alpha > \geq 0\}$. Its reductive quotient is $\mathrm{Gl}_u \times \mathrm{Gl}_u \times \cdots \times \mathrm{Gl}_u$ where the number of factors is v. The embedding into Gl_n is not the obvious one and P_η does not contain the standard Borel subgroup of upper triangular matrices.

To describe the relation between these two groups we denote by $\mathfrak{e}_1, \mathfrak{e}_2, \dots, \mathfrak{e}_n$ the standard orthonormal basis of our underlying vector space $\mathbb{R}^n$. Then we group these basis elements

$$\{\{\mathfrak{e}_1, \dots, \mathfrak{e}_v\}, \{\{\mathfrak{e}_{v+1}, \dots, \mathfrak{e}_{2v}\}, \dots, \{\mathfrak{e}_{(u-1)v+1}, \dots, \mathfrak{e}_{uv}\}\}$$

and this grouping yields a direct sum decomposition

$$\mathbb{R}^n = (\mathbb{R}\mathfrak{e}_1 \oplus \cdots \oplus \mathbb{R}\mathfrak{e}_v) \oplus (\mathbb{R}\mathfrak{e}_{v+1} \oplus \cdots \oplus \mathbb{R}\mathfrak{e}_{2v}) \oplus \cdots \oplus (\mathbb{R}\mathfrak{e}_{(u-1)v+1}, \dots, \mathbb{R}\mathfrak{e}_{uv}) =$$

$$V_1 \oplus V_2 \oplus \cdots \oplus V_u \tag{298}$$

and then $M^\vee = \mathrm{Gl}(V_1) \times \cdots \times \mathrm{Gl}(V_u)$.

We get a second grouping of the basis elements

$$\{\{\mathfrak{e}_1, \mathfrak{e}_{v+1}, \dots, \mathfrak{e}_{(u-1)v+1}\}, \{\mathfrak{e}_2, \mathfrak{e}_{v+2}, \dots, \mathfrak{e}_{(u-1)v+2}\}, \dots\}, \{\dots \mathfrak{e}_{uv}\}\} \tag{299}$$

which yields direct sum decomposition

$$\mathbb{R}^n = \Big(\mathbb{R}\mathfrak{e}_1 \oplus \mathbb{R}\mathfrak{e}_{v+1} \oplus \cdots \oplus \mathbb{R}\mathfrak{e}_{(u-1)v+1}\Big) \oplus \Big(\mathbb{R}\mathfrak{e}_2 \oplus \mathbb{R}\mathfrak{e}_{v+2} \oplus \cdots \oplus \mathbb{R}\mathfrak{e}_{(u-1)v+2}\Big) \oplus \dots$$

$$W_1 \oplus W_2 \oplus \cdots \oplus W_v \tag{300}$$

and then $M = \mathrm{Gl}(W_1) \times \mathrm{Gl}(W_2) \times \cdots \times \ \mathrm{Gl}(W_v) = \mathrm{Gl}_u \times \mathrm{Gl}_u \times \cdots \times \mathrm{Gl}_u$. The groups $M^\vee$ and M are mutual centralizers of each other.

The two groupings define two different Borel subgroups, the first one defines the standard Borel B of upper triangular matrices and the second Borel B^* fixes the flag $\{\mathfrak{e}_1\}, \{\mathfrak{e}_1, \mathfrak{e}_{v+1}\} \dots$. Let us denote by $\lambda^*, \rho^*, w^*_{u,v}, \dots$ the dominant weight with respect to B^*, the half sum of positive roots and so on. Our highest weight λ is trivial on the semi simple part of $M^\vee$ it must be of the form (235) Now we consider the highest weight for the group M

$$\begin{aligned} w^*_{u,v}(\lambda^* + \rho^*) - \rho^* = \underline{\mu}* = (a_1 + v - 1)(\gamma_1^{*,M} + \gamma_{1+u}^{*,M} + \cdots + \gamma_{1+(v-1)u}^{*,M}) + \\ (a_2 + v - 1)(\gamma_2^{*,M} + \gamma_{2+u}^{*,M} + \cdots + \gamma_{2+(v-1)u}^{*,M}) + \\ \vdots \\ (a_{u-1} + v - 1)(\gamma_{u-1}^{*,M} + \gamma_{u-1+u}^{*,M} + \cdots + \gamma_{u-1+(v-1)u}^{*,M}) + \\ -(u+1)(\gamma_u^* + \gamma_{2u}^* + \cdots + \gamma_{(v-1)u}^*) + d\delta. \end{aligned} \tag{301}$$

We choose $\sigma_\infty = \mathbb{D}_{\underline{\mu}^*}$. (See (231))

We check the lowest K_∞ type in $\mathrm{Ind}_{P^*}^G \mathbb{D}_{\underline{\mu}^*}$. To compute this lowest K_∞ type we write $M = \prod M_\nu$ where of course each $M_\nu = \mathrm{Gl}_u$. Accordingly we write $T = \prod T_\nu$. The weight $\underline{\mu}^* = \sum \mu^*_\nu$ where the semi simple part is "always the same". We apply the considerations in section 6.6.1 to the factors M_ν. We take $\nu = 1$ then

$$\mu_1^* = (a_1 + v - 1)\gamma_1^* + (a_2 + v - 1)\gamma_1^* + \cdots + (a_{u-1} + v - 1)\gamma_{u-1}^* + d^*\mathrm{det}_u$$

Inside M_1 we have the subgroup ${}^\circ M_1$ which is the reductive Levi factor of ${}^\circ P_1$ as in section 6.6.1 and we have the Kostant element $w_{1,\mathrm{un}}$. Then we consider the character

$$\tilde{\mu}_1^* = w_{1,\mathrm{un}}(\mu_1^* + \rho_1^*) - \rho_1^* = \sum_{i:i \text{ odd}} b_i^* \gamma_i^{{}^\circ M_1^{(1)}} + \tilde{\mu}_1^{*,\mathrm{ab}} \tag{302}$$

where again the b_i^* are the cuspidal parameters and they are given by

$$b_{2j-1}^* = v(u+1-2j) - 1 + \begin{cases} 2a_j + 2a_{j+1} + \cdots + 2a_{\frac{u}{2}-1} + a_{\frac{u}{2}} & \text{if } u \text{ is even} \\ 2a_j + 2a_{j+1} + \cdots + 2a_{\frac{u-1}{2}} & \text{if } u \text{ is odd} \end{cases} \tag{303}$$

The abelian part $\tilde{\mu}_1^{*,\mathrm{ab}}$ does not play any role in the following (The λ in section (6.6.1) is now μ_1^* and the $\underline{\mu}$ in formula (254) is now $\tilde{\mu}_1^*$) We renumber our basis (299)

$$\{\mathfrak{f}_1, \mathfrak{f}_2, \dots, \mathfrak{f}_{u-1}, \mathfrak{f}_u, \dots\} = \{\mathfrak{e}_1, \mathfrak{e}_{v+1}, \dots, \mathfrak{e}_{(u-1)v+1}, \mathfrak{e}_2, \dots\} \tag{304}$$

and decompose the space $\mathbb{R}^n$ into a direct sum of euclidian planes (plus a line if n is odd)

$$\mathbb{R}^n = (\mathbb{R}\mathfrak{f}_1 \oplus \mathbb{R}\mathfrak{f}_2) \oplus (\mathbb{R}\mathfrak{f}_3 \oplus \mathbb{R}\mathfrak{f}_4) \oplus \cdots \oplus (\mathbb{R}\mathfrak{f}_n).$$

and this provides a maximal anisotropic torus

$$T_c^* = \mathrm{SO}(2) \times \mathrm{SO}(2) \times \cdots \times \mathrm{SO}(2)$$

In analogy with section 6.6.2 we write

$$X^*(T_c^* \otimes \mathbb{C}) = \oplus \mathbb{Z} f_j \tag{305}$$

where f_j is defined in analogy with the e_ν in section 6.6.2.

We have

$$M = \mathrm{Gl}(\mathbb{R}\mathfrak{f}_1 \oplus \mathbb{R}\mathfrak{f}_2 \oplus \cdots \oplus \mathbb{R}\mathfrak{f}_u) \times \cdots \times \mathrm{Gl}(\mathbb{R}\mathfrak{f}_{(v-1)u+1} \oplus \cdots \oplus \mathbb{R}\mathfrak{f}_{uv})$$

and the intersection $T_c^{*,M} = T_c^* \cap M$ is a maximal anisotropic torus in M. It is equal to T_c^* if u is even. If u is odd (and $v > 1$) then it is a proper sub torus, if ${}^\circ r_u = \frac{u-1}{2}$ then

$$T_c^{*,M} = \underbrace{\mathrm{SO}(2) \times \cdots \times \mathrm{SO}(2)}_{{}^\circ r_u \text{ factors}} \quad \underbrace{\times\{\pm 1\}\times}_{\text{spot } u \text{ and } u+1} \quad \underbrace{\mathrm{SO}(2) \times \cdots \times \mathrm{SO}(2)}_{{}^\circ r_u \text{ factors}} \times\{\pm 1\}\times \tag{306}$$

where the product of signs is one. To get the torus T_c^* we have to put another $\mathrm{SO}(2)$ at the spots $(u, u+1), (2u, 2u+1), \dots$. We apply the reasoning of section (6.6.2) to the factors M_ν.

The representation $\mathbb{D}_{\mu_1^*} = \mathrm{Ind}_{{}^\circ P_\nu}^{M_1} \mathcal{D}_{\tilde{\mu}_1^*}$ contains as lowest $K_\infty^{M_\nu}$ type the representation with highest weight

$$(b_1^* + 2)f_1 + (b_3^* + 2)f_2 + \cdots + (b_{2{}^\circ r_u - 1}^* + 2)f_{{}^\circ r_u}$$

where the b_{2j-1}^* are taken from (303). This weight occurs in $\mathcal{D}_{\tilde{\mu}_1^*}$ Hence we see that as a T_c^* module the representation $\otimes \mathcal{D}_{\underline{\tilde{\mu}}_\nu^*}$ contains the weight (depending on u even or odd)

$$\begin{cases} \Big((b_1^* + 2)f_1 + (b_3^* + 2)f_2 + \cdots + (b_{2{}^\circ r_u - 1}^* + 2)f_{{}^\circ r_u}\Big) + \Big((b_1^* + 2)f_{{}^\circ r_u + 1} + \dots\Big) + \dots \\ \Big((b_1^* + 2)f_1 + (b_3^* + 2)f_2 + \cdots + (b_{2{}^\circ r_u - 1}^* + 2)f_{{}^\circ r_u - 1}\Big) + \Big((b_1^* + 2)f_{{}^\circ r_u + 1} + \dots\Big) + \dots \end{cases} \tag{307}$$

This weight is not dominant, to get a dominant weight we have to reorder the f_ν according to the size of the coefficient in front. Then we get a dominant weight

$$(b_1^*+2)(f_1^\dagger+f_2^\dagger+\cdots+f_v^\dagger)+(b_3^*+2)(f_{v+1}^\dagger+f_{v+2}^\dagger+\cdots+f_{2v}^\dagger)+\dots \tag{308}$$

and then formula (288) and the formula for the b_j^* give us the following dominant weight expressed in terms of the fundamental dominant weights

$$\sum_{\nu:\nu v\le \frac{u}{2}}(2v+e(\nu)+2a_\nu)\gamma^c_{\nu v} \tag{309}$$

This is now the weight $\mu_c(\chi_{u,v},\lambda)$ in (288). Hence we see that $\Theta_{\mu_c(\chi_{u,v},\lambda)}$ occurs with multiplicity one in $\mathbb{D}_{\underline{\mu}})$: $\mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}\mathbb{D}_{\underline{\mu}}$ and we get

Theorem 6.1. *We have a nonzero intertwining operator* : $T^{(\mathrm{loc})}(\mathbb{D}_{\underline{\mu}}) : \mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}\mathbb{D}_{\underline{\mu}}\to \mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}\mathbb{D}_{\underline{\mu}'}$ *and get a diagram*

$$\begin{array}{ccc} \mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}\mathbb{D}_{\underline{\mu}} & \stackrel{T^{(\mathrm{loc})}(D_\mu)}{\longrightarrow} & A_{\mathfrak{q}}(\lambda) \\ & & \downarrow \\ & & \mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}\mathbb{D}_{\mu'} \end{array} \tag{310}$$

The horizontal arrow is surjective, and the vertical arrow is injective. The map induced by the vertical arrow in cohomology

$$H^q(\mathfrak{g},K_\infty;A_{\mathfrak{q}}(\lambda)\otimes\mathcal{M}_\lambda)\ \longrightarrow\ H^q(\mathfrak{g},K_\infty;{}^{\mathrm{a}}\mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}\mathbb{D}_{\mu'}\otimes\mathcal{M}_\lambda)$$

is a bijection in the lowest degree of nonzero cohomology; this lowest degree is

$$q\ =\ v\left[\frac{u^2}{4}\right]+\frac{n(u-1)(v-1)}{4}.$$

Proof. We have an inclusion between the two complexes

$$\mathrm{Hom}_{K_\infty^0}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}),A_{\mathfrak{q}}(\lambda)\otimes\mathcal{M}_\lambda))\to\mathrm{Hom}_{K_\infty^0}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}),\mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}\mathbb{D}_{\mu'}\otimes\mathcal{M}_\lambda).$$

In the complex on the left all differentials are zero. It follows from the work of Kostant that we have a splitting

$$\mathrm{Hom}(\Lambda^\bullet(\mathfrak{u}_P),\mathcal{M}_\lambda))=\mathbb{H}^\bullet(\mathfrak{u}_P,\mathcal{M}_\lambda)\oplus AC^\bullet$$

where $\mathbb{H}^\bullet(\mathfrak{u}_P,\mathcal{M}_\lambda)$ is the space of harmonic forms (and this space is isomorphic to the cohomology $H^\bullet(\mathfrak{u}_P,\mathcal{M}_\lambda)$.) and where $AC^\bullet$ is an acyclic complex.

We have Delorme's formula

$$\mathrm{Hom}_{K_\infty^0}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}),\mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}\mathbb{D}_{\mu'}\otimes\mathcal{M}_\lambda)=\mathrm{Hom}_{K_\infty^M}(\Lambda^\bullet(\mathfrak{m}/\mathfrak{k}^M),\mathbb{D}_{\mu'}\otimes\mathrm{Hom}(\Lambda^\bullet(\mathfrak{u}_P),\mathcal{M}_\lambda))=$$

$$\mathrm{Hom}_{K_\infty^M}(\Lambda^\bullet(\mathfrak{m}/\mathfrak{k}^M),\mathbb{D}_{\mu'}\otimes\mathbb{H}^\bullet(\mathfrak{u}_P,\mathcal{M}_\lambda))\oplus\mathrm{Hom}_{K_\infty^M}(\Lambda^\bullet(\mathfrak{m}/\mathfrak{k}^M),\mathbb{D}_{\mu'}\otimes AC^\bullet) \tag{311}$$

The $(\mathfrak{m}/K_\infty^M)$ has a lowest $K_\infty^M)$ type $\vartheta(\mu')$, which can be computed easily from 3.1.4 and we have

$$\mathrm{Hom}_{K_\infty^M}(\Lambda^\bullet(\mathfrak{m}/\mathfrak{k}^M),\mathbb{D}_{\mu'}\otimes\mathbb{H}^\bullet(\mathfrak{u}_P,\mathcal{M}_\lambda))=\mathrm{Hom}_{K_\infty^M}(\Lambda^\bullet(\mathfrak{m}/\mathfrak{k}^M),\mathbb{D}_{\mu'}(\vartheta(\mu'))\otimes\mathbb{H}^\bullet(\mathfrak{u}_P,\mathcal{M}_\lambda)).$$

Using the formula in [Vo-Zu] for the highest weight of the lowest K_∞-type $\Theta(\mathfrak{q},\lambda)$ in $A_\mathfrak{q}(\lambda)$ we see that $\Theta(\mathfrak{q},\lambda)$ is the lowest K_∞ type in $\mathrm{Ind}_{K_\infty^M}^{K_\infty}$. This implies that the map

$$\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}),\mathbb{A}_\mathfrak{q}(\lambda)(\Theta(\mathfrak{q},\lambda)\otimes\mathcal{M}_\lambda)\to\mathrm{Hom}_{K_\infty^M}(\Lambda^\bullet(\mathfrak{m}/\mathfrak{k}^M),\mathbb{D}_{\mu'}\otimes\mathbb{H}^\bullet(\mathfrak{u}_P,\mathcal{M}_\lambda)) \tag{312}$$

is an isomorphism of vector spaces (but not of complexes). But since the complex on the right is zero in degrees $\bullet<q$ it follows that the classes in the image of $\mathrm{Hom}_{K_\infty}(\Lambda^q(\mathfrak{g}/\mathfrak{k}),A_\mathfrak{q}(\lambda)(\Theta(\mathfrak{q},\lambda))\otimes\mathcal{M}_\lambda)$ survive in cohomology.

□

We got to the global context, we have a diagram

$$\begin{array}{ccc} J_{\sigma_\infty}\otimes J_{\sigma_f}^{K_f} & \hookrightarrow & L^2_{\mathrm{disc}}(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f,\omega_{\mathcal{M}_\lambda}^{-1}|_{S(\mathbb{R})^0}) \\ \downarrow & & \downarrow\mathcal{F}^P \\ {}^{\mathrm{a}}\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}\mathbb{D}_{\mu'}\otimes V_{\sigma_f}^{K_f} & \hookrightarrow & \mathcal{A}(P(\mathbb{Q})U(\mathbb{A})\backslash G(\mathbb{A})/K_f) \end{array} \tag{313}$$

This induces maps in cohomology

$$\begin{array}{ccc} H^\bullet(\mathfrak{g},K_\infty,J_{\sigma_\infty}\otimes\mathcal{M}_\lambda)\otimes J_{\sigma_f}^{K_f} & \to & H^\bullet(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda) \\ \downarrow & & \downarrow\mathcal{F}^P \\ H^\bullet(\mathfrak{g},K_\infty,{}^{\mathrm{a}}\mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}\mathbb{D}_{\mu'}\otimes\mathcal{M}_\lambda)\otimes V_{\sigma_f}^{K_f} & \hookrightarrow & H^\bullet(\partial_P\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda) \end{array} \tag{314}$$

The left vertical arrow is non zero for $\bullet=q$, the horizontal arrow in the bottom line is injective for all values of $\bullet$ (Borel see) hence the horizontal arrow in the top line is non zero in degree $\bullet=q$.

Of course we also should investigate the horizontal arrow in the to line in all degrees, this question becomes delicate. To answer it we should invoke the results in Franke's paper [ ] or we could work with proposition (4.4) or its corollary (4.1).

In the extremal case $u=n,v=1$ the parabolic subgroup P is all of G and $A_\mathfrak{q}(\lambda)=\mathbb{D}_\lambda$. In this case, and only this case, the representation $A_\mathfrak{q}(\lambda)$ is tempered.

In the other extremal case that $u=1,v=n$ the representation J_{σ_∞} is one dimensional - (basically it is the space of constant functions twisted by a character on the group of connected components) - in this case the map in the top row is understood in terms of the topological model (Franke + Diploma students).

6.6.6 Congruences

We formulate a condition ($NUQuot$) (No unitarizable quotient) for the induced module:

The induced module $I_P^G(\sigma_f)$ as module under the Hecke- algebra does not have a non trivial quotient which admits a unitary scalar product (here it may be necessary to pass to the corresponding representation of $G(\mathbb{A}_f)$).

The negation of this condition ($UQuot$) says that for all primes p the induced module $I_P^G\sigma_p$ has a unitarizable quotient.

This condition has been discussed in [Ha-Eis] Kap. II, 2.3.

If we have ($NUQuot$) then

$$\mathrm{Hom}_{\mathcal{H}_{K_f}^G}(I_P^G(\sigma_f), H_!^\bullet(\mathcal{S}_{K_f}^G, \mathcal{M}\otimes\mathbb{C})) = 0 \tag{315}$$

this implies that the Manin-Drinfeld is valid and this implies that our above section is defined over F, i.e. we get a unique section of Hecke-modules

$$\mathrm{Eis}: H^{q-l(w)})(\mathcal{S}_{K_f^M}^M, \mathcal{M}(w\cdot\lambda)\otimes F)(\sigma_f) \to H^q(\mathcal{S}_{K_f}^G, \tilde{\mathcal{M}}_\lambda\otimes F). \tag{316}$$

Then is looks as if the second term is completely uninteresting, but in fact it is not. In the lecture notes volume [Ha-Eis] we raise the question whether it influences the structure of the integral cohomology $H^q_{\mathrm{int}}(\mathcal{S}_{K_f}^G, \tilde{\mathcal{M}}_\lambda\otimes F)$. In some cases we have convincing experimental evidence that "arithmetic" of the ratio of special values

$$\frac{1}{\Omega(\sigma_f)}\prod_a \frac{\Lambda^{\mathrm{coh}}(\sigma_f, r_a^{\mathfrak{u}_P^\vee}, <\eta_a, \tilde{\mu}^{(1)}> -ab(w,\lambda))}{\Lambda^{\mathrm{coh}}(\sigma_f, r_a^{\mathfrak{u}_P^\vee}, <\eta_a, \tilde{\mu}^{(1)}> -ab(w,\lambda)+1)} \tag{317}$$

has influence on the structure integral of the cohomology. Under certain conditions the above expression is a product of an algebraic part and the value of a motivic extension class. Primes dividing the denominator of the algebraic part may occur in the denominator of the Eisenstein class and we will have congruences (See (5.2),(135). This will be explained in the next section in the special case of the group $\mathrm{GSp}_2/\mathbb{Z}$.

6.6.7 Attaching motives to σ_f???

The condition ($NUQuot$)) will be true if λ is sufficiently regular but for non regular weights it fails. If the weight is not regular then we may have a pole of the Eisenstein series at $z=0$. This possibility has to be discussed, it can only happen if we have *(UQuot)*. But even if we have *(UQuot)* we may not have a pole.

Let us assume that we have *(UQuot)* and the Eisenstein operator is holomorphic at $z=0$. Then we may have several copies of $J(\sigma_f)$ in $H_!^\bullet(\mathcal{S}_{K_f}^G, \tilde{\mathcal{M}}_\lambda\otimes\mathbb{C})$. This defines again an isotypical submodule $H_!^\bullet(\mathcal{S}_{K_f}^G, \tilde{\mathcal{M}}_\lambda\otimes F)(\bar{\sigma_f})$. We get an exact sequence

$$0\to H_!^\bullet(\mathcal{S}_{K_f}^G, \tilde{\mathcal{M}}_\lambda\otimes F)(\bar{\sigma_f}) \to \mathcal{X}(\sigma_f)\to J(\sigma_f)\to 0 \tag{318}$$

This is a sequence of Hecke-modules over F, the section (207) provides a section over $\mathbb{C}$.

If our locally symmetric space $\mathcal{S}^G_{K_f}$ the set of complex points of a Shimura variety then we can interpret this sequence as a mixed motive. This motive has an extension class in the category of mixed Hodge-structures

$$[\mathcal{X}(\sigma_f)]_{B-dRh} \in \mathrm{Ext}^1_{B-dRh}(J(\sigma_f), H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes F)(\bar{\sigma_f})) \tag{319}$$

and in some cases we can compute this class (we have to look at a suitable bi-extension) and express it in terms of the second term in the constant term (See [MixMot-2013.pdf].)

We have seen that in many situations the space $\mathcal{S}^M_{K^M_f}$ is not the set of complex points of a Shimura variety and therefore we do not know how to attach a motive or an ℓ adic Galois representation to it. (Sometimes we know how to attach a motive to it but it is simply a Tate motive). But if it happens that the module $J(\sigma_f)$ produces a non trivial submodule $H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes F)(\bar{\sigma_f}))$ then the situation changes and we can attach a Galois-module $H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes F_\lambda)(\bar{\sigma_f}))$ to it which contains a lot of information about σ_f. Again we refer to ([MixMot-2013.pdf].) We have seen in [Ha-Eis] (3.1.4.) that this can happen.

6.6.8 The motivic interpretation of Shahidis theorem

We go back to a general submodule $\sigma_f = \sigma_f^{(1)} \times \sigma_f^{(2)} = \sigma_f \in \mathrm{Coh}(H^\bullet_{\mathrm{cusp}}(\mathcal{S}^M_{K^M_f}, \tilde{\mathcal{M}}_{w\cdot\lambda}),$ we drop the assumptions above. We assume that we can attach motives $\mathbb{M}(\sigma_f^{(1)}, r_1), \mathbb{M}(\sigma_f^{(2)}, r_1)$ where r_1 is the tautological representation. (Actually we do not need the motives it suffices to have the compatible systems of $\mathfrak{l}$-adic representations) Then we can attach the Rankin-Selberg motive to this pair

$$\mathbb{M}_{\mathrm{RS}}(\sigma_f, \mathrm{Ad}) = \mathbb{M}(\sigma_f^{(1)}, r_1) \times \mathbb{M}(\sigma_f^{(2)}, r_1)^\vee = \mathrm{Hom}(\mathbb{M}(\sigma_f^{(2)}, r_1), \mathbb{M}(\sigma_f^{(1)}, r_1)) \otimes \mathbb{Z}(-\mathbf{w}(\mu^{(2)}, r_2)) \tag{320}$$

Under the assumption of the theorem the we have $\mathbb{M}(\sigma_f^{(1)}, r_1) \xrightarrow{\sim} \mathbb{M}(\sigma_f^{(2)}, r_1)$ and we see that the Galois module $\mathrm{Hom}(\mathbb{M}(\sigma_f^{(2)}, r_1), \mathbb{M}(\sigma_f^{(1)}, r_1))$ contains a copy of $\mathbb{Z}_{\mathfrak{l}}(0)$ and therefore we get an exact sequence of Galois modules

$$0 \to \mathbb{Z}(-\mathbf{w}(\mu^{(2)}, r_2)) \to \mathbb{M}_{\mathrm{RS}}(\sigma_f, \mathrm{Ad})_{\text{ét},\mathrm{Ad}} \to \mathbb{M}^{(0)}_{\mathrm{RS}}(\sigma_f, \mathrm{Ad})_{\text{ét},\mathrm{Ad}} \to 0$$

Hence the motivic L function is a product

$$L(\mathbb{M}_{\mathrm{RS}}(\sigma_f, \mathrm{Ad})_{\text{ét},\mathrm{Ad}}, s) = L(\mathbb{Z}(-\mathbf{w}(\mu^{(2)}), s) L(\mathbb{M}^{(0)}_{\mathrm{RS}}(\sigma_f, \mathrm{Ad})_{\text{ét},\mathrm{Ad}}, s)$$

If we substitute for s the expression

$$\frac{\mathbf{w}(r_1, \mu_1^{(1)}) + \mathbf{w}(r_2, \mu_2^{(1)})}{2} - b(w, \lambda) + s = \mathbf{w}(r_2, \mu_2^{(1)}) - b(w, \lambda) + s$$

then we find

$$L(\mathbb{M}_{\mathrm{RS}}(\sigma_f, \mathrm{Ad})_{\text{ét},\mathrm{Ad}}, s) = \zeta(-b(w, \lambda) + s) L(\mathbb{M}^{(0)}_{\mathrm{RS}}(\sigma_f, \mathrm{Ad})_{\text{ét},\mathrm{Ad}}, s)$$

Then the motivic interpretation of Shahidis theorem is, that $L(\mathbb{M}^{(0)}_{\mathrm{RS}}(\sigma_f,\mathrm{Ad})_{\text{ét},\mathrm{Ad}},\mathbf{w}(r_2,\mu_2^{(1)})-b(w,\lambda)+s)$ is holomorphic at $s=0$ and non zero (this is in a sense the prime number theorem for this L function) and therefore - if we have $b(w,\lambda)=-1$- the pole comes from the first order pole of the Riemann -ζ function. If now $\sigma_f^{(1)}\times\sigma_f^{(2)}=\sigma_f$ occurs in the cuspidal cohomology then we have an inclusion

$$\mathbb{D}_\mu\times H_{\sigma_f}\hookrightarrow\mathcal{A}(M(\mathbb{Q})\backslash M(\mathbb{A})/K_f^M)$$

We form the Eisenstein intertwining operator and compose it with constant Fourier coefficient, then we get

$$\mathcal{F}^P\circ\mathrm{Eis}(s): f\mapsto f+C(\sigma,s)T^{\mathrm{loc}}(s)(f) \tag{321}$$

The operator $T^{\mathrm{loc}}(s)=T^{\mathrm{loc}}_\infty(s)\otimes\bigotimes T^{\mathrm{loc}}_p(s)$ is holomorphic at $s=0$. Under our assumptions the function $C(\sigma,s)$ has a first order pole at $s=0$ and we get a residual intertwining operator

$$\mathrm{Res}_{s=0}:\mathrm{Ind}_P^G\mathbb{D}_\mu\times H_{\sigma_f}\otimes(0)\to\mathcal{A}^{(2)}(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f) \tag{322}$$

6.6.9 Rationality results

Finally we want to discuss the case that $P\neq\Theta(P)=Q$. If this happens then $\mathcal{S}^G_{K_f}$ is never a Shimura variety. We have isotypical pieces (see (187))

$$H_!^{\bullet-l(w)}(\mathcal{S}^M_{K_f^M},\tilde{\mathcal{M}}(w\cdot\lambda)\otimes F)(\sigma_f)\oplus H_!^{\bullet-l(w')}(\mathcal{S}^{M'}_{K_f^{M'}},\tilde{\mathcal{M}}(w'\cdot\lambda)\otimes F)(\sigma'_f) \tag{323}$$

and we know that component of the Eisenstein cohomology consists of the classes

$$\{\psi_f\oplus\mathcal{L}(\sigma_f)T^{\mathrm{loc}}_f(\psi_f)\} \tag{324}$$

where $\mathcal{L}(\sigma_f)$ is an element of F and for all $\iota:F\to\mathbb{C}$ we have

$$\iota(\mathcal{L}(\sigma_f))=\frac{1}{\Omega(\iota\circ\sigma_f)}C(\sigma_\infty,\lambda)C(\iota\circ\sigma_f,\lambda) \tag{325}$$

If the factor at infinity $C(\sigma_\infty,\lambda)\neq 0$ then we get from this rationality results for the ratios of L-values. (See [Ha-Mum], [Ha-Rag]) These rationality results will be important when we discuss the arithmetic nature of the numbers in**??**

Combining the results of Borel–Garland [**?**] and Mœglin–Waldspurger [**?**] we get that the homomorphism

$$\bigoplus_{u|n}\bigoplus_{\sigma_f:\mathrm{segment}}H^\bullet(\mathfrak{g},K_\infty;A_\mathfrak{q}(\lambda)\otimes\mathcal{M}_\lambda)\otimes J_{\sigma_f}\to H^\bullet_{(2)}(\mathcal{S}^G_{K_f},\mathcal{M}_\lambda) \tag{326}$$

is surjective. This gives us the decomposition into isotypical spaces of $H^\bullet_{(2)}(\mathcal{S}^G_{K_f},\mathcal{M}_\lambda)$. We separate the cuspidal part ($v=1$) from the residual part and get

$$H^\bullet_{(2)}(\mathcal{S}^G_{K_f},\mathcal{M}_\lambda)=\bigoplus_{\pi_f:\mathrm{cuspidal}}H^\bullet_{\mathrm{cusp}}(\mathcal{S}^G_{K_f},\mathcal{M}_\lambda)(\pi_f)\ \oplus\bigoplus_{\substack{u|n\\u<n}}\bigoplus_{\sigma_f:\mathrm{segment}}\overline{H^\bullet(\mathfrak{g},K_\infty;A_\mathfrak{q}(\lambda)\otimes\mathcal{M}_\lambda)}\otimes J_{\sigma_f},$$

where the bar on top means we have gone to its image via the map in (326). It follows from the theorem of Jacquet–Shalika [?] that there are no intertwining operators between the summands.

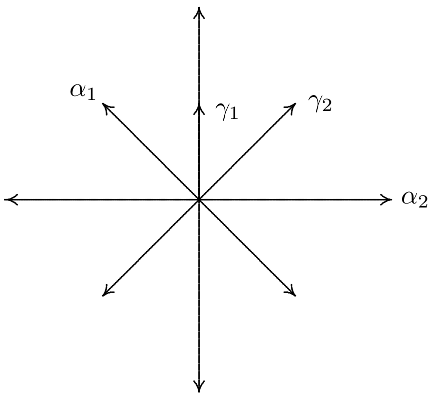
In the extremal case $u = n, v = 1$ the parabolic subgroup P is all of G and $A_{\mathfrak{q}}(\lambda) = \mathbb{D}_\lambda$. In this case and only this case the representation $A_{\mathfrak{q}}(\lambda)$ is tempered, and the lowest degree of nonvanishing cohomology is the number b_n^F. An easy computation shows that in the case $v > 1$ the number $q < b_n^F$. Then our theorem above implies that in degree q

$$H^q(\gamma, K_\infty; A_{\mathfrak{q}}(\lambda) \otimes \mathcal{M}_\lambda) \otimes J_{\sigma_f} \to H^q(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda)$$

is injective. This has also been proved by Grobner [?]. The above result, which we announced earlier (**??**), can be sharpened as in the following theorem. During the induction argument we use Thm. **??** for the reductive quotients M of the parabolic subgroups.

7 The example $G = \mathrm{Sp}_2/\mathbb{Z}$

7.1 Some notations and structural data



The maximal torus is

$$T_0/\mathbb{Z} = t = \left\{ \begin{pmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 & t_1^{-1} \end{pmatrix} \right\}$$

the simple roots are

$$\alpha_1(t) = t_1/t_2, \alpha_2(t) = t_2^2$$

and the fundamental dominant weights are

$$\gamma_1(t) = t_1, \gamma_2(t) = t_1 t_2$$

and finally we have

$$2\gamma_1^M = t_1/t_2$$

We choose a highest weight $\lambda = n_1\gamma_1 + n_2\gamma_2$ let $\mathcal{M}_\lambda$ be a resulting module for $G/\mathrm{Spec}(\mathbb{Z})$. We get the following list of Kostant representatives for the Siegel parabolic subgroup P and they provide the following list of weights.

$$\begin{aligned} 1\cdot\lambda &= \lambda = \tfrac{1}{2}(2n_2+n_1)\gamma_2 + n_1\gamma_1^{M_1} \\ s_2\cdot\lambda &= \tfrac{1}{2}(-2+n_1)\gamma_2 + (2n_2+n_1+2)\gamma_1^{M_1} \\ s_2s_1\cdot\lambda &= \tfrac{1}{2}(-4-n_1)\gamma_2 + (2+2n_2+n_1)\gamma_1^{M_1} \\ s_2s_1s_2\cdot\lambda &= \tfrac{1}{2}(-6-2n_2-n_1)\gamma_2 + n_1\gamma_1^{M_1}, \end{aligned}$$

We choose for $K_\infty \subset \mathrm{Sp}_2(\mathbb{R})$ the standard maximal compact subgroup $U(2)$, it is the centralizer of the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

which defines a complex structure. With this choice we can define $\mathcal{S}^G_{K_f} = G(\mathbb{Q})\backslash G(\mathbb{R})/K_\infty \times G(\mathbb{A}_f)/K_f$.

7.2 The cuspidal cohomology of the Siegel-stratum

We consider the cohomology groups $H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$ and the resulting fundamental exact sequence. We have the boundary stratum $\partial_P(\mathcal{S}^G_{K_f})$ with respect to the Siegel parabolic. Let us assume that we are in the unramified case, then we get

$$H^\bullet(\partial_P(\mathcal{S}^G_{K_f}), \tilde{\mathcal{M}}_\lambda) = \bigoplus_{w\in W^P} H^{\bullet - l(w)}(\mathcal{S}^M_{K^M_f}, H^{l(w)}(\mathfrak{u}_P, \mathcal{M}_\lambda)) \tag{327}$$

We look at the case $w = s_2s_1$ in this case we know how to describe the corresponding summand in terms of automorphic forms on Gl_2. We introduce the usual abbreviation $H^{l(w)}(\mathfrak{u}_P, \mathcal{M}_\lambda) = \mathcal{M}_\lambda(w\cdot\lambda)$.

Our coefficient modules are the modules attached to the highest weight

$$w\cdot\lambda = \mu = (2+2n_2+n_1)\gamma_1^{M_1} + \frac{1}{2}(-4-n_1)\gamma_2$$

Let us put $k = 4+2n_2+n_1$ and $m = \frac{1}{2}n_1$. We give the usual concrete realization for these modules as $\mathcal{M}_{2+2n_2+n_1}[n_2-3-k] = \mathcal{M}_{k-2}[n_2-3-k]$

Let us look at the space $\mathcal{S}^M_{K^M_f}$. The group $M/\mathrm{Spec}(\mathbb{Z})$ is isomorphic to Gl_2, it is the Levi-quotient of the Siegel parabolic. The group K^M_∞ is the image of $P(\mathbb{R})\cap K_\infty$ under the projection $P(\mathbb{R}) \to M(\mathbb{R})$. This is the group $\mathbb{O}(2)$ it contains the standard choice $K^M_\infty(1) = \mathrm{SO}(2)$ as a subgroup of index 2. Hence we get a covering of degree 2

$$\mathcal{S}^{\tilde{M}}_{K^M_f} = M(\mathbb{Q})\backslash M(\mathbb{R})/K^M_\infty(1) \times M(\mathbb{A}_f)/K^M_f \to \mathcal{S}^M_{K^M_f} \tag{328}$$

We get an inclusion

$$i : H^1(\mathcal{S}^M_{K^M_f}, \mathcal{M}_\lambda(w\cdot\lambda)) \hookrightarrow H^1(\mathcal{S}^{\tilde{M}}_{K^M_f}, \mathcal{M}_\lambda(w\cdot\lambda)). \tag{329}$$

On the cohomology on the right we have the action of $\mathbb{O}(2)/\mathrm{SO}(2) = \mathbb{Z}/2\mathbb{Z}$ and the cohomology decomposes into a $+$ and a $-$ eigenspace. The inclusion i provides an isomorphism of the left hand side and the $+$ eigenspace.

This inclusion is of course compatible with the action of the Hecke algebra. If we pass to a suitable extension $F/\mathbb{Q}$ we get the decompositions into isotypic subspaces if we tensor our coefficient system by F. An isomorphism type σ_f occurs with multiplicity one on the left hand side and with multiplicity two on the right hand side. Over the ring O_F the modules $H^1_{\pm,\,\mathrm{int}}(\mathcal{S}^M_{K_f^M}, \mathcal{M}_\lambda(w\cdot\lambda)_F)(\sigma_f)$ are of rank one, hence we can find locally in the base $\mathrm{Spec}(\mathcal{O}_F)$ an isomorphism

$$T^{\mathrm{arith}}(\sigma_f) : H^1_+(\mathcal{S}^{\tilde{M}}_{K_f^M}, \mathcal{M}_\lambda(w\cdot\lambda)_F))(\sigma_f) \xrightarrow{\sim} H^1_-(\mathcal{S}^{\tilde{M}}_{K_f^M}, \mathcal{M}_\lambda(w\cdot\lambda)_F))(\sigma_f) \tag{330}$$

The isomorphism given by the fundamental class (see(79) interchanges the $+$ and the $-$ eigenspace, hence we can arrange our arithmetic intertwining operator such that it satisfies

$$T^{\mathrm{arith}}(\sigma_f\otimes|\delta_f|) = T^{\mathrm{arith}}(\sigma_f\otimes|\delta_f|)^{-1} \tag{331}$$

We consider the transcendental description of the cohomology groups

$$H^1(\mathcal{S}^{\tilde{M}}_{K_f^M}, \mathcal{M}_\lambda(w\cdot\lambda)_\mathbb{C}) = \bigoplus_{\sigma_f} H^1_+(\mathcal{S}^{\tilde{M}}_{K_f^M}, \mathcal{M}_\lambda(w\cdot\lambda)_\mathbb{C})(\sigma_f) \oplus H^1_-(\mathcal{S}^{\tilde{M}}_{K_f^M}, \mathcal{M}_\lambda(w\cdot\lambda)_\mathbb{C})(\sigma_f) \tag{332}$$

We consider the standard Borel subgroup $B\subset M$ the standard split torus $T_0\subset B$ it contains our torus Z_0. We define the character

$$\chi_\mu = (k, m+2) : B(\mathbb{R}) \to \mathbb{C}^\times,\ \chi(t) = \gamma_1^M(t)^k|\gamma_2|^{m+2}.$$

It yields the induced Harish-Chandra module $I_{B(\mathbb{R})}^{M(\mathbb{R})}\chi_\mu$: We consider the functions

$$f : M(\mathbb{R}) \to \mathbb{C}; f(bg) = \chi(b)f(g); f|T_1 \text{ is of finite type }.$$

This is in fact a $(\mathfrak{m}, K_\infty^{M,0})$ -module, it contains the discrete representation $\mathcal{D}_{\chi_\mu}$. We have the decomposition

$$\mathcal{D}_{\chi_\mu} = \bigoplus_{\nu\equiv 0(2), |\nu|\geq k} F\phi_{\chi,\nu}$$

where

$$\phi_{\chi,\nu}(g) = \phi_{\chi,\nu}(b\begin{pmatrix}\cos(\phi) & \sin(\phi)\\ -\sin(\phi) & \cos(\phi\end{pmatrix}) = \chi(b)e^{2\pi i\nu\phi}.$$

Of course $K_\infty^{M,0} = T_1(\mathbb{R}) = \{e(\phi) = \begin{pmatrix}\cos(\phi) & \sin(\phi)\\ -\sin(\phi) & \cos(\phi\end{pmatrix}\}$ and we can write $e(\phi)^\nu = e^{2\pi\nu i\phi}$.

We have the well known formula for the $((\mathfrak{m}, K_\infty^{M,0})$ cohomology

$$H^1((\mathfrak{m},K_\infty^{M,0}),\mathcal{D}_{\chi_\mu}\otimes\mathcal{M}_\lambda(w\cdot\lambda))=\mathrm{Hom}_{K_\infty^{M,0}}(\Lambda^1(\mathfrak{m}/\mathfrak{k}^M),\mathcal{D}_{\chi_\mu}\otimes\mathcal{M}_\lambda(w\cdot\lambda))= \tag{333}$$

$$\mathbb{C}P_+^\vee\otimes\phi_{\chi,-k}\otimes v_{k-2}+\mathbb{C}P_-^\vee\otimes\phi_{\chi,k}\otimes v_{-k+2}=\mathbb{C}\omega_{k,m}+\mathbb{C}\bar{\omega}_{k,m} \tag{334}$$

Here $v_{k-2}=(X+iY)^{k-2}$, resp. $v_{2-k}=(X-iY)^{k-2}$ are two carefully chosen highest (resp. lowest) weight vectors with respect to the action of $K_\infty^{M,0}$. The elements $P_\pm$ are the usual elements in $\mathfrak{m}/\mathfrak{k}$. We choose a model space H_{σ_f} for σ_f i.e. a free rank one $\mathcal{O}_F$ -module on which the Hecke algebra acts by the homomorphism $\sigma_f:\mathcal{H}^M_{K_f^M}\to\mathcal{O}_F$. We also choose and embedding $\iota:F\hookrightarrow\mathbb{C}$ and an $(\mathfrak{m},K_\infty^{M,0})\times K_\infty^M\times\mathcal{H}^M_{K_f^M}$- invariant embedding

$$\Phi_\iota:\mathcal{D}_{\chi_\mu}\otimes H_{\sigma_f}\otimes_{F,\iota}\mathbb{C}\to L^2_0(M(\mathbb{Q})\backslash M(\mathbb{A})) \tag{335}$$

this is unique up to a scalar in $\mathbb{C}^\times$ because the representation is irreducible and occurs with multiplicity one in the right hand side. This yields an isomorphism

$$\Phi_\iota^1:H^1((\mathfrak{m},K_\infty^{M,0}),\mathcal{D}_{\chi_\mu}\otimes\mathcal{M}_\lambda(w\cdot\lambda))\otimes H_{\sigma_f}\otimes_{F,\iota}\mathbb{C}\xrightarrow{\sim}H^1(\mathcal{S}^{\tilde{M}}_{K_f^M},\mathcal{M}_\lambda(w\cdot\lambda)_\mathbb{C})(\iota\circ\sigma_f)$$

We observe that the element $\epsilon=\begin{pmatrix}-1&0\\0&1\end{pmatrix}\in K_\infty^M$ has the following effect

$$\mathrm{Ad}(\epsilon)(P_+)=P_-\ ,\epsilon(\phi_{\chi,k})=\phi_{\chi,-k}\ \text{ and }\epsilon(v_{k-2})=(-1)^m v_{2-k} \tag{336}$$

Hence we see that

$$\omega^{(+)}_{k,m}=\omega_{k,m}+(-1)^m\bar{\omega}_{k,m}\text{ resp. }\ \omega^{(-)}_{k,m}=\omega_{k,m}-(-1)^m\bar{\omega}_{k,m} \tag{337}$$

are generators of the $+$ and the $-$ eigenspace in $H^1(\mathfrak{m},K_\infty^{M,0},\mathcal{D}_{\chi_\mu}\otimes\mathcal{M}_\lambda(w\cdot\lambda))$. Therefore our map Φ and the choice of these generators provide isomorphisms

$$\Phi_\iota^{(+)}:H_{\sigma_f}\otimes_{F,\iota}\mathbb{C}\xrightarrow{\sim}H^1_+(\mathcal{S}^{\tilde{M}}_{K_f^M},\mathcal{M}_\lambda(w\cdot\lambda)_\mathbb{C})(\iota\circ\sigma_f), \tag{338}$$

$$\Phi_\iota^{(-)}:H_{\sigma_f}\otimes_{F,\iota}\mathbb{C}\xrightarrow{\sim}H^1_-(\mathcal{S}^{\tilde{M}}_{K_f^M},\mathcal{M}_\lambda(w\cdot\lambda)_\mathbb{C})(\iota\circ\sigma_f) \tag{339}$$

The choice of P_+,P_- and $\phi_{\chi,-\nu}$ is canonic, hence we see that the identifications depend only on Φ_ι , which is unique up to a scalar. This means that the composition

$$\begin{aligned}T^{\mathrm{trans}}(\iota\circ\sigma_f)&=\Phi_\iota^{(-)}\circ(\Phi_\iota^{(+)})^{-1}\\ :H^1_+(\mathcal{S}^{\tilde{M}}_{K_f^M},\mathcal{M}_\lambda(w\cdot\lambda)_\mathbb{C})(\iota\circ\sigma_f)&\xrightarrow{\sim}H^1_-\mathcal{S}^{\tilde{M}}_{K_f^M},\mathcal{M}_\lambda(w\cdot\lambda)_\mathbb{C})(\iota\circ\sigma_f)\end{aligned}$$

yields a second (canonical) identification between the $\pm$ eigenspaces in the cohomology. Our arithmetic intertwining operator (See (330) yields an array of intertwining operators

$$T^{\mathrm{arith}}(\sigma_f)\otimes_{F,\iota}\mathbb{C}:H^1_+(\mathcal{S}^{\tilde{M}}_{K_f^M},\mathcal{M}_\lambda(w\cdot\lambda)_F))(\sigma_f)\otimes_{F,\iota}\mathbb{C}\xrightarrow{\sim}H^1_-(\mathcal{S}^{\tilde{M}}_{K_f^M},\mathcal{M}_\lambda(w\cdot\lambda)_F))(\sigma_f)\otimes_{F,\iota}\mathbb{C} \tag{340}$$

Hence get an array of periods which compare these two arrays of intertwining operators

$$\Omega(\sigma_f,\iota)T^{\text{trans}}(\iota\circ\sigma_f) = T^{\text{arith}}(\sigma_f)\otimes_{F,\iota}\mathbb{C} \tag{341}$$

Our formula (331) tells us that we can arrange the intertwining operators such that

$$\Omega(\sigma_f\otimes|\delta_f|,\iota) = \Omega(\sigma_f,\iota)^{-1} \tag{342}$$

These periods are uniquely defined up to a unit in $\mathcal{O}_F^\times$.

7.2.1 The Eisenstein intertwining

We pick a σ_f which occurs in $H^1_!(\mathcal{S}^{\tilde{M}}_{K_f^M},\mathcal{M}_\lambda(w\cdot\lambda)_F))$, we choose a $\iota : F\hookrightarrow\mathbb{C}$ and we choose an embedding

$$\Phi_\iota : \mathcal{D}_{\chi_\mu}\otimes H_{\sigma_f}\otimes_{F,\iota}\mathbb{C}\hookrightarrow L^2_{\text{cusp}}(M(\mathbb{Q})\backslash M(\mathbb{A})) \tag{343}$$

and from this we get the Eisenstein intertwining

$$\text{Eis}\circ\Phi_\iota : \text{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}(\mathcal{D}_{\chi_\mu})\otimes H_{\sigma_f}\otimes_{F,\iota}\mathbb{C}\to\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A})) \tag{344}$$

(Here we use that $K_f = \text{GSp}_2(\hat{\mathbb{Z}})$.) Hence we get an intertwining operator

$$\text{Eis}^\bullet : \text{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), I_P^G(\mathcal{D}_{\chi_\mu})\otimes\mathcal{M}_\lambda)\otimes H_{\sigma_f}\otimes_{F,\iota}\mathbb{C}\to\text{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}),\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))\otimes\mathcal{M}_\lambda) \tag{345}$$

and this induces a homomorphism in cohomology

$$H^3(\mathfrak{g},K_\infty,I_P^G(\mathcal{D}_{\chi_\mu})\otimes\mathcal{M}_\lambda)\otimes H_{\sigma_f}\otimes_{F,\iota}\mathbb{C})\to H^3(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\lambda,\mathbb{C}}) \tag{346}$$

and we want to compose it with the restriction to the cohomology of the boundary. We have to compose it with the the constant Fourier coefficient $\mathcal{F}^P : \mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))\to\mathcal{A}(P(\mathbb{Q})U(\mathbb{A})\backslash G(\mathbb{A}))$. We know that $\mathcal{F}^P$ maps into the subspace

$$I_P^G\mathcal{D}_{\chi_\mu}\otimes H_{\sigma_f}\otimes_{F,\iota}\mathbb{C}\xrightarrow{\mathcal{F}^P} I_P^G\mathcal{D}_{\chi_\mu}\otimes H_{\sigma_f}\otimes_{F,\iota}\mathbb{C}\bigoplus I_P^G\mathcal{D}_{\chi_{\mu'}}\otimes H_{\sigma_f^{w_P}|\gamma_{P,f}|^{2f_P}}\otimes_{F,\iota}\mathbb{C} \tag{347}$$

where $\mu' = w_P w\cdot\lambda = s_2\cdot\lambda = (2+2n_2+n_1)\gamma_1^{M_1}+\frac{1}{2}(-2+n_1)\gamma_2$. More precisely we know that for $h\in I_P^G\mathcal{D}_{\chi_\mu}\otimes H_{\sigma_f}\otimes_{F,\iota}\mathbb{C}$

$$\mathcal{F}^P(h) = h + C(\sigma,0)T^{\text{loc}}(0)(h) \tag{348}$$

where $T^{\text{loc}}(0) = T^{\text{loc}}_\infty\otimes\otimes_p T^{\text{loc}}_p$. The local intertwining operator at the finite primes is normalized, it maps the standard spherical function into the standard spherical function. The operator T^{loc}_∞ will be discussed below.

Our general formula for the constant term yields for an $h = h_\infty\times h_f$

Explain in more detail

$$\mathcal{F}^P(h) = h + C(\sigma_\infty,\lambda)T_\infty^{\mathrm{loc}}(h_\infty)\frac{L^{\mathrm{coh}}(f,n_1+n_2+2)}{L^{\mathrm{coh}}(f,n_1+n_2+3)}\frac{\zeta(n_1+1)}{\zeta(n_1+2)} \times T_f^{\mathrm{loc}}(0)(h_f) \tag{349}$$

(For the following compare SecOps.pdf) We analyze the factor $C(\sigma_\infty,\lambda)T_\infty^{\mathrm{loc}}$ more precisely we study the effect of this operator on the cohomology. Let us look at the map between complexes

$$T_\infty^{\mathrm{loc},\bullet} : \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), I_P^G\mathcal{D}_{\chi_\mu}\otimes\mathcal{M}_\lambda) \to \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), I_P^G\mathcal{D}_{\chi_{\mu'}}\otimes\mathcal{M}_\lambda) \tag{350}$$

The intertwining operator $T_\infty^{\mathrm{loc}} : I_P^G\mathcal{D}_{\chi_\mu} \to I_P^G\mathcal{D}_{\chi_{\mu'}}$ has a kernel $\mathbb{D}_{\chi_\mu}$, this is a discrete series representation. We know that

$$\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}),\mathbb{D}_{\chi_\mu}\otimes\mathcal{M}_\lambda) = \mathrm{Hom}_{K_\infty}(\Lambda^3(\mathfrak{g}/\mathfrak{k}),\mathbb{D}_{\chi_\mu}\otimes\mathcal{M}_\lambda) = \tag{351}$$

$$H^3(\mathfrak{g},K_\infty,\mathbb{D}_{\chi_\mu}\otimes\mathcal{M}_\lambda) = \mathbb{C}\Omega_{2,1}\oplus\mathbb{C}\Omega_{1,2} \tag{352}$$

We have the surjective homomorphism

$$H^3(\mathfrak{g},K_\infty,\mathbb{D}_{\chi_\mu}\otimes\mathcal{M}_\lambda) \to H^3(\Lambda^3(\mathfrak{g}/\mathfrak{k}), I_P^G\mathcal{D}_{\chi_\mu}\otimes\mathcal{M}_\lambda) = H^1(\mathfrak{m},K_\infty^M,\mathcal{D}_{\chi_\mu}\otimes H^2(\mathfrak{u}_P,\mathcal{M}_\lambda) = \mathbb{C}\omega^{(3)} \tag{353}$$

the differential form $\Omega_{2,1}+\epsilon(\lambda)\Omega_{1,2}$ maps to a non zero multiple $A(\lambda)\omega^{(3)}$. (The factor $\epsilon(\lambda)$ is a sign depending on λ). We can write $\Omega_{2,1}-\epsilon(\lambda)\Omega_{1,2} = d\psi$ where

$$\psi \in \mathrm{Hom}_{K_\infty}(\Lambda^2(\mathfrak{g}/\mathfrak{k}),\mathbb{D}_{\chi_\mu}\otimes\mathcal{M}_\lambda) \tag{354}$$

and $\omega = T_\infty^{\mathrm{loc},2}(\psi) \in \mathrm{Hom}_{K_\infty}(\Lambda^2(\mathfrak{g}/\mathfrak{k}),\mathcal{D}_{\chi_{\mu'}}\otimes\mathcal{M}_\lambda)$ is a closed form, hence it provides a cohomology class. Let us denote this cohomology class by $\kappa(\omega^{(3)})$.

Choosing $\omega^{(3)}$ as a basis element and applying the Eisenstein intertwining operator (345) yields a homomorphism

$$\mathrm{Eis}^{(3)}\circ\Phi_\iota : H^1_!(\mathcal{S}^{\tilde{M}}_{K_f^M},\mathcal{M}_\lambda(w\cdot\lambda)_F))(\sigma_f\circ\iota) \to H^3(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}_\lambda}\otimes\mathbb{C}) \tag{355}$$

.

The local intertwining operator T_∞^{loc} maps $\omega^{(3)}$ to zero and hence it follows that the composition $r\circ\mathrm{Eis}^{(3)}$ is the identity, the Eisenstein intertwining operator yields a section on $H^1_!(\mathcal{S}^{\tilde{M}}_{K_f^M},\mathcal{M}_\lambda(w\cdot\lambda)_F))(\sigma_f)$. (Remember $w = s_2s_1$). If we define

$$H^3(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\lambda F})(\sigma_f) = r^{-1}(H^1_!(\mathcal{S}^{\tilde{M}}_{K_f^M},\mathcal{M}_\lambda(w\cdot\lambda)_F))(\sigma_f)) \tag{356}$$

(Induction does not play a role since the level is one) then we get the decomposition

$$H^3_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\lambda F})\oplus H^3_{\mathrm{Eis}}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\lambda F})(\sigma_f) = H^3_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\lambda F})(\sigma_f) \tag{357}$$

7.2.2 The denominator of the Eisenstein class

We restrict this decomposition to the integral cohomology (better the image of the integral cohomology in the cohomology with rational coefficients)

$$H^3_{\text{int}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda F})(\sigma_f) \supset H^3_{!,\,\text{int}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda F})(\sigma_f) \oplus H^3_{\text{int,Eis}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda F})(\sigma_f) \tag{358}$$

The image of $H^3_{\text{int,Eis}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda F})(\sigma_f)$ under r is a submodule of finite index in $H^1_{!,\,\text{int}}(\mathcal{S}^{\tilde{M}}_{K^M_f}, \mathcal{M}_\lambda(w\cdot\lambda)_F))(\sigma_f))$ and the quotient is

$$H^3_{\text{int}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda F})(\sigma_f)/(H^3_{!,\,\text{int}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda F})(\sigma_f) \oplus H^3_{\text{int,Eis}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda F})(\sigma_f)) = \\ H^1_{!\ \text{int}}(\mathcal{S}^{\tilde{M}}_{K^M_f}, \mathcal{M}_\lambda(w\cdot\lambda)_F))(\sigma_f))/\text{image}(r). \tag{359}$$

The quotient on the right hand side is $\mathcal{O}_F/\Delta(\sigma_f)$ where $\Delta(\sigma_f)$ is the denominator ideal. Tensoring the exact sequence

$$0 \to H^3_{!,\,\text{int}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda F})(\sigma_f) \oplus H^3_{\text{int,Eis}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda F})(\sigma_f) \to H^1_{\text{int}}(\mathcal{S}^{\tilde{M}}_{K^M_f}, \mathcal{M}_\lambda(w\cdot\lambda)_F))(\sigma_f)) \to \mathcal{O}_F/\Delta(\sigma_f) \to 0 \tag{360}$$

by $\mathcal{O}_F/\Delta(\sigma_f)$ yields an inclusion

$$\text{Tor}^1_{\mathcal{O}_F}(\mathcal{O}_F/\Delta(\sigma_f), \mathcal{O}_F/\Delta(\sigma_f) = \mathcal{O}_F/\Delta(\sigma_f)) \hookrightarrow H^3_{!,\,\text{int}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda F})(\sigma_f) \otimes \mathcal{O}_F/\Delta(\sigma_f) \tag{361}$$

and this explains the congruences.

7.2.3 The secondary class

We choose generators $\omega^{(3)}(\sigma_f)($ resp. $\omega^{(2)}(\sigma_f^{w_P}|\gamma_{P,f}|^{2f_P}))$ for $H^1_{\text{int}}(\mathcal{S}^{\tilde{M}}_{K^M_f}, \mathcal{M}_\lambda(w\cdot \lambda)_F))(\sigma_f)($ resp. $H^1_{\text{int}}(\mathcal{S}^{\tilde{M}}_{K^M_f}, \mathcal{M}_\lambda(s_2\cdot\lambda))(\sigma_f))$ (Perhaps we can do this only locally on $\text{Spec}(\mathcal{O}_F)$.) We may arrange these generators such that $T^{\text{arith}}(\sigma_f)(\omega^{(3)}(\sigma_f)) = \omega^{(2)}(\sigma_f^{w_P}|\gamma_{P,f}|^{2f_P})$. The isomorphism

$$\Phi^{(3)}_\iota : H^3(\mathfrak{g}, K_\infty, \mathcal{D}_{\chi_\mu} \otimes \mathcal{M}_\lambda) \otimes H_{\sigma_f} \otimes_{F,\iota} \mathbb{C} \xrightarrow{\sim} H^1_{\text{int}}(\mathcal{S}^{\tilde{M}}_{K^M_f}, \mathcal{M}_\lambda(w\cdot\lambda)_F))(\iota\circ\sigma_f) \tag{362}$$

maps

$$(\Omega_{2,1} + \epsilon(\lambda)\Omega_{1,2}) \otimes \omega^{(3)}(\iota\circ\sigma_f) \mapsto \Omega_+(\sigma_f,\iota)\omega(\sigma_f)$$

where $\Omega_+(\sigma_f,\iota)$ is a period depending on the choice of Φ_ι. The element

$$(\Omega_{2,1} - \epsilon(\lambda)\Omega_{1,2}) \otimes \omega^{(3)}(\iota\circ\sigma_f) = d\psi \otimes \omega^{(3)}(\iota\circ\sigma_f).$$

where $\psi \in \text{Hom}_{K_\infty}(\Lambda^2(\mathfrak{g}/\mathfrak{k}), I_P^G\mathcal{D}_{\chi_\mu} \otimes \mathcal{M}_\lambda)$. The operator $T^{\text{loc}}(0)$ in (348) provides a homomorphism (350)

$$T^{\text{loc},2} \otimes T^{\text{loc}}_f : \text{Hom}_{K_\infty}(\Lambda^2(\mathfrak{g}/\mathfrak{k}), I_P^G\mathcal{D}_{\chi_\mu} \otimes \mathcal{M}_\lambda) \otimes H_{\iota\circ\sigma_f} \to \text{Hom}_{K_\infty}(\Lambda^2(\mathfrak{g}/\mathfrak{k}), I_P^G\mathcal{D}_{\chi_{\mu'}} \otimes \mathcal{M}_\lambda) \otimes H_{\iota\circ\sigma_f^{w_P}}|\gamma_{P,f}|^{2f_P}))$$

Under this homomorphism the class ψ is mapped to a multiple of $\omega^{(2)}(\sigma_f^{w_P}|\gamma_{P,f}|^{2f_P}))$. We can calculate this multiple, during this calculation we see a second period $\Omega_-(\sigma_f, \iota)$ depending on Φ_ι and the ratio of these periods will be our period $\Omega(\iota \circ \sigma_f)$ in formula (341) .

This period is independent of Φ_ι. To state the final result we denote by f the modular cusp form attached to σ_f, this is a modular form with coefficients in F, then $\iota \circ f$ is a modular form with coefficients in $\mathbb{C}$. By $\Lambda(f, s)$ we denote the usual completed L -function. We get

$$C(\sigma, 0)T^{\text{loc}}(\kappa(\omega^{(3)}(\iota \circ \sigma_f)) =$$

$$\Big(\frac{1}{\Omega(\sigma_f, \iota))^{\epsilon(k,m)}}\frac{\Lambda^{\text{coh}}(\iota \circ f, n_1 + n_2 + 2)}{\Lambda^{\text{coh}}(\iota \circ f, n_1 + n_2 + 3)}\frac{1}{\zeta(-1 - n_1)}\Big)\frac{\zeta'(-n_1)}{\pi}\omega^{(2)}(\sigma_f^{w_P}|\gamma_{P,f}|^{2f_P}))$$

The factor inside the large brackets is essentially rational (it is in F and behaves invariantly under the action of the Galois group) the factor $\frac{\zeta'(-n_1)}{\pi}$ should viewed as a generator of a group of extension classes of mixed motives.

For me the most difficult part in the calculation is the treatment of the intertwining operator at ∞, this is carried out in SecOps.pdf. At the end of SecOps.pdf. I discuss the arithmetic applications and the conjectural relationship between the primes dividing the denominator of the expression in the large brackets and the denominators of the Eisenstein classes in (135)

References

[B-C] Borel-,A.Casselman, W L^2 Cohomology of Locally Symmetric Manifolds of Finite Volume ,Duke Mathematical Journal, Vol. 50, No. 3,1983

[B-G] Borel, A, Garland, H Laplacian and the Discrete Spectrum of an Arithmetic Group, American Journal of Mathematics, Vol. 105, No. 2 , pp. 309-335

[Go] Godement, R. Princeton Lecture Notes on Jacquet-Langlands

[De-Val] Deligne, P. Valeurs des Fonctions L, Proc. Symp. pure Math. 33 ,II,(1979),

[Fa-vdG] Faber, C., van der Geer, G. Sur la Cohomologie des Systèmes Locaux (preprint)

[G-J] Gelbart,S.-Jacquet,H, A relation between automorphic representations of GL(2) and GL(3). Ann. Sci. cole Norm. Sup. (4) 11 (1978), no. 4, 471–542

[Grob] Grobner, H. A cohomological injectivity result for the residual automorphic spectrum, Pacific Journal of mathematics, Vol268 ; No 1, . p. 33-46

[H-C] Harish-Chandra, Automorphic Forms on semi simple Lie Groups, SLN 62

[Ha-Cong] Harder, G. A congruence between a Siegel and an Elliptic Modular Form The 1-2-3 of Modular Forms, Springer Unitext

[Ha-Eis] Harder, G. Eisensteinkohomologie und die Konstruktion gemischter Motive, SLN 1562

[Ha-HC] Arithmetic Harish-Chandra modules. Preprint available at http://arxiv.org/abs/1407.0574

[Ha-Mum] Harder, G. Arithmetic aspects of Rank one Eisenstein cohomology, with an appendix by Don Zagier
in Cycles, Motives, Shimura Varieties (ed. by V. Srinivas),
Studies in Mathematics, TIFR.

[Ha-Mod] Harder, G., General aspects in the theory of modular symbols. Seminarberichte des Séminaire DPP, Birkhäuser, 73 - 88, 1983

[Ha-Gl2] Harder, G., Eisenstein cohomology of arithmetic groups. The case GL_2. Invent. math. **89** (1987), 37 - 118

[Ha-Gln] Harder, G., Some results on the Eisenstein cohomology of arithmetic subgroups of GL_n in Cohomology of arithmetic groups. Proceedings of a conference held at CIRM, ed.: J. P. Labesse-J. Schwermer, Springer Lecture Notes, 1447, 1990

[Ha-Ra] Harder, G. - Raghuram, A. Eisenstein Cohomology and Ratios of critical Values of Rankin-Selberg L-Functions submitted

The following items can be obtained from my home page www.math.uni-bonn.de/harder/Manuscripts/Eisenstein

[MixMot-2015.pdf] Modular Construction of Mixed Motives

[SecOps.pdf] Secondary Operations on the Cohomology of Harish-Chandra Modules

[Hi] Hida, Haruzo Congruence of cusp forms and special values of their zeta functions. Invent. Math. 63 (1981), no. 2, 225-261.

[J-L] Jacquet- Langlands, Automorphic forms on Gl_2,SLN 114

[Lau] Laumon, G. ,Sur la cohomologie supports compacts des varietes de Shimura pour GSp$(4)_Q$. Compositio Math. 105 (1997), no. 3, 267–359.

[Langlands] Langlands, R. Euler products. A James K. Whittemore Lecture in Mathematics given at Yale University, 1967. Yale Mathematical Monographs, 1.

[La-Eis] On the Functional Equations Satisfied by Eisenstein Series, Springer Lecture Notes, 544 , 1976

[P] Pink, R. On l-adic sheaves on Shimura varieties and their higher direct images in the Baily-Borel compactification. Math. Ann. 292 (1992), no. 2, 197–240.

[Sat] Satake, I. Spherical functions and Ramanujan conjecture. 1966 Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math.,Boulder, Colo., 1965) pp. 258?264 Amer. Math. Soc., Providence, R.I.

[Scholl] Scholl, A.,Motives for modular forms. Invent. Math. 100 (1990), no. 2, 419–430.

[Schwermer] Schwermer,J. On Euler products and residual Eisenstein cohomology classes for Siegel modular varieties. (English. English summary) Forum Math. 7 (1995), no. 1, 1–28.

[Wei] Weissauer, R. Endoscopy for GSp(4) and the Cohomology of Siegel Modular Threefolds Springer Lecture Notes 1968