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Cohomology of Arithmetic Groups

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Preface

Finally this is now the book on "Cohomology of Arithmetic Groups" which was announced in my two volumes "Lectures on Algebraic Geometry I and II" [40],[41]. Originally the purpose of these two volumes was to assemble basic material for this third volume. This applies especially to the chapters I-IV in the first volume, where we provide the necessary background in homological algebra.

During the years 1980-2000 I gave various advanced courses on number theory, algebraic geometry and also on "Cohomology of Arithmetic Groups" at the university of Bonn. I prepared some notes for these lectures, because there was essentially no literature covering this subject.

At some point I had the idea to use these notes as a basis for a book. It was clear that a self-contained exposition needs some preparation, we need some basic tools from homological algebra. Since the cohomology groups of arithmetic groups are sheaf cohomology groups, and since the theory of sheaves and cohomology sheaf is ubiquitous in algebraic geometry some branches grew out and I wrote the two volumes [40], [41]..

The subject has applications to number theory - actually it is part of number theory. My main concern is the relationship between special values of L-functions and the integral structure of the cohomology as module under the Hecke algebra. We can prove rationality statements for special values (Manin and Shimura), on the other hand these special values tell us something about the denominators of the Eisenstein classes. These connections was already discussed in the original notes for the special case of $\mathrm{Sl}_2(\mathbb{Z})$. and a precise result (in some sense culminating result in this special case are stated at the end of Chapter 5 In the probably removed section (but actually not removed se3ction) I discus- for a specific example - an application concerning to the structure of the Galois group. This is really number theory

For other groups than Gl_2 this relationship between special values of L-functions and the denominators of Eisenstein classes is mainly conjectural. It is one of the central themes of this book. The conjectures concerning the denominators imply congruences between eigenvalues of Hecke operators on different groups. It was extremely important for me that these conjectures on congruences got some support by experimental calculations by G. van der Geer and C. Faber and others. The discovery-or the verification- of these congruences had great influence on the content of this book.

The subject has some very interesting computational and experimental aspects. In principle there exists an algorithm which verifies the denominator conjecture in any given case. In Chapter 3 we discuss the basic steps for writing an algorithm which computes the cohomology and the Hecke endomorphisms explicitly for any specific example. Hence we can check the conjecture in such a situation. For the group $\mathrm{Sl}_2(\mathbb{Z})$ such explicit calculations have been done by my former student X.-D. Wang in his Bonn dissertation and with the help of H. Gangl I also wrote such an algorithm which is discussed in Chapter 3.

But to the best of my knowledge there are only very few other cases, where we have such an algorithm, which works in practice. For instance it is very desirable to have such an such an algorithm in the case of the group $\mathrm{GSp}_2(\mathbb{Z})$ to treat the issues raised in Chapter 9.

In the final chapter 9 I talk about the general aspects of Eisenstein cohomology, but I also want to point out that there are some very interesting open questions. The denominator question is not only an interesting problem in itself. These denominators allow us to produce non trivial elements in certain Selmer groups. This means that we can construct elements in various Selmer groups which owe their existence certain divisibility of special L-values. Such a connection between L-values and the structure of the Galois group is a central theme in number theory and starts with Kummer and continues with Herbrand, Ribet and Bloch-Kato.

I hope that this book can serve as an introduction into the field "Cohomology of Arithmetic Groups", but that it also initiates some interesting research.

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0.1 Introduction

An arithmetic group Γ is a discrete subgroups of a Lie group $G(\mathbb{R}) \subset \mathrm{Gl}_n(\mathbb{R})$ whose matrix entries satisfy certain rationality and integrality condition. The most basic example of such a group is the group $\mathrm{Sl}_n(\mathbb{Z}) \subset \mathrm{Sl}_n(\mathbb{R})$. More generally we can start from an algebraic subgroup $G/\mathbb{Q} \subset \mathrm{Gl}_n/\mathbb{Q}$, for instance the orthogonal group of a quadratic form. Then we get arithmetic groups $\Gamma \subset \mathbb{G}(\mathbb{Q}) \subset G(\mathbb{R})$ if we impose certain integrality conditions on the matrix coefficients of the elements of Γ.

For any Γ- module $\mathcal{M}$ we can define the cohomology groups $H^\bullet(\Gamma, \mathcal{M}) = \bigoplus_q H^q(\Gamma, \mathcal{M})$. These cohomology groups are abelian groups, which are defined in terms of homological algebra, they are the derived functors of the functor $\mathcal{M} \to \mathcal{M}^\Gamma (=$ invariants under Γ.)

We are mainly interested in the cohomology of a very special class of Γ-modules. We consider rational representations $\rho : G/\mathbb{Q} \to \mathcal{M}_\mathbb{Q}$, where $\mathcal{M}_\mathbb{Q}$ is a finite dimensional $\mathbb{Q}$-vector space. Then we can find finitely generated $\mathbb{Z}$ modules $\mathcal{M}$ such that $\mathcal{M}_\mathbb{Q} = \mathcal{M} \otimes_\mathbb{Z} \mathbb{Q}$ which are Γ-invariant and hence Γ-modules.

Let $K_\infty \subset G(\mathbb{R})$ be a maximal compact subgroup, for example $\mathrm{SO}(n) \subset \mathrm{Sl}_n(\mathbb{R})$. The quotient $X = \mathbb{G}(\mathbb{R})/K_\infty$ is a symmetric space, it carries a Riemannian metric which is $G(\mathbb{R})-$ invariant under the left action, it may have finitely many connected components, each connected component is diffeomorphic to $\mathbb{R}^d$, hence contractible.

Our arithmetic group Γ acts properly discontinuously on X, we can form the quotient $\Gamma\backslash X$, this quotient is an orbifold. We can pass to a suitable subgroup of finite index $\Gamma' \subset \Gamma$ such that Γ' has no non trivial elements of finite order (i.e. is torsion free). Then $\Gamma'\backslash X$ is a Riemannian manifold, it is a so called locally symmetric space. The map $\Gamma'\backslash X \to \Gamma\backslash X$ is a finite covering with some ramifications. If Γ has elements of finite order then $\Gamma\backslash X$ is only a Riemannian orbifold. These spaces $\Gamma\backslash X$ provide a very interesting class of spaces, which are of interest for differential geometers, mathematicians interested in analysis on manifolds and topologists. But they are in a sense of arithmetic origin and therefore they are of interest for number theorists.

Our Γ module $\mathcal{M}$ endows the space $\Gamma\backslash X$ with a sheaf $\tilde{\mathcal{M}}$ (section 6.2) with values in finitely generated abelian groups. If Γ is torsion free then $\tilde{\mathcal{M}}$ is a locally constant sheaf, or in other words a local system.

We introduce the sheaf- cohomology groups

$$H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}) = \bigoplus_q H^q(\Gamma\backslash X, \tilde{\mathcal{M}})$$

these cohomology groups are "essentially" the same as the above group cohomology groups, these two versions of cohomology become equal, if X is connected and Γ is torsion free. We will see that these cohomology groups are finitely generated $\mathbb{Z}-$ modules.

We have some additional structure on these cohomology groups. In general the quotient space $\Gamma\backslash X$ is not compact. We have the Borel-Serre compactification $i : \Gamma\backslash X \hookrightarrow \Gamma\backslash \bar{X}$, where i is a homotopy equivalence and $\Gamma\backslash \bar{X}$ is a manifold (orbifold) with corners. The difference set $\partial(\Gamma\backslash X) := \Gamma\backslash \bar{X} \setminus \Gamma\backslash X$ is the boundary of the Borel-Serre compactification. Moreover we will construct a "tubular" neighbourhood $\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X) \subset \Gamma\backslash X$ of "infinity" (see (1.2.8)). We may also consider the cohomology with compact supports $H_c^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}})$. and we get the fundamental long exact sequence

$$\cdots \to H_c^q(\Gamma\backslash X, \tilde{\mathcal{M}}) \xrightarrow{i_c} H^q(\Gamma\backslash X, \tilde{\mathcal{M}}) \xrightarrow{r} H^q(\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X), \tilde{\mathcal{M}}) \xrightarrow{\delta} H_c^{q+1}(\Gamma\backslash X, \tilde{\mathcal{M}}) \to \dots \tag{1}$$

We also introduce the "inner cohomology"

$$H_!^q(\Gamma\backslash X, \tilde{\mathcal{M}}) := \ker(r) = \mathrm{Im}(i_c).$$

A second structural ingredient is the Hecke algebra. We have an action of a big algebra of operators acting on all these cohomology groups and the action commutes with arrows in the fundamental exact sequence.

This is the so called Hecke algebra $\mathcal{H}($ or $\mathcal{H}_\Gamma)$, it originates from the structure of the arithmetic group Γ. The group Γ has many subgroups Γ' of finite index, for which we can construct two arrows

$$\Gamma'\backslash X \underset{p_2}{\overset{p_1}{\rightrightarrows}} \Gamma\backslash X. \tag{2}$$

Such a pair of arrows is also called a correspondence between on $\Gamma\backslash X$. Such a correspondence, together with a suitable map $u : p_1^*(\tilde{\mathcal{M}}) \to p_2^*(\tilde{\mathcal{M}})$, induces an endomorphism in the cohomology. These endomorphisms act on all the modules in the exact sequence above and are compatible with the arrows.

The basic objects of interest in this book are the various cohomology groups, which appear in the fundamental exact sequence, together with the action of the Hecke algebra $\mathcal{H}$ on them.

It is my intention is to keep the exposition as elementary as possible, the text should be readable by graduate students. We will need some background material from algebraic topology and from homological algebra (cohomology and homology of groups, spectral sequences, sheaf cohomology). This material is expounded in the first four chapters in [40], of course it can be found in many other textbooks.

In the later chapters (starting from chapter 6) we also need results and concepts from the theory of algebraic groups, the theory of symmetric spaces, arithmetic groups, and reduction theory for arithmetic groups. Furthermore we need results from the theory of representations of real semi-simple groups.

This material is somewhat more advanced, but in the in the first five chapters all these concepts and results are explained in terms in terms of special examples. Especially the sections on the general reduction theory and the Borel-Serre compactification (section (1.2.8)) could be skipped in a first reading.

For the Lie groups $\mathrm{Sl}_2(\mathbb{R})$ and $\mathrm{Sl}_2(\mathbb{C})$ and their arithmetic subgroups $\mathrm{Sl}_2(\mathbb{Z})$ and $\mathrm{Sl}_2(\mathbb{Z}[\sqrt{-1}])$ these prerequisite concepts are easy to explain and we will do so in this book. For instance if $\Gamma = \mathrm{Sl}_2(\mathbb{Z})$ or more generally a congruence subgroup of finite index the symmetric space $\mathrm{Sl}_2(\mathbb{R})/K_\infty$ is the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \Im(z) = y > 0\} = \mathrm{Sl}_2(\mathbb{R})/\mathrm{SO}(2)$. The quotient space $\Gamma\backslash\mathbb{H}$ is punctured Riemann surface. In this special case we have the Γ module $\mathcal{M}_n = \{\sum a_\nu X^\nu Y^{n-\nu} \mid a_\nu \in \mathbb{Z}\}$. We will study thecohomology groups $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n)$

and their module structure under the Hecke algebra in detail. We will prove some very specific results for these cohomology groups.

In Chapter four we discuss results from the theory of representations of the Lie- groups $\mathrm{Sl}_2(\mathbb{R})$ and $\mathrm{Sl}_2(\mathbb{C})$, and we explain the impact of these results on the cohomology. With these results at hand we formulate the famous Eichler-Shimura isomorphism, and we can sketch its proof. This Eichler-Shimura isomorphism also establishes the connection between $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}_n}) \otimes \mathbb{C}$ and the space of modular forms of weight $n+2$. In the second half of this book in Chapter 8 we discuss what is called "Representation theoretic Hodge theory" and the Eichler-Shimura theorem becomes a special case of a much more general theorem.

On the other hand we will show that the results for the special groups $\mathrm{Sl}_2(\mathbb{Z}), \mathrm{Sl}_2(\mathbb{Z}[\sqrt{d}])$, or suitable subgroups of finite index of them, have deep and interesting consequences. We will discuss the Eisenstein cohomology for these special groups and explain the interaction between special values of L-functions and the structure of the cohomology. A prototype of such a result is the formula for the denominator of the Eisenstein class (Theorem 5.1.2). It is clear that this result should be a special case of a much more general theorem. At this moment it is not clear how far these generalisations reach (See section 9.3.1).

In Chapter 5 we discuss some applications of these results to number theory, and we have to accept some even more advanced topics. We concentrate on the case that $\Gamma \subset \mathrm{Sl}_2(\mathbb{Z})$ and we will use the fact that- with a grain of salt - the quotient $\Gamma\backslash\mathbb{H}$ is the set of $\mathbb{C}$-valued points the moduli space of elliptic curves (with some additional structure). This is also explained in [40],[41].

Then for any prime ℓ the cohomology groups $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}_n}) \otimes \mathbb{Z}_\ell)$ are actually ℓ-adic etale cohomology groups, especially we get an action of the Galois-group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on these $\ell-$ adic cohomology groups. This action commutes with the action of the Hecke algebra. The insights into the structure of the cohomology groups as Hecke modules provides insights into the structure of the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, for instance we discuss the theorem of Herbrand-Ribet ([20], [87])

In Chapter 6 we study the cohomology groups of arithmetic groups in a more general framework. We start from arbitrary reductive groups $G/\mathbb{Q}$, we assume some familiarity with the theory of semi -simple real groups and the theory of symmetric spaces. There will be some overlap with the earlier chapters.

We use the adelic language, our locally symmetric spaces will be double coset spaces $\mathcal{S}^G_{K_f} = G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K_f$. Here K_f is an open compact subgroup of $G(\mathbb{A}_f)$, it the so called level subgroup. These locally symmetric spaces turn out to be disjoint unions of the previous ones.

Again define sheaves $\tilde{\mathcal{M}}$ on these spaces, this will be sheaves with values in the category of finitely generated $\mathbb{Z}$-modules. We are interested in the various cohomology groups $H^\bullet_?(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})$. in our fundamental exact sequence.(1) We know that all these cohomology groups are finitely generated $\mathbb{Z}$-modules. (Raghunathan)

Here we have to work a little bit to define the integral cohomology and to define the action of the Hecke operators on these integral cohomology groups.

In this context the Hecke -algebra becomes a restricted product of local Hecke -algebras, this means $\mathcal{H}_{K_f} = \bigotimes'_p \mathcal{H}_p$. The local algebras $\mathcal{H}_p$ have an identity. The level subgroup K_f determines a finite set $\Sigma = \Sigma_{K_f}$ of ramified primes. The sub algebra $\mathcal{H}^{(\Sigma)} = \bigotimes_{p \notin \Sigma}$ is a central sub-algebra of $\mathcal{H}_{K_f}$. For an unramified prime $p \notin \Sigma$ the structure of $\mathcal{H}_p$ is given by the Satake isomorphism. (Theorem 6.3.1).

We may pass to the rational cohomology groups $H^\bullet_?(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}} \otimes \mathbb{Q})$, these are finite dimensional $\mathbb{Q}$ vector space together with the action of $\mathcal{H}$. We will show in section 8.1.8 that the action of $\mathcal{H}$ on the inner cohomology $H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}} \otimes \mathbb{Q})$ is semi simple, i.e. each $\mathcal{H}$ invariant submodule has a $\mathcal{H}$-invariant complement. This implies that we can find a finite (normal) extension $F/\mathbb{Q}$ such that $H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}} \otimes F)$ is a direct sum of absolutely irreducible $\mathcal{H}$ module. Therefore we get an isotypical decomposition

$$H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}} \otimes F) = \bigoplus_{\pi_f \in \mathrm{Coh}_!(G, K_f, \mathcal{M})} H^\bullet_!(\mathcal{S}^G_{K_f} \tilde{\mathcal{M}} \otimes F)(\pi_f)$$

where the π_f denote isomorphism classes of absolutely irreducible $\mathcal{H}$-modules. Such an absolutely irreducible Hecke module is the restricted tensor product: $\pi_f = \otimes'_p \pi_p$. The restriction of π_f to $\mathcal{H}^{(\Sigma)}$ gives us a homomorphism $\pi^{(\Sigma)} = \bigotimes_{p \notin \Sigma} : \mathcal{H}^{(\Sigma)} \to \mathcal{O}_F$.

After that we discuss some general facts concerning these cohomology groups (Poincare duality, homology, adjunction formulas for Hecke operators) and we have a section on the Gauss-Bonnet theorem.

Chapter 7 is somewhat philosophical. We have seen in the previous Chapter 4 and we will also see in Chapter 8 how the cohomology groups after tensoring by $\mathbb{C}$ are related to the space of automorphic forms. In 1967 R. Langlands formulated a visionary program concerning automorphic forms, this is the Langlands program. In this Chapter 7 we discuss some of the aspects of this program in the context of cohomology of arithmetic groups. The main player is the Langlands dual group ${}^\vee G/\mathbb{Q}$.

The Langlands dual group ${}^\vee G/\mathbb{Q}$ has the following purpose: For any absolutely irreducible π_f which occurs non trivially in the cohomology $H^\bullet_?(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}} \otimes F)$ and any $p \notin \Sigma$ the theorem of Satake provides a canonical semi-simple conjugacy class $\omega_p(\pi_p) \in {}^\vee G(F)$. For any representation $r : {}^\vee G/\mathbb{Q} \to \mathrm{Gl}(V)$ of the algebraic group ${}^\vee G/\mathbb{Q}$ we can attach an L-function which is defined as an infinite product

$$L^{(\Sigma)}(\pi_f, r, s) := \prod_{p \notin \Sigma} \frac{1}{\det(\mathrm{Id} - r(\omega_p(\pi_p))p^{-s}|V)} = \prod_{p \notin \Sigma} L(\pi_p, r, s)$$

With some extra effort we can also attach local Euler factors $L(\pi_p, r, s)$ to the ramified primes $p \in \Sigma$ and then the L function is defined as $L(\pi_f, r, s) = \prod_p L(\pi_p, r, s)$.

A very bold prediction of the Langlands philosophy says that to any absolutely irreducible π_f which occurs somewhere in the cohomology $H^\bullet_?(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}} \otimes$

F) and any representation r we can find a motive $\{\mathbb{M}(\pi_f, r)\}$ such that we an equality of L-functions

$$L(\pi_f, r, s) = L(\mathbb{M}(\pi_f, r), s).$$

It is one of the central themes in this book to investigate the relationship between the $L-$ functions $L(\pi_f, r, s)$ (analytic properties, special values) and the structure of the integral cohomology as modules under the Hecke-algebra, a first instance is theorem 5.1.2.

In Chapter 8 we develop the analytic tools for the computation of the cohomology. Here we do not use the language of adeles. We assume that the Γ-module $\mathcal{M}$ is a $\mathbb{C}$-vector space and it is obtained from a rational representation of the underlying algebraic group. In this case one may interpret the sheaf $\tilde{\mathcal{M}}$ as the sheaf of locally constant sections in a flat bundle, and this implies that the cohomology is computable from the de-Rham-complex associated to this flat bundle. We could even go one step further and introduce a Laplace operator so that we get some kind of Hodge-theory and we can express the cohomology in terms of harmonic forms. Here we encounter serious difficulties since the quotient space $\Gamma\backslash X$ is not compact. But we will proceed in a slightly different way. Instead of doing analysis on $\Gamma\backslash X$ we work on $\mathcal{C}_\infty(\Gamma\backslash G(\mathbb{R}))$. This space is a module under the group $G(\mathbb{R})$, which acts by right translations, but we rather consider it as a module under the Lie algebra $\mathfrak{g}$ of $G(\mathbb{R})$ on which also the group K_∞ acts, it is a $(\mathfrak{g}, K)$-module.

Since our module $\mathcal{M}$ comes from a rational representation of the underlying group G, we may replace the de-Rham-complex by another complex

$$H^\bullet(\mathfrak{g}, K_\infty, \mathcal{C}_\infty(\Gamma\backslash G(\mathbb{R})) \otimes \tilde{\mathcal{M}}),$$

this complex computes the so called $(\mathfrak{g}, K)$-cohomology. The general principle will be to "decompose" the $(\mathfrak{g}, K)$-module $\mathcal{C}_\infty(\Gamma\backslash G(\mathbb{R})$ into irreducible submodules and therefore to compute the cohomology as the sum of the contributions of the individual submodules. This is a group theoretic version of the classical approach by Hodge-theory. Again we have to overcome two difficulties. The first one is that the quotient $\Gamma\backslash G(\mathbb{R})$ is not compact and hence the above decomposition does not make sense.

The second problem is that we have to understand the irreducible $(\mathfrak{g}, K)$-modules and their cohomology.

The first problem is of analytical nature, we will give some indication how this can be solved by passing to certain subspaces of the cohomology the so called cuspidal or better the inner cohomology. The central result is the Theorem 8.1.1.

This result is a partial generalisation of the theorem of Eichler-Shimura, it describes the cuspidal part of the cohomology in terms of irreducible representations occurring in the space of cusp forms and contains some information on the discrete cohomology, which is slightly weaker. (See proposition 8.1.4) We shall also give some indications how it can be proved.

We shall shall also state some general results concerning the second problem, we briefly recall the construction of the irreducible modules with non trivial $(\mathfrak{g}, K_\infty)$ cohomology.

We discuss some consequences of Theorem 8.1.1. It implies some vanishing theorems in cohomology, it implies that the inner cohomology is a semi simple module for the Hecke-algebra, and it helps to understand the $K-$-theory of algebraic number fields.

In the next section we use reduction theory-or better the description of $\overset{\bullet}{\mathcal{N}}$ $(\Gamma\backslash X), \tilde{\mathcal{M}})$- to discuss some growth conditions for cohomology classes, basically we show that cohomology classes which given by a weight can be represented by differential forms which have essentially the same weight.

In the second half of this chapter we will resume the discussion of modular symbols.

In the last chapter 9 we discuss the Eisenstein-cohomology. The theorem of Eichler-Shimura describes only a certain part of the cohomology, the Eisenstein -cohomology is meant to fill the gap, it is complementary to the cuspidal cohomology. These Eisenstein classes are obtained by an infinite summation process, which sometimes does not converge and is made convergent by analytic continuation.

In the beginning of this chapter 9 we recall the Borel-Serre compactification, we discuss the spectral sequences induced by the stratification of the Borel-Serre boundary. We continue by recalling the process of constructing Eisenstein cohomology classes by infinite summations and analytic (or meromorphic) continuation. We already discussed Eisenstein cohomology in this book for thecase of the special group $\mathrm{Sl}_2(\mathbb{R})$ in chapter 4. For the group Gl_2/K over a number field we refer to [35]. We have the general theorem of Franke [26], but I think that Franke's theorem is still far away from a final answer, there are many questions open and we have to exploit the various possibilities for applications in number theory. In the rest of this chapter we give an outline of these possible application, we formulate some results and we also formulate some speculative ideas.

Under certain conditions (if the Manin Drinfeld principle is valid) these Eisenstein cohomology classes are actually rational classes (or classes over some specific number field). Then we may for instance evaluate on certain cycles and it happens that the result is a special value of an L-function divided by a period (See for instance chapter 4.) Hence we can prove rationality results for these modified $L-$ values. This allows us to prove rationality results for special L-values. (See [37], [48]).

The central theme of this book is the understanding of the integral cohomology $H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})$ as a module under the Hecke algebra, for instance we want to understand the denominators of the Eisenstein classes.

In Chapter 9 we formulate the general principle that under suitable conditions this denominator should be related (divisible?, equal ?) to a certain special value of an L-function, which occurs in the constant term of the Eisenstein series. The prototype of such a relationship occurs in [43], (actually the "abelian" case is discussed in chapter 5).

This principle (or conjecture) can be verified (or falsified) experimentally, on the other hand there is a strategy to prove assuming certain finiteness for mixed Grothendieck motives.

Chapter 1

Basic Notions and Definitions

Affgr

1.1 Affine algebraic groups over $\mathbb{Q}$.

A linear algebraic group $G/\mathbb{Q}$ is a subgroup $G \subset \mathrm{Gl}_n$, which is defined as the set of common zeroes of a set of polynomials in the matrix coefficients, where in addition these polynomials have coefficients in $\mathbb{Q}$. Of course we cannot take just any set of polynomials, the set has to be somewhat special before its common zeroes form a group. The following examples will clarify what I mean:

1.) The group GL_n is an algebraic group itself, the set of equations is empty. It has the subgroup $\mathrm{Sl}_n \subset \mathrm{Gl}_n$, which is defined by the polynomial equation

$$\mathrm{Sl}_n = \{x \in GL_n \mid \det(x) = 1\}$$

2.) Another example is given by the orthogonal group of a quadratic form

$$f(x_1, \dots, x_n) = \sum_{i=}^{n} a_i x_i^2$$

where $a_i \in \mathbb{Q}$ and all $a_i \neq 0$ (this is actually not necessary for the following). We look at all matrices

$$\alpha = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

which leave this form invariant, i.e.

$$f(\alpha \underline{x}) = f(\underline{x})$$

for all vectors $\underline{x} = (x_1, \dots, x_n)$. This defines a set of polynomial equations for the coefficient a_{ij} of α. These α form a group, this is the linear algebraic group $SO(f)$.

3.) Instead of taking a quadratic form — which is the same as taking a symmetric bilinear form — we could take an alternating bilinear form

$$\begin{aligned}\langle \underline{x}, \underline{y}\rangle =&\langle x_1, \ldots, x_{2n}, y_1, \ldots y_{2n}\rangle = \\ &\sum_{i=1}^{n}\langle x_1 y_{i+n} - x_{i+n}y_i\rangle = f\langle \underline{x}, \underline{y}\rangle.\end{aligned}$$

This form defines the symplectic group:

$$Sp_n = \left\{\alpha \in GL_{2n} \mid \langle \alpha\underline{x}, \alpha\underline{y}\rangle = \langle \underline{x}, \underline{y}\rangle\right\}.$$

An Important remark: The reader may have observed tha we did not specify a field (or a ring) from which we take the entries of the matrices. This is done intentionally, because we may take the entries from any commutative ring R which contains the rational numbers $\mathbb{Q}$ and for which $1 \in \mathbb{Q}$ is the identity element (this means that R is a $\mathbb{Q}-$ algebra). In other words: for any algebraic group $G/\mathbb{Q} \subset GL_n$ and any $\mathbb{Q}$ algebra R we may define

$$G(R) \subset \mathrm{Gl}_n(R)$$

as the group of those matrices whose coefficients satisfy the required polynomial equations. Adopting this point of view we can say that

A linear algebraic group $G/\mathbb{Q}$ defines a functor from the category of $\mathbb{Q}$-algebras (i.e. commutative rings containing $\mathbb{Q}$) into the category of groups.

4.) Another example is obtained by the so-called ırestriction of scalars. Let us assume we have a finite extension $K/\mathbb{Q}$, we consider the vector space $V = K^n$. This vector space may also be considered as a $\mathbb{Q}$-vector space V_0 of dimension $n[K : \mathbb{Q}] = N$. We consider the group

$$GL_N/\mathbb{Q}.$$

Our original structure as a K-vector space may be considered as a map

$$\Theta : K \longrightarrow \mathrm{End}_{\mathbb{Q}}(V_0),$$

and the group $GL_n(K)$ is then the subgroup of elements in $GL_N(\mathbb{Q})$ which commute with all the elements of $\Theta(x), x \in K$. Hence we define the subgroup

$$G/\mathbb{Q} = R_{K/\mathbb{Q}}(\mathrm{Gl}_n) = \{\alpha \in \mathrm{Gl}_N \mid \alpha \text{ commutes with } \Theta(K)\}. \tag{1.1}$$

Then $G(\mathbb{Q}) = \mathrm{Gl}_n(K)$. For any $\mathbb{Q}$-algebra R we get

$$G(R) = \mathrm{Gl}_n(K \otimes_{\mathbb{Q}} R).$$

This can also be applied to an algebraic subgroup $H/K \hookrightarrow \mathrm{Gl}_n/K$, i.e. a subgroup that is defined by polynomial equations with coefficients in K.

Our definition of a linear algebraic group is a little bit provisorial. If we consider for instance the two linear algebraic groups

$$\begin{aligned}G_1/\mathbb{Q} &= \left\{\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix}\right\} \subset \mathrm{Gl}_2 \\ G_2/\mathbb{Q} &= \left\{\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{pmatrix}\right\} \subset GL_3\end{aligned}$$

then we would like to say, that these two groups are isomorphic. They are two different "realizations" of the *additive group* $G_a/\mathbb{Q}$. We see that the same linear algebraic group may be realized in different ways as a subgroup of different Gl_N's.

Of course there is a concept of linear algebraic group which does not rely on embeddings. The understanding of this concept requires a little bit of affine algebraic geometry. The drawback of our definition here is that we cannot define morphism between linear algebraic group. Especially we do not know when they are isomorphic.

We assert the reader that the general theory implies that a morphism between two algebraic groups is the same thing as a morphism between the two functors form $\mathbb{Q}$-algebras to groups. In some sense it is enough to give this functor. For instance, we have the *multiplicative group* $\mathbb{G}_m/\mathbb{Q}$ given by the functor

$$R \longrightarrow R^\times$$

and the additive group $G_a/\mathbb{Q}$ given by $R \to R^+$.

We can realise (represent is the right term) thegroup $\mathbb{G}_m/\mathbb{Q}$ as

$$\mathbb{G}_m/\mathbb{Q} = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right\} \subset \mathrm{Gl}_2$$

AGS

1.1.1 Affine group schemes

We say just a few words concerning the systematic development of the theory of linear algebraic groups. This is not directly used in the next few chapters but it will be useful later.

For any field k an affine k-algebra is a finitely generated algebra A/k, i.e. it is a commutative ring with identity, containing k, the identity of k is equal to the identity of A, which is finitely generated over k as an algebra. In other words

$$A = k[x_1, x_2, \ldots, x_n] = k[X_1, X_2, \ldots, X_n]/I,$$

where they X_i are independent variables and where I is a finitely generated ideal of polynomials in $k[X_1, \ldots, X_n]$.

Such an affine k-algebra defines a functor from the category of $k-$ algebras to the category of sets, namely $B \mapsto \mathrm{Hom}_k(A, B)$.

A structure of an *affine group scheme scheme* on A/k consists of the following data:

a) A k homomorphism $m : A \to A \otimes_k A$ (the comultiplication)
b) A k-valued point $e : A \to k$ (the identity element)
c) An inverse $inv : A \to A$,

which satisfy the following requirement: For any k-algebra B our homomorphism m induces a map

$${}^t m : \mathrm{Hom}_k(A \otimes_k A, B) = \mathrm{Hom}_k(A, B) \times \mathrm{Hom}_k(A, B) \to \mathrm{Hom}_k(A, B)$$

and we require that this induces a group structure on $\mathrm{Hom}_k(A, B)$. We also require that the k valued point e is the identity and that inv yields the inverse.

We leave it to the reader to figure out what this means for m, e, inv, especially what does associativity mean (Hint: Choose $B = A$).

An affine k-algebra A together with such a collection m, e, inv is called an *affine group scheme* $G/k = (A, m, e, inv)$. The k-algebra A is the coordinate ring, or the ring of *regular functions* of the group scheme. We will denote it by $A(G)$. The group of B/k valued points will be denoted by $G(B) = \mathrm{Hom}_k(A(G), B)$. For $g \in G(B)$ and $f \in A(G) \otimes B$ we write $g(f) = f(g)$, we evaluate the regular function at the point $g \in G(B)$.

The group $\mathbb{G}_m$ has the coordinate ring $A(\mathbb{G}_m) = k[t, t^{-1}], m(t) = t \otimes t, e(t) = 1, \mathrm{inv}(t) = t^{-1}$ and the coordinate ring of the additive group $\mathbb{G}_a$ is $A(\mathbb{G}_a) = k[x]$ and $m(x) = x \otimes 1 + 1 \otimes x, e(x) = 0, \mathrm{inv}(x) = -x$.

The group scheme Gl_n/k has the coordinate ring

$$A = k[\ldots, x_{i,j}, \ldots, y]/(\det(x_{i,j})y - 1);\ 1 \leq i, j, \leq n$$

and the comultiplication is given by

$$m(x_{i,j}) = \sum_{\nu=1}^{n} x_{i,\nu} \otimes x_{\nu,j} \tag{1.2}$$

Now we know what a homomorphism between affine group schemes is. This is a homomorphism ${}^t\phi$ between the affine algebras $A(H)$ and $A(G)$ which is compatible with the respective maps in a),b),c). A homomorphism $\phi : G \to H$ is surjective (resp. injective) if the homomorphism ${}^t\phi : A(H) \to A(G)$ is injective (resp.) surjective.

A *rational representation* of G/k is a homomorphism of group schemes $\rho : G/k \to \mathrm{Gl}_n/k$.

If for instance V/k is a vector space of dimension n then we can define the group scheme $\mathrm{Gl}(V)$, if we choose a k-basis on V, then we can identify $\mathrm{Gl}(V)/k = \mathrm{Gl}_n/k$. If G/k is any affine group scheme, we say that V/k is a G-module if we have a homomorphism $\rho : G/k \to \mathrm{Gl}(V)$. Hence we know that for any k -algebra B/k we have a homomorphism $\rho(B) : G(B) \to \mathrm{Gl}(V \otimes_k B)$. Of course this is functorial in B/k, i.e. a homomorphism $\psi : B/k \to B'/k$ induces a homomorphism $G(B) \to G(B')$.

We may also consider actions of G/k on vector spaces W/k which are not of finite dimension, here we require a certain finiteness condition. As before we have an action

$$\rho_B : G(B) \times (W \otimes B) \to W \otimes B \tag{1.3}$$

which is functorial in B/k. But now we assume in addition that for any $w \in W$ there is a finite set of elements $w_1, w_2, \ldots, w_d$ such that for any B/k and any $g \in G(B)$

$$\rho_B(g)w = \sum_{i=1}^{d} w_i \otimes b_i(g) \text{ with } b_i \in A(G).$$

It suffices to check this for the "universal" element $\mathrm{Id} \in \mathrm{Hom}_k(A(G), A(G)) = G(A(G))$, this means we have to find $w_1, w_2, \dots, w_d \in W$ such that

$$\rho_{A(G)}(\mathrm{Id})w = \sum_{i=1}^{d} h_i \otimes w_i \text{ with } h_i \in A(G).$$

This implies of course that the k-subspace $W' = \sum kw_i$ which is generated by these w_i is invariant under the action ρ and it contains w. Hence we see that our k-vector space W is a union of finite dimensional subspaces which are invariant under the action of G/k.

Therefore we say that a vector space W/k with an action of G/k is a G-module if it satisfies the above finiteness condition.

The ring of regular functions $A(G)$ is a $G \times_k G$ module: For $(g_1, g_2) \in G \times_k G(B) = G(B) \times G(B)$ the action and $f \in A(G), x \in G(B)$ the action is defined by

$$\rho(g_1, g_2)f(x) = f(g_1^{-1} x g_2).$$

We have to verify the finiteness condition. To do this we write a formula for $\rho(g_1, g_2)f \in A(G) \otimes B$. We have the comultiplication $m : A(G) \to A(G) \otimes_k A(G)$, we apply it to the first factor on the right hand side and get $m_{1,2} \circ m : A(G) \to A(G) \otimes_k A(G) \otimes_k A(G)$. Then

$$m_{1,2} \circ m(f) = \sum_{\mu} h'_\mu \otimes h_\mu \otimes h''_\mu$$

Then by definition

$$\rho(g_1, g_2)f = \sum_{\mu} h_\mu \otimes \mathrm{inv}(h'_\mu)(g_1) h''_\mu(g_2)$$

and this says that $\rho(g_1, g_2)f$ lies in the submodule generated by the h_μ.

Of course we may restrict the action to each the two factors, we simply choose $g_1 = e$,-we get the action by right translations- or we choose $g_2 = e$, this gives the action by left translations.

It is not difficult to show that for an affine group scheme we can find a collection of elements $e_0, e_1, \dots, e_r \in A(G)$ such that $e_i^2 = e_i \; \forall i, e_i e_j = 0 \; \forall i \neq j$ such that $1_A = \sum_i e_i$ and such that the subalgebras $A(G)e_i$ are integral. Then there is exactly one element (say e_0) such that $e(e_0) = 1$. Then $A(G)e_0$ is a subgroup scheme, it is called the *connected component of the identity* (See for instance [41], Chap. 7 , 7.2)

A group scheme G/k is *connected*, if its affine algebra $A(G) = A(G)e_0$ is integral, i.e. it does not have zero divisors.

Base change

If we have a field $L \supset k$ and a linear group G/k then the group $G/L = G \times_k L$ is the group over L where we forget that the coefficients of the equations are contained in k. The group $G \times_k L$ is the *base extension* from G/k to L.

Charmod

1.1.2 Tori, their character module,..

A special class of algebraic groups is given by the *tori*. We briefly recall the results of T. Ono [80].

An algebraic group T/k over a field k is called a *split torus* if it is isomorphic to a product of $\mathbb{G}_m/k$-s,

$$T/k \xrightarrow{\sim} \mathbb{G}_m^d.$$

The algebraic group T/k is called a *torus* if it becomes a split torus after a suitable finite extension of the ground field, i.e we have $T \times_k L \xrightarrow{\sim} \mathbb{G}_m^r/L$.

If we take an arbitrary separable finite field extension L/k we may consider the functor

$$R \to (L \otimes_k R)^\times.$$

It is not hard to see that this functor can be represented by an algebraic group over k, which is denoted by $R_{L/k}(\mathbb{G}_m/L)$ and called the *Weil restriction* of $\mathbb{G}_m/L$. We propose the notation

$$R_{L/k}(\mathbb{G}_m/L) = \mathbb{G}_m^{L/k} \tag{1.4}$$

The reader should try to prove that for a finite extension $\tilde{L}/L$ which is normal over $\mathbb{Q}$ we have

$$\mathbb{G}_m^{L/k} \times_k \tilde{L} \xrightarrow{\sim} (\mathbb{G}_m/\tilde{L})^{[L:k]}$$

and this shows that $\mathbb{G}_m^{L/k}$ is a torus .

A torus T/k is called *anisotropic* if is does not contain a non trivial split torus. Any torus C/k contains a maximal split torus S/k and a maximal anisotropic torus C_1/k. The multiplication induces a map

$$m : S \times C_1 \to C$$

this is a surjective (in the sense of algebraic groups) homomorphism whose kernel is a finite algebraic group. We call such map an *isogeny* and we write that $C = S \cdot C_1$, we say that C is the product of S and C_1 up to isogeny.

We give an example. Our torus $R_{L/\mathbb{Q}}(\mathbb{G}_m/L)$ contains $\mathbb{G}_m/k$ as a subtorus: For any ring R containing k we have $R^\times = \mathbb{G}_m(R) \subset (R \otimes L)^\times$. On the other and we have the norm map $N_{L/k} : (R \otimes L)^\times \to R^\times$ and the kernel defines a subgroup

$$R^{(1)}_{L/k}(\mathbb{G}_m/L) \subset R_{L/k}(\mathbb{G}_m/L)$$

and it is clear that

$$m : \mathbb{G}_m \times R^{(1)}_{L/k}(\mathbb{G}_m/L) \to R_{L/k}(\mathbb{G}_m/L)$$

has a finite kernel which is the finite algebraic group of $[L : k]$-th roots of unity.

For any torus T we define the character module as the group of homomorphisms

$$X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m).. \tag{1.5}$$

If the torus is split, i.e. $T = \mathbb{G}_m^r$ then $X^*(T) = \mathbb{Z}^r$ and the identification is given by $(n_1, n_2, \ldots, n_r) \mapsto \{(x_1, x_2, \ldots, x_r) \mapsto x_1^{n_1} x_2^{n_2} \ldots x_r^{n_r}\}$. We write the group structure on $X^*(T)$ additively, this means that $(\gamma_1 + \gamma_2)(x) = \gamma_(x)\gamma_2(x)$.

It is a theorem that for any torus T/k we can find a finite, separable, normal extension L/k such that $T \times_k L$ splits. Then it is easy to see that we have an action of the Galois group $\mathrm{Gal}(L/k)$ on $X^*(T \times_k L) = \mathbb{Z}^r$. If we have two tori $T_1/K, T_2/K$ which split over L

$$\mathrm{Hom}_k(T_1, T_2) \xrightarrow{\sim} \mathrm{Hom}_{\,\mathrm{Gal}(L/k)}(X^*(T_2 \times_k L), X^*(T_1 \times_k L)) \qquad (1.6)$$

To any $\mathrm{Gal}(L/k)-$ action on $\mathbb{Z}^n$ we can find a torus T/k which splits over L and which realises this action.

A homomorphism $\phi : T_1/k \to T_2/k$ is called an *isogeny* if $\dim(T_1) = \dim(T_2)$ and if ${}^t\phi : X^*(T_2) \to X^*(T_1)$ is injective. Then the kernel $\ker(\psi)$ is a finite group scheme of multiplicative type. If $Y \subset X^*(T_1)$ is a submodule of finite index the $Y = X^*(T_2)$ and the inclusion provides an isogeny $\psi : T_1 \to T_2$. The quotient $X^*(T_1)/Y$ is a finite constant group scheme and $\ker(\psi)$ the*dual* of his quotient.

)We also define the *cocharacter module* $\mathrm{Hom}(\mathbb{G}_m, T)$. If the torus $/k = \mathbb{G}_m^r$ then every cocharacter is the form $x \mapsto (x^{n_1}, x^{n_2}, \ldots, x^{n_r})$ It is clear that we have a pairing

$$< , >: X_*(T) \times X^*(T) \to \mathbb{Z} \text{ which is defined by } \gamma(\chi(t)) = t^{<\chi,\gamma>} \qquad (1.7)$$

1.1.3 Semi-simple groups, reductive groups,.

An important class of linear algebraic groups is formed by the *semisimple* and the *reductive* groups. (For a general reference [103].) We do not want to give the precise definition here. Roughly, a linear group is *reductive* if it is connected and if it does not contain a non trivial normal subgroup which is isomorphic to a product of groups of type G_a. A group is called semisimple, if it is reductive and does not contain a non trivial torus in its centre.

A semi-simple group G/k is simple, if it does not contain any normal subgroup of dimension > 0. Any semi-simple group is up to isogeny a product of simple groups. Any semi simple group $G/\mathbb{Q}$ contains a maximal torus $T/\mathbb{Q} \subset G/\mathbb{Q}$ such a maximal torus is equal to its own centraliser. A semi simple group is split if it contains a split maximal torus T_0/k, i.e. a maximal torus which is split. If $T/k \subset G/k$ is any (maximal) torus, then there is a finite extension $L/\mathbb{Q}$ such that $T \times_{\mathbb{Q}} L$ is split, and hence $G \times_{\mathbb{Q}} L$ is also split.

For example the groups Sl_n, Sp_n are (split) semi simple, the groups $\mathrm{SO}(f)$ are semi-simple provided $n \geq 3$. (See next subsection 1.1.5). The groups Gl_n and especially the multiplicative group $\mathrm{Gl}_1/\mathbb{Q} = \mathbb{G}_m/\mathbb{Q}$ are reductive. Any reductive group $G/\mathbb{Q}$ (or over any field of characteristic zero) has a central torus $C/\mathbb{Q}$ and this central torus contains a maximal split torus S. The derived group $G^{(1)}/\mathbb{Q}$ is semi simple and we get an isogeny

$$m : G^{(1)} \times C_1 \times S \to G$$

or briefly $G = G^{(1)} \cdot C_1 \cdot S$.

If for instance $G = R_{L/\mathbb{Q}}(\mathrm{Gl}_n/L)$ then $G^{(1)} = R_{L/\mathbb{Q}}(\mathrm{Sl}_n/L)$ and $C = R_{L/\mathbb{Q}}(\mathbb{G}_m/L)$ and this yields the product decomposition up to isogeny

$$G = G^{(1)} \cdot R^{(1)}_{L/\mathbb{Q}}(\mathbb{G}_m/L) \cdot \mathbb{G}_m. \tag{1.8}$$

For $\mathrm{Gl}_n/\mathbb{Q}$ the central torus is the group $\mathbb{G}_m/\mathbb{Q}$. The center of $\mathrm{Sl}_n/\mathbb{Q}$ is the finite group group scheme μ_n of of n-th roots of unity. The coordinate ring of μ_n is the finite algebra $A(\mu_n) = \mathbb{Q}[t]/(t^n - 1)$. Of course we may replace $\mathbb{Q}$ by any ring commutative ring R.

We can form the quotient group scheme

$$\mathrm{PGl}_n/\mathbb{Q} = (\mathrm{Gl}_n/\mathbb{G}_m)/\mathbb{Q} \xrightarrow{\sim} (\mathrm{Sl}_n/\mathbb{Q})/\mu_n \tag{1.9}$$

this is also the adjoint group of $\mathrm{Gl}_n/\mathbb{Q}$ and $\mathrm{Sl}_n/\mathbb{Q}$, i.e.

$$\mathrm{Ad}(\mathrm{Gl}_n) = \mathrm{PGl}_n = \mathrm{Gl}_n/\mathbb{G}_m = \mathrm{Sl}_n/\mu_n. \tag{1.10}$$

We could certainly drop the assumption that a reductive group should be connected, we could simple say that $G/\mathbb{Q}$ is reductive (semi-simple...) if its connected component of the identity is reductive (semi-simple...).

Another important class of semi simple groups is given by the *quasisplit* groups (see also section 1.1.7. A group $G/\mathbb{Q}$ is called quasisplit if it contains a Borel subgroup $B/\mathbb{Q} \subset G/\mathbb{Q}$. A Borel subgroup $B/\mathbb{Q}$ is a maximal solvable subgroup, it contains a maximal torus $T/\mathbb{Q} \subset B/\mathbb{Q}$, this torus is also a maximal torus in $G/\mathbb{Q}$. Then $B = U \rtimes T$ is the semidirect product of this torus and the *unipotent radical* $U/\mathbb{Q}$. We discuss a special example which is of great relevance for our subject.

Let $L/\mathbb{Q}$ be a quadratic extension, let us denote the non trivial automorphism by $a \mapsto \bar{a}$. Let V/L be a finite dimensional vector space together with a hermitian form $h : V \times_L V \to L$, i.e.

$$h(v,w) = \overline{h(w,v)};\ h(\lambda u + \mu v, w) = \lambda h(u,w) + \mu h(v,w)\ \forall u,v,w \in V, \lambda,\mu \in L.$$

Furthermore we assume that h is non degenerate, i.e. for any $v \in V, v \neq 0$ we find a $w \in V$ such that $h(v,w) \neq 0$. Then we can define the group $\mathrm{SU}(h)/\mathbb{Q}$: For any commutative $\mathbb{Q}$ -algebra R we define

$$\mathrm{SU}(h)(R) = \{g \in \mathrm{Sl}(V \otimes_{\mathbb{Q}} R \mid h(gv,gw) = h(v,w) \text{ and } \det(g) = 1\}. \tag{1.11}$$

Then $\mathrm{SU}(h)/\mathbb{Q}$ is a semi simple group over $\mathbb{Q}$. We can also define the unitary group $\mathrm{U}(h)/\mathbb{Q}$ where we drop the condition that the determinant is one and the group of hermitian similitudes $\mathrm{GU}(h)$ where

$$\mathrm{GU}(h)(R) = \{g \in \mathrm{Gl}(V \otimes_{\mathbb{Q}} R \mid h(gv,gw) = d(g)h(v,w)\ \forall v,w \in V \otimes_{\mathbb{Q}} R\}, \tag{1.12}$$

here $d : \mathrm{GU}(h) \to R_{L/\mathbb{Q}}(\mathbb{G}_m)$ is a homomorphism, the kernel of d is the group $\mathrm{U}(h)$.

We consider the special case where

$$V = Le_1 \oplus \cdots \oplus Le_n \oplus (Le_0) \oplus Lf_n \oplus \cdots \oplus Lf_1$$

the summand Le_0 is left out if $\dim_L V$ is even. The hermitian scalar product is given by

$$h_1(e_i, f_i) = h_1(f_i, e_i) = 1 \; \forall i = 1, \dots, n, \;\; (h_1(e_0, e_0) = 1)$$

and all other scalar products equal to zero. Then $\mathrm{SU}(h_1)$ is a quasi split semi simple group over $\mathbb{Q}$: The elements $t \in \mathrm{Gl}(V)$ for which

$$t = \{t : e_i \mapsto t_i e_i; t : f_i \mapsto \bar{t_i}^{-1};\; (t : e_0 \mapsto t_0 e_0 \text{ with } t_0 \bar{t_0} = 1)\}$$

are the $\mathbb{Q}$-valued points of a maximal torus $T_1/\mathbb{Q} \subset \mathrm{SU}(h_1)$. The vector space V/L comes with a natural flag

$$\mathcal{F} := \{0\} \subset Le_1 \subset \cdots \subset \oplus Le_1 \oplus \cdots \oplus Le_n \subset (Le_1 \oplus \dots Le_n + Le_0) \subset \\ (Le_1 \oplus \dots Le_n \oplus Le_0 \oplus Lf_n) \subset \dots (Le_1 \oplus \cdots \oplus Le_n \oplus Le_0 \oplus Lf_n \oplus \cdots \oplus Lf_2) \subset V. \tag{1.13}$$

Now the subgroup $B_1/\mathbb{Q} \subset \mathrm{SU}(h_1)/\mathbb{Q}$ which fixes $\mathcal{F}$ is a maximal solvable subgroup in $\mathrm{SU}(h_1)$.

1.1.4 The Lie-algebra

We need some basic facts about the Lie-algebras of algebraic groups.

For any algebraic group G/k we can consider its group of points with values in $k[\epsilon] = k[X]/(X^2)$. We have the homomorphism $k[\epsilon] \to k$ sending ϵ to zero and hence we get an exact sequence

$$0 \to \mathfrak{g} \to G(k[\epsilon]) \to G(k) \to 1.$$

The kernel $\mathfrak{g}$ is a k-vector space, if the characteristic of k is zero, then its dimension is equal to the dimension of G/k. It is denoted by $\mathfrak{g} = \mathrm{Lie}(G)$.

Let us consider the example of the group $G = SO(f)$, where $f : V \times V \to k$ is a non degenerate symmetric bilinear form. In this case an element in $G(k[\epsilon])$ is of the form $\mathrm{Id} + \epsilon A, A \in \mathrm{End}(V)$ for which

$$f((\mathrm{Id} + \epsilon A)v, (\mathrm{Id} + \epsilon A)w) = f(v, w)$$

for all $v, w \in V$. Taking into account that $\epsilon^2 = 0$ we get

$$\epsilon(f(Av, w) + f(v, Aw)) = 0,$$

i.e. A is skew with respect to the form, and $\mathfrak{g}$ is the k-vector space of skew endomorphisms. If we give V a basis and if $f = \sum x_i^2$ with respect to this basis then this means thematrix of A is skew symmetric.

If we consider $G = \mathrm{Gl}_n/k$ then $\mathfrak{g} = M_n(k)$, the Lie-bracket is given by

$$(A, B) \mapsto AB - BA \tag{1.14}$$

We have some kind of a standard basis for our Lie algebra

$$\mathfrak{g} = \bigoplus_{i=1}^{n} kH_i \oplus \bigoplus_{i,j,i\neq j} kE_{i,j} \tag{1.15}$$

where H_i (resp.$E_{i,j}$) are the matrices

$$H_i = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \ddots & & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ resp. } E_{i,j} = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \ddots & & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and the only non zero entries (=1) is at (i,i) on the diagonal (resp. and (i,j) off the diagonal.)

For the group Sl_n/k the Lie-algebra is $\mathfrak{g}^{(0)} = \{A \in M_n(k) | \operatorname{tr}(A) = 0\}$ and again we have a standard basis

$$\mathfrak{g}^{(0)} = \bigoplus_{i=1}^{n-1} k(H_i - H_{i+1}) \oplus \bigoplus_{i,j,i\neq j} kE_{i,j} \tag{1.16}$$

If $\rho : G \to \mathrm{Gl}(V)$ is a rational representation of our group G/k then it is clear from our considerations above that we have a "derivative" of this representation

drho

$$d\rho : \mathfrak{g} = \mathrm{Lie}(G/k) \to \mathrm{Lie}(\mathrm{Gl}(V)) = \mathrm{End}(V) \tag{1.17}$$

this is k-linear.

Every group scheme G/k has a very special representation, this is the *Adjoint representation*. We observe that the group acts on itself by conjugation, this is the morphism

$$Inn : G \times_k G \to G$$

which on R valued points is given by

$$Inn(g_1, g_2) \mapsto g_1 g_2 (g_1)^{-1}.$$

This action clearly induces a representation

$$\mathrm{Ad} : G/k \to \mathrm{Gl}(\mathfrak{g})$$

and this is the adjoint representation. This adjoint representation has a derivative and this is a homomorphism of k vector spaces

$$D_{\mathrm{Ad}} = \mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}).$$

We introduce the notation: For $T_1, T_2 \in \mathfrak{g}$ we put

$$[T_1, T_2] := \mathrm{ad}(T_1)(T_2).$$

Now we can state the famous and fundamental result

Theorem 1.1.1. *The map $(T_1, T_2) \mapsto [T_1, T_2]$ is bilinear and antisymmetric. It induces the structure of a Lie-algebra on $\mathfrak{g}$, i.e. we have the Jacobi identity*

$$[T_1, [T_2, T_3]] + [T_2, [T_3, T_1]] + [T_3, [T_1, T_2]] = 0.$$

We do not prove this here. In the case $G/k = \mathrm{Gl}(V)$ and $T_1, T_2 \in \mathrm{Lie}(\mathrm{Gl}(V)) = \mathrm{End}(V)$ we have $[T_1, T_2] = T_1T_2 - T_2T_1$ and in this case the Jacobi Identity is a well known identity.

On any Lie algebra we have a symmetric bilinear form (the *Killing form*)

$$B : \mathfrak{g} \times \mathfrak{g} \to k \tag{1.18}$$

which is defined by the rule

$$B(T_1, T_2) = \mathrm{trace}(\mathrm{ad}(T_1) \circ \mathrm{ad}(T_2))$$

A simple computation shows that for the examples $\mathfrak{g} = \mathrm{Lie}(\mathrm{Gl}_n)$ and $\mathfrak{g}^{(0)} = \mathrm{Lie}(\mathrm{Sl}_n)$ we have

$$B(T_1, T_2) = 2n\ \mathrm{tr}(T_1T_2) - 2\ \mathrm{tr}(T_1)\ \mathrm{tr}(T_2) \tag{1.19}$$

we observe that in case that one of the T_i is central, i.e.$= u\mathrm{Id}$ we have $B(T_1, T_2) = 0$. In the case of $\mathfrak{g}^{(0)}$ the second term is zero.

It is well known that a linear algebraic group is semi-simple if and only if the Killing form B on its Lie algebra is non degenerate.
Ref ????????
class

1.1.5 The classical groups and their realisation as split semi-simple group schemes over Spec($\mathbb{Z}$)

We will not discuss the general notion of a semi-simple group scheme over a base S, instead we will discuss the examples of classical groups and explain the main structure theorems in examples.

The group scheme $\mathrm{Sl}_n/\ \mathrm{Spec}(\mathbb{Z})$

We consider a free module M of rang n over $\mathrm{Spec}(\mathbb{Z})$. We define the group scheme $\mathrm{Sl}(M)/\ \mathrm{Spec}(\mathbb{Z})$: for any $\mathbb{Z}$ algebra R we have $\mathrm{Sl}(M)(R) = \mathrm{Sl}(M \otimes_{\mathbb{Z}} R)$.
This is clearly a semi simple group scheme over $\mathrm{Spec}(\mathbb{Z})$ because :

a) The group scheme is smooth over $\mathrm{Spec}(\mathbb{Z})$
b) For any field k -which is of course a $\mathbb{Z}$-algebra we have

$$\mathrm{Sl}(M) \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(k) = \mathrm{Sl}(M \otimes_{\mathbb{Z}} k)/\ \mathrm{Spec}(k)$$

and for any k this group scheme does not contain a normal subgroup scheme, which is isomorphic to $G_a^r/\ \mathrm{Spec}(k)$ (hence it is reductive) and its center is a finite group scheme.

If we fix a basis $e_1, e_2, \ldots, e_n$ then we get a split maximal torus $T/\ \mathrm{Spec}(\mathbb{Z})$ this is the sub group scheme which fixes the lines $\mathbb{Z}e_i$, with respect to this basis we have

$$T(R) = \{ \begin{pmatrix} t_1 & 0 & \dots & 0 \\ 0 & t_2 & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & t_n \end{pmatrix} \quad | \quad t_i \in R^\times, \prod_i t_i = 1 \tag{1.20}$$

With respect to this torus $T/\operatorname{Spec}(\mathbb{Z})$ we define root subgroups. This are smooth subgroup schemes $U \subset G$ which are isomorphic to the additive group scheme $G_a/\operatorname{Spec}(\mathbb{Z})$ and which are normalized by T. It is clear that these root subgroups are given by

$$\tau_{ij} : G_a \to \mathrm{Sl}(M) \tag{1.21}$$

$$\tau_{ij} : x \to \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & x & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.22}$$

where the entry x is placed in the i-th row and j-th collumn. Let us denote the image by $U_{\alpha_{ij}}$.

Then we get the relation

$$t\tau_{ij}(x)t^{-1} = \tau_{ij}((t_i/t_j)x)$$

(If I write such a relation then I always mean that $t, x..$ are elements in $T(R), G_a(R)...$ for some unspecified $\mathbb{Z}-$ algebra R.)

The root system

The characters

$$\alpha_{ij} : T \to G_m$$

$$\alpha_{ij} : \begin{pmatrix} t_1 & 0 & \dots & 0 \\ 0 & t_2 & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & t_n \end{pmatrix} \to t_i/t_j$$

are form the set Δ of roots in the character module of the torus. We may select a subset of positive roots

$$\Delta^+ = \{\alpha_{ij} \mid i < j\}.$$

Then the torus T and the $U_{\alpha_{ij}}$ with $\alpha_{ij} \subset \Delta^+$ stabilize the flag

$$\mathcal{F} = (0) \subset \mathbb{Z}e_1 \subset \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \subset \dots \subset M.$$

The stabilizer of the flag is a smooth sub group scheme $B/\operatorname{Spec}(\mathbb{Z})$. It is so-but not entirely obvious- that B is a maximal solvable sub group scheme. These maximal subgroup schemes are called Borel subgroups.

It is clear that the morphism

$$T \times \prod_{\alpha_{ij}, i<j} U_{\alpha_{ij}} \to B,$$

which is induced by the multiplication is an isomorphism of schemes.

The set Δ^+ of positive roots contains the subset $\pi \subset \Delta$ of simple roots $\alpha_i := t_i/t_{i+1}$. Every positive root can be written as a sum of simple roots with positive coefficients.

We consider the normaliser $N(T) \subset \mathrm{Sl}_n$, it acts by permutations on the set of submodules $\mathbb{Z}e_i$. The quotient $N(T)/T = W$ is the Weyl group, in this case it is isomorphic to the symmetric group S_n. It is easy to see that we have a positive definite, symmetric, W invariant bilinear form on $X^*(T)$ which is given by

$$< \alpha_i, \alpha_i >= 2; < \alpha_i, \alpha_{i+1} > -1, \text{ and } < \alpha_i, \alpha_j >= 0 \text{ if } |i-j| > 1 \tag{1.23}$$

All these data about the set of roots and simple roots are encoded in the Dynkin diagram

$$A_{n-1} := \alpha_1 \quad - \quad \alpha_2 \quad - \cdots - \quad \alpha_{n-1} \tag{1.24}$$

The flag variety

It is not so difficult to see that the flags form a projective scheme $\mathrm{Gr}/\operatorname{Spec}(\mathbb{Z})$. From this it follows: For any Dedekind ring A and its quotient field K we have

$$\mathrm{Gr}(K) = \mathrm{Gr}(A).$$

If A is even a discrete valuation ring then we can show easily that the group $\mathrm{Sl}_n(A)$ acts transitively on $\mathrm{Gr}(A)$.

The whole point is, that results of this type are true for arbitrary split semi simple groups $\mathcal{G}/\operatorname{Spec}(\mathbb{Z})$. This is not so easy to explain and also much more difficult to prove. But we have the series of so called classical groups and for those these results are again easy to see. (The main problem in the general approach is that we have to start from an abstract definition of a semi simple group and not from a group which is given to us in a rather explicit way like Sl_n or the classical groups)

The group scheme $\mathrm{Sp}_g/$ Spec($\mathbb{Z}$)

Now we choose again a free $\mathbb{Z}$ module M but we assume that we have a non degenerate alternating pairing

$$< \, , \, >: M \times M \to \mathbb{Z}$$

where non degenerate means: If $x \in M$ and $< x, M > \subset a\mathbb{Z}$ with some integer $a > 1$, then $x = ay$ with $y \in M$. It is well known and also very easy to prove that M is of even rank $2g$ and that we can find a basis

$$\{e_1, \dots, e_g, f_g, \dots, f_1\}$$

such that $< e_i, f_i >= - < f_i, e_i >= 1$ and all other values of the pairing on basis elements are zero.

The automorphism group scheme of $\mathbb{G} = \mathrm{Aut}((M, < \ . \ >))$ is the symplectic group $\mathrm{Sp}_g / \operatorname{Spec}(\mathbb{Z})$. Again it is easy to find out how a maximal torus must look like. With respect to our basis we can take

$$T = \{ \begin{pmatrix} t_1 & 0 & & & \dots & 0 \\ 0 & \ddots & & & \dots & 0 \\ 0 & 0 & t_g & & & 0 \\ 0 & 0 & 0 & t_g^{-1} & \dots & \\ 0 & & & & \ddots & 0 \\ & 0 & & & & t_1^{-1} \end{pmatrix} \} \tag{1.25}$$

We can say that the torus is the stabilser of the ordered collection of rank 2 submodules $\mathbb{Z}e_i, \mathbb{Z}f_i$. We can define a Borel subgroup $B/\mathbb{Z}$ which is the stabilizer of the flag

$$\mathcal{F} = (0) \subset \mathbb{Z}e_1 \subset \dots \subset \mathbb{Z}e_1 \dots \oplus \dots \mathbb{Z}e_g \subset \mathbb{Z}e_1 \dots \oplus \dots \mathbb{Z}e_g \oplus \mathbb{Z}f_g \subset \dots \subset M$$

(A flag starts with isotropic subspaces until we reach half the rank of the module. But then this lower part of the flag determines the upper half, because it is given by the orthogonal complements of the members in the lower half).

Again we can define the root subgroups (with respect to T)

$$rootsubgroup\tau_\alpha : G_a \xrightarrow{\sim} U_\alpha \subset \mathcal{G} \tag{1.26}$$

which are normalized by T. As before we have the relation

$$\tau(x)t^{-1} = \tau(\alpha(t)x), \tag{1.27}$$

where $\alpha \in \Delta \subset X^*(T)$.

Now it is not quite so easy to write down what these root subgroups are, we write down the simple positive roots in the thecase $g = 2$: We have the maximal torus

$$T = \{ \begin{pmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 & t_1^{-1} \end{pmatrix} \}$$

and we want to find one-parameter subgroups $U_\alpha \subset \mathbb{G}$ which stabilize the flag.

The one parameter subgroups corresponding to the simple roots are

$$\tau_{\alpha_1} : x \mapsto \{e_1 \mapsto e_1, e_2 \mapsto e_2 + xe_1, f_2 \mapsto f_2, f_1 \mapsto f_1 - xf_2\}$$

$$\tau_{\alpha_2} : y \mapsto \{e_1 \mapsto e_1, e_2 \mapsto e_2, f_2 \mapsto f_2 + ye_2, f_1 \mapsto f_1\}$$

where $\alpha_1(t) = t_1/t_2, \alpha_2(t) = t_2^2$. The unipotent radical is then

$$\{\begin{pmatrix} 1 & x & v & u \\ 0 & 1 & y & v - xy \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 \end{pmatrix}\}$$

From here it is not difficult to see that for all values of g the simple roots are $\alpha_i(t) = t_i/t_{i+1}$ with $1 \leq i < g$ and $a_g(t) = t_g^2$. Again we define the Weyl group W as above, We have a W invariant positive definite, symmetric bilinear form on $X^*(T)$ and for this form we have

$$\begin{gathered} < \alpha_i, \alpha_i >= 2 \text{ for } i < g \text{ and } < \alpha_g, \alpha_g >= 4, \\ < \alpha_i, \alpha_{i+1} >= -1 \text{ if } i < g - 1 \text{ and } < \alpha_{g-1}, \alpha_g >= -2 \end{gathered} \tag{1.28}$$

and all other values of the pairing between simple roots are zero.

Agin these data are encoded in the Dynkin diagram

$$C_n := \alpha_1 \quad - \quad \alpha_2 \quad - \cdots - <= \quad \alpha_g$$

See [**?**]. We will see this Dynkin diagram for $g = 3$ at the end of this book.

As before it is not so difficult to show that the flags form a smooth projective scheme $\mathcal{X}/\operatorname{Spec}(\mathbb{Z})$ (see also [book], V.2.4.3). Show that for any discrete valuation ring A the group $\mathbb{G}(A)$ acts transitively on $\mathcal{X}(A) = \mathcal{X}(K)$. It is also easy to verify the statements in 1.1.

The group scheme $\mathrm{SO}(n, n)/$ Spec($\mathbb{Z}$)

We can play the same game with symmetric forms. Let M together with its basis as above, we replace g by n. But now we take the quadratic form F

$$F : M \to \mathbb{Z}$$

which is defined by

$$F(x_1 e_1 \cdots + x_n e_n + y_n f_n + \cdots + y_1 f_1) = \sum x_i y_i$$

and all other values of the pairing on basis elements are zero. We define the group scheme of isomorphisms but in addition we require thedeterminant is one. Hence

$$\mathrm{SO}(n, n)/\operatorname{Spec}(\mathbb{Z}) = \operatorname{Aut}(M, F, \det = 1).$$

The maximal torus and the flags look pretty much the same as in the previous case. But the set of roots looks different. For $n = 2$ the torus and the unipotent radical are given by

$$T = \{\begin{pmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 & t_1^{-1} \end{pmatrix}\}, \; U = \{\begin{pmatrix} 1 & x & y & -xy \\ 0 & 1 & 0 & -y \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 \end{pmatrix}\}.$$

The system of positive roots consists of two roots $\alpha_1(t) = t_1/t_2, \alpha_2(t) = t_1t_2$. This is the Dynkin diagram $A_1 \times A_1$ hence there exists a homomorphism (isogeny) between group schemes over $\mathrm{Spec}(\mathbb{Z})$:

$$\mathrm{Sl}_2 \times \mathrm{Sl}_2 \to \mathrm{SO}(2,2).$$

It is an amusing exercise to write down this isogeny.

Another even more interesting excercise is the computation of the roots for the torus (here $n = 3$)

$$T = \{ \begin{pmatrix} t_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & t_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & t_3^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & t_1^{-1} \end{pmatrix} \}. \tag{1.29}$$

In this case we have the root subgroups

$$\tau_{\alpha_1} : x \mapsto \begin{pmatrix} 1 & x & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tau_{\alpha_2} : x \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & x & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -x & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\tau_{\alpha_3} : x \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & x & 0 & 0 \\ 0 & 0 & 1 & 0 & -x & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where

$$\alpha_1(t) = t_1/t_2,\ \alpha_2(t) = t_2/t_3,\ \alpha_3(t) = t_2t_3$$

Use the result of this computation to show that we have an isogeny

$$\mathrm{Sl}_4 \to \mathrm{SO}(3,3).$$

How can we give a linear algebra interpretation of this isogenies?

If we now consider the maximal torus (1.25) and put (1.29) into the middle then we see that the simple roots are

$$\alpha_i(t) = t_i/t_{i+1} \text{ for } i = 1, \dots n-1, \text{ and } \alpha_n(t) = t_{n-1}t_n \tag{1.30}$$

which gives us the Dynkin diagram (wird noch korrigiert!)

$$D_n := \alpha_1 \quad - \quad \alpha_2 \quad - \cdots - \alpha_{n-2} \begin{matrix} \alpha_{n-1} \\ \\ \alpha_n \end{matrix} \tag{1.31}$$

The group scheme $\mathrm{SO}(n+1,n)/$ **Spec**$(\mathbb{Z})$

Of course we can also consider quadratic forms in an odd number of variables. We take a free $\mathbb{Z}$-module of rank $2n+1$ with a basis

$$\{e_1, \dots, e_n, h, f_n, \dots, f_1\}.$$

On this module we consider the quadratic form

$$F : M \to \mathbb{Z}$$

$$F(\sum x_i e_i + zh + \sum y_i f_i) = \sum x_i y_i + z^2.$$

From this quadratic fom we get the bilinear form

$$B(u,v) = F(u+v) - F(u) - F(v).$$

We have the relation

$$F(u) = 2B(u,u),$$

hence we can reconstruct the quadratic form from the bilinear form if we extend $\mathbb{Z}$ to a larger ring where 2 is invertible.

We consider the automorphism scheme

$$\mathcal{G}/\operatorname{Spec}(\mathbb{Z}) = \mathrm{SO}(n+1,n)/\operatorname{Spec}(\mathbb{Z}) = \mathrm{Aut}(M, F, det = 1)/\operatorname{Spec}(\mathbb{Z})$$

and I claim that this is indeed a semi simple group scheme over $\operatorname{Spec}(\mathbb{Z})$. To see this I strongly recommend to discuss the case $n = 1$.

We have of course the maximal torus

$$T = \{ \begin{pmatrix} t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t^{-1} \end{pmatrix} \}.$$

It is also the stabiliser of the collection of three subspaces $\mathbb{Z}e, \mathbb{Z}h, \mathbb{Z}f$, here we use the determinant condition.

Now one has to discuss the root subgroups with respect to this torus.

From this we can derive that we have an isogeny

$$\mathrm{Sl}_2 \to \mathrm{SO}(2,1)$$

It is also interesting to look at the case $n = 2$. In this case we can compare the root systems of Sp_2 and $\mathrm{SO}(3,2)$ they are isomorphic. Now it is a general theorem in the theory of split semi simple group schemes that the root system determines the group scheme up to isogeny. Hence weshould be able to construct an isogeny between Sp_2 and $\mathrm{SO}(3,2)$. Who cthean do it?

For an arbitrary value of n we get the Dynkin diagram

$$B_n := \alpha_1 \quad - \quad \alpha_2 \quad - \cdots - => \quad \alpha_n \tag{1.32}$$

The element w_0.

Finally we have a short look at the automorphism groups of the Dynkin diagrams.

For the Dynkin diagram of type A the group of automorphisms is trivial if $n = 1$ and for $n > 1$ the group of automorphism is $\mathbb{Z}/2\mathbb{Z}$, the non trivial element ϵ exchanges the roots α_i and α_{n-1-i}.

For the diagrams B_n, C_n the automorphism group is trivial,

For the Dynkin diagram D_n and $n > 4$ the group of automorphisms is $\mathbb{Z}/2\mathbb{Z}$ and the non trivial automorphism ϵ fixes the simple roots α_i with $1 \leq i \leq n-2$ and interchanges α_{n-1}, α_n.

For $n = 4$ the automorphism group is the symmetric group S_3 and it acts by permutations on the three simple roots $\alpha_1, \alpha_3, \alpha_4$.

The Weyl group $W = N(T)/T$ acts simply transitively on the set of Borel subgroups $B' \supset T$. Hence there is a unique element $w_0 \in W$ which sends our Borel subgroup B into its opposite B^- this is the group whose simple roots are the roots $-\alpha_i$.

If the automorphism group of the Dynkin diagram is trivial we have $w_0 = -1$, i.e. it ats by multiplication by -1 on $X^*(T)$.

For the diagram A_n and $n > 1$ the element $w_0 = -\epsilon$ and therefore not equal to -1.

For the diagram D_n the element $w_0 = -1$ if n is even and equal to $-\epsilon$ if n is odd.

The element w_0 will play an important role later in this book.

The dominant fundamental weights

The Weyl group $W = N(T)/T$ acts by conjugation on the character module $X^*(T)$ and there a positive definite symmetric bilinear form $< \, , \, >: X^*(T) \times X^*(T) \to \mathbb{Q}$ which is invariant invariant under this action.. The Weyl group is generated by the reflections

$$s_{\alpha_i} : \gamma \mapsto \gamma - \frac{2 < \gamma, \alpha_i >}{< \alpha_i, \alpha_i >} \alpha_i \tag{1.33}$$

and this implies that $\frac{2<\gamma,\alpha_i>}{<\alpha_i,\alpha_i>} \in \mathbb{Z}$. Of course we have $\alpha_i \in X^*(T)$ for all simple roots, the sublattice $\bigoplus \mathbb{Z}\alpha_i$ it is of finite index in $X^*(T)$. To this sublattice belongs a torus T^{ad} and an isogeny $\psi : T \to T^{(\text{ad})}$. The kernel $\ker(\psi) = \mu$ is the centre of our group scheme $G/\mathbb{Z}$ and the quotient $G/\mu = G^{(\text{ad})}$ is the *adjoint* group.

In $X^*(T) \otimes \mathbb{Q}$ we have the elements γ_i which are defined by

$$\frac{2 < \gamma_i, \alpha_i >}{< \alpha_j, \alpha_j >} = \delta_{i,j} \tag{1.34}$$

these elements are dominant fundamental weights. The lattice $\bigoplus \mathbb{Z}\gamma_i$ contains $X^*(T)$ as a sublattice of finite index. It provides a torus $T^{(\text{sc})}$ and an isogeny

$\psi_1 : T^{(\mathrm{sc})} \to T$. This torus is the maximal torus in a semi simple group scheme $G^{((\mathrm{sc}))}/\mathbb{Z}$ which admits an isogeny

$$\psi_1 : G^{(\mathrm{sc})} \to G \tag{1.35}$$

whose kernel is $\ker(\psi_1) \subset T^{(\mathrm{sc})}$. The group $G^{(\mathrm{sc})}/\mathbb{Z}$ is the *simply connected cover* of $G/\operatorname{Spec}(\mathbb{Z})$.

The abstract group $G^{(\mathrm{sc})}(k)$

We want to show that the abstract group $G^{(\mathrm{sc})}(k)$ is generated by the groups $U_\alpha(k)$.

For any root α we can consider the two root subgroups $U_\alpha, U_{-\alpha}$. It is easy to see -at least in our examples above - that these root subgroups generate a subgroup $H_\alpha \subset G$, this is the smallest subgroup which contains $U_\alpha, U_{-\alpha}$. This subgroup is either PSl_2 or Sl_2. Then $T^{(\alpha)} = H_\alpha \cap T$ is a maximal torus in H_α.

If our group $G = G^{((\mathrm{sc})}$ is simply connected then $H_\alpha = \mathrm{Sl}_2$ and we define define the coroot $\alpha^\vee \in X_*(T^{(\mathrm{sc})})$ by $\alpha^\vee : \mathbb{G}_m \to T^{(\alpha)}$ and $< \alpha^\vee, \alpha >= 2$. We have the relation $< \alpha^\vee, \gamma_j >= \delta_{i,j}$ and this implies that the $\alpha_i^\vee$ form a basis of $X_*(T^{(\mathrm{sc})})$. This in turn implies that the map given by multiplication

$$m : \prod_i \alpha_i^\vee(\mathbb{G}_m) \xrightarrow{\sim} T^{(\mathrm{sc})}. \tag{1.36}$$

is an isomorphism.

Now it is easy to see that for any field k the abstract group $\mathrm{Sl}_2(k)$ is generated by the two root subgroups $U_\alpha(k), U_{-\alpha}(k)$. Combined with the observation above this implies that $T^{(\mathrm{sc})}(k)$ is contained in the subgroup which is generated by the subgroups $U_{\alpha_i}(k), U_{-\alpha_i}(k)$. Now we recall the Bruhat decomposition. The unipotent radical U_+ of B is equal to the product $U_+ = \prod_{\alpha\in\Delta^+} U_\alpha$ and the same holds for $U_= \prod_{\alpha\in\Delta^-} U_\alpha$. The Bruhat decomposition tells us that the multiplication $m : U_- \times T^{(\mathrm{sc})} \times U_+ \to G$ provides an isomorphism of the left hand side with an Zariski-open $\mathcal{V} \subset G$, (this is the Big Cell). This means that we get a bijection

$$U_-(k) \times T^{(\mathrm{sc})}(k) \times U_+(k) \xrightarrow{\sim} \mathcal{V}(k). \tag{1.37}$$

Our previous arguments imply that $\mathcal{V}(k)$ lies in the subgroup generated by the $U_\alpha(k)$. But then it is clear that $G^{(\mathrm{sc})}(k)$ is generated by the $U_\alpha(k)$.

1.1.6 k-forms of algebraic groups

For the following concepts and results on Galois cohomology we also refer to [83] and [96].

Exercise: 1) *Consider the following two quadratic forms over $\mathbb{Q}$:*

$$f(x,y,z) = x^2 + y^2 - z^2\ ,\ f_1(x,y,z) = x^2 + y^2 - 3z^2.$$

Prove that the first form is isotropic. This means there exists a vector $(a,b,c) \in \mathbb{Q}^3 \setminus \{0\}$ with

$$f(a,b,c) = 0.$$

Show that the second form is anisotropic, i.e. it has no such vector.

2) Prove that the two linear algebraic group $G/\mathbb{Q} = \mathrm{SO}(f)/\mathbb{Q}$ *and* $G_1/\mathbb{Q} = \mathrm{SO}(f_1)/\mathbb{Q}$ *cannot be isomorphic. (Hint: This is not so easy since we did not define when two groups are isomorphic.)*

Here is some advice: In general we call an element $e \neq u \in G(\mathbb{Q})$ *unipotent if it is unipotent in* $\mathrm{Gl}_n(\mathbb{Q})$ *where we consider* $G/\mathbb{Q} \hookrightarrow \mathrm{Gl}_n/\mathbb{Q}$*. It turns out that this notion of unipotence does not depend on the embedding.*

Now it is possible to show that our first group $G(\mathbb{Q}) = \mathrm{SO}(f)(\mathbb{Q})$ *has unipotent elements, and* $G_1(\mathbb{Q})$ *does not. Hence these two groups cannot be isomorphic.*

3) Prove that the two algebraic groups $G \times_{\mathbb{Q}} \mathbb{R}$ *and* $G_1 \times_{\mathbb{Q}} \mathbb{R}$ *are isomorphic, and therefore the two groups* $G(\mathbb{R})$ *and* $G_1(\mathbb{R})$ *are isomorphic.*

In this example we see, that we may have two groups $G/k, G_1/k$ which are not isomorphic but which become isomorphic over some extension L/k. Then we say that the groups are *k-forms* of each other. To determine the different forms of a given group G/k is sometimes difficult one has to use the concepts of Galois cohomology. For a separable normal extension L/k we have the almost tautological description

$$G(k) = \{g \in G(L) | \sigma(g) = g \text{ for all elements in the Galois group } \quad \mathrm{Gal}(L/k)\}.$$

Now let we can consider the functor $\mathrm{Aut}(G)$: It attaches to any field extension L/k the group of automorphisms $\mathrm{Aut}(G)(L)$ of the algebraic group $G \times_k L$. We denote this action by $g \mapsto \sigma(g) = g^\sigma$. Note that this notation gives us the rule $g^{(\sigma\tau)} = (g^\tau)^\sigma$. A 1-cocycle of $\mathrm{Gal}(L/k)$ with values in $\mathrm{Aut}(G)$ is a map $c : \sigma \mapsto c_\sigma \in \mathrm{Aut}(G)(L)$ which satisfies the cocycle rule

$$c_{\sigma\tau} = c_\sigma c_\tau^\sigma \tag{1.38}$$

Now we define a new action of $\mathrm{Gal}(L/k)$ on $G(L)$: An element σ acts by

$$g \mapsto c_\sigma g^\sigma c_\sigma^{-1}$$

We define a new algebraic group G_1/k: For any extension E/k we have an action of $\mathrm{Gal}(L/k)$ on $E \otimes_k L$ and we put

$$G_1(E) = \{g \in G(E \otimes_k L) | g = c_\sigma g^\sigma c_\sigma^{-1}\} \tag{1.39}$$

For the trivial cocycle $\sigma \mapsto 1$ this gives us back the original group.

It is plausible and in fact not very difficult to show that $E \to G_1(E)$ is in fact represented by an algebraic group G_1/k. This group is clearly a k-form of G/k.

We can define an equivalence relation on the set of cocycles, we say that

$$\{\sigma \mapsto c_\sigma\} \sim \{\sigma \mapsto c'_\sigma\}$$

if and only if we can find a $a \in G(L)$ such that

$$c'_\sigma = a^{-1} c_\sigma a^\sigma \text{ for all } \sigma \in \mathrm{Gal}(L/k)$$

We define $H^1(L/k, \mathrm{Aut}(G))$ as the set of 1-cocycles modulo this equivalence relation. If we have a larger normal separable extension $L' \supset L \supset k$ then we get an inclusion $H^1(L/k, \mathrm{Aut}(G)) \hookrightarrow H^1(L'/k, \mathrm{Aut}(G))$. If $\bar{k}_s$ is a separable closure of k we can form the limit over all finite extensions $k \subset L \subset \bar{k}_s$ and put

$$H^1(\bar{k}_s/k, \mathrm{Aut}(G)) = \lim_{\rightarrow} H^1(L/k,\ \mathrm{Aut}(G))$$

This set is isomorphic to the set of isomorphism classes of k-forms of G/k.

If L/k is a cyclic extension and if $\sigma \in \mathrm{Gal}(/k)$ is a generator, then a cocycle $c :\ \mathrm{Gal}(L/k) \to \mathrm{Aut}(G)(L)$ is determined by its value $g = c(\sigma) \in \mathrm{Aut}(G)(L)$. But we have to satisfy the cocycle relation. We have a useful little

Lemma 1.1.1. *The assigment $\sigma \mapsto c(\sigma) = g$ provides a 1-cocycle if and only iff*

$$\mathrm{Norm}(g) = gg^{\sigma} \ldots g^{\sigma^{n-1}} = Id$$

and

$$H^1(\ Gal(L/k, Aut(G)(L)) = \{g \in Aut(G)(L) |\ \mathrm{Norm}(g) = Id\}/hgh^{-\sigma} \sim g\}.$$

Proof. Straightforward calculation □

We may apply the same concepts in a slightly different situation. A $k-$ algebra $\mathcal{D}$ over the field k is called a *central simple algebra*, if it has a unit element $\neq 0$, if it is finite dimensional over k, if its centre is k (embedded via the unit element) and if it has no non trivial two sided ideals. It is a classical theorem, that such an algebra over a separably closed field k_s is isomorphic to a full matrix algebra $M_n(k_s)$. Hence we can say that over an arbitrary field k any central simple algebra of dimension n^2 is a k-forms of $M_n(k)$.

For any algebraic group G/k we may consider the adjoint group $\mathrm{Ad}(G)$, this is the quotient of G/k by its center. It can be shown, that this is again an algebraic group over k. It is clear that we have an embedding

$$\mathrm{Ad}(G) \to\ \mathrm{Aut}(G)$$

which for any $g \in \mathrm{Ad}(G)(L)$ is given by

$$g \mapsto \{x \mapsto g^{-1}xg\}.$$

A k-form G_1/k of a group G/k is called an *inner k-form*, if it is in the image of

$$H^1(\bar{k}_s/k, \mathrm{Ad}(G)) \to H^1(\bar{k}_s/k, \mathrm{Aut}(G)).$$

We call a semi simple group G/k *anisotropic* , if it does not contain a non trivial split torus (See exercise in (1.1.6)) In our example below the group of elements of norm 1 is always semi simple and anisotropic if and only if $D(a, b)$ is a field.

I want to give an example, we consider the algebraic group $\mathrm{Gl}_2/\mathbb{Q}$ we consider two integers $a, b \neq 0$, for simplicity we assume that b is not a square. Then we have the quadratic extension $L = \mathbb{Q}(\sqrt{b})$, let σ be its non trivial automorphism. The element $\begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}$ defines the inner automorphism

$$\mathrm{Ad}(\begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}) : g \mapsto \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix} g \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}^{-1}$$

of the group Gl_2, Then $\sigma \mapsto \mathrm{Ad}(\begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix})$ and $\mathrm{Id}_{\mathrm{Gal}(L/k)} \mapsto \mathrm{Id}_{\mathrm{Aut}(\mathrm{Gl}_2)(L)}$ is a 1-cocycle and we get a $\mathbb{Q}$ form of our group.

Hence we get a $\mathbb{Q}$ form $G_1 = G(a,b)/\mathbb{Q}$ of our group Gl_2. It is an inner form.

Now we can see easily that group of rational points of our above group $G(a,b)(\mathbb{Q})$ is the multiplicative group of a central simple algebra $D(a,b)/\mathbb{Q}$. To get this algebra we consider the algebra $M_2(L)$ of (2,2)-matrices over L. We define

$$D(a,b) = \{x \in M_2(L) | x = \mathrm{Ad}(\begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}) x^\sigma \mathrm{Ad}(\begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix})^{-1}\}. \tag{1.40}$$

We have an embedding of the field L into this algebra, which is given by

$$u \mapsto \begin{pmatrix} u & 0 \\ 0 & u^\sigma \end{pmatrix}$$

Let u_b the image of $\sqrt{b}$ under this map. We also have the element $u_a = \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}$ in this algebra.

Now I leave it as an exercise to the reader that as a $\mathbb{Q}$ vector space

$$D(a,b) = \mathbb{Q} \oplus \mathbb{Q}u_b \oplus \mathbb{Q}u_a \oplus \mathbb{Q}u_a u_b$$

We have the relation $u_a^2 = a, u_b^2 = b, u_a u_b = -u_b u_a$.

Of course we should ask ourselves: When is $D(a,b)$ split, this means isomorphic to $M_2(\mathbb{Q})$? To answer this question we consider the norm homomorphism, which is defined by

$$x+yu_b+zu_a+wa_a u_b \mapsto (x+yu_b+zu_a+wa_a u_b)(x-yu_b-zu_a-wa_a u_b) = x^2-y^2b-z^2a+w^2ab.$$

It is easy to see that $D(a,b)$ splits if and only if we can find a non zero element whose norm is zero.

If we do this over $\mathbb{R}$ as base field and if we take $a=-1, b=-1$ then we get the Hamiltonian quaternions, which is non split.

We may also look at the p-adic completions $\mathbb{Q}_p$ of our field. Then it is not difficult to see that $D(a,b)$ splits over $\mathbb{Q}_p$ if $p \neq 2$ and $p \nmid ab$. Hence it is clear that there is only a finite number of primes p for which $D(a,b)$ does not split.

If we consider $\mathbb{R}$ as completion at the infinite place, and the $\mathbb{Q}_p$ as the completions at the finite places, then we have

The algebra $D(a,b)$ splits if and only if it splits at all places. The number of places where it does not split is always even.

The first assertion is the so called Hasse-Minkowski principle, the second assertion is essentially equivalent to the quadratic reciprocity law.

Construction of division algebras and anisotropic groups

We give some indication how to construct anisotropic groups over $\mathbb{Q}$ (or even overn any number field). We choose a cyclic extension $L/\mathbb{Q}$ of degree n and we pick a number $a \in \mathbb{Q}^\times$, let $A(a) \in \mathrm{Gl}_n(\mathbb{Q})$ be the following matrix

$$A(a) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & \vdots & \ddots & \\ 0 & 0 & \dots & 0 & 1 \\ a & 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.41}$$

Let $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$ be a generator then $\sigma^\nu \mapsto A(a)^\nu \mod \mathbb{G}_m$ is a homomorphism from $\mathrm{Gal}(L/\mathbb{Q})$ to $\mathrm{PGl}_n(\mathbb{Q})$ and since $A(a) \in \mathrm{Gl}_n(\mathbb{Q})$ this is also a 1-cocycle $c : \mathrm{Gal}(L/K) \to \mathrm{PGl}_n(\mathbb{Q}) := \{\sigma^\nu \mapsto A(a)^\nu\}$. It defines a cohomology class $[A(a)] \in H^1(L/\mathbb{Q}, \mathrm{Ad}(\mathrm{Gl}_n)$ and hence an inner $\mathbb{Q}$-form $G/\mathbb{Q}$ of $\mathrm{Gl}_n/\mathbb{Q}$. In Galois cohomology we have the boundary map

$$\delta : H^1(L/\mathbb{Q}, \mathrm{Ad}(\mathrm{Gl}_n) \to H^2(L/\mathbb{Q}, \mathbb{G}_m) = \mathbb{Q}^\times/N_{L/\mathbb{Q}}(L^\times)$$

and it is clear that

$$\delta([A(a)]) = a \in \mathbb{Q}^\times/N_{L/\mathbb{Q}}(L^\times)$$

Now it is well known that the $\mathbb{Q}$ -form $G/\mathbb{Q}$ of $\mathrm{Gl}_n/\mathbb{Q}$ is anisotropic if and only if the class $a \in \mathbb{Q}^\times/N_{L/\mathbb{Q}}(L^\times)$ is an element of order n. We know from algebraic number theory that there are infinitely many primes p which are inert, i.e. p is unramified in L and the prime ideal (p) stays prime in the ring of integers $\mathcal{O}_L$. Then it easy to see that the order of $p \in \mathbb{Q}^\times/N_{L/\mathbb{Q}}(L^\times)$ is n. Hence we see that the set of isomorphism classes of anisotropic $\mathbb{Q}$ forms of $\mathrm{Gl}_n/\mathbb{Q}$ is abundant.

Obviously the group $M_n(\mathbb{Q})^\times = \mathrm{Gl}_n((\mathbb{Q})$ and we also know that any automorphism of $M_n((\mathbb{Q})^\times$ is inner, hence $\mathrm{Aut}(M_n(\mathbb{Q})) = \mathrm{PGl}_n(\mathbb{Q})$ Therefore the isomorphism classes of ($\mathbb{Q}$-forms of $M_n(\mathbb{Q})$ are equal to the set $H^1(\mathbb{Q}, \mathrm{PGl}_n)$. Such a $\mathbb{Q}$-form $\mathcal{D}/\mathbb{Q}$ is a central simple algebra over $\mathbb{Q}$. The central simple algebra $\mathcal{D}$ defined by the class $[A(a)]$ can be described explicitly:

It contains the field $L/\mathbb{Q}$ as a maximal commutative subalgebra and it is generated by L and another element $a_\sigma \in \mathcal{D}$ which satisfies the following relations

$$\forall x \in L \text{ we have } a_\sigma x \alpha_\sigma^{-1} = \sigma(x)\ ;\ a_\sigma^n = a$$

If we modify a_σ and put $a'_\sigma = a_\sigma y$ with $y \in L^\times$ then the first relation still holds and the second relation becomes $(a'_\sigma)^n = aN_{L/\mathbb{Q}}(y)$. Hence the isomorphism class of $\mathcal{D}$ is determined by the class $a \in \mathbb{Q}^\times/N_{L/\mathbb{Q}}(L^\times)$. It is easy to see that for $a = 1$ the central simple algebra is equal to the endomorphism ring of the $\mathbb{Q}$ vector space $L/\mathbb{Q}$. (This is the linear independence of the elements σ^ν in $\mathrm{End}(L/\mathbb{Q})$.)

1.1.7 Quasisplit $\mathbb{Q}$-forms

We recall that a semi-simple group $G/\mathbb{Q}$ is quasisplit, if contains a Borel subgroup $B/\mathbb{Q}$. This Borel subgroup contains its unipotent radical $U/\mathbb{Q}$ and a maximal torus $T/\mathbb{Q}$. Two such maximal tori $T/\mathbb{Q}, T_1/\mathbb{Q}$ are conjugate by an element

$u \in U(\mathbb{Q})$. Let $G_0/\mathbb{Q}$ by a split group which is a $\mathbb{Q}$-form of $G/\mathbb{Q}$. We pick a maximal split torus $T_0/\mathbb{Q}$ and a Borel $B_0/\mathbb{Q} \supset T_0/\mathbb{Q}$. Then we see that the triple $(G, B, T)/\mathbb{Q}$ is a $\mathbb{Q}$-form of $(G_0, B_0, T_0)/\mathbb{Q}$. Hence it can by constructed from a 1-cocycle representing a cohomology class $\xi \in H^1(\mathbb{Q}, \mathrm{Aut}(((G_0, B_0, T_0)))$, where of course $\mathrm{Aut}(((G_0, B_0, T_0))$ is the subgroup of $\mathrm{Aut}(G_0)$ which fixes T_0, B_0. Obviously we have an exact sequence

$$1 \to T_0^{(\mathrm{ad})} \to \mathrm{Aut}(((G_0, B_0, T_0)) \to \mathrm{Autext}((G_0, B_0, T_0)) \to 1, \tag{1.42}$$

here $\mathrm{Autext}((G_0, B_0, T_0))$ is the very "small" group of automorphisms of the Dynkin diagram Φ. This is also the subgroup of $\mathrm{Aut}(X^*(T_0))$ which leaves the set Δ^+ of positive roots invariant. We could say $Autext((G_0, B_0, T_0)) = \mathrm{Aut}(X^*(T_0), \Delta^+))$

It is well known- and easy to see in the examples of classical groups- that this sequence has a section $s_0 : \mathrm{Autext}((G_0, B_0, T_0)) \to \mathrm{Aut}((G_0, B_0, T_0))$ and this gives us a map in Galois cohomology

$$\begin{aligned} s_0^\bullet : H^1(\mathbb{Q}, \mathrm{Autext}((G_0, B_0, T_0)) = \ & \mathrm{Hom}(\ \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \mathrm{Autext}((G_0, B_0, T_0)) \\ & \to H^1(\mathbb{Q}, \mathrm{Autext}((G_0)) \end{aligned} \tag{1.43}$$

Hence we see that the isomorphism classes of quasisplit $\mathbb{Q}$ -forms of $G_0/\mathbb{Q}$ are given homomorphisms $\psi : \ \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Autext}((G_0)$. The maximal torus $T/\mathbb{Q} \subset B/\mathbb{Q}$ is not split (unless $G/\mathbb{Q}$ is spilt). Hence there is a finite normal extension $F_0/\mathbb{Q}$ such that $T \times_{\mathbb{Q}} F_0$ splits, we assume that $F_0/\mathbb{Q}$ is minimal. i.e. $\mathrm{Gal}(F_0\mathbb{Q}) \subset \mathrm{Aut}(X^*(T \times_{\mathbb{Q}} F_0), \Delta^+)$. We see that a quasisplit form of $G_0/\mathbb{Q}$ is given by a finite normal extension $F_0/\mathbb{Q}$ and a injection $\psi :\ \mathrm{Gal}(F_0/\mathbb{Q}) \hookrightarrow \mathrm{Aut}(X^*(T_0), \Delta^+)$.

In the special case $G_0/\mathbb{Q} = \mathrm{Sl}_n/\mathbb{Q}$ with $T_0/\mathbb{Q}, B_0/\mathbb{Q}$ being the standard diagonal torus and the standard Borel subgroup of upper triangular matrices this looks as follows: We have the element

$$w_0 = \begin{pmatrix} 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 1 & 0 \\ & & \vdots & \ddots & \\ 0 & 1 & \dots & 0 & 0 \\ \pm 1 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathrm{Sl}_n(\mathbb{Q}) \tag{1.44}$$

this element w_0 conjugates B_0 into its opposite B_0^- the group of lower triangular matrices. The standard Cartan involution $\Theta : g \to^t g^{-1}$ does the same and therefore the composition $\mathrm{Ad}(w_0) \circ \Theta$ is an automorphism of $G_0/\mathbb{Q}$ which fixes B_0, T_0. It is an outer automorphism if $n \geq 3$ and gives us the non trivial element of $\mathrm{Autext}(G_0)$. Hence we get a 1-cocycle if choose a quadratic extension $L/\mathbb{Q}$ and send the non trivial element in $\mathrm{Gal}(L/\mathbb{Q})$ to $\mathrm{Ad}(w_0) \circ \Theta$.

We leave it an exercise to the reader to show the$\mathbb{Q}$ form obtained from this cocycle (cohomology class) is isomorphic to the above group $\mathrm{SU}(h_1)/\mathbb{Q}$.

An important class of quasi split groups is given by the groups $G/\mathbb{Q} = R_{F_0/\mathbb{Q}}(G_0)$ where $F_0/\mathbb{Q}$ is a finite extension of $\mathbb{Q}$ and G_0/F_0 is a split group. If

$B_0/F_0 \subset G_0$ is a Borel subgroup then $B = R_{F_0/Q}(B_0)$ is a Borel subgroup in $G/\mathbb{Q}$. Let $F \supset F_0$ be a normal closure of F_0 then

$$G \times_{\mathbb{Q}} F = \prod_{\iota : F_0 \to F} G_0 \times_{F_0, \iota} F \tag{1.45}$$

where ι runs over the set Σ of maps from F_0 to F. The Galois group acts on the product via the action on Σ.

1.1.8 Structure of semisimple groups over $\mathbb{R}$ and the symmetric spaces

We need some information concerning the structure of the group $G_\infty = G(\mathbb{R})$ for semisimple groups over $G/\mathbb{R}$. We will provide this information simply by discussing a series of examples.

Of course the group $G(\mathbb{R})$ is a topological group, actually it is even a Lie group. This means it has a natural structure of a $\mathcal{C}_\infty$ -manifold with respect to this structure. Instead of $G(\mathbb{R})$ we will very often write G_∞. Let G^0_∞ be the connected component of the identity in G_∞. It is an open subgroup of finite index. We will discuss the

Theorem of E. Cartan: *The group G^0_∞ always contains a maximal compact subgroup $K_\infty \subset G^0_\infty$ and all maximal compact subgroups are conjugate under G^0_∞. The quotient space $X = G^0_\infty/K_\infty$ is again a $\mathcal{C}_\infty$-manifold. It is diffeomorphic to an $\mathbb{R}^N$ and carries a Riemannian metric which is invariant under the operation of G^0_∞ from the left. It has sectional curvature ≤ 0 and therefore any two points can be joined by a unique geodesic. The maximal compact subgroup $K \subset G^0_\infty$ is connected and equal to its own normalizer. Therefore the space X can be viewed as the space maximal compact subgroups in G^0_∞. See for instance [52].*

For any maximal compact subgroup $K_x \subset G_\infty$ exists an unique automorphism Θ_x with $\Theta_x^2 = e$ such that $K_x = \{g \in G^0_\infty \, |\Theta(g) = g\}$, this is the Cartan involution corresponding to K_x. The Cartan involutions are in one-to one correspondence with the maximal compact subgroups.

A Cartan involution Θ_x induces an involution also called Θ_x on the Lie algebra $\mathfrak{g}_\mathbb{R}$ of G_∞ and we get a decomposition into $\pm$ eigenspaces

$$\mathfrak{g} = \mathfrak{k}_x \oplus \mathfrak{p}_x;\ \mathfrak{k}_x = \{U \in \mathfrak{g} | \Theta_x(U) = U\}\,; \mathfrak{p}_x = \{V \in \mathfrak{g} | \Theta_x(V) = -V\}$$

where of course $\mathfrak{k}_x$ is the Lie algebra of K_x. The differential of the action of G_∞ on $G(\mathbb{R})/K_x$ provides an isomorphism $D_x : \mathfrak{p}_x \xrightarrow{\sim} T^X_x$ (then tangent space at x). For $V_1, V_2 \in \mathfrak{p}_x$ we have $[V_1, V_2] \in \mathfrak{k}_x$ the map $R : \mathfrak{p}_x \times \mathfrak{p}_x \to \mathfrak{k}_x$ is the curvature tensor. The $\mathbb{R}$-vector space $\mathfrak{g}_c := \mathfrak{k}_x + \sqrt{-1}\mathfrak{p}_x \subset \mathfrak{g} \otimes_\mathbb{R} \mathbb{C}$ is a Lie algebra, for $U_1 + \sqrt{-1}V_1, U_2 + \sqrt{-1}V_2 \in \mathfrak{g}_c$ we get for the Lie-bracket

$$[U_1 + \sqrt{-1}V_1, U_2 + \sqrt{-1}V_2] = [U_1, U_2] - [V_1, V_2] + \sqrt{-1}([U_1, V_2] + [U_2, V_1]) \in \mathfrak{g}_c$$

To this Lie algebra $\mathfrak{g}_c$ corresponds an algebraic group $G_c/\mathbb{R}$ which is a $\mathbb{R}$-form of $G/\mathbb{R}$, the group $G_c(\mathbb{R})$ is compact. The group $G_c/\mathbb{R}$ is called the compact dual of $G/\mathbb{R}$. On $G_c/\mathbb{R}$ we have only one Cartan involution $\Theta = Id$.

This theorem is fundamental. To illustrate this theorem we consider a series of examples:

The groups $\mathrm{Sl}_n(\mathbb{R})$ and $\mathrm{Gl}_n(\mathbb{R})$:

The group $\mathrm{Sl}_d(\mathbb{R})$ is connected. If $K \subset \mathrm{Sl}_d(\mathbb{R})$ is a closed compact subgroup, I claim that we can find a positive definite quadratic form $f : \mathbb{R}^n \to \mathbb{R}$, such that $K \subset SO(f, \mathbb{R})$ Snce the group $SO(f, \mathbb{R})$ itself is compact, it is maximal compact. Two such forms f_1, f_2 define the same maximal compact K_∞ subgroup if there is a $\lambda > 0$ in $\mathbb{R}$ such that $\lambda f_1 = f_2$. We say that f_1 and f_2 are *conformally equivalent.*

This is rather clear, if we believe the first assertion about the existence of f. The existence of f is also easy to see if one believes in the theory of integration on K. This theory provides a positive invariant integral

$$\begin{array}{ccc} \mathcal{C}_c(K) & \longrightarrow & \mathbb{R} \\ \varphi & \longrightarrow & \int\limits_K \varphi(k)dk \end{array}$$

with $\int \varphi > 0$ if $\varphi \geq 0$ and not identically zero (positivity), $\int \varphi(kk_0)dk = \int \varphi(k_0k)dk = \int \varphi(k)dk$ (invariance). To get our form f we start from any positive definite form f_0 on $\mathbb{R}^n$ and put

$$f(\underline{x}) = \int_K f_0(k\underline{x})dk.$$

A positive definite quadratic form on $\mathbb{R}^n$ is the same as a symmetric positive definite bilinear form. Hence the space of positive definite forms is the same as the space of positive definite symmetric matrices

$$\tilde{X} = \left\{A = (a_{ij}) \mid A =^t A, A > 0\right\}.$$

Hence we can say that the space of maximal compact subgroups in $\mathrm{Sl}_n(\mathbb{R})$ is given by

$$X = \tilde{X}/\mathbb{R}^*_{>0}.$$

It is easy to see that a maximal compact subgroup $K_\infty \subset \mathrm{Sl}_d(\mathbb{R})$ is equal to its own normalizer (why?). If we view X as the space of positive definite symmetric matrices with determinant equal to one, then the action of $\mathrm{Sl}_d(\mathbb{R})$ on $X = \mathrm{Sl}_d(\mathbb{R})/K$ is given by

$$(g, A) \longrightarrow g\ A\ {}^t g,$$

and if we view it as the space of maximal compact subgroups, then the action is conjugation.

There is still another interpretation of the points $x \in X$. In our above interpretation a point was a symmetric, positive definite bilinear form $< \, , \, >_x$ on $\mathbb{R}^n$ up to a homothety. From this we get a transposition $g \mapsto {}^{t_x}g$, which is defined by the rule $< gv, u >_x = < v, {}^{t_x}gu >_x$ and from this we get the involution

$$\Theta_x : g \mapsto ({}^{t_x}g)^{-1} \tag{1.46}$$

Then the corresponding maximal compact subgroup is

$$K_x = \{g \in \mathrm{Sl}_n(\mathbb{R}) | \Theta_x(g) = g\} \tag{1.47}$$

This involution Θ_x is a Cartan involution, it also induces an involution also called Θ_x on the Lie-algebra and it has the property that (See 1.18)

$$(u, v) \mapsto B(u, \Theta_x(v)) = B_{\Theta_x}(u, v) \tag{1.48}$$

is negative definite. This bilinear form is K_x invariant. All these Cartan involutions are conjugate.

If we work with $\mathrm{Gl}_n(\mathbb{R})$ instead then we have some freedom to define the symmetric space. In this case we have the non trivial center $\mathbb{R}^\times$ and it is sometimes useful to define

$$X = \mathrm{Gl}_n(\mathbb{R})/\mathrm{SO}(\mathbb{R}) \cdot \mathbb{R}^\times_{>0}, \tag{1.49}$$

then our symmetric space has two components, a point is pair (Θ_x, ϵ) where ϵ is an orientation. If we do not divide by $\mathbb{R}^\times_{>0}$ then we multiply the Riemannian manifold X by a flat space and we get the above space $\tilde{X}$.

A Cartan involution on $\mathrm{Gl}_n(\mathbb{R})$ is an involution which induces a Cartan involution on $\mathrm{Sl}_n(\mathbb{R})$ and which is trivial on the center.

Proposition 1.1.1. *The Cartan involutions on* $\mathrm{Gl}_n(\mathbb{R})$ *are in one to one correspondence to the euclidian metrics on* $\mathbb{R}^n$ *up to conformal equivalence.*

Finally we recall the Iwasawa decomposition. Inside $\mathrm{Gl}_n(\mathbb{R})$ we have the standard Borel- subgroup $B(\mathbb{R})$ of upper triangular matrices and it is well known that

$$\mathrm{Gl}_n(\mathbb{R}) = B(\mathbb{R}) \cdot \mathrm{SO}(\mathbb{R}) \cdot \mathbb{R}^\times_{>0} \tag{1.50}$$

and hence we see that $B(\mathbb{R})$ acts transitively on X.

The compact dual of $\mathrm{Sl}_n(\mathbb{R})$

If $G/\mathbb{R}$ is a semi simple group, then $G_c/\mathbb{R}$ is a $\mathbb{R}$-form of $G/\mathbb{R}$. Therefore we find a cohomology class $\xi_c \in H^1\mathbb{C}/(\mathbb{R}, \mathrm{Aut}(G))$ corresponding to G_c. It is clear from the Theorem of Cartan how we get a cocycle representing this class: We choose a Cartan involution $\Theta \in \mathrm{Aut}(G)$, the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ is cyclic of order SO(2 let $\mathbf{c}$ be the generator (the complex conjugation). Then $\mathbf{c} \mapsto \mathbf{c} \circ \Theta$ yields a 1-cocycle in $C^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathrm{Aut}(G)(\mathbb{C}))$. (Lemma 1.1.1) and this 1-cocycle represents the class ξ_c.

This means for the group $\mathrm{Sl}_n/\mathbb{R}$ that

$$G_c(\mathbb{R}) = \{g \in \mathrm{Sl}_n(\mathbb{C}) | \mathbf{c}({}^t g^{-1}) = g\}$$

and if we go back to the usual notion and write $\mathbf{c}(g) = \bar{g}$ then we get

$$G_c(\mathbb{R}) = \{g \in \mathrm{Sl}_n(\mathbb{C}) | {}^t\bar{g}g = \mathrm{Id}\} = \mathrm{SU}(n)$$

Here of course $\mathrm{SU}(n) = \mathrm{SU}(h_c)$ where $h_c(z_1, z_2, \ldots, z_n) = \sum_{i=1}^n z_i \bar{z}_i$ is the standard positive definite hermitian form on $\mathbb{C}^n$.

We know that for $G/\mathbb{R} = \mathrm{Sl}_n/\mathbb{R}$ and $n > 2$ the Cartan involution Θ is the generator of $\mathrm{Aut}(G)/\mathrm{Ad}(G)$ and hence it is clear that ξ_c is not in the image of $H^1(\mathbb{C}/\mathbb{R}, \mathrm{Ad}(G)) \to H^1(\mathbb{C}/(\mathbb{R}, \mathrm{Aut}(G))$. This means that in this case $G_c/\mathbb{R} = \mathrm{SU}(n)/\mathbb{R}$ is not an inner $\mathbb{R}$ -form of $\mathrm{Sl}_n/\mathbb{R}$, in turn this also means that $\mathrm{Sl}_n/\mathbb{R}$ is not an inner $\mathbb{R}$ -form of $\mathrm{SU}(n)/\mathbb{R}$.

In this context the following general proposition is of importance

Proposition 1.1.2. *A semi simple group scheme $G/\mathbb{R}$ is an inner $\mathbb{R}$ form of its compact dual $G_c/\mathbb{R}$ if an only if*
a) The Cartan involution Θ of $G/\mathbb{R}$ is an inner automorphism of $G/\mathbb{R}$.
b) The group $G/\mathbb{R}$ has a compact maximal torus $T_c/\mathbb{R} \subset G/\mathbb{R}$.

Give a name to this class of groups ? Examples?

The Arakelow- Chevalley scheme $(\mathrm{Gl}_n/\mathbb{Z}, \Theta_0)$

We start from the free lattice $L = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \cdots \oplus \mathbb{Z}e_n$ and we think of $\mathrm{Gl}_n/\mathbb{Z}$ as the scheme of automorphism of this lattice. If we choose an euclidian metric $< \, , \, >$ on Ł $\otimes \mathbb{R}$, then we call the pair $(L, < \, , \, >)$ an *Arakelow vector bundle.* From the (conformal class of) metric we get a Cartan involution Θ. on $\mathrm{Gl}_n(\mathbb{R})$, and the pair $(\mathrm{Gl}_n/\mathbb{Z}, \Theta)$ is an *Arakelow group scheme*

We may choose the standard euclidian metric with respect to the given basis, i.e. $< e_i, e_j >= \delta_{i,j}$. theresulting Cartan involution is the standard one: $\Theta_0 : g \mapsto ({}^t g)^{-1}$. This pair $(\mathrm{Gl}_n/\mathbb{Z}, \Theta_0)$ is called an Arakelow- Chevalley scheme. (In a certain sense the integral structure of $\mathrm{Gl}_n/\mathbb{Z}$ and the choice of the Cartan involution are "optimally adapted")

In this case we find for our basis elements in (1.15)

$$B_{\Theta_0}(H_i, H_j) = -2n\delta_{i,j} + 2; B_{\Theta_0}(E_{i,j}, E_{k,l}) = -2n\delta_{i,k}\delta_{j,l} \qquad (1.51)$$

hence the $E_{i,j}$ are part of an orthonormal basis.

We propose to call a pair $(L, < \, , \, >_x)$ an Arakelow vector bundle over $\mathrm{Spec}(\mathbb{Z}) \cup \{\infty\}$ and $(\mathrm{Gl}_n, \Theta_x)$ an Arakelow group scheme. The Arakelow vector bundles modulo conformal equivalence are in one-to one correspondence with the Arakelow group schemes of type Gl_n.

The group $\mathrm{Sl}_d(\mathbb{C})$

We now consider the group $G/\mathbb{R}$ whose group of real points is $G(\mathbb{R}) = \mathrm{Sl}_d(\mathbb{C})$ (see 1.1 example 4)).

A completely analogous argument as before shows that the maximal compact subgroups are in one to one correspondence to the positive definite hermitian forms on $\mathbb{C}^n$ (up to multiplication by a scalar). Hence we can identify the space of maximal compact subgroups to the space of positive definite hermitian matrices

$$X = \left\{ A \mid A ={}^t \overline{A} \, , \; A > 0 \; , \; \det A = 1 \right\} .$$

The action of $\mathrm{Sl}_d(\mathbb{C})$ by conjugation on the maximal compact subgroups becomes

$$A \longrightarrow g \; A \; {}^t\overline{g}$$

on the space of matrices.

The orthogonal group:

The next example we want to discuss is the orthogonal group of a non degenerate quadratic form

$$f(x_1, \ldots, x_n) = x_1^2 + \ldots + x_m^2 - x_{m+1}^2 - \ldots - x_n^2,$$

since at this moment we consider only groups over the real numbers, we may assume that our form is of this type. In this case one has the usual notation

$$\mathrm{SO}(f, \mathbb{R}) = \mathrm{SO}(m, n-m).$$

Of course we can use the same argument as before and see that for any maximal compact subgroup $K \subset \mathrm{SO}(f, \mathbb{R})$ we may find a positive definite form ψ

$$\psi : \mathbb{R}^n \longrightarrow \mathbb{R}$$

such that $K = \mathrm{SO}(f, \mathbb{R}) \cap \mathrm{SO}(\psi, \mathbb{R})$. But now we cannot take all forms ψ, i.e. only special forms ψ provide maximal compact subgroup.

We leave it to the reader to verify that any compact subgroup K fixes an orthogonal decomposition $\mathbb{R}^n = V_+ \oplus V_-$ where our original form f is positive definite on V_+ and negative definite on V_-. Then we can take a ψ which is equal to f on V_+ and equal to $-f$ on V_-.

Exercise 3 *a) Let $V/\mathbb{R}$ be a finite dimensional vector space and let f be a symmetric non degenerate form on V. Let $K \subset \mathrm{SO}(f)$ be a compact subgroup. If f is not definite then the action of K on V is not irreducible.*

b) We can find a K invariant decomposition $V = V_- \oplus V_+$ such that f is negative definite on V_- and positive definite on V_+

In this case the structure of the quotient space $G(\mathbb{R})/K$ is not so easy to understand. We consider the special case of the form

$$x_1^2 + \ldots + x_n^2 - x_{n+1}^2 = f(x_1, \ldots, x_{n+1}).$$

We consider in $\mathbb{R}^{n+1}$ the open subset

$$X_- = \{v = (x_1 \ldots x_{n+1}) \mid f(v) < 0\}.$$

It is clear that this set has two connected components, one of them is

$$X_-^+ = \{v \in X_- \mid x_{n+1} > 0\}$$

Since it is known that $\mathrm{SO}(n, 1)$ acts transitively on the vectors of a given length, we find that $\mathrm{SO}(n, 1)$ cannot be connected. Let $G_\infty^0 \subset \mathrm{SO}(n, 1)$ be the subgroup leaving X_-^+ invariant.

Now it is not to difficult to show that for any maximal compact subgroup $K_\infty \subset G_\infty^0$ we can find a ray $\mathbb{R}_{>0}^* \cdot v \subset X_-^{(+)}$ which is fixed by K_∞.

(Start from $v_0 \in X_-^{(+)}$ and show that $R_{>0}^* K_\infty v_0$ is a closed convex cone in $X_-^{(+)}$. It is K_∞ invariant and has a ray which has a "centre of gravity" and this is fixed under K_∞.)

For a vector $v = (x_1, \dots, x_{n+1}) \in X_-^{(+)}$ we may normalise the coordinate x_{n+1} to be equal to one; then the rays $\mathbb{R}_{>0}^+ v$ are in one to one correspondence with the points of the ball disc

$$\overset{\circ}{D}_n = \left\{(x_1, \dots, x_n) \mid x_1^2 + \dots + x_n^2 < 1\right\} \subset X_-^{(+)}. \quad (1.52)$$

This tells us that we can identify the set of maximal compact subgroups $K_\infty \subset G_\infty^0$ with the points of this ball. The first conclusion is that $G_\infty^0/K_\infty \simeq D^n$ is topologically a cell (diffeomorphic to $\mathbb{R}^n$). Secondly we see that for a $v \in X_-^+$ we have an orthogonal decompositon with respect to f

$$\mathbb{R}^{n+1} = \langle v \rangle + \langle v \rangle^\perp,$$

and the corresponding maximal compact subgroup is the orthogonal group on $\langle v \rangle^\perp$.

The space X_-^+ is often called the n-dimensional hyperbolic space $\mathbb{H}_n$.

Give Cartan Involutions?

1.1.9 Special low dimensional cases

1) We consider the (semi-simple) group $\mathrm{Sl}_2(\mathbb{R})$. It acts on the upper half plane

$$\mathbb{H} = \{z \mid z \in \mathbb{C}, \Im(z) > 0\}$$

by

$$(g, z) \longrightarrow \frac{az+b}{cz+b}, \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}_2(\mathbb{R}).$$

It is clear that the stabiliser of the point $i \in \mathbb{H}$ is the standard maximal compact subgroup

$$K_\infty = \mathrm{SO}(2) = \left\{ \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \right\}.$$

Hence we have $\mathbb{H} = \mathrm{Sl}_2(\mathbb{R})/K_\infty$. But this quotient has also been realized as the space of symmetric positive definite 2×2-matrices with determinant equal to one

$$z = \left\{ \begin{pmatrix} y_1 & x_1 \\ x_1 & y_2 \end{pmatrix} \mid y_1 y_2 - x_1^2 = 1, y_1 > 0 \right\}.$$

It is clear how to find an isomorphism between these two explicit realizations. The map

$$\begin{pmatrix} y_1 & x_1 \\ x_1 & y_2 \end{pmatrix} \longrightarrow \frac{i + x_1}{y_2}$$

is compatible with the action of $\mathrm{Sl}_2(\mathbb{R})$ on both sides and sends the identity $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ to the point i.

If we start from a point $z \in \mathbb{H}$ the corresponding metric is as follows: We identify the lattices $\langle 1, z \rangle = \{a + bz \mid a, b \in \mathbb{Z}\} = \Omega$ to the lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$ by sending $1 \to \binom{1}{0}$ and $z \to \binom{0}{1}$. The standard euclidian metric on $\mathbb{C} = \mathbb{R}^2$ induces a metric on $\Omega \subset \mathbb{C}$, and this metric is transported to $\mathbb{R}^2$ by the identification $\Omega \otimes \mathbb{R} \to \mathbb{R}^2$. Hence the symmetric matrix will be $\begin{pmatrix} 1 & x \\ x & z\bar{z} \end{pmatrix}$.

We may also start from the (reductive) group $\mathrm{Gl}_2(\mathbb{R})$, it has the centre $C(\mathbb{R}) = \{\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}\}$. Let $C(\mathbb{R})^{(0)}$ be the connected component of the identity of $C(\mathbb{R})$. In this case we define $K_\infty = \mathrm{SO}(2) \times C(\mathbb{R})^{(0)}$. Then the quotient

$$\mathrm{Gl}_2(\mathbb{R})/K_\infty = \mathbb{H} \cup \mathbb{H}_-$$

where $\mathbb{H}_-$ is the lower half plane.

2) The two groups $\mathrm{Sl}_2(\mathbb{R})$ and $\mathrm{PSl}_2(\mathbb{R})^{(0)} = \mathrm{Sl}_2(\mathbb{R})/\{\pm\mathrm{Id}\}$ give rise to the same symmetric space. The group $P\mathrm{Sl}_2(\mathbb{R})$ acts on the space $M_2(\mathbb{R})$ of 2×2-matrices by conjugation (the group $\mathrm{Gl}_2(\mathbb{R})$ acts by conjugation and the centre acts trivially) and leaves invariant the space

$$\{A \in M_2(\mathbb{R}) \mid \mathrm{trace}(A) = 0\} = M_2^0(\mathbb{R}).$$

On this three-dimensional space we have a symmetric quadratic form

$$\begin{aligned} B &: M_2^0(\mathbb{R}) \longrightarrow \mathbb{R} \\ B &: A \mapsto \frac{1}{2}\,\mathrm{trace}\,(A^2) \end{aligned}$$

and with respect to the basis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{1.53}$$

this form is $x_1^2 + x_2^2 - x_3^2$.

Hence we see that $\mathrm{SO}(M_2^0(\mathbb{R}), B) = \mathrm{SO}(2,1)$, and hence we have an isomorphism between $P\mathrm{Sl}_2(\mathbb{R})$ and the connected component of the identity $G_\infty^0 \subset \mathrm{SO}(2,1)$. Hence we see that our symmetric space $\mathbb{H} = \mathrm{Sl}_2(\mathbb{R})/K_\infty = P\mathrm{Sl}_2(\mathbb{R})/\overline{K}_\infty$ can also be realised as disc

$$D = \{(x_1, x_2) \mid x_1^2 + x_2^2 < 1\}$$

where we normalized $x_3 = 1$ on $X_-^{(+)}$ as in 1.52) .

The group $\mathbf{Sl}_2(\mathbb{C})$.

Recall that in this case the symmetric space is given by the positive definite hermitian matrices

$$A = \left\{ \begin{pmatrix} y_1 & z \\ \overline{z} & y_2 \end{pmatrix} \mid \det(A) = 1, y_1 > 0 \right\}.$$

In this case we have also a realization of the symmetric space as an upper half space. We send

$$\begin{pmatrix} y_1 & w \\ \overline{w} & y_2 \end{pmatrix} \longmapsto \left(\frac{w}{y_2}, \frac{1}{y_2} \right) = (z, \zeta) \in \mathbb{C} \times \mathbb{R}_{>0}$$

The inverse of this isomorphism is given by

$$(z,\zeta) \mapsto \begin{pmatrix} \zeta + z\bar{z}/\zeta & z/\zeta \\ \bar{z}/\zeta & 1/\zeta \end{pmatrix}$$

As explained earlier, the action of $\mathrm{Gl}_2(\mathbb{C})$ on the maximal compact subgroup given by conjugation yields the action

$$G(\mathbb{R}) \times X \longrightarrow X,$$

$$(g, A) \longrightarrow gA^t\bar{g},$$

on the hermitian matrices. Translating this into the realization as an upper half space yield the slightly scaring formula

$$G \times (\mathbb{C} \times \mathbb{R}_{>0}) \longrightarrow \mathbb{C} \times \mathbb{R}_{>0},$$

$$(g,(z,\zeta)) \longrightarrow \left(\frac{(az+b)\,\overline{(cz+d)} + a\bar{c}\,\zeta^2}{(cz+d)\,\overline{(cz+d)} + c\bar{c}\,\zeta^2} \;,\; \frac{\zeta}{(cz+d)\,\overline{(cz+d)} + c\bar{c}\,\zeta^2} \right)$$

. Here X is the three dimensional hyperbolic space $\mathbb{H}_3$.

1.3.4. The Riemannian metric: It was already mentioned in the statement of the theorem of Cartan that we always have a G^0_∞ invariant Riemannian metric on X. It is not to difficult to construct such a metric, which in many cases is rather canonical.

In the general case we observe that the maximal compact subgroup is the stabilizer of the point $x_0 = e \cdot K_\infty \in G^0_\infty/K_\infty = X$. Hence it acts on the tangent space of x_0, and we can construct a K_∞-invariant positive definite quadratic form on this tangent sapce. Then we use the action of G^0_∞ on X to transport this metric to an arbitrary point in X: If $x \in X$ we find a g so that $x = gx_0$, it defines an isomorphism between the tangent space at x_0 and the tangent space at x. Hence we get a quadratic form on the tangent space at x, which will not depend on the choice of $g \in G^0_\infty$. In our examples this metric is always unique up to scalars.

a) In the case of the group $\mathrm{Sl}_n(\mathbb{R})$ we may take as a base point $x_0 \in X$ the identity $\mathrm{Id} \in \mathrm{Sl}_n(\mathbb{R})$. The corresponding maximal compact subgroup is the orthogonal group $\mathrm{SO}(n)$. The tangent space at Id is given by the space

$$\mathrm{Sym}^0_n(\mathbb{R}) = T^X_{\mathrm{Id}}$$

of symmetric matrices with trace zero. On this space we have the form

$$Z \longrightarrow \mathrm{trace}(Z^2),$$

which is positive definite (a symmetric matrix has real eigenvalues). It is easy to see that the orthogonal group acts on this tangent space by conjugation, hence the form is invariant.

b) A similar argument applies to the group $G_\infty = \mathrm{Sl}_d(\mathbb{C})$. Again the identity Id is a nice positive definite hermitian matrix. The tangent space consists of the hermitian matrices

$$T^X_{\mathrm{Id}} = \left\{ A \mid A =^t \bar{A} \text{ and } \mathrm{tr}(A) = 0 \right\},$$

and the invariant form is given by

$$A \longrightarrow \operatorname{tr}(A\overline{A}).$$

c) In the case of the group $G^0_\infty \subset \mathrm{SO}(f)(\mathbb{R})$ where f is the quadratic form

$$f(x_1, \ldots, x_{n+1}) = x_1^2 + \ldots + x_n^2 - x_{n+1}^2.$$

We realized the symmetric space as the open ball

$$\overset{\circ}{D}_n = \{(x_1, \ldots, x_n) \mid x_1^2 + \ldots + x_n^2 < 1\}.$$

The orthogonal group $\mathrm{SO}(n,1)$ is the stabilizer of $0 \in \overset{\circ}{D}_n$, and hence it is clear that the Riemannian metric has to be of the form

$$h(x_1^2 + \ldots + x_n^2)(dx_1^2 + \ldots dx_n^2)$$

(in the usual notation). A closer look shows that the metrics has to be

$$\frac{dx_1^2 + \ldots + dx_n^2}{\sqrt{1 - x_1^2 - \ldots - x_n^2}}.$$

In our two low dimensional spacial examples the metric is easy to determine. For the action of the group $\mathrm{Sl}_2(\mathbb{R})$ on the upper half plane $\mathbb{H}$ we observe that for any point $z_0 = x + iy \in \mathbb{H}$ the tangent vectors $\frac{\partial}{\partial x}|_{z_0}$, $\frac{\partial}{\partial y}|_{z_0}$ form a basis of the tangent spaces at z_0.

If we take $z_0 = i$ then the stabilizer is the group $\mathrm{SO}(2)$ and for

$$e(\varphi) = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}.$$

We have

$$e(\varphi) \cdot \left(\frac{\partial}{\partial x}\,|_i\right) = \cos 2\varphi \cdot \frac{\partial}{\partial x}\,|_i + \sin 2\varphi \cdot \frac{\partial}{\partial y}\,|_i$$
$$e(\varphi) \left(\frac{\partial}{\partial y}\,|_i\right) = \sin 2\varphi \cdot \frac{\partial}{\partial x}\,|_i + \cos 2\varphi \cdot \frac{\partial}{\partial y}\,|_i.$$

Hence we find that $\frac{\partial}{\partial x}\,|_i$ and $\frac{\partial}{\partial y}\,|_i$ have to be orthogonal and of the same length.

Now the matrix

$$\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$$

sends i into the point $z = x + iy$. It sends $\frac{\partial}{\partial x}\,|_i$ and $\frac{\partial}{\partial y}\,|_i$ into $y \cdot \frac{\partial}{\partial x}\,|_z$ and $y \cdot \frac{\partial}{\partial y}\,|_z$, and hence we must have for our invariant metric

$$\langle \frac{\partial}{\partial x}\,|_z, \frac{\partial}{\partial y}\,|_z \rangle = 0 \ ; \ \langle \frac{\partial}{\partial x}\,|_z, \frac{\partial}{\partial x}\,|_z \rangle = \frac{1}{y^2} \ ; \ \langle \frac{\partial}{\partial y}\,|_z, \frac{\partial}{\partial y}\,|_z \rangle = \frac{1}{y^2},$$

and this is in the usual notation the metric

$$ds^2 = \frac{1}{y^2}(dx^2 + dy^2). \tag{1.54}$$

A completely analogous argument yields the metric

$$ds^2 = \frac{1}{\zeta^2}\,(d\zeta^2 + dx^2 + dy^2) \tag{1.55}$$

for the space $\mathbb{H}_3$.

1.2 Arithmetic groups

If we have a linear algebraic group $G/\mathbb{Q} \hookrightarrow GL_n$ we may consider the group $\Gamma = G(\mathbb{Q}) \cap GL_n(\mathbb{Z})$. This is the first example of an *arithmetic* group. It has the following fundamental property:

Proposition: *The group* Γ *is a discrete subgroup of the topological group* $G(\mathbb{R})$*.*

This is rather easily reduced to the fact that $\mathbb{Z}$ is discrete in $\mathbb{R}$. Actually our construction provides a big family of arithmetic groups. For any integer $m > 0$ we have the homomorphism of reduction $\mod m$, namely

$$GL_n(\mathbb{Z}) \longrightarrow GL_n(\mathbb{Z}/m\mathbb{Z}).$$

The kernel $GL_n(\mathbb{Z})(m)$ of this homomorphism has finite index in $GL_n(\mathbb{Z})$ and hence the intersection $\Gamma' = GL_n(\mathbb{Z})(m) \cap \Gamma$ has finite index in Γ.

Definition 2.1.: *A subgroup* Γ'' *of* Γ *is called a congruence subgroup, if we can find an integer* m *such that*

$$GL_n(\mathbb{Z})(m) \cap \Gamma \subset \Gamma'' \subset \Gamma.$$

At this point a remark is in order. We explained already that a linear algebraic group $G/\mathbb{Q}$ may be embedded in different ways into different groups GL_n, i.e.

$$G \begin{array}{ll} \hookrightarrow & GL_{n_1} \\ \\ \hookrightarrow & GL_{n_2} \end{array}$$

In this case we may get two different congruence subgroups

$$\Gamma_1 = G(\mathbb{Q}) \cap GL_{n_1}(\mathbb{Z}), \Gamma_2 = G(\mathbb{Q}) \cap GL_{n_2}(\mathbb{Z}).$$

It is not hard to show that in such a case we can find an $m > 0$ such that

$$\begin{array}{l} \Gamma_1 \supset \Gamma_2 \cap GL_{n_2}(\mathbb{Z})(m) \\ \Gamma_2 \supset \Gamma_1 \cap GL_{n_1}(\mathbb{Z})(m) \end{array} \quad .$$

From this we conclude that the notion of congruence subgroup does not depend on the way we realized the group $G/\mathbb{Q}$ as a subgroup in the general linear group.

Now we may also define the notion of an *arithmetic* subgroup. A subgroup $\Gamma' \subset G(\mathbb{Q})$ is called arithmetic if for any congruence subgroup $\Gamma \subset G(\mathbb{Q})$ the group $\Gamma' \cap \Gamma$ is of finite index in Γ' and Γ. (We say that Γ' and Γ are commensurable.) By definition all congruence subgroups are arithmetic subgroups.

The most prominent example of an arithmetic group is the group

$$\Gamma = \mathrm{Sl}_2(\mathbb{Z}).$$

Another example is obtained as follows. We defined for any number field $K/\mathbb{Q}$ the group

$$G/\mathbb{Q} = R_{K/\mathbb{Q}}(\mathrm{Sl}_d)$$

for which $G(\mathbb{Q}) = \mathrm{Sl}_d(K)$. If $\mathcal{O}_K$ is the ring of integers in K, then $\Gamma = \mathrm{Sl}_d(\mathcal{O}_K)$ (and also $\tilde{\Gamma} = GL_n(\mathcal{O}_K)$) is a congruence (and hence arithmetic) subgroup of $G(\mathbb{Q})$.

It is very interesting that the groups $\Gamma = \mathrm{Sl}_2(\mathbb{Z})$ and $\mathrm{Sl}_2(\mathcal{O}_K)$ for imaginary quadratic $K/\mathbb{Q}$ always contain arithmetic subgroups $\Gamma' \subset \Gamma$ which are not congruence subgroups. This means that in general the class of arithmetic subgroups is larger than the class of congruence subgroups. We will prove this assertion in **Non Congruence subgroups**).

If only the group $G(\mathbb{R})$ is given (as the group of real points of a group $G/\mathbb{R}$ or perhaps only as a Lie group, then the notion of arithmetic group $\Gamma \subset G(\mathbb{R})$ is not defined. The notion of an arithmetic subgroup $\Gamma \subset G(\mathbb{R})$ requires the choice of a group scheme $G/\mathbb{Q}$ such that the group $G(\mathbb{R})$ is the group of real points of this group over $\mathbb{Q}$. The exercise in 1.1.2. shows that different $\mathbb{Q}$- forms provide different arithmetic groups.

Exercise 2 *If $\gamma \in Gl_n(\mathbb{Z})$ is a nontrivial torsion element and if $\gamma \equiv \mathrm{Id} \mod m$ then $m = 1$ or $m = 2$. In the latter case the element γ is of order 2.*

This implies that for $m \geq 3$ the congruence subgroup $Gl_n(\mathbb{Z})(m)$ of $Gl_n(\mathbb{Z})$ is torsion free.

This implies of course that any arithmetic group has a subgroup of finite index, which is torsion free.

Affgroup

1.2.1 Affine group schemes over $\mathbb{Z}$

There is a slightly more sophisticated view of arithmetic groups. In our book [41] section 7.5.6 and on p. 50,51 we discuss briefly the general notion of a group scheme over an arbitrary base scheme S. An affine group scheme over $G/\mathbb{Z}$ is a finitely generated $\mathbb{Z}$-algebra $A(G)$ together with a comultiplication $m : A(G) \to A(G) \otimes A(G)$. For any $\mathbb{Z}$ -algebra B (commutative and with identity) the comultiplication m induces a multiplication on the B-valued points

$$^t m : \mathrm{Hom}_{\mathrm{alg}}(A, B) \times \mathrm{Hom}_{\mathrm{alg}}(A, B) \to \mathrm{Hom}_{\mathrm{alg}}(A, B)$$

and the requirement is that this multiplication defines a group structure on $G(B) = \mathrm{Hom}_{\mathrm{alg}}(A, B)$. In educated language : $G/\mathbb{Z}$ is a functor from the category of affine schemes into the category of groups.

For instance we can define the group scheme $\mathrm{Gl}_n/\mathbb{Z}$. The affine algebra is

$$A(\mathrm{Gl}_n) = \mathbb{Z}[X_{11}, X_{12}, \ldots, X_{1n}, X_{21}, \ldots, X_{nn}, Y]/(Y \det(X_{ij} - 1)$$

Then the group $\mathrm{Gl}_n(\mathbb{Z})$ of $\mathbb{Z}$-valued points of $\mathrm{Gl}_n/\mathbb{Z}$ is our group $Gl_n(\mathbb{Z})$.

If $G/\mathbb{Q} \subset \mathrm{Gl}_n/\mathbb{Q}$ is a subgroup, then the affine algebra $A(G) = A(\mathrm{Gl}_n)\otimes\mathbb{Q}/I$, where I is an ideal in $A(\mathrm{Gl}_n)\otimes\mathbb{Q}$. Since $G/\mathbb{Q}$ is a subgroup this ideal must satisfy

$$m_{\mathrm{Gl}_n}(I) \subset A(\mathrm{Gl}_n) \otimes \mathbb{Q} \otimes I + I \otimes A(\mathrm{Gl}_n) \otimes \mathbb{Q}.$$

Let $J = A(\mathrm{Gl}_n) \cap I$, then it is easy to check that the comultiplication of $A(\mathrm{Gl}_n)$ satisfies

$$m_{\mathrm{Gl}_n}(J) \subset A(\mathrm{Gl}_n) \otimes J + J \otimes A(\mathrm{Gl}_n)$$

and this tells us that m_{Gl_n} induces a comultiplication

$$m : A(\mathrm{Gl}_n)/J \to A(\mathrm{Gl}_n)/J \otimes A(\mathrm{Gl}_n)/J$$

which provides a group scheme structure. This means that we have extended the group scheme $G/\mathbb{Q}$ to a group scheme over $\mathcal{G}/\mathbb{Z}$. The affine algebra $A(\mathcal{G}) = A(\mathrm{Gl}_n)/J$. This extension depends on the choice of the embedding into $\mathrm{Gl}_n/\mathbb{Q}$ and it is called the *flat extension.* Then the base extension $\mathcal{G} \times_{\mathbb{Z}} \mathbb{Q} = G/\mathbb{Q}$, this base extension is called the *generic fiber* of $\mathcal{G}/\mathbb{Z}$.

We now may understand our arithmetic group $\Gamma = G(\mathbb{Q}) \cap \mathrm{Gl}_n(\mathbb{Z})$ as the group $\mathcal{G}(\mathbb{Z})$ of $\mathbb{Z}$ valued points of a group scheme over $\mathbb{Z}$. Since we know what $\mathcal{G}(\mathbb{Z}/m\mathbb{Z})$ is we can define congruence subgroups Γ_H as inverse images of subgroups $H \subset \mathcal{G}(\mathbb{Z}/m\mathbb{Z})$ under the projection $\mathcal{G}(\mathbb{Z}) \to \mathcal{G}(\mathbb{Z}/m\mathbb{Z})$.

There is the special class of semi-simple or reductive group schemes. Roughly speaking an affine group scheme $G/\mathbb{Z}$ is semi-simple (resp. reductive) , if its generic fiber $G \times_{\mathbb{Z}} \mathbb{Q}$ is semi-simple (resp. reductive) and if for all primes p the group scheme $G \times_{\mathbb{Z}} \mathbb{F}_p$ (the reduction $\mod p$) is a semi-simple ((resp. reductive)) group scheme over $\mathbb{F}_p$.

Of course the simplest example of a semi-simple (resp. reductive) group (scheme) over $\mathbb{Z}$ is the group $\mathrm{Sl}_n/\mathbb{Z}$ (resp. $\mathbb{G}_n/\mathbb{Z}$).

We can also construct semi-simple group-schemes by taking flat extensions of orthogonal (resp. symplectic) groups over $\mathbb{Q}$, (see section1.2.1, example 2) and 3). Here the symmetric (resp. alternating) form has to satisfy certain arithmetic conditions (See chap4.pdf).

lattices

1.2.2 Γ -modules

We consider modules $\mathcal{M}$ (i.e. abelian groups) with an action of Γ, we want to discuss briefly discuss some special classes of such Γ-modules.

The most important classes of Γ-modules are the modules of *arithmetic origin.* To construct such modules we realise our arithmetic group as $\Gamma = G(\mathbb{Q}) \cap \mathrm{Gl}_n(\mathbb{Z})$. Then we take any rational representation $\rho : G/\mathbb{Q} \to \mathrm{Gl}(V)$, where V is a finite dimensional $\mathbb{Q}-$ vector space. Now we look for finitely generated submodules $\mathcal{M} \subset V$ such that $\mathcal{M} \otimes \mathbb{Q} = V$ which are invariant under the action of Γ. Such a module is a Γ-module of arithmetic origin.

It is not to difficult to show that given any finitely generated module $\mathcal{M}'$ which is a full sublattice, i.e. $\mathcal{M}' \otimes \mathbb{Q} = V$, we can find a congruence subgroup $\Gamma_1 \subset \Gamma$ such that $\Gamma_1 \mathcal{M}' = \mathcal{M}'$. Then

$$\mathcal{M} = \bigcap_{\gamma \in \Gamma/\Gamma_1} \gamma \mathcal{M}'$$

is a Γ module (of arithmetic origin).

A second class of Γ modules are those of *congruence origin*. To get such a module we simply pick a congruence subgroup $\Gamma(N) \subset \Gamma$ and then we simply look at finitely generated abelian groups V with an action of $\Gamma/\Gamma(N)$ on V.

We get some important examples of Γ modules of congruence origin if we start from a Γ-module $\mathcal{M}$ of arithmetic origin. Then we choose an integer N and consider the Γ module $\mathcal{M} \otimes \mathbb{Z}/N\mathbb{Z}$. On this module $\Gamma(N)$ acts trivially, hence this module is a $\Gamma/\Gamma(N)$ module of congruence origin.

We go back to the more sophisticated point of view above, our arithmetic group is the group $\Gamma = \mathcal{G}(\mathbb{Z})$ of $\mathbb{Z}$ valued points of the flat extension $\mathcal{G}/\mathbb{Z}$.

Now we pick a torsion free finitely generated module $\mathcal{M}$, we know what it means that $\mathcal{M}$ is a $\mathcal{G}/\mathbb{Z}$ module: It simply means that for any commutative ring B with identity we have a B-linear action of $\mathcal{G}(B)$ on the B-module $\mathcal{M} \otimes B$, or in other words we have a homomorphism $\mathcal{G}(B) \to \mathrm{Gl}_B(\mathcal{M} \otimes_{\mathbb{Z}} B)$. Of course we require that this action is functorial in B.

For this book -especially for the first half- the group scheme $\mathrm{Gl}_2/\mathbb{Z}$ plays a dominant role. In this case the irreducible representations of $\mathrm{Gl}_2 \times_{\mathbb{Z}} \mathbb{Q}$ are well known. We consider the $\mathbb{Q}$ vector space of homogenous polynomials in two variables and of degree n

$$\mathcal{M}_{n,\mathbb{Q}} := \{P(X,Y) = \sum_{\nu=0}^{n} a_\nu X^\nu Y^{n-\nu} | a_\nu \in \mathbb{Q}\}. \tag{1.56}$$

We choose an integer m define an action of $\mathrm{Gl}_2(\mathbb{Q})$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} P(X,Y) = P(aX+cY, bX+dY)\det(\begin{pmatrix} a & b \\ c & d \end{pmatrix})^m, \tag{1.57}$$

this gives us the $\mathrm{Gl}_2/\mathbb{Q}$-module $\mathcal{M}_{n,\mathbb{Q}}[m]$.

But now it is easy to get $\mathrm{Gl}_2/\mathbb{Z}$-modules, we simply define

$$\mathcal{M}_n := \{P(X,Y) = \sum_{\nu=0}^{n} a_\nu X^\nu Y^{n-\nu} | a_\nu \in \mathbb{Z}\} \tag{1.58}$$

and then we define the $\mathrm{Gl}_2/\mathbb{Z}$ modules $\mathcal{M}_n[m]$ by the same formula as above. If n is even we will sometimes with the module $\mathcal{M}[-\frac{n}{2}]$. (See following remark).

At this point a small remark is in order. If look at $\mathcal{M}_n[m]$ only as $\mathrm{Gl}_2(\mathbb{Z})$-module then the module "knows" what n is, clearly $n = \mathrm{rank}(\mathcal{M}_n) - 1$. But this $\mathrm{Gl}_2(\mathbb{Z})$- module does not "know" what m is. The only information we get is

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} P = (-1)^m P$$

and from this we only get the value of $m \mod 2$. But if we consider $\mathcal{M}_n[m]$ as module for the group scheme $\mathrm{Gl}_2/\mathbb{Z}$ then the module also knows the value of m because then we know

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} P = \alpha^m P$$

for any $\alpha \in R^\times$ in any commutative ring R with identity. If n is even we may consider the module $\mathcal{M}_n[-\frac{n}{2}]$, this is a module for $\mathrm{PGl}_2/\mathbb{Z} = \mathrm{Gl}_2/\mathbb{G}_m$.

In section 4.1.1 we discuss the corresponding situation for groups $\mathrm{Gl}_2(\mathbb{Z}[\sqrt{-d}])$.

1.2.3 The locally symmetric spaces

We start from a semisimple group $G/\mathbb{Q}$. To this group we attached the group of real points $G(\mathbb{R}) = G_\infty$. In G_∞ we have the connected component G_∞^0 of the identity and in this group we choose a maximal compact subgroup K_∞. The quotient space $X = G_\infty/K_\infty$ is a symmetric space which now may have several connected components. On this space we have the action of an arithmetic group Γ.

We have a fundamental fact:

The action of Γ on X is properly discontinuous, i.e. for any point $x \in X$ there exists an open neighbourhood U_x such that for all $\gamma \in \Gamma$ we have

$$\gamma U_x \cap U_x = \emptyset \quad \text{or} \quad \gamma x = x.$$

Moreover for all $x \in X$ the stabilizer

$$\Gamma_x = \{\gamma \mid \gamma x = x\}$$

is finite.

This is easy to see: If we consider the projection $p : G(\mathbb{R}) \to G(\mathbb{R})K_\infty = X$, then the inverse image $p^{-1}(U_x)$ of a relatively compact neighbourhood U_x of $x = g_0K_\infty$ is of the form $V_{g_0} \cdot K_\infty$, where V_{g_0} is a relatively compact neighbourhood of g_0. Hence we look for the solutions of the equation

$$\gamma vk = v'k', \gamma \in \Gamma, v, v' \in V_{g_0}, k, k' \in K_\infty.$$

Since Γ is discrete in $G(\mathbb{R})$ there are only finitely many possibilities for γ and they can be ruled out by shrinking U_x with the exception of those γ for which $\gamma x = x$. If $\gamma x = x$ this means that $\gamma g_0 K_\infty = g_0 K_\infty$ and hence $\gamma \in \Gamma \cap g_0 K_\infty g_0^{-1}$ this intersection is a compact discrete set, hence finite.

If Γ has no torsion then the projection

$$\pi : X \longrightarrow \Gamma\backslash X$$

is locally a $\mathcal{C}_\infty$-diffeomorphism. To any point $x \in \Gamma\backslash X$ and any point $\tilde{x} \in \pi^{-1}(x)$ we find a neighbourhood $U_{\tilde{x}}$ such that

$$\pi : U_{\tilde{x}} \xrightarrow{\sim} U_x.$$

Hence the space $\Gamma\backslash X$ inherits the Riemannian metric and the quotient space is a *locally symmetric space* .

If our group Γ has torsion, then a point $\tilde{x} \in X$ may have a nontrivial stabilizer $\Gamma_{\tilde{x}}$. Then it is not difficult to prove that $\tilde{x}$ has a neighbourhood $U_{\tilde{x}}$ which is invariant under $\Gamma_{\tilde{x}}$ and that for all $\tilde{y} \in U_{\tilde{x}}$ the stabilizer $\Gamma_{\tilde{y}} \subset \Gamma_{\tilde{x}}$. This gives us a diagram

$$\begin{array}{ccc} U_{\tilde{x}} & \longrightarrow & \Gamma_{\tilde{x}}\backslash U_{\tilde{x}} = U_x \\ \downarrow & & \downarrow \\ X & \xrightarrow{\pi} & \Gamma\backslash X \end{array}$$

i.e. the point $x \in \Gamma\backslash X$ has a neighbourhood which is the quotient of a neighbourhood $U_{\tilde{x}}$ by a finite group.

In this case the quotient space $\Gamma\backslash X$ may have singularities. Such spaces are called orbifolds. They have a natural stratification. Any point x defines a Γ conjugacy class $[\Gamma_{\tilde{x}}]$ of finite subgroups $\Gamma_{\tilde{x}} \subset \Gamma$. On the other hand a conjugacy class $[c]$ of finite subgroups $H \subset \Gamma$ defines the (non empty) subset (stratum) $\Gamma\backslash X([c])$ of those points $x \in \Gamma\backslash X$ for which $\Gamma_{\tilde{x}} \in [c]$.

These strata are easy to describe. We observe that for any finite $H \subset \Gamma$ the fixed point set X^H intersected with a connected component of X is contractible. Let $x_0 \in X^H$ be a point with $\Gamma_{x_0} = H$. Then any other point $x \in X^H$ is of the form $x = gx_0$ with $g \in G(\mathbb{R})$. This implies that $g \in N(H)(\mathbb{R})$, where $N(H)$ is the normaliser of H, it is an algebraic subgroup. Then $N(H)(\mathbb{R}) \cap K_\infty = K_\infty^H$ is compact subgroup, put $\Gamma^H = \Gamma \cap N(H)(\mathbb{R})$, and we get an embedding

$$\Gamma^H\backslash X^H \hookrightarrow \Gamma\backslash X.$$

This space contains the open subset $(\Gamma^H\backslash X^H)^{(0)}$ of those x where $H \in [\Gamma_{\tilde{x}}]$ and this is in fact the stratum attached to the conjugacy class of H.

We have an ordering on the set of conjugacy classes, we have $[c_1] \leq [c_2]$ if for any $H_1 \in [c_1]$ there exists a subgroup $H_2 \in [c_2]$ such that $H_1 \subset H_2$. These strata are not closed, the closure $\overline{\Gamma\backslash X([c])}$ is the union of lower dimensional strata.

If we start investigating the stratification above we immediately hit upon number theoretic problems. Let us pick a prime p and we consider the group $\Gamma = \mathrm{Sl}_{p-1}[\mathbb{Z}]$ and the ring of p-th roots of unity $\mathbb{Z}[\zeta_p]$ as a $\mathbb{Z}$-module is free of rank $p-1$ and hence we get an element

$$\zeta_p \in \mathrm{Sl}(\mathbb{Z}[\zeta_p]) = \mathrm{Sl}_{p-1}(\mathbb{Z})$$

and hence a cyclic subgroup of order p. But clearly we have many conjugacy classes of elements of order p in Γ because any ideal $\mathfrak{a}$ is a free $\mathbb{Z}$-module. If we want to understand the conjugacy classes of elements of order p or the conjugacy classes of cyclic subgroups of order p in $\mathrm{Sl}_{p-1}(\mathbb{Z})$ we need to understand the ideal class group. In the next section we will discuss some simple examples.

These quotient spaces $\Gamma\backslash X$ attract the attention of various different kinds of mathematicians. They provide interesting examples of Riemannian manifolds and they are intensively studied from that point of view. On the other hand number theoretic data enter into their construction. Hence any insight into the structure of these spaces contains number theoretic information. This is the main theme of this book.

It is not difficult to see that any arithmetic group Γ contains a normal congruence subgroup Γ' which does not have torsion. This can be deduced easily from the exercise .... at the end of this section. Hence we see that $\Gamma'\backslash X$ is a Riemannian manifold which is a finite cover of $\Gamma\backslash X$ with covering group Γ/Γ'.

We discuss special examples below.

1.2.4 Low dimensional examples

We consider the action of the group $\Gamma = \mathrm{Sl}_2(\mathbb{Z}) \subset \mathrm{Sl}_2(\mathbb{R})$ on the upper half plane

$$X = \mathbb{H} = \{z \mid \Im(z) = y > 0\} = \mathrm{Sl}_2(\mathbb{R})/\mathrm{SO}(2).$$

We want to describe the quotient $\Gamma\backslash\mathbb{H}$, for this purpose we construct further down the fundamental domain $\mathcal{F}$

As we explained in .section 1.1.9 we may consider the point $z = x + iy$ as a positive definite euclidian metric on $\mathbb{R}^2$ up to a positive scalar. We saw already that this metric can be interpreted as the metric on $\mathbb{C}$ induced on the lattice $\Omega = \langle 1, z\rangle$. The action of $\mathrm{Sl}_2(\mathbb{Z})$ on the upper half plane corresponds to changing the basis $1, z$ of Ω into another basis and then normalising the first vector of the new basis to length equal one. This means that under the action of $\mathrm{Sl}_2(\mathbb{Z})$ we may achieve that the first vector 1 in the lattice is of shortest length. In other words $\Omega = \langle 1, z\rangle$ where now $|z| \geq 1$.

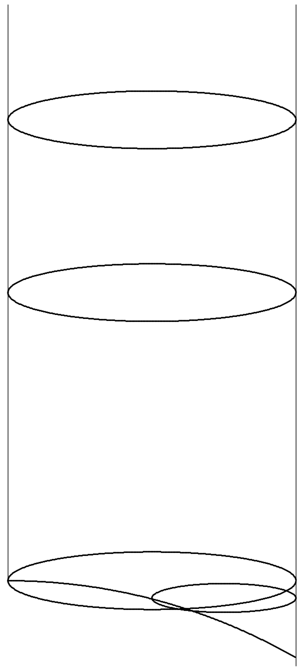
Since we can change the basis by $1 \to 1$ and $z \to z + n$. We still have $|z + n| \geq 1$. Hence see that this condition implies that we can move z by these translation into the strip $-1/2 \leq \Re(z) \leq 1/2$ and since 1 is still the shortest vector we end up in the classical fundamental domain:

$$\mathcal{F} = \{z| - 1/2 \leq \Re(z) \leq 1/2, |z| \geq 1\} \tag{1.59}$$

Two points $z_1, z_2 \in \mathcal{F}$ are inequivalent under the action of $\mathrm{Sl}_2(\mathbb{Z})$ unless they differ by a translation. i.e.

$$z_1 = -\frac{1}{2} + it \ , \ z_2 = z_1 + 1 = \frac{1}{2} + it,$$

or we have $|z_1| = 1$ and $z_2 = -\frac{1}{z_1}$. Hence the quotient $\mathrm{Sl}_2(\mathbb{Z})\backslash\mathbb{H}$ is given by the following picture



The circles are actually the images of horizontal lines $iy + x$ where $x \in \mathbb{R}$ or $x \in [9, 1]$ in the quotient. The picture is a little bit misleading because it does not reflect the Riemannian metric: The circumference of the circle at level iy is $\frac{1}{y}$.

It turns out that this quotient is actually a Riemann surface, i.e. the finite stabilisers at i and ρ do not produce singularities. As a Riemann surface the quotient is the complex plane or better the projective line $\mathbb{P}^1(\mathbb{C})$ minus the point at infinity.

It is clear that the points i and $\rho = +\frac{1}{2} + \frac{1}{2}\sqrt{-3}$ in the upper half plane are -up to conjugation by an element $\gamma \in \mathrm{Sl}_2(\mathbb{Z})$- the only points with non-trivial stabiliser . Actually the stabilisers are given by

$$\Gamma_i = \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\} \quad , \quad \Gamma_\rho = \left\{ \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \right\}.$$

The second example is given by the group $\Gamma = \mathrm{Sl}_2(\mathbb{Z}[i]) \subset \mathrm{Sl}_2(\mathbb{C}) = G_\infty = R_{\mathbb{C}/\mathbb{R}}(Gl_2/\mathbb{C})(\mathbb{R})$ (See(1.1) . Here we should remember that the choice of G_∞ allows a whole series of arithmetic groups. For any imaginary quadratic extension $K = \mathbb{Q}(\sqrt{-d})$ with $\mathcal{O}_K$ as its ring of integers we may embed K into $\mathbb{C}$ and get

$$\mathrm{Sl}_2(\mathcal{O}_K) = \Gamma \subset G_\infty.$$

If the number d becomes larger then the structure of the group Γ becomes more and more complicated. We discuss only the simplest case.

We will construct a fundamental domain for the action of Γ on the three-dimensional hyperbolic space $\mathbb{H}_3 = \mathbb{C} \times \mathbb{R}_{>0}$.

We identify $\mathbb{H}_3$ with the space of positive definite hermitian matrices

$$X = \{A \in M_2(\mathbb{C}) \mid A = {}^t\overline{A}, A > 0, \det(A) = 1\}.$$

We consider the lattice

$$\Omega = \mathbb{Z}[i] \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathbb{Z}[i] \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

in $\mathbb{C}^2$ and view A as a hermitian metric on $\mathbb{C}^2$ where $\mathbb{C}/\Omega$ has volume 1. Let $e_1' = \binom{\alpha}{\beta}$ be a vector of shortest length. We can find a second vector $e_2' = \binom{\gamma}{\delta}$ so that $\det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = 1$. This argument is only valid because $\mathbb{Z}[i]$ is a principal ideal domain. We consider the vectors $e_2' + \nu e_1'$ where $\nu \in \mathbb{Z}[i]$. We have

$$\langle e_2' + \nu e_1', e_2' + \nu e_1' \rangle_A = \langle e_2' + \nu e : 1' \rangle_A + \nu \langle e_1', e_2' \rangle_A + \overline{\nu} \langle e_2', e_1' \rangle_a + \nu\overline{\nu} \langle e_1', e_1' \rangle_A.$$

Since we have theeuclidean algorithm in $\mathbb{Z}[i]$ we can choose ν such that

$$-\frac{1}{2}\langle e_1', e_1' \rangle \leq \mathrm{Re}\langle e_1', e_2' \rangle_A, \Im\langle e_1', e_2' \rangle_A \leq \frac{1}{2}\langle e_1', e_1' \rangle_A.$$

If we translate this to the action of $\mathrm{Sl}_2(\mathbb{Z}[i])$ on $\mathbb{H}_3$ then we find that every point $x = (z; \zeta) \in \mathbb{H}_3$ is equivalent to a point in the domain

$$\tilde{F} = \{(z, \zeta) \mid -\frac{1}{2} \leq \mathrm{Re}(z), \Im(z) \leq \frac{1}{2}; z\overline{z} + \zeta^2 \geq 1\}.$$

Since we have still the action of the matrix $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ we even find a smaller fundamental domain

$$F = \{(z, \zeta) \mid -\frac{1}{2} \leq \mathrm{Re}(z), \Im(z) \leq \frac{1}{2}; z\overline{z} + \zeta^2 \geq 1 \text{ and } \mathrm{Re}(z) + \Im(z) \geq 0\}.$$

I want to discuss also the extension of our considerations to the case of the reductive group $\mathrm{Gl}_2(\mathbb{C})$. In such a case we have to enlarge the maximal compact

Figure 1.1: Fundamental Domain

subgroup. In this case the group $\tilde{K} = \mathrm{Sl}_1(2)\cdot\mathbb{C}^* = K\cdot\mathbb{C}^*$ is a good choice where $\mathbb{C}^*$ is the centre of $\mathrm{Gl}_2(\mathbb{C})$. Then we get

$$\mathbb{H}_3 = \mathrm{Sl}_2(\mathbb{C})/K = \mathrm{Gl}_2(\mathbb{C})/\tilde{K}$$

i.e. we have still the same symmetric space. But the group $\tilde{\Gamma} = \mathrm{Gl}_2(\mathbb{Z}[i])$ is still larger. We have an exact sequence

$$1 \to \Gamma \to \tilde{\Gamma} \to \{i^\nu\} \to 1.$$

The centre $Z_{\tilde{\Gamma}}$ of $\tilde{\Gamma}$ is given by the matrices $\left\{\begin{pmatrix} i^v & 0 \\ 0 & i^v \end{pmatrix}\right\}$. The centre Z_Γ has index 2 in $Z_{\tilde{\Gamma}}$. Since the centre acts trivially on the symmetric space, hence the above fundamental domain will be "cut into two halfes" by the action of $\tilde{\Gamma}$. the matrices $\begin{pmatrix} i^v & 0 \\ 0 & 1 \end{pmatrix}$ induce rotation of $\nu\cdot 90°$ around the axis $z=0$ and therefore it becomes clear that the region

$$F_0 = \{(z,\zeta) \mid 0 \le \Im(z), \mathrm{Re}(z) \le \frac{1}{2}, z\overline{z}+\zeta^2 \ge 1\}$$

is a fundamental domain for $\tilde{\Gamma}$.

The translations $z \to z+1$ and $z \to z+i$ identify the opposite faces of F. This induces an identification on F_0, namely

$$\left(\frac{1}{2}+iy,\zeta\right) \longrightarrow \left(-\frac{1}{2}+iy,\zeta\right) \longrightarrow \left(y+\frac{i}{2},\zeta\right).$$

On the bottom of the domain F_0, namely

$$F_0(1) = \{(z,\zeta) \in F_0 \mid z\overline{z}+\zeta^2 = 1\}$$

we have the further identification

$$(z,\zeta) \longrightarrow (i\overline{z},\zeta).$$

Hence we see that the quotient space $\tilde{\Gamma}\backslash\mathbb{H}_3$ is given by the following figure.

I want to discuss the fixed points and the stabilizers of the fixed points of $\tilde{\Gamma}$. Before I can do that, I need some simple facts concerning the structure of Gl_2.

The group $\mathrm{Gl}_2(K)$ acts upon the projective line $\mathbb{P}^1(K) = (K^2 \setminus \{0\})/K^*$. We write

$$\mathbb{P}^1(K) = (K) \cup \{\infty\}\ ;\ K(xe_1+e_2) = x, Ke_1 = \infty.$$

It is quite clear that the action of $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{Gl}_2(K)$ is given by

$$gx = \frac{\alpha x+\beta}{\gamma x+\delta}.$$

The action of $\mathrm{Gl}_2(K)$ on $\mathbb{P}^1(K)$ is transitive. For a point $x \in \mathbb{P}^1(K)$ the stabilizer B_x is clearly a linear subgroup of Gl_2/K. If $x=\infty$, then this stabilizer is the subgroup

$$B_\infty = \left\{\begin{pmatrix} a & u \\ 0 & b \end{pmatrix}\right\},$$

and for $x = 0$ we get

$$B_0 = \left\{ \begin{pmatrix} a & 0 \\ u & b \end{pmatrix} \right\}.$$

It is clear that these subgroups B_x are conjugate under the action of $\mathrm{Gl}_2(K)$. They are in fact maximal solbable subgroups of Gl_2.

If we have two different points $x_1, x_2 \in \mathbb{P}^1(K)$, then this corresponds to a choice of a basis where the basis vectors are only determined up to scalars. Then the intersection of the two groups $B_{x_1} \cap B_{x_2}$ is a so-called maximal torus. If we choose $x_1 = Ke_1$, $x_2 = Ke_2$, then

$$B_{x_1} \cap B_{x_2} = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right\}.$$

Any other maximal torus of the form B_{x_1}, B_2 is conjugate to T_0 under $\mathrm{Gl}_2(K)$.

Now we assume $K = \mathbb{C}$. We compactify the three dimensional hyperbolic space by adding $\mathbb{P}^1(\mathbb{C})$ at infinity, i.e.

$$\mathbb{H}_3 \hookrightarrow \overline{\mathbb{H}}_3 = \mathbb{H}_3 \cup \mathbb{P}^1(\mathbb{C}) = \mathbb{C} \times \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

(The reader should verify that there is a natural topology on $\overline{\mathbb{H}}_3$ for which the space is compact and for which $\mathrm{Gl}_2(\mathbb{C})$ acts continuously.)

Now let us assume that $a \in \mathrm{Gl}_2(\mathbb{C})$ is an element which has a fixed point on $\mathbb{H}_3$ and which is not central. Since it lies in a maximal compact subgroup times $\mathbb{C}^x$ we see that this element a can be diagonalized

$$a \longrightarrow g_0\ a\ g_0^{-1} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} = a'$$

with $\alpha \neq \beta$ and $|\alpha/\beta| = 1$.

Then it is clear that the fixed point set for a' is the line

$$\mathrm{Fix}\ (a') = \{(0, \zeta) \mid \zeta \in \mathbb{R}_{>0}\},$$

i.e. we do not get an isolated fixed point but a full fixed line.

The element a' has the two fixed points $\infty, 0$ in $\mathbb{P}^1(\mathbb{C})$, and hence ist defines the torus $T_0(\mathbb{C})$. Then it is clear that

$$\mathrm{Fix}(a') = \{(0, \zeta) \mid \zeta > 0\} = T_0(\mathbb{C}) \cdot (0, 1)$$

i.e. the fixed point set is an orbit under the action of $T_0(\mathbb{C})$.

1.2.5 Fixed point sets and stabilizers for $\mathrm{Gl}_2(\mathbb{Z}[i]) = \tilde{\Gamma}$

If we want to describe the stabilizers up to conjugation, we can focus our attention on F_0.

If we have an element $\gamma \in \tilde{\Gamma}$, γ not central and if we assume that γ has fixed points on $\mathbb{H}_3$, then we know that γ defines a torus $T_\gamma = \mathrm{centralizer}_{\mathrm{Gl}_2}(\gamma) =$ stabilizer of $x_\gamma, x_{\gamma'} \in \mathbb{P}^1(\mathbb{C})$. This torus is defined over $\mathbb{Q}(i)$, but it is not necessarily diagonalizable over $\mathbb{Q}(i)$, it may be that the coordinates of $x_\gamma, x_{\gamma'}$ lie in a quadratic extension of $F/\mathbb{Q}(i)$. This is the quadratic extension defined by the eigenvalues of γ.

We look at the edges of the fundamental domain F_0. We saw that they consist of connected pieces of the straight lines

$$G_1 = \{(z,\zeta) \mid z = 0\}, G_2 = \{(z,\zeta) \mid z = \frac{1}{2}\}, G_3 = \{(z,\zeta) \mid z = \frac{1+i}{2}\},$$

and the circles (these circles are euclidean circles and geodesics for the hyperbolic metric)

$$D_1 = \{(z,\zeta) \mid z\overline{z} + \zeta^2 = 1, \Im(z) = \mathrm{Re}(z)\}, D_2 = \{(z,\zeta) \mid z\overline{z} + \zeta^2 = 1, \Im(z) = 0\},$$

$$D_3 = \{(z,\zeta) \mid z\overline{z} + \zeta^2 = 1, \mathrm{Re}(z) = \frac{1}{2}\}.$$

The pair of points $(\infty, (z_0, 0)) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ has as its stabilizer

$$T_{z_0}(\mathbb{C}) = \begin{pmatrix} 1 & z_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & -z_0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & z_0(\beta - \alpha) \\ 0 & \beta \end{pmatrix},$$

the straight line $\{(z_0, \zeta) \mid \zeta > 0\}$ is an orbit u nder $T_{z_0}(\mathbb{C})$ and it consists of fixed points for

$$T_{z_0}(\mathbb{C})(1) = \left\{ \begin{pmatrix} \alpha & z_0(\beta - \alpha) \\ 0 & \beta \end{pmatrix} \middle| \alpha/\beta \in S^1 \right\}.$$

We can easily compute the pointwise stabilizer of G_1, G_2, G_3 in $\tilde{\Gamma}$. They are

$$\begin{aligned} \tilde{\Gamma}_{G_1} &= \left\{ \begin{pmatrix} i^\nu & 0 \\ 0 & i^\mu \end{pmatrix} \right\} = \left\{ \begin{pmatrix} i^\nu & 0 \\ 0 & i \end{pmatrix} \right\} \cdot z_{\tilde{\Gamma}} \\ \Gamma_{\tilde{G}_2} &= \left\{ \begin{pmatrix} i^\nu & \frac{1-i^\nu}{2} \\ 0 & 1 \end{pmatrix} \middle| \frac{1-i^\nu}{2} \in \mathbb{Z}[i] \right\} \cdot Z_{\tilde{\Gamma}} = \left\{ \begin{pmatrix} \pm 1 & \frac{1 \pm 1}{2} \\ 0 & 1 \end{pmatrix} \right\} \cdot Z_{\tilde{\Gamma}} \\ \Gamma_{\tilde{G}_3} &= \left\{ \begin{pmatrix} i^\nu & \frac{(1-i^\nu)(1+i)}{2} \\ 0 & 1 \end{pmatrix} \right\} \cdot Z_{\tilde{\Gamma}}, \end{aligned}$$

where in the last case we have to take into account that $\frac{(1-i^\nu)(1+i)}{2} \in \mathbb{Z}[i]$ for all ν.

Hence modulo the centre $Z_{\tilde{\Gamma}}$ these stabilizers are cyclic groups of order $4, 2, 4$.

The arcs D_i are also pointwise fixed under the action of certain cyclic groups, namely

$$\begin{aligned} D_1 =& \mathrm{Fix}\left(\begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix} \right) \\ D_2 =& \mathrm{Fix}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \\ D_3 =& \mathrm{Fix}\left(\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \right), \end{aligned}$$

and we check easily that these arcs are geodesics joining the following points in the boundary

D_1 runs from $\sqrt{i}$ to $-\sqrt{i}$

D_2 runs from i to $-i$

D_3 runs from $e = e^{\frac{1\pi i}{6}} = e^{\frac{\pi i}{3}}$ to $\overline{\rho}$.

The corresponding tori are

$$\begin{aligned} T_1 =& \text{Stab}(-1,1) = \left\{ \begin{pmatrix} \alpha & i\beta \\ \beta & \alpha \end{pmatrix} \right\} \\ T_2 =& \text{Stab}(-\sqrt{i},\sqrt{i}) = \left\{ \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \right\} \\ T_3 =& \text{Stab}(\rho,\overline{\rho}) = \left\{ \begin{pmatrix} \delta-\beta & \beta \\ -\beta & \delta \end{pmatrix} \right\}. \end{aligned}$$

The torus T_2 splits over $\mathbb{Q}(i)$, the other two tori split over an quadratic extension of $\mathbb{Q}(i)$.

Now it is not difficult anymore to describe the finite stabilizers and the corresponding fixed point sets. If $x \in \mathbb{H}_3$ for which the stabilizer is bigger than $Z_{\tilde{\Gamma}}$, then we can conjugate x into F_0. It is very easy to see that x cannot lie in the interior of F_0 because then we would get an identification of two points nearby x and hence still in F_0 under $\tilde{\Gamma}$.

If x is on one of the lines D_1, D_2, D_3 or on one of the arcs G_1, G_2, G_3 but not on the intersection of two of them, then the stabilizer Γ_x is equal to $Z_{\tilde{\Gamma}}$ times the cyclic group we attached to the line or the arc earlier. Finally we are left with the three special points

$$\begin{aligned} x_{12} =& D_1 \cap D_2 \cap G_1 = \{(0,1)\} \\ x_{13} =& D_1 \cap D_3 \cap G_3 = \left\{ \left(\frac{1+i}{2}, \frac{\sqrt{2}}{2} \right) \right\} \\ x_{23} =& D_2 \cap D_3 \cap G_2 = \left\{ \left(\frac{1}{2}, \frac{\sqrt{3}}{2} \right) \right\}. \end{aligned}$$

In this case it is clear that the stabilizers are given by

$$\begin{aligned} \tilde{\Gamma}_{x_{12}} =& \langle \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} \rangle = D_4 \\ \tilde{\Gamma}_{x_{13}} =& \langle \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} i & 1 \\ 0 & 1 \end{pmatrix} \rangle = S_4 \\ \tilde{\Gamma}_{x_{23}} =& \langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \rangle = S_3. \end{aligned}$$

1.2.6 Compactification of $\Gamma\backslash X$

Our two special low dimensional examples show clearly that the quotient spaces $\Gamma\backslash X$ are not compact in general. There exist various constructions to compactify them.

If, for instance, $\Gamma \subset \text{Sl}_2(\mathbb{Z})$ is a subgroup of finite index, then the quotient $\Gamma\backslash\mathbb{H}$ is a Riemann surface. It can be embedded into a compact Riemann surface by adding a finite number of points. this is a special case of a more general theorem of Satake and Baily-Borel: If the symmetric space X is actually hermitian symmetric (this means it has a complex structure) then we have the structure of a quasi-projective variety on $\Gamma\backslash X$. This is the so-called Baily-Borel compactification. It exists only under special circumstances.

I will discuss the process of compactification in some more detail for our special low dimensional examples.

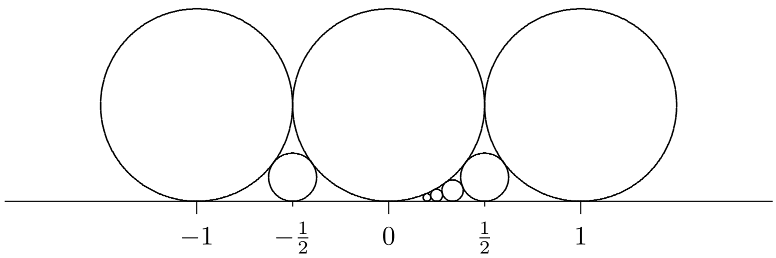
Compactification of $\mathrm{Sl}_2(\mathbb{Z})\backslash\mathbb{H}$ by adding points

Let $\Gamma \subset \mathrm{Sl}_2(\mathbb{Z})$ be any subgroup of finite index. The group Γ acts on the rational projective line $\mathbb{P}^1(\mathbb{Q})$. We add it to the upper half plane and form

$$\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q}),$$

and we extend the action of Γ to this space. Since the full group $\mathrm{Sl}_2(\mathbb{Z})$ acts transitively on $\mathbb{P}^1(\mathbb{Q})$ we find that Γ has only finitely many orbits on $\mathbb{P}^1(\mathbb{Q})$.

Now we introduce a topology on $\overline{\mathbb{H}}$. We defined a system of neighbourhoods of points $\frac{p}{q} = r \in \mathbb{P}^1(\mathbb{Q})$. We define the Farey circles $S\left(c, \frac{p}{q}\right)$ which touch the real axis in the point $r = p/q$ $(p,q) = 1$ and have the radius $\frac{c}{2q^2}$. For $c = 1$ we get the picture



Let us denote by $D\left(c, \frac{p}{q}\right) = \cup_{c':0<c'\leq c} S\left(c', \frac{p}{q}\right)$ the Farey disks. For $c \to 0$ these Farey disks $D\left(c, \frac{p}{q}\right)$ define a system of neighbourhoods of the point $r = p/q$. The Farey disks at $\infty \in \mathbb{P}^1(\mathbb{Q})$ are given by the regions

$$D(T, \infty) = \{z \mid \Im(z) \geq T\}.$$

It is easy to check that an element $\gamma \in \mathrm{Sl}_2(\mathbb{Z})$ which sends $\infty \in \mathbb{P}^1(\mathbb{Q})$ into the point $r = \frac{p}{q}$ sends $D(T, \infty)$ to $D\left(\frac{1}{T}, \frac{p}{q}\right)$. These Farey disks $D(c, r)$ do not meet provided we take $c < 1$. The considerations in 1.6.1 imply that the complement of the union of Farey disks is relatively compact modulo Γ, and since Γ has finitely many orbits on $\mathbb{P}^1(\mathbb{Q})$, we see easily that

$$Y_\Gamma = \Gamma\backslash\overline{\mathbb{H}}$$

is compact (which means of course also Hausdorff).

It is essential that the set of Farey circles $D(c, r)$ and $D\left(\frac{1}{c}, \infty\right)$ is invariant under the action of Γ on the one hand and decomposes into several connected components (which are labeled by the point $r \in \mathbb{P}^1(\mathbb{Q})$) on the other hand. Hence

$$\Gamma\backslash\bigcup_r D(c, r) = \bigcup \Gamma_{r_i}\backslash D(c, r_i)$$

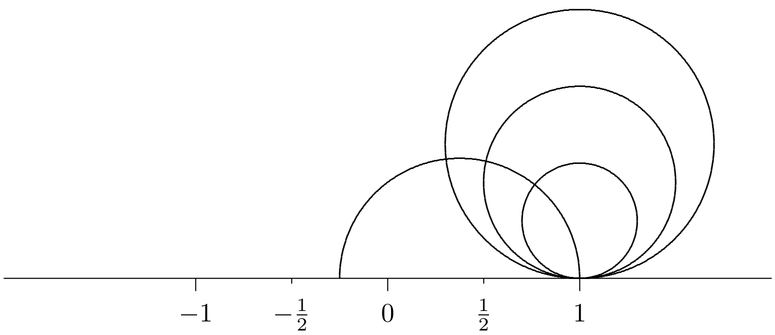
where r_i is a set of representatives for the action of Γ on $\mathbb{P}^1(\mathbb{Q})$ and where Γ_{r_i} is the stabilizer of r_i in Γ.

It is now clear that $\Gamma_{r_i}\backslash D(c, r_i)$ is holomorphically equivalent to a punctured disc and hence the above compactification is obtained by filling the point into this punctured disc and this makes it clear that Y_Γ is a Riemann surface.

BSC

1.2.7 The Borel-Serre compactification of $\mathrm{Sl}_2(\mathbb{Z})\backslash\mathbb{H}$

There is another construction of a compactification. We look at the disks $D(c, r)$ and divide them by the action of Γ_r. For any point $y \in S(c', r) - \{r\}$ there exists a unique geodesic joining r and y, passing orthogonally through $S(c', r)$ and hitting the projective line in another point y_∞ ($= -1/4$ in the picture below)



If $r = \infty$, then this system of geodesics is given by the vertical lines $\{y \cdot I + x \mid x \in \mathbb{R}\}$.. This allows us to write the set

$$D(c, r) - \{r\} = X_{\infty,r} \times [c, 0)$$

where $X_{\infty,r} = \mathbb{P}^1(\mathbb{R}) - \{r\}$. The stabilizer Γ_r acts $D(c, r)$ and on the right hand side of the identification it acts on the first factor, the quotient $\Gamma_r\backslash X_{\infty,r}$ is a circle. Hence we can compactify the quotient

$$\Gamma_r\backslash D(c, r) - \{r\} \hookrightarrow \Gamma_r\backslash X_{\infty,r} \times [c, 0].$$

This gives us a second way to compactify $\Gamma\backslash\mathbb{H}$, we apply this process to a finite set of representatives of $\mathbb{P}^1(\mathbb{Q}) \mod \Gamma$.

There is a slightly different way of looking at this. We may form the union

$$\mathbb{H} \cup \bigcup_r X_{\infty,r} = \tilde{\mathbb{H}}$$

and topologize it in such a way that

$$D(c, r) = X_{\infty,r} \times [c, 0) \subset X_{\infty,r} \times [c, 0] \tag{1.60}$$

is a local homeomorphism. Then we see that the compactification above is just the quotient $\Gamma\backslash\tilde{\mathbb{H}}$ and the boundary is simply

$$\partial(\Gamma\backslash\bar{\mathbb{H}}) = \Gamma\backslash \bigcup_{r\in\mathbb{P}^1(\mathbb{Q}} X_{\infty,r}. \tag{1.61}$$

This compactification is called the Borel-Serre compactification. Its relation to the Baily-Borel is such that the latter is obtained by the former by collapsing the circles at infinity to a point.

It is quite clear that a similar construction applies to the action of a group $\Gamma \subset \mathrm{Sl}_2(\mathbb{Z}[i])$ on the three-dimensional hyperbolic space. The Farey circles will be substituted by spheres $S(c,\alpha)$ which touch the complex plane $\{(z,0) \mid z \in \mathbb{C}\} \subset \overline{\mathbb{H}}_3$ in the point $(\alpha,0), \alpha \in \mathbb{P}^1(\mathbb{Q}(i))$ and for $\alpha = \infty$ the Farey sphere is the horizontal plane $S(\infty,\zeta_0) = \{(z,\zeta_0) \mid z \in \mathbb{C})$. An element $\gamma \in \Gamma$ which maps $(0,\infty)$ to α maps $S(\infty,\zeta_0)$ to $S(c,\alpha)$, where $c = 1/\zeta_0$. For a given α we may identify the different spheres if we vary c and for any point $\alpha \in \mathbb{P}^1(\mathbb{Q}(i))$ we define $X_{\infty,\alpha} = \mathbb{P}^1(\mathbb{C}) \setminus \{\alpha\}$. Again we can identify

$$D(c,\alpha) \setminus \{\alpha\} = X_{\infty,\alpha} \times (0,c] \subset \overline{D(c,\alpha) \setminus \{\alpha\}} = \partial(\Gamma\backslash\mathbb{H}) = X_{\infty,\alpha} \times [0,c]$$

The stabilizer Γ_α acts on $D(c,\alpha) \setminus \{\alpha\}$ and again this yields an action on the first factor. If we choose $\alpha = \infty$ then

$$\Gamma_\infty = \{\begin{pmatrix} \zeta & a \\ 0 & \zeta^{-1} \end{pmatrix} |\zeta \text{ root of unity}, a \in M_\infty\}$$

where M_∞ is a free rank 2 module in $\mathbb{Z}[i]$. If ζ does not assume the value i then $\Gamma_\infty\backslash X_{\infty,\infty}$ is a two-dimensional torus, a product of two circles. If ζ assumes the value i then $\Gamma_\infty\backslash X_{\infty,\infty}$ is a two dimensional sphere. If course we get the same result for an arbitrary α.

Then we get an action of the group Γ on $\tilde{\mathbb{H}}_3 = \mathbb{H}_3 \cup \bigcup_{\alpha\in\mathbb{P}^1(K)} \overline{D(c,\alpha) \setminus \{\alpha\}}$ and the quotient is compact.

theset of orbits of Γ on $\mathbb{P}^1(\mathbb{Q}(i))$ is finite, these orbits are called the cusps.

BSC0

1.2.8 The Borel-Serre compactification, reduction theory of arithmetic groups

This section could be skipped in a first reading. For the particular groups $\mathrm{Sl}_2/\mathbb{Q}$ or $\mathrm{Sl}_2(\mathbb{Z}[\sqrt{-d})$ this compactification has been discussed in detail in the previous sections. A reader who is interested in the specific applications to number theory which will be discussed in the following chapters 2-5 only needs the results from section 1.2.7.

The Borel-Serre compactification works in complete generality for any semi-simple or reductive group $G/\mathbb{Q}$. To explain it, we need the notion of a parabolic subgroup of $G/\mathbb{Q}$.

A subgroup $P/\mathbb{Q} \hookrightarrow G/\mathbb{Q}$ is parabolic if the quotient variety in the sense of algebraic geometry is a projective variety. We mentioned already earlier that for the group $\mathrm{Gl}_2/\mathbb{Q}$ we have an action of Gl_2 on the projective line $\mathbb{P}^1$ and the stabilizers B_x of the points $x \in \mathbb{P}^1(\mathbb{Q})$ are the so-called Borel subgroups of $\mathrm{Gl}_2/\mathbb{Q}$. They are maximal solvable subgroups and

$$\mathrm{Gl}_2/B_x = \mathbb{P}^1,$$

hence they are also parabolic.

More generally we get parabolic subgroups of $\mathrm{Gl}_n/\mathbb{Q}$, if we choose a flag on the vector space $V = \mathbb{Q}^n = \mathbb{Q}e_1 \oplus \cdots \oplus \mathbb{Q}e_n$. This is an increasing sequence of subspaces

$$\mathcal{F} : (0) = \{(0)\} = V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_k = V.$$

The stabilizer P of such a flag is always a parabolic subgroup; the quotient space

$$G/P = \text{Variety of all flags of the given type},$$

where the type of the flag is the sequence of the dimensions $n_i = \dim V_i$.

These flag varieties (the Grassmannians) are smooth projective schemes over Spec($\mathbb{Z}$) and this implies that any flag $\mathcal{F}$ is induced by a flag

$$\mathcal{F}_{\mathbb{Z}} : (0) = \{(0)\} = L_0 \subset L_1 \subset L_2 \subset \ldots \subset L_k = L = \mathbb{Z}^n \tag{1.62}$$

where $L_i = V_i \cap L$, and of course $L_i \otimes \mathbb{Q} = V_i$. This is the elementary fact which will be used later.

If our group $G/\mathbb{Q}$ is the orthogonal group of a quadratic form

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} a_i x_i^2$$

with $a_i \in K^*$. Then we have to replace the flags by sequences of subspaces

$$\mathcal{F} : 0 \subset W_1 \subset W_2 \ldots \qquad\qquad \subset W_2^{\perp} \subset W_1^{\perp} \subset V,$$

where the W_i are isotropic spaces for the form f, i.e. $f \mid W_i \equiv 0$, and where the $W_i^{\perp}$ are the orthogonal complements of the subspaces. Again the stabilizers of these flags are the parabolic subgroups defined over $\mathbb{Q}$.

Especially, if the form f is anisotropic over $\mathbb{Q}$, i.e. there is no non-zero vector $\underline{x} \in K^n$ with $f(\underline{x}) = 0$, then the group $G/\mathbb{Q}$ does not have any parabolic subgroup over $\mathbb{Q}$. This equivalent to the fact that $G(\mathbb{Q})$ does not have unipotent elements.

These parabolic subgroups always have a unipotent radical U_P which is always the subgroup which acts trivially on the successive quotients of the flag. The unipotent radical is a normal subgroup, the quotient $P/U_P = M$ is a reductive group again, it is called the Levi-quotient of P.

We go back to the group $\mathrm{Gl}_n/\mathbb{Q}$. It contains the standard maximal torus whose R valued points are

$$T_0(R) = \{\underline{t} = \begin{pmatrix} t_1 & 0 & \ldots & 0 \\ 0 & t_2 & \ldots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & t_n \end{pmatrix} \quad \mid \quad t_i \in R^{\times}\} \tag{1.63}$$

It is a subgroup of the Borel subgroup (maximal solvable subgroup or minimal parabolic subgroup) whose R-valued points are

$$B_0(R) = \{\underline{b} = \begin{pmatrix} t_1 & u_{1,2} & \dots & u_{1,n} \\ 0 & t_2 & \dots & u_{2,n} \\ 0 & 0 & \ddots & u_{n-1,n} \\ 0 & 0 & 0 & t_n \end{pmatrix} \quad | \quad t_i \in R^\times\} \tag{1.64}$$

and its unipotent radical U_0 consists of those $b \in B_0$ where all the $t_i = 1$. This unipotent radical contains the one dimensional root subgroups

$$U_{i,j} = \{\begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & x & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\}, x \in R \tag{1.65}$$

where $i < j$, these one dimensional subgroups are isomorphic to the one dimensional additive group $\mathbb{G}_a$. They are normalized by the torus, for an element $t \in T(R)$ and $x_{i,j} \in U_{i,j}(R) = R$ we have

$$\underline{t}x_{i,j}\underline{t}^{-1} = t_i/t_j x_{i,j}. \tag{1.66}$$

For $i = 1, \dots, n, j = 1, \dots, n, i \neq j$ (resp. $i < J$) the characters $\alpha_{i,j}(\underline{t}) = t_i/t_j$ are called the roots (resp. positive roots) of T_0 in Gl_n. We denote these systems of roots by Δ^{Gl_n} (resp)$\Delta_+^{\mathrm{Gl}_n}$. The one dimensional subgroups $U_{i,j}, i \neq j$ are called the root subgroups.

Inside the set of positive roots we have the set of simple roots

$$\pi = \pi^{\mathrm{Gl}_n} = \{\alpha_{1,2}, \dots, \alpha_{i,i+1}, \dots, \alpha_{n-1,n}\} \tag{1.67}$$

If we pass to the semi-simple subgroup $\mathrm{Sl}_n/\mathbb{Q}$ then the torus and the Borel-subgroup has to be replaced by $T_0^{(1)}, B_0^{(1)}$, where we have $\prod_i t_i = 1$. The system of roots does not change, we have $\pi = \pi^{\mathrm{Gl}_n} = \pi^{\mathrm{Sl}_n}$.

We change the notation slightly, for $i = 1, \dots, n-1$ we define $\alpha_i := \alpha_{i,i+1}$ then for $i < j$ we get $\alpha_{i,j} = \alpha_i + \dots \alpha_{j-1}$, and $\pi = \{\alpha_1, \alpha_2, \dots, \alpha_{n-1}\}$

The Borel subgroup B_0 is the stabilizer of the "complete" flag

$$\{0\} \subset \mathbb{Q}e_1 \subset \mathbb{Q}e_1 \oplus \mathbb{Q}e_2 \subset \dots \subset \mathbb{Q}e_1 \oplus \mathbb{Q}e_2 \oplus \dots \oplus \mathbb{Q}e_n, \tag{1.68}$$

the parabolic subgroups $P_0 \supset B_0$ are the stabilizers of "partial" flags

$$\{0\} \subset \mathbb{Q}e_1 \oplus \dots \oplus \mathbb{Q}e_{n_1} \subset \mathbb{Q}e_1 \oplus \dots \oplus \mathbb{Q}e_{n_1} \oplus \mathbb{Q}e_{n_1+1} \oplus \dots \oplus \mathbb{Q}e_{n_1+n_2} \subset \dots \subset \mathbb{Q}^n. \tag{1.69}$$

The parabolic subgroup P_0 also acts on the direct sum of the successive quotients

$$(\mathbb{Q}e_1 \oplus \dots \oplus \mathbb{Q}e_{n_1}) \bigoplus (\mathbb{Q}e_{n_1+1} \oplus \dots \oplus \mathbb{Q}e_{n_1+n_2}) \bigoplus \dots \tag{1.70}$$

and this yields a homomorphism RP

$$r_{P_0} : P_0 \to M_0 = \mathrm{Gl}_{n_1} \times \mathrm{Gl}_{n_2} \times \dots \tag{1.71}$$

hence M_0 is the Levi quotient of P_0. By definition the unipotent radical U_{P_0} of P_0 is the kernel of r_0. The semi-simole component will be $M_0^{(1)} = \mathrm{Sl}_{n_1} \times \mathrm{Sl}_{n_2} \times \dots$

A parabolic subgroups $P_0 \supset B_0$ defines a subset

$$\Delta^{P_0} = \{\alpha_{i,j} \in \Delta^{\mathrm{Gl}_n} \mid U_{i,j} \subset P_0\}$$

and the set decomposes int two sets

$$\Delta^{M_0} = \{\alpha_{i,j} \mid U_{i,j} \text{ and } U_{j,i} \subset \Delta^{P_0}\};\ \Delta^{U_{P_0}} = \Delta^{P_0} \setminus \Delta^{M_0}. \tag{1.72}$$

Intersecting this decomposition with the set π^{Gl_n} yields a disjoint decomposition

$$\pi^{\mathrm{Gl}_n} = \pi^{M_0} \cup \pi^U \tag{1.73}$$

where $\pi^U = \{\alpha_{n_1}, \alpha_{n_1+n_2}, \dots, \}$. In turn any such decomposition of π^{Gl_n} yields a well defined parabolic $P_0 \supset B_0$.

We define the *index* of a parabolic subgroup this is the number

$$d(P) = \#\pi^U \tag{1.74}$$

The proper maximal parabolic subgroups are the ones with $d(P) = 1$.

If we choose another maximal split torus T_1 and a Borel subgroup $B_1 \supset T_1$ then this amounts to the choice of a second ordered basis $v_1, v_2, \dots, v_n$ the v_i are given up to a non zero scalar factor. We can find a $g \in \mathrm{Gl}_n(\mathbb{Q})$ which maps $e_1, e_2, \dots, e_n$ to $v_1, v_2, \dots, v_n$, and hence we can conjugate the pair (B_0, T_0) to (B_1, T_1) and hence the parabolic subgroups containing B_0 into the parabolic subgroups containing B_1. The conjugating element g also identifies

$$i_{T_0,B_0,T_1,B_1} : X^*(T_0) \xrightarrow{\sim} X^*(T_1)$$

and this identification does not depend on the choice of the conjugating element g. This allows us to identify the two set of positive simple roots $\pi^{\mathrm{Gl}_n} \subset X^*(T_0)$ and $\pi \subset X^*(T_1)$. Eventually we can speak of the set π of simple roots of Gl_n. Hence we have the fundamental fact

The $\mathrm{Gl}_n(\mathbb{Q})$ *conjugacy classes of parabolic subgroups* $P/\mathbb{Q}$ *are in one to one correspondence with the subsets* $\pi' = \pi^M$. *Then number of elements in* $\pi \setminus \pi' = \pi^U$ *is called the rank of* P, *the set* π' *is called the type of* P.

We will denote the unipotent radical of P by U_P and thereductive quotient of P by U_P will be denoted by $M_P = P/U_P$. Then $\pi' = \pi^{M_P}$. We will also use a slightly different notation: If P is given then we also use $U(= U_P)$ for the unipotent radical and $M = P/U$ for the reductive quotient.

We formulated this result for $\mathrm{Gl}_n/\mathbb{Q}$ but we can replace $\mathbb{Q}$ by any field k and Gl_n by any reductive group G/k. We have to define the system of relative simple positive roots π^G for any G/k (See [B-T]).

The group G/k itself is also a parabolic subgroup it corresponds to $\pi' = \pi$. We decide that we do not like it and hence we consider only proper parabolic subgroups $P \neq G$, i.e. $\pi' \neq \emptyset$. We can define the Grassmann variety $\mathrm{Gr}^{[\pi']}$ of parabolic subgroups of type π' This is a smooth projective variety and $\mathrm{Gr}^{[\pi']}(\mathbb{Q})$ is the set of parabolic subgroups of type π'.

There is always a unique minimal conjugacy class it corresponds to $\pi' = \emptyset$. (In our examples this minimal class is given by the maximal flags, i.e. those flags where the dimension of the subspaces increases by one at each step (until we reach a maximal isotropic space in the case of an orthogonal group)). The (proper) maximal parabolic subgroups are those for which $\pi' = \pi \setminus \{\alpha_i\}$, i.e. $\pi^{U_{P_i}} = \{\alpha_i\}$

For any parabolic subgroup $P/\mathbb{Q} \subset G/\mathbb{Q}$ we consider the character module $X^*(P) := \mathrm{Hom}(P/\mathbb{Q}, \mathbb{G}_m)$. Since we do not have any non trivial homomorphisms from the unipotent U_P to $\mathbb{G}_m$ we have $\mathrm{Hom}(P/\mathbb{Q}, \mathbb{G}_m) = \mathrm{Hom}(M_P, \mathbb{G}_m)$.

The reductive quotient $M_P = M_P^{(1)} \cdot C_P$ where C_P is the central torus und $M_P^{(1)}$ the semi-simple part (the derived group). The quotient $M_P/M_P^{(1)} = C'_P$ is a torus and $C_P \to C'_P$ is an isogeny. Hence we have

$$\mathrm{Hom}(P/\mathbb{Q}, \mathbb{G}_m) \otimes \mathbb{Q} = \mathrm{Hom}(M_P, \mathbb{G}_m) \otimes \mathbb{Q} = \mathrm{Hom}(C_P, \mathbb{G}_m) \otimes \mathbb{Q} = \mathrm{Hom}(C'_P, \mathbb{G}_m) \otimes \mathbb{Q} \tag{1.75}$$

For a maximal parabolic subgroup P of type $\pi' = \{\alpha_i\}$ we consider the module $\mathrm{Hom}(P, \mathbb{G}_m) \otimes \mathbb{Q} \subset X^*(T) \otimes \mathbb{Q}$. Of course it always contains the determinant and

$$\mathrm{Hom}(P, \mathbb{G}_m) \otimes \mathbb{Q} = \mathbb{Q}\gamma_i \oplus \mathbb{Q}\det$$

where γ_i is

$$\gamma_i(t) = (\prod_{\nu=1}^{\nu=i} t_\nu) \det(t)^{-i/n}. \tag{1.76}$$

These γ_i are called the dominant fundamental weights.

If our maximal parabolic subgroup is $P/\mathbb{Q}$ is defined as the stabilizer of a flag $0 \subset W \subset V = \mathbb{Q}^n$, then the unipotent radical is $U = \mathrm{Hom}(V/W, W)$. An element $y \in P(\mathbb{Q})$ induces linear maps $y_W, y_{V/W}$ and hence $\mathrm{Ad}(y)$ on $U = \mathrm{Hom}(V/W, W)$. We get a character $\gamma_P(y) = \det(\mathrm{Ad}(y)) \in \mathrm{Hom}(P, \mathbb{G}_m)$ which is called the sum of the positive roots. An easy computation shows that

$$n\gamma_i = \gamma_P \tag{1.77}$$

We add points at infinity to our symmetric space: We consider the disjoint union $\cup_{\pi \neq \pi_G} \mathrm{Gr}^{[\pi']}(\mathbb{Q})$ and form the space

$$\overline{X} = X \cup \bigcup_{\pi' \neq \emptyset} \mathrm{Gr}^{[\pi']}(\mathbb{Q}).$$

This is the analogue of or $\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$ in our first example, it is now more complicated because we have several Grassmannians, and we also have maps

$$r_{\pi_1,\pi_2} \mathrm{Gr}^{[\pi_1]}(\mathbb{Q}) \to \mathrm{Gr}^{[\pi_2]}(\mathbb{Q}) \text{ if } \pi_2 \subset \pi_1.$$

Our first aim is to put a topology on this space such that $\Gamma\backslash\overline{X}$ becomes a compact Hausdorff space.

In our first example we interpreted the Farey circle $D\left(c, \frac{p}{q}\right)$ with $0 < c < 1$ as an open subset of points in $\mathbb{H}$, which are close to the point $\frac{p}{q}$, but far away from any other point in $\mathbb{P}^1(\mathbb{Q})$.

The point of reduction theory is that for any parabolic $P \in \mathrm{Gr}^{[\pi']}(\mathbb{Q})$ (here we also allow $P = G$) we will define open sets

$$X^P(\underline{c}_{\pi'}, r(\underline{c}_{\pi'})) \subset X \tag{1.78}$$

which depend on certain parameters $\underline{c}_{\pi'}, r(\underline{c})_{\pi'}$ The points in $X^P(\underline{c}_{\pi'}, r(\underline{c}_{\pi'}))$ should be viewed as the points, which are "very close" to the parabolic subgroup P (controlled by $\underline{c}_{\pi'}$) but "keep a certain distance" (controlled by $r(\underline{c}_{\pi'})$) to the parabolic subgroups $Q \not\supset P$. They are the analogues of the Farey circles. We will see:

a)This system of open sets is invariant under the action $\mathrm{Gl}_n(\mathbb{Z})$

b) For $P = G$ the set $X^G(\emptyset, r_0)$ is relatively compact modulo the action of $\mathrm{Gl}_n(\mathbb{Z})$.

c) Any subgroup $\Gamma \subset \mathrm{Gl}_n(\mathbb{Z})$ has only finitely many orbits on any $\mathrm{Gr}^{[\pi']}(\mathbb{Q})$

d) For a suitable choice of theparameters $\underline{c}_{\pi'}$, and $r(c_{\pi'})$ we have :

$$X = \bigcup_P X^P(\underline{c}_{\pi'}, r(c_{\pi'})) = X^G(\emptyset, r_0) \cup \bigcup_{P:P\text{proper}} X^P(\underline{c}_{\pi'}, r(\underline{c}_{\pi'}))$$

and if P and P_1 are conjugate and $P \neq P_1$ then $X^P(\underline{c}_{\pi'}, r(\underline{c}_{\pi'})) \cap X^{P_1}(\underline{c}_{\pi'}, r(\underline{c}_{\pi'})) = \emptyset$.

Let us assume that we have constructed such a system of open sets, then c) and d) impliy that for a given type π' we have

$$\Gamma \backslash \bigcup_{P:\text{type}(\pi')=\pi} X^P(\underline{c}_{\pi'}, r(\underline{c}_{\pi'})) = \bigcup \Gamma_{P_i} \backslash X^{P_i}(\underline{c}_{\pi'}, r(\underline{c}_{\pi'}))$$

where $\{\dots, P_i, \dots\} = \Sigma(\pi, \Gamma)$ is a set of representatives of $\mathrm{Gr}^{[\pi']}(\mathbb{Q})$ modulo the action of Γ and $\Gamma_{P_i} = \Gamma \cap P_i(\mathbb{Q})$.

This tells us that we have a covering

$$\Gamma \backslash X = \Gamma \backslash X^G(\emptyset, r_0) \cup \bigcup_{\pi' \neq \emptyset} \bigcup_{P \in \Sigma(\pi', \Gamma)} \Gamma_P \backslash X^P(\underline{c}_{\pi'}, r(\underline{c}_{\pi'})) \tag{1.79}$$

The philosophy of reduction theory is that $\Gamma \backslash X^G(\emptyset, r_0)$ *is relatively compact and that we have an explicit description of the sets* $\Gamma_P \backslash X^P(\underline{c}_{\pi'}, r(\underline{c}_{\pi'}))$ *as fiber bundles with nil manifolds as fiber over the locally symmetric spaces* $\Gamma_M \backslash X^M$.

We give the definition of the sets $X^P(\underline{c}_{\pi'}, r(\underline{c}_{\pi'}))$. We stick to the case that $G = \mathrm{Gl}_n/\mathbb{Q}$ and $\Gamma \subset \Gamma_0 = \mathrm{Gl}_n(\mathbb{Z})$ is a (congruence) subgroup of finite index. We defined the positive definite bilinear form (See 1.48)

$$\tilde{B}_{\Theta_x} = -\frac{1}{2n} B_{\Theta_x} : \mathfrak{g}_{\mathbb{R}} \times \mathfrak{g}_{\mathbb{R}} \to \mathbb{R}$$

and we have the identification $\mathfrak{g}_{\mathbb{R}} \xrightarrow{\sim} T_e^{\mathbb{G}(\mathbb{R})}$, and hence we get a euclidian metric on the tangent space $T_e^{\mathbb{G}(\mathbb{R})}$ at the identity e. This extends to a left invariant Riemannian metric on $G(\mathbb{R})$, we denote it by $d_{\Theta_x} s^2$. Hence we get a volume form $d^{\Theta_x}_{\mathrm{vol_H}}$ on any closed subgroup $H(\mathbb{R}) \subset G(\mathbb{R})$.

For any point $x \in X$ and any parabolic subgroup $P/\mathbb{Q}$ with unipotent radical $U/\mathbb{Q})$ we define

$$p_P(P,x) = \mathrm{vol_U}^{\Theta_x}(\Gamma_0 \cap U(\mathbb{R}))\backslash U(\mathbb{R})) \tag{1.80}$$

For the Arakelow-Chevalley scheme $(\mathrm{Gl}_n/\mathbb{Z}, \Theta_0)$ See(1.1.8) we have that $\tilde{B}_{\Theta_0}(E_{i,j}) = 1$. We have by construction

$$U_{i,j}(\mathbb{Z})\backslash U_{i,j}(\mathbb{R}) = \mathbb{R}/\mathbb{Z} \tag{1.81}$$

and under this identification $E_{i,j}$ maps to $\frac{\partial}{\partial x}$. Hence we get

$$d^{\Theta_0}_{\mathrm{vol_{U_{i,j}}}}(U_{i,j}(\mathbb{Z})\backslash U_{i,j}(\mathbb{R})) = 1$$

and from this we get immediately

Proposition 1.2.1. *For any parabolic subgroup P_0 containing the torus T_0 we have*

$$p_P(P_0, \Theta_0) = 1.$$

Let $(L, <\ ,\ >_x)$ be an Arakelow vector bundle and $(\mathrm{Gl}_n, \Theta_x)$ the corresponding Arakelow group scheme (of type Gl_n) let

$$\mathcal{F}_{\mathbb{Z}} : (0) = \{(0)\} = L_0 \subset L_1 \subset L_2 \subset \ldots \subset L_k = L = \mathbb{Z}^n$$

be a flag and $P/\mathbb{Z}$ the corresponding parabolic subgroup. Then we have the homomorphism

$$r_P : P/\operatorname{Spec}(\mathbb{Z}) \to M/\mathbb{Z} = \prod_{i=1}^{i=k} \mathrm{Gl}(L_i/L_{i-1}) \tag{1.82}$$

with kernel $U_P/\mathbb{Z}$. The metric $<\ ,\ >_x$ on $L \otimes \mathbb{R}$ yields an orthogonal decomposition

$$L \otimes \mathbb{R} = \bigoplus_{i=1}^{i=k} L_i/L_{i-1} \otimes \mathbb{R}$$

and hence an Arakelow bundle structure $(L_i/L_{i-1}, (\Theta_x)_i)$ for all i, and therefore an Arakelow group scheme structure on $M/\mathbb{Z}$.

Hence we get

Proposition 1.2.2. *If (Gl_n, Θ) is an Arakelow group scheme then Θ induces an Arakelow group scheme structure Θ^M on any reductive quotient $M = P/U$.*

Definition : A pair $(\mathrm{Gl}_n/\mathbb{Z}, \Theta)$ is called stable (resp. semi stable) if for any proper parabolic subgroup $P/\mathbb{Q} \subset \mathrm{Gl}_n/\mathbb{Q}$ we have

$$p_P(P, \Theta) > 1 \tag{1.83}$$

In our example in (1.2.6) the stable points are those outside the union of the closed Farey circles.

To get a better understanding of these numbers we have to do some computations with roots and weights. Let us start from an Arakelow vector bundle $(L = \mathbb{Z}^d, <\ ,\ >)$ and let us assume that L is equipped with a complete flag

$$\mathcal{F}_0 = \{)\} = L_0 \subset L_1 \subset \cdots \subset L_{d-1} \subset L_d \tag{1.84}$$

which defines a Borel subgroup $B/\mathbb{Z}$. The quotients $(L_i/L_{i-1}, <\ ,\ >_i)$ are Arakelow line bundles over $\mathbb{Z}$ or in a less sophisticated language they are free modules of rank one and the generating vector $\bar{e}_i$ has a length $\sqrt{< \bar{e}_i, \bar{e}_i >_i}$. This length is of course also the volume of $(L_i/L_{i-1} \otimes \mathbb{R})/(L_i/L_{i-1})$.

The unipotent radical $U/\mathbb{Z} \subset B/\mathbb{Z}$ has a filtration $\{(0)\} \subset V_1 \subset \ldots, V_{n(n-1)/2-1} \subset V_{n(n-1)/2} = U$ by normal subgroups, the successive quotients are isomorphic to $\mathbb{G}_a$ and the torus $T = B/U$ acts by a positive root $\alpha_{i,j}$ and this is a one to one correspondence between the subquotients and the positive roots. Then it is clear: If ν corresponds to (i,j) then

$$(V_\nu/V_{\nu+1}, \Theta_\nu) = (L_i/L_{i-1}, <\ ,\ >_i) \otimes (L_j/L_{j-1}, <\ ,\ >_j)^{-1}. \tag{1.85}$$

Moreover the quotients $(V_\nu/V_{\nu+1}, \Theta_\nu)$ depend only on the conformal class of $<\ ,\ >$ and hence only on the resulting Cartan involution Θ.

The unipotent subgroup $U/\mathbb{Z}$ contains the one parameter subgroup $U_{i,j}/Z$ and this one parameter subgroup maps isomorphically to $(V_\nu/V_{\nu+1})$. Hence our construction defines the Arakelow line bundle $(U_{i,j}, \Theta_{i,j})$.

If we now define $n_{\alpha_{i,j}}(B,x) = \text{vol}_{\Theta_{i,j}}(U_{i,j}(\mathbb{R})/U_{i,j}(\mathbb{Z}))$ then it is clear that

$$p_B(B,x) = \prod_{i<j} n_{\alpha_{i,j}}(B,x) \tag{1.86}$$

If $P \supset B$ then its unipotent radical $U_P \subset U$ and we defined the set Δ^{U_P} as the set of positive roots for which $U_{i,j} \subset U_P$. Then we have

$$p_P(B,x) = \prod_{(i,j)\in\Delta^{U_P}} n_{\alpha_{i,j}}(B,x) \tag{1.87}$$

Here it is important to notice the right hand side does not depend on the choice of $B \subset P$.

We follow a convention and put $2\rho_P = \sum_{(i,j)\in\Delta^{U_P}} \alpha_{i,j}$ so that ρ_P is the half sum of positive roots in in the unipotent radical. Formula (1.77) tells us that for any maximal parabolic subgroup P_{i_0} rhoP

$$2\rho_{P_{i_0}} = \sum_{i\le i_0, j\ge i_0+1} \alpha_{i,j} = n\gamma_{i_0}. \tag{1.88}$$

For any $\gamma = \sum \alpha_{i,i+1} \otimes z_i \in X^*(T) \otimes \mathbb{C}$ we define the homomorphism

$$|\gamma| : T(\mathbb{R}) \to \mathbb{C}^\times : |\gamma| : t \to \prod_i |\alpha_{i,i+1}(t)|^{z_i} \tag{1.89}$$

Since the numbers $n_{\alpha_{i,j}}(B,x)$ are positive real numbers we define for any

$$n_\gamma(B,x) = \prod_{i=1}^{n-1} n_{\alpha_{i,j}}(B,x). \tag{1.90}$$

Here we see that the second argument is a Borel-subgroup B. But if the above character $\gamma : B(\mathbb{R}) \to \mathbb{R}^\times_{>0}$ extends to a character $\gamma : P(\mathbb{R}) \to \mathbb{R}^\times_{>0}$ then we can define

$$n_\gamma(P,x) := n_\gamma(B,x)$$

and this number only depends on P and not on the Borel subgroup $B \subset P$. The characters in $\gamma \in X^*(T)$ for which $|\gamma|$ extend to $P(\mathbb{R})$are exactly the linear combinations (See (1.92) below) $\gamma = \sum_{\alpha_i \in \pi^U} x_i\gamma_i$. The characters $\gamma_P = \sum_{\alpha_i \in \pi^U} r_i\gamma_i$ where the $r_i > 0$ are rational numbers. Let P_i be the maximal parabolic subgroup of type $\pi \setminus \{\alpha_i\}$ containing P then the above formula implies that

$$p_P(P,x) = \prod_{\alpha_i \in \pi^U} n_{\gamma_i}(P_i,x)^{r_i} = \prod_{\alpha_i \in \pi^U} p_{P_i}(P_i,x)^{\frac{r_i}{n}} \tag{1.91}$$

This tells us

The Arakelow scheme $(\mathrm{Gl}_n/\mathbb{Z},\Theta)$ *is stable if for all maximal parabolic subgroups* $p_{P_i}(P_i,\Theta) = n_{\gamma_i}(P_i,\Theta)^n > 1$.

We need a few more formulas relating roots and weights. For any parabolic subgroup we have the division of the set of simple roots into two parts

$$\pi = \pi^M \cup \pi^{U_P}.$$

This induces a splitting of the character module split

$$X^*(T) \otimes \mathbb{Q} = \bigoplus_{\alpha_i \in \pi^M} \mathbb{Q}\alpha_i \oplus \bigoplus_{\alpha_i \in \pi^{U_P}} \mathbb{Q}\gamma_i \tag{1.92}$$

where γ_i is the dominant fundamental weight attached to α_i (See (1.76)).

If now $\alpha_i \in \pi^{U_P}$ then we can project α_i to the second component, this projection

$$\alpha_i^P = \alpha_i + \sum_{\alpha_\nu \in \pi^M} c_{i,\nu}\alpha_\nu \tag{1.93}$$

Here an elementary - but not completely trivial - computation shows that

$$c_{i,\nu} \geq 0 \tag{1.94}$$

Since $\alpha_i^P \in \oplus \bigoplus_{\alpha_i \in \pi^{U_P}} \mathbb{Q}\gamma_i$ these characters extend to $P(\mathbb{R})$ and hence $n_{\alpha_i^P}(P,x)$ is defined.

We state the two fundamental theorems of reduction theory

Theorem 1.2.1. *For any Arakelow group scheme* $(\mathrm{Gl}_n, \Theta_x)$ *we can find a Borel subgroup* $B \subset \mathrm{Gl}_n$ *for which*

$$n_{\alpha_i}(B, \Theta_x) = n_{\alpha_i}(B, x) < \frac{2}{\sqrt{3}} \text{ for all } i = 1, \ldots, n-1$$

Theorem 1.2.2. *For any Arakelow group scheme* (Gl_n, Θ) *we can find a a unique parabolic subgroup* P *such that for all* $\alpha_i \in \pi^{U_P}$ *we have*

$$n_{\alpha_i^P}(P, \Theta) < 1 \text{ and the reductive quotient } (M, \Theta^M) \text{ is semi stable.}$$

The first theorem is due to Minkowski, the second theorem is proved in [Stu], [Gray].

This parabolic subgroup is called the canonical destabilizing group. We denote it by $P(x)$, if (G, x) is semi stable then $P(x) = G$. This gives us a dissection of X into the subsets

$$X = \bigcup_{P\text{: parabolic subgroups of } G/\mathbb{Q}} X^{[P]} = \{x \in X \mid P(x) = P\} \tag{1.95}$$

Clearly $\gamma X^{[P]} = X^{[\gamma P \gamma^{-1}]}$, if we divide by the group Γ the we get

$$\Gamma\backslash X = \bigcup_{P \in \mathrm{Par}(\Gamma)} \Gamma_P \backslash X^{[P]} \tag{1.96}$$

where Par (Γ) is a set of representatives of Γ conjugacy classes of parabolic subgroups of $\mathrm{Gl}_n/\mathbb{Q}$. This is a decomposition of $\Gamma\backslash X$ into a disjoint union of subsets. The subset $\Gamma\backslash X^{[\mathrm{Gl}_n]}$ is compact, it is the set of semi stable pairs (x, Gl_n), the subsets $\Gamma_P \backslash X^{[P]}$ for $P \neq G$ are in a certain sense "open in some directions" and "closed in some other direction". Therefore this decomposition is not so useful for the study of cohomology groups.

Do remedy this we introduce larger subsets. For a real number $r, 0 < r < 1$ we define Gstable

$$X^{\mathrm{Gl}_n}(r) = \{x \in X \mid n_{\gamma_\alpha}(P(x), x) > r, \text{ for all } \alpha \in \pi^{U_{P(x)}}). \tag{1.97}$$

It contains the set of semi-stable (Gl_n, x) If we choose $r < 1$ but close to one then some of the elements in $X^{\mathrm{Gl}_n}(r)$ may be unstable but only a "little bit ".

Together with the first theorem this has a consequence

Proposition 1.2.3. *The quotient* $X^{\mathrm{Gl}_n}(r) = \Gamma\backslash X^{\mathrm{Gl}_n}(r)$ *is relatively compact open subset of* $\Gamma\backslash X$, *It contains the set of semi-stable* (Gl_n, x).

We start from a parabolic subgroup P and let $M = P/U_P$ be its Levi-quotient. Our considerations above also apply to $M/\mathbb{Q}$. The group $P(\mathbb{R})$ acts transitively on X and we put (See (1.82))

$$X^M = U_P(\mathbb{R})\backslash X \text{ and let } q_M : X \to X^M \text{ be the projection },$$

here $X^M = M(\mathbb{R})/K_\infty^M$ where K_∞^M is the image of $P(\mathbb{R}) \cap K_\infty$ in $M(\mathbb{R})$. Let $S \subset M$ be the maximal split torus in the center of M then we define

$$X^{M^{(1)}} := M(\mathbb{R})/K_\infty^M \cdot S^{(0)}(\mathbb{R}) \tag{1.98}$$

where of course $S^{(0)}(\mathbb{R})$ is the connected component of the identity of $S(\mathbb{R})$, For a simple roots $\alpha \in \pi^M$, a Borel subgroup $\bar{B} \subset M/\mathbb{Q}$ and a point $x^M = q_M(x)$ we can define the numbers $n_\alpha(\bar{B}, x^M)$ essentially in the same way as before and clearly

$$n_\alpha(\bar{B}, x^M) = n_\alpha(B, x)$$

if B is the inverse image of $\bar{B}$.

We have to be a little bit careful with the numbers $p_{\bar{Q}}(\bar{Q}, x^M)$ because the for the inverse image Q the unipotent radical U_Q is larger than $U_{\bar{Q}}$. Therefore we have to look at the dominant fundamental weights $\gamma_\alpha^M \in \bigoplus_{\alpha_i \in \pi^M} \mathbb{Q}\alpha_i$, and formulate the stability condition for x^M in terms of these γ_α^M :

The point x^M is stable, if for all $\alpha_i \in \pi^M$ the inequality $n_{\gamma^M_{\alpha_i}}(\bar{P}_{\alpha_i}, x^M,) > 1$ holds. Again we denote the destabilizing group by $P(x^M)$.

Hence we see that for a number $r_M < 1$ we can define regions

$$X^M(r_M) = \{x^M | n_{\gamma^M_{\alpha_i}}(\bar{P}_{\alpha_i}, x^M) > r_M \text{ whenever } \bar{P}_{\alpha_i} \supset \bar{P}(x^M)\} \tag{1.99}$$

We choose numbers $0 < c_P < 1$, furthermore we choose a number $r(c_P) < 1$ and define

$$^*X^P(c_P, r(c_P)) = \{x | \; n_{\alpha^P}(P, x) < c_P \text{ for all } \alpha \in \pi^{U_P}; x^M \in X^M(r(c_P))\} \tag{1.100}$$

Proposition 1.2.4. *For a given $r(c_P) < 1$ we can find numbers c_P such that for any $x \in {}^*X^P(c_P, r(\underline{c}_P))$ the destabilising parabolic subgroup $P(x) \subset P$. The same is true in the other direction: For a given $0 < c_P < 1$ we can find $r < 1$ such that for $x \in {}^*X^P(c_P, r))$ the destabilising parabolic subgroup $P(x) \subset P$.*

To see this we have to look at the destabilising subgroup $\bar{Q} \subset (M, x_M)$. Its inverse image $Q \subset P$ is a parabolic subgroup of Gl_n. The reductive quotient $(\bar{M}, x_{\bar{M}})$ of Q is semi- stable. We want to show that Q is the destabilising parabolic of (Gl_n, x). We have to show that

$$n_{\alpha^Q}(Q, x) < 1 \; \forall \; \alpha \in \pi^{U_Q} = \pi^{U_P} \cup \pi^{U_{\bar{Q}}}.$$

For $\alpha \in \pi^{U_{\bar{Q}}}$ this is true by definition. For $\alpha \in \pi^{U_P}$ we have

$$\alpha^P = \alpha + \sum_{\beta \in \pi^M} a_{\alpha,\beta}\beta \text{ and } \alpha^Q = \alpha + \sum_{\beta' \in \pi^{\bar{M}}} a'_{\alpha,\beta'}\beta,$$

where $a_{\alpha,\beta} \geq 0$. The roots $\beta \in \pi^{U_{\bar{Q}}}$ can be expressed in terms of the $\beta^{\bar{Q}} = \beta^Q$:

$$\beta^Q = \beta + \sum_{\beta' \in \pi^{\bar{M}}} a^*_{\beta,\beta'}\beta' \tag{1.101}$$

and hence

$$\alpha^Q = \alpha^P - \sum_{\beta \in \pi^{U_{\bar{Q}}}} a_{\alpha,\beta}\beta^Q + \sum_{\beta' \in \pi^{\bar{M}}} c_{\alpha\beta'}\beta'. \tag{1.102}$$

The last sum is zero because $\alpha^Q, \alpha^P, \beta^Q$ are orthogonal to the module $\oplus_{\beta'}\mathbb{Z}\beta'$.

We get the relation

$$n_{\alpha^Q}(Q,x) = n_{\alpha^P}(P,x,) \cdot \prod_{\beta\in\pi^{U_{\bar{Q}}}} n_{\beta^Q}(Q,x)^{-a_{\alpha,\beta}}. \tag{1.103}$$

Now it comes down to show that [wc]

$$\begin{gathered} n_{\alpha^P}(P,x) < c_\alpha,\ \forall\ \alpha\in\pi^{U_P} \text{ and } n_{\beta^Q}(Q,x) > r,\ \forall\beta\ \in\pi^{U_{\bar{Q}}} \\ \implies n_{\alpha^Q}(P,x) < 1\,;\forall\ \alpha\in\pi^{U_P} \end{gathered} \tag{1.104}$$

This is certainly true if either the c_α are small enough or if r is sufficiently close to one. In this case we say that (c_P, r) is well chosen.

Therefore we define

$$X^P(c_P, r(c_P)) = \{x \in {}^*X^P(c_P, r(c_P)) | P(x) \subset P\} \tag{1.105}$$

we have $X^P(c_P, r(c_P)) = {}^*X^P(c_P, r(c_P))$, if $(c_P, r(c_P))$ is well chosen.

We claim that we can find a family of parameters

$$(\ldots, (c_P, r(c_P)), \ldots)_{P\text{: parabolic over } \mathbb{Q}}$$

where $(c_P, r(c_P))$ only depend on the type of P, such that we get a covering [COV]

$$X = \bigcup_P X^P(c_P, r(c_P))) \tag{1.106}$$

and hence

$$\Gamma\backslash X = \Gamma\backslash \bigcup_P X^P(c_P, r(c_P)) = \bigcup_{P\in\mathrm{Par}(\Gamma)} \Gamma_P\backslash X^P(c_P, r(c_P))$$

We change the notation slightly, since these numbers only depend on the type $\pi' = \pi^M = t(P)$ we replace c_P by $c_{\pi'}$ and $r(c_P)$ by $r(c_{\pi'})$.

To prove the claim we choose a number $0 < c_\emptyset < 1$. In this case $r_0 = r(c_\emptyset)$ can be any number. Then we choose a number $0 < r_1 < c_\emptyset$. For any $\pi_i = \{\alpha_i\}$ we choose a $c_{\pi_i} < 1$ such that (c_{π_i}, r_1) is well chosen. We continue and chose $0 < r_2 < c_{\pi_i}$ for all i and for any two element subset $J \subset \pi$ we choose numbers $0 < c_J < 1$ such that (c_J, r_2) is well chosen. This goes until we reach top parabolic.

Now we get a covering of X by the open sets $X^P(c_\pi, r(\pi))$. To see this we pick a point $x \in X$, we have to show that it lies in at least one of the sets $X^P(c_P, r(c_P))$. If it is not in $X^{\mathrm{Gl}_n}(r_{n-1})$ then we find a maximal parabolic P_i such that $n_{\alpha_i}(P_i, x) < c_{\pi\setminus\{\alpha_i\}}$. We project x to the point $x^{M_i} \in X^{M_i}$. If this point is in $X^{M_i}(r_{n-2})$ then $x \in X^{P_i}(c_{\pi\setminus\{\alpha_i\}}, r_{n-2})$ and we are done. If not we apply our argument above to x^{M_i} and $\pi' = \pi \setminus \{\alpha_i\}$. We continue the same reasoning and at latest it stops for $\pi' = \emptyset$.

We have a very explicit description of these sets $\Gamma_P\backslash X^P(c_{\pi'}, r(c_{\pi'}))$. We consider the evaluation map

$$\begin{gathered} n^{\pi_{U_P}} : \Gamma_P\backslash X^P(c_{\pi'}, r(c_{\pi'})) \to \prod_{\alpha\in\pi_{U_P}}(0, c_\alpha) \\ x \mapsto (\dots, n_{\alpha^P}(P, x), \dots)_{\alpha\in\pi_{U_P}} \end{gathered} \tag{1.107}$$

Of course we also have the homomorphism

$$|\alpha^{\pi_{U_P}}| : P(\mathbb{R}) \to \{\dots, |\alpha^P|, \dots\}_{\alpha\in\pi_{U_P}} \tag{1.108}$$

and the multiplication by an element $y \in P(\mathbb{R})$ induces an isomorphisms of the fibers

$$(n^{\pi_{U_P}})^{-1}(c_1) \xrightarrow{\sim} (n^{\pi_{U_P}})^{-1}(c_2) \text{ if } |\alpha^{\pi_{U_P}}|(y)\cdot c_1 = c_2$$

where the multiplication is taken componentwise. This identification depends on the choice of y.

To get a canonical identification we use the geodesic action which is introduced in the paper by Borel and Serre. We define an action of $A = (\prod_{\alpha\in\pi\backslash\pi'} \mathbb{R}^\times_{>0})$ on X. This action depends on P and we denote it by

$$(a, x) \mapsto a \bullet x \tag{1.109}$$

A point $x \in X$ defines a Cartan involution Θ_x and then the parabolic subgroup P^{Θ_x} of $G \times \mathbb{R}$ is opposite to $P \times \mathbb{R}$ and $P \times \mathbb{R} \cap P^{\Theta_x} = M_x$ is a Levi factor, the projection $P \to M$ induces an isomorphism

$$\phi_x : M \times \mathbb{R} \xrightarrow{\sim} M_x. \tag{1.110}$$

The character $\alpha^{\pi'}$ induces an isomorphism

$$s_x : A \xrightarrow{\sim} S_x(\mathbb{R})^{(0)}$$

where S_x is the maximal Hence we $S_x(\mathbb{R})^{(0)}$ is the connected component of the identity of the center $M_x(\mathbb{R}) \cap \mathrm{Sl}_n(\mathbb{R})$ and we put

$$a \bullet x = s_x(a)x$$

We have to verify that this is indeed an action. This is clear because for the Cartan-involution $\Theta_{a\bullet x}$ we obviously have

$$P^{\Theta_x} = P^{\Theta_{a\bullet x}}.$$

It is also clear that this action commutes with the action of $P(\mathbb{R})$ on X because

$$ys_x(a)x = s_{yx}(a)yx \text{ for all } y \in P(\mathbb{R}), x \in X.$$

It follows from the construction that the semigroup $A_- = \{\dots, a_\nu, \dots\}$- where $0 < a_\nu \le 1$ - acts via the geodesic action on $X^P(c_\pi, r(\underline{c}_{\pi'}))$ and that for $a \in A_-$ we get an isomorphism

$$(n^{\pi^{U_P}})^{-1}(c_1) \xrightarrow{\sim} (n^{\pi_{U_P}})^{-1}(ac_1).$$

This yields a decomposition

$$X^P(c_{\pi'}, r(\underline{c}_{\pi'})) = (n^{\pi^{U_P}})^{-1}(c_0) \times \prod_{\alpha\in\pi'} (0, c_\alpha]$$

where c_0 is an arbitrary point in the product.

Since we know that $|\alpha^{\pi'}|$ is trivial on Γ_P and since the action of P commutes with the geodesic action we conclude

$$\Gamma_P\backslash X^P(c_{\pi'}, r(\underline{c}_{\pi'})) = \Gamma_P\backslash(n^{\pi'})^{-1}(c_0) \times \prod_{\alpha\in\pi'} (0, c_\alpha] \tag{1.111}$$

Let $P^{(1)}(\mathbb{R}) = \ker(|\alpha^{\pi_{U_P}}|)$ then the fiber $(n^{\pi'})^{-1}(c_0)$ is a homogenous space under $P^{(1)}(\mathbb{R})$. We have the symmetric space X^M attached to M, to be precise this is

$$X^M = M(\mathbb{R})/K_\infty$$

We have the projection map $p_{P,M} : X \to X^M$ where X^M is the space of Cartan involutions on the reductive quotient M. Hence we get a map

$$p^*_{P,M} = p_{P,M} \times n^{\pi_{U_P}} : X \to X^M \times \prod_{\alpha\in\pi'} (0, c_\alpha] \tag{1.112}$$

The geodesic action only acts on the second factor of the product $X^M \times \prod_{\alpha\in\pi'}(0, c_\alpha]$, the map $p^*_{P,M}$ commutes with the geodesic action.

The group $U_P(\mathbb{R})$ acts simply transitively on the fibers of this projection, and hence

$$q_{P,M} : \Gamma_P\backslash X^P(c_{\pi'}, r(\underline{c}_{\pi'})) \to \Gamma_M\backslash X^M(r(\underline{c}_P)) \times \prod_{\alpha\in\pi'} (0, c_\alpha] \tag{1.113}$$

is a fiber bundle with fiber isomorphic $\Gamma_U\backslash U(\mathbb{R})$. If we pick a point $x \in \Gamma_M\backslash X^M(r(\underline{c}_P))\times \prod_{\alpha\in\pi'}(0, c_\alpha]$ then the identification of $q^{-1}_{P,M}()$ with $\Gamma_U\backslash U(\mathbb{R})$ depends on the choice of a point $\tilde{x} \in X^P(c_{\pi'}, r(\underline{c}_{\pi'}))$ which maps to x.

This can now be compactified, we define the closure

$$\overline{\Gamma_P\backslash X^P(c_{\pi'}, r(\underline{c}_P)} := \Gamma_P\backslash(n^{\pi^{U_P}})^{-1}(c_0) \times \prod_{\alpha\in\pi_G\backslash\pi} [0, c_{\pi'}], \tag{1.114}$$

and

$$\partial\overline{\Gamma_P\backslash X^P(c_{\pi'}, \Omega_\pi)} = \overline{\Gamma_P\backslash X^P(c_{\pi'}, \Omega_\pi)} \setminus \Gamma_P\backslash X^P(c_\pi, \Omega_\pi) \tag{1.115}$$

this is equal to

$$\partial\overline{\Gamma_P\backslash X^P(c_{\pi'}, r(\underline{c}_P)} = \Gamma_P\backslash(n^{\pi^{U_P}})^{-1}(c_0) \times \partial(\prod_{\nu\in\pi_G\backslash\pi} [0, c_\pi])$$

where of course $\partial(\prod_{\nu\in\pi_G\backslash\pi}[0, c_\pi]) \subset \prod_{\nu\in\pi_G\backslash\pi}[0, c_\pi]$ is the subset where at least one of the coordinates is equal to zero.

We form the disjoint union of of these boundaries over the π and set of representatives of Γ conjugacy classes, this is a compact space. Now there is still a minor technical point. If we have two parabolic subgroups $Q \subset P$ then the intersection $X^P(\underline{c}_P, r(\underline{c}_P) \cap X^Q(\underline{c}_Q, r(\underline{c}_Q)) \neq \emptyset$. If we now have points

$$x \in \partial\overline{\Gamma_P\backslash X^P(c_\pi, r(\underline{c}_P)}, y \in \partial\overline{\Gamma_Q\backslash X^Q(c_{\pi'}, r(\underline{c}_{P'})}$$

then we identify these two points if we have a sequence of points $\{x_n\}_{n\in\mathbb{N}}$ which lies in the intersection $X^P(c_\pi, r(\underline{c}_P)) \cap X^Q(c_{\pi'}, r(\underline{c}_{P'}))$ and which converges to x in $\overline{\Gamma_P\backslash X^P(c_\pi, r(\underline{c}_P)}$ and to y in $\overline{\Gamma_Q\backslash X^Q(c_{\pi'}, r(\underline{c}_{P'})}$. A careful inspection shows that this provides an equivalence relation $\sim$, and we define

$$\partial(\Gamma\backslash X) = \bigcup_{\pi', P\in \mathrm{Par}(\Gamma)} \partial\overline{\Gamma_P\backslash X^P(c_\pi, r(\underline{c}_P)}/\sim \tag{1.116}$$

and the Borel-Serre compactification will be the manifold with corners

$$\overline{\Gamma\backslash X} = \Gamma\backslash(X \cup \bigcup_{P:P\text{proper}} \overline{X^P(\underline{c}_{\pi'}, r(\underline{c}_P))}. \tag{1.117}$$

We define a "tubular" neighbourhood of the boundary we put

$$\mathcal{N}(\Gamma\backslash X)(\mathbf{c}) = \Gamma\backslash \bigcup_{P:P\text{proper}} \overline{X^P(\underline{c}_{\pi'}, r(\underline{c}_P))} \tag{1.118}$$

where $\mathbf{c}$ stands for the collection of parameters $c_{\pi'}, r(c_P)$. Then we define the "punctured tubular" neighbourhood as ptub

$$\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X)(\mathbf{c}) = \Gamma\backslash \bigcup_{P:P\text{proper}} X^P(\underline{c}_{\pi'}, r(\underline{c}_P)) = \Gamma\backslash X \cap \mathcal{N}(\Gamma\backslash X) \tag{1.119}$$

Eventually we want to use the above covering as a tool to understand cohomology (See section 8.1.9)) For this it is also necessary to understand the intersections

$$X^{P_1}(c_{\pi_1}, r(\underline{c}_{\pi_1})) \cap \cdots \cap X^{P_\nu}(c_{\pi_k}, r(\underline{c}_{\pi_k})) \tag{1.120}$$

Our proposition 1.2.4 implies that for any point x in the intersection the destabilizing parabolic subgroup $P(x) \subset P_1 \cap \cdots \cap P_k$. Hence we see that the above intersection can only be non empty if $Q = P_1 \cap \cdots \cap P_k$ is a parabolic subgroup. Then $\pi^{U_Q} = \cup_{\nu=1}^k \pi^{U_{P_\nu}}$. Let M be the reductive quotient of Q.

Now we look at the product $\prod_{\alpha\in\pi^{U_Q}} \mathbb{R}_{>0}^\times$, here it seems to be helpful to identify it - using the logarithm - with $\mathbb{R}^{d_Q}$:

$$\log : \prod_{\alpha\in\pi^{U_Q}} \mathbb{R}_{>0}^\times \xrightarrow{\sim} \mathbb{R}^{d_Q} \tag{1.121}$$

We consider the map

$$\begin{aligned} N^Q &: X^{P_1}(c_{\pi_1}, r(\underline{c}_{\pi_1})) \cap \cdots \cap X^{P_k}(c_{\pi_k}, r(\underline{c}_{\pi_k})) \to \mathbb{R}^{d_Q} \\ N^Q &: x \mapsto (\ldots, -\log(n_{\alpha^Q}(Q,x)), \ldots)_{\alpha^Q\in\pi^{U_Q}} \end{aligned} \tag{1.122}$$

Consider a point $x \in X^{P_\nu}(c_{\pi_\nu}, r(\underline{c}_{\pi_\nu}))$, for $\alpha \in \pi^{U_{P_\nu}}$ we have

$$-\log(n_{\alpha^{P_\nu}}(P_\nu, x)) \geq -\log(c_{\pi_\nu})$$

We can express $-\log(n_{\alpha^{P_\nu}}(P_\nu, x))$ as a linear combination of the $-\log(n_{\alpha^Q}(Q, x)$, with $\alpha \in \pi^{U_Q}$. This means that the root $\alpha \in \pi^{U_{P_\nu}}$ defines a half space $H^+_\nu(\alpha)$ in $\mathbb{R}^{d_Q}$ and $N^Q(x) \subset H^+_\nu(\alpha)$ in $\mathbb{R}^{d_Q}$.

Now we assume that x is in the intersection (1.120). For the roots $\alpha \in \pi \setminus \pi^{U_{P_\nu}}$ we have the condition (1.99). For the roots $\alpha \in \pi^{U_Q} \setminus \pi^{U_{P_\nu}}$ this yields

$$-\log(n_{\gamma_\alpha^{M_\nu}}(P_\nu, x)) \leq -\log(r(\pi_\nu)).$$

Therefore we see that the image of N^Q is contained in the intersection of a finite number of half spaces, which are obtained from a finite family of hyperplanes. These hyperplanes depend on the parameters $c_{\pi_\nu}, r(\pi_\nu)$, let us call this intersection $C(\underline{c}, \underline{r})$, it is a convex -possibly empty- subset of $\mathbb{R}^{d_Q}$.

We investigate the restriction

$$N^Q : X^{P_1}(c_{\pi_1}, r(\underline{c}_{\pi_1})) \cap \cdots \cap X^{P_\nu}(c_{\pi_k}, r(\underline{c}_{\pi_k})) \to C(\underline{c}, \underline{r})$$

We observe that the unipotent radical $U_Q(\mathbb{R})$ acts by left translations on the intersection, we get a diagram

$$\begin{array}{ccc} X^{P_1}(c_{\pi_1}, r(\underline{c}_{\pi_1})) \cap \cdots \cap X^{P_\nu}(c_{\pi_k}, r(\underline{c}_{\pi_k})) & \to & C(\underline{c}, \underline{r}) \\ \downarrow p_M & & \\ X^M \times \mathbb{R}^{d_Q} & \to & \mathbb{R}^{d_Q} \end{array} \tag{1.123}$$

Now it is clear from the definitions that the image of p_M is a set

$$\mathrm{Im}(p_M) = \Omega^M(\underline{c}, \underline{r}) \times C(\underline{c}, \underline{r})$$

where $\Omega^M(\underline{c}, \underline{r}) \subset X^M$ is a subset containing the set $X^{M,st}$ of semi stable points and is described by certain inequalities as in (1.97). This subset is Γ_M invariant and $\Gamma_M \backslash \Omega^M(\underline{c}, \underline{r})$ is relatively compact.

Hence we see that we have essentially the same situation as in (1.113). The map

$$q_M : X^{P_1}(c_{\pi_1}, r(\underline{c}_{\pi_1})) \cap \cdots \cap X^{P_\nu}(c_{\pi_k}, r(\underline{c}_{\pi_k})) \to \Gamma_M \backslash \Omega^M(\underline{c}, \underline{r}) \times C(\underline{c}, \underline{r}) \tag{1.124}$$

is a fiber bundle with fiber isomorphic to $\Gamma_{U_Q} \backslash U_Q(\mathbb{R})$.

In the following we refer to the book of S. Helgason [52].

We mention an important property of the sets $X^P(c'_\pi, r(c_P))$. We assume that our symmetric space X is connected, then it is well known that it is convex, any two points $p, q \in X$ can be joined by a unique geodesic $[p, q]$. We say that a subset $U \subset X$ is convex if for any two points $p, q \in U$ also the geodesic $[p, q] \subset U$.

Proposition 1.2.5. *Let $\Omega \subset \Omega^M(\underline{c}, \underline{r})$ be a convex subset. Then the inverse image $p_M^{-1}(\Omega \times C(\underline{c}, \underline{r}))$ is a convex subset of $X^{P_1}(c_{\pi_1}, r(\underline{c}_{\pi_1})) \cap \cdots \cap X^{P_\nu}(c_{\pi_k}, r(\underline{c}_{\pi_k}))$*

Proof. The assertion is easily reduced to the following:

Let P be a maximal parabolic subgroup, let M be its reductive quotient, let α be the simple root not in π^M and $\Omega \subset X^{M^{(1)}}$. Then the set for any choice of We choose a $c_\alpha > 0$ and claim that $X^P(c_\alpha, \Omega) = \{x \in X \mid n_{\alpha^P}(P, x) \leq c_\alpha\,; q_M(x) \in \Omega\}$ is convex .

To see this we pick a point $x \in X^P(c_\alpha, \Omega)$, let T_x^X be the tangent space at x. The action of $G(\mathbb{R})$ on X gives us a surjective map $D_x : \mathfrak{g}_\mathbb{R} \to T_{x_0}^X$ and this induces an isomorphism $D_x : \mathfrak{g}_\mathbb{R}/\mathfrak{k}_x \xrightarrow{\sim} T_x^X$, here of course $\mathfrak{k}_x$ is the Lie-algebra of K_x. We get the well known Cartan decomposition of the Lie-algebra

$$\mathfrak{g}_\mathbb{R} = \mathfrak{k}_x \oplus \mathfrak{p}_x \text{ where } \mathfrak{p}_x = \{V \in \mathfrak{g}_\mathbb{R} \mid \Theta_x(V) = -V\} \tag{1.125}$$

and we get the isomorphism $D_x : \mathfrak{p}_x \xrightarrow{\sim} T_x^X$. Starting from our parabolic subgroup P we get a finer decomposition of $\mathfrak{p}_x$.

Let $\mathfrak{P}_\mathbb{R}$ be theLie algebra of $P\times\mathbb{R}$. The intersection $P\times\mathbb{R} \cap \Theta_x(P\times\mathbb{R}) = M_x$ and we get for the Lie algebras $\mathfrak{m}_x = \mathfrak{m}^{(0)} \oplus \mathfrak{a}$ and this gives the finer decoposition $\mathfrak{m}_x = \mathfrak{k}_x^M \oplus \mathfrak{p}^{(M_x} \oplus \mathfrak{a}$ and then

$$\mathfrak{p}_x = \mathfrak{p}^{(M_x)} \oplus \mathfrak{a} \oplus \{V - \Theta_x(V)\}_{V\in\mathfrak{u}} \tag{1.126}$$

where $V \in \mathfrak{u}_\mathbb{R}$ and $\mathfrak{a} = \mathbb{R}Y_A$. We normalise Y_A such that $d\alpha^P(Y_A) = 1$.Then we can write a tangent vector T_x^X as image of

$$Y = Y_M + aY_A + (V - \theta(V));$$

We know that there is a unique geodesic $c : \mathbb{R} \to X$ starting at x with $c'(t) = Y$ The theorem 3.3 in Chapter IV in [52] says that this geodesic is $c(t) = \exp(tY) \cdot x$. A tedious computation using the Iwasawa decomposition and the Campell-Hausdorff formula shows that

$$-\log(n_{\alpha^P}(\exp(tY) \cdot x) = -\log(n_{\alpha^P}(x)) + at - a^2 q(Y_A, V)t^2 \tag{1.127}$$

where $q(Y_A, , V)$ is a positive definite form in V.

If now $x_1 \in X^P(c_\alpha, \Omega)$ is a second point, We find a tangent vector $Y = Y_M + aY_A + (V - \theta(V))$ such that $t \mapsto \exp(tY) \cdot x$ is the geodesic joining x and $x_1 = \exp(Y) \cdot x$. If we project these two points to $X^{M^{(1)}}$ then the images $\bar{x}, \bar{x_1} \in \Omega$ and $\exp(t(Y_M)\bar{x}$ is the geodesic in $X^{M^{(1)}}$. and hence for $t \in [0, 1]$ we have $\exp(t(Y_M)\bar{x}$. But now

$$-\log(n_{\alpha^P}(x)) \geq -\log(c_\alpha);\ -\log(n_{\alpha^P}(\exp(Y)\cdot x) = -\log((n_{\alpha^P}(x_1) \geq -\log(c_\alpha).$$

Since the second derivative is always > 0 (see(1.127) it follows that $-\log(n_{\alpha^P}(\exp(tY)\cdot x) \geq -\log(c_\alpha)\ \forall\, t \in [0, 1]$.

□

We formulated the main theorems of reduction theory only for $\mathrm{Gl}_n/\mathbb{Q}$ because we did not want to use to much from the theory of reductive groups (for instance [10]). But actually these results extend to general reductive groups, basically in the same formulation. Especially we get

Theorem 1.2.3. *(Borel-Harish-Chandra): If $G/\mathbb{Q}$ is an anisotropic reductive group and $\Gamma \subset G(\mathbb{Q})$ is an arithmetic subgroup then*

$$\Gamma\backslash X = \Gamma\backslash G(\mathbb{R})/K_\infty$$

is compact.

Chapter 2

The Cohomology groups

2.1 Cohomology of arithmetic groups as cohomology of sheaves on $\Gamma\backslash X$.

We are now in the position to unify — at least for the special case of arithmetic groups — the two cohomology theories from our chapter II and chapter IV in [40].

We start from a semi simple group $G/\mathbb{Q}$ and we choose an arithmetic congruence subgroup $\Gamma \subset G(\mathbb{Q})$. Let $X = G(\mathbb{R})/K_\infty$ as before. A second datum will be a Γ- module $\mathcal{M}$, in principle this can be any Γ- module.

To such a $\Gamma-$ module we attach a sheaf $\tilde{\mathcal{M}}$ on $\Gamma\backslash X$. This sheaf has values in the category of abelian groups.For any open subset $U \subset X$ we have to define the group of sections $\tilde{\mathcal{M}}(U)$. We start from the projection

$$\pi : X \longrightarrow \Gamma\backslash X \tag{2.1}$$

and define sheaf

$$\tilde{\mathcal{M}}(U) = \{f : \pi^{-1}(U) \to \mathcal{M} \mid f \text{ is locally constant } f(\gamma u) = \gamma f(u)\}. \tag{2.2}$$

This is clearly a sheaf. For any point $x \in \Gamma\backslash X$ we can find a neighbourhood V_x with the following property: We choose a point $\tilde{x} \in \pi^{-1}(x)$, then $\tilde{x}$ has a convex $\Gamma_{\tilde{x}}$-invariant neighbourhood $U_{\tilde{x}}$, for which $\gamma U_{\tilde{x}} \cap U_{\tilde{x}} \neq \emptyset \iff \gamma \notin \Gamma_{\tilde{x}}$. Then we put $V_x = \Gamma_{\tilde{x}}\backslash U_{\tilde{x}}$. We call such a neighbourhood V_x an *orbiconvex* neighbourhood. It is clear that

$$\tilde{\mathcal{M}}(V_x) = \mathcal{M}^{\Gamma_{\tilde{x}}}.$$

Since x has a cofinal system of neighbourhoods of this kind, we see that we get an isomorphism

$$j_{\tilde{x}} : \tilde{\mathcal{M}}(V_x) = \tilde{\mathcal{M}}_x \tilde{\longrightarrow} \mathcal{M}^{\Gamma_{\tilde{x}}}.$$

If we are in the special case that Γ has no fixed points then we can cover $\Gamma\backslash X$ by open sets U so that $\tilde{\mathcal{M}}/U$ is isomorphic to a constant sheaf $\underline{\mathcal{M}}_U$. These sheaves are called *local systems* If we have fixed points we call them *orbilocal systems*

Sometimes we will denote the functor, which sends $\mathcal{M}$ to $\tilde{\mathcal{M}}$ by

$$\mathrm{sh}_\Gamma : \mathbf{Mod}_\Gamma \to \mathcal{S}_{\Gamma\backslash X},$$

this may be useful if we are dealing with varying subgroups Γ.

The motivations for these constructions are

1) The spaces $\Gamma\backslash X$ are interesting examples of so-called locally symmetric spaces (provided Γ has no torsion). Hence they are of interest for differential geometers and analysts.

2) If we have some understanding of the geometry of the quotient space $\Gamma\backslash X$ we gain some insight into the structure of Γ. This will become clear when we discuss the examples in (2.1.4)

3) The cohomology groups $H^\bullet(\Gamma, \mathcal{M})$ are closely related and in many cases even isomorphic to the sheaf cohomology groups $H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}})$. Again the geometry provides tools to compute these cohomology groups in some cases (again 2.1.4).

4) If the Γ-module $\mathcal{M}$ is a $\mathbb{C}$-vector space and obtained from a rational representation of $G/\mathbb{Q}$, then we can apply analytic tools to get insight (de Rham cohomology, Hodge theory See Chapter 8).

2.1.1 The relation between $H^\bullet(\Gamma, \mathcal{M})$ and $H^\bullet(\Gamma\backslash X, \mathcal{M})$

For the following we refer to [40] Chapter 2. In this section we assume that X is connected.

The functor

$$\mathcal{M} \to H^0(\Gamma\backslash X, \tilde{\mathcal{M}}) = \mathcal{M}^\Gamma.$$

is a functor from the category of $\Gamma-$ modules to the category **Ab** of abelian groups. We can write our functor $\mathcal{M} \to \mathcal{M}^\Gamma$ as a composition of

$$\mathrm{sh}_\Gamma : \mathcal{M} \longrightarrow \tilde{\mathcal{M}} \text{ and } H^0 : \tilde{\mathcal{M}} \to H^0(\Gamma\backslash X, \tilde{\mathcal{M}}).$$

We want to apply the composition rule from [40] 4.6.4.

In a first step we have to convince ourselves that sh_Γ sends injective Γ-modules to acyclic sheaves.

In [40], 2.2.4. we constructed the induced Γ -module $\mathrm{Ind}_{\{1\}}^\Gamma \mathcal{M}$, for any Γ module $\mathcal{M}$. This is the module of all functions $f : \Gamma \to \mathcal{M}$ and $\gamma_1 \in \Gamma$ acts on this module by $(\gamma_1 f)(\gamma) = f(\gamma\gamma_1)$. The map

$$m \mapsto f_m = \{\gamma \mapsto \gamma m\} \tag{2.3}$$

is an injective $\Gamma-$ module homomorphism.

In a first step we prove that for any such induced module the sheaf $\mathrm{sh}_\Gamma(\mathrm{Ind}_{\{1\}}^\Gamma \mathcal{M})$. is acyclic. We have a little

Lemma 2.1.1. *Let us consider the projection $\pi : X \to \Gamma\backslash X$ and the constant sheaf $\underline{\mathcal{M}}_X$ on X. Then we have a canonical isomorphism of sheaves*

$$\pi_*(\underline{\mathcal{M}}_X) \xrightarrow{\sim} \widetilde{Ind_{\{1\}}^{\Gamma}}\mathcal{M}.$$

Proof. This is rather obvious. Let us consider a small neighbourhood U_x of a point x, such that $\pi^{-1}(U_x)$ is the disjoint union of small contractible neighbourhoods $U_{\tilde{x}}$ for $\tilde{x} \in \pi^{-1}(x)$. Then for all points $\tilde{x}$ we have $\underline{\mathcal{M}}_X(U_{\tilde{x}}) = \mathcal{M}$ and

$$\pi_*(\underline{\mathcal{M}}_X)(U_x) = \prod_{\tilde{x}\in\pi^{-1}(x)} \mathcal{M}.$$

On the other hand

$$\widetilde{\mathrm{Ind}_{\{1\}}^{\Gamma}}\mathcal{M}(U_x) = \left\{ h : \pi^{-1}(U_x) \to \ \mathrm{Ind}_{\{1\}}^{\Gamma}\mathcal{M} \mid h \text{ is locally constant } h(\gamma u) = \gamma h(u) \right\}$$

For $u \in \pi^{-1}(U_x)$ the element $h(u)$ itself is a map

$$h(u) : \Gamma \longrightarrow \mathcal{M},$$

and $(\gamma h(u))(\gamma_1) = h(u)(\gamma_1\gamma)$ (here $\gamma_1 \in \Gamma$ is the variable.) Hence we know the function $u \to h(u)$ from $\pi^{-1}(U_x)$ to $\mathrm{Ind}_{\{1\}}^{\Gamma}\mathcal{M}$ if we know its value $h(u)(1)$ and this value can be prescribed on the connected components of $\pi^{-1}(U_x)$. On these connected components it is constant, we may take its value at $\tilde{x}$ and hence

$$h \longrightarrow (\ldots, h(\tilde{x})(1), \ldots)_{\tilde{x}\in\pi^{-1}(x)}$$

yields the desired isomorphism.

Now acyclicity is clear.. We apply example d) in [40], 4.6.3 to this situation. The fibre of π is a discrete space and hence

$$\pi_*(\underline{\mathcal{M}}_X) = \widetilde{\mathrm{Ind}_{\{1\}}^{\Gamma}}\mathcal{M}$$

and $R^q(\pi_*)(\underline{\mathcal{M}}_X) = 0$ for $q > 0$. Therefore the spectral sequence yields

$$H^q(X, \underline{\mathcal{M}}_X) = H^q(\Gamma\backslash X, \pi_*(\underline{\mathcal{M}}_X)) = H^q\left(\Gamma\backslash X,\ \widetilde{\mathrm{Ind}_{\{1\}}^{\Gamma}}\mathcal{M}\right),$$

and since X is a cell, we see that this is zero for $q \geq 1$. □

We apply this to the case that $\mathcal{M} = \mathcal{I}$ is an injective Γ-module. Clearly we can always embed $\mathcal{I} \longrightarrow \mathrm{Ind}_{\{1\}}^{\Gamma}\mathcal{I}$. But this is now a direct summand; hence it follows from the acyclicity of $\widetilde{\mathrm{Ind}_{\{1\}}^{\Gamma}}\mathcal{I}$ that also $\tilde{\mathcal{I}}$ must be acyclic.

Hence we can apply the composition rule and get spectral sequence with E_2 term

$$H^p(\Gamma\backslash X, R^q(\mathrm{sh}_\Gamma)(\mathcal{M})) \Rightarrow H^n(\Gamma, \mathcal{M}).$$

The edge homomorphism yields a homomorphism

$$H^n(\Gamma\backslash X, \mathrm{sh}_\Gamma(\mathcal{M})) \to H^n(\Gamma, \mathcal{M}) \tag{2.4}$$

which in general is neither injective nor surjective.

We have seen in section (1.2.2) that -under our assumption that $G/\mathbb{Q}$ is semisimple- the stabilisers Γ_x are finite. This implies hat the stalks $R^q(\mathrm{sh}_\Gamma)(\mathcal{M})_x = H^q(\Gamma_{\tilde{x}}, \mathcal{M})$ for $q > 0$ are torsion groups actually they are annihilated by $\#\Gamma_x$. This implies that the edge homomorphism has finite kernel and kokernel.

In this book we are mainly interested in the cohomology groups $H^n(\Gamma\backslash X, \mathrm{sh}_\Gamma(\mathcal{M}))$ and not so much in the group cohomology $H^\bullet(\Gamma, \mathcal{M})$.

2.1.2 Functorial properties of cohomology

We investigate the functorial properties of the cohomology with respect to the change of Γ. If $\Gamma' \subset \Gamma$ is a subgroup of finite index, then we have, the functor

$$\mathbf{Mod}_\Gamma \longrightarrow \mathbf{Mod}_{\Gamma'},$$

which is obtained by restricting the Γ-module structure to Γ'. Since for any Γ-module $\mathcal{M}$ we have $\mathcal{M}^\Gamma \longrightarrow \mathcal{M}^{\Gamma'}$, we obtain a homomorphism

$$\mathrm{res}\ : H^i(\Gamma, \mathcal{M}) \longrightarrow H^i(\Gamma', \mathcal{M}).$$

We give an interpretation of this homomorphism in terms of sheaf cohomology. We have the diagram

$$\begin{array}{ccc} & X & \\ \pi_{\Gamma'} \swarrow & & \searrow \pi_\Gamma \\ \pi_1 = \pi_{\Gamma,\Gamma'} : \Gamma'\backslash X & \longrightarrow & \Gamma\backslash X \end{array}$$

and a Γ-module $\mathcal{M}$ produces sheaves $\mathrm{sh}_\Gamma(\mathcal{M}) = \tilde{\mathcal{M}}$ and $\mathrm{sh}_{\Gamma'}(\mathcal{M}) \tilde{=} \mathcal{M}'$ on $\Gamma'\backslash X$ and $\Gamma\backslash X$ respectively. It is clear that we have a homomorphism

$$\pi_1^*(\tilde{\mathcal{M}}) \longrightarrow \tilde{\mathcal{M}}'.$$

To get this homomorphism we observe that for $y_1 \in \Gamma'\backslash X$ we have $\pi_1^*(\tilde{\mathcal{M}})_{y_1} = \tilde{\mathcal{M}}_{\pi_1(y_1)}$, and this is

$$\{f : \pi^{-1}(\pi_1(y)) \to \mathcal{M} \mid f(\gamma\tilde{y}) = \gamma f(\tilde{y}) \text{ for all } \gamma \in \Gamma, \tilde{y} \in \pi^{-1}(\pi(y_1))\}$$

and

$$\tilde{\mathcal{M}}'_{y_1} = \{fg : (\pi')^{-1}(y_1) \to \mathcal{M} \mid f(\gamma'\tilde{y}) = \gamma' f(\tilde{y}) \text{ for all } \gamma \in \Gamma', \tilde{y} \in (\pi')^{-1}(y_1)\},$$

and if we pick a point $\tilde{y} \in (\pi')^{-1}(y_1) \subset \pi^{-1}(\pi_1(y_1))$ then

$$\pi_1^*(\mathcal{M})_{y_1} \simeq \mathcal{M}^{\Gamma_{\tilde{y}_1}} \subset \tilde{\mathcal{M}}'_{y_1} = \mathcal{M}^{\Gamma'_{\tilde{y}_1}}.$$

Hence we get (or define) our restriction homomorphism as

$$H^i(\Gamma\backslash X, \mathrm{sh}_\Gamma(\mathcal{M})) \longrightarrow H^i(\Gamma'\backslash X, \pi_1^*(\mathrm{sh}_\Gamma(\mathcal{M})) \longrightarrow H^i(\Gamma'\backslash X, \mathrm{sh}_{\Gamma'}(\mathcal{M})). \quad (2.5)$$

There is also a map in the opposite direction.

Since the fibres of π_1 are discrete we have

$$H^i(\Gamma'\backslash X, \tilde{\mathcal{M}}) \xrightarrow{\sim} H^i(\Gamma\backslash X, \pi_{1,*}(\tilde{\mathcal{M}})).$$

But the same reasoning as in the previous section yields an isomorphism

$$\pi_{1,*}(\tilde{\mathcal{M}}) \xrightarrow{\sim} \widetilde{\mathrm{Ind}_{\Gamma'}^{\Gamma}\mathcal{M}}.$$

Hence we get an isomorphism

$$H^i(\Gamma'\backslash X, \tilde{\mathcal{M}}) \xrightarrow{\sim} H^i(\Gamma\backslash X, \widetilde{\mathrm{Ind}_{\Gamma'}^{\Gamma}\mathcal{M}}) \tag{2.6}$$

which is well known as Shapiro's lemma. But we have a Γ-module homomorphism

$$e : \mathrm{Ind}_{\Gamma'}^{\Gamma}\mathcal{M} \longrightarrow \mathcal{M}$$

which sends an $f : \Gamma \to \mathcal{M}$, in $f \in \mathrm{Ind}_{\Gamma'}^{\Gamma}\mathcal{M}$ to the sum

$$\mathrm{tr}(f) = \sum \gamma_i^{-1} f(\gamma_i)$$

where the γ_i are representatives for the classes of $\Gamma'\backslash\Gamma$. This homomorphism induces a map in the cohomology. We get a composition

$$\pi_{1,\bullet} : H^i(\Gamma'\backslash X, \tilde{\mathcal{M}}) \longrightarrow H^i(\Gamma\backslash X, \tilde{\mathcal{M}}). \tag{2.7}$$

It is not difficult to check that

$$\pi_{1,\bullet} \circ \pi_1^{\bullet} = [\Gamma : \Gamma']. \tag{2.8}$$

2.1.3 How to compute the cohomology groups $H^i(\Gamma\backslash X, \tilde{\mathcal{M}})$?

The Čech complex of an orbiconvex Covering

We consider a point $\tilde{x} \in X$ and an open neighbourhood $\tilde{U}_{\tilde{x}} \subset X$. We say that $\tilde{U}_{\tilde{x}}$ is an *orbiconvex* neighbourhood of $\tilde{x}$ if

a) The set $\tilde{U}_{\tilde{x}}$ is convex, i.e. for any two points in $\tilde{x}_1, \tilde{x}_2 \in \tilde{U}_{\tilde{x}}$ the geodesic joining $\tilde{x}_1$ and $\tilde{x}_2$ lies in $\tilde{U}_{\tilde{x}}$.

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b) We have $\gamma\tilde{U}_{\tilde{x}} \cap \tilde{U}_{\tilde{x}} = \emptyset$ unless $\gamma\tilde{x} = \tilde{x}$ and in this case we even have $\gamma\tilde{U}_{\tilde{x}} = \tilde{U}_{\tilde{x}}$.

A family of orbiconvex neighbourhoods $\{\tilde{U}_{\tilde{x}_i}\}_{i=1,\dots,r}$ ($\tilde{x}_1, \dots, \tilde{x}_r$ set of points) will be called an *orbiconvex covering,* if

$$\bigcup_{i=1}^{r} \bigcup_{\gamma\in\Gamma} \gamma\tilde{U}_{\tilde{x}_i} = X. \tag{2.9}$$

We will show later that we can always find a finite orbiconvex covering of X.

If now $\{\tilde{U}_{\tilde{x}_i}\}_{i=1,\dots,r}$ is an orbiconvex covering we put $U_{x_i} = \pi(\tilde{U}_{\tilde{x}_i})$, and then we get finite covering by open sets

$$\bigcup_{x_i} U_{x_i} = \Gamma\backslash X.$$

We abbreviate and use the usual notation $\mathfrak{U} = \{U_{x_i}\}$ for this orbiconvex covering of $\Gamma\backslash X$.

We will see further down that the intersections $U_{\underline{i}} = U_{x_{i_1}} \cap U_{x_{i_2}} \cap \cdots \cap U_{x_{i_q}}$ are acyclic, i.e. $H^k(U_{\underline{i}}, \tilde{\mathcal{M}}) = 0$ for $k > 0$. This implies that the Čzech complex (See [40], Chap. 4)

$$C^\bullet(\mathfrak{U}, \tilde{\mathcal{M}}) \quad := 0 \to \bigoplus_{i\in I} \tilde{\mathcal{M}}(U_{x_i}) \xrightarrow{d_0} \bigoplus_{i<j} \tilde{\mathcal{M}}(U_{x_i} \cap U_{x_j}) \to \tag{2.10}$$

computes the cohomology.

For the implementation on a computer we need to resolve the definition of the spaces of sections and the definition of the boundary maps. (By this I mean that we have to write explicitly

$$\tilde{\mathcal{M}}(U_{\underline{i}}) = \bigoplus_{\eta} \mathcal{M}_\eta$$

where η runs through an index set and the $\mathcal{M}_\eta$ are explicit subspaces of $\mathcal{M}$ and then we have to write down certain explicit linear maps $\mathcal{M}_\eta \to \mathcal{M}_{\eta'}$.)

We have to be aware that the intersections $U_{\underline{i}}$ are not necessarily connected. We write $U_{\underline{i}} = \cup U_\eta$ as the union of its connected components, we have to choose a connected component $\tilde{U}_\eta$ in $\pi^{-1}(U_\eta)$ for each value of η. Then the evaluation of a section $m \in \tilde{\mathcal{M}}(U_{\underline{i}})$ on these $\tilde{U}_\eta$ yields an isomorphism

$$\bigoplus_\eta ev_{\tilde{U}_\eta} : \bigoplus_\eta \tilde{\mathcal{M}}(U_\eta)(= \mathcal{M}(U_{\underline{i}}) \xrightarrow{\sim} \bigoplus_\eta \mathcal{M}^{\Gamma_\eta}.$$

If we replace $\tilde{U}_\eta$ by $\gamma\tilde{U}_\eta$ then we get for $m \in \tilde{\mathcal{M}}(\pi(\tilde{U}_\eta))$ the equality

$$\gamma ev_{\tilde{U}_\eta}(m) = ev_{\gamma\tilde{U}_\eta} \tag{2.11}$$

In degree zero the U_{x_i} are connected and this gives for the first term of the complex

$$ev_{U_{x_i}} : \tilde{\mathcal{M}}(U_{x_i}) \xrightarrow{\sim} \mathcal{M}^{\Gamma_{\tilde{x}_i}}. \tag{2.12}$$

The computation the second term is a little bit more delicate. We have to understand the connected components of $U_{x_i} \cap U_{x_j}$. To get these connected components we have to find the elements $\gamma \in \Gamma$ for which

$$\tilde{U}_{\tilde{x_i}} \cap \gamma(\tilde{U}_{\tilde{x_j}}) \neq \emptyset \tag{2.13}$$

It is clear that this gives us a finite set $G_{i,j}$ of elements $\gamma \in \Gamma/\Gamma_{x_j}$. We have a little lemma

Lemma 2.1.2. *The images $\pi(\tilde{U}_{\tilde{x_i}} \cap \gamma(\tilde{U}_{\tilde{x_j}}))$ are the connected components of $U_{x_i} \cap U_{x_j}$, two elements γ, γ_1 give the same connected component if and only if $\gamma_1 \in \Gamma_{x_i}\gamma\Gamma_{x_j}$.*

Let $F_{i,j} \subset G_{i,j}$ be a set of representatives for the action of Γ_{x_1} on $G_{i,j}$ this set can be identified to the set of connected components. Of course the set $\tilde{U}_{\tilde{x_i}} \cap \gamma(\tilde{U}_{\tilde{x_j}})$ may have a non trivial stabilizer $\Gamma_{i,j,\gamma}$ and then we get an identification

$$\oplus_{\gamma\in F_{i,j}} ev_{\tilde{U}_{x_i}\cap\gamma\tilde{U}_{x_j}} : \tilde{\mathcal{M}}(U_{x_i} \cap U_{x_j}) \xrightarrow{\sim} \bigoplus_{\gamma\in F_{i,j}} \mathcal{M}^{\Gamma_{i,j,\gamma}} \tag{2.14}$$

This is now an explicit (i.e. digestible for a computer) description of the second term in our complex above. We still need to give the explicit formula for d_0 in the complex

$$0 \to \bigoplus_{i\in I} \mathcal{M}^{\Gamma_{\tilde{x}_i}} \xrightarrow{d_0} \bigoplus_{i<j} \bigoplus_{\gamma\in F_{i,j}} \mathcal{M}^{\Gamma_{i,j,\gamma}} \tag{2.15}$$

Looking at the definition it is clear that this map is given by

$$(\dots, m_i, \dots, m_j, \dots) \mapsto (\dots, m_i - \gamma m_j, \dots) \tag{2.16}$$

Here we have to observe that $\gamma \in \Gamma/\Gamma_{x_j}$ but this does not matter since $m_j \in \mathcal{M}^{\Gamma_{\tilde{x}_j}}$. So we have an explicit description of the beginning of the Čech complex. A little reasoning shows of course that a different choice $F'_{i,j}$ of the representatives provides an isomorphic complex.

Now it is clear, how to proceed. At first we have to understand the combinatorics of the covering $\mathfrak{U} = \{U_{x_i}\}_{i\in I}$. We consider sets

$$G_{\underline{i}} = \{\underline{\gamma} = (e, \gamma_1, \dots, \gamma_q) | \gamma_i \in \Gamma/\Gamma_{x_i}; \tilde{U}_{\tilde{x}_0} \cap \dots \cap \gamma_i \tilde{U}_{\tilde{x_i}} \cap \gamma_q \tilde{U}_{\tilde{x}_q} \neq \emptyset\}$$

on these sets we have an action of Γ_{x_0} by multiplication from the left. Again let $F_{\underline{i}}$ be a system of representatives modulo the action of Γ_{x_0}.

We abbreviate

$$\tilde{U}_{\underline{i},\underline{\gamma}} = \tilde{U}_{\tilde{x}_0} \cap \dots \cap \gamma_i \tilde{U}_{\tilde{x_i}} \cap \gamma_q \tilde{U}_{\tilde{x}_q},$$

let $\Gamma_{\underline{i},\underline{\gamma}}$ be the stabiliser of $\tilde{U}_{\underline{i},\underline{\gamma}}$.

The images $\pi(\tilde{U}_{\underline{i},\underline{\gamma}})$ under the projection map π are the connected components $\pi(\tilde{U}_{\underline{i},\underline{\gamma}}) = U_{\underline{i},\underline{\gamma}} \subset U_{\underline{i}} = U_{x_{i_0}} \cap \dots \cap U_{x_{i_\nu}} \cap \dots U_{x_{i_q}}$. On the other hand each set $\tilde{U}_{\underline{i},\underline{\gamma}}$ is a connected component in $\pi^{-1}(U_{\underline{i},\underline{\gamma}})$. We get an isomorphism

$$\bigoplus_{\underline{\gamma}\in F_{\underline{i}}} ev_{\tilde{U}_{\underline{i},\gamma}} : \tilde{\mathcal{M}}(U_{\underline{i}}) = \tilde{\mathcal{M}}(U_{x_{i_0}} \cap \dots \cap U_{x_{i_\nu}} \cap \dots U_{x_{i_q}}) \xrightarrow{\sim} \bigoplus_{\underline{\gamma}\in F_{\underline{i}}} \mathcal{M}^{\Gamma_{\underline{i},\underline{\gamma}}}. \tag{2.17}$$

We need to give explicit formulas for the boundary maps

$$\bigoplus_{\underline{i}\in I^q} \tilde{\mathcal{M}}(U_{\underline{i}}) \xrightarrow{d_q} \bigoplus_{\underline{i}\in I^{q+1}} \tilde{\mathcal{M}}(U_{\underline{i}}).$$

Abstractly this boundary operator is defined as follows: We look at pairs $\underline{i} \in I^{q+1}, \underline{i}^{(\nu)} \in I^q$ where $\underline{i}^{(\nu)}$ is obtained from $\underline{i}$ by deleting the ν-th entry. Then we have $U_{\underline{i}} \subset U_{\underline{i}^{(\nu)}}$ and from this we get the resulting restriction homomorphism $R_{\underline{i}^{(\nu)},\underline{i}} : \tilde{\mathcal{M}}(U_{\underline{i}^{(\nu)}}) \to \tilde{\mathcal{M}}(U_{\underline{i}})$. Then

$$d_q = \sum_{\underline{i}} \sum_{\nu=0}^{q} (-1)^\nu R_{\underline{i}^{(\nu)},\underline{i}} \tag{2.18}$$

and hence we have to give an explicit description of $R_{\underline{i}^{(\nu)},\underline{i}}$ with respect to the isomorphism in the diagram (2.17).

We pick two connected components $\pi(\tilde{U}_{\underline{i},\underline{\gamma}}) \subset U_{\underline{i}})$ and $\pi(\tilde{U}_{\underline{i}^{(\nu)},\underline{\gamma}'} \subset U_{\underline{i}^{(\nu)}})$, then we know that

$$\tilde{U}_{\underline{i},\underline{\gamma}} \subset \tilde{U}_{\underline{i}^{(\nu)},\underline{\gamma}'} \iff \exists\, \eta_{\gamma,\gamma'} \in \Gamma \text{ such that } \eta_{\gamma,\gamma'}\gamma'_\mu = \gamma_\mu \text{ for all } \mu \neq \nu$$

and then the restriction of $R_{\underline{i}^{(\nu)},\underline{i}}$ to these two components is given by

$$\begin{array}{ccc} \tilde{\mathcal{M}}(\pi(\tilde{U}_{\underline{i}^{(\nu)},\underline{\gamma}'})) & \xrightarrow{ev_{\tilde{U}_{\underline{i}^{(\nu)},\underline{\gamma}'}}} & \mathcal{M}^{\Gamma_{\underline{i}^{(\nu)},\underline{\gamma}'}} \\ \downarrow R_{\underline{i}^{(\nu)},\underline{i}} & & \downarrow \eta_{\gamma,\gamma'} \\ \tilde{\mathcal{M}}(\pi(\tilde{U}_{\underline{i},\underline{\gamma}})) & \xrightarrow{ev_{\tilde{U}_{\underline{i},\underline{\gamma}}}} & \mathcal{M}^{\Gamma_{\underline{i},\underline{\gamma}}} \end{array} \tag{2.19}$$

Here the two horizontal maps are isomorphisms, we observe that $\eta_{\gamma,\gamma'}$ is unique up to an element in $\Gamma_{\underline{i}^{(\nu)},\underline{\gamma}'}$ and hence the vertical arrow $\eta_{\gamma,\gamma'}$ is well defined.

Hence we conclude:

Once we have found a a finite orbiconvex covering of $\Gamma\backslash X$, we can write down an explicit complex, which computes the cohomology groups $H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}})$.

We may also look at this situation from a different point of view: If $x \in X$ is any point and $\Gamma_x \subset \Gamma$ its stabilizer, then we define the induced Γ module

$$\mathrm{Ind}_{\Gamma_x}^{\Gamma}\mathbb{Z} := \{f : \Gamma \to \mathbb{Z} \mid f \text{ has finite support and } f(a\gamma) = f(\gamma),\ \forall a \in \Gamma_x, \gamma \in \Gamma\} \tag{2.20}$$

If V_x is an open neighbourhood of x which satisfies b) an c) then we have $\pi^{-1}(\pi(V_x)) = \bigcup_{\gamma\in\Gamma/\Gamma_x} \gamma V_x$ and

$$\pi^*(\tilde{\mathcal{M}})(\bigcup_{\gamma\in\Gamma/\Gamma_x} \gamma V_x) = \mathrm{Hom}(\mathrm{Ind}_{\Gamma_x}^{\Gamma}\mathbb{Z}, \mathcal{M}).$$

We have the covering

$$\tilde{\mathfrak{U}} = \bigcup_{i,\gamma\in\Gamma/\Gamma_{\tilde{x}_i}} \gamma \tilde{U}_{\tilde{x}_i} = X$$

of the symmetric space. The Čzech-complex $C^\bullet(\tilde{\mathfrak{U}}, \pi^*(\tilde{\mathcal{M}}))$ computes the cohomology groups $H^q(X, \pi^*(\tilde{\mathcal{M}}))$ which are trivial for $q > 0$. Our considerations above yield

$$C^\bullet(\tilde{\mathfrak{U}}, \pi^*(\tilde{\mathcal{M}})) = 0 \to \bigoplus_{i=1}^{r} \mathrm{Hom}(\mathrm{Ind}_{\Gamma_{x_i}}^{\Gamma}\mathbb{Z}, \mathcal{M}) \xrightarrow{d^1} \bigoplus_{i<j,\tilde{x}_{i,j}} \mathrm{Hom}(\mathrm{Ind}_{\Gamma_{\tilde{x}_{i,j}}}^{\Gamma}\mathbb{Z}, \mathcal{M}) \xrightarrow{d^2} \dots$$

Now it is easy see that the boundary maps are induced by maps between the induced modules

$$\xrightarrow{\delta^2} \bigoplus_{i<j,\tilde{x}_{i,j}} \mathrm{Ind}_{\Gamma_{\tilde{x}_{i,j}}}^{\Gamma} \mathbb{Z} \xrightarrow{\delta^1} \bigoplus_{i=1}^{r} \mathrm{Ind}_{\Gamma_{\tilde{x}_i}}^{\Gamma} \mathbb{Z} \to 0,$$

where for $f \in \bigoplus \mathrm{Ind}_{\Gamma_{\tilde{x}_J}}^{\Gamma} \mathbb{Z}$, in degree ν and $\omega \in C^{\nu-1}(\tilde{\mathfrak{U}}, \pi^*(\tilde{\mathcal{M}}))$ the relation $\omega(\delta^\nu(f)) = d^{\nu-1}(\omega)(f)$ defines δ^ν. We get an augmented complex

$$P^\bullet :=\to \bigoplus_{\tilde{x}_J} \mathrm{Ind}_{\Gamma_{\tilde{x}_J}}^{\Gamma} \mathbb{Z} \to \cdots \to \bigoplus_{\tilde{x}_{i,j}} \mathrm{Ind}_{\Gamma_{\tilde{x}_{i,j}}}^{\Gamma} \mathbb{Z} \to \bigoplus_{i=1}^{r} \mathrm{Ind}_{\Gamma_{\tilde{x}_i}}^{\Gamma} \mathbb{Z} \to \mathbb{Z} \to 0 \tag{2.21}$$

and since $C^\bullet(\tilde{\mathfrak{U}}, \pi^*(\tilde{\mathcal{M}}))$ is acyclic in degree > 0, we get that $P^\bullet$ is an acyclic resolution of the trivial module $\mathbb{Z}$.

Let $N = \prod_i \#\Gamma_{\tilde{x}_i}$ and $R := \mathbb{Z}[\frac{1}{N}]$ then the $R[\Gamma]$ module $\mathrm{Ind}_{\Gamma_x}^{\Gamma} \otimes R$ is a direct summand in $R[\Gamma]$ and hence a projective $R[\Gamma]$ module. This implies of course that

$$P^\bullet \otimes R =\to \bigoplus_{\tilde{x}_J} \mathrm{Ind}_{\Gamma_{\tilde{x}_J}}^{\Gamma} R \to \cdots \to \bigoplus_{\tilde{x}_{i,j}} \mathrm{Ind}_{\Gamma_{\tilde{x}_{i,j}}}^{\Gamma} R \to \bigoplus_{i=1}^{r} \mathrm{Ind}_{\Gamma_{\tilde{x}_i}}^{\Gamma} R \to R \to 0 \tag{2.22}$$

is indeed a projective resolution of the trivial Γ -module R. Therefore we know that

$$H^\bullet(\Gamma, \mathcal{M}_R) = H^\bullet(0 \to \mathrm{Hom}_\Gamma(\bigoplus_{i=1}^{r} \mathrm{Ind}_{\Gamma_{\tilde{x}_i}}^{\Gamma} R, \mathcal{M}_R) \to \bigoplus_{i<j,\tilde{x}_{i,j}} \mathrm{Hom}_\Gamma(\mathrm{Ind}_{\Gamma_{\tilde{x}_{i,j}}}^{\Gamma} R, \mathcal{M}_R) \to) \tag{2.23}$$

where now on the left hand side we have the group cohomology.

If we do not tensor by R then the Čzech-complex

$$0 \to \bigoplus_{i=1}^{r} \mathrm{Hom}_\Gamma(\mathrm{Ind}_{\Gamma_{x_i}}^{\Gamma} \mathbb{Z}, \mathcal{M}) \to \bigoplus_{i<j,i,j} \mathrm{Hom}_\Gamma(\mathrm{Ind}_{\Gamma_{\tilde{x}_{i,j}}}^{\Gamma} \mathbb{Z}, \mathcal{M}) \to \ldots \tag{2.24}$$

is isomorphic to the Čzech complex (2.10) and it computes the sheaf cohomology $H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}})$.

It follows from reduction theory that

Theorem 2.1.1. *We can construct a finite covering* $\Gamma\backslash X = \bigcup_{i\in E} U_{x_i} = \mathfrak{U}$ *by orbiconvex sets.*

Proof. This is rather clear. We start from the covering by the sets $X^P(c'_\pi, r(c'_\pi))$.) The set of "almost stable" points $X^G(r) \subset X$ is relatively compact modulo Γ. For any point $\tilde{x} \in X$ we look at the minimum distance

$$d(\tilde{x}) := \min_{\gamma\in\Gamma\backslash\Gamma_{\tilde{x}}} d(\tilde{x}, \gamma\tilde{x}).$$

since the action of Γ is properly discontinuous this minimum distance $d(\tilde{x}) > 0$. Let $D(\tilde{x}, d(\tilde{x})/2) := \{\tilde{y} | d(\tilde{y}, \tilde{x}) < d(\tilde{x})/2\}$, (-the Dirichlet-ball around $\tilde{x}$-) then $D(\tilde{x}, d(\tilde{x})/2)$ is an orbiconvex neighbourhood of $\tilde{x}$. Then we can find finitely many points $\tilde{x}_1, \dots, \tilde{x}_r$ such that

$$\bigcup_{i=1}^{r} \bigcup_{\gamma \in \Gamma} \gamma D(\tilde{x}_i, d(\tilde{x}_i)/2)) \supset X^G(r).$$

We have to find a covering for the $X^P(c_P, r(c_P)))$. We recall the fibration (See (1.112))

$$p^*_{P,M} : X^P(c_{\pi'}, r(\underline{c}_{\pi'})) \to X^M(r(\underline{c}_{\pi'})) \times \prod_{\alpha \in \pi'} (0, c_\alpha].$$

We apply our previous argument and find a finite covering

$$\bigcup_{i=1}^{s} \bigcup_{\gamma \in \Gamma_M} \gamma D(\tilde{y}_i, d(\tilde{y}_i)/2)) \supset X^M(r(\underline{c}'_\pi)).$$

We pick a point $\underline{c}_0 \in \prod_{\alpha \in \pi'} (0, c_\alpha]$ then the inverse image $(p^*_{P,M})^{-1}(D(\tilde{y}_i, d(\tilde{y}_i)/2)) \times \underline{c}_0$ is relatively compact and we can find an orbiconvex covering $\{\mathfrak{V}\{V_{\tilde{x}_\mu}\}$ of this set. Then the products $V_{\tilde{x}_\mu} \times \prod_{\alpha \in \pi'} (0, c_\alpha]$ provide an orbiconvex covering of $X^P(c_{\pi'}, r(\underline{c}_{\pi'}))$. Of course these sets are not (relatively) compact anymore.

□

I think that it is a very important problem to write algorithms which compute the cohomology effectively. The main goal would be to collect experimental data which may suggest conjectures or give support for conjectures which come from different sources. We come back to this aspect in the following chapter 3 and also in the final chapter 9.

I do not claim that my proposal using orbiconvex coverings provides an acceptable solution to this problem. It solves the problem in principle but it not clear how far it reaches in practice. We see that the fixed points create some problems if we want to write down the explicit complexes. But it is certainly no solution to avoid these problems by passing to a congruence subgroup. Then the number of members in the covering growth rapidly and the complexes become much bigger. For the groups $\mathrm{Sl}_n(\mathbb{Z})$ (for some small values of n and for some congruence subgroups) various authors have computed the cohomology (Ash, Stevens, Gunnels) using the Voronoi decomposition of the cone of positive definite symmetric matrices.

A first step would be to find effectively an optimal orbiconvex covering $\{U_{\tilde{x}_\nu}\}$ of the set $X^G(r)$ of almost stable points. The covering sets must not necessarily be Dirichlet balls. We could proceed and apply this also to the different $X^M(r(\underline{c}_{\pi'}))$ and find orbiconvex covers $\{V^M_{\tilde{y}_\mu}\}$ for them. Then we may consider the inverse images $(p^*_{P,M})^{-1}(V^M_{\tilde{y}_\mu} \times \prod_{\alpha \in \pi'} (0, c_\alpha]) = \tilde{V}^M_{\tilde{y}_\mu}$. This family of sets $\{\{\gamma U_{\tilde{x}_\nu}\}, \dots, \gamma_1 \tilde{V}^M_{\tilde{y}_\mu}, \dots\}$ provide a covering of X by open sets, hence the images under the projection provide a covering

$$\mathfrak{W} = \{W_i\}_{i \in I} = \{\{U_{x_\nu}\}, \dots, \{\tilde{V}^M_{y_\mu}\}, \dots\}$$

of $\Gamma\backslash X$, here the index set I is the union of the $x_]\tilde{x}_\nu, \dots, {}^M y_{\tilde{y}_\mu}$.

Of course we have a problem: The sets $\tilde{V}^M_{\tilde{y}_\mu}$ are not acyclic anymore, so we can not use the Čzech complex of this covering for the computation of the cohomology. But we know that

$$\tilde{V}^M_{\tilde{y}_\mu} \to V^M_{\tilde{y}_\mu} \times \prod_{\alpha\in\pi'} (0, c_\alpha]$$

is a fiber bundle with fiber $U(\mathbb{Z})\backslash U(\mathbb{R})$, Since the base $V^M_{\tilde{y}_\mu} \times \prod_{\alpha\in\pi'}(0, c_\alpha]$ is acyclic we know that

$$H^\bullet(\tilde{V}^M_{\tilde{y}_\mu}) \xrightarrow{\sim} \mathbb{H}^\bullet(U(\mathbb{Z})\backslash U(\mathbb{R}), \tilde{\mathcal{M}}) \tag{2.25}$$

and we have a good understanding of the cohomology on the right. If for instance we tensor by the rationals the Theorem of Kostant (See section 8.1.9) gives us a complete description of the cohomology $H^\bullet(U(\mathbb{Z})\backslash U(\mathbb{R}), \tilde{\mathcal{M}} \otimes \mathbb{Q})$.

For $\underline{i} \in I^{p+1}$ we put $\mathfrak{W}_{\underline{i}} = W_{i_0} \cap W_{i_1} \cap \dots \cap\ W_{i_p}$ Now we follow [40], 4.6.6, for any $q \geq 0$ write the Čzech complexe

$$C^\bullet(\mathfrak{W}, \mathcal{H}^q) :=\to \prod_{\underline{i}\in I^{p+1}} H^q(W_{\underline{i}}) \to \prod_{\underline{i}\in I^{p+2}} H^q W_{\underline{i}}) \tag{2.26}$$

and then we know that we get a spectral sequence

$$H^p(C^\bullet(\mathfrak{W}, \mathcal{H}^q)) = E_1^{p,q} \implies H^{p+q}(\Gamma\backslash X, \tilde{\mathcal{M}}) \tag{2.27}$$

Lowdim

2.1.4 Special examples in low dimensions.

We consider the group $\Gamma = \mathrm{Sl}_2(\mathbb{Z})/\{\pm\mathrm{Id}\}$ and its action on the upper half plane $\mathbb{H}$. We want to investigate the cohomology groups $H^i(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})$ for any module Γ-module $\mathcal{M}$. Let $p : \mathbb{H} \to \Gamma\backslash\mathbb{H}$ be the projection. We have the two special points i and ρ in $\mathbb{H}$ they are up to conjugation by Γ the only points which have a non trivial stabilizer. We construct two nice orbiconvex neighbourhoods of these two points. The stabilizers Γ_i, resp. Γ_ρ are cyclic and generated by the two elements

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, R = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

respectively.

We begin with i. We consider the strip $V_i = \{z| - 1/2 < \Re(z) < 1/2\}$, the element S maps the two vertical boundary lines $\Re(z) = \pm\frac{1}{2}$ into geodesic circles starting from 0 and ending in ± 2. Then the intersection $\tilde{U}_i = V_i \cap S(V_i)$ is an orbiconvex neighbourhood of i.

Let us look at ρ. We consider the strip $V_\rho = \{z \mid -0 < \Re(z) < 1\}$ and now we define $\tilde{U}_\rho = V_\rho \cap R(V_\rho) \cap R^2(V_\rho)$. This is a nice orbiconvex neighbourhood of ρ.

Now it is clear that these two sets provide an orbiconvex covering of $\mathbb{H}$, if $U_i = p(\tilde{U}_i), U_\rho = p(\tilde{U}_\rho)$ then

$$\Gamma\backslash\mathbb{H} = U_i \cup U_\rho. \tag{2.28}$$

We have $\tilde{\mathcal{M}}(U_i) = \mathrm{sh}_\Gamma(\mathcal{M})(U_i) = \mathcal{M}^{\Gamma_i}, \tilde{\mathcal{M}}(U_\rho) = \mathcal{M}^{\Gamma_\rho}$ and hence the cohomology groups are given by the cohomology of the complex

$$0 \to \mathcal{M}^{\Gamma_i} \oplus \mathcal{M}^{\Gamma_\rho} \to \mathcal{M} \to 0 \tag{2.29}$$

Then $H^0(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) = \mathcal{M}^\Gamma = \mathcal{M}^{\Gamma_i} \cap \mathcal{M}^{\Gamma_\rho}$. Since this is true for any Γ module we easily conclude that Γ is generated by Γ_i, Γ_ρ. And we get

$$H^1(\mathrm{Sl}_2(\mathbb{Z})\backslash\mathbb{H}, \tilde{\mathcal{M}}_{\mathbb{Z}}) = \mathcal{M}/(\mathcal{M}^{(<S>} \oplus \mathcal{M}^{<R>}), \tag{2.30}$$

and the cohomology vanishes in higher degrees.

Exercise 1: *Let* $\Gamma' \subset \Gamma = \mathrm{Sl}_2(\mathbb{Z})/\pm\mathrm{Id}$ *be a subgroup of finite index. Prove*
ii) We have (Shapiros lemma)

$$H^1(\Gamma'\backslash\mathbb{H}, \mathbb{Z}) = H^1(\Gamma\backslash\mathbb{H}, \widetilde{\mathrm{Ind}_{\Gamma'}^{\Gamma}\mathbb{Z}}).$$

These cohomology groups are free of rank

$$[\Gamma : \Gamma'] - n_i - n_\rho + 1$$

where n_i *(resp.* n_ρ*) is the number of orbits of* Γ_i *(resp.* Γ_ρ*) on* $\Gamma'\backslash\Gamma$*. If* Γ' *is torsion free then*

$$\mathrm{rank}(H^1\Gamma\backslash\mathbb{H}, \widetilde{\mathrm{Ind}_{\Gamma'}^{\Gamma}\mathbb{Z}}) = \frac{1}{6}[\Gamma : \Gamma'] + 1$$

The Euler-characteristic of $\Gamma'\backslash\mathbb{H}$ *is* $\frac{1}{6}[\Gamma : \Gamma']$.

Exercise 2: *Let* $\mathcal{M}_n$ *be the* $\mathrm{Sl}_2(\mathbb{Z})$*-module of homogenous polynomials in the two variables* X, Y *and coefficients in* $\mathbb{Z}$*. (See 1.2.2). We have the usual action of* $\mathrm{Sl}_2(\mathbb{Z})$ *on this module by*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} P(X,Y) = P(aX + cY, bX + dY).$$

these modules define a sheaf $\tilde{\mathcal{M}}_n$ *on* $\Gamma\backslash\mathbb{H}$*, and we want to investigate their cohomology groups.*

Prove:
i) If n *is odd, then* $\mathcal{M}_n = 0$.
Hence we assume $n \geq 2$ *and* n *even from now on.*

ii) For $n > 0$ *we have* $H^0(\Gamma\backslash\mathbb{H}, \mathcal{M}_n l) = 0$.

iii) If we tensorize by $\mathbb{Q}$ *, then* $H^1(\Gamma\backslash\mathbb{H}, \mathcal{M}_n \otimes \mathbb{Q})$ *is a vector space of rank* $n - 1 - 2\left[\frac{n}{4}\right] - 2\left[\frac{n}{6}\right]$.

Hint: *Diagonalise the action of* Γ_i *and* Γ_ρ *on* $\mathcal{M}_n \otimes \overline{\mathbb{Q}}$ *separately and look at the eigenspaces. To say it differently: Over* $\bar{\mathbb{Q}}$ *we can conjugate the matrices* $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ *into the diagonal maximal torus* $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ *, and then look at the decomposition of* $\mathcal{M}_n$ *into weight spaces.*

iv) Investigate the torsion in $H^1(\Gamma\backslash\mathbb{H}, \mathcal{M}_n)$. *(Start from the sequence* $0 \to \mathcal{M}_n \to \mathcal{M}_n \to \mathcal{M}_n/\ell\mathcal{M}_n \to 0$.*)*

v) Now we consider $\Gamma = \mathrm{Sl}_2(\mathbb{Z})$. *The two matrices* $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ *and* $R = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$ *are generators of the stabilisers of* i *and* ρ *respectively.*

We take for our module $\mathcal{M}$ *the cyclic group* $\mathbb{Z}/12\mathbb{Z}$, *consider the spectral sequence*

$$H^p(\Gamma\backslash\mathbb{H}, R^q(\mathrm{sh}_\Gamma)(\mathbb{Z}/12\mathbb{Z})).$$

Show that $H^0(\Gamma\backslash\mathbb{H}, R^1(\mathrm{sh}_\Gamma)(\mathbb{Z}/12\mathbb{Z}) = \mathbb{Z}/12\mathbb{Z}$. *Show that the differential*

$$H^0(\Gamma\backslash\mathbb{H}, R^1(\mathrm{sh}_\Gamma)(\mathbb{Z}/12\mathbb{Z})) \to H^2(\Gamma\backslash\mathbb{H}, \mathrm{sh}_\Gamma(\mathbb{Z}/12\mathbb{Z}))$$

vanishes and conclude

$$H^1(\Gamma, \mathbb{Z}/12\mathbb{Z}) = \mathbb{Z}/12\mathbb{Z}.$$

Here it seems that our method to compute the cohomology is very successful, but this is misleading. The topological space $\Gamma\backslash\mathbb{H}$ is simply to simple.

The group $\Gamma = \mathrm{Gl}_2(\mathbb{Z}[i])$

A similar computation can be made up to compute the cohomology in the case of $\tilde{\Gamma} = \mathrm{Gl}_2(\mathbb{Z}[i]))$. We have the three special points x_{12}, x_{13} and x_{23} (See(1.2.5), and we choose closed sets A_{ij} containing these points which just leave out a small open strip containing the opposite face. If $\tilde{A}_{ij}$ is a connected component of the inverse image of A_{ij} in $\mathbb{H}_3$, then

$$A_{ij} = \Gamma_{ij}\backslash\tilde{A}_{ij}.$$

The intersections $A_{ij} \cap A_{i'j'} = A_\nu$ are closed sets. They are of the form

$$A_\nu = \Gamma_\nu\backslash\tilde{A}_\nu$$

where Γ_ν is the stabilizer of the arc joining x_{ij} and $x_{i'j'}$. The restrictions of our sheaves $\tilde{\mathcal{M}}$ to the A_{ij} and A_ν and to $A = A_{12} \cap A_{23} \cap A_{13}$ are acyclic and hence we get a complex

$$0 \longrightarrow \tilde{\mathcal{M}} \longrightarrow \bigoplus_{(i,j)} \tilde{\mathcal{M}}_{A_{ij}} \longrightarrow \bigoplus \tilde{\mathcal{M}}_{A_\nu} \longrightarrow \tilde{\mathcal{M}}_A \longrightarrow 0 \qquad (2.31)$$

where the $\tilde{\mathcal{M}}_?$ are the restrictions of $\tilde{\mathcal{M}}$ to ? and then extended to the space again.

Hence we find that our cohomology groups are equal to the cohomology groups of the complex

$$0 \longrightarrow \bigoplus_{(i,j)} \mathcal{M}^{\Gamma_{ij}} \xrightarrow{d^1} \bigoplus_{\nu} \mathcal{M}^{\Gamma_\nu} \xrightarrow{d^2} \mathcal{M} \longrightarrow 0 \tag{2.32}$$

with boundary maps

$$\begin{aligned} d^1 :&(m_{12}, m_{13}, m_{23}) \longmapsto (m_{12} - m_{13}, m_{23} - m_{12}, m_{13} - m_{23}) \\ d^2 :&(m_1, m_2, m_3) \longmapsto m_1 + m_2 + m_3. \end{aligned}$$

If we take for instance $\tilde{\mathcal{M}} = \mathbb{Z}$ then we get $H^0(\tilde{\Gamma}\backslash\mathbb{H}_3, \mathbb{Z}) = \mathbb{Z}$ and $H^i(\tilde{\Gamma}\backslash\mathbb{H}_3, \mathbb{Z}) = 0$ for $i > 0$ as it should be.

Here we do not get a satisfying answer to our question. For instance we are not able to read off the dimensions of the cohomology groups from the complex

Homology, Cohomology with compact support and Poincaré duality.

Here we have to use the theory of compactifications. For any locally symmetric space we can embed $\Gamma\backslash X$ into its Borel-Serre compactification

$$i : \Gamma\backslash X \longrightarrow \Gamma\backslash\overline{X}_{BS},$$

and this process was explained in detail for our low dimensional examples. Especially we give an explicit description of a neighbourhood of a point $x \in \partial(\Gamma\backslash\overline{X}_{BS})$. If we have a sheaf $\tilde{\mathcal{M}}$ on $\Gamma\backslash X$, we can extend it to the compactification by using the functor i_*. We get a sheaf

$$i_*(\tilde{\mathcal{M}}) \quad \text{on} \quad \Gamma\backslash\overline{X}_{BS},$$

it is clear from the description of a neighbourhood of a point in the boundary, that i_* is exact. (This is not true for the Baily-Borel compactification.)

Our construction $\mathcal{M} \to \tilde{\mathcal{M}}$ can be extended to the action of Γ on $\overline{X}_{BS}$ and clearly

$$i_*(\tilde{\mathcal{M}}) = \text{ result of the construction } \mathcal{M} \to \tilde{\mathcal{M}} \text{ on } \Gamma\backslash\overline{X}_{BS}.$$

Hence we get from our general results in Chapter I, ... that

$$H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}) = H^\bullet(\Gamma\backslash\overline{X}_{BS}, i_*(\tilde{\mathcal{M}})).$$

But we have another construction of extending the sheaf $\tilde{\mathcal{M}}$ from $\Gamma\backslash X$ to $\Gamma\backslash\overline{X}_{BS}$. This is the so called extension by zero. We define the sheaf $i_!(\tilde{\mathcal{M}})$ on $\Gamma\backslash\overline{X}_{BS}$ by giving the stalks. For $x \in \Gamma\backslash\overline{X}_{BS}$ we put

$$i_!(\tilde{\mathcal{M}})_x = \begin{cases} \tilde{\mathcal{M}}_x & \text{if} \quad x \in \Gamma\backslash X \\ 0 & \text{if} \quad x \notin \Gamma\backslash X \end{cases}.$$

It is clear that $i_!$ is an exact functor sending sheaves on $\Gamma\backslash X$ to sheaves on $\Gamma\backslash\overline{X}_{BS}$, and we have for an arbitrary sheaf

$$H^0(\Gamma\backslash\overline{X}_{BS}, i_!(\mathcal{F})) = H^0_c(\Gamma\backslash X, \mathcal{F})$$

where $H^0_c(\Gamma\backslash X, \mathcal{F})$ is the abelian group of those sections $s \in H^0(\Gamma\backslash X, \mathcal{F})$ for which the support

$$\text{supp }(s) = \{x \mid s_x \neq 0\}$$

is compact.

Hence we define the cohomology with compact supports as

$$H^q_c(\Gamma\backslash X, \mathcal{F}) = H^q(\Gamma\backslash\overline{X}_{BS}, i_!\mathcal{F})).$$

If $\tilde{\mathcal{M}}$ is a sheaf on $\Gamma\backslash X$ which is obtained from a Γ-module $\mathcal{M}$, then it is quite clear that

$$H^0_c(\Gamma\backslash X, \tilde{\mathcal{M}}) = 0,$$

provided our quotient $\Gamma\backslash X$ is not compact.

The cohomology with compact supports is actually related to the homology of the group: I want to indicate that we have a natural isomorphism

$$H_i(\Gamma, \mathcal{M}) \simeq H^{d-i}_c(\Gamma\backslash X, \tilde{\mathcal{M}})$$

under the assumption that X is connected and the orders of the stabilizers are invertible in R.

This is the analogous statement to the theorem .... which we discussed when we introduced cohomology.

Our starting point is the fact that the projective Γ-modules have analogous vanishing properties as the induced modules.

Lemma 2.1.3. *Let us assume that Γ acts on the connected symmetric space X. If P if a projective module then*

$$H^i_c(\Gamma\backslash X, \tilde{P}) = \begin{cases} 0 & \textit{if} \quad i \neq \dim X \\ \\ P_\Gamma & \textit{if} \quad i = \dim X. \end{cases}$$

Let us believe this lemma. Then it is quite clear that

$$H_i(\Gamma, \mathcal{M}) \simeq H^{d-i}_c(\Gamma\backslash X, \tilde{P}),$$

because both sides can be computed from a projective resolution.

2.1.5 The homology as singular homology

We have still another description of the homology. We form the singular chain complex

$$\to C_i(X) \to C_{i-1}(X) \to \ldots \to C_0(X) \to 0.$$

This is a complex of Γ-modules, and we can form the tensor product with $\mathcal{M}$. We get a complex of Γ-modules

$$\xrightarrow{d_{i+1}} C_i(X) \otimes \mathcal{M} \xrightarrow{d_i} C_{i-1}(X) \otimes \mathcal{M} \longrightarrow \dots .$$

We define the chain complex

$$C_\bullet(\Gamma\backslash X, \underline{\mathcal{M}}),$$

simply as the resulting complex of Γ-coinvariants. The homology groups are defined as

$$H_i(\Gamma\backslash X, \underline{\mathcal{M}}) = \ker(d_i)/\mathrm{Im}(d_{i+1}) \tag{2.33}$$

The cosheaves

The symbol $\underline{\mathcal{M}}$ should be interpreted as the cosheaf attached to our Γ-module, this is an object which is dual to the sheaf $\tilde{\mathcal{M}}$. For a point $\bar{x} \in \Gamma\backslash X$ costalk $\underline{\mathcal{M}}_{\bar{x}}$ is given as follows: As in (2.2) we consider the projection $\pi_\Gamma : X \to \Gamma\backslash X$ and maps with finite support

$$\mathcal{C}(\bar{x}, \mathcal{M}) := \{f : \pi_\Gamma^{-1}(\bar{x}) \to \mathcal{M}\}. \tag{2.34}$$

On this module we have an action of Γ which is given by $\boxed{\text{act}}$

$$(\gamma f)(x) = \gamma(f(\gamma^{-1}x). \tag{2.35}$$

Then our costalk is given by the coinvariants

$$\underline{\mathcal{M}}_{\bar{x}} = \mathcal{C}(\bar{x}, \mathcal{M})_\Gamma = \mathcal{C}(\bar{x}, \mathcal{M})/\{f - \gamma f, \gamma \in \Gamma, f \in \mathcal{C}(\bar{x}, \mathcal{M})\} \tag{2.36}$$

We have the homomorphism $\int : \mathcal{M}_{\bar{x}} \to \mathcal{M}$ which is given by summation $f \mapsto \sum_{x\in\pi_\Gamma^{-1}(\bar{x})} f(x)$ and this induces an isomorphism $\boxed{\text{invint}}$

$$\int : \mathcal{C}(\bar{x}, \mathcal{M})_\Gamma \xrightarrow{\sim} \underline{\mathcal{M}}_{\bar{x}} \tag{2.37}$$

We pick a point $x \in \pi_\Gamma^{-1}(\bar{x})$ and an open neighbourhood U_x of x such that $\gamma U_x \cap U_x \neq \emptyset$ implies $\gamma \in \Gamma_x$. We consider the space $\mathcal{C}(\bar{x}, x, \mathcal{M})$ of those maps, which are supported in in the point x. This space is of course equal to $\mathcal{M}$ and the composition

$$\delta_x : \mathcal{C}(\bar{x}, x, \mathcal{M}) \to \mathcal{C}(\bar{x}, \mathcal{M}) \to \underline{\mathcal{M}}_{\bar{x}}$$

induces an isomorphism

$$\delta_x : \mathcal{M}_{\Gamma_x} \xrightarrow{\sim} \underline{\mathcal{M}}_{\bar{x}} \tag{2.38}$$

If we pick a second point $\bar{y} \in \pi_\Gamma(U_x)$ and a $y \in \pi_\Gamma^{-1}(\bar{y}) \cap U_x$ then clearly $\Gamma_y \subset \Gamma_x$ and therefore we get a specialization map

$$r_{\bar{y},\bar{x}} : \underline{\mathcal{M}}_{\bar{y}} \to \underline{\mathcal{M}}_{\bar{x}}. \tag{2.39}$$

Now it becomes clear why these objects are called cosheaves. For the sheaf $\tilde{\mathcal{M}}$ we get in the corresponding situation a map in the opposite direction

$$\tilde{\mathcal{M}}_{\bar{x}} \to \tilde{\mathcal{M}}_{\bar{y}} \tag{2.40}$$

as a specialization map between the stalks of $\tilde{\mathcal{M}}$. An element $f^* \in \underline{\mathcal{M}}_{\bar{x}}$ can be represented as an array refcos

$$f^* = \{\dots, f(x), \dots\}_{x\in\pi^{-1}(\bar{x})} \tag{2.41}$$

where $f(x) \in (\underline{\mathcal{M}}_{\bar{x}})_{\Gamma_x}$ and $f(\gamma x) = \gamma f(x)$.

Now we can give a different description of the group of i-chains $C_i(\Gamma\backslash X, \underline{\mathcal{M}})$: An i-chain with values in the cosheaf $\underline{\mathcal{M}}$ is of the form $\sigma\otimes f$ where $\sigma : \Delta^i \to \Gamma\backslash X$ is a continuous (differentiable) map from the i dimensional simplex Δ^i to $\Gamma\backslash X$ and where f is a section in the cosheaf, i.e. $f_x \in \underline{\mathcal{M}}_{\sigma(x)}$ and where f_x varies continuously. (This means: If $\sigma(y)$ specializes to $\sigma(x)$ then $r_{\sigma(y),\sigma(x)}(f_y) = f_x$.)

Then $C_i(\Gamma\backslash X, \underline{\mathcal{M}})$ is the free abelian group generated by these i chains with values in $\underline{\mathcal{M}}$). Then the boundary maps d_i are defined in the usual way and we get a slightly different description of the homology groups $H_i(\Gamma\backslash X, \underline{\mathcal{M}})$.

But we may choose for our module $\mathcal{M}$ simply the group ring. Then

$$(C_\bullet(X) \otimes \mathbb{Z}[\Gamma])_\Gamma \simeq C_\bullet(X),$$

and hence we have, since X is a cell, that

$$H_i(\Gamma\backslash X, \underline{\mathbb{Z}[\Gamma]}) = 0 \qquad \text{for} \qquad i > 0.$$

On the other hand we have

$$H_0(\Gamma\backslash X, \underline{\mathcal{M}}) = \mathcal{M}_\Gamma.$$

This follows directly from looking at the complex

$$(C_1(X) \otimes \mathcal{M})_\Gamma \longrightarrow (C_0(X) \otimes \mathcal{M})_\Gamma.$$

First of all we observe that 0-cycles

$$x_1 \otimes m - x_0 \otimes m$$

are boundaries since X is pathwise connected. On the other hand we have that

$$x_0 \otimes m - \gamma x_0 \otimes \gamma m \in C_0(X) \otimes \mathcal{M}$$

becomes zero if we go to the coinvatiants and this implies the assertion.

If we have in addition that the orders of the stabilizers are invertible in R than it is clear that a short exact sequence of R-Γ-modules

$$0 \longrightarrow \mathcal{M}' \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}'' \longrightarrow 0$$

leads to an exact sequence of complexes

$$0 \longrightarrow C_\bullet(\Gamma\backslash X, \underline{\mathcal{M}'}) \longrightarrow C_\bullet(\Gamma\backslash X, \underline{\mathcal{M}}) \longrightarrow C_\bullet(\Gamma\backslash X, \underline{\mathcal{M}''}) \longrightarrow 0,$$

and hence to a long exact cohomology sequence

$$H_i(\Gamma\backslash X, \underline{\mathcal{M}}') \longrightarrow H_i(\Gamma\backslash X, \underline{\mathcal{M}}) \longrightarrow H_i(\Gamma\backslash X, \underline{\mathcal{M}}'') \longrightarrow H_{i-1}(\Gamma\backslash X, \underline{\mathcal{M}}').$$

Now it is clear that

$$H_i(\Gamma, \mathcal{M}) \simeq H_i(\Gamma\backslash X, \underline{\mathcal{M}}) \simeq H_c^{d-i}(\Gamma\backslash X, \tilde{\mathcal{M}}).$$

fundex

2.1.6 The fundamental exact sequence

By construction we have the exact sequence

$$0 \to i_!(\tilde{\mathcal{M}}) \to i_*(\tilde{\mathcal{M}}) \to i_*(\tilde{\mathcal{M}})/i_!(\tilde{\mathcal{M}}) \to 0$$

of sheaves and clearly $i_*(\mathcal{M})/i_!(\mathcal{M})$ is simply the restriction of $i_*(\tilde{\mathcal{M}})$ to the boundary extended by zero to the entire space. This yields the *fundamental exact sequence*

$$\to H^{q-1}(\partial(\Gamma\backslash X), \tilde{\mathcal{M}}) \to H_c^q(\Gamma\backslash X, \tilde{\mathcal{M}}) \to H^q(\Gamma\backslash \bar{X}, \tilde{\mathcal{M}}) \xrightarrow{r} H^q(\partial(\Gamma\backslash X), \tilde{\mathcal{M}}) \to \dots \tag{2.42}$$

We define the "inner cohomology" inncoh

$$H_!^q(\Gamma\backslash X, \tilde{\mathcal{M}}) := \mathrm{Im}(H_c^q(\Gamma\backslash X, \tilde{\mathcal{M}}) \to H^q(\Gamma\backslash \bar{X}, \tilde{\mathcal{M}})) = \ker H^q(\Gamma\backslash \bar{X}, \tilde{\mathcal{M}}) \xrightarrow{r} H^q(\partial(\Gamma\backslash X), \tilde{\mathcal{M}}) \tag{2.43}$$

(This a little bit misleading because these groups are not honest cohomology groups, they are not the cohomology groups of a space with coefficients in a sheaf. An exact sequence of sheaves $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ does not provide an exact sequence for these $H_!$ groups.)

In the special case that the underlying group $G/\mathbb{Q}$ is anisotropic the fundamental exact sequence becomes trivial, in this case the quotient $\Gamma\backslash X$ is compact and we have

$$H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}) = H_c^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}) = H_!^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}).$$

Many authors prefer to consider the case of a compact quotient $\Gamma\backslash X$, but I think we loose some very interesting phenomena if we concentrate on this case. On the other hand we do not need to read the next subsection. Also readers who are more interested in the low dimensional cases and the more specific results in these cases may well skip reading the next subsection.

The cohomology of the boundary

We want to have a slightly different look at this sequence. We recall the covering (See 1.118,1.119)

$$\Gamma\backslash X = \Gamma\backslash X(r) \cup \overset{\bullet}{\mathcal{N}} (\Gamma\backslash X) = \Gamma\backslash X(r) \cup \bigcup_{P:P \text{proper}} \Gamma_P\backslash X^P(\underline{c}_{\pi'}, r(\underline{c}_{\pi'})) \tag{2.44}$$

where the union runs over Γ conjugacy classes of parabolic subgroups over $\mathbb{Q}$ and $\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X)$ is a punctured tubular neighbourhood of ∞, i.e. the boundary of the Borel-Serre compactification.

It is well known (See for instance [book] vol I , 4.5) that from a covering $\Gamma\backslash X = \bigcup_i V_i$ we get a Čzech complex and a spectral sequence with $E_1^{p,q}$- term

$$\prod_{\underline{i}=\{i_0,i_1\dots,i_p\}} H^q(V_{\underline{i}},\tilde{\mathcal{M}}) \tag{2.45}$$

where $V_{\underline{i}} = V_{i_0}\cap\cdots\cap V_{i_p}$. The boundary in the Čzech complex gives us the differential

$$d_1^{p,q}: \prod_{\underline{i}=\{i_0,i_1\dots,i_p\}} H^q(V_{\underline{i}},\tilde{\mathcal{M}}) \to \prod_{\underline{j}=\{j_0,j_1\dots,j_{p+1}\}} H^q(V_{\underline{j}},\tilde{\mathcal{M}}) \tag{2.46}$$

Here we work with the alternating Čzech complex, we also assume that we have an ordering on the set of simple positive roots. If such a $V_{\underline{i}}$ is non empty then it of the form $\Gamma_Q\backslash X^Q(C(\underline{\tilde{c}}))$.

We return to the diagram (2.47), on the left hand side we can divide by Γ_Q. We have the map which maps a Cartan involution on X to a Cartan-involution on M. Then we get a diagram

$$\begin{array}{ccc} f^\dagger: X^Q(C(\underline{\tilde{c}})) & \to & X^M(r)\times C_{U_Q}(\underline{\tilde{c}}) \\ \downarrow p_Q & & \downarrow p_M \\ f:\Gamma_Q\backslash X^Q(C(\underline{\tilde{c}})) & \to & \Gamma_M\backslash X^M(r)\times C_{U_Q}(\underline{\tilde{c}})) \end{array} \tag{2.47}$$

where the bottom line is a fibration. To describe the fiber in a point $\tilde{x}$ we pick a point $x\in(p_m\circ f^\dagger)^{-1}$. Then $U_Q(\mathbb{R})$ acts simply transitively on the fiber $(f^\dagger)^{-1}(f^\dagger(x))$ hence $U_Q(\mathbb{R}) = (f^\dagger)^{-1}(f^\dagger(x))$. Then $p_Q: U_Q(\mathbb{R})\to\Gamma_{U_Q}\backslash U_Q(\mathbb{R})$ yields the identification $i_x:\Gamma_{U_Q}\backslash U_Q(\mathbb{R}) \xrightarrow{\sim} f^{-1}(\tilde{x})$. If we replace x by $\gamma x = x_1$ with $\gamma\in\Gamma_{U_Q}$ then we get $i_{x_1} = \mathrm{Ad}(\gamma)\circ i_x$ where for $u\in U_{U_Q}$ $\mathrm{Ad}(\gamma)(u) = \gamma u\gamma^{-1}$ where for $u\in U_Q(\mathbb{R})$, under this action of Γ_Q.

We have the spectral sequence

$$H^p(\Gamma_M\backslash X^M(r), R^q f_*(\tilde{\mathcal{M}})) \Rightarrow H^{p+q}(\Gamma_Q\backslash X^Q(C(\underline{c}_{\pi_1},\dots,c_{\pi_\nu})),\tilde{\mathcal{M}})$$

and clearly $R^q f_*(\tilde{\mathcal{M}})$ is a locally constant sheaf. This sheaf is easy to determine. Under the above identification we get an isomorphism

$$i_x^\bullet: H^\bullet(\Gamma_{U_Q}\backslash U_Q(\mathbb{R}),\tilde{\mathcal{M}})) \xrightarrow{\sim} R^\bullet(\tilde{\mathcal{M}})_{\tilde{x}}.$$

The adjoint action $\mathrm{Ad}:\Gamma_Q\to\mathrm{Aut}(\Gamma_{U_Q}\backslash U_Q(\mathbb{R}))$ induces an action of Γ_Q on the cohomology $H^\bullet((\Gamma_{U_Q}\backslash U_Q(\mathbb{R})),\tilde{\mathcal{M}})$. Since the functor cohomology is the derived functor of taking Γ_{U_Q} invariants it follows that the restriction of Ad to Γ_{U_Q} acts trivially on $H^\bullet(\Gamma_{U_Q}\backslash U_Q(\mathbb{R}),\tilde{\mathcal{M}})$. Consequently $H^\bullet((\Gamma_{U_Q}\backslash U_Q(\mathbb{R})),\tilde{\mathcal{M}})$ is a Γ_M- module. We get

$$R^\bullet f_*(\tilde{\mathcal{M}}) \xrightarrow{\sim} H^\bullet(\widetilde{\Gamma_{U_Q}\backslash U_Q}(\mathbb{R}),\tilde{\mathcal{M}})$$

and hence our spectral sequence becomes

vEst

$$H^p(\Gamma_M\backslash X^M(r), H^\bullet(\widetilde{\Gamma_{U_Q}\backslash U_Q(\mathbb{R})}, \tilde{\mathcal{M}})) \Rightarrow H^{p+q}(\Gamma_Q\backslash X^Q(C(\underline{\tilde{c}})), \tilde{\mathcal{M}}) \qquad (2.48)$$

We can take the composition $r_Q \circ f$. Then it is obvious that for any point $c_0 \in C_{U_Q}(\underline{\tilde{c}}))$ the restriction map

$$H^\bullet(X^Q(C(\underline{\tilde{c}})), \tilde{\mathcal{M}}) \to H^\bullet(X^Q((r_Q \circ f)^{-1}(c_0), \tilde{\mathcal{M}}) \qquad (2.49)$$

is an isomorphism. On the other hand it is clear that we may vary our parameter $\underline{\tilde{c}}$ we may assume that the $C_{U_Q}(\underline{\tilde{c}})$ go to infinity. Then we may enlarge the parameter r without violating the assumptions in proposition 1.2.3. Hence we get that the inclusion $\Gamma_Q\backslash X^Q(C(\underline{\tilde{c}})) \subset \Gamma_Q\backslash X^Q$ induces an isomorphism in cohomology

$$H^\bullet(\Gamma_Q\backslash X^Q(C(\underline{\tilde{c}}), \tilde{\mathcal{M}}) \xrightarrow{\sim} H^\bullet(\Gamma_Q\backslash X, \tilde{\mathcal{M}}) \qquad (2.50)$$

We choose a total ordering on the set of Γ conjugacy classes of parabolic subgroups, i.e. we enumerate them by a finite interval of integers $[1, N]$. We also enumerate the set of simple roots $\{\alpha_1, \dots, \alpha_d)$ in our special case $\alpha_i = \alpha_{i,i+1}$. For any conjugacy class $[P]$ we define the type of P to be $t(P) = \pi^{U_P}$ the subset of unipotent simple roots and $d(P) = \#\pi^{U_P}$ the cardinality of this set. If $P_{i_1}, \dots, P_{i_r}$ are maximal, $i_1 < i_2 \dots < i_r$ and if $P_{i_1}\cap, \dots \cap P_{i_r} = Q$ is a parabolic subgroup then we require that $t(P_{i_1}) < \dots < t(P_{i_r})$.

The indexing set $\mathrm{Par}(\Gamma)$ of our covering is the Γ conjugacy classes of parabolic subgroups over $\mathbb{Q}$. If we have a finite set $[P_{i_0}], [P_{i_1}], \dots, [P_{i_p}]$ of conjugacy classes then we say $[Q] \in [P_{i_0}], [P_{i_1}], \dots, [P_{i_p}]$ if we can find representatives $P'_{i_\nu} \in [P_{i_\nu}]$ and $Q' \in [Q]$ such that $Q' = P'_{i_0} \cap \dots \cap P'_{i_p}$.

Hence we see that the $E_1^{\bullet,q}$ complex in our spectral sequence (2.46) is given by

$$\prod_i H^q(\Gamma_{Q_i}\backslash X^{Q_i}(C(\underline{\tilde{c}})), \tilde{\mathcal{M}}) \to \prod_{i<j} \prod_{[R]\in[Q_i]\cap[Q_j]} H^q(\Gamma_R\backslash X^R(C(\underline{\tilde{c}})), \tilde{\mathcal{M}}) \to \qquad (2.51)$$

this obtained from our covering (1.119). Now we replace our covering by a simplicial space, i.e. we consider the diagram of maps between spaces

$$\mathfrak{Par} := \prod_i \Gamma_{Q_i}\backslash X \overset{p_1}{\underset{p_2}{\leftleftarrows}} \prod_{i<j} \prod_{[R]\in[Q_i]\cap Q_j]} \Gamma_R\backslash X \begin{smallmatrix}\longleftarrow\\ \longleftarrow\\ \longleftarrow\end{smallmatrix} \qquad (2.52)$$

this yields a spectral sequence with $E_1^{\bullet,q}$ term

$$\prod_i H^q(\Gamma_{Q_i}\backslash X, \tilde{\mathcal{M}}) \xrightarrow{d^{(0)}} \prod_{i<j} \prod_{[R]\in[P_i]\cap[P_j]} H^q(\Gamma_R\backslash X^R, \tilde{\mathcal{M}}) \xrightarrow{d^{(1)}} \qquad (2.53)$$

Our covering also yields a simplicial space which is a subspace of (2.52) we get a map from (2.46) to (2.53) and this map is an isomorphism of complexes.

We replace $\mathfrak{Par}$ by another simplicial complex

$$\mathfrak{Parmax} := \prod_{[P]:d(P)=1} \Gamma_P\backslash X \underset{p_2}{\overset{p_1}{\longleftarrow}} \prod_{[Q]:d(Q)=2} \Gamma_Q\backslash X \begin{matrix}\longleftarrow\\ \longleftarrow\\ \longleftarrow\end{matrix} \tag{2.54}$$

We have an obvious projection $\Pi : \mathfrak{Par} \to \mathfrak{Parmax}$ which induces a homomorphism

$$\begin{array}{ccccc} \prod_i H^q(\Gamma_{Q_i}\backslash X, \tilde{\mathcal{M}}) & \xrightarrow{d^{(0)}} & \prod_{i<j} \prod_{[R]\in[P_i]\cap[P_j]} H^q(\Gamma_R\backslash X^R, \tilde{\mathcal{M}}) & \xrightarrow{d^{(1)}} \\ \uparrow & & \uparrow & \\ \prod_{[P]:d(P)=1} H^q(\Gamma_P\backslash X, \tilde{\mathcal{M}}) & \xrightarrow{d^{(0)}} & \prod_{[R]:d(R)=2} H^q(\Gamma_R\backslash X^R, \tilde{\mathcal{M}}) & \xrightarrow{d^{(1)}} \end{array} \tag{2.55}$$

and an easy argument in homological algebra shows that this induces an isomorphism in cohomology or in other words an isomorphism of the $E_2^{p,q}$ terms of the two spectral sequences.

We had the covering

$$\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X) = \bigcup_{P:P\text{proper}} \Gamma_P\backslash X^P(\underline{c}_{\pi'}, r(\underline{c}_{\pi'})) \tag{2.56}$$

which gives us the spectral sequence converging to $H^\bullet(\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X), \tilde{\mathcal{M}})$ with

$$E_1^{p,q} = \bigoplus_{i_0<i_1<\cdots<i_p} \bigoplus_{[Q]\in[P_{i_0}]\cap[P_{i_1}]\cap\cdots\cap[P_{i_p}]} H^q(\Gamma_Q\backslash X^Q(\underline{c}_{\pi'}, r(\underline{c}_{\pi'}), \tilde{\mathcal{M}})) \tag{2.57}$$

Our covering of $\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X)$ gives us a simplicial space $\mathfrak{Cov}(\overset{\bullet}{\mathcal{N}})\Gamma\backslash X)$ and we have maps

$$\mathfrak{Cov}(\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X)) \hookrightarrow \mathfrak{Par} \to \mathfrak{Parmax}. \tag{2.58}$$

We saw that the resulting maps induced an isomorphism in the $E_2^{p,q}$ terms of the spectral sequences. Hence we see that $\mathfrak{Parmax}$ yields a spectral sequence

$$E_1^{p,q} = \bigoplus_{[P]:d(P)=p+1} H^q(\Gamma_P\backslash X, \tilde{\mathcal{M}}) \Rightarrow H^{p+q}(\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X), \tilde{\mathcal{M}})) \tag{2.59}$$

At this point we want to raise an interesting question

Does this spectral sequence degenerate at $E_2^{p,q}$ level?

The author of this book is hoping that the answer to this question is no! And this is so for interesting reasons! We come back to this question when we discuss the Eisenstein cohomology.

The complement of $\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X)$ is a relatively compact open set $V \subset \Gamma\backslash X$, this set contains the stable points. We define $\tilde{\mathcal{M}}^!_V = i_{V,!}(\tilde{\mathcal{M}})$ then we get an exact sequence

$$0 \to \tilde{\mathcal{M}}^!_V \to \tilde{\mathcal{M}} \to \tilde{\mathcal{M}}/\tilde{\mathcal{M}}^!_V \to 0 \tag{2.60}$$

and $\tilde{\mathcal{M}}/\tilde{\mathcal{M}}^!_V$ is obviously the extension of the restriction of $\tilde{\mathcal{M}}$ to $\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X)$ and the extended by zero to $\Gamma\backslash X$. We claim (easy proof later) that

$$H^\bullet_c(\Gamma\backslash X, \tilde{\mathcal{M}}) = H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}^!_V) \tag{2.61}$$

and this gives us again the fundamental exact sequence fux

$$H^{q-1}(\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X), \tilde{\mathcal{M}}) \to H^q(\Gamma\backslash X, \tilde{\mathcal{M}}^!_V) \to H^q(\Gamma\backslash X, \tilde{\mathcal{M}}) \to H^q(\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X), \tilde{\mathcal{M}}) \to \tag{2.62}$$

2.1.7 How to compute the cohomology groups $H^q_c(\Gamma\backslash X, \tilde{\mathcal{M}})$

We apply the considerations in 4.8 from [40] Again we cover $\Gamma\backslash X$ by orbiconvex open neighbourhoods U_{x_i}, and now we define

$$\tilde{\mathcal{M}}^!_{\underline{x}} = (i_{\underline{x}})_! i^*_{\underline{x}}(\tilde{\mathcal{M}}).$$

These sheaves have properties, which are dual to those of the sheaves $\tilde{\mathcal{M}}_{\underline{x}}$. If $\underline{x} = (x_1, \dots, x_s)$ and if we add another point $\underline{x}' = (x_1, \dots, x_s, x_{s+1})$ then we have the restriction $\tilde{\mathcal{M}}_{\underline{x}} \to \tilde{\mathcal{M}}_{\underline{x}'}$, which were used to construct the Čech resolution.

Now let $d = \dim(X)$. For the ! sheaves we get (See [40] , loc. cit.) get a morphism $\tilde{\mathcal{M}}^!_{\underline{x}'} \to \tilde{\mathcal{M}}^!_{\underline{x}}$. For $\underline{x} = (x_1, \dots, x_s)$ we define the degree $d(\underline{x}) = d+1-s$. Then we construct the Čech-coresolution (See [40] 4.8.3)

$$\to \prod_{\underline{x}:d(\underline{x})=q} \tilde{\mathcal{M}}^!_{\underline{x}} \to \cdots \to \prod_{(x_i,x_j)} \tilde{\mathcal{M}}^!_{x_i,x_j} \to \prod_{x_i} \tilde{\mathcal{M}}^!_{x_i} \to i_!(\tilde{\mathcal{M}}) \to 0. \tag{2.63}$$

Now we have a dual statement to the proposition with label **acyc**

Proposition: **(acyc!)** *If* $d = \dim(X)$ *then*

$$H^q(U_{\underline{\tilde{x}}}, \tilde{\mathcal{M}}^!_{\underline{x}}) = \begin{cases} \mathcal{M}_{\Gamma_{\tilde{y}}} & q = d \\ 0 & q \neq d \end{cases}$$

Hence the above complex of sheaves provides a complex of modules

$$C^\bullet_!(\mathfrak{U}, \tilde{\mathcal{M}}) :=$$

$$\to \textstyle\prod_{\underline{x}:d(\underline{x})=q} H^d(U_{\underline{x}}, \tilde{M}^!_{\underline{x}}) \to \cdots \to \prod_{(x_i,x_j)} H^d(U_{x_i,x_j}, \tilde{\mathcal{M}}^!_{x_i,x_j}) \to \prod_{x_i} \tilde{H}^d(U_{x_i}, \tilde{\mathcal{M}}^!_{x_i}) \to 0. \tag{2.64}$$

and then

$$H^q(\Gamma\backslash X, i_!(\tilde{\mathcal{M}})) = H^q_c(\Gamma\backslash X, \tilde{\mathcal{M}}) = H^q(C^\bullet_!(\mathfrak{U}, \tilde{\mathcal{M}})). \tag{2.65}$$

Now let us assume that $\mathcal{M}$ is a finitely generated module over some commutative noetherian ring R with identity. Then clearly all our cohomology groups will be R-modules.

Our Theorem A above implies

Theorem (Raghunathan) *Under our general assumptions all the cohomology groups* $H^q_c(\Gamma\backslash X, \tilde{\mathcal{M}}), H^q(\Gamma\backslash X, \tilde{\mathcal{M}}), H^q_!(\Gamma\backslash X, \tilde{\mathcal{M}}), H^q(\partial(\Gamma\backslash X), \tilde{\mathcal{M}})$ *are finitely generated* R *modules.*

2.1.8 Modified cohomology groups

Most of the time our module $\mathcal{M}$ will be a finitely generated $\mathbb{Z}$ module and the theorem of Raghunathan says that the cohomology groups are also finitely generated $\mathbb{Z}$ modules. Sometimes we replace $\mathbb{Z}$ ring of integers $\mathcal{O}_F$ of a finite extension $F/\mathbb{Q}$ and then we will even invert some finite numbers of primes. Hence we our coefficient modules will be finitely generated R-modules where $\mathcal{O}_F \subset R \subset F$. In any case these rings R will be Dedekind rings.

Starting from the fundamental exact sequence we have introduced the modified cohomology groups $H^q_!(\ \ )$. There is a second process of modification: If $H^\bullet(\ \ )$ is any of these cohomology groups then Hint

$$H^\bullet(\ \ )_{\text{ int}} := H^\bullet(\ \ )/\text{Tors} = \text{Im}(H^\bullet(\ \ ) \to H^\bullet(\ \ ) \otimes \mathbb{Q}) \tag{2.66}$$

We have to discuss a minor problem: These two processes of modification do not quite commute. This is due to the fact that the resulting sequence

$$\to H^{q-1}(\partial(\Gamma\backslash X), \tilde{\mathcal{M}}_R)_{\text{ int}} \to H^q_c(\Gamma\backslash X, \tilde{\mathcal{M}}_R)_{\text{ int}} \xrightarrow{j} H^q(\Gamma\backslash \bar{X}, \tilde{\mathcal{M}}_R)_{\text{ int}} \xrightarrow{r} H^q(\partial(\Gamma\backslash X), \tilde{\mathcal{M}}_R)_{\text{ int}}$$

is not necessarily exact anymore. Clearly we have $H^q_!(\Gamma\backslash X, \tilde{\mathcal{M}}_R)_{\text{ int}} = \text{Im}(j)$ and if we now define $H^q\Gamma\backslash X, \tilde{\mathcal{M}}_R)_{\text{ int},!} :=\ \ker(r)$ then we have

$$H^q_!(\Gamma\backslash X, \tilde{\mathcal{M}}_R)_{\text{ int}} \subset H^q\Gamma\backslash X, \tilde{\mathcal{M}}_R)_{\text{ int},!} \tag{2.67}$$

but this inclusion may be proper. The following proposition is an elementary exercise in homological algebra. supqbd

Proposition 2.1.1. *The quotient* $H^q(\Gamma\backslash X, \tilde{\mathcal{M}}_R)_{\text{ int},!}/H^q_!(\Gamma\backslash X, \tilde{\mathcal{M}}_R)_{\text{ int}}$ *is finite and isomorphic to a subquotient of* $H^q(\partial(\Gamma\backslash X), \tilde{\mathcal{M}}_R)$

We will discuss an example in section 3.3.1

This may be a good place to introduce some terminology. If X is a torsion free, finitely generated R -module and we have a direct sum of submodules $X \supset \oplus_\nu X_\nu$ then we say that this direct sum is a *decomposition up to isogeny* if the quotient $X \supset / \oplus_\nu X_\nu$ is a torsion module and if for any ν the quotient X/X_ν is torsion free. Sometimes we also call this a *saturated* decomposition (see section 6.3.9).

2.1.9 The case $\Gamma = \mathbf{Sl}_2(\mathbb{Z})$

In this book we study intensively the special case $\Gamma = \mathrm{Sl}_2(\mathbb{Z})$. In this case we can formulate and prove some very specific results, especially we understand the denominators of the Eisenstein classes (Theorem 3.78).

In the following $\mathcal{M}$ can be any Γ-module. We investigate the fundamental exact sequence for this special group.

Of course we start again from our covering $\Gamma\backslash\mathbb{H} = U_i \cup U_\rho$. The cohomology with compact supports is the cohomology of the complex (see 2.64)

$$0 \to H^2(U_i \cap U_\rho, \tilde{\mathcal{M}}^!_{i,\rho}) \to H^2(U_i\ , \tilde{\mathcal{M}}^!_i) \oplus H^2(U_\rho, \tilde{\mathcal{M}}^!_\rho) \to 0.$$

Now we have $H^2(U_i \cap U_\rho, \tilde{\mathcal{M}}^!_{i,\rho}) = \mathcal{M}, H^2(U_i\ , \tilde{\mathcal{M}}^!_i) = \mathcal{M}_{\Gamma_i} = \mathcal{M}/(\mathrm{Id} - S)M, H^2(U_\rho\ , \tilde{\mathcal{M}}^!_\rho) = M_{\Gamma_\rho} = \mathcal{M}/(\mathrm{Id} - R)\mathcal{M}$ and hence we get the complex

$$0 \to \mathcal{M} \to \mathcal{M}_{\Gamma_i} \oplus \mathcal{M}_{\Gamma_\rho} \to 0$$

and from this we obtain

$$H^1(\Gamma\backslash\mathbb{H}, i_!(\mathcal{M})) = \ \ker(\mathcal{M} \to (M/(\mathrm{Id} - S)\mathcal{M} \oplus \mathcal{M}/(\mathrm{Id} - R)\mathcal{M}))$$

and

$$H^0(\Gamma\backslash\mathbb{H}, i_!(\mathcal{M})) = 0, H^2(\Gamma\backslash\mathbb{H}, i_!(\mathcal{M})) = \mathcal{M}_\Gamma.$$

We discuss the fundamental exact sequence in this special case. To do this we have to understand the cohomology of the boundary $H^\bullet(\partial(\Gamma\backslash\mathbb{H}), \tilde{M})$. We discussed the Borel-Serre compactification and saw that in this case we get this compactification if we add a circle at infinity to our picture of the quotient. But we may as well cut the cylinder at any level $c > 1$, i.e. we consider the level line $\mathbb{H}(c) = \{z = x + ic | z \in \mathbb{H}\}$ and divide this level line by the action of the translation group

$$\Gamma_U = \{\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} | n \in \mathbb{Z}\} = \{\begin{pmatrix} \epsilon & n \\ 0 & \epsilon \end{pmatrix} | n \in \mathbb{Z}, \epsilon = \pm 1\}/\{\pm \mathrm{Id}\}.$$

But this quotient is homotopy equivalent to the cylinder

$$\Gamma_U\backslash\mathbb{H} \simeq \Gamma_U\backslash\mathbb{H}(c).$$

We apply our general consideration on cohomology of arithmetic groups to this situation and find

$$H^\bullet(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}) = H^\bullet(\Gamma_U\backslash\mathbb{H}, \mathrm{sh}_{\Gamma_U}(\mathcal{M})) = H^\bullet(\Gamma_U\backslash\mathbb{H}(c), \mathrm{sh}_{\Gamma_U}(\mathcal{M})).$$

This cohomology is easy to compute. The group Γ_U is generated by the element $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. It is rather clear that

$$H^0(\Gamma_U\backslash\mathbb{H}, \mathrm{sh}_{\Gamma_U}(\mathcal{M})) = \mathcal{M}^{\Gamma_U}, H^1(\Gamma_U\backslash\mathbb{H}, \mathrm{sh}_{\Gamma_U}(\mathcal{M})) = \mathcal{M}_{\Gamma_U} = \mathcal{M}/(\mathrm{Id} - T)\mathcal{M}.$$

Then our fundamental exact sequence becomes (See(2.30)) fundexsq

$$0 \to \mathcal{M}^{\Gamma} \to \mathcal{M}^{\Gamma_U} \to \ker(\mathcal{M} \to (\mathcal{M}/(\mathrm{Id}-S)\mathcal{M} \oplus \mathcal{M}/(\mathrm{Id}-R)\mathcal{M})) \xrightarrow{j}$$
$$\mathcal{M}/(\mathcal{M}^{\Gamma_i} \oplus \mathcal{M}^{\Gamma_\rho}) \xrightarrow{r} \mathcal{M}/(\mathrm{Id}-T)\mathcal{M} \to \mathcal{M}_{\Gamma} \to 0 \tag{2.68}$$

Now it may come as a little surprise to the readers, that we can formulate a little exercise which is not entirely trivial

Exercise: *Write down explicitly all the arrows in the above fundamental sequence*

We give the answer without proof. I change notation slightly and work with the matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, R = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

and we have the relation

$$RS = T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Then $\Gamma_i =< S >, \Gamma_\rho =< R >$. The map

$$\mathcal{M}/(\mathcal{M}^{<S>} \oplus \mathcal{M}^{<R>}) \to \mathcal{M}/(\mathrm{Id}-T)\mathcal{M}$$

is given by

$$m \mapsto m - Sm$$

We have to show that this map is well defined: If $m \in \mathcal{M}^{<S>}$ then $m \mapsto 0$. If $m \in \mathcal{M}^{<R>}$ then

$$m - Sm = m - SR^{-1}m = m - Tm$$

and this is zero in $\mathcal{M}/(\mathrm{Id}-T)\mathcal{M}$.

The map

$$\ker(\mathcal{M} \to (\mathcal{M}/(\mathrm{Id}-S)\mathcal{M} \oplus \mathcal{M}/(\mathrm{Id}-R)\mathcal{M})) \to \mathcal{M}/(\mathcal{M}^{<S>} \oplus \mathcal{M}^{<R>})$$

is a little bit delicate. We pick an element m in the kernel, hence we can write it as

$$m = m_1 - Sm_1 = m_2 - R^{-1}m_2$$

and send $m \mapsto m_1 - m_2$ (Here we have to use the orientation). If we modify m_1, m_2 to $m_1' = m_1 + n_1, m_2' = m_2 + n_2$ then $m_1' - m_2'$ gives the same element in $\mathcal{M}/(\mathcal{M}^{<S>} \oplus \mathcal{M}^{<R>})$.

This answer can only be right if $m_1 - m_2$ goes to zero under the map r, i.e. we have to show that

$$m_1 - m_2 - S(m_1 - m_2) \in (\mathrm{Id}-T)\mathcal{M}$$

We compute

$$m_1 - m_2 - S(m_1 - m_2) = m - m_2 + Sm_2 = m - m_2 + R^{-1}m_2 - R^{-1}m_2 + Sm_2 =$$
$$-R^{-1}m_2 + Sm_2 = -T^{-1}Sm_2 + Sm_2 \in (\mathrm{Id}-T)\mathcal{M}$$

Finally we claim that the map $\mathcal{M}^{<T>} \to \ker(\mathcal{M} \to (\mathcal{M}/(\mathrm{Id}-S)\mathcal{M} \oplus \mathcal{M}/(\mathrm{Id}-R)\mathcal{M}))$ is given by $m \mapsto m - Sm = m - R^{-1}T^{-1}m = m - R^{-1}m$.

There is still another element of structure. The map $c : z \mapsto -\bar{z}$ induces an (differentiable) involution of $\mathbb{H}$. We put $S_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ then $\gamma cz = cS_1\gamma S_1^{-1}z$ and therefore c induces an involution on $\Gamma\backslash\mathbb{H}$. We get an isomorphism of cohomology groups

$$c^{(1)} : H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) \xrightarrow{\sim} H^1(\Gamma\backslash\mathbb{H}, c_*(\tilde{\mathcal{M}})) \tag{2.69}$$

The direct image sheaf $c_*(\tilde{\mathcal{M}})$ is by definition the sheaf attached to the Γ module $\mathcal{M}^{(S_1)}$: This module is equal to $\mathcal{M}$ as an abstract module, but the action is twisted by a conjugation by the above matrix S_1, i.e.

$$\gamma * m = S_1\gamma S_1^{-1}m \tag{2.70}$$

Now we assume that $\mathcal{M}$ is actually a $\mathrm{Gl}_2(\mathbb{Z})$ module. Then the map $m \to S_1m$ provides an isomorphism $\mathcal{M}^{(S_1)} \xrightarrow{\sim} \mathcal{M}$ and hence we get in involution on the cohomology groups

$$c^\bullet : H^\bullet(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) \to H^\bullet(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) \tag{2.71}$$

We have the explicit description of the cohomology groups $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})$ and we can compute this involution in terms of this description. We observe that the matrix SS_1 fixes the two points i, ρ and hence the two open sets U_i, U_ρ of the covering. Hence it also fixes $\mathcal{M}^{\Gamma_i}$ and $\mathcal{M}^{\Gamma_\rho}$ and therefore the map $m \mapsto SS_1m$ induces an involution on $\mathcal{M}/\mathcal{M}^{\Gamma_i} \oplus \mathcal{M}^{\Gamma_\rho} = H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})$ and this is our map $c^{(1)}$.

The cohomology has a $+$ and a $-$ eigen submodule under this involution, and

$$H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) \supset H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})_+ \oplus H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})_-, \tag{2.72}$$

the sum of the two eigen modules has finite index which is a power of 2.

Poincare' duality

We assume that our Γ module $\mathcal{M}$ is a finitely generated and locally free module over R , where R is a Dedekind ring or a field. We assume $\frac{1}{2} \in R$. In section 6.3.11 we discuss Poincare duality in greater generality, here we consider the pairing (see 6.104)

$$H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})_{\text{ int}} \times H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}^\vee)_{\text{ int},!} \to H^2_!(\Gamma\backslash\mathbb{H}, R) = R \tag{2.73}$$

It is clear that the involution c induces multiplication by -1 on $H^2_!(\Gamma\backslash\mathbb{H}, R)$. On the other hand we have the decompositions of the above cohomology groups into $\pm$ eigen modules. The pairings of the $+,+$ parts and the $-,-$ give zero and then we get pairings

$$\begin{aligned} H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})_{\text{ int},+} \times H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}^\vee)_{\text{ int},!,-} \to R \\ H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})_{\text{ int},!,+} \times H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}^\vee)_{\text{ int},-} \to R \end{aligned} \tag{2.74}$$

both of them are partially non degenerate.

If we have $\mathcal{M} = \mathcal{M}^\vee$ then we get eqrank

$$rank(H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})_{\text{ int},+}) = rank(H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})_{\text{ int},-}) \tag{2.75}$$

Final remark: The reader may get the impression that - at least in the case $\Gamma = \mathrm{Sl}_2(\mathbb{Z})$-it is easy to compute the cohomology, but the contrary is true. In the case $\Gamma = \mathrm{Sl}_2(\mathbb{Z})/\pm\mathrm{Id}$ we found formulae for the rank of the cohomology groups, this seems to be a satisfactory answer, but it is not. The point is that in the next section we will introduce the Hecke operators, these Hecke operators form an algebra of endomorphisms of the cohomology groups. It is a fundamental question (see further down) to understand the cohomology as a module under the action of this Hecke algebra. It is not so easy to write down the effect of a Hecke operator on a module like $\mathcal{M}/(\mathcal{M}^{\Gamma_i} + \mathcal{M}^{\Gamma_\rho})$. We will discuss an explicit example in section 3.3.

The situation is even less satisfying if we consider the case $\Gamma = \mathrm{Gl}_2(\mathbb{Z}[\mathbf{i}])$. In this case our coefficient systems are obtained from highest weight modules $\tilde{\mathcal{M}}_\lambda$(, here $\lambda = n_1\gamma_1 + n_2\gamma_2 + \delta$ is a highest weight see section 4.1 (4.5). These modules are defined over $\mathbb{Z}[\,\mathbf{i}]$. It turns out that we are unable to read off the dimensions of the individual groups $H^i(\Gamma\backslash\mathbb{H}_3, \tilde{\mathcal{M}}_\lambda)$ from the complex in (2.1.4) . Of course we can compute them in any given case, but our approach does not give any kind of theoretical insight.

For instance using tools from analysis we can prove a vanishing theorem

$$H^i_!(\Gamma\backslash\mathbb{H}_3, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{Q}) = 0 \text{ if } n_1 \neq n_2 \tag{2.76}$$

Such a result can not be obtained from studying the complex.

The complex computes the $\mathbb{Z}[\mathbf{i}]$ modules $H^i_!(\Gamma\backslash\mathbb{H}_3, \tilde{\mathcal{M}}_\lambda)$ and therefore it also computes the torsion. Again it seems to be difficult to derive general theorems, except the obvious finiteness assertions.

Chapter 3

Hecke Operators

3.1 The construction of Hecke operators

We mentioned already that the cohomology and homology groups of an arithmetic group have an additional structure: They are modules for the Hecke algebra. The following description of the Hecke algebra is somewhat provisorial, we get a richer Hecke algebra, if we work in the adelic context (See Chapter 6). But the description here is more intuitive.

We start from the arithmetic group $\Gamma \subset G(\mathbb{Q})$ and an arbitrary Γ-module $\mathcal{M}$. The module $\mathcal{M}$ is also a module over a ring R which in the beginning may be simply $\mathbb{Z}$. More generally is R may the ring of integers in an algebraic number field, where we also inverted a finite number of primes.

At this point it is better to have a notation for this action

$$\Gamma \times \mathcal{M} \to \mathcal{M}, (\gamma, m) \mapsto r(\gamma)(m)$$

where now $r : \Gamma \to \mathrm{Aut}_R(\mathcal{M})$.

If we have a subgroup $\Gamma' \subset \Gamma$ of finite index, then we constructed maps

$$\begin{aligned} \pi^{\bullet}_{\Gamma',\Gamma} :& H^{\bullet}(\Gamma\backslash X, \tilde{\mathcal{M}}) \longrightarrow H^{\bullet}(\Gamma'\backslash X, \tilde{\mathcal{M}}) \\ \pi_{\Gamma',\Gamma,\bullet} :& H^{\bullet}(\Gamma'\backslash X, \tilde{\mathcal{M}}) \longrightarrow H^{\bullet}(\Gamma\backslash X, \tilde{\mathcal{M}}) \end{aligned}$$

(see section 2.1.2). We abbreviate $r(\gamma)m = \gamma m$.

We pick an element $\alpha \in G(\mathbb{Q})$. The group

$$\Gamma(\alpha^{-1}) = \alpha^{-1}\Gamma\alpha \cap \Gamma$$

is a subgroup of finite index in Γ and the conjugation by α induces an isomorphism

$$\mathrm{inn}(\alpha) : \Gamma(\alpha^{-1}) \longrightarrow \Gamma(\alpha).$$

We get an isomorphism

$$j(\alpha) : \Gamma(\alpha^{-1})\backslash X \longrightarrow \Gamma(\alpha)\backslash X$$

which is induced by the map $x \longrightarrow \alpha x$ on the space X. This yields an isomorphism of cohomology groups

$$j(\alpha)^{\bullet} : H^{\bullet}(\Gamma(\alpha^{-1})\backslash X, \tilde{\mathcal{M}}) \longrightarrow H^{\bullet}(\Gamma(\alpha)\backslash X, j(\alpha)_*(\tilde{\mathcal{M}})).$$

We compute the sheaf $j(\alpha)_*(\tilde{\mathcal{M}})$. For a point $x \in \Gamma(\alpha)\backslash X$ we have $j(\alpha)_*(\tilde{\mathcal{M}})_x = \tilde{\mathcal{M}}_{x'}$ where $j(\alpha)(x') = x$. We have the projection $\pi_{\Gamma(\alpha^{-1})} : X \to \Gamma(\alpha^{-1})\backslash X$, and the definition yields

$$\tilde{\mathcal{M}}_{x'} = \left\{ s : \pi^{-1}_{\Gamma(\alpha^{-1})}(x') \to \mathcal{M} \mid s(\gamma m) = \gamma s(m) \text{ for all } \gamma \in \Gamma(\alpha^{-1}) \right\} \tag{3.1}$$

The map $z \longrightarrow \alpha z$ provides an identification $\pi^{-1}_{\Gamma(\alpha^{-1})}(x') \xrightarrow{\sim} \pi^{-1}_{\Gamma(\alpha)}(x)$ in terms of this fibre we can describe the stalk at x as

$$(\alpha)_*(\tilde{\mathcal{M}})_x = \left\{ s : \pi^{-1}_{\Gamma(\alpha)}(x) \to \mathcal{M} \mid s(\gamma v) = \alpha^{-1}\gamma\alpha s(v) \text{ for all } \gamma \in \Gamma(\alpha) \right\}. \tag{3.2}$$

Hence we see: We may use α to define a new $\Gamma(\alpha)$-module $\mathcal{M}^{(\alpha)}$: The underlying abelian group of $\mathcal{M}^{(\alpha)}$ is $\mathcal{M}$ but the operation of $\Gamma(\alpha)$ is given by

$$(\gamma, m) \longrightarrow (\alpha^{-1}\gamma\alpha)m = \gamma *_\alpha m.$$

Then the sheaf $j(\alpha)_*(\tilde{\mathcal{M}})$ is equal to $\tilde{\mathcal{M}}^{(\alpha)}$. Then every element

$$u_\alpha \in \operatorname{Hom}_{\Gamma(\alpha)}(\mathcal{M}^{(\alpha)}, \mathcal{M})$$

defines a map $\tilde{u}_\alpha : j(\alpha)_*(\tilde{\mathcal{M}}) \to \tilde{\mathcal{M}}$. Now we get a commuting diagram

$$\begin{array}{ccccc} H^\bullet(\Gamma(\alpha^{-1})\backslash X, \tilde{\mathcal{M}}) & \xrightarrow{j(\alpha)^\bullet} & H^\bullet(\Gamma(\alpha)\backslash X, j(\alpha)_*(\tilde{\mathcal{M}})) & \xrightarrow{\tilde{u}^\bullet_\alpha} & H^\bullet(\Gamma(\alpha)\backslash X, \mathcal{M}) \\ \uparrow \pi^\bullet & & & & \downarrow \pi_\bullet \\ H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}) & & \xrightarrow{\qquad T(\alpha, u_\alpha) \qquad} & & H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}) \end{array}, \tag{3.3}$$

where the operatorin the bottom line is the Hecke operator.

The Hecke operator depends on two data:
a) the element $\alpha \in G(\mathbb{Q})$,

b) the choice of $u_\alpha \in \operatorname{Hom}_{\Gamma(\alpha)}(\mathcal{M}^{(\alpha)}, \mathcal{M})$.

It is not difficult to show that the operator $T(\alpha, u_\alpha)$ only depends on the double coset $\Gamma\,\alpha\,\Gamma$, provided we adapt the choice of u_α. To be more precise if

$$\alpha_1 = \gamma_1\alpha\gamma_2 \qquad\qquad \gamma_1, \gamma_2 \in \Gamma,$$

then we have an obious bijection

$$\Phi_{\gamma_1,\gamma_2} : \operatorname{Hom}_{\Gamma(\alpha)}(\mathcal{M}^{(\alpha)}, \mathcal{M}) \longrightarrow \operatorname{Hom}_{\Gamma(\alpha_1)}(\mathcal{M}^{\alpha_1)}, \mathcal{M})$$

which is given by

$$\Phi_{\gamma_1,\gamma_2}(u_\alpha) = u_{\alpha_1} = \gamma_1 u_\alpha \gamma_2.$$

The reader will verify without difficulties that

$$T(\alpha, u_\alpha) = T(\alpha_1, u_{\alpha_1}).$$

(Verify this for H^0 and then use some kind of resolution (See next section))

The choice of u_α may be delicate in some situations. There is are cases where we have also a canonical choice of u_α. The first case is that our Γ-module $\mathcal{M}$ is of arithmetic origin. In this case $G(\mathbb{Q})$ acts upon $\mathcal{M}_\mathbb{Q} = \mathcal{M} \otimes \mathbb{Q}$. Then the canonical choice of an

$$u_{\alpha,\mathbb{Q}} : \mathcal{M}_\mathbb{Q}^{(\alpha)} \longrightarrow \mathcal{M}_\mathbb{Q},$$

is given by $u_\alpha : m \mapsto \alpha m$. Hence we can speak of the Hecke-opertor $T(\alpha)$: $H^\bullet(\Gamma\backslash X, \mathcal{M}_\mathbb{Q}) \to H^\bullet(\Gamma\backslash X, \mathcal{M}_\mathbb{Q})$.

But if we return to the R-module sheaf $\mathcal{M}$ this morphism $u_{\alpha,\mathbb{Q}}$ will not necessarily map the lattice $\mathcal{M}^{(\alpha)}$ into $\mathcal{M}$. Clearly we can find a rational number $d(\alpha) > 0$ for which

$$d(\alpha) \cdot u_{\alpha,\mathbb{Q}} : \mathcal{M}^{(\alpha)} \longrightarrow \mathcal{M} \text{ and } d(\alpha) \cdot u_{\alpha,\mathbb{Q}}(\mathcal{M}^{(\alpha)}) \not\subset b\mathcal{M} \text{ for any integer } b > 1.$$

Then $u_\alpha = d(\alpha) \cdot u_{\alpha,\mathbb{Q}}$ is called the *normalised choice*, and then $T(\alpha, u_\alpha)$ will be the *normalised Hecke operator*.

The canonical choice defines endomorphisms on the rational cohomology, i.e. the cohomology with coefficients in $\tilde{\mathcal{M}}_\mathbb{Q}$ whereas the normalised Hecke operators induce endomorphism of the integral cohomology. The normalised choice and the canonical choice differ only by a scalar factor. We will resume this theme in section 6.3.2.

In the second case we assume that $\Gamma_0 = G(\mathbb{Z})$, let $\Gamma(N) \subset \Gamma_0$ be the full congruence subgroup mod N. Then $\Gamma_0/\Gamma(N) \subset G(\mathbb{Z}/N\mathbb{Z})$ Now we assume that $\mathcal{V}$ is a $G(\mathbb{Z}/N\mathbb{Z})$ module, in section 1.2.2 we calledl this a module of congruence origin. Then we have some constraints on the choice of elements α. We introduce the semi local ring $\mathbb{Z}_{(N)}$ where invert all primes not dividing N. Now we pick our elements $\alpha \in G(\mathbb{Z}_{(N)}$ Since we have the homomorphism $\mathbb{Z}[_{(N)} \to \mathbb{Z}/N\mathbb{Z}$ our module $\mathcal{V}$ is also a $G(\mathbb{Z}_{(N)})$ module. Therefore we can simply choose $u_\alpha := m \mapsto \alpha m$.

We see that we can construct many endomorphisms $T(\alpha, u_\alpha) : H^\bullet(\Gamma\backslash X, \tilde{\mathcal{V}}) \to H^\bullet(\Gamma\backslash X, \tilde{\mathcal{V}})$. These endomorphisms will generate an algebra

$$\mathcal{H}_{N,\tilde{\mathcal{V}}} \subset \operatorname{End}(H^\bullet(\Gamma\backslash X, \tilde{\mathcal{V}})). \tag{3.4}$$

This is the so-called Hecke algebra. We can also define endomorphisms $T(\alpha, u_\alpha)$ on the cohomology with compact supports, on the inner cohomology and the cohomology of the boundary. Since the operators are compatible with all the arrows in the fundamental exact sequence we denote them by the same symbol.

The Hecke algebra also acts on the inner cohomology $H_!^q(\Gamma\backslash X, \tilde{\mathcal{M}})$. Of course we may tensorize our coefficient system with any number field $L \supset \mathbb{Q}$, then we write $\mathcal{M}_L = \mathcal{M} \otimes L$..

We state without proof the following fundamental theorem :

He-ss

Theorem 3.1.1. *Let $\mathcal{M}$ be a module of arithmetic origin. For any extension $L/\mathbb{Q}$ the $\mathcal{H}_\Gamma \otimes L$ module $H^q_!(\Gamma\backslash X, \tilde{\mathcal{M}}_L)$ is semi simple, i.e. a direct sum of irreducible $\mathcal{H}_\Gamma$ modules.*

The proof of this theorem will be discussed in Chapter 8 (section 8.1.8) it requires some input from analysis. We give a brief sketch. We tensorize our coefficient system by $\mathbb{C}$, i.e. we consider $\mathcal{M}_L \otimes_L \mathbb{C} = \mathcal{M}_\mathbb{C}$. Let us assume that Γ is torsion free. First of all start from the well known fact, that the cohomology $H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}_\mathbb{C})$ can be computed from the de-Rham-complex

$$H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}_\mathbb{C}) = H^\bullet(\Omega^\bullet \otimes \tilde{\mathcal{M}}_\mathbb{C}(\Gamma\backslash X)).$$

We introduces some specific positive definite hermitian form on $\mathcal{M}_\mathbb{C}$ and this allows us to define a hermitian scalar product between two $\tilde{\mathcal{M}}_\mathbb{C}$ -valued p-forms

$$< \omega_1, \omega_2 >= \int_{\Gamma\backslash X} \omega_1 \wedge *\omega_2,$$

provided one of the forms is compactly supported.

This will allow us a positive definite scalar product on $H^p_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{n,\mathbb{C}})$, We apply theorem 8.1.1 , this theorem tells us that we can find representatives ω_1^h, ω_2^h which are harmonic (they satisfy certain differential equations) and then

$$< [\omega_1], [\omega_2] >:= \int_{\Gamma\backslash X} \omega_1^h \wedge *\omega_2^h, \tag{3.5}$$

defines a positive definite hermitian scalar product on $H^q_!(\Gamma\backslash X, \tilde{\mathcal{M}}_\mathbb{C})$. Finally we show that $\mathcal{H}_\Gamma$ is self adjoint with respect to this scalar product, and then semi-simplicity follows from the standard argument.

For the groups $\Gamma \subset \mathrm{Sl}_2(\mathbb{Z})$ and the cohomology groups $H^1_!(\Gamma\backslash\mathbb{H}, \mathcal{M}_n \otimes \mathbb{C})$ these harmonic representatives are given linear combinations of holomorphic and antiholomorphic cusp forms of weight $n+2$ (See 4.1.7). The scalar product on this space of modular forms is given by the by the Peterson scalar product (see section 4.1.8.)

3.1.1 Commuting relations

We want to say some words concerning the structure of the Hecke algebra.

To begin we discuss the action of the Hecke-algebra on $H^0(\Gamma\backslash X, \tilde{\mathcal{M}})$. We do this since we defined the cohomology in terms of injective (or acyclic) resolutions and therefore the general results concerning the structure of the Hecke algebra can be reduced to this special case.

If we have a Γ-module $\mathcal{M}$ and if we look at the diagram defining the Hecke operators, then we see that we get in degree 0

$$\begin{array}{ccc} \mathcal{M}^{\Gamma(\alpha^{-1})} & \longrightarrow (\mathcal{M}^{(\alpha)})^{\Gamma(\alpha)} \xrightarrow{u_\alpha} & \mathcal{M}^{\Gamma(\alpha)} \\ \uparrow & & \downarrow \\ \mathcal{M}^\Gamma & \xrightarrow{T(\alpha, u_\alpha)} & \mathcal{M}^\Gamma \end{array}$$

where the first arrow on the top line is induced by the identity map $\mathcal{M} \to \mathcal{M}^{(\alpha)} = \mathcal{M}$ and the second by a map $u_\alpha \in \mathrm{Hom}_{\mathbf{Ab}}(\mathcal{M},\mathcal{M})$ which satisfies $u_\alpha((\alpha\gamma\alpha^{-1})m) = \gamma u_\alpha(m)$. Recalling the definition of the vertical arrow on the right, we find

$$T(\alpha, u_\alpha)(v) = \sum_{\gamma\in\Gamma/\Gamma(\alpha)} \gamma \cdot u_\alpha(v).$$

We are interested to get formulae for the product of Hecke operators, so, for instance, we would like to show that under certian assumptions on α, β and certain adjustment of u_α, u_β and $u_{\alpha\beta}$ we can show

$$T(\alpha, u_\alpha) \cdot T(\beta, u_\beta) = T(\beta, u_\beta) \cdot T(\alpha, u_\alpha) = T(\alpha\beta, u_{\alpha\beta}).$$

It is easy to see what the conditions are if we want such a formula to be true. We look at what happens in H^0. For $v \in \mathcal{M}^\Gamma$ we get

$$T(\alpha, u_\alpha) \cdot T(\beta, u_\beta)(v) = \sum_{\gamma\in\Gamma/\Gamma(\alpha)} \gamma u_\alpha(\sum_{\eta\in\Gamma/\Gamma(\beta)} \eta u_\beta(v))$$

We assume that the following three conditions hold

(i) for each η we can find an $\eta' \in \Gamma$ such that

$$\eta' \circ u_\alpha = u_\alpha \circ \eta,$$

(ii) The elements $\gamma\eta'$ form a system of representatives for $\Gamma/\Gamma(\alpha\beta)$

(iii) $u_\alpha u_\beta(v) = u_\beta u_\alpha(v) = u_{\alpha\beta}(v)$.

Then we get

$$T(\alpha, u_\alpha) \cdot T(\beta, u_\beta)(v) = \sum_{\gamma\in\Gamma/\Gamma(\alpha)} \sum_{\eta'\in\Gamma/\Gamma(\beta)} \gamma\eta' u_\alpha u_\beta(v) = \sum_{\xi\in\Gamma/\Gamma(\alpha\beta)} \xi u_{\alpha\beta}(v) =$$

$$T(\alpha\beta, u_{\alpha\beta})(v)$$

We want to explain in a special case that we may have relations like the one above.

Let S be a finite set of primes, let $|S|$ be the product of these primes. Then we define $\Gamma_S = G(\mathbb{Z}[\frac{1}{|S|}])$. We say that $\alpha \in G(\mathbb{Q})$ has support in S if $\alpha \in G(\mathbb{Z}[\frac{1}{|S|}])$.

We take the group $\Gamma = \mathrm{Sl}_d(\mathbb{Z})$, and we take two disjoint sets of primes S_1, S_2. For the group Γ one can prove the so-called strong approximation theorem which asserts that for any natural number m the map

$$\mathrm{Sl}_d(\mathbb{Z}) \longrightarrow \mathrm{Sl}_d(\mathbb{Z}/m\mathbb{Z})$$

is surjective. (This special case is actually not so difficult. The theorem holds for many other arithmetic groups, for instance for simply connected Chevalley schemes over Spec($\mathbb{Z}$).)

We consider the case

$$\alpha = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_d \end{pmatrix} \in \Gamma_{S_1}, \beta = \begin{pmatrix} b_1 & & & \\ & b_2 & & \\ & & \ddots & \\ & & & b_d \end{pmatrix} \in \Gamma_{S_2},$$

where $a_d|a_{d-1}\dots|a_1$ and $b_d|b_{d-1}|\dots|b_1$. It is clear that we can find integers n_1 and n_2 which are only divisible by the primes in S_1 and S_2 respectively, so that

$$\Gamma(n_i) \subset \Gamma(\alpha^{-1}), \Gamma(n_2) \subset \Gamma(\beta^{-1}),$$

where the $\Gamma(n_i)$ are the full congruence subgroups $\mod n_i$. Since we have

$$\mathrm{Sl}_d(\mathbb{Z}/n\mathbb{Z}) = \mathrm{Sl}_d(\mathbb{Z}/n_1\mathbb{Z}) \times \mathrm{Sl}_d(\mathbb{Z}/n_2\mathbb{Z})$$

we get

$$\Gamma/\Gamma(\alpha^{-1}\beta^{-1}) \xrightarrow{\sim} \Gamma/\Gamma(\alpha^{-1}) \times \Gamma/\Gamma(\beta^{-1}).$$

On the right hand side we can chose representatives γ for $\Gamma/\Gamma(\alpha^{-1})$ which satisfy $\gamma \equiv \mathrm{Id} \mod n_2$ and η for $\Gamma/\Gamma(\beta^{-1})$ which satisfy $\eta \equiv \mathrm{Id} \mod n_1$. Then the products $\gamma\eta$ will form a system of representatives for $\Gamma/\Gamma(\alpha^{-1}\beta^{-1})$. But then we clearly have $u_\alpha\eta = \eta u_\alpha$ and we see that (i) and (ii) above are true. Then we can put $u_{\alpha\beta} = u_\alpha u_\beta$.

We consider the case that our module $\mathcal{M}$ is a R-lattice in $\mathcal{M}_\mathbb{Q}$, where $\mathcal{M}_\mathbb{Q}$ is a rational $G(\mathbb{Q})$-module. Then we saw that we can write

$$u_\alpha = d(\alpha) \cdot \alpha$$

where $d(\alpha)$ will be a product of powers of the primes p dividing n_1 and an analogous statement can be obtained for β and n_2.

Since we have $\alpha\beta = \beta\alpha$ and since clearly $d(\alpha)d(\beta) = d(\alpha\beta)$ we also get the commutation relation.

So far we only proved this relation only for the action on $H^0(\Gamma\backslash X, \tilde{\mathcal{M}})$. If we want to prove it for cohomology in higher degrees, we have to choose an acyclic resolution

$$0 \longrightarrow \mathcal{M} \longrightarrow A^0 \longrightarrow A^1 \longrightarrow \dots = 0 \longrightarrow \mathcal{M} \longrightarrow A^\bullet$$

and compute the cohomology from this resolution. We have to extend the maps u_α, u_β to this complex

$$\begin{array}{ccccc} 0 \longrightarrow & \mathcal{M}^{(\alpha)} & \longrightarrow & (A^\bullet)^{(\alpha)} \\ & \Big\downarrow u_\alpha & & \Big\downarrow u_\alpha^{(\bullet)} \\ 0 \longrightarrow & \mathcal{M} & \longrightarrow & A^\bullet, \end{array}$$

and we have to prove that the relation

$$u_\alpha \eta u_\beta = \eta' u_\alpha u_\beta = \eta' u_{\alpha\beta}$$

also holds on the complex. Once we can prove this, it becomes clear that the commutation rule also holds in higher degrees.

We choose the special resolution

$$\begin{gathered} 0 \to \mathcal{M} \to \operatorname{Ind}^\bullet(\mathcal{M}) = \\ 0 \longrightarrow \mathcal{M} \longrightarrow \operatorname{Ind}_{\{1\}}^\Gamma \mathcal{M} \longrightarrow \operatorname{Ind}_{\{1\}}^\Gamma(\operatorname{Ind}_{\{1\}}^\Gamma \mathcal{M}/\mathcal{M}) \longrightarrow \end{gathered} \tag{3.6}$$

It is clear that if suffices to show: If we selected the u_α, u_β in such a way that we have the condition (i), (ii) and (iii) above satisfied, then we can choose extensions $u_\alpha, u_\beta, u_{\alpha\beta}$ to $\operatorname{Ind}_{\{1\}}^\Gamma \mathcal{M}$ so that (i), (ii) and (iii) are also satisfied. Once we have done this we can proceed by induction.

In other words we have the diagram of $\Gamma(\alpha)$-modules

$$\begin{array}{ccccc} 0 \longrightarrow & \mathcal{M}^{(\alpha)} & \longrightarrow & (\operatorname{Ind}_{\{1\}}^\Gamma \mathcal{M})^{(\alpha)} \\ & \downarrow u_\alpha & & \downarrow ? \\ 0 \longrightarrow & \mathcal{M} & \longrightarrow & \operatorname{Ind}_{\{1\}}^\Gamma \mathcal{M}, \end{array}$$

and we are searching for a suitable vertical arrow ?. The horizontal arrows are given by (as before see (2.3)) by $i : m \longrightarrow f_m : \{\gamma \longrightarrow \gamma m\}$.

We make another assumption concerning our α, β. We assume that there exists an automorphism Θ of $G/\mathbb{Q}$ such that $\Theta(\alpha) = \alpha^{-1}, \Theta(\beta) = \beta^{-1}$ and $\Theta\Gamma = \Gamma$. This assumption is certainly fulfilled in the case above, we simply take $\Theta(g) =^t g^{-1}$, i.e. transpose inverse.

We choose representatives $\xi_1, \dots, \xi_r$ for $\Gamma/\Gamma(\alpha^{-1})$, then $\Theta\xi_1, \dots, \Theta\xi_r$ is a system of representatives for $\Gamma/\Gamma(\alpha)$. To define the vertical arrow $? = u_\alpha^{(0)}$ we require

$$u_\alpha^{(0)}(f)(\Theta\xi_\nu) = u_\alpha(f(\xi_\nu)) \ \ \forall\, \nu = 1, \dots, r$$

and this yields a unique $\Gamma(\alpha)$- module isomorphism, for all $\gamma \in \Gamma(\alpha)$ we must have

$$u_\alpha^{(0)}(f)(\Theta\xi_\nu\gamma) = u_\alpha(f(\xi_\nu\alpha^{-1}\gamma\alpha) \ \ \forall\, \nu = 1, \dots, r.$$

Iterating this construction gives us the $u_\alpha^{(\bullet)}$, by construction these morphisms satisfy (i), (ii), (iii). Since the complex $H^0(\Gamma\backslash X, \tilde{\operatorname{Ind}(\mathcal{M})})$ computes the cohomology groups $H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}})$ the commutation rules hold in all degrees.

HSO

3.1.2 More relations between Hecke operators

We look at the algebra of Hecke operators in the special case that $G/\mathbb{Z} = \mathrm{Gl}_2/\mathbb{Z}$, we consider the action on $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})$ where $\Gamma = \mathrm{Sl}_2(\mathbb{Z})$, we assume n even and $\mathcal{M} = \mathcal{M}[-\frac{n}{2}]$. This has the effect that the centre of $G/\mathbb{Z}$ acts trivially on $\mathcal{M}$ and this makes life simpler.

We attach a Hecke operator to any coset $\Gamma\alpha\Gamma$ where $\alpha \in \mathrm{Gl}_2^+(\mathbb{Q})$ (i.e. $\det(\alpha) > 0$, we want α to act on the upper half plane). Then α and $\lambda\alpha$ with $\lambda \in \mathbb{Q}^*$ define the same operator. Hence we may assume that the matrix entries of α are integers. The theorem of elementary divisors asserts that the double cosets

$$\Gamma \cdot M_n(\mathbb{Z})_{\det \neq 0} \cdot \Gamma \subset \mathrm{Gl}_2^+(\mathbb{Q})$$

are represented by matrices of the form

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

where $b \mid a$. But here we can divide by b, and we are left with the matrix

$$\alpha = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad a \in \mathbb{N}.$$

We can attach a Hecke operator to this matrix provided we choose u_α. We see that α induces on the basis vectors of our module $\mathcal{M}$

$$X^\nu Y^{n-\nu} \longrightarrow a^{\nu - n/2} \cdot X^\nu Y^{n-\nu}.$$

Hence we see that we have the following natural choice for u_α

$$u_\alpha : P(X.Y) \longrightarrow a^{n/2} \alpha \cdot P(X,Y).$$

(See the general discussion of the Hecke operators)

Hence we get a family of endomorphisms

$$T\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, u_{\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}}\right) = T(a) \tag{3.7}$$

of the cohomology $H^i(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})$ We have seen already that we have $T_a T_b = T_{ab}$ if a, b are coprime.

Hence we have to investigate the local algebra $\mathcal{H}_p$ which is generated by the

$$T_{p^r} = T\left(\begin{pmatrix} p^r & 0 \\ 0 & 1 \end{pmatrix}, u_{\begin{pmatrix} p^r & 0 \\ 0 & 1 \end{pmatrix}}\right)$$

for the special case of the group $\Gamma = \mathrm{Sl}_2(\mathbb{Z})$ and the coefficient system $\mathcal{M} = \mathcal{M}_n[-\frac{n}{2}]$. To do this we compute the product

$$T_{p^r} \cdot T_p = T\left(\begin{pmatrix} p^r & 0 \\ 0 & 1 \end{pmatrix}, u_{\alpha_p^r}\right) \cdot T\left(\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, u_{\alpha_p}\right)$$

where the u'_{α^r} are the canonical choices.

Again we investigate first what happens in degree zero, i.e. on $H^0(\Gamma\backslash\mathbb{H}, \tilde{I})$ here I is any Γ-module. Let $\alpha = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, \xi \in H^0(\Gamma\backslash X, \tilde{I})$ then

$$T(\alpha^r, u_{\alpha^r}) T(\alpha, u_\alpha)\xi = (\sum_{\gamma \in \Gamma/\Gamma(\alpha^r)} \gamma u_{\alpha^r})(\sum_{\eta \in \Gamma/\Gamma(\alpha)} \eta u_\alpha)(\xi)$$

We have the classical system of representatives

$$\Gamma/\Gamma(\alpha^r) = \bigcup_{j \mod p^r} \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} \Gamma(\alpha^r) \quad \bigcup \quad \bigcup_{j' \mod p^{r-1}} \begin{pmatrix} 1 & 0 \\ j'p & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Gamma(\alpha^r),$$

and our product of Hecke operators becomes

$$(\sum_{j \mod p^r} \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} + \sum_{j' \mod p^{r-1}} \begin{pmatrix} 1 & 0 \\ j'p & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} u_{\alpha^r})((\sum_{j_1 \mod p} \begin{pmatrix} 1 & j_1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})u_\alpha)(\xi) =$$

$$[\sum_{j \mod p^r, j_1 \mod p} \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} u_{\alpha^r} \begin{pmatrix} 1 & j_1 \\ 0 & 1 \end{pmatrix} u_\alpha)(\xi)$$

$$+ \sum_{j' \mod p^{r-1}, j_1 \mod p} \begin{pmatrix} 1 & 0 \\ j'p & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} u_{\alpha^r} \begin{pmatrix} 1 & j_1 \\ 0 & 1 \end{pmatrix} u_\alpha(\xi)]+$$

$$+[\sum_{j \mod p^r} \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} u_{\alpha^r} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})u_\alpha(\xi)+$$

$$(\sum_{j' \mod p^{r-1}} \begin{pmatrix} 1 & 0 \\ j'p & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} u_{\alpha^r} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} u_\alpha(\xi)]$$

Now we have to assume t u_{α^ν} satisfy commutation rules

$$\begin{gathered} u_{\alpha^r} u_\alpha = u_{\alpha^{r+1}} \\ u_{\alpha^r} \begin{pmatrix} 1 & j_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & j_1 p^r \\ 0 & 1 \end{pmatrix} u_{\alpha^r} \\ u_{\alpha^r} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} u_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = c_I(p) u_{\alpha^{r-1}} \end{gathered} \tag{3.8}$$

where $c_I(p)$ is a non zero integer. If we exploit the first two commutation relation then we get as the sum in the first $[\dots]$

$$[\textstyle\sum_{j \mod p^r, j_1 \mod p} \begin{pmatrix} 1 & j + p^r j_1 \\ 0 & 1 \end{pmatrix}$$

$$\textstyle\sum_{j' \mod p^{r-1}, j_1 \mod p} \begin{pmatrix} 1 & 0 \\ (j' + p^{r-1} j_1)p & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}] u_{\alpha^{r+1}}(\xi)] \tag{3.9}$$

$$= T(p^{r+1}, u_{\alpha^{r+1}})(\xi).$$

To compute the contribution of the second $[\dots]$ we observe that $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \Gamma$ and hence we have $w\xi = \xi$.Then the second commutation relation yields for the sum of the terms in the second $[\dots]$

$$c_I(p)(\sum_{j \mod p^r} \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} + \sum_{j' \mod p^{r-1}} \begin{pmatrix} 1 & 0 \\ j'p & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) u_{\alpha^{r-1}}(\xi). \quad (3.10)$$

We observe that for $j \equiv 0 \mod p^{r-1}$ we get

$$\begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} \alpha^{r-1}(\xi) = u_{\alpha^{r-1}} \begin{pmatrix} 1 & \frac{j}{p^{r-1}} \\ 0 & 1 \end{pmatrix} (\xi) = u_{\alpha^{r-1}}(\xi)$$

and in case $r > 1$ for $j' \equiv 0 \mod p^{r-2}$

$$\begin{pmatrix} 1 & 0 \\ j'p & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) u_{\alpha^{r-1}}(\xi) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) u_{\alpha^{r-1}} \begin{pmatrix} 1 & \frac{pj'}{p^{r-1}} \\ 0 & 1 \end{pmatrix} (\xi) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) u_{\alpha^{r-1}}(\xi). \quad (3.11)$$

Here we used again (3.8) and $\xi \in H^0(\Gamma\backslash X, \tilde{I})$. In other words in the summation (3.10) the first term only depends on $j \mod p^{r-1}$ and the second only on $j' \mod p^{r-2}$. For $r > 1$ this yields for the second term (3.10)

$$pc_I(p)(\sum_{j \mod p^{r-1}} \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} + \sum_{j' \mod p^{r-2}} \begin{pmatrix} 1 & 0 \\ j'p & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) u_{\alpha^{r-1}}(\xi) = pc_I(p) T(p^{r-1})\xi$$

If $r = 1$ the value for (3.10) is $c_I(p)(p+1)u_{\alpha^0}$ and hence we get the general formula

$$T_{p^r} \cdot T_p = T_{p^{r+1}} + (p + \epsilon(p)) c_I(p) T_{p^{r-1}} \quad (3.12)$$

where $\epsilon(r) = 0$ if $r > 1$ and $\epsilon(r) = 1$ for $r = 1$.

This formula is valid for all values of $r \geq 0$ if we put $T_{p^{-1}} = 0$.

We want to know what this means for the action on $H^1(\Gamma\backslash\mathbb{H}, \mathcal{M})$, we start again from our special resolution. (3.6). A simple calculation gives that the u_{α^r} satisfy the relations (3.8) with $c_{\mathcal{M}}(p) = p^n$. Hence we get for the action on $H^1(\Gamma\backslash\mathbb{H}, \mathcal{M})$

$$T_{p^r} \cdot T_p = T_{p^{r+1}} + p^{n+1} T_{p^{r-1}} + \epsilon(r)) p^n T_{p^{r-1}} \quad (3.13)$$

where $\epsilon(r) = 0$ if $r > 1$ and $\epsilon(r) = 1$ for $r = 1$.

Interlude

We assume that a majority of the readers has seen Hecke operators in the context of modular forms and also has seen formulas for these Hecke operators acting on spaces of modular forms, which look very similar to the formulas above. (See [95], [51]) This is of course not accidental, in the following chapter we will discuss the Eichler-Shimura isomorphism, which provides an injection of the space of modular forms of weight k into the cohomology $H^1(\Gamma\backslash\mathbb{H}, \mathcal{M}_{k-2} \otimes \mathbb{C})$). (See Thm. 4.1.3). This is a Hecke-module isomorphism and this explains the relation between the classical Hecke operators and the "cohomological" Hecke operators.

There is a slight difference between the formulas here and in [**?**], the reason is that our T_{p^r} differ slightly from the classical Hecke operators. But we always have T_p defined as above is equal to T_p in (3.1.2).

We want to stress that in this text so far -except in the introduction- there is no mentioning of modular forms, this is intentional.

End Interlude

This can be generalised. We choose an integer $N > 1$ and we take as our arithmetic group the full congruence group $\Gamma = \Gamma(N)$. For any prime $p \nmid N$ the $T(\alpha, u_\alpha)$ with $\alpha \in \mathrm{Gl}_2^+(\mathbb{Z}[1/p])$ form a commutative subalgebra $\mathcal{H}_p$ which is generated by T_p. This is the so called *unramified* Hecke algebra.

For $p|N$ we can also consider the $T(\alpha, u_\alpha)$ with $\alpha \in \mathrm{Gl}_2^+(\mathbb{Z}[1/p])$. They will also generate a local algebra $\mathcal{H}_p$ of endomorphisms in any of our cohomology groups, but this algebra will not necessarily be commutative. But if we have two different primes p, p_1 then we saw that the $\mathcal{H}_p, \mathcal{H}_{p_1}$ commute with each other. All these algebras $\mathcal{H}_p$ have an identity element e_p, we form the algebra

$$\mathcal{H}_\Gamma = \bigotimes_p{}' \mathcal{H}_p$$

where the superscript indicates that a tensor $h_f = \bigotimes_p h_p \in \mathcal{H}_\Gamma$ has a a factor e_p for almost all p. (See also further down section 6.3.3) This algebra acts on all our cohomology groups. We recall that the action of $\mathcal{H}_\Gamma$ on the inner cohomology groups is semi-simple (See Thm. 3.1.1). This has important consequences, which we discuss after a brief recapitulation of the theory of semi simple modules.

3.2 Some results on semi-simple $\mathcal{A}$- modules

We fix a field L and its algebraic closure $\bar{L}$, for simplicity we assume that the characteristic of L is zero, or that L is perfect. We consider an L-algebras $\mathcal{A}$, not necessarily commutative but with identity. We need a few results and concepts from the theory on finite dimensional vector spaces V/L with an action of $\mathcal{A}$, i.e equipped with a homomorphism $\mathcal{A} \to \mathrm{End}_L(V)$.

Such an $\mathcal{A}$ module V is called *irreducible* if it does not contain an $\mathcal{A}$ invariant proper submodule $W \subset V$, i.e $\{0\} \neq W \neq V$. It is called *absolutely irreducible* if $\mathcal{A} \otimes \bar{L}$ module $V \otimes \bar{L}$ is irreducible. We say that V is *indecomposable* if it can not be written as the direct sum of two non zero submodules. An irreducible module is also indecomposable.

We say that the action of $\mathcal{A}$ on V is semi-simple, if the action of $\mathcal{A} \otimes \bar{L}$ on $V \otimes \bar{L}$ is semi simple and this means that any $\mathcal{A}$ submodule $W \subset V \otimes \bar{L}$ has a complement, i.e. we can find an $\mathcal{A}$-submodule $W^\perp \subset V \otimes \bar{L}$ such that $V \otimes \bar{L} = W \oplus W^\perp$.

Then it is clear that we get a decomposition indexed by a finite set E

$$V \otimes \bar{L} = \bigoplus_{i \in E} W_i$$

where the W_i are (absolutely) irreducible submodules. In general this decomposition will not be unique. For any two W_i, W_j of these submodules we have (Schur's lemma)

$$\mathrm{Hom}_{\mathcal{A}}(W_i, W_j) = \begin{cases} \bar{L} & \text{if they are isomorphic as } \mathcal{A} \text{ -modules} \\ 0 & \text{else} \end{cases}$$

We decompose the indexing set $E = E_1 \cup E_2 \cup .. \cup E_k$ according to isomorphism types. For any E_ν we choose an $\mathcal{A}$ module $W_{[\nu]}$ of this given isomorphism type. Then by definition

$$\mathrm{Hom}_{\mathcal{A}}(W_{[\nu]}, W_j) = \begin{cases} \bar{L} & \text{if } j \in E_\nu \\ 0 & \text{else} \end{cases}.$$

Now we define $H_{[\nu]} = \mathrm{Hom}_{\mathcal{A}}(W_{[\nu]}, V \otimes \bar{L})$ we get an inclusion $H_{[\nu]} \otimes W_{[\nu]} \hookrightarrow V \otimes \bar{L}$. The image X_ν will be an $\mathcal{A}$ submodule, which is a direct sum of copies of $W_{[\nu]}$, it is the unique such submodule.

We get a direct sum decomposition

$$V \otimes \bar{L} = \bigoplus_{\nu} \bigoplus_{i \in E_\nu} W_i = \bigoplus_{\nu} X_\nu$$

then this last decomposition is easily seen to be unique, it is called the *isotypical decomposition*.

If V is a semi simple $\mathcal{A}$ module then any submodule $W \subset V$ also has a complement (this is not entirely obvious because by definition only $W \otimes_L \bar{L}$ has a complement in $V \otimes_L \bar{L}$. But a small moment of meditation gives us that finding such a complement is the same as solving an inhomogeneous system of linear equations over L. If this system has a solution over $\bar{L}$ it also has a solution over L.) Therefore we also can decompose the $\mathcal{A}$ module V into irreducibles. Again we can group the irreducibles according to isomorphism types and we get the *isotypical* decomposition

$$V = \bigoplus_{i \in E} U_i = \bigoplus_{\nu} \bigoplus_{i \in E_\nu} U_i = \bigoplus_{\nu} Y_\nu. \tag{3.14}$$

But of course a summand U_i may become reducible if we extend the scalars to $\bar{L}$ (See examplple below). Since it is clear that for any two $\mathcal{A}$- modules V_1, V_2 we have

$$\mathrm{Hom}_{\mathcal{A}}(V_1, V_2) \otimes \bar{L} = \mathrm{Hom}_{\mathcal{A} \otimes \bar{L}}(V_1 \otimes \bar{L}, V_2 \otimes \bar{L})$$

we know that we get the isotypical decomposition of $V \otimes \bar{L}$ by taking the isotypical decomposition of the $Y_\nu \otimes \bar{L}$ and then taking the direct sum over ν.

Example: Let L_1/L be a finite extension of degree > 1, then we put $\mathcal{A} = L_1$ and $V = L_1$, the action is given by multiplication. Clearly V is irreducible, but $V \otimes \bar{L}$ is not. If L_1/L is separable then the module is semisimple, otherwise it is not.

We have a classical result:

Proposition 3.2.1. *Let V be a semi simple $\mathcal{A}$ module. Then the following assertions are equivalent*
i) The $\mathcal{A}$ module V is absolutely irreducible
ii) The image of $\mathcal{A}$ in the ring of endomorphisms is $End(V)$
iii) The vector space of $\mathcal{A}$ endomorphisms $End_{\mathcal{A}}(V) = L$.

This can be an exercise for an algebra class. Where do we need the assumption that V is semi simple?

We return to our algebra $\mathcal{A}$ over L.Let V be an irreducible semi-simple $\mathcal{A}$ -module, which is not necessarily absolutely irreducible. Let I_V be the two sided ideal which annihilates V, i.e. the kernel of $\mathcal{A} \to \mathrm{End}_L(V)$. Let $\mathcal{C}_L$ be the centre of $\mathcal{A}/I_V$. This centre is a field, because any $c \in \mathcal{C}_L$ is either zero or an isomorphism, in other words V is a $\mathcal{C}_L$ vector space. The $\mathcal{C}_L$-algebra $\mathcal{A}/I_V$ is a central simple algebra. There is a central division algebra $\mathcal{D}/\mathcal{C}_L$ such that $\mathcal{A}/I_V \xrightarrow{\sim} M_r(\mathcal{D})$, this is the algebra of (r,r) matrices with coefficients in $\mathcal{D}$. This algebra has exactly one -up to isomorphism- non zero irreducible module, this is the module of column vectors $\mathcal{D}^r$, the algebra acts by multiplication from the left. Let us denote this module by $\mathcal{X}[\mathcal{A}/I_V]$

Theorem 3.2.1. *The extension $\mathcal{C}_L/L$ is separable. Let L_1/L be a normal closure of $\mathcal{C}_L$. Then we have the isotypical decomposition*

$$V \otimes_L L_1 = \bigoplus_{\sigma:\mathcal{C}_L \to L_1} V \otimes_{\mathcal{C}_L,\sigma} L_1 \tag{3.15}$$

The Galois group $Gal(L_1/L)$ permutes the summands simply transitively. The $\mathcal{A}/I_V \otimes_{\mathcal{C}_L,\sigma} L_1$ module $V \otimes_{\mathcal{C}_L,\sigma} L_1$ is isomorphic to the standard module $\mathcal{X}[\mathcal{A}/I_V \otimes_{\mathcal{C}_L,\sigma} L_1]$.

Here $M_r(\mathcal{D})$ is the L_1 algebra of (r,r) matrices with coefficients in $\mathcal{D}$. This is essentially the classical Wedderburn theorem.

Proposition 3.2.2. *For any semi -simple $\mathcal{A}$ module V we can find a finite extension L_2/L such that the irreducible sub modules in the decomposition into irreducibles are absolutely irreducible.*

Clear, we have to take an extension which splits $\mathcal{D}$.

If V is any $\mathcal{A}$ module- not necessarily semi simple but finite dimensional over L-then there is a finite extension L_2/L and a filtration

$$\{0)\} \subset V_1 \subset V_2 \subset \cdots \subset V_{r-1} \subset V \otimes_L L_2$$

such that the successive quotients V_i/V_{i-1} are absolutely irreducible. A very elementary argument shows that the set of isomorphism types occurring in this filtrations does not depend on the filtration, let us denote this set of isomorphism types by $\mathrm{Spec}_V(\mathcal{A} \otimes L_2)$.

We say that an $\mathcal{A}$- sub module $W \subset V$ is *complete in* V if the two sets $\mathrm{Spec}_W(\mathcal{A} \otimes L_2)$ and $\mathrm{Spec}_{V/W}(\mathcal{A} \otimes L_2)$ are disjoint. We have the simple

Proposition 3.2.3. *a) If V is a semi simple $\mathcal{A}$-module and if $W \subset V$ is complete in V then we have a canonical splitting $V = W \oplus W'$.*

b) If V is not necessarily semi simple but if $\mathcal{A}$ is commutative instead then any $W \subset V$ which is complete in V also has a canonical complement W', i.e. $V = W \oplus W'$.

Proof. For the second assertion we observe that an absolutely irreducible $\mathcal{A}$ module U is simply one dimensional over L_2 and given by a homomorphism $\pi : \mathcal{A} \to L_2$, i.e. it is an eigenspace for $\mathcal{A}$. □

Let us call such a decomposition a *decomposition into complete summands.*

Let us now assume that we have two algebras $\mathcal{A}, \mathcal{B}$ acting on V, let us assume that these two operations commute i.e. for $A \in \mathcal{A}, B \in \mathcal{B}, v \in V$ we have $A(Bv) = B(Av)$. This structure is the same as having a $\mathcal{A} \otimes_L \mathcal{B}$ structure on V. Let us assume that $\mathcal{A}$ acts semi simply on V and let us assume that the irreducible $\mathcal{A}$ submodules of V are absolutely irreducible. Then it is clear that the isotypical summands $Y_\nu = \bigoplus W_i$ are invariant under the action $\mathcal{B}$. Now we pick an index i_0 then the evaluation maps gives us a homomorpism

$$W_{i_0} \otimes \ \mathrm{Hom}_{\mathcal{A}}(W_{i_0}, Y_\nu) \to Y_\nu.$$

Under our assumptions this is an isomorphism. Then we see that we get

$$V = \bigoplus_\nu W_{i_\nu} \otimes \ \mathrm{Hom}_{\mathcal{A}}(W_{i_0}, Y_\nu)$$

where i_ν is any element in E_ν and where $\mathcal{A}$ acts upon the first factor and $\mathcal{B}$ acts upon the second factor via the action of $\mathcal{B}$ on Y_ν.

We apply this to our Hecke algebra $\mathcal{H}_\Gamma = \bigotimes_p{}' \mathcal{H}_p$ and consider its action on $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})$. We anticipate the theorem that this action is semi-simple. Hence we can find a finite normal extension $F/\mathbb{Q}$ such that we get an isotypical decomposition

$$H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes F) = \bigoplus_{\pi_f} H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes F)(\pi_f), \tag{3.16}$$

here π_f is the isomorphism type of an absolutely irreducible $\mathcal{H}$ module over F. We can realise this module by a vector space H_{π_f}/F with an absolutely irreducible action of $\mathcal{H}_\Gamma$ on it. Then $H_{\pi_f} = \bigotimes' H_{\pi_p}$ where H_{π_p} is an absolutely $\mathcal{H}_p$ module. For almost all primes $H_{|pi_p}$ is one dimensional and simply a homomorphism

$$\pi_p : \mathcal{H}_p \to F \text{ which is determined by its value } \pi_f(T_p) \in F \tag{3.17}$$

HEOP

3.2.1 Explicit formulas for the Hecke operators, a general strategy.

In the following section we discuss the Hecke operators and for numerical experiments it is useful to have an explicit procedure to compute them in a given case. The main obstruction to get such an explicit procedure is to find an explicit way to compute the arrow $j^\bullet(\alpha)$ in the top line of the diagram (3.3). (we change notation $j(\alpha)$ to $m(\alpha)$).

Let us assume that we have computed the cohomology groups on both sides by means of orbiconvex coverings $\mathfrak{V} : \cup_{i\in I} V_{y_i} = \Gamma(\alpha^{-1})\backslash X$ and $\mathfrak{U} : \cup_{j\in J} U_{y_j} = \Gamma(\alpha)\backslash X$.

The map $m(\alpha)$ is an isomorphism between spaces and hence $m(\alpha)(\mathfrak{V})$ is an acyclic covering of $\Gamma(\alpha)\backslash X$. This induces an identification

$$C^\bullet(\mathfrak{V}, \tilde{\mathcal{M}}) = C^\bullet(m(\alpha)(\mathfrak{V}), \tilde{\mathcal{M}}^{(\alpha)})$$

and the complex on the right hand side computes $H^\bullet(\Gamma(\alpha)\backslash X, \tilde{\mathcal{M}}^{(\alpha)})$. But this cohomology is also computable from the complex $C^\bullet(\mathfrak{U}, \tilde{\mathcal{M}}^{(\alpha)})$. We take the disjoint union of the two indexing sets $I \cup J$ and look at the covering $m_\alpha(\mathfrak{V}) \cup \mathfrak{U}$. (To be precise: We consider the disjoint union $\tilde{I} = I \cup J$ and define a covering $\mathfrak{W}_{\mathrm{i}}$ indexed by $\tilde{I}$. If $i \in \tilde{I}$ then $W_i = m(\alpha)(V_{y_i})$ and if $i \in J$ then we put $W_i = U_{x_i}$. We get a diagram of Čzech complexes Čzech

$$\begin{array}{ccccc} \to & \bigoplus_{\underline{i}\in I^q} \tilde{\mathcal{M}}^{(\alpha)}(W_{\underline{i}}) & \to & \bigoplus_{\underline{i}\in I^{q+1}} \tilde{\mathcal{M}}^{(\alpha)}(W_{\underline{i}}) \to \\ & \uparrow & & \uparrow \\ \to & \bigoplus_{\underline{i}\in \tilde{I}^q} \tilde{\mathcal{M}}^{(\alpha)}(W_{\underline{i}}) & \to & \bigoplus_{\underline{i}\in \tilde{I}^{q+1}} \tilde{\mathcal{M}}^{(\alpha)}(W_{\underline{i}}) \to \\ & \downarrow & & \downarrow \\ \to & \bigoplus_{\underline{i}\in J^q} \tilde{\mathcal{M}}^{(\alpha)}(W_{\underline{i}}) & \to & \bigoplus_{\underline{i}\in J^{q+1}} \tilde{\mathcal{M}}^{(\alpha)}(W_{\underline{i}}) \to \end{array} \tag{3.18}$$

The sets $I^\bullet, J^\bullet$ are subsets of $\tilde{I}^\bullet$ and the up- and down-arrows are the resulting projection maps. We know that these up- and down-arrows induce isomorphisms in cohomology.

Hence we can start from a cohomology class $\xi \in H^q(\Gamma(\alpha)\backslash X, \tilde{\mathcal{M}}^{(\alpha)})$, we represent it by a cocycle

$$c_\xi \in \bigoplus_{\underline{i}\in I^q} \tilde{\mathcal{M}}^{(\alpha)}(W_{\underline{i}}).$$

Then we can find a cocycle $\tilde{c}_\xi \in \bigoplus_{\underline{i}\in \tilde{I}^q} \tilde{\mathcal{M}}^{(\alpha)}(W_{\underline{i}})$ which maps to c_ξ under the uparrow. To get this cocycle we have to do the following: our cocycle c_ξ is an array with components $c_\xi(\underline{i})$ for $\underline{i} \in I^q$. We have $d_q(c_\xi) = 0$. To get $\tilde{c}_\xi$ we have to give the values $\tilde{c}_\xi(\underline{i})$ for all $\underline{i} \in \tilde{I}^q \setminus I^q$. We must have

$$d_q \tilde{c}_\xi = 0.$$

this yields a system of linear equations for the remaining entries. We know that this system of equations has a solution -this is then our $\tilde{c}_\xi$ - and this solution is unique up to a boundary $d_{q-1}(\xi')$. Then we apply the downarrow to $\tilde{c}_\xi$ and get a cocycle $c_\xi^\dagger$, which represents the same class ξ but this class is now represented by a cocycle with respect to the covering $\mathfrak{U}$. We apply the map $\tilde{u}^\alpha : \tilde{\mathcal{M}}^{(\alpha)} \to \tilde{\mathcal{M}}$ to this cocycle and then we get a cocycle which represents the image of our class ξ under T_α.

In the following section we discuss the explicit computation of a Hecke operator in a very specific situation. We start from our computation in section (2.1.4) and write down some $H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}})$ explicitly. On these modules we give explicit procedures to compute a Hecke operator. We get some supply of data and we look for some interesting laws or we try to verify some conjectures (see (3.78)).

3.3 Hecke operators for Gl_2:

For the rest of this chapter we discuss a very specific case. The algebraic group scheme will be $\mathrm{Gl}_2/\mathbb{Z}$. The symmetric space will be

$$X = \mathrm{Gl}_2(\mathbb{R})/K_\infty \text{ where } K_\infty = \mathrm{SO}(2) \times \{\begin{pmatrix} t & 0 \\ 0 & t) \end{pmatrix} |t \in \mathbb{R}^\times, t > 0\}.$$

Then the space X is the union of an upper and a lower half plane. We choose $\tilde{\Gamma} = \mathrm{Gl}_2(\mathbb{Z})$, then

$$\tilde{\Gamma}\backslash G_\infty/K_\infty = \Gamma\backslash\mathbb{H},$$

where $\Gamma = \mathrm{Sl}_2(\mathbb{Z})$ and $\mathbb{H}$ is the upper half plane. Earlier we defined the Γ-modules $\mathcal{M}_n[m]$ (See1.2.2), in the following we put $\mathcal{M} = \mathcal{M}_n[0]$.

We refer to Chapter 2 2.1.3. We have the two open sets $\tilde{U}_{\mathbf{i}}$, resp. $\tilde{U}_\rho \subset \mathbb{H}$, they are fixed under

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } R = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix},$$

respectively. We also will use the elements

$$T_+ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \; S_1^+ = T_- S T_-^{-1} = \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix} \in \Gamma_0^+(2)$$

$$T_- = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \; S_1^- = T_+ S T_+^{-1} = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} \in \Gamma_0^-(2)$$

The elements S_1^+ and S_1^- are elements of order four, i.e. $(S_1^+)^2 = (S_1^-)^2 = -\mathrm{Id}$, the corresponding fixed points are $\frac{\mathbf{i}+1}{2}$ and $\mathbf{i}+1$ respectively. Hence S_1^- fixes the sets $\alpha\tilde{U}_{\frac{\mathbf{i}+1}{2}}$ and $\tilde{U}_{\mathbf{i}+1}$, this is the only occurrence of a non trivial stabilizer.

3.3.1 The boundary cohomology

It is easier to compute the action of the Hecke operator T_p on the cohomology of the boundary, i. e. to compute the endomorphism

$$T_p : H^1(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}) \to H^1(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}).$$

We know (see 2.68) that $H^1(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}) = \mathcal{M}/(1-T_+)\mathcal{M}$, we collect some easy facts concerning this module. For $n \geq k \geq 0$ we define the submodules

$$\mathcal{M}^{(k)} = \mathbb{Z}\, X^k Y^{n-k} \oplus \mathbb{Z}\, X^{k+1} Y^{n-k-1} \oplus \cdots \oplus \mathbb{Z}\, X^n$$

for $k = 0$(resp. $k = n$) we have $\mathcal{M}^{(0)} = \mathcal{M}$(resp. $\mathcal{M}^{(n)} = \mathbb{Z}\ X^n$). These modules are invariant under the action of T_+ we have $(1-T_+)\mathcal{M}^{(k)} \subset \mathcal{M}^{(k+1)}$, and $\mathcal{M}^{(k)}/\mathcal{M}^{(k+1)} \xrightarrow{\sim} \mathbb{Z}$. The map $(1-T_+)$ induces a map

$$\partial_k : \mathcal{M}^{(k)}/\mathcal{M}^{(k+1)} \to \mathcal{M}^{(k+1)}/\mathcal{M}^{(k+2)}$$

which is given by multiplication with $n-k$. Hence it is clear that

$$\mathcal{M}/(1-T_+)\mathcal{M} = \mathbb{Z}[Y^n] \oplus \mathcal{M}^{(1)}/(1-T_+)\mathcal{M}$$

and the second summand is a finite module. The filtration of $\mathcal{M}$ by the $\mathcal{M}^{(k)}$ induces a filtration on $H^1(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}})$, we put

$$H^1(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}})^{(k)} := \mathrm{Im}(H^1(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}^{(k)}) \to H^1(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}) \qquad (3.19)$$

Then pn1

Proposition 3.3.1. *For* $k > 0$ *the quotient*

$$H^1(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}})^{(k)}/H^1(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}})^{(k+1)} \xrightarrow{\sim} \mathbb{Z}/(n-k+1)\mathbb{Z}$$

The Hecke operator T_p *acts on* $H^1(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}})^{(k)}/H^1(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}})^{(k+1)}$ *by multiplication with* $p^k + p^{n-k+1}$. *Especially we have*

$$T_p[Y^n] = (p^{n+1}+1)[Y^n]$$

Proof. We introduce the polynomials

$$\epsilon_k(X,Y) := X^n \tfrac{Y}{X}(\tfrac{Y}{X}-1)\dots(\tfrac{Y}{X}-k+1) = X^n \textstyle\prod_{\nu=0}^{k-1}(\tfrac{Y}{X}-\nu) = k!X^n \binom{\frac{Y}{X}}{k} =$$

$$X^{n-k}(Y-X)\dots(Y-(k-1)X) = X^{n-k}Y^k + \dots + (-1)^k k! X^n$$

Obviously these $\epsilon_k(X,Y)$ form a basis of $\mathcal{M}$. Pascal's rule for binomial coefficient says $\binom{\frac{Y}{X}+1}{k} = \binom{\frac{Y}{X}}{k} + \binom{\frac{Y}{X}}{k-1}$ and this yields

$$T_+\epsilon_k(X,Y) = \epsilon_k(X, X+Y) = \epsilon_k(X,Y) + k\epsilon_{k-1}(X,Y)$$

and from this we get

$$\mathcal{M}/(1-T_+)\mathcal{M} = \mathbb{Z}\epsilon_n(X,Y) \oplus \bigoplus_{k=n-1}^{0} (\mathbb{Z}/(k+1)\mathbb{Z})\epsilon_k(X,Y) \qquad (3.20)$$

this is the first assertion.

We pick a prime p and investigate the action of T_p on $H^1(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}})$. We recall the definition of the Hecke operator: We start from the matrix $\alpha = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ and consider the diagram (3.3) adapted to our situation

$$\begin{array}{ccccc} H^1(\partial(\Gamma(\alpha^{-1})\backslash\mathbb{H}), \tilde{\mathcal{M}}) & \xrightarrow{j(\alpha)^{(1)}} & H^1(\partial(\Gamma(\alpha)\backslash\mathbb{H}), j(\alpha)_*(\tilde{\mathcal{M}})) & \xrightarrow{\tilde{u}_\alpha^{(1)}} & H^1(\partial(\Gamma(\alpha)\backslash\mathbb{H}), \mathcal{M}) \\ \uparrow \pi^{(1)} & & & & \downarrow \pi_{(1)} \\ H^1(\Gamma\backslash X, \tilde{\mathcal{M}}) & & \xrightarrow{T(\alpha,u_\alpha)} & & H^1(\partial(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) \end{array} \qquad (3.21)$$

The group $\Gamma(\alpha^{-1}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} | c \equiv 0 \mod p\}$, it acts on $\mathbb{P}^1(\mathbb{Q})$ and has two orbits which can be represented by ∞ and 0. The stabilisers of these two cusps are $\Gamma_\infty = \{\pm \mathrm{Id}\; T_+^\nu\}$ and $\Gamma_0 = \{\pm \mathrm{Id}\; T_-^{p\nu}\}$ respectively. Hence we get

$$H^1(\partial(\Gamma(\alpha^{-1})\backslash\mathbb{H}), \tilde{\mathcal{M}}) = \mathcal{M}/(\mathrm{Id} - T_+)\mathcal{M} \oplus \mathcal{M}/(\mathrm{Id} - T_-^p)\mathcal{M} \tag{3.22}$$

We identify $H^1(\partial(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) = \mathcal{M}/(\mathrm{Id} - T_+)\mathcal{M} \xrightarrow{w_0} \mathcal{M}/(\mathrm{Id} - T_-)\mathcal{M}$ where the last arrow is induced by the map $m \mapsto w_0 m$ with $w_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then

$$\pi^{(1)}(m) = (m, \sum_{j=0}^{p-1} \begin{pmatrix} 1 & 0 \\ j & 1 \end{pmatrix} w_0 m). \tag{3.23}$$

Therefore the composition

$$u_\alpha^{(1)} \circ j(\alpha)^{(1)} : \mathcal{M}/(\mathrm{Id}-T_+)\mathcal{M} \oplus \mathcal{M}/(\mathrm{Id}-T_-^p)\mathcal{M} \to \mathcal{M}/(\mathrm{Id}-T_+^p)\mathcal{M} \oplus \mathcal{M}/(\mathrm{Id}-T_-)\mathcal{M}$$

is given by $u_\alpha^{(1)} \circ j(\alpha)^{(1)}(m_\infty, m_0) \mapsto (\alpha m_\infty, \alpha m_0)$. and $\pi_{(1)}((n_\infty, n_0)) = n_\infty + w_0 n_0$. This yields

$$T_p(m) = \alpha m + w_0 \alpha w_0^{-1} \sum_{j=0}^{p-1} \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} m.$$

On $\mathcal{M}^{(k)}/\mathcal{M}^{(k+1)}$ the element $\begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix}$ acts as identity, α is multiplication by p^k and $w_0 \alpha w_0^{-1}$ is multiplication p^{n-k}. □

Here we encounter a situation where the quotient $H^1(\Gamma\backslash\mathbb{H}, \mathcal{M})_{\text{ int},!}/H^1_!(\Gamma\backslash\mathbb{H}, \mathcal{M})_{!\text{ int}}$ may become non trivial and somewhat interesting (see(2.66)). We have to consider the exact sequence

$$0 \to H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) \to H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) \xrightarrow{r} H^1(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}). \tag{3.24}$$

Our cohomology groups may have some torsion $\mathcal{T}_1 \subset H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}), \mathcal{T}_2 \subset H^1(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}})$ and the map r maps the torsion $\mathcal{T}_1$ to a submodule $r(\mathcal{T}_1) \subset \mathcal{T}_2$. But it will happen that $r(r^{-1}(\mathcal{T}_2))$ is strictly larger than $r(\mathcal{T}_1)$ this means that some non torsion elements are mapped to torsion elements under r. By definition $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})_{\text{ int},!} = r^{-1}(\mathcal{T}_2)$ and therefore

$$H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})_{\text{ int},!}/H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})_{\text{ int}} = r(r^{-1}(\mathcal{T}_2)/\mathcal{T}_1) \tag{3.25}$$

This has been investigated extensively by Taiwang Deng in [22].

Let $\pi_1 : \mathbb{H} \to \Gamma\backslash\mathbb{H}$ be the projection. We get a covering $\Gamma\backslash\mathbb{H} = \pi_1(\tilde{U}_{\mathbf{i}}) \cup \pi_1(\tilde{U}_\rho) = U_i \cap U_\rho$. From this covering we get the Čzech complex

$$\begin{array}{ccccc} 0 \to & \tilde{\mathcal{M}}(U_{\mathbf{i}}) \oplus \tilde{\mathcal{M}}(U_\rho) & \to & \tilde{\mathcal{M}}(U_{\mathbf{i}} \cap U_\rho) & \to 0 \\ & \downarrow ev_{\tilde{U}_{\mathbf{i}}} \oplus ev_{\tilde{U}_\rho} & & \downarrow ev_{\tilde{U}_{\mathbf{i}} \cap \tilde{U}_\rho} & \\ & \mathcal{M}^{<S>} \oplus \mathcal{M}^{<R>} & \to & \mathcal{M} & \to 0 \end{array} \tag{3.26}$$

and this gives us our formula for the first cohomology

$$H^1(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}) = \mathcal{M}/(\mathcal{M}^{<S>}\oplus\mathcal{M}^{<R>}) \tag{3.27}$$

We want to discuss the Hecke operator T_2. To do this we pass to the subgroups

$$\begin{aligned}\Gamma_0^+(2) &= \{\begin{pmatrix} a & b\\ c & d\end{pmatrix} \mid c\equiv 0 \mod 2\}\\ \Gamma_0^-(2) &= \{\begin{pmatrix} a & b\\ c & d\end{pmatrix} \mid b\equiv 0 \mod 2\}\end{aligned} \tag{3.28}$$

we form the two quotients and introduce the projection maps $\pi_2^{\pm}:\mathbb{H}\to\Gamma_0^{\pm}(2)\backslash\mathbb{H}$. We have an isomorphism between the spaces

$$\Gamma_0^+(2)\backslash\mathbb{H}\xrightarrow{\alpha_2}\Gamma_0^-(2)\backslash\mathbb{H}$$

which is induced by the map $m_2 : z\mapsto \begin{pmatrix} 2 & 0\\ 0 & 1\end{pmatrix} z = 2z$. This map induces an isomorphism

$$\alpha_2^\bullet : H^1(\Gamma_0^+(2)\backslash\mathbb{H},\tilde{\mathcal{M}})\xrightarrow{\sim} H^1(\Gamma_0^-(2)\backslash\mathbb{H},\tilde{\mathcal{M}}^{(\alpha)}). \tag{3.29}$$

We also have the map between sheaves $u_2 : m\mapsto \begin{pmatrix} 2 & 0\\ 0 & 1\end{pmatrix} m$ and the composition with this map induces a homomorphism in cohomology

$$H^1(\Gamma_0^+(2)\backslash\mathbb{H},\tilde{\mathcal{M}})\xrightarrow{u_2^\bullet\circ\alpha_2^\bullet} H^1(\Gamma_0^-(2)\backslash\mathbb{H},\tilde{\mathcal{M}}). \tag{3.30}$$

This is the homomorphism we need for the computation of the Hecke operator; it is easy to define but it may be difficult in practice to compute it.

Each of the spaces $\Gamma_0^+(2)\backslash\mathbb{H},\Gamma_0^-(2)\backslash\mathbb{H}$ has two cusps which can be represented by the points $\infty, 0\in\mathbb{P}^1(\mathbb{Q})$. The stabilizers of these two cusps in $\Gamma_0^+(2)$ resp. $\Gamma_0^-(2)$ are

$$< T_+ > \times\{\pm\mathrm{Id}\} \text{ and } < T_-^2 > \times\{\pm\mathrm{Id}\}\subset\Gamma_0^+(2)$$

resp.

$$< T_+^2 > \times\{\pm\mathrm{Id}\} \text{ and } < T_- > \times\{\pm\mathrm{Id}\}\subset\Gamma_0^-(2)$$

the factor $\{\pm\mathrm{Id}\}$ can be ignored. Then we get

$$\begin{aligned}H^1(\partial(\Gamma_0^+(2)\backslash\mathbb{H}),\tilde{\mathcal{M}}) &\xrightarrow{\sim}\mathcal{M}/(\mathrm{Id}-T_+)\mathcal{M}\oplus\mathcal{M}/(\mathrm{Id}-T_-^2)\mathcal{M}\\ H^1(\partial(\Gamma_0^-(2)\backslash\mathbb{H}),\tilde{\mathcal{M}}) &\xrightarrow{\sim}\mathcal{M}/(\mathrm{Id}-T_+^2)\mathcal{M}\oplus\mathcal{M}/(\mathrm{Id}-T_-)\mathcal{M}.\end{aligned}$$

But now it is obvious that α maps the cusp ∞ to ∞ and 0 to 0 and then it is also clear that for the boundary cohomology the map

$$\alpha_2^\bullet : \mathcal{M}/(\mathrm{Id}-T_+)\mathcal{M}\oplus\mathcal{M}/(\mathrm{Id}-T_-^2)\mathcal{M}\to\mathcal{M}/(\mathrm{Id}-T_+^2)\mathcal{M}\oplus\mathcal{M}/(\mathrm{Id}-T_-)\mathcal{M}$$

is simply the map which is induced by $u_2 : \mathcal{M} \to \mathcal{M}$. If we ignore torsion then the individual summands are infinite cyclic.

Our module $\mathcal{M}$ is the module of homogenous polynomials of degree n in 2 variables X, Y with integer coefficients. Then the classes $[Y^n], [X^n]$ of the polynomials Y^n (resp.) X^n are generators of $(\mathcal{M}/(\mathrm{Id}-T_+^\nu)\mathcal{M})/\mathrm{tors}($ resp. $(\mathcal{M}/(\mathrm{Id}-T_+^\nu)\mathcal{M})/\mathrm{tors})$ where $\nu = 1($ resp. $2.)$ Then we get for the homomorphism $\alpha_2^\bullet$

$$\alpha_2^\bullet : [Y^n] \mapsto [Y^n], \alpha_2^\bullet : [X^n] \mapsto 2^n[X^n]. \tag{3.31}$$

Nochmal ansehen

3.3.2 The explicit description of the cohomology

We give the explicit description of the cohomology $H^1(\Gamma_0^+(2)\backslash\mathbb{H}, \tilde{\mathcal{M}})$. We introduce the projections

$$\mathbb{H} \xrightarrow{\pi_2^+} \Gamma_0^+(2)\backslash\mathbb{H};\ \mathbb{H} \xrightarrow{\pi_2^-} \Gamma_0^-(2)\backslash\mathbb{H}$$

and get the covering $\mathfrak{U}_2$

$$\Gamma_0^+(2)\backslash\mathbb{H} = \pi_2^+(\tilde{U}_{\mathbf{i}}) \cup \pi_2^+(T_-\tilde{U}_{\mathbf{i}}) \cup \pi_2^+(\tilde{U}_\rho) = \pi_2^+(\tilde{U}_{\mathbf{i}}) \cup \pi_2^+(\tilde{U}_{\frac{\mathbf{i}+1}{2}}) \cup \pi_2^+(\tilde{U}_\rho)$$

where we put $T_-\tilde{U}_{\mathbf{i}} = \tilde{U}_{\frac{\mathbf{i}+1}{2}}$. Our set $\{x_\nu\}$ of indexing points is $\mathbf{i}, \frac{\mathbf{i}+1}{2}, \rho$, we put $U_{x_i}^+ = \pi_2^+(\tilde{U}_{x_i})$. Note $T_- \notin \Gamma_0^+(2), T_+ \in \Gamma_0^+(2)$.

Again the cohomology is computed by the complex

$$0 \to \tilde{\mathcal{M}}(U_{\mathbf{i}}^+) \oplus \tilde{\mathcal{M}}(T_-\tilde{U}_{\mathbf{i}}^+) \oplus \tilde{\mathcal{M}}(U_\rho^+) \to \tilde{\mathcal{M}}(U_{\mathbf{i}}^+ \cap U_\rho^+) \oplus \tilde{\mathcal{M}}(T_-\tilde{U}_{\mathbf{i}}^+ \cap U_\rho^+) \to 0$$

we have to identify the terms as submodules of some $\bigoplus \mathcal{M}$ and write down the boundary map explicitly. We have

$$\begin{array}{ccc} \tilde{\mathcal{M}}(U_{\mathbf{i}}^+) \oplus \tilde{\mathcal{M}}(U_{\frac{\mathbf{i}+1}{2}}^+) \oplus \tilde{\mathcal{M}}(U_\rho^+) & \xrightarrow{d_0} & \tilde{\mathcal{M}}(U_i^+ \cap U_\rho^+) \oplus \tilde{\mathcal{M}}(U_{\frac{\mathbf{i}+1}{2}}^+ \cap U_\rho^+) \\ \downarrow ev_{\tilde{U}_{\mathbf{i}}} \oplus ev_{T_-\tilde{U}_{\mathbf{i}}} \oplus ev_{\tilde{U}_\rho} & & \downarrow ev_{\tilde{U}_{\mathbf{i}}\cap\tilde{U}_\rho} \oplus ev_{\tilde{U}_{\mathbf{i}}\cap T_+^{-1}\tilde{U}_\rho} \oplus ev_{T_-\tilde{U}_{\mathbf{i}}\cap\tilde{U}_\rho} \\ \mathcal{M} \oplus \mathcal{M}^{<S_1^+>} \oplus \mathcal{M} & \xrightarrow{\bar{d}_0} & \mathcal{M} \oplus \mathcal{M} \oplus \mathcal{M} \end{array} \tag{3.32}$$

where the vertical arrows are isomorphisms. The boundary map $\bar{d}_0$ in the bottom row is given by

$$(m_1, m_2, m_3) \mapsto (m_1 - m_3, m_1 - T_+^{-1}m_3, m_1 - m_2) = (x, y, z)$$

We may look at the (isomorphic) sub complex where $x = z = 0$ and $m_1 = m_2 = m_3$ then we obtain the complex

$$0 \to \mathcal{M}^{<S_1^+>} \to \mathcal{M} \to 0;\ m_2 \mapsto m_2 - T_+^{-1}m_2$$

which provides an isomorphism

$$H^1(\Gamma_0^+(2)\backslash\mathbb{H}, \tilde{\mathcal{M}}) \xrightarrow{\sim} \mathcal{M}/(\mathrm{Id} - T_+^{-1})\mathcal{M}^{<S_1^+>}. \tag{3.33}$$

A simple computation shows that the cohomology class represented by the class (x, y, z) is equal to the class represented by $(0, y-x+T_+^{-1}z-z, 0)$ we write

$$[(x, y, z)] = [(0, y - x + T_+^{-1}z - z, 0)] \tag{3.34}$$

3.3.3 The map to the boundary cohomology

We have the restriction map for the cohomology of the boundary

$$\begin{array}{ccc} H^1(\Gamma_0^+(2)\backslash\mathbb{H}, \tilde{\mathcal{M}}) & \xrightarrow{\sim} & \mathcal{M}/(\mathrm{Id} - T_+^{-1})\mathcal{M}^{<S_1^+>} \\ \downarrow & & r^+ \oplus r^- \downarrow \\ H^1(\partial(\Gamma_0^+(2)\backslash\mathbb{H}), \tilde{\mathcal{M}}) & \xrightarrow{\sim} & \mathcal{M}/(\mathrm{Id} - T_+)\mathcal{M} \oplus \mathcal{M}/(\mathrm{Id} - T_-^2)\mathcal{M} \end{array} \tag{3.35}$$

we give a formula for the second vertical arrow. We represent a class $[m]$ by an element $m \in \mathcal{M}$ and send m to its class in each the two summands, respectively. This is well defined, for r^+ it is obvious, while for r^- we observe that if $m = x - T_+^{-1}x$ and $S_1^+x = x$ then $m = x - T_+^{-1}S_1^+x = x - T_-^2x$.

Restriction and Corestriction

Now we have to give explicit formulas for the two maps π^*, π_* in the big diagram on p. 50 in Chap2.pdf. Here we should change notation: The map π in Chap.2 will now be denoted by :

$$\varpi_2^+ : \Gamma_0^+(2)\backslash\mathbb{H} \to \Gamma\backslash\mathbb{H} \tag{3.36}$$

We have the two complexes which compute the cohomology $H^1(\Gamma_0^+(2)\backslash\mathbb{H}, \tilde{\mathcal{M}})$ and $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})$, and we have defined arrows between them. We realized these two complexes explicitly in (3.32) resp. (3.26) and we have

$$\begin{array}{ccc} \tilde{\mathcal{M}}(U_i^+) \oplus \tilde{\mathcal{M}}(U_{\frac{i+1}{2}}^+) \oplus \tilde{\mathcal{M}}(U_\rho^+) & \xrightarrow{d_0} & \tilde{\mathcal{M}}(U_i^+ \cap U_\rho^+) \oplus \tilde{\mathcal{M}}(U_{\frac{i+1}{2}}^+ \cap U_\rho^+) \\ (\varpi_2^+)^{(0)} \uparrow \downarrow (\varpi_2^+)_{(0)} & & (\varpi_2^+)^{(1)} \uparrow \downarrow (\varpi_2^+)_{(1)} \\ \tilde{\mathcal{M}}(U_i) \oplus \tilde{\mathcal{M}}(U_\rho) & \xrightarrow{d_0} & \tilde{\mathcal{M}}(U_i \cap U_\rho) \end{array} \tag{3.37}$$

and in terms of our explicit realization in diagram (3.32) this gives

$$\begin{array}{ccc} \mathcal{M} \oplus \mathcal{M}^{<S_1>} \oplus \mathcal{M} & \xrightarrow{d_0} & \mathcal{M} \oplus \mathcal{M} \oplus \mathcal{M} \\ (\varpi_2^+)^{(0)} \uparrow \downarrow (\varpi_2^+)_{(0)} & & (\varpi_2^+)^{(1)} \uparrow \downarrow (\varpi_2^+)_{(1)} \\ \mathcal{M}^{<S>} \oplus \mathcal{M}^{<R>} & \xrightarrow{d_0} & \mathcal{M} \end{array} \tag{3.38}$$

Looking at the definitions we find

$$\begin{aligned}(\varpi_2^+)^{(0)} &: (m_1, m_2) \mapsto (m_1, T_- m_1, m_2)\\ (\varpi_2^+)_{(0)} &: (m_1, m_2, m_3) \mapsto (m_1 + Sm_1 + T_-^{-1} m_2, (1 + R + R^2) m_3)\end{aligned} \tag{3.39}$$

and we check easily that the composition $(\varpi_2^+)_{(0)} \circ (\varpi_2^+)^{(0)}$ is the multiplication by 3 as it should be, since this is the index of $\Gamma_0(2)^+$ in Γ.

For the two arrows in degree one we find

$$\begin{aligned}(\varpi_2^+)^{(1)} &: m \mapsto (m, Sm, T_- m)\\ (\varpi_2^+)_{(1)} &: (m_1, m_2, m_3) \mapsto (m_1 + Sm_2 + T_-^{-1} m_3)\end{aligned} \tag{3.40}$$

We apply equation (3.34) and we see that $(\varpi_2^+)^{(1)}(m)$ is represented by

$$[(\varpi_2^+)^{(1)}(m)] = [0, Sm + T_+^{-1} T_- m - m - T_- m, 0] \tag{3.41}$$

We do the same calculation for $\Gamma_0^-(2)$. As before we start from a covering

$$\Gamma_0^-(2)\backslash\mathbb{H} = \pi_2^-(\tilde{U}_{\mathbf{i}}) \cup \pi_2^-(T_+\tilde{U}_{\mathbf{i}}) \cup \pi_2^-(\tilde{U}_\rho) = \pi_2^-(\tilde{U}_{\mathbf{i}}) \cup \pi_2^-(\tilde{U}_{i+1}) \cup \pi_2^-(\tilde{U}_\rho)$$

and as before we put $U_{y_\nu}^- = \pi_2^-(\tilde{U}_{y_\nu})$. In this case $\tilde{U}_{i+1} = T_+\tilde{U}_i$ is fixed by $S_1^- = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} \in \Gamma_0^-(2)$ and we get a diagram for the Čzech complex

$$\begin{array}{ccc} \tilde{\mathcal{M}}(U_{\mathbf{i}}^-) \oplus \tilde{\mathcal{M}}(U_{\mathbf{i}+1}^-) \oplus \tilde{\mathcal{M}}(U_\rho^-) & \xrightarrow{d_0} & \tilde{\mathcal{M}}(U_{\mathbf{i}}^- \cap U_\rho^-) \oplus \tilde{\mathcal{M}}(U_{\mathbf{i}+1}^- \cap U_\rho^-) \\ ev_{\tilde{U}_{\mathbf{i}}} \oplus ev_{\tilde{U}_{\mathbf{i}+1}} \downarrow \oplus ev_{\tilde{U}_\rho} & & ev_{\tilde{U}_{\mathbf{i}} \cap \tilde{U}_\rho} \oplus ev_{\tilde{U}_{\mathbf{i}} \cap T_-^{-1}\tilde{U}_\rho} \downarrow \oplus ev_{\tilde{U}_{\mathbf{i}+1} \cap \tilde{U}_\rho} \\ \mathcal{M} \oplus \mathcal{M}^{<S_1^->} \oplus \mathcal{M} & \xrightarrow{\bar{d}_0} & \mathcal{M} \oplus \mathcal{M} \oplus \mathcal{M} \end{array} \tag{3.42}$$

Again we can modify this complex and get

$$H^1(\Gamma_0^-(2)\backslash\mathbb{H}, \tilde{\mathcal{M}}) \xrightarrow{\sim} \mathcal{M}/(\mathrm{Id} - T_-^{-1})\mathcal{M}^{<S_1^->}. \tag{3.43}$$

We compute the arrows $(\varpi_2^-)^*, (\varpi_2^-)_*$ in degree one

$$\begin{aligned}(\varpi_2^-)^{(1)} &: m \mapsto (m, Sm, T_+ m),\\ (\varpi_2^-)_{(1)} &: (m_1, m_2, m_3) \mapsto (m_1 + Sm_2 + T_+^{-1} m_3).\end{aligned} \tag{3.44}$$

The computation of $\alpha_2^\bullet$.

We recall our isomorphism α between the spaces and the resulting isomorphism (3.29). The identity map of the module $\mathcal{M}$ and the isomorphism α on the space identifies the two complexes

$$\begin{array}{ccc} \tilde{\mathcal{M}}(U_{\mathbf{i}}^+) \oplus \tilde{\mathcal{M}}(U_{\frac{\mathbf{i}+1}{2}}^+) \oplus \tilde{\mathcal{M}}(U_\rho^+) & \xrightarrow{d_0} & \tilde{\mathcal{M}}(U_{\mathbf{i}}^+ \cap U_\rho^+) \oplus \tilde{\mathcal{M}}(U_{\frac{\mathbf{i}+1}{2}}^+ \cap U_\rho^+) \\ \tilde{\mathcal{M}}^{(\alpha)}(\alpha(U_{\mathbf{i}}^+)) \oplus \tilde{\mathcal{M}}^{(\alpha)}(\alpha(U_{\frac{\mathbf{i}+1}{2}}^+)) \oplus \tilde{\mathcal{M}}^{(\alpha)}(\alpha(U_\rho^+)) & \xrightarrow{d_0} & \tilde{\mathcal{M}}^{(\alpha)}(\alpha(U_{\mathbf{i}}^+ \cap U_\rho^+)) \oplus \tilde{\mathcal{M}}^{(\alpha)}(\alpha(U_{\frac{\mathbf{i}+1}{2}}^+ \cap U_\rho^+)) \end{array} \tag{3.45}$$

and if we consider their explicit realization then this identification is given by the equality of $\mathbb{Z}$ modules $\mathcal{M} = \mathcal{M}^{(\alpha)}$. This equality of complexes expresses the identification (3.29). We can compute the cohomology $H^1(\Gamma_0^-(2)\backslash\mathbb{H}, \tilde{\mathcal{M}}^{(\alpha)})$ from any of the two coverings

$$\begin{array}{l} \Gamma_0^-(2)\backslash\mathbb{H} = \alpha(U_{\mathbf{i}}^+) \cup \alpha(U_{\frac{\mathbf{i}+1}{2}}^+) \cup \alpha(U_\rho^+) = U_{x_1} \cup U_{x_2} \cup U_{x_3} \\ \text{and} \\ \quad \Gamma_0^-(2)\backslash\mathbb{H} = U_{\mathbf{i}}^- \cup U_{\mathbf{i}+1}^- \cup U_\rho^- = U_{x_4} \cup U_{x_5} \cup U_{x_6}. \end{array} \tag{3.46}$$

We have to pick a class $\xi \in H^1(\Gamma_0^-(2)\backslash\mathbb{H}, \tilde{\mathcal{M}}^{(\alpha)})$ and represent it by a cocycle

$$c_\xi \in \bigoplus_{1\le i<j\le 3} \tilde{\mathcal{M}}^{(\alpha)}(U_{x_i} \cap U_{x_j})$$

(The cocycle condition is empty since $U_{x_1} \cap U_{x_2} \cap U_{x_3} = \emptyset$.)

Then we have to produce a cocycle

$$c_\xi^{(\alpha)} \in \bigoplus_{4\le i<j\le 6} \tilde{\mathcal{M}}^{(\alpha)}(U_{x_i} \cap U_{x_j})$$

which represents the same class.

To get this cocycle we write down the three complexes

$$\begin{array}{ccc} \bigoplus_{1\le i<j\le 3} \tilde{\mathcal{M}}^{(\alpha)}(U_{x_i} \cap U_{x_j}) & \to & 0 \\ \uparrow & & \\ \bigoplus_{1\le i<j\le 6} \tilde{\mathcal{M}}^{(\alpha)}(U_{x_i} \cap U_{x_j}) & \to & \bigoplus_{1\le i<j<k\le 6} \tilde{\mathcal{M}}^{(\alpha)}(U_{x_i} \cap U_{x_j} \cap U_{x_k}) \\ \downarrow & & \\ \bigoplus_{4\le i<j\le 6} \tilde{\mathcal{M}}^{(\alpha)}(U_{x_i} \cap U_{x_j}) & \to & 0 \end{array} \tag{3.47}$$

for our cocycle c_ξ we find a cocycle $c_\xi^\dagger$ in the complex in the middle which maps to c_ξ under the upwards arrow and this cocycle is unique up to a coboundary. Then we project it down by the downwards arrow, i.e. we only take its $4 \le i < j \le 6$ components, and this is our cocycle $c_\xi^{(\alpha)}$.

We write down these complexes explicitly. For any pair $\underline{i} = (i,j), i < j$ of indices we have to compute the set $\mathcal{F}_{\underline{i}}$. We drew some pictures and from these pictures we get (modulo errors) the following list (of lists):

$$\begin{array}{cccc} \mathcal{F}_{1,2} = \emptyset & \mathcal{F}_{1,3} = \{\mathrm{Id}, T_+^{-2}\} & \mathcal{F}_{1,4} = \{\mathrm{Id}\} & \mathcal{F}_{1,5} = \{\mathrm{Id}, T_+^{-2}\} \\ \mathcal{F}_{1,6} := \{\mathrm{Id}, T_-^{-1}\} & \mathcal{F}_{2,3} = \{\mathrm{Id}\} & \mathcal{F}_{2,4} = \{\mathrm{Id}, T_-\} & \mathcal{F}_{2,5} = \{\mathrm{Id}\} \\ \mathcal{F}_{2,6} = \{\mathrm{Id}\} & \mathcal{F}_{3,4} = \{\mathrm{Id}, T_+^2\} & \mathcal{F}_{3,5} = \{\mathrm{Id}\} & \mathcal{F}_{3,6} = \{\mathrm{Id}, S_1^-\} \\ \mathcal{F}_{4,5} = \emptyset & \mathcal{F}_{4,6} = \{\mathrm{Id}, T_-^{-1}\} & \mathcal{F}_{5,6} = \{\mathrm{Id}\} & \end{array} \tag{3.48}$$

Now we have to follow the rules in the first section and we can write down an explicit version of the diagram (3.47) . We refer to section 2.1.3 and get

$$\begin{array}{ccc} \bigoplus_{1\le i<j\le 3} \bigoplus_{\gamma\in\mathcal{F}_{i,j}} (\mathcal{M}^{(\alpha)})^{\Gamma_{i,j,\gamma}} & \to & 0 \\ \uparrow & & \\ \bigoplus_{1\le i<j\le 6} \bigoplus_{\gamma\in\mathcal{F}_{i,j}} (\mathcal{M}^{(\alpha)})^{\Gamma_{i,j,\gamma}} & \to & \bigoplus_{1\le i<j<k\le 6} \bigoplus_{\gamma\in\mathcal{F}_{i,j,k}} (\mathcal{M}^{(\alpha)})^{\Gamma_{i,j,k,\gamma}} \\ \downarrow & & \\ \bigoplus_{4\le i<j\le 6} \bigoplus_{\gamma\in\mathcal{F}_{i,j}} (\mathcal{M}^{(\alpha)})^{\Gamma_{i,j,\gamma}} & \to & 0 \end{array} \tag{3.49}$$

Here we have to interpret this diagram. The module $\mathcal{M}^{(\alpha)}$ is equal to $\mathcal{M}$ as an abstract module, but an element $\gamma \in \Gamma_0^-(2)$ acts by the twisted action (See ChapII, 2.2)

$$m \mapsto \gamma *_\alpha m = \alpha^{-1}\gamma\alpha * m$$

here the $*$ denotes the original action. Hence we have to take the invariants $(\mathcal{M}^{(\alpha)})^{\Gamma_{i,j,\gamma}}$ with respect to this twisted action. In our special situation this has very little effect since almost all the $\Gamma_{i,j,\gamma}$ are trivial, except for the intersection $\alpha(\tilde{U}_{\frac{\mathbf{i}+1}{2}}) \cap \tilde{U}_{\mathbf{i}}$ in which case $\Gamma_{i,j,\gamma} =< S_1^- >$. Hence

$$(\mathcal{M}^{(\alpha)})^{<S_1^->} = \mathcal{M}^{<S_1^+>}.$$

Each of the complexes in (3.49) compute the cohomology group $H^1(\Gamma_0^-(2)\backslash\mathbb{H}, \tilde{\mathcal{M}})$ and the diagram gives us a formula for the isomorphism in (3.29). To get $u_\alpha^\bullet$ in (3.29) we apply the multiplication $m_2{:}m \mapsto \alpha m$ to the complex in the middle and the bottom. Then the cocycle c_ξ^α is now an element in $\bigoplus \tilde{\mathcal{M}}^{(\alpha)}$ and αc_ξ^α represents the cohomology class $u_\alpha^\bullet(\xi) \in H^1(\Gamma_0^-(2)\backslash\mathbb{H}, \tilde{\mathcal{M}})$.

Now it is clear how we can compute the Hecke operator

$$T_2 = T_{\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}} : \mathcal{M}/(\mathcal{M}^{<S>} \oplus \mathcal{M}^{<R>}) \to \mathcal{M}/(\mathcal{M}^{<S>} \oplus \mathcal{M}^{<R>})$$

We pick a representative $m \in \mathcal{M}$ of the cohomology class. We apply $(\varpi_2^+)^{(1)}$ in the diagram (3.38) to it and this gives the element $(Sm, m, T_-m) = c_\xi$. We apply the above process to compute $c_\xi^{(\alpha)}$. Then $\alpha c_\xi^{(\alpha)} = (m_1, m_2, m_3)$ is an element in $\tilde{\mathcal{M}}(U_{\mathbf{i}}^- \cap U_\rho^-) \oplus \tilde{\mathcal{M}}(U_{\mathbf{i}+1}^- \cap U_\rho^-)$ and this module is identified with $\mathcal{M} \oplus \mathcal{M} \oplus \mathcal{M}$ by the vertical arrow in (3.42). To this element we apply the trace

$$(\varpi_2^-)_{(1)}(m_1, m_2, m_3) = m_1 + m_2 + T_+^{-1} m_3$$

and the latter element in $\mathcal{M}$ represents the class $T_2([m])$.

We have written a computer program which for a given $\mathcal{M} = \mathcal{M}_n$, i.e. for a given even positive integer n, computes the module $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})$ and the endomorphism T_2 on it.

Looking our data we discovered the following (surprising?) fact: We consider the isomorphism in equation (3.29). We have the explicit description of the cohomology in (3.33)

$$H^1(\Gamma_0^+(2)\backslash\mathbb{H}, \tilde{\mathcal{M}}) \xrightarrow{\sim} \mathcal{M}/(\mathrm{Id} - T_+^{-1})\mathcal{M}^{<S_1^+>}$$

and

$$H^1(\Gamma_0^-(2)\backslash\mathbb{H}, \tilde{\mathcal{M}}^{(\alpha)}) \xrightarrow{\sim} \mathcal{M}/(\mathrm{Id} - T_-^{-1})(\mathcal{M}^{(\alpha)})^{<S_1^->}$$

We know that we may represent any cohomology class by a cocycle

$$c_\xi = (0, c_\xi, 0) \in \mathcal{M}^{(\alpha)}(\pi_2^-(\alpha(U_{\mathbf{i}}) \cap \alpha(U_\rho)) \oplus \mathcal{M}^{(\alpha)}(\pi_2^-(\alpha(U_{\mathbf{i}}) \cap \alpha(T_+^{-1}U_\rho)) \oplus \mathcal{M}^{(\alpha)}(\pi_2^-(\alpha(U_{\frac{\mathbf{i}+1}{2}}) \cap \alpha(T_+^{-1}U_\rho))$$

so it is non zero only in the middle component and then it is simply an element in $\mathcal{M}$. If we now look at our data, then it seems to by so that $c_\xi^{(\alpha)}$ is also non zero only in the middle, hence

$$c_\xi^{(\alpha)} \in (0, c'_\xi, 0) \in 0 \oplus \mathcal{M}^{(\alpha)}(\pi_2^-(U_{\mathbf{i}} \cap T_-^{-1}U_\rho)) \oplus 0$$

hence it is also in $\mathcal{M}^{(\alpha)}$ and then our data seem to suggest that

$$c'_\xi = c_\xi$$

Hence we see that the homomorphism in equation (3.30) is simply given by

$$X^\nu Y^{n-\nu} \mapsto 2^\nu X^\nu Y^{n-\nu}.$$

Is there a kind of homotopy argument (- 2 moves continuously to 1?-)-, which explains this?

We get an explicit formula for the Hecke operator T_2 : We pick an element $m \in \mathcal{M}$ representing the class $[m]$. We send it by $(\varpi_2^+)^{(1)}$ to $H^1(\Gamma_0^+(2)\backslash\mathbb{H}, \tilde{\mathcal{M}})$, i.e.

$$(\varpi_2^+)^{(1)} : m \mapsto (m, Sm, T_-m) \tag{3.50}$$

We modify it so that the first and the third entry become zero see(3.34)

$$[(m, Sm, T_-m)] = [(0, Sm - m + T_+^{-1}T_-m - T_-m, 0)] \tag{3.51}$$

To the entry in the middle we apply $M_2 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and then apply $(\varpi_2^-)_{(1)}$ and get

$$T_2([m]) = [S \cdot M_2(Sm - m + T_+^{-1}T_- m - T_- m)] \tag{3.52}$$

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3.3.4 The first interesting example

We give an explicit formula for the cohomology in the case of $\mathcal{M} = \mathcal{M}_{10}$. We define the sub-module

$$\mathcal{M}^{\rm tr} = \bigoplus_{\nu=0}^{5} \mathbb{Z} Y^{10-\nu} X^{\nu}$$

and we have the truncation operator trunc : $\mathcal{M} \to \mathcal{M}^{\rm tr}$

$$\text{trunc} : Y^{10-\nu}X^{\nu} \mapsto \begin{cases} Y^{10-\nu}X^{\nu} & \text{if } \nu \leq 5, \\ (-1)^{\nu+1} Y^{\nu} X^{10-\nu} & \text{else,} \end{cases},$$

it identifies the quotient module $\mathcal{M}/\mathcal{M}^{<S>}$ to $\mathcal{M}^{\rm tr}$. To get the cohomology we have to divide by the relations coming from $\mathcal{M}^{<R>}$, i.e. we have to divide by the submodule trunc$(\mathcal{M}^{<R>})$ The module of these relations is generated by

$$\begin{aligned} \mathrm{R}_1 &= 10Y^9X + 20Y^7X^3 + Y^5X^5 \\ \mathrm{R}_2 &= 9Y^8X^2 - 36Y^7X^3 + 14Y^6X^4 - 45Y^5X^5 \\ \mathrm{R}_3 &= 8Y^7X^3 + 10Y^5X^5 \end{aligned}$$

and then

$$H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) = \bigoplus_{\nu=0}^{5} \mathbb{Z}Y^{10-\nu}X^{\nu}/\{\mathrm{R}_1, \mathrm{R}_2, \mathrm{R}_3\} \tag{3.53}$$

Discuss complex conjugation

We simplify the notation and put $e_\nu = Y^\nu X^{n-\nu}$. Using R_1 we can eliminate $e_5 = -10e_9 - 20e_7$ and then

$$H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) = \bigoplus_{\nu=10}^{\nu=6} \mathbb{Z}e_\nu/\{-50e_9 + 9e_8 - 96e_7 + 14e_6, -100e_9 - 192e_7\} \tag{3.54}$$

introduce a new basis $\{f_{10}, f_9, f_8, f_7, f_6, f_5\}$ of the $\mathbb{Z}$ module $\mathcal{M}^{\rm tr}$:

$$\begin{gathered} f_{10} = e_{10}; f_8 = -2e_8 - 3e_6; f_6 = 9e_8 + 14e_6 \\ f_9 = -12e_9 - 23e_7; f_7 = 25e_9 + 48e_7; f_5 = 10e_9 + 20e_7 + e_5 \end{gathered} \tag{3.55}$$

and hence in the quotient we get $\bar{f}_5 = 0$ and $2\bar{f}_7 = \bar{f}_6$ and therefore

$$H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) = \mathbb{Z}\bar{f}_{10} \oplus \mathbb{Z}\bar{f}_9 \oplus \mathbb{Z}\bar{f}_8 \oplus \mathbb{Z}/(4)\bar{f}_7 \tag{3.56}$$

(If we invert the primes < 12 then we can work with e_{10}, e_9, e_8 and in cohomology $e_6 = -\frac{9}{14}e_8, e_5 = \frac{5}{12}e_9, e_7 = -\frac{25}{48}e_9$.)

If we can apply the above procedure to compute the action of T_2 on cohomology we get the following matrix for T_2 :

$$T_2 = \begin{pmatrix} 2049 & -68040 & 0 & 0 \\ 0 & -24 & 0 & 0 \\ 0 & 0 & -24 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \tag{3.57}$$

Hence we see that T_2 is non trivial on the torsion subgroup. If we divide by the torsion then the matrix reduces to a (3,3)-matrix and this matrix gives us the endomorphism on the "integral" cohomology which is defined in generality by

$$H^\bullet_{\mathrm{int}}(\Gamma\backslash X, \tilde{\mathcal{M}}) = H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}})/\mathrm{tors} \subset H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}_{\mathbb{Q}}). \tag{3.58}$$

Here we should be careful: the functor $H^\bullet \to H^\bullet_{\mathrm{int}}$ is not exact. In our case we get (perhaps up to a little piece of 2-torsion) exact sequences of Hecke modules

$$\begin{array}{ccccccc} 0 & \to & \mathbb{Z}\bar{f}_9 \oplus \mathbb{Z}\bar{f}_8 & \to \mathbb{Z}\bar{f}_{10} \oplus \mathbb{Z}\bar{f}_9 \oplus \mathbb{Z}\bar{f}_8 \xrightarrow{r} & \mathbb{Z}\bar{f}_{10} \to & & 0 \\ & & \| & \| & \| & & \\ 0 & \to & H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) \to & H^1_{\mathrm{int}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) \xrightarrow{r} & H^1_{\mathrm{int}}(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}) \to & & 0 \end{array} \tag{3.59}$$

where $T_2(\bar{f}_{10}) = (2^{11}+1)\bar{f}_{10}$. If we tensor by $\mathbb{Q}$ then we can find an unique element - *the Eisenstein class*- $f^\dagger_{10} \in H^1_{\mathrm{int}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{Q})$ which maps to $\bar{f}_{10}$ and which satisfies $T_2(f^\dagger_{10}) = (2^{11}+1)f^\dagger_{10}$. This element is not necessarily integral, in our case an easy computation shows that $f^\dagger \notin H^1_{\mathrm{int}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})$, but $691 f^\dagger \in H^1_{\mathrm{int}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})$. This means that 691 is the denominator of $f^\dagger_{10}$, i.e. 691 is the *denominator of the Eisenstein class* $f^\dagger_{10}$.

Hence we see that

$$H^1_{\mathrm{int}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) \supset H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) \oplus \mathbb{Z}(691 f^\dagger) \tag{3.60}$$

the quotient of these modules is isomorphic to $\mathbb{Z}/691\mathbb{Z}$.

The exact sequence $\mathcal{X}_{10}$ in (3.59) is an exact sequence of modules for the Hecke algebra $\mathcal{H} \supset \mathbb{Z}[T_2]$ and hence it yields an element

$$[\mathcal{X}_{10}] \in \mathrm{Ext}^1_{\mathcal{H}}(\mathbb{Z}f_{10}, H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})), \tag{3.61}$$

and an easy calculation shows that this Ext^1 group is cyclic of order 691 and that it is generated by $\mathcal{X}_{10}$.

We look at the action of the full Hecke algebra $\mathcal{H}$ on these cohomology groups. It turns out that for any prime p the Hecke operator T_p acts by the eigenvalue $p^{11}+1$ on f_{10}(see proposition 3.3.1). We will also see that a simple argument using Poincare duality and the self adjointness of the Hecke operators shows that

T_p acts by multiplication by a scalar $\tau(p)$ on the inner cohomology .

Then we can conclude

For all primes p *we have*

$$\tau(p) \equiv p^{11} + 1 \mod 691$$

3.3.5 Interlude: Ramanujan's $\Delta(z)$

We want to stress that the previous considerations are of purely algebraic and combinatorial nature, no analysis is involved. In the next chapter we will use analytic methods -especially we will use the results from the theory modular forms- to obtain some further insight into the structure of the cohomology groups. In our special case here it comes down to the following.

In his paper [85] Ramunujan introduced the function

$$\Delta(z) = e^{2\pi iz} \prod_{n=1}^{\infty} (1 - e^{2\pi inz})^{24}$$

this is the unique (up to a non zero scalar) cusp form of weight 12 for $\mathrm{Sl}_2(\mathbb{Z})$, (See [95]). We can expand

$$\Delta(z) = e^{2\pi iz} - 24e^{4\pi iz} + 252e^{6\pi iz} + \cdots + a_n e^{2n\pi iz} + \ldots$$

The coefficients satisfy (conjectured by Ramanujan) the following recursions

$$\begin{aligned} a_{n_1 n_2} &= a_{n_1} a_{n_2} \text{ if } n_1, n_2 \text{ are coprime;} \\ a_{p^r} &= a_p a_{p^{r-1}} + p^{11} a_{p^{r-2}} \text{ if } p \text{ is a prime and } r \geq 2 \end{aligned} \tag{3.62}$$

These recursion formulas for the coefficients of the expansion were proved by Mordell [75] (essentially by using Hecke operators) and later by Hecke in a more general framework.

In the next section we discuss the Eichler-Shimura isomorphism (see 4.1.7) which in this special case it implies that for any prime p we have $a_p = \tau(p)$. Therefore we define the Ramanujan τ function by $\tau(n) = a_n$. With this definition of $\tau(n)$ Ramanujan proved the famous congruence $\tau(p) \equiv p^{11} + 1 \mod 691$.

Ramanujan also made the famous conjecture saying that for all primes p we have the inequality

$$\tau(p) \leq 2\, p^{\frac{11}{2}}$$

This inequality implies of course that for all primes p (and especially for $p = 2$) $\tau(p) \neq p^{11} + 1$ and this implies that any Hecke operator T_p provides a canonical splitting into eigenspaces $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{Q}) = H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{Q}) \oplus \mathbb{Q} f_{10}$. This is the simplest instance where the Manin-Drinfeld principle works.

Other congruences

It is easy to check that $H^1(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}})$ and $H^2_c(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}})$ do not have 5 or 7 torsion. Therefore we have (see Prop. 3.3.1,)

$$\mathbb{Z}/10\mathbb{Z}\epsilon_9(X,Y)\oplus\mathbb{Z}/5\mathbb{Z}\epsilon_4(X,Y)\oplus\mathbb{Z}/7\mathbb{Z}\epsilon_6(X,Y)\subset r(r^{-1}(\mathcal{T}_2))/\mathcal{T}_1 \qquad (3.63)$$

and this implies the well known congruences

$$\tau(p)\equiv p^{10}+p\equiv p^6+p^5 \mod 5;\ \tau(p)\equiv p^7+p^5 \mod 7 \qquad (3.64)$$

[105] [22] These congruences are called congruences of *local origin* whereas the congruence mod 691 is a congruence of *global origin.*

End of interlude

Reduction mod 691

We can go one step further and reduce mod 691. Since there is at most 2 torsion we get an exact sequence of Hecke-modules

$$0\to H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}\otimes\mathbb{F}_{691})\to H^1_{\mathrm{int}}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}\otimes\mathbb{F}_{691})\xrightarrow{r} H^1_{\mathrm{int}}(\partial(\Gamma\backslash\mathbb{H}),\tilde{\mathcal{M}}\otimes\mathbb{F}_{691})\to 0. \qquad (3.65)$$

The matrix giving the Hecke operator mod 691 becomes

$$T_2=\begin{pmatrix}667 & 369 & 0\\ 0 & 667 & 0\\ 0 & 0 & 667\end{pmatrix} \qquad (3.66)$$

This implies that the extension class $[\mathcal{X}_{10}\otimes\mathbb{F}_{691}]$ is a element of order 691, and hence 691 divides the order of $[\mathcal{X}_{10}]$ and hence divides the order of the denominator of the Eisenstein class.

Of course we may also consider the Hecke operator T_p then the corresponding matrix will be

$$T_p=\begin{pmatrix}p^{11}+1 & t^{(p)} & 0\\ 0 & \tau(p) & 0\\ 0 & 0 & \tau(p)\end{pmatrix} \qquad (3.67)$$

and we know that $(p^{11}+1-\tau(p))x=t^{(p)}$ has no solution with $x\in\mathbb{Z}_{(691)}$. But then it may happen that the above sequence (3.65) splits as T_p-module sequence, this happens exactly when we have $t^{(p)}\equiv 0 \mod 691$. But this implies that we have the stronger congruence

$$p^{11}+1-\tau(p)\equiv 0 \mod 691^2 \qquad (3.68)$$

On the other hand it is clear that the sequence splits as T_p module sequence if and only of this stronger congruence holds. For the curious reader we mention that this happens for $p=3559$ and for the first ten thousand primes it happens 13 times and 13 is roughly equal to 10000/691.

At the end of Chapter 5 we presume the result of Deligne which says that we have an action of the Galois group on $H^1_{\mathrm{int}}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}\otimes\mathbb{F}_{691})$. We will see that the structure of this cohomology as a module for the Hecke algebra has interesting consequences for this action (See Theorem 5.1.5).

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3.3.6 The general case

Now we describe the general case $\mathcal{M} = \mathcal{M}_n$ where n is an even integer. We define $\mathcal{M}^{\mathrm{tr}}$ as above, if $n/2$ is even, then we leave out the summand $X^{n/2}Y^{n/2}$, we get

$$\mathcal{M}^{\mathrm{tr}} = \mathcal{M}/\mathcal{M}^{<S>}.$$

This gives us for the cohomology and the restriction to the boundary cohomology

$$\begin{array}{ccc} H^1(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}) & \xrightarrow{\sim} & \mathcal{M}^{\mathrm{tr}}/\mathrm{Rel} \\ \downarrow & & \downarrow \\ H^1(\partial(\Gamma\backslash\mathbb{H}),\tilde{\mathcal{M}}) & \xrightarrow{\sim} & \mathcal{M}/(\mathrm{Id}-T)\mathcal{M}. \end{array} \tag{3.69}$$

We have the basis

$$e_n = \mathrm{trunc}(Y^n), e_{n-1} = \mathrm{trunc}(Y^{n-1}X), \dots, \begin{cases} Y^{n/2}X^{n/2} & n/2 \text{ odd} \\ 0 & \text{else} \end{cases}$$

for $\mathcal{M}^{\mathrm{tr}}$. Let us put $n_2 = n/2$ or $n/2-1$. Then the algorithm *Smithnormalform* provides a second basis $f_n = e_n, f_{n-1}, \dots, f_{n_2}$ such that the module of relations becomes

$$d_n f_n = 0, d_{n-1}f_{n-1} = 0, \dots, d_t f_t = 0, \dots, d_{n_2} f_{n_2} = 0$$

where $d_{n_2}|d_{n_2+1}|\dots|d_n$. We have $d_n = d_{n-1} = \cdots = d_{n-2s} = 0$ where $2s+1 = \dim H^1(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}})\otimes\mathbb{Q}$ and $d_{n-2s-1} \neq 0$.

Now we have written a computer program which for a given n gives us an explicit matrix for T_2, it is of the form

$$T_2(f_i) = \sum_{j=n}^{j=n_2} t_{i,j}^{(2)} f_j \tag{3.70}$$

where we have (the numeration of the rows and columns is downwards from n to n_2)

$$\begin{array}{c} t_{\nu,n}^{(2)} = 0 \text{ for } \nu < n \text{ and } t_{i,j}^{(2)} \in \mathrm{Hom}(\mathbb{Z}/(d_i),\mathbb{Z}/(d_j)) \\ \text{and } t_{i,j}^{(2)} = 0 \text{ for } i \geq n-2s, j < n-2s \end{array} \tag{3.71}$$

If we divide by the torsion we get for the restriction map to the boundary cohomology

$$H^1(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}})_{\mathrm{int}} = \bigoplus_{\nu=n}^{n-2s}\mathbb{Z}f_\nu \xrightarrow{r} H^1(\partial(\Gamma\backslash\mathbb{H}),\tilde{\mathcal{M}})_{\mathrm{int}} = \mathbb{Z}Y^n \tag{3.72}$$

where $f_n \mapsto Y^n$ and $T_2(Y^n) = (2^{n+1}+1)Y^n$. Now we will find that the endomorphism $T_2 - (2^{n+1}+1)\mathrm{Id}$ of $H^1_!(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}})_{\mathrm{int}}$ is injective..

Comment : We can verify this of course for any given n experimentally. But this assertion follows from the *Manin-Drinfeld principle.* This principle exploits the fact that we have estimates for the eigenvalues of T_2 (or more generally for

T_p on $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{C})$. These estimates say that for all primes p we always have the Ramanujan inequality

$$|\pi_f(T_p)| \leq p^{\frac{n+1}{2}} : \tag{3.73}$$

(This is a very deep theorem which has been proved by Deligne)

This implies that $2^{n+1}+1$ can not be an eigenvalue of T_2 on $H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}\otimes\mathbb{C})$ and this proves the injectivity. This implies that we can find a vector

$$\mathrm{Eis}_n = f_n + \sum_{\nu=n-1}^{\nu=n-2s} x_\nu f_\nu, \;\; x_\nu \in \mathbb{Q} \tag{3.74}$$

which is an eigenvector for T_2 i.e.

$$T_2(\;\mathrm{Eis}_n) = (2^{n+1}+1)\;\mathrm{Eis}_n. \tag{3.75}$$

The least common multiple $\Delta(n)$ of the denominators of the x_ν is the denominator of the Eisenstein class, it is the smallest positive integer for which

$$\Delta(n)\mathrm{Eis}_n \in H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})_{\mathrm{int}}. \tag{3.76}$$

This denominator is of great interest and our computer program allows us to compute it for any given not to large n. We simply have to compute the x_ν. We know that $T_2(f_n) = (2^{n+1}+1)f_n + \sum_{\mu=n-1}^{\mu=n-2s} t^{(2)}_{n,\mu} f_\mu$ and then the x_ν are the unique solution of

$$\sum_{\nu=n-1}^{\nu=n-2s} ((2^{n+1}+1)\delta_{\nu,\mu} - t^{(2)}_{\nu,\mu})x_\nu = t^{(2)}_{n,\mu}; \{\mu = n-1, \ldots, n-2s\} \tag{3.77}$$

With the help of H. Gangl we carried the computation of the x_ν and hence the $\Delta(n)$ and we found for some not too large values of n (roughly $n \leq 150$) that

$$\Delta(n) = \mathrm{numerator}(\zeta(-1-n)). \tag{3.78}$$

Here of course $\zeta(s)$ is the Riemann ζ function, it is well known that for any even positive integer n the value $\zeta(-1-n)$ is a rational number, hence it makes sense to speak of the numerator. A prime number p is called*irregular* if it divides the numerator of such a value of the Riemann ζ function. The most famous irregular prime is $p = 691$ we have

$$\zeta(-11) = \frac{691}{32760}.$$

Actually 3.78 is a theorem, we will give a proof in Chapter 5 (Theorem 5.1.2).

The reader might argue, why do you make such efforts to find out some experimental evidence for something you know to be true?

There are several reasons for doing this, but the main motivation is the following. The Theorem 5.1.2 is hopefully a special case of a much more general ensemble of assertions. The problem to determine (estimate) denominators of Eisenstein classes is ubiquitous in the cohomology of arithmetic groups. And we have many cases where we have conjectures relating these denominators to special values of L-functions. Some further examples will be discussed in Chapter 9 (See also [43]) But in many of these cases the methods to prove theorems like Theorem 5.1.2 fail.

On the other hand we explained in section 3.2.1 that in any given case we can compute the denominator -in principle- . Therefore it seems to be of interest to develop algorithms which compute the cohomology and the action of Hecke operators explicitly in given cases and verify or falsify these conjectures. A general strategy for such an algorithm has been outlined in section 3.2.1 and H. Gangl and I wrote a toy model program in the above case.

We are aware that these algorithms may become very slow for more general reductive groups, and it is very likely that we need clever new ideas to achieve this task. Finally I want to say that in many cases the resulting congruences have been checked for certain finite sets of primes (see also Chapter 9).

3.3.7 Localisation at a prime ℓ

We will see later the we should not consider the denominator of the Eisenstein class as a number but rather as an ideal. Hence we are only interested in the decomposition into prime ideals, i.e. for a prime ℓ we want to know the exact power of ℓ which divides $\Delta(n)$. To achieve this we replace in the considerations above the coefficient system $\tilde{\mathcal{M}}$ by $\tilde{\mathcal{M}}_{(\ell)} := \tilde{\mathcal{M}} \otimes \mathbb{Z}_{(\ell)}$, here $\mathbb{Z}_{(\ell)} \subset \mathbb{Q}$ is the local ring at ℓ. Then our cohomology modules will be finitely generated $\mathbb{Z}_{(\ell)}$-modules $H^\bullet(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{(\ell)})$.

ℓ -ordinary endomorphisms

In this subsection we fix a prime ℓ we consider finitely generated modules M over the local ring $\mathbb{Z}_{(\ell)} \subset \mathbb{Q}$. We consider such a module together with an endomorphism $\Phi : M \to M$. Then

Proposition 3.3.2. *We have a canonical decomposition into Φ submodules $M = M_{\,ord} \oplus M_{nilpt}$ where M_{nilpt} is the maximal submodule such that*

$$\bigcap_k \Phi^k(M_{nilpt}) = \{0\}$$

Proof. This is rather obvious if M is a finite, i.e. a torsion module. If M is a free $\mathbb{Z}_{(\ell)}$ module then we find a finite, normal extension $F/\mathbb{Q}$ such that $M \otimes F$ can be decomposed into generalised eigenspaces

$$M \otimes F = \bigoplus_\mu M[\mu] \ ; \ M[\mu] \neq 0$$

where $\mu \in \mathcal{O}_F \otimes \mathbb{Z}_{(\ell)}$, $M[\mu] := \{m \in M \otimes F \mid (\Phi - \mu \cdot \mathrm{Id})^k m = 0 \text{ for some } k > 0\}$ The Galois group acts on the set of eigenvalues μ. We consider the set of primes $\mathfrak{l}_1, \dots, \mathfrak{l}_\mathfrak{f} \subset \mathcal{O}_F$ which lie above ℓ, the Galois group $\mathrm{Gal}(F/\mathbb{Q})$ acts transitively on this set. We say that μ is ordinary if there is a prime $\mathfrak{l}_\mu$ such that $\mu \notin \mathfrak{l}_\mu$, the set of ordinary eigenvalues is invariant under the action of the Galois group. We get a decomposition into

$$M \otimes F = \bigoplus_{\mu \text{ ordinary}} M[\mu] \oplus \bigoplus_{\mu \text{ not ordinary}} M[\mu]$$

the two summands are invariant under the action of the Galois group. We put

$$M_{\text{ ord}} = \bigoplus_{\mu \text{ is a unit}} M[\mu] \cap M \text{ and } M_{\text{nilpt}} = \bigoplus_{\mu \text{ is not a unit}} M[\mu] \cap M.$$

Because of the Galois invariance it is clear that $M \otimes \mathbb{Q} = M_{\text{ ord}} \otimes \mathbb{Q} \oplus M_{\text{nilpt}} \otimes \mathbb{Q}$. But a little bit of semi-local algebra shows that actually $M = M_{\text{ ord}} \oplus M_{\text{nilpt}}$ and this decomposition has the desired properties. □

We call $M_{\text{ ord}}$ the ordinary part with respect to Φ and ℓ and we call $M_{\text{ ord}}$ an ℓ-ordinary Φ module. Of course the functor $M \to M_{\text{ ord}}$ is exact.

This has some nice consequences for our considerations above. Since the functor $X \to X_{\text{ int}}$ is not exact the surjectivity in (3.72) is problematic, because $H^2_c(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) \neq 0$. But if we localise our fundamental exact sequence

$$H^1_c(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{(\ell)}) \to H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{(\ell)}) \to H^1(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}_{(\ell)}) \to H^2_c(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}_{(\ell)}) \tag{3.79}$$

and choose for Φ the Hecke operator T_ℓ then it follows from our computations in section 3.3.1 that T_ℓ acts nilpotently on $H^2_c(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}_{(\ell)})$, and therefore $H^2_{c,\text{ ord}}(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}_{(\ell)}) = 0$. We get the exact sequence

$$H^1_{c,\text{ ord}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{(\ell)}) \to H^1_{\text{ord}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{(\ell)}) \to H^1_{\text{ord}}(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}_{(\ell)}) \to 0. \tag{3.80}$$

It follows from our earlier computations (prop. 3.3.1) that $H^1_{\text{ord}}(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}_{(\ell)}) = \mathbb{Z}_{(\ell)}[Y^n]$. Then we get for all Hecke operators $T_p[Y^n] = (p^{n+1}+1)[Y^n]$, we denote this Hecke-module by $\mathbb{Z}_{(\ell)}[n]$.

Now we can replace the sequence (3.72) by the above sequence if we want to study the power $\ell^{\delta_\ell(n)} = \Delta_\ell(n)$ in $\Delta(n)$.

The $\mathrm{Gl}_2/\mathbb{Z}$ module $\mathcal{M}_n$ contains the submodule

$$\mathcal{M}_n^\flat = \{\sum a_\nu \binom{n}{\nu} X^\nu Y^{n-\nu} \mid a_\nu \in \mathbb{Z}\} \tag{3.81}$$

(see 4.1.1) , this is actually the smallest submodule of $\mathcal{M}_n$ which contains X^n. Then we consider the cohomology $H^\bullet(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n^\flat)$ and again we can ask for the denominator of the Eisenstein class. Here the method of localising at ℓ provides a simple answer. We consider the exact sequence of coefficients

$$0 \to \tilde{\mathcal{M}}_n^\flat \otimes \mathbb{Z}_{(\ell)} \to \tilde{\mathcal{M}}_n \otimes \mathbb{Z}_{(\ell)} \to \tilde{\mathcal{M}}_n/\tilde{\mathcal{M}}_n^\flat \otimes \mathbb{Z}_{(\ell)} \to 0.$$

Now it follows easily from the definition that the Hecke operator T_ℓ acts nilpotently on the cohomology modules $H^\bullet(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n/\tilde{\mathcal{M}}_n^\flat \otimes \mathbb{Z}_{(\ell)})$ and hence we see that

$$H^1_{\text{ord},?}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n^\flat \otimes \mathbb{Z}_{(\ell)}) \xrightarrow{\sim} H^1_{\text{ord},?}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{Z}_{(\ell)}) \tag{3.82}$$

is an isomorphism. This implies that the denominator of the Eisenstein class does not depend on the choice of the coefficient system.

At this point it seems to be appropriate to use some homological algebra. We consider the exact sequence of modules for the Hecke algebra $\mathcal{H}$

$$\mathcal{X}_n := 0 \to H^1_{\mathrm{ord},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{Z}_{(\ell)}) \to H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{Z}_{(\ell)}) \quad \to \mathbb{Z}_{(\ell)}[n] \to 0. \tag{3.83}$$

We consider the sequence $\mathrm{Hom}_{\mathcal{H}}(\mathbb{Z}_{(\ell)}[n], \mathcal{X}_n)$ which is not exact anymore, this sequence yields a long exact sequence, we are interested in the boundary map $\boxed{\mathrm{Ext}}$

$$\begin{gathered} \to \mathrm{Hom}_{\mathcal{H}}(\mathbb{Z}_{(\ell)}[n], H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{Z}_{(\ell)}) \to \\ \to \mathrm{Hom}_{\mathcal{H}}(\mathbb{Z}_{(\ell)}[n], \mathbb{Z}_{(\ell)}[n]) \xrightarrow{\delta} \mathrm{Ext}^1_{\mathcal{H}}(\mathbb{Z}_{(\ell)}[n], H^1_{\mathrm{ord},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{Z}_{(\ell)})) \to \end{gathered} \tag{3.84}$$

It is clear that the boundary map δ maps the identity element $\mathbf{1} \in \mathrm{Hom}_{\mathcal{H}}(\mathbb{Z}_{(\ell)}[n], \mathbb{Z}_{(\ell)}[n])$ to an element of order $\Delta_\ell(n)$, in other words ∂_1 maps $\mathrm{Hom}_{\mathcal{H}}(\mathbb{Z}_{(\ell)}[n], \mathbb{Z}_{(\ell)}[n])$ to a cyclic subgroup of the Ext^1 of order $\Delta_\ell(n)$.

We introduce the Eisenstein ideal $\mathcal{IE} \subset \mathcal{H}$, this is the ideal which is generated the elements $(p^{n+1}+1)\mathrm{Id} - T_p$, where p runs through all primes. It is not difficult to see that

$$\begin{gathered} \mathrm{Ext}^1_{\mathcal{H}}(\mathbb{Z}_{(\ell)}[n], H^1_{\mathrm{ord},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{Z}_{(\ell)}) = \\ H^1_{\mathrm{ord},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{Z}_{(\ell)}) / \mathcal{IE} H^1_{\mathrm{ord},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{Z}_{(\ell)}). \end{gathered} \tag{3.85}$$

Now we choose a prime p and look at the sub algebra $\mathbb{Z}[T_p] \subset \mathcal{H}$ which is generated by the Hecke operator T_p. We consider the exact sequence (3.84) but we change the subscript $\mathcal{H}$ to $Z[T_p]$. As before the map

$$\mathrm{Hom}_{\mathbb{Z}[T_p]}(\mathbb{Z}_{(\ell)}[n], \mathbb{Z}_{(\ell)}[n]) \xrightarrow{\delta} \mathrm{Ext}^1_{\mathbb{Z}[T_p]]}(\mathbb{Z}_{(\ell)}[n], H^1_{\mathrm{ord},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{Z}_{(\ell)})) \tag{3.86}$$

maps the identity element $\mathbf{1}$ to an element to an element of order $\Delta_\ell(n)$. But now it follows from the definition of δ_1 that

$$\delta(\mathbf{1}) = \mathbf{t}_n^{(p)} = \{\dots, t_{n,\nu}^{(p)}, \dots\}_\nu \mod (\mathrm{Id}(p^{n+1}+1) - T_p) H^1_{\mathrm{ord},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{Z}_{(\ell)})) \tag{3.87}$$

Hence we see that we simply have to compute the order of $\mathbf{t}_n^{(p)}$ in $\mathrm{Ext}^1_{\mathbb{Z}[T_p]]}\mathbb{Z}_{(\ell)}[n], H^1_{\mathrm{ord},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{Z}_{(\ell)}))$ for just one prime p.

3.3.8 Computing mod ℓ

Of course the coefficients $t_{\nu,\mu}^{(p)}$ will become very large if n or p becomes larger, hence we can verify (3.78) only in a very small range of degrees n. But if we only want to verify that $\ell|\Delta_\ell(n)$ then it is sufficient to compute the coefficients $t^{(p)}_{\nu,\mu}$ modulo ℓ and to check whether $\mathbf{t}^{(p)}_n$ represents a non zero class in $\mathrm{Ext}^1_{\mathbb{Z}[T_p]]}(\mathbb{F}_\ell[n], H^1_{\mathrm{ord},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{F}_\ell)$. Hence we see: We have $\ell|\Delta_\ell(n)$ if for some choice of p the equation

$$\sum_{\nu=n-1}^{\nu\equiv n-2s}((p^{n+1}+1)\delta_{\nu,\mu} - t^{(p)}_{\nu,\mu})x_\nu \equiv t^{(p)}_{n,\mu} \mod \ell \tag{3.88}$$

has no solution. But now the coefficients are elements in $\mathbb{F}_\ell$ and this reduces the computational complexity considerably.

We have to be careful, it may and will happen that $t^{(p)}_{n,\mu} \mod \ell$ is zero (for some values of p) but still $\ell|\Delta_\ell(n)$.

higher

Higher powers of ℓ

This reasoning can also be applied if we look at higher powers of ℓ dividing a numerator of a $\zeta(-1-n))$. Let us assume that $\ell^{\delta_\ell(n)}||\mathrm{numerator}(\zeta(-1-n))$. We have to show that $\ell^{\delta_\ell(n)}$ divides the lcm of the denominators of the x_ν in equation (3.77). If we assume that $\mathbf{t}^{(p)}_n$ is not zero modulo ℓ then this follows if we show that the equation

$$\sum_{\nu=n-1}^{\nu\equiv n-2s}((p^{n+1}+1)\delta_{\nu,\mu} - t^{(p)}_{\nu,\mu})x_\nu \equiv \ell^{\delta_\ell(n)-1}t^{(2)}_{n,\mu} \mod \ell^{\delta_\ell(n)} \tag{3.89}$$

has no solution in $\mathbb{Z}/(\ell^{\delta_\ell(n)})$. Then the class

$$[\mathcal{X}_n \otimes \mathbb{Z}/\ell^{\delta_\ell(n)}\mathbb{Z}] \in \mathrm{Ext}^1_{\mathcal{H}}((\mathbb{Z}/\ell^{\delta_\ell(n)}\mathbb{Z})(-1-n), H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes (\mathbb{Z}/\ell^{\delta_\ell(n)}\mathbb{Z}))$$

has exact order $\ell^{\delta_\ell(n)}$.

If ℓ is an irregular prime, then there is always an even positive integer n_0 such that $n_0 < p-1$ such that $\ell|\zeta(-1-n_0)$. One does not know any pair(ℓ, n) with $n_0 < \ell - 1$ such that we even have $\ell^2|\zeta(-1-n_0)$. But if we drop the assumption $n < \ell - 1$ then we may find arbitrary high powers of ℓ dividing $\zeta(-1-n)$ (See also section 3.3.11) We have some examples

$$\zeta(-31) \equiv 0 \mod 37;\ \zeta(-283) \equiv 0 \mod 37^2;$$

$$\zeta(-37579) \equiv 0 \mod 37^3;\ \zeta(-1072543) \equiv 0 \mod 37^4; \dots$$

$$\zeta(-43) \equiv 0 \mod 59;\ \zeta(-913) \equiv 0 \mod 59^2$$

$$\zeta(-23) \equiv 0 \mod 103\ ;\zeta(-227) \equiv 0 \mod 103^2$$

We verified (3.78) in the cases $(37, 282), (59, 912), (103, 226)$ using our program with Gangl. The case $(59, 912)$ used roughly 18 hours, our algorithm becomes very slow if n becomes large.

3.3.9 The denominator and the congruences

For the following we assume that (3.78) is correct. We discuss the denominator of the Eisenstein class in this special case. In [Talk-Lille] this is discussed in a more abstract way, so here we treat basically the simplest example of 4.3 in [Talk-Lille]. Remember that in this section $\tilde{\mathcal{M}} = \tilde{\mathcal{M}}_n^\flat$, or $\tilde{\mathcal{M}} = \tilde{\mathcal{M}}_n$ for some even positive integer n.

The fundamental exact sequence provides the short exact sequence fuex

$$0 \to H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}) \to H^1_{\mathrm{int}}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}) \xrightarrow{r} H^1_{\mathrm{int}}(\partial(\Gamma\backslash\mathbb{H}),\tilde{\mathcal{M}}) \to 0 \tag{3.90}$$

It is clear that the restriction map r is surjective because it is surjective if we localise at primes. We have $H^1_{\mathrm{int}}(\partial(\Gamma\backslash\mathbb{H}),\tilde{\mathcal{M}}) = \mathbb{Z}e_n$ and $T_2(e_n) = (2^{n+1}+1)e_n$. We get a saturated decomposition into Hecke modules

$$H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}) \oplus \mathbb{Z}\tilde{e}_n \subset H^1_{\mathrm{int}}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}) \tag{3.91}$$

where $T_2\tilde{e}_n = (2^{n+1}+1)\tilde{e}_n$ and $r(\tilde{e}_n) = \Delta(n)e_n$ and

$$H^1_{\mathrm{int}}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}})/(H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}) \oplus \mathbb{Z}\tilde{e}_n) = \mathbb{Z}/\Delta(n)\mathbb{Z}. \tag{3.92}$$

If $e_n^\dagger \in H^1_{\mathrm{int}}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}})$ maps to e_n then we can write

$$e_n^\dagger = r(\frac{y' + \tilde{e}_n}{\Delta(n)}) \tag{3.93}$$

and the element $y' \in H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}})$ is unique up to an element in $\Delta(n)H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}})$.

Hence

Theorem 3.3.1. *The Hecke module $H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}) \otimes \mathbb{Z}/\Delta(n)\mathbb{Z}$ contains a cyclic submodule $\mathbb{Z}/\Delta(n)\mathbb{Z}(-1-n)$ on which for all primes p the Hecke operator T_p acts by the eigenvalue $p^{n+1}+1 \mod \Delta(n)$*

Proof. The submodule is simply the cyclic submodule generated by y'. □

We discuss some consequences of this theorem. We anticipate some results from the following chapter and from chapter 5. These results can be formulated in terms of the concepts and the language we used up to here, but the proofs require tools from analysis.

We mentioned already the theorem that the cohomology $H^1_!(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}\otimes\mathbb{Q})$ is semi-simple as module for the Hecke-algebra(Thm.3.1.1). This theorem implies that we can find a finite normal field extension $F/\mathbb{Q}$ such that we have an isotypical decomposition (see3.16) decoFint

$$H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}) \otimes F = \bigoplus_{\pi_f} H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}\otimes F)(\pi_f). \tag{3.94}$$

Here π_f runs over a finite set of homomorphisms $\pi_f : \mathcal{H} \to \mathcal{O}_F$. We also have the action of the complex conjugation on the cohomology (See sect. (**??**)). The complex conjugation commutes with the action of the Hecke algebra and under this action each eigenspace decomposes into a + and a −eigenspace. In the following chapter 4 we will prove the famous multiplicity one theorem which says that the spaces $H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}\otimes F)(\pi_f)_\pm$ are one dimensional. Let us denote the set of $\pi_f : \mathcal{H}\to\mathcal{O}_F$ which occur with positive multiplicity (then 2) in the above decomposition by $\mathrm{Coh}_!(n)$.

We know that

$$H^1(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}})\otimes F = H^1_!(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}})\otimes F\oplus Fe_n$$

where $T_pe_n = (p^{n+1}+1)e_n$. Let $\pi_f^{\mathrm{Eis}} : \mathcal{H}\to\mathbb{Z}$ be the homomorphism $\pi_f^{\mathrm{Eis}} : T_p\to p^{n+1}+1$, then $\mathrm{Coh}(n)=\mathrm{Coh}_!(n)\cup\{\pi_f^{\mathrm{Eis}}\}$.

We make a list $\{\pi_{1,f},\dots,\pi_{r,f}\}$ of the elements in $\mathrm{Coh}_!(n)$. . This decomposition induces a Jordan-Hölder filtration on the integral cohomology JH

$$(0)\subset\mathcal{JH}^{(1)}H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_{\mathcal{O}_F})\subset\mathcal{JH}^{(2)}H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_{\mathcal{O}_F})\subset\cdots\subset\mathcal{JH}^{(r)}H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_{\mathcal{O}_F}) \tag{3.95}$$

Here the first step $\mathcal{JH}^{(1)}H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_{\mathcal{O}_F}) = H^1(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}\otimes|\mathcal{O}_F)(\pi_{1,f})$, where the subquotients a locally free $\mathcal{O}_F$ modules of rank 2 and after tensoring with F they become isomorphic to the corresponding $\pi_{j,f}$ eigenspace.

We choose a prime ℓ which divides $\Delta(n)$, let $\ell^{\delta_\ell(n)}||\Delta(n)$. Let $\mathfrak{l}$ be a prime in $\mathcal{O}_F$ which lies above ℓ. If e_ℓ is the ramification index then we have

$$\{0\}\subset\mathcal{O}_F/\mathfrak{l}^{e_\ell\delta_\ell(n)}(-1-n)\subset H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_{\mathcal{O}_F})\otimes\mathcal{O}_F/\mathfrak{l}^{e_\ell\delta_\ell(n)} \tag{3.96}$$

The above Jordan-Hölder filtration induces a Jordan-Hölder filtration on the cohomology $\mod \mathfrak{l}^{e_\ell\delta_\ell(n)}$ we have JHmod

$$\{0\}\subset\mathcal{JH}^{(1)}H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_{\mathcal{O}_F})\otimes\mathcal{O}_F/\mathfrak{l}^{e_p\delta_\ell(n)}\subset\mathcal{JH}^{(2)}\dots \tag{3.97}$$

where again the successive subquotients $\overline{\mathcal{JH}^{(\nu)}}H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_{\mathcal{O}_F})$ are free $\mathcal{O}_F/\mathfrak{l}^{e_p\delta_\ell(n)}$ modules of rank 2.

cong1

Theorem 3.3.2. *We can find* $\pi_{f,1},\pi_{f,2}\dots,\pi_{f,r}$ *and numbers* $f_1>0, f_2>0,\dots,f_r>0$ *in the above filtration such that* $\sum f_i = e_\ell\delta_\ell(n)$ *and we have the congruence*

$$\pi_{f,i}(T_p)\equiv p^{n+1}+1 \mod \mathfrak{l}^{f_i} \tag{3.98}$$

for all primes p.

Proof. We look at the map from our cyclic submodule into the top Jordan-Hölder quotient

$$\mathcal{O}_F/\mathfrak{l}^{e_\ell\delta_\ell(n)}(-1-n)\to\overline{\mathcal{JH}^{(r)}}H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_{\mathcal{O}_F}) \tag{3.99}$$

This map has a kernel $\mathfrak{l}^{g_r}$ and the image in the Jordan-Hölder quotient is the cyclic sub module $\mathfrak{l}^{e_\ell\delta_\ell(n)-g_r}=\mathfrak{l}^{f_r}$. The Hecke operator T_p acts on the Jordan-Hölder quotient quotient by multiplication by $\pi_{f,r}(T_p)$ and on the cyclic submodule by multiplication by $p^{n+1}+1$. Hence we get $\pi_{f,r}(T_p)\equiv p^{n+1}+1 \mod \mathfrak{l}^{f_r}$ Now we get an embedding $\mathfrak{l}^{g_r}\hookrightarrow\mathcal{JH}^{(r-1)}H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_{\mathcal{O}_F})\otimes\mathcal{O}_F/\mathfrak{l}^{e_\ell\delta_\ell(n)}$ and we apply the same reasoning to this embedding. This process stops if the embedded cyclic sub module becomes trivial. This proves the claim. □

But now we have to be aware that the Jordan-Hölder filtration does not split, if we define

$$H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_{\mathcal{O}_F})(\pi_f)=H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_{\mathcal{O}_F})\cap H^1(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}\otimes F)(\pi_f)$$

then we get a saturated decomposition (decomposition up to isogeny) satdeco

$$H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_{\mathcal{O}_F})\supset\bigoplus_{\pi_f\in\mathrm{Coh}_!(n)}H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_{\mathcal{O}_F})(\pi_f) \tag{3.100}$$

Here we encounter another interesting problem:

What can we say about the structure of the quotient if we divide the left hand side by the right hand side.

We can formulate some more or less plausible assertions which we can verify experimentally, but which are very difficult to prove. We definitely have to use methods which go far beyond the very elementary tools we used so far. For instance we verified experimentally (3.78) for a certain range of values of n but our proof in Chapter 5 requires some analysis.

in the following we choose a prime p, the role of the two primes p and ℓ will be exchanged. In a first step we consider the cohomology mod p we are mainly interested in the ordinary part. We start from the exact sequence of Γ modules

$$0\to\mathcal{M}_n\xrightarrow{\times p}\mathcal{M}_n\to\mathcal{M}_n\otimes\mathbb{F}_p\to 0. \tag{3.101}$$

Here we want to assume that $p>3$ then we get the resulting exact sequence of sheaves and hence a long exact sequence of cohomology groups

$$\begin{gathered}0\to(\mathcal{M}_n^\Gamma)_{\mathrm{\ ord}}\xrightarrow{\times p}(\mathcal{M}_n^\Gamma)_{\mathrm{\ ord}}\to(\mathcal{M}_n\otimes\mathbb{F}_p)_{\mathrm{\ ord}}\to\\ \to H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H},\mathcal{M}_n)\xrightarrow{\times p}H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H},\mathcal{M}_n)\to H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H},\mathcal{M}_n\otimes\mathbb{F}_p)\to 0\end{gathered} \tag{3.102}$$

and we can break this sequence into pieces

$$0\to(\mathcal{M}_n^\Gamma)_{\mathrm{\ ord}}\xrightarrow{\times p}(\mathcal{M}_n^\Gamma)_{\mathrm{\ ord}}\to(\mathcal{M}_n\otimes\mathbb{F}_p)^\Gamma_{\mathrm{ord}}\to H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H},\mathcal{M}_n)[p]\to 0 \tag{3.103}$$

and

$$0\to H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H},\mathcal{M}_n)[p]\to H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H},\mathcal{M}_n)\xrightarrow{\times p}H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H},\mathcal{M}_n)\to H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H},\mathcal{M}_n\otimes\mathbb{F}_p)\to 0 \tag{3.104}$$

where of course $\dots[p]$ means kernel of the multiplication by p and the far most 0 on the right is the vanishing of H^2.

We analyse these two sequences and get

ordtorfree

Theorem 3.3.3. *The cohomology $H^1_{ord}(\Gamma\backslash\mathbb{H},\mathcal{M}_n)$ is p- torsion free unless we have $n>0$ and $n\equiv 0 \mod p(p-1)$. The cohomology groups $H^1_{c,\ ord}(\Gamma\backslash\mathbb{H},\mathcal{M}_n)$ are always torsion free and $H^2_{c,\ ord}(\Gamma\backslash\mathbb{H},\mathcal{M}_n)=0$*

Proof. We consider the polynomial ring in two variables $\mathbb{F}_p[X,Y]$. On this ring we have the action of $\mathrm{Sl}_2(\mathbb{Z})$. It is an old theorem of L.E. Dickson that the ring of invariants is generated by the two polynomials

$$f_1 = X^pY - XY^p \text{ and } f_2 = \frac{X^{p^2-1} - Y^{p^2-1}}{X^{p-1} - Y^{p-1}} = X^{(p-1)p} + X^{(p-1)(p-1)}Y^{p-1} + \dots \tag{3.105}$$

Now every element in $(\mathcal{M}_n \otimes \mathbb{F}_p)^{\Gamma}_{\mathrm{ord}}$ is a sum of monomials $f_1^a f_2^b$ where $a(p+1) + bp(p-1) = n$. We see that

$$u_{\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}} = u_\alpha : \mathcal{M}_n^{(\alpha)} \to \mathcal{M}_n$$

multiplies f_1 with a multiple of p and hence we see that all the monomials with $a > 0$ are multiplied by a multiple of p. This means that $(\mathcal{M}_n \otimes \mathbb{F}_p)^{\Gamma}_{\mathrm{ord}} \neq 0$ if and only if $n = bp(p-1)$. If $n = 0$ the map $\mathcal{M}_n^\Gamma = \mathbb{Z}_p \to (\mathcal{M}_n \otimes \mathbb{F}_p)^\Gamma$ is surjective if $n > 0$ we have $\mathcal{M}_n^\Gamma = 0$ and hence the theorem.

For the assertions concerning the compactly supported cohomology we have to recall that $H^2_c(\Gamma\backslash\mathbb{H}, \mathcal{M}_n) = (\mathcal{M}_n)_\Gamma = \mathcal{M}_n/I_\Gamma\mathcal{M}_n$ [book vol I, section 2 and 4.8.5]. We check easily that $X^n, Y^n \in I_\Gamma\mathcal{M}_n$ and the assertion is clear. □

We now briefly discuss some interesting questions concerning the cohomology groups $H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \mathcal{M}_n \otimes \mathbb{F}_p)$. We assume that we are not in the exceptional case that $n \equiv 0 \mod p(p-1)$ hence we know that

$$H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \mathcal{M}_n \otimes \mathbb{F}_p) = H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \mathcal{M}_n) \otimes \mathbb{F}_p \tag{3.106}$$

We can find a finite extension $\mathbb{F}_{p^r}/\mathbb{F}_p$ such that decomodp

$$H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \mathcal{M}_n \otimes \mathbb{F}_{p^r}) = \bigoplus_{\bar{\pi_f}} H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \mathcal{M}_n \otimes \mathbb{F}_{p^r})\{\bar{\pi_f}\} \tag{3.107}$$

where $\bar{\pi}_f : \mathcal{H} \to \mathbb{F}_{p^r}$ is a homomorphism and

$$H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \mathcal{M}_n \otimes \mathbb{F}_{p^r})\{\bar{\pi}_f\} = \{x \in H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \mathcal{M}_n \otimes \mathbb{F}_{p^r}) \mid (T_\ell - \bar{\pi}_f(T_\ell))^N x = 0\} \tag{3.108}$$

is a generalised eigenspace. Such an generalised eigenspace has a *socle*, this is the space

$$H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \mathcal{M}_n \otimes \mathbb{F}_{p^r})(\pi_f) = \{x \in H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \mathcal{M}_n \otimes \mathbb{F}_{p^r}) \mid (T_\ell - \bar{\pi}_f(T_\ell))x = 0\} \tag{3.109}$$

(Note the difference between ( ) and { }.) We know that the inner cohomology $H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{Q})$ is a semi simple Hecke module, but we can not expect anymore that the inner cohomology $H^1_{\mathrm{ord},!}(\Gamma\backslash\mathbb{H}, \mathcal{M}_n \otimes \mathbb{F}_{p^r})$ is semi-simple. For the restriction oft the decomposition (3.107) to the inner cohomology semi simplicity means that the generalised eigenspaces are always equal to their socle.

We choose a prime $\mathfrak{p} \subset \mathcal{O}_F$ above p and assume $\mathcal{O}_F/\mathfrak{p} = \mathbb{F}_{p^r}$. Let $F_\mathfrak{p}$ be the completion of F at $\mathfrak{p}$, let $\mathcal{O}_\mathfrak{p}$ its ring of integers. We consider the reduction maps $\boxed{\text{redukdiag}}$

$$\begin{array}{ll} H^1_{!,\ \mathrm{ord}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathcal{O}_\mathfrak{p}) & \to H^1_{!,\ \mathrm{ord}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{F}_{p^r}) \\ \downarrow & \downarrow \\ H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathcal{O}_\mathfrak{p}) & \to H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{F}_{p^r}) \end{array} \tag{3.110}$$

under this map an eigenspace maps into the socle,, i.e.

$$r_{\pi_f} : H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathcal{O}_\mathfrak{p})(\pi_f) \to H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{F}_{p^r})(\bar{\pi}_f). \tag{3.111}$$

The image $r_{\pi_f}(H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathcal{O}_\mathfrak{p})(\pi_f))$ is a $\mathbb{F}_{p^r}$ vector space of dimension 2.

We are interested in the fibres of the surjective map

$$R_\mathfrak{p} : \mathrm{Coh}_!(n) \to \mathrm{Cohmodp}_!(n)\ ;\ \pi_f \mapsto \bar{\pi}_f$$

For a subset $\Sigma \subset \{R_\mathfrak{p}^{-1}\bar{\pi}_f\}$ we define

$$\begin{gathered} H^1_{!,\ \mathrm{ord}}(\Gamma\backslash\mathbb{H}, \mathcal{M}_n \otimes \mathcal{O}_\mathfrak{p})\{\Sigma\} = \\ (\bigoplus_{\pi_f \in \Sigma} H^1_{!,\ \mathrm{ord}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes F)(\pi_f)) \cap H^1_{!\,ord}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathcal{O}_\mathfrak{p}). \end{gathered} \tag{3.112}$$

Given Σ we put $\Sigma' = R_\mathfrak{p}^{-1}(\bar{\pi}_f) \setminus \Sigma$ and we say that Σ (or Σ') is *closed* if we get a direct sum decomposition

$$H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathcal{O}_\mathfrak{p}) = H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathcal{O}_\mathfrak{p})\{\Sigma\} \oplus H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathcal{O}_\mathfrak{p})\{\Sigma'\} \tag{3.113}$$

and consequently we say that the fibre $R_\mathfrak{p}^{-1}(\bar{\pi}_f)$ (or simply $\bar{\pi}_f$) is *connected* if $\emptyset$ and the fibre itself are the only closed subsets. If π_f and $\pi'_f \in R_p^{-1}(\bar{\pi}_f)$ then we say that they are *inner congruent* . We have an easy proposition

Proposition 3.3.3. *If $R_\mathfrak{p}^{-1}(\bar{\pi}_f)$ is not connected, then the dimension of the socle $H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{F}_{p^r})^{(1)}\{\bar{\pi}_f\}$ over $\mathbb{F}_{p^r}$ is ≥ 4*

Proof. This is rather clear. If we have a non trivial direct sum decomposition as above then we also get a direct sum decomposition

$$H^1_{!,\ \mathrm{ord}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n) \otimes \mathbb{F}_{p^r} = H^1_{!,\ \mathrm{ord}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n) \otimes \mathbb{F}_{p^r}\{\Sigma\} \oplus H^1_{!,\ \mathrm{ord}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n) \otimes \mathbb{F}_{p^r}\{\Sigma'\} \tag{3.114}$$

Now any $\pi_f \in \Sigma$ (resp.$\pi'_f \in \Sigma'$) provides provides a two dimensional $\mathbb{F}_{p^r}$ vector space $r_{\pi_f}(H^1_{!,\ \mathrm{ord}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathcal{O}_F)(\pi_f))$ (resp $r_{\pi'_f}(H^1_{!,\ \mathrm{ord}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathcal{O}_F)(\pi'_f))$. These two vector spaces lie in the socle and in two different summands. $\square$

We say that $\bar{\pi}_f$ occurs *with weak multiplicity one* if the dimension of the socle $\dim H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{F}_{p^r})(\bar{\pi}_f) = 2$, this socle is the direct sum of the $\pm$ eigenspaces under the complex conjugation. Our above theorem implies that then the fibre of $\bar{\pi}_f$ must be connected.

The (plain) multiplicity of $\bar{\pi}_f$ is just the number of elements in the fibre $R_{\mathfrak{p}}^{-1}(\bar{\pi}_f)$ this is the number $m(\bar{\pi}_f) = \frac{1}{2} \dim H_!^(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{F}_2)\{\bar{\pi}_f\}$. One might expect that a $\bar{\pi}_f$ occurs with weak multiplicity one happens "very frequently". But it seems to be very difficult to say something substantial in this direction. We will come back to this issue.(See discussion of the Wieferich dilemma).

We are especially interested in the case of the Eisenstein homomorphism $\bar{\pi}_f^{\mathrm{Eis}} : T_\ell \to \ell^{n+1} + 1 \mod p$. It certainly occurs in the cohomology $\mod p$ and it follows from theorems 3.3.1 and 5.1.2 that $\bar{\pi}_f^{\mathrm{Eis}}$ occurs in Cohnodp$_!(n)$ if $p|\zeta(-1-n)$.

Here it is very tempting to ask whether or not $\bar{\pi}_f^{\mathrm{Eis}}$ *always occurs with weak multiplicity one in the inner cohomology.*

This question can be checked experimentally. For any prime ℓ we look at the operator T_ℓ on $H_!^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{F}_p)$. We compute the characteristic polynomial

$$P_\ell(X) = \det(X\mathrm{Id} - ((\ell^{n+1}+1)\mathrm{Id} - T_\ell) \mid H^1_{\mathrm{ord},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n) \otimes \mathbb{F}_p) = Q_\ell(X)^2 \tag{3.115}$$

The characteristic polynomial is a square because the $\pm$ eigenspaces are isomorphic as Hecke modules. Our computer program with Gangl gives us an explicit expression for $Q_2(X)$ for a large number of pairs (n,p). We are interested in pairs (n,p) with $p \mid \zeta(-1-n)$. Then we find

$$Q_2(X) = a_1(n,p)X + a_2(n,p)X^2 \dots \tag{3.116}$$

Then we found $a_1(n,p) \not\equiv 0 \mod p$ for $n \leq 200$. This means that in these cases $\bar{\pi}_f^{\mathrm{Eis}}$ occurs with multiplicity one. This is in no way surprising, we expect that $p|\zeta(-1-n)$ and $p|a_1(n,p)$ will be a very rare event. Tobias Berger drew my attention to the paper [2] in which the authors consider the same problem in a slightly different context. They show that $p|a_1(n,p)$ happens only once for $p < 10^5$ and this is the case $p = 547, n = 484$.

Assume we have such a pair (p,n) and we know that in addition that $p \not| a_2(n,p)$. Then the Hecke operator $T_\ell^{\mathrm{Eis}} = T_\ell - (\ell^{n+1}+1)\mathrm{Id}$ acts nilpotently on the 4 dimensional space $H^1_{\mathrm{ord},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n) \otimes \mathbb{F}_p)\{\bar{\pi}_f^{\mathrm{Eis}}\}$ and we can ask the next question:

Is T_ℓ^{Eis} *the zero operator? (For all choices of* ℓ*). Then this means that* $\bar{\pi}_f^{\mathrm{Eis}}$ *has weak multiplicity 2.*

If we restrict T_ℓ^{Eis} to one of the two-dimensional $\pm$ subspaces then T_ℓ^{Eis} is a nilpotent endomorphism, i.e. a nilpotent (2,2) matrix with entries in $\mathbb{F}_p$. If I am not mistaken then there are exactly p^2 such matrices and the zero matrix is just one of them. So one might argue that the probability for $T_\ell^{\mathrm{Eis}} = 0$ is roughly $\frac{1}{p^2}$. So with a high probability the answer to the above question is NO. Since the probability that $p|a_1(n,p)$ is also very small it may be safe to conjecture that $\bar{\pi}_f^{\mathrm{Eis}}$ always occurs weakly with multiplicity one.

Of course I checked the case $(484, 547)$ for the operator T_2 and indeed the answer was NO!

We briefly return to theorem 3.3.2. Assume we have a pair (n,p) with $p^\delta|\zeta(-1-n)$ and in addition $p|a_1(n,p)$, we also assume $p \not| a_2(n,p)$, Then we expect that $\bar{\pi}_f^{\text{Eis}}$ occurs weakly with multiplicity one. Let $F_{\mathfrak{p}}/\mathbb{Q}_p$ a smallest extension such that we get a decomposition

$$H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes F_{\mathfrak{p}}) = H^1_1(\mathbb{H}, \tilde{\mathcal{M}}_n \otimes F_{\mathfrak{p}})(\pi_{f,1}) \oplus H^1_!(\mathbb{H}, \tilde{\mathcal{M}}_n \otimes F_{\mathfrak{p}})(\pi_{f,2}).$$

This extension is either trivial or a ramified quadratic extension of $\mathbb{Q}_p$. The probability that we are in the first case is again very low, so let us assume that $F_{\mathfrak{p}} = \mathbb{Q}_p[\sqrt{p}]$ We have the inclusion $j : \mathbb{Z}/p^\delta(-n-1) \hookrightarrow H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{Z}_p/p^\delta)$, The filtration has 2-steps

$$\{0\} \subset H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{Z}_p/p^\delta)(\pi_{f,1}) \subset H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{Z}_p/p^\delta) \tag{3.117}$$

and let f_1 the smallest integer such that $p^{f_1}\mathbb{Z}/p^\delta(-n-1) \hookrightarrow H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{Z}_p)(\pi_{f,1})$. Then our previous argument yields the congruence $\pi_{f,1}(T_\ell) \equiv \ell^{n+1}+1 \mod p^{\delta-f_1}$. Then the inclusion j yields the cyclic submodule $\mathbb{Z}/p^{f1}$ in the quotient. This yields the congruence $\pi_{f,2}(T_\ell) \equiv \ell^{n+1}+1 \mod p^{f_1}$ Now we invoke our assumption that $\bar{\pi}_f$ occurs weakly with multiplicity one and this implies that we can find a Hecke operator $T^{\text{Eis}}_{\ell_1}$ which maps the above cyclic module $\mathbb{Z}/p^{f1}$ into $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{Z}_p/p^\delta)(\pi_{f,1})$ and hence to the cyclic submodule $p^{f_1}Z/p^\delta \subset \mathbb{Z}/p^\delta)$ Hence we can conclude that $2f_1 \leq \delta$.

Hence we see we see that under the above assumptions we have a congruence

$$\pi_{f,1}(T_\ell) \equiv \ell^{n+1} + 1 \mod \mathfrak{p}^\delta$$

Of course the same applies to $\pi_{f,2}$.

With a little bit of luck we can check the assumptions using the explicit computation of T_2.

It is not very difficult to produce examples of $\bar{\pi}_f \in \text{Coh}_!(n)$ where we have $\dim H^1_{\text{ord},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{F}_{p^r})\{\bar{\pi}_f\} > 2$. If we take $n = 22$, then we know that $H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{12}\otimes\mathbb{Q})$ is of dimension four, then the $\pm$ eigenspaces are of dimension 2. We can compute T_2 and find for the smallest field that decomposes the cohomology (see 3.94) $F = \mathbb{Q}(\sqrt{144169})$, this was of course known to Hecke. In this case $\text{Coh}_!(22)$ consists of 2 elements, which are conjugate under the Galois group $\text{Gal}(F/\mathbb{Q})$. If π_f is one of the elements in $\text{Coh}_!(22)$ then our program yields $\pi_f(T_2) = -12(-45 + \sqrt{144169})$ (of course this value can be looked up in any table for modular forms) hence we see that for any prime $p > 3$ the image of ring $\mathbb{Z}_{(p)}[T_2] = \mathcal{O}_F \otimes \mathbb{Z}_{(p)}$. This implies that we do not have inner congruences except for $p = 144169$.

But now our program with Gangl provides an explicit matrix for $T_2 \mod 144169$ and this matrix has only one eigenvalue $\mod 144169$.

This matrix for T_2 *is not a diagonal matrix, and this implies that the fibre is connected and hence* π_f *occurs weakly with multiplicity one.*

Essentially the same happens if we look at the six values $n = 22, 26, 28, 30, 32, 36$ for which the degree of the splitting field F is 2. It will happen that the discriminant is not a prime, so we will have inner congruences modulo several primes.

3.3.10 L-values, weak multiplicity one and connectedness

in the previous section we investigated questions regarding the structure of the integral cohomology as module for the Hecke algebra. We discussed the denominator of the Eisenstein classes and our experimental data suggested an answer in terms of special values of L-functions. (See 3.78 and also (5.1.2).)

We briefly mention some results about the questions concerning inner congruences which we contemplated in the previous section. To state these results we have to anticipate the notion of $L-$ functions attached to a $\pi_f \in \mathrm{Coh}_!(n)$ and we anticipate the theorems on special values. We still localise at a prime p and we only look at the ordinary part. We apply the results from section 5.1.2 .

Assume we have two elements $\pi_f, \pi'_f \in R_{\mathfrak{p}}^{-1}(\bar{\pi}_f)$ We say that these two elements are *linked* if they lie in the same connected component. I refer to Theorem 5.1.1

Theorem 3.3.4. *The Hecke modules $\pi_f, \pi'_f \in R_{\mathfrak{p}}^{-1}(\bar{\pi}_f)$ are linked if and only if*

$$\frac{Z(\nu)}{\Omega(\epsilon(\nu)\times\pi_f)}\Lambda^{\mathrm{coh}}(\pi, n+1-\nu) \equiv \frac{Z(\nu)}{\Omega(\epsilon(\nu)\times\pi'_f)}\Lambda^{\mathrm{coh}}(\pi', n+1-\nu) \mod \mathfrak{p} \tag{3.118}$$

for all $\nu = 0, 1, \ldots, n$ and where $Z(\nu) = 1$ for $\nu \neq 0, n$ and $Z(0) = Z(n) =$ numerator$(\zeta(-1-n))$ and $\epsilon(\nu) = \pm$ depending on the parity of ν.

(The factor $Z(\nu)$ is needed to make the expressions integers) This is a slightly strengthened version of a theorem of Vatsal ([107]). We do not prove it in this book, it is not to difficult to prove using the results in ...

We have a second theorem which is due to Hida. We pick a $\pi_f \in R_{\mathfrak{p}}^{-1}(\bar{\pi}_f)$ and we say that π_f is *isolated* if $\{\pi_f\}$ is open. Then Hida's theorem says ([53])

Theorem 3.3.5. *The Hecke module $\pi_f \in R_{\mathfrak{p}}^{-1}(\bar{\pi}_f)$ is isolated if and only if*

$$\frac{\Lambda^{\mathrm{coh}}(\pi_f, \mathrm{Sym}^2, 1)}{\Omega(+\otimes\pi_f)\Omega(-\otimes\pi_f)} \notin \mathfrak{p} \tag{3.119}$$

3.3.11 p-adic interpolation

p-adic-zeta

The p-adic ζ-function

Let p be an irregular prime, i.e. $p \mid \zeta(-1-n_0)$, here we assume $0 < n_0 < p-1$. We consider $\zeta(-1-n) = \zeta(-1-n_0-\alpha(p-1))$ as function in the variable $\alpha \in \mathbb{N}$ and we want to find values $n = -1-n_0-\alpha(p-1)$ such that $\zeta(-1-n)$ is divisible by higher powers of p. We know that there exist a p-adic $\zeta-$ function (See [66],[54],[109]) and this function has an expansion

p-appr

$$\zeta(-1-n) = \zeta(-1-n_0-\alpha(p-1)) \equiv \zeta(-1-n_0) + a(n_0,1)\alpha p + a(n_0,2)\alpha^2 p^2 \ldots \tag{3.120}$$

where the coefficients $a(n_0,\nu)\in\mathbb{Z}_p$ they are only defined $\mod p$. If now $p\not|a(n_0,1)$ then we can apply Newton's method and we find a converging sequence $\alpha_1,\alpha_2,\dots$ such that

$$\alpha_\nu\equiv\alpha_{\nu+1}\mod p^\nu \text{ and } \zeta(-1-n_0-\alpha_\nu(p-1))\equiv 0\mod p^{\nu+1} \quad (3.121)$$

The sequence converges to a zero α_∞ of the p-adic $\zeta-$ function.

It is not always possible to raise the power of p which divides $\zeta(--1-n_0-\alpha(p-1))$. If for instance $p^2\not|\zeta(-1-n_0)$ and in addition $p|a(n_0,1)$ then we never find a higher power of p dividing some $\zeta(-1-n_0-\alpha(p-1))$. Again a naive probabilistic argument suggests that is an extremely rare event that this happens, but the argument also suggests that such a prime exists (See section on the Wieferich Dilemma).

Again we write $n=n_0+(p-1)\alpha$ where we assume $0<n_0<p-1$ and we want to study how the cohomology $H^1_{\text{ord}}(\Gamma\backslash\mathbb{H},\mathcal{M}_n\otimes\mathbb{Z}_p)$ varies with n. We know for instance that the denominator of the Eisenstein class may become larger. We have seen already that

$$H^1_{\text{ord}}(\Gamma\backslash\mathbb{H},\mathcal{M}_n)\otimes\mathbb{Z}/p^r\mathbb{Z}\xrightarrow{\sim}H^1_{\text{ord}}(\Gamma\backslash\mathbb{H},\mathcal{M}_n\otimes\mathbb{Z}/p^r) \quad (3.122)$$

Now we have the following theorem which is due to Hida

interpol

Theorem 3.3.6. *If $n=n_0+(p-1)\alpha, n'=n_0+(p-1)\alpha'$ and $\alpha\equiv\alpha'\mod p^{r-1}$, (i.e. $n\equiv n'\mod (p-1)p^{r-1}$) then we have a canonical Hecke invariant isomorphism*

$$\Phi(n,n')_r: H^1_{ord}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_n\otimes\mathbb{Z}/p^r)\xrightarrow{\sim}H^1_{ord}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_{n'}\otimes\mathbb{Z}/p^r). \quad (3.123)$$

This system of isomorphisms is consistent with change of the parameter α,α' and r.

Proof. See paper on interpolation. □

We find a finite extension $F_\mathfrak{p}/\mathbb{Q}_p$ such that we have a decomposition into eigenspaces

$$H^1_{\text{ord}}(\Gamma\backslash\mathbb{H},\mathcal{M}_n\otimes F)=\bigoplus_{\pi_f}H^1_{!,\text{ ord}}(\Gamma\backslash\mathbb{H},\mathcal{M}_n\otimes F)(\pi_f)\oplus Fe_n \quad (3.124)$$

where the first summation on the right hand side goes over those $\pi_f\in\text{Coh}_!(n)$ for which $\pi_f(T_p)$ is a unit in $\mathcal{O}_\mathfrak{p}$, the ring of integers in F. Let us denote this set by $\text{Coh}^{(n)}_{!,\text{ord}}$. Then the full summation goes over the set $\text{Coh}^{(n)}_{\text{ord}}=\text{Coh}^{(n)}_{!,\text{ord}}\cup\{\pi_f^{\text{Eis}}\}$. Intersecting this decomposition with $H^1_{\text{ord}}(\Gamma\backslash\mathbb{H},\mathcal{M}_n\otimes\mathcal{O}_\mathfrak{p})$ gives us a submodule of finite index

$$H^1_{\text{ord}}(\Gamma\backslash\mathbb{H},\mathcal{M}_n\otimes\mathcal{O}_\mathfrak{p})\supset\bigoplus_{\pi_f}H^1_{!,\text{ ord}}(\Gamma\backslash\mathbb{H},\mathcal{M}_n\otimes\mathcal{O}_\mathfrak{p})(\pi_f)\oplus\mathcal{O}_\mathfrak{p}e_n \quad (3.125)$$

and this also gives us a Jordan-Hölder filtration as in (3.95).

For $\bar\pi_f\in\text{Cohmodp}_!(n)$ we define

$$H^1_{!,\text{ ord}}(\Gamma\backslash\mathbb{H},\mathcal{M}_n\otimes\mathcal{O}_\mathfrak{p})\{\bar\pi_f\}:=H^1_{!,\text{ ord}}(\Gamma\backslash\mathbb{H},\mathcal{M}_n\otimes\mathcal{O}_\mathfrak{p})\{R_\mathfrak{p}^{-1}\{\bar\pi_f)\} \quad (3.126)$$

and then we get a direct sum decomposition

$$H^1_{!,\,\mathrm{ord}}(\Gamma\backslash\mathbb{H},\mathcal{M}_n\otimes\mathcal{O}_\mathfrak{p}) = \bigoplus_{\bar\pi_f} H^1_{!,\,\mathrm{ord}}(\Gamma\backslash\mathbb{H},\mathcal{M}_n\otimes\mathcal{O}_\mathfrak{p})\{\bar\pi_f\} \tag{3.127}$$

and for any $\bar\pi_f$ we get a decomposition up to isogeny

decoEis

$$H^1_{!,\,\mathrm{ord}}(\Gamma\backslash\mathbb{H},\mathcal{M}_n\otimes\mathcal{O}_\mathfrak{p})\{\bar\pi_f\} \supset \bigoplus_{\pi_f\in R_\mathfrak{p}^{-1}(\bar\pi_f)} H^1_{!,\,\mathrm{ord}}(\Gamma\backslash\mathbb{H},\mathcal{M}_n\otimes\mathcal{O}_\mathfrak{p})(\pi_f) \tag{3.128}$$

We are mainly interested in the Hecke module $H^1_{!,\,\mathrm{ord}}(\Gamma\backslash\mathbb{H},\mathcal{M}_n\otimes\mathcal{O}_\mathfrak{p})\{\bar\pi_f^{\mathrm{Eis}}\}$ how this varies if α varies. Our theorem above implies that the Hecke module $H^1_{!,\,\mathrm{ord}}(\Gamma\backslash\mathbb{H},\mathcal{M}_n\otimes\mathcal{O}_\mathfrak{p}/\mathfrak{p})$ does not depend on α.

Let us now assume that $\bar\pi_f^{\mathrm{Eis}}$ has multiplicity one. Then we know that the decomposition (**??**) applied to $\bar\pi_f^{\mathrm{Eis}}$ has only one summand $\pi_{!,f}^{\mathrm{Eis}}$, We still assume that $n = n_0 + (p-1)\alpha$,

Vand

Theorem 3.3.7. *If $p^{\delta_\ell(n)} \mid \zeta(-1-n)$ we have the congruence*

$$\pi_f(T_\ell) \equiv \ell^{n+1}+1 \mod p^{\delta_\ell(n)}\ \forall\ \text{primes } \ell$$

Finally we get $\pi_f(T_\ell)\in\mathbb{Z}_p$ for all primes ℓ and hence we may take $\mathcal{O}_\mathfrak{p}=\mathbb{Z}_p$. We can find a basis f_0,f_1,f_2 of $H^1_{ord}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_n)\{\bar\pi_f^{\mathrm{Eis}}\}$ where

a) f_1,f_2 form a basis of $H^1_{!,\,ord}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_n)$ and f_0 maps to a generator of $H^1(\partial(\Gamma\mathbb{H}),\tilde{\mathcal{M}}_n\otimes\mathbb{Z}_p)$

b)The complex conjugation c acts by $c(f_i)=(-1)^{i+1}f_i$

and finally
c) the matrix ${T_\ell}^{ord}$ with respect to this basis satisfies

$${T_\ell}^{ord} \equiv \begin{pmatrix} \ell^{n+1}+1 & 0 & t^{(\ell)} \\ 0 & \ell^{n+1}+1 & 0 \\ 0 & 0 & \ell^{n+1}+1 \end{pmatrix} \mod p^{\delta_\ell(n)}$$

Proof. Clear □

Now we assume that $p \not| a(1,n_0)$ and we choose a sequence $\alpha_0=0,\alpha_1,\dots,\alpha_\nu$ as in (3.121) then we get Hecke-module maps

$$H^1_{!,\mathrm{ord}}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_{n_{\nu+1}}\otimes\mathbb{Z}_p/p^{\nu+1}) \to H^1_{!,\mathrm{ord}}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_{n_{\nu+1}}\otimes\mathbb{Z}_p/p^{\nu}) \xrightarrow{\sim} H^1_{!,\mathrm{ord}}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_{n_\nu}\otimes\mathbb{Z}_p/p^{\nu}) \tag{3.129}$$

The sequence n_ν converges to an p-adic integer n_∞, we can form the projective limit and define

$$H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_{n_\infty}) = \lim_{\leftarrow} H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_{n_\nu}\otimes\mathbb{Z}/p^\nu\mathbb{Z}) \tag{3.130}$$

Under our assumptions $H^1_{\mathrm{ord},!}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_{n_\infty})\{\bar\pi_f^{\mathrm{Eis}}\}$ is a free $\mathbb{Z}_p$-module of rank 3. The Hecke operators T_ℓ^{ord} acts on $H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_{n_\nu}\otimes\mathbb{Z}/p^\nu\mathbb{Z})$ by a matrix of the shape as in theorem 3.3.7, and the eigenvalues on the diagonal are

$$\ell^{n_\nu+1}+1=\ell^{n_0+(p-1)\alpha_\nu}+1 \mod p^\nu$$

For $\ell\neq p$ we write

$$\ell^{p-1}=1+p\delta(\ell),\delta_p(\ell)\in\mathbb{N}$$

and then

$$\ell^{n_0+(p-1)\alpha_\nu}=\ell^{n_0}(1+\delta_p(\ell)p)^{\alpha_\nu}=l^{n_0}(1+\alpha_\nu p\delta_p(\ell)+\binom{\alpha}{2}\alpha_\nu^2p^2\delta_p(\ell)^2\dots$$

We see that we can define $\ell^{n_0+\alpha(p-1)}$ for any $\alpha\in\mathbb{Z}_p$ and then clearly $\lim_{\nu\to\infty}\ell^{n_\nu}=\ell^{n_\infty}$. Hence we see that T_ℓ^{ord} acts on $H^1_{\mathrm{ord}}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_{n_\infty})$ by the matrix

$$T_\ell^{\mathrm{ord}}\equiv\begin{pmatrix}\ell^{n_\infty+1}+1 & 0 & t^{(\ell)}\\ 0 & \ell^{n_\infty+1}+1 & 0\\ 0 & 0 & \ell^{n_\infty+1}+1\end{pmatrix}$$

where we have $t^{(\ell)}\neq 0$. (This follows from earlier arguments)

If we drop the assumption that $\bar\pi_f^{\mathrm{Eis}}$ has multiplicity one then the situation becomes definitely much more complicated. We believe that this a rare event, we have seen that for $p<10^5$ this happens only once. But on the other hand our naive probabilistic argument suggests that it should happen again. But then the same probabilistic argument suggests that it never happens that $m(\bar\pi_f^{\mathrm{Eis}})>2$.

Therefore we make the assumption that $m(\bar\pi_f^{\mathrm{Eis}})=2$. We look at that the decomposition (3.134) applied to $\bar\pi_f=\bar\pi_f^{\mathrm{Eis}}$. We consider the characteristic polynomial

$$\det(T_\ell^{\mathrm{Eis}}-\mathrm{Id}X|H^1_!(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}\otimes\mathbb{Z}_p)\{\bar\pi_f^{\mathrm{Eis}})\}=(b_0(n,p)+b_1(n,p)X+X^2)^2. \tag{3.131}$$

Our assumption $m(\bar\pi_f^{\mathrm{Eis}})=2$ implies that $b_1(n,p),b_0(n,p)\equiv 0 \mod p$. Let us write

$$P^{\mathrm{Eis}}(T_\ell,n,,X):=b_0(n,p)+b_1(n,p)X+X^2 \tag{3.132}$$

If we now find an ℓ such that $p^2\not| \,b_0(n,p)$, then it is clear that we need a quadratic extension $F_{\mathfrak{p}}=\mathbb{Q}_p[\sqrt{\epsilon p}]$ (ϵ a unit in $\mathbb{Z}_p^\times$) if we want a decomposition into eigen spaces (3.124). If $\mathcal{O}_{\mathfrak{p}}$ is the ring of integers in $F_{\mathfrak{p}}$ then we get a decomposition up to isogeny into two conjugate eigenspaces

$$H^1_!(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_n\otimes\mathcal{O}_{\mathfrak{p}})/\{\bar\pi_f^{\mathrm{Eis}}\}\supset H^1_!(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_n\otimes\mathcal{O}_{\mathfrak{p}})(\pi_f)\oplus H^1_!(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}\otimes\mathcal{O}_{\mathfrak{p}})({}^\sigma\pi_f) \tag{3.133}$$

where σ is the non trivial element in the Galois group of $F_{\mathfrak{p}}/\mathbb{Q}_p$.

Under our assumptions the quotient of the left hand side by the right hand side is isomorphic to $H^1_!(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}\otimes\mathcal{O}_{\mathfrak{p}}/\mathfrak{p})(\bar\pi_f^{\mathrm{Eis}})$. For all ℓ' we have the congruence $\pi_f(T_{\ell'})\equiv{}^\sigma\pi_f(T_{\ell'}) \mod \mathfrak{p}$. We recall that $n=n_0+\alpha(p-1)$. It is clear that the condition $p^2 \not| b_0(n,p)$ only depends on $\alpha \mod p$. Therefore we should expect that for a fixed α the "probability" that $p^2 | b_2(n,p)$ is $1/p$. but there may be a value of α for which this divisibility holds.

We return to our sequence $\alpha_0, \alpha_1, \ldots$ see((3.121)). We assume that $p^2 \not| b_0(n_0 + \alpha_1(p-1), p)$. Then it is easy to see that for $\nu \geq 1$ we have the decomposition up to isogeny (3.133)

$$H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{n_\nu} \otimes \mathcal{O}_\mathfrak{p})\{\bar{\pi}_f^{\mathrm{Eis}}\} \supset H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{n_\nu} \otimes \mathcal{O}_\mathfrak{p})(\pi_f^{(\nu)}) \oplus H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{n_\nu} \otimes \mathcal{O}_\mathfrak{p})({}^\sigma\pi_f^{(\nu)}) \tag{3.134}$$

We have the inclusion $\mathbb{Z}_p/(p^{\nu+1})(-n_\nu - 1) \hookrightarrow H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{n_\nu} \otimes \mathbb{Z}/(p^{\nu+1}))$ and tensoring by $\mathcal{O}_\mathfrak{p}/\mathfrak{p}^{2(n_\nu+1)}$ yields the inclusion

$$j_{\mathfrak{p},\nu} : \mathcal{O}_\mathfrak{p}/\mathfrak{p}^{2(\nu+1)}(-n_\nu - 1) \hookrightarrow H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{n_\nu} \otimes \mathcal{O}_\mathfrak{p}/(\mathfrak{p}^{2(\nu+1)}))$$

We have $H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathcal{O}_\mathfrak{p}/(\mathfrak{p}^{(2(\nu+1))}))(\pi_f) \subset H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathcal{O}_\mathfrak{p}/(\mathfrak{p}^{(2(\nu+1))}))$ this is the first step in the Jordan-Hölder filtration and the quotient by this first step is isomorphic to $H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathcal{O}_\mathfrak{p}/(\mathfrak{p}^{(2(\nu+1))}))({}^\sigma\pi_f)$. We repeat the argument in the proof of Theorem 3.3.2 and we conclude that we have congruences

$$\pi_f^{(\nu)}(T_\ell) \equiv \ell^{n_\nu+1} + 1 \mod \mathfrak{p}^{(\nu+1)}\ ;^\sigma \pi_f^{(\nu)}(T_\ell) \equiv \ell^{n_\nu+1} + 1 \mod \mathfrak{p}^{(\nu+1)} \tag{3.135}$$

Again we can pass to the limit $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{n_\infty})\{\bar{\pi}_f^{\mathrm{Eis}}\}$, this limit has a three step Jordan-Hölder filtration, the top quotient is $\mathbb{Z}_p(-n_\infty - 1) \otimes \mathcal{O}_\mathfrak{p}$ -this is the cohomology of the boundary. For the eigenvalues on the middle step and the bottom we also get the same limit . Hence -if we choose our basis as before, i.e. we take the action of the complex conjugation into account, then we get for the matrix of the Hecke operator-restricted to the Eisenstein part-

$$T_\ell^\infty = \begin{pmatrix} \ell^{n_\infty+1}+1 & 0 & t^{(\ell)} & 0 & s^{(\ell)} \\ 0 & \ell^{n_\infty+1}+1 & 0 & u^{(\ell)} & 0 \\ 0 & 0 & \ell^{n_\infty+1}+1 & 0 & v^{(\ell)} \\ 0 & 0 & 0 & \ell^{n_\infty+1}+1 & 0 \\ 0 & 0 & 0 & 0 & \ell^{n_\infty+1}+1 \end{pmatrix} \tag{3.136}$$

where the non zero entries are units for a suitable choice of ℓ.

In the special case (484,547) the number $\alpha_1 = 100$ we have $\zeta(-485 - 100 * 546) = \zeta(-55085) \equiv 0 \mod 547^2$. Earlier we checked $547^2 \not| b_0(484, 547)$ for the Hecke operator T_2. But how can we ever check $547^2 \not| b_0(55084, 547)$? The matrices become too big.

Our chances that $547^2 \mid b_0(55084, 547)$ are 1/547.

But there is a way out. I think it is possible to prove that we have an expansion

$$b_0(484 + \alpha 546, 547) \equiv b_0(484, 547) + \alpha 547 b_0'(484, 457) \mod 547^2. \tag{3.137}$$

(This should follow from the general results which we announced in [42] and which we still hope to prove in a paper with J. Mahnkopf).

Using our program with Gangl for T_2 and with the help of A.Weisse we computed $T_2 \mod p^2$ for the cases $\alpha = 0, 1, 2$,the program still works in reasonable time in these cases. We found (of course everything $\mod 547^2$)

$$\begin{aligned} b_0(484) &= 547 \times 10 \\ b_0(484+546, 547) &= 547 \times 174 = 547(10+164), \\ b_0(484+2\times 546, 547) &= 547 \times 338 = 547(10+2\times 164) \end{aligned} \tag{3.138}$$

Hence we expect $b_0'(484, 547) = 164 \mod 547$ and assuming the above linearity we get $b_0(55084, 547) = 547(10 + 100 \times 164) = 547 \times 16410$.

To my great surprise $16410 = 2 \times 3 \times 5 \times 547$!!!

Perhaps I was just stupid and a closer look shows that there is an obvious reason that this must be so

Hence we have to compute $P^{\text{Eis}}(T_2, 484 + a * 546, X) \mod 547^3$ for some small values of a. We computed the zeroes

$$\lambda(T_2, a) = \alpha * 457 \pm \sqrt{\beta * 547} \mod 547^3$$

of the quadratic factor $P^{\text{Eis}}(T_2, 484 + a * 546, X)$ for $a = 0, 1, 2$. We found

$$\begin{aligned} \lambda(T_2, 0) &= 268381 * 547 \pm \sqrt{537 * 547} \mod 547^3 \\ \lambda(T_2, 1) &= 189064 * 547 \pm \sqrt{251993 * 547} \mod 547^3 \\ \lambda(T_2, 2) &= 13475 * 547 \pm \sqrt{169232 * 547} \mod 547^3 \end{aligned} \tag{3.139}$$

we see that these roots lie in a ramified quadratic extension of $Z/(547^3)$. From the roots we get the coefficients of $P^{\text{Eis}}(T_2, 484 + a * 546, X)$

$$\begin{aligned} b_0(484, 547) &= (268381 * 547)^2 - 537 * 547 = 37406595 \mod 547^3 \\ b_0(484+546, 547) &= (189064 * 547)^2 - 251993 * 547 = 135636855 \mod 547^3 \\ b_0(484+2*546, 547) &= (13475 * 547)^2 - 169232 * 547 = 91742840 \mod 547^3 \\ b_1(484, 547) &= 2 * 268381 * 547 \mod 547^3 \\ b_1(484+546, 547) &= 2 * 189064 * 547 \mod 547^3 \end{aligned} \tag{3.140}$$

We now hope that in the still to be written paper with J. Mahnkopf we will show that we have an expansion

$$b_0(484 + \alpha 546, 547) = b_0(484, 547) + b_0'(484)\alpha 547 + b_0''(484)\alpha^2 547^2 \mod 547^3 \tag{3.141}$$

where $b_0'(484), b_0''(484)$ are numbers $\mod 547$, which we can compute from the three values above. We easily get

$$b_0'(484) = 164\ ;\ b_0''(484) = 24 \tag{3.142}$$

We do the same for $b_1(484 + \alpha * 546) = b_1(484) + 543 * 547 \mod 547^2$. Now the discriminant is Discriminant $\Delta(\alpha, 547) = -b_0(484 + \alpha 546, 547) + b_1(484 + \alpha * 546, 457)^2$ and if we believe in the interpolation formula we get

$$\Delta(100, 547) = 286 * 547^2 \mod 547^3 \tag{3.143}$$

Hence we see that

$$\lambda(T_2, 100) = 547(238096 \pm \frac{1}{2}\sqrt{286}) \mod 547^3 \tag{3.144}$$

Now we check easily that 286 is not a square $\mod 547$ and hence we see the roots now lie in the unramified quadratic extension $\mathbb{Z}/(547^3)[\sqrt{2}]$. Now we put again $\mathcal{O}_\mathfrak{p} = \mathbb{Z}_{547}[\sqrt{2}]$ let us put $p = 547$ then $\mathfrak{p} = (p)$ We consider the sequence $\alpha_0, \alpha_1, \ldots, \alpha_\nu$ (see 3.121). As before we get for any $\nu \geq 1$ (a decomposition up to isogeny (3.134)

$$\begin{gathered} H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{n_1} \otimes \mathcal{O}_\mathfrak{p})\{\bar{\pi}_f^{\mathrm{Eis}}\} \supset \\ H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{n_1} \otimes \mathcal{O}_\mathfrak{p})(\pi^{(\nu)})_f) \oplus H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{n_1} \otimes \mathcal{O}_\mathfrak{p}))({}^\sigma\pi_f^{(\nu)}). \end{gathered} \tag{3.145}$$

We argue as before, our embedding $\mathbb{Z}/p^{\nu+1}(-1-n_\nu) \hookrightarrow H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{n_1} \otimes \mathbb{Z}/(p^{\nu+1}))$ provides the Galois invariant embedding $\mathcal{O}_\mathfrak{p}/(p^{\nu+1})(-1-n_\nu) \hookrightarrow H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{n_\nu} \otimes \mathcal{O}_\mathfrak{p}/(p^{\nu+1})$. There is a largest number $f_\nu \leq \nu$ such that the submodule $p^{f_\nu}\mathcal{O}_\mathfrak{p}/p^{\nu+1}(-1-n_\nu)$ embeds into the submodule $H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{n_1} \otimes \mathcal{O}_\mathfrak{p}/(p^{\nu+1}))(\pi_f^{(\nu})$. But then we get an injection

$$\begin{gathered} \mathcal{O}_\mathfrak{p}/(p^{\nu+1})(-1-n_\nu)/p^\mu\mathcal{O}_\mathfrak{p}/(p^{f_\nu})(-1-n_\nu) \hookrightarrow \\ H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{n_\nu} \otimes \mathcal{O}_\mathfrak{p}/(p^{\nu+1})/H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{n_1} \otimes \mathcal{O}_\mathfrak{p})(\pi_f^{(\nu)}) \otimes \mathcal{O}_\mathfrak{p}/(p^{\nu+1}) \end{gathered} \tag{3.146}$$

The module in the bottom line is isomorphic to $H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{n_1} \otimes \mathcal{O}_\mathfrak{p})({}^\sigma\pi_f^{(\nu}) \otimes \mathcal{O}_\mathfrak{p}/(p^\nu)$ and hence we get congruences

$$\pi_f^{(\nu)}(T_\ell) \equiv \ell^{n_\nu+1} + 1 \mod p^{f_\nu}\ ,\ {}^\sigma\pi_f^{(\nu)}(T_\ell) \equiv \ell^{n_\nu+1} + 1 \mod p^{\nu+1-f_\nu} \tag{3.147}$$

Since these congruences are invariant under the action of the Galois group we get congruences

$$\pi_f^{(\nu)}(T_\ell) \equiv \ell^{n_\nu+1} + 1 \mod p^{[\frac{\nu+2}{2}]}\ ,\ {}^\sigma\pi_f^{(\nu)}(T_\ell) \equiv \ell^{n_\nu+1} + 1 \mod p^{[\frac{\nu+2}{2}]} \tag{3.148}$$

where $[x]$ denotes the usual Gauss bracket.

The Galois group

It is a fundamental fact that we have an action of the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on the modules $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{Z}_p/p^\delta), H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathcal{O}_\mathfrak{p}), H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{n_\infty})\{\bar{\pi}_f^{\mathrm{Eis}}\}$ this action commutes with the action of the Hecke algebra. Hence we get interesting representations of the Galois group, these representations have been studied by many people. (See for instance [74], [87],[105] or [47]).

We will be explain this in a very cursory manner in section 5.1.5.

3.3.12 The Wieferich dilemma

In 1909 the student Arthur Wieferich proved the following

If for a prime $p > 2$ the equation $x^p + y^p + z^p = 0$ has a solution in integers with $xyz \not\equiv 0 \mod p$ then $2^{p-1} - 1 \equiv 0 \mod p^2$

Of course we now that $2^{p-1} - 1 \equiv 0 \mod p$ but checking a few of small primes suggests that the residue class $w(p) = \frac{2^{p-1}-1}{p} \mod p$ can just be any number mod p. Hence we expect that it is a rare event that this residue class is zero, the "probability" is $\frac{1}{p}$. Later these primes ere called Wieferich primes. At the present moment it seems that there are only two Wieferich primes $< 6.7 \times 10^5$.,

But poor Arthur could not show that he had proved the first case of Fermat for infinitely many prime exponents p because he could not show that there are infinitely many non Wieferich primes (and we still do not know it now):

This kind of phenomenon was not new and well known at the time when Wieferich proved his theorem (Simply Google: The first case of Fermat's Theorem) but Wieferich's case is a striking example because it is so easy to state. That is the reason why we propose to call it the Wieferich dilemma.

In this book we encounter the Wieferich dilemma at several occasions. If $p|\zeta(-1-n)$ we raised the question whether $\bar{\pi}_f^{\mathrm{Eis}}$ occurs with multiplicity one. We saw that this is exactly the case if $a_1(n,p) \not\equiv 0 \mod p$. Again we may argue that the probability that $p|a_1(n,p)$ is $\frac{1}{p}$ so we expect this to be a rare event that $\bar{\pi}_f^{\mathrm{Eis}}$ occurs with higher multiplicity. Actually we learned from [2] that there is only one exception for $p < 10^5$. On the other hand we believe that $\sum_{p\ irregular} \frac{1}{p}$ is divergent so there is a certain chance that some larger such a prime exists.

Assume that there is a prime p for which $\bar{\pi}_f^{\mathrm{Eis}}$ occurs with higher multiplicity. Then we may ask whether it occurs weakly with multiplicity one and again a probability argument shows that the probability that this is not so is $\frac{1}{p^2}$. This now suggests that it may always occurs weakly with multiplicity one.

Here I want to make a metamathematical statement.

It is very well conceivable that $\bar{\pi}_f^{\mathrm{Eis}}$ always occurs with weak multiplicity one, but we will never find a proof. But it is simply true because the probability that $\bar{\pi}_f^{\mathrm{Eis}}$ occurs with higher weak multiplicity is so small. It is simply "true" without a proof.

In this chapter 3 we discuss some questions concerning the structure of the cohomology of arithmetic groups as module under the Hecke algebra. We we execute computations and experiments to support and suggest certain hypotheses. But we only considered a very special example.

But there is much wider range where can ask questions and make hypotheses.

We drop our assumption that we are in the totally unramified situation, this means that we can replace $\Gamma_0 = \mathrm{Sl}_2(\mathbb{Z})$ by a (normal) congruence subgroup $\Gamma \subset \Gamma_0$. We choose a free $\mathbb{Z}-$ module of finite rank $\mathcal{V}$ with an action of Γ_0/Γ, i.e. we have a representation

$$\rho_{\mathcal{V}} : \Gamma_0/\Gamma \to \mathrm{Gl}(\mathcal{V})$$

we assume that the matrix $-\mathrm{Id}$ acts by a scalar $\rho_{\mathcal{V}}(-\mathrm{Id}) = \pm\mathrm{Id}$. The Γ_0-modules $\mathcal{M}_n \otimes \mathcal{V}$ provide sheaves $\widetilde{\mathcal{M}_n \otimes \mathcal{V}}$, here we assume that $\rho_{\mathcal{V}}(-\mathrm{Id}) \equiv n \mod 2$.Again we study the cohomology groups and especially we can study the fundamental exact sequence

$$\to H^1_c(\Gamma_0\backslash\mathbb{H}, \widetilde{\mathcal{M}_n \otimes \mathcal{V}}) \to H^1(\Gamma_0\backslash\mathbb{H}, \widetilde{\mathcal{M}_n \otimes \mathcal{V}}) \xrightarrow{r} H^1(\partial(\Gamma_0\backslash\mathbb{H}), \widetilde{\mathcal{M}_n \otimes \mathcal{V}}) \tag{3.149}$$

On these cohomology groups we have the action of Hecke operators $T(\alpha, u_\alpha)$. Here we have to be a little carful. Our group Γ contains a full congruence group $\Gamma(N)$. Then we take the elements $\alpha \in \mathrm{Gl}_2(\mathbb{Z}_{(N)})$ and our $u_\alpha = u_\alpha^{\mathcal{M}} \otimes u_\alpha^{\mathcal{V}}$ (See section 3.1).

We may for instance choose a positive integer N and we consider the congruence subgroup $\Gamma_0(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}_2(\mathbb{Z}) | c \equiv 0 \mod N\}$. Let $\Gamma_1(N) \subset \Gamma_0(N)$ be the subgroup where $a \equiv 1 \mod N$ then $\Gamma_0(N)/\Gamma_1(N) = (\mathbb{Z}/N\mathbb{Z})^\times$. We choose a character $\chi : \Gamma_0(N)/\Gamma_1(N) \to \mathbb{Z}[\zeta_N]^\times$ and consider the induced representation

$$\mathcal{V}_\chi = \mathrm{Ind}_{\Gamma_0(N)}^{\Gamma_0}\chi = \{f : \Gamma_0 \to \mathbb{Z}[\zeta_N] \mid f(\gamma x) = \chi(\gamma)f(x); \gamma \in \Gamma_0(N), x \in \Gamma_0\}.$$

It an interesting task to extend the results and the experimental computations from the previous chapters to these Hecke modules. It should not be too difficult to generalise the results in section 3.3.1 for the Hecke modules $H^1(\partial(\Gamma_0\backslash\mathbb{H}), \widetilde{\mathcal{M}_n \otimes \mathcal{V}_\chi})$. Then we can formulate the denominator question again.

In this case the denominator should be related to the L values $L(\chi, -1-n)$, It is certainly interesting to collect some data, which might allow us to formulate more precise hypotheses. For this purpose one has to extend the algorithm for T_2 to this new situation.

At this point we ignore (or forget) that analytic methods (Eisenstein cohomology) also provide some tools to understand the denominator. (see refdenomEis)

I think it is even more interesting to investigate the multiplicity questions. Now our cohomology groups are $\mathbb{Z}[\zeta_N]$ modules, let us assume that we have a prime ideal $\mathfrak{p}|L(\chi, -1-n)$. We assume $\mathfrak{p} \not| N$ and let $\mathcal{O}_{\mathfrak{p}} = \mathbb{Z}[\zeta_N]_{\mathfrak{p}}$ be the completion at $\mathfrak{p}$. Then we can should be able to define the direct summand (see3.108)

$$H^1(\Gamma_0\backslash\mathbb{H}, \mathcal{M}_n \widetilde{\otimes \mathcal{V}} \otimes \mathcal{O}_{\mathfrak{p}})\{\pi_{f,\chi}^{\mathrm{Eis}}\} \subset H^1_{\mathrm{ord}}(\Gamma_0\backslash\mathbb{H}, \mathcal{M}_n \widetilde{\otimes \mathcal{V}} \otimes \mathcal{O}_{\mathfrak{p}}) \tag{3.150}$$

and we want this direct summand as Hecke module. Earlier we have done some experimental computation in the unramified case ($N = 1$) and probabilistic arguments let us make some conjectures. But now the we have many more cases (vary the character χ and the primes involved are much smaller so we have some chances to falsify the analogous conjectures once we have some ramification.

I think that here is a wide field for interesting experiments.

Chapter 4

Representation Theory, Eichler-Shimura Isomorphism

HC

4.1 Harish-Chandra modules with cohomology

In Chapter 8 we will give a general discussion of the tools from representation theory and analysis which help us to understand the cohomology of arithmetic groups. Especially in Chapter 8 section 9.5 we will recall the results of Vogan-Zuckerman on the cohomology of Harish-Chandra modules.

Here we specialise these results to the specific cases $G = \mathrm{Gl}_2(\mathbb{R})$ (case A)) and $G = \mathrm{Gl}_2(\mathbb{C})$ (case B)). For the general definition of Harish-Chandra modules and for the definition of $(\mathfrak{g}, K_\infty)$ cohomology we refer to (8.1.2)

Mlambda

4.1.1 The finite rank highest weight modules

We consider the case A), in this case our group $G/\mathbb{R}$ is the base extension of the reductive group scheme $\mathcal{G} = \mathrm{Gl}_2/\operatorname{Spec}(\mathbb{Z})$. In principle this a pretentious language. At this point it simply means that we can speak of $\mathcal{G}(R)$ for any commutative ring R with identity and that $\mathcal{G}(R)$ depends functorially on R.

(Sometimes in the following we will replace Spec($\mathbb{Z}$) by $\mathbb{Z}$.) Then $\mathcal{G}^{(1)}/\mathbb{Z}$ is the kernel of the determinant map det : $\mathcal{G}/\mathbb{Z} \to \mathbb{G}_m/\mathbb{Z}$. We have the standard maximal torus $\mathcal{T}/\mathbb{Z}$ and choose the Borel subgroup $\mathcal{B}/\mathbb{Z} \supset \mathcal{T}/\mathbb{Z}$ to be the group of upper triangular matrices. Let $X^*(\mathcal{T}) = X^*(T \times \mathbb{C})$ be the character module This character module is $\mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ where

$$e_i : \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \mapsto t_i \tag{4.1}$$

Any character can be written as $\lambda = n\gamma + d\det$ where $\gamma = \frac{e_1 - e_2}{2} (\notin X^*(\mathcal{T})\ !), \det = e_1 + e_2$ and where $n \in \mathbb{Z}, d \in \frac{1}{2}\mathbb{Z}$ and $n \equiv 2d \mod 2$. We assume that λ is dominant, i.e. $n \geq 0$.

To any such character $\lambda = n\gamma + d\det$ we want to attach a highest weight module $\mathcal{M}_\lambda$. We consider the $\mathbb{Z}-$ module of polynomials

$$\mathcal{M}_n = \{P(X,Y) \mid P(X,Y) = \sum_{\nu=0}^{n} a_v X^\nu Y^{n-\nu}, a_\nu \in \mathbb{Z}\}.$$

To a polynomial $P \in \mathcal{M}_n$ we attach the regular function (see 1.1.1)

$$f_P(\begin{pmatrix} x & y \\ u & v \end{pmatrix}) = P(u,v)\det(\begin{pmatrix} x & y \\ u & v \end{pmatrix})^{-\frac{n}{2}+d}, \tag{4.2}$$

then

$$f_P(\begin{pmatrix} t_1 & w \\ 0 & t_2 \end{pmatrix}\begin{pmatrix} x & y \\ u & v \end{pmatrix}) = t_2^n (t_1 t_2)^{-\frac{n}{2}+d} f_P(\begin{pmatrix} x & y \\ u & v \end{pmatrix}) = \lambda^-(\begin{pmatrix} t_1 & w \\ 0 & t_2 \end{pmatrix}) f_P(\begin{pmatrix} x & y \\ u & v \end{pmatrix}) \tag{4.3}$$

where $\lambda^- = -n\gamma + d\det$. On this module of regular functions the group scheme $\mathcal{G}/\mathbb{Z}$ acts by right translations:

$$\rho_\lambda(\begin{pmatrix} a & b \\ c & d \end{pmatrix})(f_P)(\begin{pmatrix} x & y \\ u & v \end{pmatrix})) = f_P(\begin{pmatrix} x & y \\ u & v \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix})$$

This is now the highest weight module $\mathcal{M}_\lambda$ for the group scheme $\mathcal{G}/\mathbb{Z}$. The highest weight vector is f_{X^n}, clearly we have

$$\rho_\lambda(\begin{pmatrix} t_1 & w \\ 0 & t_2 \end{pmatrix})(f_{X^n}) = \lambda(\begin{pmatrix} t_1 & w \\ 0 & t_2 \end{pmatrix}) f_{X^n}$$

In the following we change the notation, instead of f_P we will simply write P.

Comment: When we say that $\mathcal{M}_\lambda$ is a module for the group scheme $\mathcal{G}/\mathbb{Z}$ we mean nothing more than that for any commutative ring R with identity we have an action of $\mathcal{G}(R)$ on $\mathcal{M}_n \otimes R$, which is given by (4.2) and depends functorially on R. We can "evaluate" at $R = \mathbb{Z}$ and get the $\Gamma = \mathrm{Gl}_2(\mathbb{Z})$ module $\mathcal{M}_{\lambda,\mathbb{Z}}$. (Actually we should not so much distinguish between the $\mathrm{Gl}_2(\mathbb{Z})$ module $\mathcal{M}_{\lambda,\mathbb{Z}}$ and $\mathcal{M}_\lambda$) Of course we have have seen these $\mathrm{Gl}_2(\mathbb{Z})$ modules before, they are of course equal to the modules $\mathcal{M}_n[d - \frac{n}{2}]$ in section 1.2.2.

Remark: There is a slightly more sophisticated interpretation of this module. We can form the flag manifold $\mathcal{B}\backslash\mathcal{G} = \mathbb{P}^1/\mathbb{Z}$ and the character λ yields a line bundle $\mathcal{L}_{\lambda^-}$. The group scheme $\mathcal{G}$ is acting on the pair $(\mathcal{B}\backslash\mathcal{G}, \mathcal{L}_{\lambda^-})$ and hence on $H^0(\mathcal{B}\backslash\mathcal{G}, \mathcal{L}_{\lambda^-})$ which is tautologically equal to $\mathcal{M}_\lambda$ (Borel-Weil theorem).

We can do essentially the same in the case B). In this case we start from an imaginary quadratic extension $F/\mathbb{Q}$ and let $\mathcal{O} = \mathcal{O}_F \subset F$ its ring of integers. We form the group scheme $\mathcal{G}/\mathbb{Z} = R_{\mathcal{O}/\mathbb{Z}}(\mathrm{Gl}_2/\mathcal{O})$. Again $\mathcal{G}^{(1)}/\mathbb{Z}$ will be the kernel of $\det : \mathcal{G}/\mathbb{Z} \to \mathcal{Z}/\mathbb{Z} = R_{\mathcal{O}/\mathbb{Z}}(\mathbb{G}_m)$. Then $\mathcal{G}(\mathcal{O}) = \mathrm{Gl}_2(\mathcal{O}\otimes\mathcal{O}) \subset \mathrm{Gl}_2(\mathcal{O}) \times \mathrm{Gl}_2(\mathcal{O})$. The base change of the maximal torus $T/\mathbb{Q} \subset \mathcal{G} \times_{\mathbb{Z}} \mathbb{Q}$ is the product $T_1 \times T_2/F$ where the two factors are the standard maximal tori in the two factors Gl_2/F.

We get for the character module CHMsplit

$$X^*(T \times F) = X^*(T_1) \oplus X^*(T_2) = \{n_1\gamma_1 + d_1 \det\} \oplus \{n_2\gamma_2 + d_2\bar{\det}\} \tag{4.4}$$

where we have to observe the parity conditions $n_1 \equiv 2d_1 \mod 2, n_2 \equiv 2d_2 \mod 2$.

Then the same procedure as in case A) provides a free $\mathcal{O}$- module $\mathcal{M}_\lambda$ with an action of $\mathcal{G}(\mathbb{Z})$ on it. To get this module and to see this action we embed the group $\mathcal{G}(\mathbb{Z}) = \mathrm{Gl}_2(\mathcal{O})$ into $\mathrm{Gl}_2(\mathcal{O}) \times \mathrm{Gl}_2(\mathcal{O})$ by the map $g \mapsto (g, \bar{g})$ where $\bar{g}$ is of course the conjugate. If now our $\lambda = n_1\gamma_1 + d_1 \det_1 + n_2\gamma_2 + d_2\det_2 = \lambda_1 + \lambda_2$ then we have our two $\mathrm{Gl}_2(\mathcal{O})$ modules $\mathcal{M}_{\lambda_1,\mathcal{O}}, \mathcal{M}_{\lambda_2,\mathcal{O}}$ and this provides the $\mathrm{Gl}_2(\mathcal{O}) \times \mathrm{Gl}_2(\mathcal{O})$- module $\mathcal{M}_{\lambda_1,\mathcal{O}} \otimes \mathcal{M}_{\lambda_2,\mathcal{O}}$, our $\mathcal{M}_{\lambda,\mathcal{O}}$ is is now simply the restriction of this tensor product module to $\mathcal{G}(\mathbb{Z})$. Sometimes we will also write our character as the sum of the semi simple component and the central component, i.e.

$$\lambda = \lambda^{(1)} + \delta = (n_1\gamma_1 + n_2\gamma_2) + (d_1\det_1 + d_2\det_2) \tag{4.5}$$

The relevant term is the semi simple component, the central component is not important at all, it only serves to fulfill the parity condition. If we restrict the representation $\mathcal{M}_\lambda$ to $\mathcal{G}^{(1)}/\mathbb{Z}$ then the dependence on d disappears. In other words representations with the same semi simple highest weight component only differ by a twist, the role played by δ is marginal.

At this point we notice that the module $\mathcal{M}_{\lambda,\mathcal{O}}$ is only a module over $\mathcal{O}$. We may also say that $\mathcal{M}_{\lambda,\mathcal{O}} \otimes F$ is an absolutely irreducible highest weight module for the group $\mathcal{G} \otimes_{\mathcal{O}} F = \mathrm{Gl}_2 \times \mathrm{Gl}_2/F$, this representation "is defined " over F. But in the special case that $\lambda_1 = \lambda_2$ we have an action of the Galois group $\mathrm{Gal}(F/\mathbb{Q})$: If c is the non trivial element in this Galois group then

$$c((\sum a_\nu X^\nu Y^{n-\nu}) \otimes (\sum_\mu b_\mu \bar{X}^\mu \bar{Y}^{n-\mu}) = (\sum_\mu c(b_\mu) X^\mu Y^{n-\mu})(\sum c(a_\nu) \bar{X}^\nu \bar{Y}^{n-\nu})$$

and for $g \in \mathcal{G}(\mathcal{O}), m \in \mathcal{M}_\lambda$ we have

$$c(g)c(m) = c(gm)$$

and therefore it is clear that the $\mathbb{Z}$ module $(\mathcal{M}_\lambda)^{(c)}$ is a module for $\mathcal{G}/\mathbb{Z}$.

We return to $\mathrm{Gl}_2/\mathbb{Z}$. Given $\lambda = \lambda^{(1)} + \delta$ we define the dual character as $\lambda^\vee = \lambda^{(1)} - \delta$. For our finite dimensional modules we have

$$\mathcal{M}_\lambda^\vee \otimes \mathbb{Q} \xrightarrow{\sim} \mathcal{M}_{\lambda^\vee} \otimes \mathbb{Q} \tag{4.6}$$

If we consider the modules over the integers the above relation is not true. We define the submodule duallambda

$$\mathcal{M}_n^\flat = \{P(X,Y) \mid P(X,Y) = \sum_{\nu=0}^{n} \binom{n}{\nu} a_\nu X^\nu Y^{n-\nu}, a_\nu \in \mathbb{Z}\}. \tag{4.7}$$

This is a submodule of $\mathcal{M}_n$ and the quotient $\mathcal{M}_n/\mathcal{M}_n^\flat$ is finite. It is also clear that this submodule is invariant under $\mathrm{Sl}_2/\mathbb{Z}$. We introduce some notation

$$e_\nu := X^\nu Y^{n-\nu} \text{ and } e_\nu^\flat := \binom{n}{\nu} X^{n-\nu} Y^\nu, \tag{4.8}$$

then the e_ν(resp. $e_\nu^\flat$) for a basis of $\mathcal{M}_n$(resp. $\mathcal{M}_n^\flat$).

An easy calculation shows that the pairing pairMn

$$< , >_{\mathcal{M}}: (e_\nu, e_\mu^\flat) \mapsto \delta_{\nu,\mu} \tag{4.9}$$

is non degenerate over $\mathbb{Z}$ and invariant under $\mathrm{Sl}_2/\mathbb{Z}$. We can also define the twisted actions of $\mathcal{G}/\mathbb{Z}$. Of course we can define the twisted modules $\mathcal{M}_\lambda^\vee$ and then we get a $\mathcal{G}/\mathbb{Z}$ invariant non degenerate pairing over $\mathbb{Z}$:

$$< , >_{\mathcal{M}}: \mathcal{M}_{\lambda^\vee}^\flat \times \mathcal{M}_\lambda \to \mathbb{Z}$$

In other words

$$(\mathcal{M}_\lambda)^\vee = \mathcal{M}_{\lambda^\vee}^\flat$$

We always consider $\mathcal{M}_\lambda^\flat$ as the above submodule of $\mathcal{M}_\lambda$.

prinseries

4.1.2 The principal series representations

We consider the two real algebraic groups $G = \mathrm{Gl}_2/\mathbb{R}$(case A)) and $G = R_{\mathbb{C}/\mathbb{R}}(\mathrm{Gl}_2/\mathbb{C})$ (case B). Let $T/\mathbb{R}$,(resp. $B/\mathbb{R}$) be the standard diagonal torus (resp. Borel subgroup of upper triangular matrices). Let us put $C/\mathbb{R} = \mathbb{G}_m$ (resp. $R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$). We have the determinant $\det : G/\mathbb{R} \to C/\mathbb{R}$ and moreover $C/\mathbb{R} = center(G/\mathbb{R})$. If we restrict the determinant to the center then this becomes the map $z \mapsto z^2$. The kernel of the determinant is denoted by $G^{(1)}/\mathbb{R}$, of course $G^{(1)} = \mathrm{Sl}_2$, resp. $R_{\mathbb{C}/\mathbb{R}}(\mathrm{Sl}_2/\mathbb{C})$. Let us denote by $\mathfrak{g}, \mathfrak{g}^{(1)}, \mathfrak{t}, \mathfrak{b}, \mathfrak{z}$ the corresponding Lie-algebras.

The Cartan decompositions

In both cases we fix a maximal compact compact subgroup $K_\infty \subset G^{(1)}(\mathbb{R})$:

$$K_\infty = e(\phi) = \{\begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix} |\phi \in \mathbb{R}\} \text{ and } K_\infty = \{\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} |\alpha\bar{\alpha} + \beta\bar{\beta} = 1\} \tag{4.10}$$

We define extensions $\tilde{K}_\infty = Z(\mathbb{R})^{(0)} K_\infty$, where of course $Z(\mathbb{R})^{(0)}$ is the connected component of the identity. In both cases the group K_∞ is the group of fixed points under the Cartan involution Θ_0 which is given by

$$\Theta_0 : g \mapsto^t g^{-1} \text{ resp. } g \mapsto^t \bar{g}^{-1} \text{ i.e. } \Theta_0(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} \bar{d} & -\bar{c} \\ -\bar{b} & \bar{a} \end{pmatrix}. \tag{4.11}$$

This involution induces an involution on $\mathfrak{g}^{(1)}$ we can extend it to an involution acting on $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}^{(1)}$, we let it act trivially on $\mathfrak{z}$. Then the fixed point Lie algebra $\tilde{\mathfrak{k}} = \mathfrak{z} \oplus \mathfrak{k} \subset \mathfrak{z} \oplus \mathfrak{g}^{(1)}$ is the Lie-algebra of $\tilde{K}_\infty$.

Here are some arithmetic considerations, they may not be so relevant, but further down we make some choices of a basis in some of these algebras, and these choices can be justified by these arithmetic considerations.

We can write our group scheme $G/\mathbb{R}$ as a base extension of a group scheme $\mathcal{G}/\mathbb{Z}$, i.e. $G/\mathbb{R} = \mathcal{G} \times_{\mathbb{Z}} \mathbb{R}$. For this we simply take $\mathcal{G}/\mathbb{Z} = \mathrm{Gl}_2/\mathbb{Z}$ in case A). In case B) we take $\mathcal{G}/\mathbb{Z} = R_{\mathbb{Z}[i]/\mathbb{Z}}(\mathrm{Gl}_2/\mathbb{Z}[i])$. In the case A) this gives a reductive group scheme over $\mathbb{Z}$, in case B) it is only a flat group scheme, but the base extension $\mathcal{G} \times_{\mathbb{Z}} \mathbb{Z}[1/2]$ is reductive. (This group scheme over $\mathbb{Z}$ is not semi-simple since $\mathbb{Z}[i]$ is ramified at the prime 2.)

Now it is clear that Θ_0 is actually an automorphism of $\mathcal{G}/\mathbb{Z}$ and then it follows that the scheme of fixed points is again a group scheme $\mathcal{K}/\mathbb{Z}$. If we define $R = \mathbb{Z}[1/2]$ then $\mathcal{K} \times_{\mathbb{Z}} R$ is actually eductive. (If we replace $\mathbb{Z}[i]$ by the ring of integers of another imaginary quadratic extension, we have to modify R accordingly.)

Consequently we see that the all the above Lie-algebras are defined over R, hence they actually are free R modules, we denote them by $\mathfrak{g}_R$ and so on.

The Cartan Θ_0 involution induces an involution on the Lie algebras $\mathfrak{g}_R, \mathfrak{g}_R^{(1)}$, the module decomposes into a + and a − eigenspace CaDec

$$\mathfrak{g}_R = \tilde{\mathfrak{k}}_R \oplus \mathfrak{p}_R \text{ and } \mathfrak{g}_R^{(1)} = \mathfrak{k}_R \oplus \mathfrak{p}_R, \tag{4.12}$$

The + eigenspaces $\tilde{\mathfrak{k}}_R, \mathfrak{k}_R$ are the Lie-algebras of $\tilde{\mathcal{K}}, \mathcal{K}$, both summands in the decompositions are $\tilde{\mathcal{K}}$-modules.

The Lie-algebra $\mathfrak{b}_R$ is not stable under Θ_0, it is clear that the intersection

$$\mathfrak{b}_R \cap \Theta_0(\mathfrak{b}_R) = \mathfrak{t}_R,$$

where $\mathfrak{t}_R$ is the Lie-algebra of the standard maximal torus $\mathcal{T}/R \subset \mathcal{G}/R$. This torus is a product (up to isogeny) $\mathcal{T}/R = \mathcal{Z} \cdot \mathcal{T}^{(1)}/R$.

In case A) the torus $\mathcal{T}^{(1)}/R \xrightarrow{\sim} \mathbb{G}_m/R$ and the Cartan involution Θ_0 acts by $t \mapsto t^{-1}$. Therefore it acts by -1 on $\mathfrak{t}_R^{(1)}$. We write

$$\mathfrak{t}_R^{(1)} = R\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = RH \tag{4.13}$$

the generator H is unique up to an element in $R^\times$, i.e. up to a sign and a power of 2.

In case B) the torus $\mathcal{T}^{(1)}/R$ is (up to isogeny) a product $\mathcal{T}_s^{(1)} \cdot \mathcal{T}_c^{(1)}/R$ the Cartan involution Θ_0 acts by $t \to t^{-1}$ on the split component $\mathcal{T}_s^{(1)}$ and by the identity on $\mathcal{T}_c^{(1)}$. The Lie-algebra decomposes accordingly into two summands of rank one:

$$\mathfrak{t}_R^{(1)} = R\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus R\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = RH \oplus RH_i.$$

In both cases the group scheme $\mathcal{K}$ acts on $\mathfrak{p}_R$ by the adjoint action, we can describe this action explicitly.

In case A) the group scheme $\mathcal{K}$ is the following group of matrices

$$\mathcal{K} = \{\alpha = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} | a^2 + b^2 = 1\}$$

this is a torus over R which splits over $R[i]$. We have

$$\mathfrak{p}_R = RH \oplus R \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = RH \oplus RV$$

and $\mathrm{Ad}(\alpha)(H) = (a^2 - b^2)H - 2abV,\ \mathrm{Ad}(\alpha)(V) = 2abH + (a^2 - b^2)V$. Since the torus splits over $\mathbb{Z}[i]$ we can decompose $\mathfrak{p} \otimes R[i]$ into weight spaces, we introduce the basis elements

$$P_+ := H - V \otimes i,\ P_- := H + V \otimes i \in \mathfrak{p} \otimes R[i]$$

then Ppm

$$\mathrm{Ad}(\alpha)P_+ = (a + bi)^2 P_+,\ \mathrm{Ad}(\alpha)P_- = (a - bi)^2 P_- \tag{4.14}$$

Hence we get - in case A) -the decomposition

$$\mathfrak{g}_R^{(1)} = \mathfrak{k}_R \oplus \mathfrak{p}_R = R \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus RP_+ \oplus RP_- = RY \oplus RP_+ \oplus RP_- \tag{4.15}$$

where the generators are unique up to an element in $R[i]^\times$.

In case B) the group scheme $\mathcal{K}/R$ is semi simple, it contains $\mathcal{T}_c^{(1)}/R$ as maximal torus. The two $\mathcal{K}/R$ modules $\mathfrak{k}_R$ and $\mathfrak{p}_R$ are highest weight modules of rank 3, since 2 is invertible in R they are even isomorphic. Again we can decompose them into rank one weight spaces and give almost canonical generators for these weight spaces. basisfkfp The Lie algebra

$$\mathfrak{k}_R = RH_i \oplus R \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus R \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = RH_i \oplus RY \oplus F_i. \tag{4.16}$$

We introduce the elements $P_{c+} = Y - F_i \otimes i,\ P_{c-} = Y + F_i \otimes i$ and then

$$\mathfrak{k}_R \otimes R[i] = R[i]H_i \oplus R[i]P_{c+} \oplus R[i]P_{c-}. \tag{4.17}$$

This is the decomposition into weight spaces under the action of $\mathcal{T}_c^{(1)}/R$, the element $\alpha = \begin{pmatrix} x & 0 \\ 0 & \bar{x} \end{pmatrix}$ acts via the adjoint action

$$\mathrm{Ad}(\alpha)P_{c+} = x^2 P_{c+}\ ,\ \mathrm{Ad}(\alpha)H_i = H_i\ ,\ \mathrm{Ad}(\alpha)P_{c-} = x^{-2}P_{c-}.$$

Essentially the same can be done for $\mathfrak{p}_R \otimes R[i]$. We define

$$P_{p,+} = V - \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \otimes i,\ P_{p,-} = V + \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \otimes i$$

then we get the weight decomposition basisfp

$$\mathfrak{p}_R \otimes R[i] = R[i]P_{\mathfrak{p},+} \oplus R[i]H \oplus R[i]P_{\mathfrak{p},-} \tag{4.18}$$

Rational characters vs continuous characters

Our aim is to to construct certain irreducible (differentiable) representations of $G(\mathbb{R})$ together with their "algebraic skeleton" the associated Harish-Chandra modules.

For any torus $T/\mathbb{R}$ we consider the group of (continuous) characters $\mathrm{Hom}(T(\mathbb{R}),\mathbb{C}^\times)$, we write this group multiplicatively, i.e. $\chi_1\cdot\chi_2(x)=\chi_1(x)\chi_2(x)$. We also have defined the group of (rational) characters $X^*(T\times_\mathbb{R}\mathbb{C},\mathbb{G}_m)$ (See Chap. 1, 1.5), and we have the evaluation map

$$X^*(T\times_\mathbb{R}\mathbb{C},\mathbb{G}_m)\xrightarrow{ev}\mathrm{Hom}(T(\mathbb{R}),\mathbb{C}^\times);\ ev:\gamma\mapsto\gamma_\mathbb{R}=\{x\mapsto\gamma(x)\} \qquad (4.19)$$

Since we wrote the group of (rational) characters additively we have

$$(\gamma_1+\gamma_2)_\mathbb{R}=\gamma_{1\mathbb{R}}\cdot\gamma_{2\mathbb{R}}.$$

We also introduce the character $|\gamma|:=\{x\mapsto|\gamma_R(x)|_\mathbb{C}\}$ where of course $|a|_\mathbb{C}=a\bar{a}$.

4.1.3 The induced representations

We start from a continuous homomorphism (a character) $\chi:T(\mathbb{R})\to\mathbb{C}^\times$, of course this can also be seen as a character $\chi:B(\mathbb{R})\to\mathbb{C}^\times$. This allows us to define the induced module

$$I_B^G\chi:=\{f:G(\mathbb{R})\to\mathbb{C}\mid f\in\mathcal{C}_\infty(G(\mathbb{R})),f(bg)=\chi(b)f(g),\ \forall\, b\in B(\mathbb{R}),g\in G(\mathbb{R})\} \qquad (4.20)$$

where we require that f should be $\mathcal{C}_\infty$. Then this space of functions is a $G(\mathbb{R})$ -module, the group $G(\mathbb{R})$ acts by right translations: For $f\in I_B^G\chi, g\in G(\mathbb{R})$ we put

$$R_g(f)(x)=f(xg)$$

If modify our character χ by a character $\delta\circ\det$ where $\delta:Z(\mathbb{R})\to\mathbb{C}^\times$ then the central character gets multiplied by δ^2.

We know that $G(\mathbb{R})=B(\mathbb{R})\cdot\tilde{K}_\infty$. This implies that a function $f\in I_B^G\chi$ is determined by its restriction to K_∞. In other words we have an identification of vector spaces Iwasawa

$$I_B^G\chi=\{f:\tilde{K}_\infty\to\mathbb{C}\mid f(t_ck)=\chi(t_c)f(k),t_c\in\tilde{K}_\infty\cap B(\mathbb{R}),k\in\tilde{K}_\infty\}. \qquad (4.21)$$

The center acts by the central character ω_χ, the restriction of χ to $Z(\mathbb{R})$.

We put $T_c=B(\mathbb{R})\cap\tilde{K}_\infty$ and define χ_c to be the restriction of χ to T_c. Then the module on the right in the above equation can be written as $I_{T_c}^{\tilde{K}_\infty}\chi_c$. By its very definition $I_{T_c}^{\tilde{K}_\infty}\chi_c$ is only a K_∞ module. Inside $I_{T_c}^{\tilde{K}_\infty}\chi_c$ we have the submodule of vectors of finite type

$${}^\circ I_{T_c}^{\tilde{K}_\infty}\chi_c:=\{f\in I_{T_c}^{\tilde{K}_\infty}\chi_c\mid \text{ the translates } R_k(f) \text{ lie in a finite dimensional subspace}\} \qquad (4.22)$$

Here it suffices to consider only the translates $R_k(f)$ for $k \in K_\infty$ because $Z(\mathbb{R})^{(0)}$ acts by the character ω_χ. The famous Peter-Weyl theorem tells us that all irreducible representations (satisfying some continuity condition) are finite dimensional and occur with finite multiplicity in $I_{T_c}^{\tilde{K}_\infty}\chi_c$ and therefore we get

$$ {}^\circ I_{T_c}^{\tilde{K}_\infty}\chi_c = \bigoplus_{\vartheta \in \hat{K}_\infty} V_\vartheta^{m(\vartheta)} = \bigoplus_{\vartheta \in \hat{K}_\infty} {}^\circ I_{T_c}^{\tilde{K}_\infty}\chi_c[\vartheta] \tag{4.23} $$

where $\hat{K}_\infty$ is the set of isomorphism classes of irreducible representations of K_∞, where V_ϑ is an irreducible module of type ϑ and where $m(\vartheta)$ is the multiplicity of ϑ in ${}^\circ I_{T_c}^{\tilde{K}_\infty}\chi_c$. Of course ${}^\circ I_{T_c}^{\tilde{K}_\infty}\chi_c$ is a submodule $I_B^G\chi$, but this submodule is not invariant invariant under the operation of $G(\mathbb{R})$, in other words if $0 \neq f \in {}^\circ I_{T_c}^{\tilde{K}_\infty}\chi_c$ and $g \in G(\mathbb{R})$ a sufficiently general element then $R_g(f) \notin {}^\circ I_{T_c}^{\tilde{K}_\infty}\chi_c$.

We can differentiate the action of $G(\mathbb{R})$ on $I_B^G\chi$. We have the well known exponential map $\exp : \mathfrak{g} = \mathrm{Lie}(G/\mathbb{R}) \to G(\mathbb{R})$ and for $f \in I_B^G, X \in \mathfrak{g}$ we define

$$ Xf(g) = \lim_{t \to 0} \frac{f(g\exp(tX)) - f(g)}{t} \tag{4.24} $$

and it is well known and also easy to see, that this gives an action of the Lie-algebra on I_B^G, we have $X_1(X_2f) - X_2(X_1f) = [X_1, X_2]f$. The Lie-algebra is a K_∞ module under the adjoint action and is obvious that for $f \in {}^\circ I_{T_c}^{\tilde{K}_\infty}\chi_c[\vartheta]$ the element Xf lies in $\bigoplus_{\vartheta \in \hat{K}_\infty} {}^\circ I_{T_c}^{\tilde{K}_\infty}\chi_c[\vartheta']$ where ϑ' runs over the finitely many isomorphism types occurring in $V_\vartheta \otimes \mathfrak{g}$. Hence

Proposition 4.1.1. *The submodule* ${}^\circ I_{T_c}^{\tilde{K}_\infty}\chi_c \subset I_B^G\chi$ *is invariant under the action of* $\mathfrak{g}$.

The submodule ${}^\circ I_{T_c}^{\tilde{K}_\infty}\chi_c$ together with this action of $\mathfrak{g}$ will now be denoted by $\mathfrak{I}_B^G\chi$. We should think of this module as the algebraic skeleton of $I_B^G\chi$.

Such a module will be called a $(\mathfrak{g}, K_\infty)$ - module or a Harish-Chandra module this means that we have an action of the Lie-algebra $\mathfrak{g}$, an action of K_∞ and these two actions satisfy some obvious compatibility conditions.

We also observe that ${}^\circ I_{T_c}^{\tilde{K}_\infty}\chi_c$ is also invariant under right translation R_z for $z \in Z(\mathbb{R})$. Hence we can extend the action of K_∞ to the larger group $\tilde{K}_\infty = K_\infty \cdot Z(\mathbb{R})$. Then $\mathfrak{I}_B^G\chi$ becomes a $(\mathfrak{g}, \tilde{K}_\infty)$ module. Finally observe that in the case A) the element complexcon

$$ \mathbf{c} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \notin \tilde{K}_\infty, \tag{4.25} $$

clearly $R_{\mathbf{c}}$ induces an involution on $\mathfrak{I}_B^G$. We could also say that we can enlarge K_∞(resp. $\tilde{K}_\infty$) to subgroups $K_\infty^*(resp.\tilde{K}_\infty^*)$ which contain $\mathbf{c}$ and contain K_∞ resp. $\tilde{K}_\infty$ as subgroups of index two. Then $\mathfrak{I}_B^G\chi$ also becomes a $(\mathfrak{g}, \tilde{K}_\infty^*)$ module.

These $(\mathfrak{g}, \tilde{K}_\infty)$ modules $\mathfrak{I}_B^G\chi$ are called the *principal series modules.* We have the following important

Theorem 4.1.1. *For any irreducible Harish-Chandra module*$(\mathfrak{g}, \tilde{K}_\infty)$ *we can find a* χ *such that we have an embedding of* $(\mathfrak{g}, \tilde{K}_\infty)$*-modules*

$$i : \mathcal{V} \hookrightarrow \mathfrak{I}_B^G \chi$$

This is actually a much more general theorem and applies *mutatis mutandis* to all reductive groups over $\mathbb{R}$. In the following we will see, that in our situation we only have a very short list of submodules of the $\mathfrak{I}_B^G \chi$ and we get a complete list of irreducible Harish-Chandra modules.

We denote the restriction of χ to the central torus $Z = \{\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}\}$ by ω_χ. Then $Z(\mathbb{R})$ acts on $\mathfrak{I}_B^G \chi$ by the *central character* character ω_χ, i.e. $R_z(f) = \omega_\chi(z) f$. Once we fix the central character, then there is no difference between $(\mathfrak{g}, \tilde{K}_\infty)$ and $(\mathfrak{g}, K_\infty)$ modules. Hence we always assume that ω_χ is fixed.

The decomposition into K_∞-types

Kutypes

We look briefly at the K_∞-module ${}^\circ I_{T_c}^{\tilde{K}_\infty} \chi_c$. In case A) the group

$$K_\infty = \mathrm{SO}(2) = \{\begin{pmatrix} \cos(\varphi) & \sin(\phi) \\ -\sin(\varphi) & \cos(\varphi) \end{pmatrix} = e(\varphi)\} \tag{4.26}$$

and $T_c = K_\infty^T = T(\mathbb{R}) \cap K_\infty$ is cyclic of order two with generator $e(\pi)$. Then χ_c is given by an integer $\mod 2$, i.e. $\chi_c(e(\varphi)) = (-1)^m$. For any $n \equiv m \mod 2$ we define $\psi_n \in \mathfrak{I}_B^G \chi$ by

$$\psi_n(e(\phi))) = e^{in\phi} \tag{4.27}$$

and then decoKuA

$$\mathfrak{I}_B^G \chi = \bigoplus_{k \equiv m \mod 2} \mathbb{C}\psi_k \tag{4.28}$$

In the case B) the maximal compact subgroup is

$$U(2) \subset G(\mathbb{R}) = R_{\mathbb{C}/\mathbb{R}}(\mathrm{Gl}_2/\mathbb{C})(\mathbb{R}) \subset \mathrm{Gl}_2(\mathbb{C}) \times \mathbb{G}_2(\mathbb{C})$$

this is the group of real points of the reductive group $U(2)/\mathbb{R}$. The intersection

$$T_c = K_\infty^T = T(\mathbb{R}) \cap K_\infty = \{\begin{pmatrix} e^{2\pi i \varphi_1} & 0 \\ 0 & e^{2\pi i \varphi_2} \end{pmatrix} = e(\underline{\phi})\}.$$

The base change $U(2) \times \mathbb{C} = \mathrm{Gl}_2/\mathbb{C}$ and $T_c \times \mathbb{C}$ becomes the standard maximal compact torus. The irreducible finite dimensional $U(2)$-modules are labelled by dominant highest weights $\lambda_c = n\gamma_c + d\det \in X^*(T_c \times \mathbb{C})$ (See section (4.1.1), here again $n \geq 0, n \in \mathbb{Z}, n \equiv 2d \mod 2$ and $\gamma_c(e(\underline{\phi})) = e^{i(\phi_1 - \phi_2)/2}$.)

We denote these modules by $\mathcal{M}_{\lambda_c}$ after base change to $\mathbb{C}$ they become the modules $\mathcal{M}_{\lambda,\mathbb{C}}$.

As a subgroup of $G(\mathbb{R}) \subset \mathrm{Gl}_2(\mathbb{C}) \times \mathbb{G}_2(\mathbb{C})$ our torus is

$$T_c = \{\begin{pmatrix} e^{2\pi i\varphi_1} & 0 \\ 0 & e^{2\pi i\varphi_2}\end{pmatrix} \times \begin{pmatrix} e^{-2\pi i\varphi_1} & 0 \\ 0 & e^{-2\pi i\varphi_2}\end{pmatrix}\} \xrightarrow{\sim} \{\begin{pmatrix} e^{2\pi i\varphi_1} & 0 \\ 0 & e^{2\pi i\varphi_2}\end{pmatrix}\} \tag{4.29}$$

and the restriction of χ to T_c is of the form

$$\chi_c(e(\underline{\phi})) = e^{ia\phi_1 + ib\phi_2} = e^{\frac{a-b}{2}(\phi_1-\phi_2)} e^{\frac{a+b}{2}(\phi_1+\phi_2)}. \tag{4.30}$$

and this character is $(a-b)\gamma_c + \frac{a+b}{2}\det$. Then we know

decoKuB

$$^\circ I_{T_c}^{\tilde{K}_\infty} \chi_c = \mathfrak{I}_B^G \chi = \bigoplus_{\mu_c = k\gamma_c + \frac{a+b}{2}\det; k \equiv (a-b) \mod 2; k \geq |a-b|} \mathcal{M}_{\mu_c} \tag{4.31}$$

IndInt

4.1.4 Intertwining operators

Let $N(T)$ the normalizer of $T/\mathbb{R}$, the quotient $W = N(T)/T$ is a finite group scheme. The in our case the group $W(\mathbb{R})$ is cyclic of order 2 and generated by

$$w_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

In case A) we have $W(\mathbb{R}) = W(\mathbb{C})$ in case B) we have

$$G \times_{\mathbb{R}} \mathbb{C} = (\mathrm{Gl}_2 \times \mathrm{Gl}_2)/\mathbb{C}\ ;\ T \times_{\mathbb{R}} \mathbb{C} = T_1 \times T_2\ ;\ \text{ and } W(\mathbb{C}) = \mathbb{Z}/2 \times \mathbb{Z}/2,$$

where the two factors are generated by $s_1 = (w_0, 1), s_2 = (1, w_0)$. The group $W(\mathbb{R})$ is the group of real points of the Weyl group, the group $W = W(\mathbb{C})$ is the Weyl group or the absolute Weyl group.

We introduces the special character

$$|\rho|_{\mathbb{R}} : \begin{pmatrix} t_1 & u \\ 0 & t_2 \end{pmatrix} \to |\frac{t_1}{t_2}|^{\frac{1}{2}},$$

here the absolute value $|t|$ is the usual absolute value if we are in case A) and $|z| = z\bar{z}$ for $z \in \mathbb{C}$, i.e if we are in case B). The group $W(\mathbb{R})$ acts on $T(\mathbb{R})$ by conjugation and hence it also acts on the group $\mathrm{Hom}(T(\mathbb{R}), \mathbb{C}^\times)$ of characters, we denote this action by $\chi \mapsto \chi^w$. We write this group of characters multiplicatively an we define the twisted action

$$w \cdot \chi = (\chi|\rho|)^w |\rho_{\mathbb{R}}|^{-1}$$

We recall some well known facts

i) We have a non degenerate $(\mathfrak{g}, K_\infty)$ invariant pairing

$$\mathfrak{I}_B^G \chi \times \mathfrak{I}_B^G \chi^{w_0} |\rho|^2_{\mathbb{R}} \to \mathbb{C}\omega_\chi^2 \text{ given by } (f_1, f_2) \mapsto \int_{K_\infty} f_1(k) f_2(k) dk \tag{4.32}$$

We define the dual $\mathfrak{I}_B^{G,\vee}\chi$ of a Harish-Chandra as a submodule of $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{I}_B^G\chi, \mathbb{C})$, it consists of those linear maps which vanish on almost all K_∞ types. It is clear that this is again a $(\mathfrak{g}, K_\infty)$-module. The above assertion can be reformulated

ii) We have an isomorphism of $(\mathfrak{g}, K_\infty)$ modules

$$\mathfrak{I}_B^G\chi(\omega_\chi \circ \det)^{\cdot -1} \to \mathfrak{I}_B^{G,\vee}\chi^{w_0}|\rho|_{\mathbb{R}}^2 \tag{4.33}$$

The group $T(\mathbb{R}) = T_c \times (\mathbb{R}_{>0}^\times)^2$ and hence we can write any character χ in the form char

$$\chi(t) = \chi_c(t)|t_1|^{z_1}|t_2|^{z_2} = |\frac{t_1}{t_2}|^{\frac{z_1-z_2}{2}}|t_1t_2|^{\frac{z_1+z_2}{2}} \tag{4.34}$$

where $z_1, z_2 \in \mathbb{C}$. We put $z = z_1 - z_2$ and $\zeta = z_1 + z_2$ The relevant variable is z.

For $f \in \mathfrak{I}_B^G\chi, g \in G(\mathbb{R})$ we consider the integral

$$T_\infty^{\mathrm{loc}}(f)(g) = \int_{U(\mathbb{R})} f(w_0ug)du \tag{4.35}$$

It is well known and easy to check that these integrals converge absolutely and locally uniformly for $\Re(z) >> 0$ and provide an intertwining operator

$$T_\infty^{\mathrm{loc}}(\chi^{w_0}, z) : \mathfrak{I}_B^G\chi^{w_0}|\rho|_{\mathbb{R}}^z \to \mathfrak{I}_B^G\chi|\rho|_{\mathbb{R}}^2|\rho|_{\mathbb{R}}^{-z.} \tag{4.36}$$

it is also not hard to see that they extend to meromorphic functions in the entire $\mathbb{C}^2$. To see this we recall the decomposition into K_∞ types

$$\mathfrak{I}_B^G\lambda_{\mathbb{R}}^{w_0}|\rho|_{\mathbb{R}}^z = \bigoplus_{\vartheta \in \hat{K}_\infty} {}^\circ I_{T_c}^{\tilde{K}_\infty}\chi_c[\vartheta] = \bigoplus_{\vartheta \in \hat{K}_\infty} \mathfrak{I}_B^G\lambda_{\mathbb{R}}^{w_0}|\rho|_{\mathbb{R}}^z[\vartheta]$$

and our intertwining operator is a direct sum of linear maps between finite dimensional vector spaces

$$c(\lambda_{\mathbb{R}}^{w_0}, z, \vartheta) : \mathfrak{I}_B^G\chi^{w_0}|\rho|_{\mathbb{R}}^z[\vartheta] \to \mathfrak{I}_B^G\chi|\rho|_{\mathbb{R}}^2|\rho|_{\mathbb{R}}^{-z}[\vartheta]$$

The finite dimensional vector spaces do not depend on z and the $c(\lambda_{\mathbb{R}}^{w_0}|\rho|_{\mathbb{R}}^z, \vartheta)$ can be expressed in terms of values of the $\Gamma-$ function. Especially they are meromorphic functions in the variable z (See sl2neu.pdf,). For any $z_0 \in \mathbb{C}$ where we have a pole we can find an integer $m \geq 0$ such that

$$(z - z_0)^m T_\infty^{\mathrm{loc}}(\chi^{w_0}, z) : \mathfrak{I}_B^G\chi^{w_0} \to \mathfrak{I}_B^G\chi|\rho|_{\mathbb{R}}^2$$

is a non zero intertwining operator and this is now our regularized operator $T_\infty^{\mathrm{loc,reg}}(\chi^{w_0})$.

iii) The regularized values define non zero intertwining operators

$$T_\infty^{\mathrm{loc},reg}(\chi^{w_0}, z) : \mathfrak{I}_B^G\chi \to \mathfrak{I}_B^G\chi^{w_0}|\rho|_{\mathbb{R}}^2 \tag{4.37}$$

These operators span the one dimensional space of intertwining operators

$$\mathrm{Hom}_{(\mathfrak{g},K_\infty)}(\mathfrak{I}_B^G\chi, \mathfrak{I}_B^G w_0 \cdot \chi).$$

Finally we discuss the question which of these representations are unitary. This means that we have to find a pairing

$$\psi : \mathfrak{I}_B^G\chi \times \mathfrak{I}_B^G\chi \to \mathbb{C} \tag{4.38}$$

which satisfies

a) it is linear in the first and conjugate linear in the second variable

b) It is positive definite, i.e. $\psi(f,f) > 0\, \forall f \in \mathfrak{I}_B^G\chi$

c) It is invariant under the action of K_∞ and Lie-algebra invariant under the action of $\mathfrak{g}$, i.e. we have

For $f_1, f_2 \in \mathfrak{I}_B^G\chi$ and $X \in \mathfrak{g}$ we have $\psi(Xf_1, f_2) + \psi(f_1, Xf_2) = 0.$

We are also interested in quasi-unitatry modules. This is notion is perhaps best explained if and instead of c) we require

d) There exists a continuous homomorphism (a character) $\eta : G(\mathbb{R}) \to \mathbb{R}^\times$ such that $\psi(gf_1, gf_2) = \eta(g)\psi(f_1, f_2),\ \forall g \in G(\mathbb{R}), f_1 f_2 \in \mathfrak{I}_B^G\chi$.

It is clear that a non zero pairing ψ which satisfies a) and c) is the same thing as a non zero $(\mathfrak{g}, K_\infty)$-module linear map

$$i_\psi : \mathfrak{I}_B^G\chi \to \overline{(\mathfrak{I}_B^G\chi)^\vee} \tag{4.39}$$

this means that i_ψ is a conjugate linear map from $\mathfrak{I}_B^G\chi$ to $(\mathfrak{I}_B^G\chi)^\vee$. The map i_ψ and the pairing ψ are related by the formula $\psi(v_1, v_2) = i_\psi(v_2)(v_1)$.

Of course we know that (See (4.33))

$$\overline{(\mathfrak{I}_B^G\chi)^\vee} \xrightarrow{\sim} \mathfrak{I}_B^G\overline{\chi^{w_0}|\rho|^2_{\mathbb{R}}\delta_\chi^{-1}} \tag{4.40}$$

and we find such an i_ψ if

$$\chi = \overline{\chi^{w_0}|\rho|^2_{\mathbb{R}}\delta_\chi^{-1}} \text{ or } \chi^{w_0}|\rho|^2_{\mathbb{R}} = \overline{\chi^{w_0}|\rho|^2_{\mathbb{R}}\delta_\chi^{-1}} \tag{4.41}$$

We write our χ in the form (4.34). A necessary condition for the existence of a hermitian form is of course that all $|\omega_\chi(x)| = 1$ for $x \in Z(\mathbb{R})$ and this means that $\Re(z_1 + z_2) = 0$, hence we write

$$z_1 = \sigma + i\tau_1, z_2 = -\sigma + i\tau_2 \tag{4.42}$$

Then the two conditions in (4.41) simply say

$$(\mathrm{un}_1) : \sigma = \frac{1}{2} \text{ or } (\mathrm{un}_2) : \tau_1 = \tau_2 \text{ and } \chi_c = \chi_c^{w_0} \tag{4.43}$$

In both cases we can write down a pairing which satisfies a) and c). We still have to check b). In the first case, i.e. $\sigma = \frac{1}{2}$ we can take the map $i_\psi = \mathrm{Id}$ and then we get for $f_1, f_2 \in \mathfrak{I}_B^G\chi$ the formula

$$\psi(f_1, f_2) = \int_{K_\infty} f_1(k)\overline{f_2(k)}dk \tag{4.44}$$

and this is clearly positive definite. These are the representation of the unitary principal series.

In the second case we have to use the intertwining operator in (4.37) and write

$$\psi(f_1, f_2) = T_\infty^{\text{loc,reg}}(f_2)(f_1) \tag{4.45}$$

Now it is not clear whether this pairing satisfies b). This will depend on the parameter σ. We can twist by a character $\eta : Z(\mathbb{R}) \to \mathbb{C}^\times$ and achieve that $\chi_c = 1, \tau_1 = \tau_2 = 0$. We know that for $\sigma = \frac{1}{2}$ the intertwining operator T_∞^{loc} is regular at χ and since in addition under these conditions $\mathfrak{I}_B^G\chi$ is irreducible we see that

$$T_\infty^{\text{loc}}(\chi) = \alpha \text{ Id with } \alpha \in \mathbb{R}_{>0}^\times \tag{4.46}$$

Since we now are in case A) and B) at the same time we see that the two pairings defined by the rule in case (un_1) and (un_2) differ by a positive real number hence the pairing defined in (4.45) is positive definite if $\sigma = \frac{1}{2}$.

But now we can vary σ. It is well known that $\mathfrak{I}_B^G\chi$ stays irreducible as long as $0 < \sigma < 1$ (See next section) and since $T_\infty^{\text{loc}}(\chi)(f)(f)$ varies continuously we see that (4.45) defines a positive definite hermitian product on $\mathfrak{I}_B^G\chi$ as long as $0 < \sigma < 1$. This is the supplementary series. What happens if we leave this interval will be discussed in the next section.

nontriv

4.1.5 Reducibility and representations with non trivial cohomology

As usual we denote by $\rho \in X^*(T) \otimes \mathbb{Q}$ the half sum of positive roots we have $\rho = \gamma($ resp. $\rho = \gamma_1 + \gamma_2 \in X^*(T) \otimes \mathbb{Q})$ in case A) (resp. B)).

For any character $\lambda \in X^*(T \times \mathbb{C})$ the character $\lambda_\mathbb{R}$ provides a homomorphism $B(\mathbb{R}) \to T(\mathbb{R})$ and hence we get the Harish-Chandra modules $\mathfrak{I}_B^G\lambda_\mathbb{R}$, which are of special interest for us because these are the only ones with non trivial cohomology. We just mention the fact that $\mathfrak{I}_B^G\chi$ is always irreducible unless $\chi = \lambda_\mathbb{R}$ for some λ. (See sl2neu.pdf, Condition (red)).

We return to the situation discussed in section (4.1.1), especially we reintroduce the field $F/\mathbb{Q}$. Then we have $X^*(T \times F) = X^*(T \times \mathbb{C})$ and hence $\lambda \in X^*(T \times F)$. We assume that λ is dominant, i.e. $n \geq 0$ in case A) or $n_1, n_2 \geq 0$ in case B). Now we realise our modules $\mathcal{M}_\lambda$ as submodules in the algebra of regular functions on $\mathcal{G}/\mathbb{Z}$: If we look at the definition (See (4.3)) we see immediately that $\mathcal{M}_{\lambda,\mathbb{C}} \subset \mathfrak{I}_B^G\lambda_\mathbb{R}^{w_0}$ and hence we get an exact sequence of $(\mathfrak{g}, K_\infty)$ modules seq

$$0 \to \mathcal{M}_{\lambda,\mathbb{C}} \to \mathfrak{I}_B^G\lambda_\mathbb{R}^{w_0} \xrightarrow{r} \mathcal{D}_\lambda \to 0 \tag{4.47}$$

Hence we see that $\mathfrak{I}_B^G\lambda_\mathbb{R}^{w_0}$ is not irreducible. We can also look at the dual sequence. Here we recall that we wrote $\lambda = n\gamma + d\det$. We consider the dual sequence. Clearly $\mathcal{M}_{\lambda,\mathbb{C}}^\vee = \mathcal{M}_{\lambda-2d\det,\mathbb{C}}$, if we twist the dual sequence by $\det^{2d}$ then dual sequence becomes

$$0 \to \mathcal{D}_\lambda^\vee \otimes \det_\mathbb{R}^{2d} \to (\mathfrak{I}_B^G\lambda_\mathbb{R}^{w_0})^\vee \otimes \det_\mathbb{R}^{2d} \to \mathcal{M}_{\lambda,\mathbb{C}} \to 0 \tag{4.48}$$

Equation (4.33) yields $(\mathfrak{I}_B^G\lambda_{\mathbb{R}}^{w_0})^\vee \otimes \det_{\mathbb{R}}^{2d} \xrightarrow{\sim} \mathfrak{I}_B^G\chi|\rho|_{\mathbb{R}}^2$ and our second sequence becomes

$$0 \to \mathcal{D}_\lambda^\vee \otimes \det_{\mathbb{R}}^{2d} \to \mathfrak{I}_B^G\lambda_{\mathbb{R}}|\rho|_{\mathbb{R}}^2 \to \mathcal{M}_{\lambda,\mathbb{C}} \to 0, \tag{4.49}$$

we put $\mathcal{D}_{\lambda^\vee} := \mathcal{D}_\lambda^\vee \otimes \det_{\mathbb{R}}^{2d}$.

Now we consider the two middle terms in the two exact sequences (4.47,4.49) above. The equation (4.37) claims that we have two non zero *regularized* intertwining operators

$$T_\infty^{\text{loc,reg}}(\lambda_{\mathbb{R}}^{w_0}) : \mathfrak{I}_B^G\lambda_{\mathbb{R}}^{w_0} \to \mathfrak{I}_B^G\lambda_{\mathbb{R}}|\rho|_{\mathbb{R}}^2 \;\; ; T_\infty^{\text{loc,reg}}(\lambda_{\mathbb{R}}|\rho|_{\mathbb{R}}^2) : \mathfrak{I}_B^G\lambda_{\mathbb{R}}|\rho|_{\mathbb{R}}^2 \to \mathfrak{I}_B^G\lambda_{\mathbb{R}}^{w_0} \tag{4.50}$$

If we now look more carefully at our two regularized intertwining operators above then a simple computation yields (see sl2neu.pdf)

Proposition 4.1.2. *The kernel of $T_\infty^{loc,\text{reg}}(\lambda_{\mathbb{R}}^{w_0})$ is $\mathcal{M}_{\lambda,\mathbb{C}}$ and this operator induces an isomorphism*

$$\bar{T}(\lambda_R) : \mathcal{D}_\lambda \xrightarrow{\sim} \mathcal{D}_\lambda^\vee \otimes \det_{\mathbb{R}}^{2d}$$

Remember λ is dominant.

The kernel of $T_\infty^{loc,\text{reg}}(\lambda_{\mathbb{R}}|\rho|_{\mathbb{R}}^2)$ is $\mathcal{D}_\lambda^\vee \otimes \det_{\mathbb{R}}^{2d}$ and it induces an isomorphism of $\mathcal{M}_{\lambda,\mathbb{C}}$.

The module $\mathfrak{I}_B^G\chi$ is reducible if $T_\infty^{loc,\text{reg}}(\chi)$ not an isomorphism and this happens if an only if $\chi = \lambda_{\mathbb{R}}$ or $\lambda_{\mathbb{R}}^{w_0}|\rho|_{\mathbb{R}}^2$ and λ dominant. (There is one exception to the converse of the above assertion, namely in the case A) and $\sigma = \frac{1}{2}$ and $\chi_c^{w_0} \neq \chi_c$.)

Unitarity

For us it is of relevance to know whether we have a positive definite hermitian form on the $(\mathfrak{g}, K_\infty)$-modules $\mathcal{D}_\lambda$. To discuss this question we treat the cases A) and B) separately.

We look at the decomposition into K_∞-types. (See (4.28)) In case A) (See (4.28)) it is clear that $\mathcal{M}_{\lambda,\mathbb{C}}$ is the direct sum of the K_∞ types $\mathbb{C}\psi_l$ with $|l| \leq n$. Hence $\boxed{\text{KTA}}$

$$\mathcal{D}_\lambda = \bigoplus_{k \leq -n-2, k\equiv d(2)} \mathbb{C}\psi_k \oplus \bigoplus_{k \geq n+2, k\equiv d(2)} \mathbb{C}\psi_k = \mathcal{D}_\lambda^- \oplus \mathcal{D}_\lambda^+ \tag{4.51}$$

Proposition 4.1.3. *The representations $\mathcal{D}_\lambda^-, \mathcal{D}_\lambda^+$ are irreducible, these are the discrete series representations. The element* **c** *interchanges $\mathcal{D}_\lambda^-, \mathcal{D}_\lambda^+$, hence $\mathcal{D}_\lambda$ is an irreducible $(\mathfrak{g}, \tilde{K}_\infty^*)$ module.*

The operator $\bar{T}(\lambda_R)$ induces a quasi-unitary structure on the $(\mathfrak{g}, \tilde{K}_\infty)$-module $\mathcal{D}_\lambda$. The two sets of K_∞ types occurring in $\mathcal{M}_{\lambda,\mathbb{C}}$ and in $\mathcal{D}_\lambda$ (resp.) are disjoint.

Proof. Remember that as a vector space $\mathcal{D}_\lambda^\vee \otimes \det_{\mathbb{R}}^{2d} = \mathcal{D}_\lambda^\vee$, only the way how $\tilde{K}_\infty$ acts is twisted by $\det_{\mathbb{R}}^{2d}$. (???).Then the form

$$h_\psi(f_1, f_2) = T_\infty^{\text{loc,reg}}(\lambda_{\mathbb{R}}^{w_0})(f_2)(f_1) \tag{4.52}$$

defines a quasi invariant hermitian form. It is positive definite (for more details see sl2neu.pdf). □

A similar argument works in case B).We restrict the $\mathrm{Gl}_2(\mathbb{C})\times\mathrm{Gl}_2(\mathbb{C})$ module $\mathcal{M}_{\lambda,\mathbb{C}}$ to $U(2)\times U(2)$ then it becomes the highest weight module $\mathcal{M}_{\lambda_c} = \mathcal{M}_{\lambda_{1,c}}\otimes \mathcal{M}_{\lambda_{2,c}}$. (See4.1.1) Under the action of $U(2) \subset U(2)\times U(2)$ it decomposes into $U(2)$ types according to the Clebsch-Gordan formula CG

$$\mathcal{M}_{\lambda_c}|_{U(2)} = \bigoplus_{\mu_c = k\gamma_c + \frac{d_1+d_2}{2}\det;\ k\equiv(n_1-n_2) \mod 2;\ n_1+n_2\geq k\geq |n_1-n_2|} \mathcal{M}_{\mu_c} \tag{4.53}$$

Hence we get KTB

$$\mathcal{D}_{\lambda_c}|_{U(2)} = \bigoplus_{\mu_c = k\gamma_c + \frac{d_1+d_2}{2}\det;\ k\equiv(n_1-n_2) \mod 2;\ k\geq n_1+n_2+2} \mathcal{M}_{\mu_c} \tag{4.54}$$

Again we have unit

Proposition 4.1.4. *The operator* $T_\infty^{loc,\mathrm{reg}}(\lambda_{\mathbb{R}}^{w_0})$ *induces an isomorphism*

$$\bar{T}(\lambda_R) : \mathcal{D}_\lambda \xrightarrow{\sim} \mathcal{D}_\lambda^\vee \otimes \det{}_{\mathbb{R}}^{2d}$$

The $(\mathfrak{g}, K_\infty)$ *modules are irreducible.*

The operator $T_\infty^{loc,\mathrm{reg}}(\lambda_{\mathbb{R}}^{w_0})$ *induces the structure of a quasi-unitary module on* $\mathcal{D}_\lambda$ *if and only if* $n_1 = n_2$. *This is the only case when we have a quasi-unitary structure on* $\mathcal{D}_\lambda$. *The two sets of* K_∞ *types occurring in* $\mathcal{M}_{\lambda,\mathbb{C}}$ *and in* $\mathcal{D}_\lambda$ *(resp.) are disjoint.*

The Weyl W group acts on T by conjugation, hence on $X^*(T\times\mathbb{C})$ and we define the twisted action by

$$s\cdot\lambda = s(\lambda+\rho)-\rho \tag{4.55}$$

Given a dominant λ we may consider the four characters $w\cdot\lambda, w\in W(\mathbb{C}) = W$ and the resulting induced modules $\mathfrak{I}_B^G w\cdot\lambda_{\mathbb{R}}$. We observe (notation from (4.1.1))

$$\begin{aligned} s_1\cdot(n_1\gamma + d_1\det + n_2\bar{\gamma} + d_2\overline{\det}) &= (-n_1-2)\gamma + d_1\det + n_2\bar{\gamma} + d_2\overline{\det}) \\ s_2\cdot(n_1\gamma + d_1\det + n_2\bar{\gamma} + d_2\overline{\det}) &= n_1\gamma + d_1\det + (-n_2-2)\bar{\gamma} + d_2\overline{\det}) \end{aligned} \tag{4.56}$$

Looking closely we see that the K_∞ types occurring in $\mathfrak{I}_B^G s_1\cdot\lambda$ or $\mathfrak{I}_B^G s_2\cdot\lambda$ are exactly those which occur in $\mathcal{D}_\lambda$. This has a simple explanation, we have exiso

Proposition 4.1.5. *For a dominant character* λ *we have isomorphisms between the* $(\mathfrak{g}, K_\infty)$ *modules*

$$\mathcal{D}_\lambda \xrightarrow{\sim} \mathfrak{I}_B^G s_1\cdot\lambda,\ \mathcal{D}_\lambda \xrightarrow{\sim} \mathfrak{I}_B^G s_2\cdot\lambda. \tag{4.57}$$

The resulting isomorphism $\mathfrak{I}_B^G s_1\cdot\lambda \xrightarrow{\sim} \mathfrak{I}_B^G s_2\cdot\lambda$ *is of course given by* $T_\infty^{\mathrm{loc}}(s_1\cdot\lambda)$.

Interlude: Here we see a fundamental difference between the two cases A) and B). In the second case the infinite dimensional subquotients of the induced representations are again induced representations. In the case A) this is not so, the representations $\mathcal{D}_\lambda^\pm$ are not isomorphic to representations induced from the Borel subgroup.

These representation $\mathcal{D}_\lambda^\pm$ are called *discrete series* representations and we want to explain briefly why. Let G be the group of real points of a reductive group over $\mathbb{R}$ for example our $G = G(\mathbb{R})$, here we allow both cases. Let Z be the center of G, it can be written as $Z_0(\mathbb{R}) \cdot Z_c$ where Z_c is maximal compact and $Z_0 = (\mathbb{R}_{>0}^\times)^t$. Let $\omega^{(0)} : Z_0 \to \mathbb{R}_{>0}^\times$ be a character. Then we define the space

$$\mathcal{C}_\infty(G, \omega_R) := \{f \in \mathcal{C}(G) \mid f(zg) = \omega^{(0)}(z)f(g) \,; \forall z \in Z_0, g \in G\} \qquad (4.58)$$

and we define the subspace

$$L^2_\infty(G, \omega_R) := \{f \in \mathcal{C}_\infty(G, \omega_R) \mid \int_G f(g)\overline{f(g)}(\omega^{(0)}(g))^{-2} dg < \infty\} \qquad (4.59)$$

where of course dg is a Haar measure. As usual $L^2(G, \omega_R)$ will be the Hilbert space obtained by completion. This Hilbert space only depends in a very mild way on the choice of $\omega^{(0)}$ we can find a character $\delta : G \to \mathbb{R}_{>0}^\times$ such that $\omega^{(0)}\delta|_{Z_0} = 1$. Then $f \mapsto f\delta$ provides an isomorphism $L^2(G, \omega^{(0)}) \xrightarrow{\sim} L^2(G/Z_0)$.

We have an action of $G \times G$ on $L^2(G, \omega^{(0)})$ by left and right translations. Then Harish-Chandra has investigated the question how this "decomposes" into irreducible submodules. Let $\hat{G}_{\omega^{(0)}}$ be the set of isomorphism classes of irreducible unitary representations of G.

Harish-Chandra shows that there exist a positive measure μ on $\hat{G}_{\omega^{(0)}}$ and a measurable family H_ξ of irreducible unitary representations of G such that

$$L^2(G, \omega_{\mathbb{R}}) = \int_{\hat{G}_{\omega_{\mathbb{R}}}} H_\xi \otimes \overline{H_\xi}\ \mu(d\xi) \qquad (4.60)$$

(If instead of a semi simple Lie group we take a finite group G then this is the fundamental theorem of Frobenius that the group ring $\mathbb{C}[G] = \oplus_\theta V_\theta \otimes V_\theta^\vee$ where V_θ are the irreducible representations.)

If we are in the case A), the sets consisting of just one point $\{\mathcal{D}_\lambda^\pm\}$ have strictly positive measure, i.e. $\mu(\{\mathcal{D}_\lambda^\pm\}) > 0$. This means that the irreducible unitary $G \times G$ modules $\mathcal{D}_\lambda^\pm \otimes \mathcal{D}_{\lambda^\vee}^\pm$ occur as direct summand (i.e. discretely in $L^2(G)$.).

Such irreducible direct summands do not exist in the case B), in this case for any $\xi \in \hat{G}$ we have $\mu(\{\xi\}) = 0$.

End Interlude

We return to the sequences (4.47),(4.49). We claim that both sequences do do not split as sequences of $(\mathfrak{g}, K_\infty)$-modules. Of course it follows from the above proposition that these sequences split canonically as sequence of K_∞ modules. But one sees easily that complementary summand is not invariant under the action of $\mathfrak{g}$. This means that the sequence provides a non trivial classes in $\mathrm{Ext}^1_{(\mathfrak{g},K_\infty)}(\mathcal{D}_\lambda, \mathcal{M}_{\lambda,\mathbb{C}})$.

The general principles of homological algebra teach us that we can understand these extension groups in terms of relative Lie-algebra cohomology. Let $\mathfrak{k}$ resp. $\tilde{\mathfrak{k}}$ be the Lie-algebras of K_∞ resp. $\tilde{K}_\infty$ the group $\tilde{K}_\infty$ acts on $\mathfrak{g}, \tilde{\mathfrak{k}}$ via the adjoint action (see 1.1.4)

We start from a $(\mathfrak{g}, \tilde{K}_\infty)$ module $\mathfrak{I}_B^G\chi$ and a module $\mathcal{M}_{\lambda,\mathbb{C}}$. Our first goal is to compute the cohomology $H^\bullet(\mathfrak{g}, K_\infty, \mathfrak{I}_B^G\chi \otimes \mathcal{M}_{\lambda,\mathbb{C}})$ which is defined as the cohomology of the complex (See 8.1.2, (8.3))

$$\mathrm{Hom}_{\tilde{K}_\infty}(\Lambda^\bullet(\mathfrak{g}/\tilde{\mathfrak{k}}), \mathfrak{I}_B^G\chi \otimes \mathcal{M}_{\lambda,\mathbb{C}}). \tag{4.61}$$

Here we only assume that $\chi : T(\mathbb{R}) \to \mathbb{C}^\times$ is any character, we will see that there is only one χ for which we have non trivial cohomology.

There is an obvious condition for the complex to be non zero. The group $Z(\mathbb{R}) \subset \tilde{K}_\infty$ acts trivially on $\mathfrak{g}/\mathfrak{k}$ and hence we see that the complex is trivial unless we have

$$\omega_\chi^{-1} = \lambda_\mathbb{R}|_{Z(\mathbb{R})^{(0)}} \tag{4.62}$$

we assume that this relation holds.

We will derive a formula for these cohomology modules, which is a special case of a formula of Delorme. which will be discussed in greater generality in Chapter 9. An element $\omega \in \mathrm{Hom}_{\tilde{K}_\infty}(\Lambda^n(\mathfrak{g}/\tilde{\mathfrak{k}}), \mathfrak{I}_B^G\chi \otimes \mathcal{M}_{\lambda,\mathbb{C}})$ attaches to any n tuple $v_1, \dots, v_n$ of elements in $\mathfrak{g}/\tilde{\mathfrak{k}}$ an element

$$\omega(v_1, \dots, v_n) \in \mathfrak{I}_B^G\chi \otimes \mathcal{M}_{\lambda,\mathbb{C}} \tag{4.63}$$

such that $\omega(\mathrm{Ad}(k)v_1, \dots, \mathrm{Ad}(k)v_n) = k\omega(v_1, \dots, v_n)$ for all $k \in \tilde{K}_\infty$.

By construction

$$\omega(v_1, \dots, v_n) = \sum f_\nu \otimes m_\nu \text{ where } f_\nu \in \mathfrak{I}_B^G\chi, m_\nu \in \mathcal{M}_{\lambda,\mathbb{C}}$$

and f_ν is a function in $\mathcal{C}_\infty$ which is determined by its restriction to $\tilde{K}_\infty$ (and this restriction is $\tilde{K}_\infty$ finite). We can evaluate this function at the identity $e_G \in G(\mathbb{R})$ and then

$$\omega(v_1, \dots, v_n)(e_G) = \sum f_\nu(e) \otimes m_\nu \in \mathbb{C}\chi \otimes \mathcal{M}_{\lambda,\mathbb{C}}$$

The $\tilde{K}_\infty$ invariance (4.63) implies that ω is determined by this evaluation at e_G. Let $\tilde{K}_\infty^T = T(\mathbb{R}) \cap \tilde{K}_\infty = Z(\mathbb{R}) \cdot T_c$. Then it is clear that

$$\omega^* : \{v_1, \dots, v_n\} \mapsto \omega(v_1, \dots, v_n)(e_G) \tag{4.64}$$

is an element in

$$\omega^* \in \mathrm{Hom}_{\tilde{K}_\infty^T}(\Lambda^n(\mathfrak{g}/\tilde{\mathfrak{k}}), \mathbb{C}\chi \otimes \mathcal{M}_{\lambda,\mathbb{C}}) \tag{4.65}$$

and we have: The map $\omega \mapsto \omega^*$ is an isomorphism of complexes $\boxed{\text{iso1}}$.

$$\mathrm{Hom}_{\tilde{K}_\infty}(\Lambda^\bullet(\mathfrak{g}/\tilde{\mathfrak{k}}), \mathfrak{I}_B^G\chi \otimes \mathcal{M}_{\lambda,\mathbb{C}}) \xrightarrow{\sim} \mathrm{Hom}_{\tilde{K}_\infty^T}(\Lambda^\bullet(\mathfrak{g}/\tilde{\mathfrak{k}}), \mathbb{C}\chi \otimes \mathcal{M}_{\lambda,\mathbb{C}}) \tag{4.66}$$

The Lie algebra $\mathfrak{g}$ can be written as a sum of $\mathfrak{c}$ invariant submodules

$$\mathfrak{g} = \mathfrak{b} + \tilde{\mathfrak{k}} = \mathfrak{t} + \mathfrak{u} + \tilde{\mathfrak{k}} \tag{4.67}$$

in case B) this sum is not direct, we have $\mathfrak{b} \cap \tilde{\mathfrak{k}} = \mathfrak{t} \cap \tilde{\mathfrak{k}} = \mathfrak{c}$ and hence we get the direct sum decomposition into $\tilde{K}_\infty^T$-invariant subspaces

$$\mathfrak{g}/\tilde{\mathfrak{k}} = \mathfrak{t}/\mathfrak{c} \oplus \mathfrak{u}. \tag{4.68}$$

We get an isomorphism of complexes isodel

$$\mathrm{Hom}_{\tilde{K}_\infty}(\Lambda^\bullet(\mathfrak{g}/\tilde{\mathfrak{k}}), \mathfrak{I}_B^G\chi \otimes \mathcal{M}_{\lambda,\mathbb{C}}) \xrightarrow{\sim} \mathrm{Hom}_{\tilde{K}_\infty^T}(\Lambda^\bullet(\mathfrak{t}/\tilde{\mathfrak{k}}), \mathbb{C}\chi \otimes \mathrm{Hom}(\Lambda^\bullet(\mathfrak{u}), \mathcal{M}_{\lambda,\mathbb{C}})) \tag{4.69}$$

the complex on the left is isomorphic to the total complex of the double complex on the right. The next step is the computation of the cohomology of the complex $\mathrm{Hom}(\Lambda^\bullet(\mathfrak{u}), \mathcal{M}_{\lambda,\mathbb{C}})$.

Case A). We have $\mathfrak{u} = \mathbb{Q}E_+$ where $E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and our module $\mathcal{M}_{\lambda,\mathbb{Q}}$ has a decomposition into weight spaces

$$\mathcal{M}_{\lambda,\mathbb{Q}} = \bigoplus_{\nu=0}^{\nu} \mathbb{Q}X^{n-\nu}Y^\nu = \bigoplus_{\mu=-n,\mu\equiv n(2)}^{\mu=n} \mathbb{Q}e_\mu. \tag{4.70}$$

The torus $T^{(1)} = \{\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\}$ acts on $e_\mu = X^{n-\nu}Y^\nu$ by

$$\rho_\lambda(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix})e_\mu = t^\mu e_\mu \tag{4.71}$$

We also have the action of the Lie algebra on $\mathcal{M}_{\lambda,\mathbb{Q}}$ and by definition we get

$$d(\rho_\lambda)(E_+)e_\mu = E_+e_\mu = \frac{n-\mu}{2}e_{\mu+2} \tag{4.72}$$

Now we can write down our complex $\mathrm{Hom}(\Lambda^\bullet(\mathfrak{u}), \mathcal{M}_{\lambda,\mathbb{C}})$ very explicitly. Let $E_+^\vee \in \mathrm{Hom}(\mathfrak{u}, \mathbb{Q})$ be the element $E_+^\vee(E_+) = 1$ then the complex becomes

$$0 \to \bigoplus_{\mu=-n,\mu\equiv n(2)}^{\mu=n} \mathbb{Q}e_\mu \xrightarrow{d} \bigoplus_{\mu=-n,\mu\equiv n(2)}^{\mu=n} \mathbb{Q}E_+^\vee \otimes e_\mu \to 0 \tag{4.73}$$

where $d(e_\mu) = \frac{n-\mu}{2}E_+^\vee \otimes e_{\mu+2}$. This gives us a decomposition of our complex into two sub complexes

$$\mathrm{Hom}(\Lambda^\bullet(\mathfrak{u}), \mathcal{M}_{\lambda,\mathbb{C}}) = \mathbb{H}^\bullet(\mathfrak{u}, \mathcal{M}_{\lambda,\mathbb{Q}}) \oplus AC^\bullet \tag{4.74}$$

where $AC^\bullet$ as acyclic (it has no cohomology) and

$$\mathbb{H}^\bullet(\mathfrak{u}, \mathcal{M}_{\lambda,\mathbb{Q}}) = \{0 \to \mathbb{Q}\ e_n \xrightarrow{d} \mathbb{Q}\ E_+^\vee \otimes e_{-n} \to 0\}, \tag{4.75}$$

where the differential d is zero. Hence we get

$$H^\bullet(\mathfrak{u}, \mathcal{M}_{\lambda,\mathbb{Q}}) = H^\bullet(\mathrm{Hom}(\Lambda^\bullet(\mathfrak{u}), \mathcal{M}_{\lambda,\mathbb{Q}})) = \mathbb{H}^\bullet(\mathfrak{u}, \mathcal{M}_{\lambda,\mathbb{Q}}). \tag{4.76}$$

We notice that the torus T acts on $H^\bullet(\mathfrak{u},\mathcal{M}_{\lambda,\mathbb{Q}})$ (The Borel subgroup B acts on the complex $\mathrm{Hom}(\Lambda^\bullet(\mathfrak{u}),\mathcal{M}_{\lambda,\mathbb{Q}})$ but since the Lie algebra cohomology is the derived functor of taking invariants under U (elements annihilated by $\mathfrak{u}$) it follows that this action is trivial on U). Now it is clear that (4.69) yields

$$H^\bullet(\mathfrak{g},K_\infty,\mathfrak{I}_B^G\chi\otimes\mathcal{M}_{\lambda,\mathbb{C}}) \xrightarrow{\sim} H^\bullet(\mathfrak{t},K_\infty^T,\mathbb{C}\chi\otimes\mathbb{H}^\bullet(\mathfrak{u},\mathcal{M}_{\lambda,\mathbb{Q}})) \tag{4.77}$$

Hence we see that T acts by the character λ on $\mathbb{Q}\ e_n = H^0(\mathfrak{u},\mathcal{M}_{\lambda,\mathbb{Q}})$ and by the character $\lambda^- - \alpha = w_0\cdot\lambda = \lambda^{w_0}-2\rho$ on $\mathbb{Q}\ E_+^\vee\otimes e_{-n} = H^1(\mathfrak{u},\mathcal{M}_{\lambda,\mathbb{Q}})$. Here we see the simplest example of the famous theorem of Kostant which will be discussed in Chap. 8 section 8.1.9

Then our cohomology groups $H^\bullet(\mathfrak{t},K_\infty^T,\mathbb{C}\chi\otimes\mathbb{H}^\bullet(\mathfrak{u},\mathcal{M}_{\lambda,\mathbb{Q}}))$ are given as the cohomology groups of the double complex with entries $\mathrm{Hom}_{K_\infty^T}(\Lambda^p(\mathfrak{t}/\mathfrak{k})\mathbb{C}\chi\otimes \mathbb{H}^q(\mathfrak{u},\mathcal{M}_{\lambda,\mathbb{Q}})$ where $p=0,1,q=0,1$ and where the differentials in direction q are zero. We have to compute the cohomology of the complexes

$$0\to\ \mathrm{Hom}_{K_\infty^T}(\Lambda^0(\mathfrak{t}/\mathfrak{k}),\mathbb{C}\chi\otimes\mathbb{H}^q(\mathfrak{u},\mathcal{M}_{\lambda,\mathbb{Q}})) \xrightarrow{d}\ \mathrm{Hom}_{K_\infty^T}(\Lambda^1(\mathfrak{t}/\mathfrak{k}),\mathbb{C}\chi\otimes\mathbb{H}^q(\mathfrak{u},\mathcal{M}_{\lambda,\mathbb{Q}}))\to 0 \tag{4.78}$$

In this complex we drop the subscript K_∞^T then both spaces in the complex are one dimensional and the differential is up to a non zero factor multiplication by $d\chi(H)+d(w\cdot\lambda)(H)$ and hence we have zero cohomology unless we have $d\chi(H)+d(w\cdot\lambda)(H)=0$. Hence we see (observe that $q=l(w)$)

$$H^\bullet(\mathfrak{t},K_\infty^T,\mathbb{C}\chi\otimes\mathbb{H}^q(\mathfrak{u},\mathcal{M}_{\lambda,\mathbb{Q}}))\neq 0 \implies \chi|T(\mathbb{R})^{(0)} = (w\cdot\lambda)_{\mathbb{R}}^{-1}|T(\mathbb{R})^{(0)}.$$

(If χ is the infinity component of a global character $\tilde{\chi}$ on the idele class group then we will say that $\tilde{\chi}$ is of type $w\cdot\lambda$ (see section 6.3.8))

We now reintroduce the subscript K_∞^T. Since clearly $K_\infty^T\cdot T(\mathbb{R})^{(0)} = T(\mathbb{R})$ we see that we have non trivial cohomology if and only if $\chi=(w\cdot\lambda)_{\mathbb{R}}^{-1}$. Putting everything together we see

$$H^{\bullet+l(w)}(\mathfrak{g},K_\infty,\mathfrak{I}_B^G\chi\otimes\mathcal{M}_{\lambda,\mathbb{Q}}) = \begin{cases}\mathbb{H}^{l(w)}(\mathfrak{u},\mathcal{M}_{\lambda,\mathbb{Q}}))\wedge\Lambda^\bullet(\mathfrak{t}/\mathfrak{k})^\vee & \text{if } \chi=(w\cdot\lambda)_{\mathbb{R}}^{-1}\\ 0 & \text{else}\end{cases} \tag{4.79}$$

Now we tensorize the sequence (4.47) with the dual $\mathcal{M}_{\lambda^\vee}$ we get an exact sequence of $(\mathfrak{g},K_\infty)$ modules and we look at the resulting long exact sequence in cohomology. We know that $H^1(\mathfrak{g},K_\infty,\mathcal{M}_\lambda\otimes M_{\lambda^\vee})=0$ and then we look at the piece

$$0\to H^1(\mathfrak{g},K_\infty,\mathfrak{I}_B^G\lambda^{w_0}\otimes\mathcal{M}_{\lambda^\vee})\to H^1(\mathfrak{g},K_\infty,\mathcal{D}_\lambda\otimes\mathcal{M}_{\lambda^\vee})\to H^2(\mathfrak{g},K_\infty,\mathcal{M}_\lambda\otimes\mathcal{M}_{\lambda^\vee})\to 0 \tag{4.80}$$

We have seen and we know that the two extreme terms are equal to $\mathbb{C}$ and then we get easily

$$H^1(\mathfrak{g},K_\infty,\mathcal{D}_\lambda\otimes\mathcal{M}_{\lambda^\vee}) = \mathbb{C}\oplus\mathbb{C} \tag{4.81}$$

and vanishes in all other degrees.

Of course we can get this last result easily if we look at the complex $\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathcal{D}_\lambda \otimes \mathcal{M}_{\lambda^\vee})$ which in this situation collapses to

$$0 \to \mathrm{Hom}_{K_\infty}(\Lambda^1(\mathfrak{g}//\mathfrak{k}, \mathcal{D}_\lambda \otimes \mathcal{M}_{\lambda^\vee}) \to 0 \to .., \tag{4.82}$$

in section 4.1.11 we give explicit elements $\omega^\dagger_\pm \in \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathcal{D}_\lambda \otimes \mathcal{M}_{\lambda^\vee})$ which form a basis for this space.

We discuss the case B). Again we want that our group $G/\mathbb{R} = R_{\mathbb{C}/\mathbb{R}}(\mathrm{Gl}_2/\mathbb{C})$ is a base change from a group $G/\mathbb{Q}$ denoted by the same letter. We need an imaginary quadratic extension $F/\mathbb{Q}$ and put $G/\mathbb{Q} = R_{F/\mathbb{Q}}(\mathrm{Gl}_2/F)$. We choose a dominant weight $\lambda = \lambda_1 + \lambda_2 = n_1\gamma_1 + d_1 \det_1 + n_2\gamma_2 + d_2 \det_2$ and then $\mathcal{M}_{\lambda,F} = \mathcal{M}_{\lambda_1,F} \otimes \mathcal{M}_{\lambda_2,F}$ is an irreducible representation of $G \times_\mathbb{Q} F = \mathrm{Gl}_2 \times \mathrm{Gl}_2/F$. Now we have $\mathfrak{u} \otimes F = FE^1_+ \oplus FE^2_+$. Then basically the same computation yields:

The cohomology $H^\bullet(\mathfrak{u}, \mathcal{M}_{\lambda,F})$ is equal the complex

$$\mathbb{H}^\bullet(\mathfrak{u}, \mathcal{M}_{\lambda,F}) = \{0 \to Fe^{(1)}_{n_1} \otimes Fe^{(2)}_{n_2} \xrightarrow{d} FE^{1,\vee}_+ \otimes e^{(1)}_{-n_1} \otimes e^{(2)}_{n_2} \oplus FE^{1,\vee}_+ \otimes e^{(1)}_{n_1} \otimes E^{2,\vee}_+ \otimes e^{(2)}_{-n_2} \xrightarrow{d} FE^{1,\vee}_+ \otimes e^{(1)}_{-n_1} \otimes E^{2,\vee}_+ \otimes e^{(2)}_{-n_2} \to 0\} \tag{4.83}$$

where all the differentials are zero. The torus T acts by the weights

$$\lambda \text{ in degree } 0,\ s_1 \cdot \lambda, s_2 \cdot \lambda \text{ in degree } 1,\ w_0 \cdot \lambda \text{ in degree } 2 \tag{4.84}$$

and we have a decomposition into one dimensional weight spaces

$$H^\bullet(\mathfrak{u}, \mathcal{M}_{\lambda,F}) = \bigoplus_{w \in W(\mathbb{C})} H^\bullet(\mathfrak{u}, \mathcal{M}_{\lambda,F})(w \cdot \lambda)$$

We go back to (4.69) and get a homomorphism of complexes

$$\mathrm{Hom}_\mathfrak{c}(\Lambda^\bullet(\mathfrak{g}/\tilde{\mathfrak{k}}), \mathbb{C}\chi \otimes \mathcal{M}_{\lambda,\mathbb{C}}) \to \mathrm{Hom}_{\tilde{K}_\infty \otimes T}(\Lambda^\bullet(\mathfrak{t}/\tilde{\mathfrak{k}}), \mathbb{C}\chi \otimes \mathbb{H}^\bullet(\mathfrak{u}, \mathcal{M}_{\lambda,\mathbb{C}})) \tag{4.85}$$

which induces an isomorphism in cohomology so that finally

$$H^\bullet(\mathfrak{g}, K_\infty, \mathfrak{I}^G_B\chi \otimes \mathcal{M}_{\lambda,\mathbb{C}}) \xrightarrow{\sim} H^\bullet(\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{t}/\tilde{\mathfrak{k}}), \mathbb{C}\chi \otimes H^\bullet(\mathfrak{u}, \mathcal{M}_{\lambda,\mathbb{C}})) \tag{4.86}$$

and combining this with the results above we get cohlam

Theorem 4.1.2. *If we can find an element $w \in W(\mathbb{C})$ such that $\chi^{-1} = w \cdot \lambda_\mathbb{R}$ then*

$$H^\bullet(\mathfrak{g}, K_\infty, \mathfrak{I}^G_B\chi \otimes \mathcal{M}_{\lambda,\mathbb{C}}) \xrightarrow{\sim} H^{l(w)}(\mathfrak{u}, \mathcal{M}_{\lambda,\mathbb{C}})(w \cdot \lambda) \otimes \Lambda^\bullet(\mathfrak{t}/\tilde{\mathfrak{k}})^\vee$$

If there is no such w then the cohomology is zero.

Proof. Our torus $T(\mathbb{R}) = \mathfrak{c} \times \{\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}; t \in \mathbb{R}^\times_{>0}\} = \mathfrak{c} \times A$. Hence we see that $\dim \mathfrak{t}/\tilde{\mathfrak{k}} = 1$, and the element $H_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Of course we must have that

$\chi^{-1}\cdot\lambda_{\mathbb{R}}|_{\mathfrak{c}}$ is the trivial character. The second factor A does acts on $\mathbb{C}\chi$ by the character $\chi(t)=t^z$ and on $H^{l(w)}(\mathfrak{u},\mathcal{M}_{\lambda,\mathbb{C}})(w\cdot\lambda)$ by $t\mapsto t^{m(w)}$. Differentiating we get for the complex

$$0\to H^{l(w)}(\mathfrak{u},\mathcal{M}_{\lambda,\mathbb{C}})(w\cdot\lambda)\to\mathbb{C}\otimes H_0^{\vee}\otimes H^{l(w)}(\mathfrak{u},\mathcal{M}_{\lambda,\mathbb{C}})(w\cdot\lambda)\to 0 \tag{4.87}$$

where the differential is multiplication by $m(w)+z$. Hence we see that the cohomology is trivial unless $m(w)+z=0$, but this means $\chi^{-1}=w\cdot\lambda_{\mathbb{R}}$. □

4.1.6 The cohomology of the modules $\mathcal{M}_{\lambda,\mathbb{C}}$, $\mathcal{D}_\lambda$ and the cohomology of unitary modules

Let $\mathrm{Irr}(G,K_\infty)$ be the set of isomorphism classes of irreducible $(\mathfrak{g},K_\infty)$-Harish-Chandra-modules, we are a little bit pedantic, if $\mathcal{V}$ is such an irreducible module, then its isomorphism class is $[\mathcal{V}]$. For any dominant λ we define the sets

$$\mathrm{Coh}(\lambda)=\{[\mathcal{V}]\in\mathrm{Irr}(G,K_\infty)\mid H^\bullet(\mathfrak{g},K_\infty,\mathcal{V}\otimes\mathcal{M}_{\lambda,\mathbb{C}})\neq 0\} \tag{4.88}$$

We also define $\mathrm{Coh}_2(\lambda)$, this are those $[\mathcal{V}]$ which in addition are unitary. This definition makes sense in greater generality (see 8.25). In our special case there these sets are very small. Remember that we have a fixed central character ω.

At first we determine the finite dimensional elements in $\mathrm{Coh}(\lambda)$. Of course $\mathcal{M}_{\lambda,\mathbb{C}}$ itself is a Harish-Chandra module and it follows from Wigner's lemma that $H^\bullet(\mathfrak{g},K_\infty,\mathcal{M}_{\lambda,\mathbb{C}}))=0$ unless $\lambda^{(1)}=0$, i.e. $\mathcal{M}_{\lambda,\mathbb{C}}$ is one dimensional. Then it follows from Clebsch- Gordan that

Proposition 4.1.6. *In case A)*

$$\begin{gathered}H^0(\mathfrak{g},K_\infty,\mathcal{M}_{\lambda^\vee,\mathbb{C}}\otimes\mathcal{M}_{\lambda,\mathbb{C}})=H^2(\mathfrak{g},K_\infty,\mathcal{M}_{\lambda^\vee,\mathbb{C}}\otimes\mathcal{M}_{\lambda,\mathbb{C}})=\mathbb{C},\\ H^1(\mathfrak{g},K_\infty,\mathcal{M}_{\lambda^\vee,\mathbb{C}}\otimes\mathcal{M}_{\lambda,\mathbb{C}})=0\end{gathered} \tag{4.89}$$

In case B)

$$\begin{gathered}H^0(\mathfrak{g},K_\infty,\mathcal{M}_{\lambda^\vee,\mathbb{C}}\otimes\mathcal{M}_{\lambda,\mathbb{C}})=H^3(\mathfrak{g},K_\infty,\mathcal{M}_{\lambda^\vee,\mathbb{C}}\otimes\mathcal{M}_{\lambda,\mathbb{C}})=\mathbb{C},\\ H^1(\mathfrak{g},K_\infty,\mathcal{M}_{\lambda^\vee,\mathbb{C}}\otimes\mathcal{M}_{\lambda,\mathbb{C}})=H^2(\mathfrak{g},K_\infty,\mathcal{M}_{\lambda^\vee,\mathbb{C}}\otimes\mathcal{M}_{\lambda,\mathbb{C}})=0\end{gathered} \tag{4.90}$$

Here we take notice of a point, which plays a role if it comes to questions concerning orientability. In case A) we can twist the $G(\mathbb{R})$ module $\mathcal{M}_{\lambda^\vee,\mathbb{C}}$ by the sign character $\eta: g\mapsto \mathrm{sgn}(\det(g))$, it has the same central character. Obviously the twisted module $\mathcal{M}_{\lambda^\vee,\mathbb{C}}\otimes\eta$ provides the same $(\mathfrak{g},K_\infty)$-module. But this depends on the choice of K_∞, if we replace K_∞ by the larger group K_∞^* (see section **??**) then the $(\mathfrak{g},K_\infty^*)$ modules $\mathcal{M}_{\lambda^\vee,\mathbb{C}}$ and $\mathcal{M}_{\lambda^\vee,\mathbb{C}}\otimes\eta$ are not isomorphic. If we replace in the above proposition K_∞ by K_∞^* and $\mathcal{M}_{\lambda^\vee,\mathbb{C}}$ by $\mathcal{M}_{\lambda^\vee,\mathbb{C}}\otimes\eta$, then the cohomology vanishes in all degrees.

Small remark: In general it is sapient to work with a connected K_∞ or $\tilde{K}_\infty$ and then keep track of the action of K_∞^* on $H^\bullet(\mathfrak{g},K_\infty),\mathcal{V}\otimes\mathcal{M}_{\lambda,\mathbb{C}})$.

Again we start from a dominant character λ. Then our considerations yield that in case A)

$$\mathrm{Coh}(\lambda^\vee)=\{\mathcal{M}_{\lambda,\mathbb{C}},\mathcal{D}_\lambda^+,\mathcal{D}_\lambda^-\} \tag{4.91}$$

we even have $\mathcal{D}_\lambda^+, \mathcal{D}_\lambda^- \in \mathrm{Coh}_2(\lambda^\vee)$ and $\mathcal{M}_{\lambda,\mathbb{C}} \in \mathrm{Coh}_2(\lambda^\vee)$ if and only if $\lambda^{(1)} = 0$.

For some reason we call $\{\mathcal{D}_\lambda^+, \mathcal{D}_\lambda^-\} = \mathrm{Coh}_{\mathrm{cusp}}(\lambda^\vee)$ and $\{\mathcal{M}_{\lambda,\mathbb{C}}\} = \mathrm{Coh}_{\mathrm{Eis}}(\lambda^\vee)$

in case B) we take the tensor product of the exact sequence (4.47) by $\mathcal{M}_{\lambda^\vee,\mathbb{C}}$ and we get a long exact sequence of $(\mathfrak{g}, K_\infty)$ cohomology modules (we insert the values for $H^\bullet(\mathfrak{g}, K_\infty, \mathcal{M}_{\lambda,\mathbb{C}} \otimes \mathcal{M}_{\lambda^\vee,\mathbb{C}})$)

$$0 \to \mathbb{C} \to H^0(\mathfrak{g}, K_\infty, \mathfrak{I}_B^G \lambda_{\mathbb{R}}^{w_0} \otimes \mathcal{M}_{\lambda^\vee,\mathbb{C}}) \xrightarrow{r^0} H^0(\mathfrak{g}, K_\infty, \mathcal{D}_\lambda \otimes \mathcal{M}_{\lambda^\vee,\mathbb{C}})(=0)$$
$$\to 0 \to H^1(\mathfrak{g}, K_\infty, \mathfrak{I}_B^G \lambda_{\mathbb{R}}^{w_0} \otimes \mathcal{M}_{\lambda^\vee,\mathbb{C}}) \xrightarrow{r^1} H^1(\mathfrak{g}, K_\infty, \mathcal{D}_\lambda \otimes \mathcal{M}_{\lambda^\vee,\mathbb{C}}) \to$$
$$0 \to H^2(\mathfrak{g}, K_\infty, \mathfrak{I}_B^G \lambda_{\mathbb{R}}^{w_0} \otimes \mathcal{M}_{\lambda^\vee,\mathbb{C}}) \xrightarrow{r^2} H^2(\mathfrak{g}, K_\infty, \mathcal{D}_\lambda \otimes \mathcal{M}_{\lambda^\vee,\mathbb{C}})$$
$$\to \mathbb{C} \to H^3(\mathfrak{g}, K_\infty, \mathfrak{I}_B^G \lambda_{\mathbb{R}}^{w_0} \otimes \mathcal{M}_{\lambda^\vee,\mathbb{C}}) \xrightarrow{r^3} 0 \tag{4.92}$$

The homomorphisms r^1, r^2 are isomorphisms and all the $H^1, H^2 = \mathbb{C}$. Hence we see that in this case

$$\mathrm{Coh}(\lambda^\vee) = \{\mathcal{M}_{\lambda,\mathbb{C}}, \mathcal{D}_\lambda\} \tag{4.93}$$

and

$$\mathrm{Coh}_2(\lambda^\vee) = \begin{cases} \{\mathcal{M}_{\lambda,\mathbb{C}}, \mathcal{D}_\lambda\} & \text{if } \lambda^{(1)} = 0 \\ \{\mathcal{D}_\lambda\} & \text{if } n_1 = n_2 > 0 \end{cases} \tag{4.94}$$

EiShiso

4.1.7 The Eichler-Shimura Isomorphism

We want to apply these facts about representation theory to the study of cohomology groups $H^\bullet(\Gamma\backslash X, \mathcal{M}_{\lambda,\mathbb{C}})$ where now Γ is a congruence subgroup of $\mathrm{Gl}_2(\mathbb{Z})$ or $\mathrm{Gl}_2(\mathcal{O})$. (Discuss also quaternionic case- perhaps)

We start again from a dominant weight $\lambda = n\gamma + d\det \in X^*(T \times \mathbb{C})$. For every $(\mathfrak{g}, K_\infty)$ invariant homomorphism $\Psi_\lambda : \mathfrak{I}_B^G w \cdot \lambda_{\mathbb{R}} \to \mathcal{C}_\infty(\Gamma\backslash G(\mathbb{R}))$ induces a homomorphism

$$\Psi_\Lambda : \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\tilde{\mathfrak{k}}), \mathfrak{I}_B^G w \cdot \lambda_{\mathbb{R}} \otimes \mathcal{M}_{\lambda^\vee,\mathbb{C}}) \to \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\tilde{\mathfrak{k}}), \mathcal{C}_\infty(\Gamma\backslash G(\mathbb{R})) \otimes \mathcal{M}_{\lambda^\vee,\mathbb{C}}) \tag{4.95}$$

We will show in section 8.1.3 Proposition 8.1.1 that the complex on the right is isomorphic to the de-Rham complex:

$$\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\tilde{\mathfrak{k}}), \mathcal{C}_\infty(\Gamma\backslash G(\mathbb{R}) \otimes \mathcal{M}_{\lambda^\vee,\mathbb{C}}) \xrightarrow{\sim} \Omega^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}_{\lambda^\vee,\mathbb{C}}) \tag{4.96}$$

This de-Rham complex computes the cohomology and hence we get an homomorphism gkdeR

$$\Psi_\lambda^\bullet : H^\bullet(\mathfrak{g}, K_\infty, \mathfrak{I}_B^G w \cdot \lambda_{\mathbb{R}} \otimes \mathcal{M}_{\lambda^\vee,\mathbb{C}}) \to H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}_{\lambda^\vee,\mathbb{C}}) \tag{4.97}$$

We denote by $\omega^{(0)}$ the restriction of the central character of $\mathfrak{I}_B^G w \cdot \lambda_{\mathbb{R}}$ to the subgroup Z_0. and we introduce the spaces

$$\begin{array}{ll} \mathcal{E}^{(\infty)}(\lambda, w, \Gamma) = & \mathrm{Hom}_{(\mathfrak{g},K_\infty)}(\mathfrak{I}_B^G w \cdot \lambda_{\mathbb{R}},\ \mathcal{C}_\infty(\Gamma\backslash G(\mathbb{R}), \omega^{(0)}) \\ \quad\cup & \\ \mathcal{E}^{(2)}(\lambda, w, \Gamma) = & \mathrm{Hom}_{(\mathfrak{g},K_\infty)}(\mathfrak{I}_B^G w \cdot \lambda_{\mathbb{R}},\ \mathcal{C}^{(2)}_\infty(\Gamma\backslash G(\mathbb{R}), \omega^{(0)}) \end{array} \tag{4.98}$$

where the superscript $^{(2)}$ means square integrable.(See 8.14). It is clear from the results in Chapter 8 that the spaces $\mathcal{E}^{(?)}$ are finite dimensional. We get two maps in cohomology

$$\Phi^? : \mathcal{E}^?(\lambda, w, \Gamma) \otimes H^\bullet(\mathfrak{g}, K_\infty, \mathfrak{I}_B^G w \cdot \lambda_{\mathbb{R}} \otimes \mathcal{M}_{\lambda^\vee,\mathbb{C}}) \to H^\bullet(\Gamma\backslash X, \mathcal{M}_{\lambda^\vee,\mathbb{C}}) \tag{4.99}$$

Of course the module $\mathcal{E}^{(2)}(\lambda, w, \lambda) = 0$ unless $\mathfrak{I}_B^G w \cdot \lambda_{\mathbb{R}}$ has a non trivial quotient module which admits a positive definite quasi unitary $(\mathfrak{g}, K_\infty)$ invariant metric. This means that $\mathcal{E}^{(2)}(\lambda, w\cdot\lambda) \neq 0$ implies that in case B) the coefficients satisfy ul

$$n_1 = n_2,\ \text{i.e.}\lambda = n(\gamma_1 + \gamma_2) + d_1 \det + d_2 \det, \tag{4.100}$$

we will say that λ is unitary if this condition is fulfilled. Then the results in section (4.1.5) yield that these irreducible quasi unitary quotient modules are $\mathcal{D}_\lambda^\pm$ in case A) and $\mathcal{D}_\lambda$ in case B) .

Hence it is clear that a $\Psi_\lambda \in \mathcal{E}^{(2)}(\lambda, w \cdot \lambda)$ must vanish on the finite dimensional submodule $\mathcal{M}_\lambda$ if $n > 0$ and hence we under this condition

$$\mathcal{E}^{(2)}(\lambda, w \cdot \lambda) = \mathrm{Hom}_{\mathfrak{g},K_\infty}(\mathcal{D}_\lambda, \mathcal{C}^{(2)}_\infty(\Gamma\backslash G(\mathbb{R}), \omega^{(0)}$$

In the first two cases we know that

We have the fundamental ESI

Theorem 4.1.3. *(Eichler-Shimura Isomorphism) Assume λ unitary, then in degree 1 in case A, (resp. degree 1,2 in case B) the map*

$$\Phi^{(2)} : \mathcal{E}^{(2)}(\lambda, w, \Gamma) \otimes H^\bullet(\mathfrak{g}, K_\infty, \mathcal{D}_\lambda \otimes \mathcal{M}_{\lambda^\vee,\mathbb{C}}) \to H^\bullet_!(\Gamma\backslash X, \mathcal{M}_{\lambda^\vee,\mathbb{C}}) \tag{4.101}$$

is an isomorphism.

If we are in the third case, i.e. $n = 0$, and if $\lambda^2|_{\Gamma\cap Z} = 1$ then $\mathrm{Hom}_{(\mathfrak{g},K_\infty)}(\mathbb{C}[\tilde{\lambda}],\ \mathcal{C}_\infty(G(\mathbb{R}))$ is one dimensional and generated by $\Phi_\lambda : 1 \mapsto \tilde{\lambda}$. The map

$$\mathbb{C}\Phi_\lambda \otimes H^\bullet(\mathfrak{g}, K_\infty, \mathbb{C}[\tilde{\lambda}] \otimes \mathbb{C}[\tilde{\lambda}^\vee]) \to H^\bullet(\Gamma\backslash X, \mathcal{M}_{\lambda^\vee,\mathbb{C}} \otimes \mathbb{C}) \tag{4.102}$$

is an isomorphism in degree zero and zero in all other degrees.

For the case A).we want to relate this to the classical formulation The group $\mathrm{Sl}_2(\mathbb{R})$ acts transitively on the upper half plane $\mathbb{H} = \mathrm{Sl}_2(\mathbb{R})/\mathrm{SO}(2)$. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $z \in \mathbb{H}$ we put $j(g, z) = cz + d$. To any

$$\Phi \in \mathrm{Hom}_{(\mathfrak{g},K_\infty)}(\mathcal{D}_\lambda^+,\ \mathcal{C}^{(2)}_\infty(\Gamma\backslash G(\mathbb{R}), \omega^{(0)}))$$

we attach a function $f^{\Phi}_{n+2}:\mathbb{H}\to\mathbb{C}$: We write $z=gi$ with $g\in \mathrm{Sl}_2(\mathbb{R})$ and put holWh

$$f^{\Phi}_{n+2}(z)=\Phi(\psi_{n+2})(g)j(g,i)^{n+2} \tag{4.103}$$

An easy calculation shows that f^{Φ}_{n+2} is well defined and holomorphic (slzweineu.pdf)p.25-26) and for $\gamma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in \mathrm{Sl}_2(\mathbb{Z})$ it satisfies

$$f^{\Phi}_{n+2}(\gamma z)=(cz+d)^{n+2}f^{\Phi}_{n+2}(z) \tag{4.104}$$

The condition that $\Phi(\psi_{n+2})(g)$ is square integrable implies that f_{n+2} is a holomorphic cusp form of weight $n+2=k$. It is a special case of the theorem of Gelfand-Graev that this provides an isomorphism GelfGraev

$$\mathrm{Hom}_{(\mathfrak{g},K_\infty)}(\mathcal{D}^+_\lambda,\ \mathcal{C}^{(2)}_\infty(\Gamma\backslash G(\mathbb{R}))\xrightarrow{\sim} S_k(\Gamma) \tag{4.105}$$

where of course $S_k(\Gamma)$ is the space of holomorphic cusp forms for Γ.

We can do the same thing with $\mathcal{D}^-_\lambda$ then we land in the spaces of anti holomorphic cusp forms, these two spaces are isomorphic under conjugation. Combining this with our results above gives the classical formulation of the Eichler-Shimura theorem:

We have a canonical isomorphism

$$S_k(\Gamma)\oplus\overline{S_k(\Gamma)}\xrightarrow{\sim} H^1_!\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_{\lambda^\vee,\mathbb{C}}) \tag{4.106}$$

Hier eventuell complexe conjugation Action von $\pi_0()$ und auch noch Bianchi

There is an analogous formulation in case where we have to work with Bianchi modular forms.

4.1.8 Petersson scalar product and semi simplicity

Earlier in chapter 3 we stated a general theorem 3.1.1 which in this case says that $H^1_!\Gamma\backslash\mathbb{H},\mathcal{M}_{\lambda^\vee,\mathbb{C}})$ is a semi-simple module for the Hecke algebra, we gave an outline of the proof. In this case the hermitian scalar product is obtained from the Petersson scalar product on $S_k(\Gamma)$. For two cusp forms $f,g\in S_k(\Gamma)$ this scalar product is given by

$$<f,g>=\int_{\Gamma\backslash\mathbb{H}} f(z)\overline{g(z)}y^{n+2}i\frac{dz\wedge d\bar{z}}{y^2}$$

For this metric the Hecke operators are self adjoint, and from this it follows that $S_k(\Gamma)$ is semi simple as Hecke module.

We can decompose into eigenspaces

$$H^1_!(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_{\lambda^\vee,F})=\bigoplus_{\pi_f} H^1_!(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_{\lambda^\vee,F})(\pi_f) \tag{4.107}$$

where $\pi_f:\mathcal{H}\to F$ is a homomorphism. In this case we know that each π_f which occurs actually occurs with multiplicity 2 (it occurs with multiplicity one in $S_k(\Gamma)$ and $\overline{S_k(\Gamma)}$)

For any embedding $\iota : F \hookrightarrow \mathbb{C}$ we know the Ramanujan-Petersson conjecture, which says

$$\text{For all primes } p \text{ we have } |\iota(\pi_f(T_p))| \leq 2\, p^{\frac{n+1}{2}} \tag{4.108}$$

and again we can conclude that we get a canonical splitting of Hecke-modules

$$H^1\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{\lambda^\vee,F}) = H^1_!\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{\lambda^\vee,F}) \oplus F\ \mathrm{Eis}_n \tag{4.109}$$

where $T_p(\ \mathrm{Eis}_n) = (p^{n+1}+1)\ \mathrm{Eis}_n$. (The eigenvalue of T_p on Eis_n is different from the eigenvalues of T_p on $H^1_!\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{\lambda^\vee,F})$ (Manin-Drinfeld principle) and then a standard linear argument gives us the splitting.) Of course we could also say that the Hecke-module $H^1_!\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{\lambda^\vee,F})$ is complete in $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{\lambda^\vee,F})$.

How do we get such Ψ_λ? In our special situation we get them from Fourier-expansions of Whittaker functions and this will be explained next.

Whittloc

4.1.9 Local Whittaker models

We recall some fundamental results from representation theory of groups $\mathrm{Gl}_2(\mathbb{Q}_p)$. Let $F/\mathbb{Q}$ be a finite extension $\mathbb{Q}$. An admissible representation of $\mathrm{Gl}_2(\mathbb{Q}_p)$ is an action of $\mathrm{Gl}_2(\mathbb{Q}_p)$ on a F-vector space V which fulfills the following two additional requirements

a) For any open subgroup $K_p \subset \mathrm{Gl}_2(\mathbb{Z}_p)$ the space of fixed vectors V^{K_p} is finite dimensional.

b) For any $v \in V$ we find an open subgroup $K_p \subset \mathrm{Gl}_2(\mathbb{Z}_p)$ such that $v \in V^{K_p}$.

We say that V is a $\mathrm{Gl}_2(\mathbb{Q}_p)$ module, we denote the action of $\mathrm{Gl}_2(\mathbb{Q}_p)$ on V by $(g,v) \mapsto gv$. In addition we want to assume that our module has a central character, this means that the center $Z(\mathbb{Q}_p) = \mathbb{Q}_p^\times$ acts by a character $\omega_V : Z(\mathbb{Q}_p) \to F^\times$. Such a module is called irreducible if it does not contain a non trivial invariant submodule.

Again we dispose of a Hecke algebra, given K_p we consider the space of functions

$$\mathcal{H}_{K_p} = \{f : \mathrm{Gl}_2(\mathbb{Q}_p) \to F \mid f(zg) = \omega_V^{-1}(z)f(g)\ ;\ f \text{ has compact support} \mod Z(\mathbb{Q}_p)\}$$

this gives as an algebra by convolution and this algebra acts on V^{K_p} by

$$f * v = \int_{\mathrm{Gl}_2(\mathbb{Q}_p)/Z(\mathbb{Q}_p)} f(x)xvdx$$

(See also section 6.3.3.) We normalize the measure dx such that it gives volume one to K_p.

We recall - and explain the meaning of - the fundamental fact that each isomorphism class of admissible irreducible modules has a unique Whittaker model. We assume that $F \subset \mathbb{C}$, then we define the (additive) character PSI

$$\psi_p : \mathbb{Q}_p \to \mathbb{C}^\times;\ \psi_p\ :\ a/p^m \mapsto e^{\frac{2\pi i a}{p^m}} \tag{4.110}$$

it is clear that the kernel of ψ_p is $\mathbb{Z}_p$. Since we have $U(\mathbb{Q}_p) = \mathbb{Q}_p$ we can view ψ_p as a character $\psi_p : U(\mathbb{Q}_p) \to \mathbb{C}^\times$. We introduce the space

$$\mathcal{C}_{\psi_p}(\mathrm{Gl}_2(\mathbb{Q}_p)) = \{f : \mathrm{Gl}_2(\mathbb{Q}_p) \to \mathbb{C} | f(ug) = \psi_p(u)f(g)\}$$

where in addition we require that our f is invariant under a suitable open subgroup $K_f \subset \mathrm{Gl}_2(\mathbb{Z}_p)$. The group $\mathrm{Gl}_2(\mathbb{Q}_p)$ acts on this space by right translation the action is not admissible but satisfies the above condition b) .

Now we can state the theorem about existence and uniqueness of the Whittaker model

Whittp

Theorem 4.1.4. *For any infinite dimensional, absolutely irreducible admissible* $\mathrm{Gl}_2(\mathbb{Q}_p)$ *-module* V *we find a non trivial (of course invariant under* $\mathrm{Gl}_2(\mathbb{Q}_p)$*) homomorphism*

$$\Psi : V \to \mathcal{C}_{\psi_p}(\mathrm{Gl}_2(\mathbb{Q}_p)), \tag{4.111}$$

it is unique up to multiplication by a non zero scalar.

Proof. We refer to the literature, [57], [28] □

Spherical representations, their Whittaker model and the Euler factor

An absolutely irreducible $\mathrm{Gl}_2(\mathbb{Q}_p)$ module is called spherical or unramified if for $K_p = \mathrm{Gl}_2(\mathbb{Z}_p)$ we have $V^{K_p} \neq \{0\}$. In this case it is known that (*Reference*)

$$\dim_F(V^{\mathrm{Gl}_2(\mathbb{Z}_p)}) = 1; V^{\mathrm{Gl}_2(\mathbb{Z}_p)} = Fh_0. \tag{4.112}$$

The Hecke algebra $\mathcal{H}_{K_p}$ is commutative and generated by the two double cosets

$$T_p = \mathrm{Gl}_2(\mathbb{Z}_p) \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \mathrm{Gl}_2(\mathbb{Z}_p) \text{ and } C_p = \mathrm{Gl}_2(\mathbb{Z}_p) \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}. \tag{4.113}$$

The space $V^{\mathrm{Gl}_2(\mathbb{Z}_p)}$ is an absolutely irreducible module for $\mathcal{H}_{K_p}$ hence it is of rank one, let ψ_0 be a generator. Our two operators act by scalars on V^{K_p}, we write

$$T_p(h_0) = \pi_V(T_p)h_0 \text{ and } C_p(h_0) = \pi_V(C_p)h_0 \tag{4.114}$$

The module V is completely determined by these two eigenvalues, of course $\pi_V(C_p) = \omega_V(C_p)$.

We can formulate this a little bit differently. Let π_p an isomorphism type of our $\mathrm{Gl}_2(\mathbb{Q}_p)$ module V. Then our theorem above asserts that there is a unique $\mathrm{Gl}_2(\mathbb{Q}_p)$ -module

$$\mathcal{W}(\pi_p) \subset \mathcal{C}_{\psi_p}(\mathrm{Gl}_2(\mathbb{Q}_p)) \tag{4.115}$$

with isomorphism-type equal to $\pi_p \times_F \mathbb{C}$. We call this module the Whittaker realization of π_p. If our isomorphism type is unramified then the resulting homomorphism of $\mathcal{H}_p$ to F is also denoted by π_p.

We have the spherical vector $h^{(0)}_{\pi_p} \in \mathcal{W}(\pi_p)^{\mathrm{Gl}_2(\mathbb{Z}_p)}$ which is unique up to a scalar. Since $\mathrm{Gl}_2(\mathbb{Q}_p) = U(\mathbb{Q}_p)T(\mathbb{Q}_p)\mathrm{Gl}_2(\mathbb{Z}_p)$ this spherical vector is determined by its restriction to $T(\mathbb{Q}_p)$. We have a formula for this restriction. First of all we observe that

$$h^{(0)}_{\pi_p}(\begin{pmatrix} p^n & 0 \\ 0 & p^m \end{pmatrix}) = \pi_p(C_p^m)h^{(0)}_{\pi_p}(\begin{pmatrix} p^{n-m} & 0 \\ 0 & 1 \end{pmatrix}). \tag{4.116}$$

We claim that $h^{(0)}_{\pi_p}(\begin{pmatrix} p^n & 0 \\ 0 & 1 \end{pmatrix}) = 0$ if $n < 0$. To see this we look at the equalities

$$h^{(0)}_{\pi_p}(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p^n & 0 \\ 0 & 1 \end{pmatrix}) = \psi_p(u)h^{(0)}_{\pi_p}(\begin{pmatrix} p^n & 0 \\ 0 & 1 \end{pmatrix}) = h^{(0)}_{\pi_p}(\begin{pmatrix} p^n & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & p^{-n}u \\ 0 & 1 \end{pmatrix})$$

and we can find an element $u \in \mathbb{Q}_p$ such that $p^{-n}u \in \mathbb{Z}_p$ and $\psi_p(u) \neq 1$, this implies the claim. We exploit the eigenvalue equation $T_p(h^{(0)}_{\pi_p}) = \pi_p(T_p)h^{(0)}_{\pi_p}$, we write the double coset $K_p \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} K_p$ as union of right K_p cosets

$$K_p \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} K_p = \bigcup_{x \in \mathbb{Z}/p\mathbb{Z}} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} K_p \bigcup \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} K_p.$$

Clearly

$$h^{(0)}_{\pi_p}(\begin{pmatrix} p^n & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}) = h^{(0)}_{\pi_p}(\begin{pmatrix} p^{n+1} & 0 \\ 0 & 1) \end{pmatrix})$$
$$h^{(0)}_{\pi_p}(\begin{pmatrix} p^n & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}) = \pi_p(C_p)h^{(0)}_{\pi_p}(\begin{pmatrix} p^{n-1} & 0 \\ 0 & 1 \end{pmatrix})$$

and this implies the recursion formula recurs

$$\pi_p(T_p)h^{(0)}_{\pi_p}(\begin{pmatrix} p^n & 0 \\ 0 & 1 \end{pmatrix}) = \pi_p(C_p)h^{(0)}_{\pi_p}(\begin{pmatrix} p^{n-1} & 0 \\ 0 & 1 \end{pmatrix}) + \begin{cases} ph^{(0)}_{\pi_p}(\begin{pmatrix} p^{n+1} & 0 \\ 0 & 1) \end{pmatrix}) & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases} \tag{4.117}$$

We can normalize $h^{(0)}_{\pi_p}(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) = 1$, then the values for $n > 0$ follow from the recursion.

There is a more elegant way writing this recursion. For our unramified π_p we define the local Euler factor Euler

$$L(\pi_p, s) = \frac{1}{1 - \pi_p(T_p)p^{-s} + p\pi_p(C_p)p^{-2s}} \tag{4.118}$$

We expand this into a power series in p^{-s} and an elementary calculation shows that Mellin

$$L(\pi_p, s) = \sum_{n=0}^{\infty} h^{(0)}_{\pi_p}(\begin{pmatrix} p^n & 0 \\ 0 & 1 \end{pmatrix})p^n p^{-ns} \tag{4.119}$$

Whittaker models for Harish-Chandra modules

We also have a theory of Whittacker models for the irreducible Harish-Chandra modules studied in section 4.1. The unipotent radical $U(\mathbb{R}) = \mathbb{R}$ resp. $U(\mathbb{R}) = \mathbb{C}$. Again we fix characters $\psi_\infty : U(\mathbb{R}) \to \mathbb{C}^\times$ we put

$$\psi_\infty(x) = \begin{cases} e^{-2\pi i x} & \text{in case A)} \\ e^{-2\pi i(x+\bar{x})} & \text{in case B)} \end{cases} \tag{4.120}$$

and as in the p-adic case we define

$$\mathcal{C}_{\psi_\infty}(G(\mathbb{R})) = \{f : G(\mathbb{R}) \to \mathbb{C} | f(ug) = \psi_\infty(u)f(g), fi \text{ is } \mathcal{C}_\infty\}$$

Then we have again Whittinf

Theorem 4.1.5. *For any infinite dimensional, absolutely irreducible admissible* $\mathrm{Gl}_2(\mathbb{R})$ *-module* V *we find a non trivial (of course invariant under* $\mathrm{Gl}_2(\mathbb{R})$*) homomorphism*

$$\Psi : V \to \mathcal{C}_{\psi_\infty}(G(\mathbb{R})), \tag{4.121}$$

This homomorphism is unique up to a scalar. The image of V *under the homomorphism* Ψ *will be denoted by* $\tilde{V}$.

Proof. Again we refer to the literature. [28]. □

Hence we can say that for any isomorphism class π_∞ of irreducible infinite dimensional Harish-Chandra modules we have a unique Whittaker model $\mathcal{W}(\pi_\infty) \subset \mathcal{C}_{\psi_\infty}(G(\mathbb{R}))$. In the book of Godement we find explicit formulae for these Whittaker functions.

Actually it is easy to write down such maps $\tilde{\Psi}_\pm$ resp. $\tilde{\Psi}$ explicitly for our induced modules, we start from a dominant weight $\lambda = n\gamma + \delta$ (resp. $n_1\gamma_1 + n_2\gamma_2 + \delta$ where $n \geq 0, n_1, n_2 \geq 0$. We define

$$\mathcal{F} : \mathfrak{I}_B^G \lambda_\mathbb{R} |\rho|_\mathbb{R}^2 \to \mathcal{C}_{\psi_\infty}(G(\mathbb{R}))$$

by the integral

$$\mathcal{F}(f)(g) = \int_{U(\mathbb{R})} f(wug)\psi_\infty(-u)du,$$

there is no problem with convergence as long $n > 0, n_1, n_2 > 0$. If one of these numbers is zero then there is a tiny difficulty to overcome, we ignore it. In any case we get an isomorphism

$$\mathcal{F} : \mathfrak{I}_B^G \lambda_\mathbb{R} |\rho|_\mathbb{R}^2 \xrightarrow{\sim} \mathfrak{I}_B^{G,\dagger} \lambda_\mathbb{R} |\rho|_\mathbb{R}^2 \tag{4.122}$$

i.e. we will denote elements or spaces which lie in a Whittaker model by $?^\dagger$.

We consider the case A). Let n be even. We consider induced module $\mathfrak{I}_B^G \lambda_\mathbb{R} \rho_\mathbb{R}^2 = \bigoplus_{\nu \equiv 0(2)} \mathbb{C}\phi_{\lambda,\nu}$, (See 4.28we have the exact sequence (See seqd

$$0 \to \mathcal{D}_{\lambda^\vee}^+ \oplus \mathcal{D}_{\lambda^\vee}^- \to \mathfrak{I}_B^G \lambda_\mathbb{R} \rho_\mathbb{R}^2 \to \mathcal{M}_\lambda \to 0$$

We have the Whittaker map

$$\mathcal{F} : \mathfrak{I}_B^G \lambda_\mathbb{R} \rho_\mathbb{R}^2 \to \mathcal{C}_\psi(G(\mathbb{R}))$$

which is defined by

$$\mathcal{F}(\phi_{\lambda,\nu})(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) := \int_{-\infty}^{\infty} \phi_{\lambda,\nu}(w \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) e^{2\pi i x} dx$$

We write the Cartan decomposition

$$w \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -t & -x \end{pmatrix} = \begin{pmatrix} \frac{t}{\sqrt{t^2+x^2}} & * \\ 0 & \sqrt{t^2+x^2} \end{pmatrix} \begin{pmatrix} \frac{-x}{\sqrt{t^2+x^2}} & \frac{t}{\sqrt{t^2+x^2}} \\ \frac{-t}{\sqrt{t^2+x^2}} & \frac{-x}{\sqrt{t^2+x^2}} \end{pmatrix}$$

and a straightforward computation gives us that we have to evaluate

$$\mathcal{F}(\phi_{\lambda,\nu})(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) = t^{\frac{n}{2}+1} \int_{-\infty}^{\infty} \frac{e^{2\pi i x}}{(x+ti)^{n/2+\nu/2+1}(x-ti)^{n/2-\nu/2+1}} dx$$

This can be done by the Residue theorem, we integrate from $-R <<$ to R and then from $R >> 0$ back to $-R$ along the circle in the upper half plane. Our function has only one pole in the upper half plane, namely for $x = ti$ and therefore

$$\int_{-\infty}^{\infty} \phi_{\lambda,\nu}(w \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) e^{2\pi i x} dx = t^{\frac{n}{2}+1} \operatorname{Res}_{x=ti} \frac{e^{2\pi i x}}{(x+ti)^{n/2+\nu/2+1}(x-ti)^{n/2-\nu/2+1}}$$

If we put $z := x - ti$ then our integral becomes

$$(2i)^{-n/2-\nu/2-1} t^{-\nu/2} e^{-2\pi t} \operatorname{Res}_{z=0} \frac{e^{2\pi i z}}{(1+\frac{z}{2ti})^{n/2+\nu/2+1} z^{n/2-\nu/2+1}} = P_{\lambda,\nu}(t) e^{-2\pi t},$$

where $P_{\lambda,\nu}(t)$ is a Laurent polynomial in $\mathbb{C}[t,t^{-1}]$. This polynomial is zero if $\nu \geq n+2$ and this implies that $\mathcal{F}$ maps $\mathcal{D}^+_{\lambda^\vee}$ to zero.

Therefore our map $\mathcal{F}$ induces an injection

$$\mathfrak{I}_B^G \lambda_{\mathbb{R}} \rho_{\mathbb{R}}^2 / \mathcal{D}^+_{\lambda^\vee} \hookrightarrow \mathcal{C}_\psi(G(\mathbb{R}))$$

this is of course an intertwining operator. The module $\mathcal{D}^-_{\lambda^\vee} \subset \mathfrak{I}_B^G \lambda_{\mathbb{R}} \rho_{\mathbb{R}}^2 / \mathcal{D}^+_{\lambda^\vee}$ it has $\phi_{\lambda,-n-2}$ as a lowest weight vector. We compute
$cF(\phi_{\lambda,-n-2})$, then t he nasty factor $(1+\frac{z}{2ti})^{n/2+\nu/2+1}$ is equal to one in this case and hence we have up to a non zero constant

$$\mathcal{F}(\phi_{\lambda,-n-2})(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) = c_\lambda t^{\frac{n}{2}+1} e^{-2\pi t}.$$

In the case A) the we the two discrete irreducible series representations $\mathcal{D}^+_{\lambda^\vee}, \mathcal{D}^-_{\lambda^\vee}$ attached to a dominant weight λ. We have their Whittaker model

$$\mathcal{F}_\pm : \mathcal{D}^\pm_{\lambda^\vee} \hookrightarrow \mathcal{C}_{\psi_\infty}(\mathrm{Gl}_2(\mathbb{R})). \tag{4.123}$$

The group $(\mathrm{Gl}_2(\mathbb{R})$ has the two connected components $\mathrm{Gl}_2(\mathbb{R})^+, \mathrm{Gl}_2(\mathbb{R})^-$,($\det > 0, \det < 0$) and we have

$$\mathcal{F}_+(\mathcal{D}^+_{\lambda^\vee}) = \mathcal{D}^{+,\dagger}_{\lambda^\vee} \text{ is supported on } \mathrm{Gl}_2(\mathbb{R})^+, \mathcal{D}^{-,\dagger}_{\lambda^\vee} \text{ is supported on } \mathrm{Gl}_2(\mathbb{R})^- \tag{4.124}$$

Under the isomorphism $\tilde{\Psi}_\pm$ the elements $\psi_{\pm(n+2)}$ (See (4.27)) are mapped to functions $\tilde{\psi}_{\pm(n+2)}$. We can normalize $\tilde{\Psi}_\pm$ such that tpsin

$$\psi^\dagger{}_{n+2}(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) = \begin{cases} t^{\frac{n}{2}+1}e^{-2\pi t} & \text{if } t > 0 \\ 0 & \text{else} \end{cases} \tag{4.125}$$

and $\psi^\dagger{}_{-n-2}$ is given by the corresponding formula.

We discuss the same issue for the group $\mathrm{Gl}_2(\mathbb{C})$ later in section 4.1.11

Whitt

Global Whittaker models, Fourier expansions and multiplicity one

We also have global Whittaker models. To define them we recall some results from Tate's thesis ([106]). We introduce the ring of adeles $\mathbb{A} = \mathbb{A}_\mathbb{Q}$, we write it as a product $\mathbb{A} = \mathbb{Q}_\infty \times \mathbb{A}_f = \mathbb{R} \times \mathbb{A}_f$. The ring of finite adeles contains the compact subring $\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$ of integral adeles.

We define a global character $\psi : U(\mathbb{A})/U(\mathbb{Q}) = \mathbb{A}/\mathbb{Q} \to \mathbb{C}^\times$ as the product psiq

$$\psi(x_\infty, \dots, x_p, \dots) = \psi_\infty(x_\infty) \prod_p \psi_p(x_p) \tag{4.126}$$

where the local components ψ_v are as above, we have to check that ψ is trivial on $U(\mathbb{Q})$. (See [106], "note the minus sign") For any $a \in \mathbb{Q}$ we define $\psi^{[a]}(x) = \psi(ax)$, so $\psi = \psi^{[1]}$. In ([106]) it is shown that the map

$$\mathbb{Q} \to \ \mathrm{Hom}(\mathbb{A}/\mathbb{Q}, \mathbb{C}^\times);\ a \mapsto \psi^{[a]} \tag{4.127}$$

is an isomorphism between $\mathbb{Q}$ and the character group of $\mathbb{A}/\mathbb{Q}$. Hence we know that for any reasonable function $h : \mathbb{A}/\mathbb{Q} \to \mathbb{C}$ we have a Fourier expansion Fouex

$$h(\underline{u}) = \sum_{a \in \mathbb{Q}} \hat{h}(a)\psi(a\underline{u}) \tag{4.128}$$

where $\hat{h}(a) = \int_{\mathbb{A}/\mathbb{Q}} h(\underline{u})\psi(-a\underline{u})d\underline{u}$, and where $\mathrm{vol}_{d\underline{u}}(\mathbb{A}/\mathbb{Q}) = 1$. Then we put

$$\mathcal{C}_\psi(\mathrm{Gl}_2(\mathbb{R}) \times \mathrm{Gl}_2(\mathbb{A}_f)/K_f)) = \{f : \mathrm{Gl}_2(\mathbb{R}) \times \mathrm{Gl}_2(\mathbb{A}_f)/K_f \to \mathbb{C} | f(\underline{u}\underline{g}) = \psi(\underline{u})f(\underline{g})\}$$

this is a module for $\mathrm{Gl}_2(\mathbb{R}) \times \bigotimes' \mathcal{H}_p$

Let us start from the Harish-Chandra module $\pi_\infty = \mathcal{D}^+_\lambda$ and a homomorphism $\pi_f = \otimes' \pi_p : \otimes' \mathcal{H}_p \to F$ from the unramified Hecke algebra to F. Here $F/\mathbb{Q}$ is a finite extension of $\mathbb{Q}$. We assume it comes with an embedding $\iota : F \hookrightarrow \mathbb{C}$, i.e. we also may it consider as a subfield of $\mathbb{C}$.

We still assume for simplicity that $K_f = \mathrm{Gl}_2(\hat{\mathbb{Z}})$.

The results on Whittaker-models imply that we have a unique Whittaker-model

$$\mathcal{W}(\pi) = \mathcal{W}(\pi_\infty) \otimes \mathbb{C}h^{(0)}_{\pi_f} \subset \mathcal{C}_\psi(\mathrm{Gl}_2(\mathbb{R}) \times \mathrm{Gl}_2(\mathbb{A}_f)/K_f) \tag{4.129}$$

for our isomorphism class $\pi = \pi_\infty \times \pi_f$. Here of course $h^{(0)}_{\pi_f} = \otimes h^{(0)}_{\pi_p}$.

We return to Theorem 4.1.3. On the space $\mathcal{C}^{(2)}_\infty(\Gamma\backslash G(\mathbb{R}), \omega^{(0)}))$ we have the action of the unramified Hecke algebra. To see this action we start from the observation that the map $\mathrm{Gl}_2(\mathbb{Q}) \to \mathrm{Gl}_2(\mathbb{A}_f)/K_f$ (Chap. III , 1.5) is surjective and hence

$$\mathrm{Gl}_2(\mathbb{Z})\backslash\mathrm{Gl}_2(\mathbb{R}) \xrightarrow{\sim} \mathrm{Gl}_2(\mathbb{Q})\backslash\mathrm{Gl}_2(\mathbb{R}) \times \mathrm{Gl}_2(\mathbb{A}_f)/K_f \tag{4.130}$$

and hence

$$\mathcal{C}^{(2)}_\infty(\mathrm{Gl}_2(\mathbb{Z})\backslash\mathrm{Gl}_2(\mathbb{R})) = \mathcal{C}^{(2)}_\infty(\mathrm{Gl}_2(\mathbb{Q})\backslash\mathrm{Gl}_2(\mathbb{R}) \times \mathrm{Gl}_2(\mathbb{A}_f)/K_f) \tag{4.131}$$

and the space on the right is a $\mathrm{Gl}_2(\mathbb{R}) \times \bigotimes' \mathcal{H}_p$ module. Now we consider the $\pi = \pi_\infty \times \pi_f$ isotypical submodule $\mathcal{C}^{(2)}_\infty(\mathrm{Gl}_2(\mathbb{Q})\backslash\mathrm{Gl}_2(\mathbb{R}) \times \mathrm{Gl}_2(\mathbb{A}_f)/K_f)(\pi) \subset \mathcal{C}^{(2)}_\infty(\mathrm{Gl}_2(\mathbb{Q})\backslash\mathrm{Gl}_2(\mathbb{R}) \times \mathrm{Gl}_2(\mathbb{A}_f)/K_f)$.

We have the famous Theorem which in the case $\Gamma = \mathrm{Sl}_2(\mathbb{Z})$ is due to Hecke

multone

Theorem 4.1.6. *If $\mathcal{C}^{(2)}_\infty(\mathrm{Gl}_2(\mathbb{Q})\backslash\mathrm{Gl}_2(\mathbb{R}) \times \mathrm{Gl}_2(\mathbb{A}_f)/K_f)(\pi) \neq 0$ then have a canonical isomorphism*

$$\mathcal{F} : \mathcal{W}(\pi) \xrightarrow{\sim} \mathcal{C}^{(2)}_\infty(\mathrm{Gl}_2(\mathbb{Q})\backslash\mathrm{Gl}_2(\mathbb{R}) \times \mathrm{Gl}_2(\mathbb{A}_f)/K_f)(\pi) \tag{4.132}$$

especially we know that π occurs with multiplicity one.

Proof. We give the inverse of $\mathcal{F}$. Given a function

$$h \in \mathcal{C}^{(2)}_\infty(\mathrm{Gl}_2(\mathbb{Q})\backslash\mathrm{Gl}_2(\mathbb{R}) \times \mathrm{Gl}_2(\mathbb{A}_f)/K_f)(\pi)$$

we define

$$h^\dagger((g_\infty, \underline{g}_f)) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} h(\underline{u}\underline{g})\overline{\psi(\underline{u})}d\underline{u} \tag{4.133}$$

it is clear that $h^\dagger(g_\infty, \underline{g}_f) \in \mathcal{W}(\pi)$. It follows from the theory of automorphic forms that h is actually in the space of cusp forms, this means that the constant Fourier coefficient $\int_{U(\mathbb{Q})\backslash U(\mathbb{A})} h(\underline{u}\underline{g})d\underline{u} = 0$ and hence our Fourier expansion yields ((4.128), evaluated at $u = 0$)

$$h(\underline{g}) = \sum_{a\in\mathbb{Q}^\times} \int_{U(\mathbb{A})/U(\mathbb{Q})} h(\underline{u}\underline{g})\psi^{[a]}(\underline{u})d\underline{u} \tag{4.134}$$

The measure $d\underline{u}$ is invariant under multiplication by $a \in \mathbb{Q}^\times$ and hence a individual term in the summation is

$$\int_{U(\mathbb{A})/U(\mathbb{Q})} h(\begin{pmatrix} 1 & \underline{u} \\ 0 & 1 \end{pmatrix}\underline{g})\psi(\begin{pmatrix} 1 & a\underline{u} \\ 0 & 1 \end{pmatrix})d\underline{u} = \int_{U(\mathbb{A})/U(\mathbb{Q})} h(\begin{pmatrix} 1 & a^{-1}\underline{u} \\ 0 & 1 \end{pmatrix}\underline{g})\psi(\begin{pmatrix} 1 & \underline{u} \\ 0 & 1 \end{pmatrix})d\underline{u} \tag{4.135}$$

Now

$$\begin{pmatrix} 1 & a^{-1}\underline{u} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \underline{u} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$

Since h is invariant under the action of $G(\mathbb{Q})$ from the left we find

$$\int_{U(\mathbb{A})/U(\mathbb{Q})} h(\underline{u}\underline{g})\psi^{[a]}(\underline{u})d\underline{u} = h^\dagger(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix})g_\infty)h_f^\dagger(\underline{a}_f)\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}(g_\infty, \underline{g}_f)) \quad (4.136)$$

We evaluate at $\underline{g} = (g_\infty, e)$ then

$$h^\dagger(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}(g_\infty, e)) = h^\dagger(\begin{pmatrix} a_\infty & 0 \\ 0 & 1 \end{pmatrix}g_\infty, \begin{pmatrix} \underline{a}_f & 0 \\ 0 & 1 \end{pmatrix}) \quad (4.137)$$

For a fixed g_∞ the function $\underline{g}_f \mapsto h^\dagger(g_\infty, \underline{g}_f)$ is up to a factor equal to $h^{(0)}_{\pi_f} = \bigotimes'_p h^{(0)}_{\pi_p}$ and hence we find

$$h^\dagger(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}(g_\infty, e)) = h^\dagger(\begin{pmatrix} a_\infty & 0 \\ 0 & 1 \end{pmatrix}g_\infty, e)h^{(0)}_{\pi_f}(\begin{pmatrix} \underline{a}_f & 0 \\ 0 & 1 \end{pmatrix}) \quad (4.138)$$

The recursion formulae (4.117),(4.119) imply that $h^{(0)}_{\pi_f}(\begin{pmatrix} \underline{a}_f & 0 \\ 0 & 1 \end{pmatrix}) = 0$ unless $a \in \mathbb{Z}$.

We restrict our functions to $\mathrm{Gl}_2^+(\mathbb{R})$, i.e. we take $g_\infty \in \mathrm{Gl}_2(\mathbb{R})^+$ and we remember that our representationπ_∞ is $\mathcal{D}^+_{\lambda^\vee}$. Then we know that for $h_\infty \in \mathcal{D}^+_{\lambda^\vee}$ the value $h^\dagger(\begin{pmatrix} a_\infty & 0 \\ 0 & 1 \end{pmatrix}g_\infty, e) = 0$ if $a_\infty < 0$ and hence

$$h^\dagger(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}(g_\infty, e)) = h^\dagger(\begin{pmatrix} a_\infty & 0 \\ 0 & 1 \end{pmatrix}g_\infty, \begin{pmatrix} \underline{a}_f & 0 \\ 0 & 1 \end{pmatrix}) = 0 \text{ unless } a > 0, a \in \mathbb{Z},$$

and our Fourier expansion (4.128) becomes Fexpl

$$h(\underline{g}) = \sum_{a=1}^{\infty} h^\dagger(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}(g_\infty, e))h^{(0)}_{\pi_f}(\begin{pmatrix} \underline{a}_f & 0 \\ 0 & 1 \end{pmatrix}) \quad (4.139)$$

□

We notice that there is never any problem with convergence. The Whittaker functions $h^\dagger_\infty$ always decay very rapidly at infinity. We write $g_\infty = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}k$ with $k \in K_\infty$, then it is easy to see

$$|h^\dagger_\infty(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}g_\infty| < P(t)e^{-2\pi t}$$

where $P(t)$ is a polynomial in t. This implies that the series is really very rapidly converging (See remark below).

Now we choose for the component at infinity the function $h^\dagger_\infty = \tilde{\psi}_{n+2}$ and we compute the corresponding holomorphic cusp form h^Φ under the Eichler-Shimura isomorphism. We have the formula (4.103)

$$h^\Phi(z) = h^\Phi(x+iy) = h(\begin{pmatrix} y^{\frac{1}{2}} & \frac{x}{y^{\frac{1}{2}}} \\ 0 & y^{-\frac{1}{2}} \end{pmatrix})j(\begin{pmatrix} y^{\frac{1}{2}} & \frac{x}{y^{\frac{1}{2}}} \\ 0 & y^{-\frac{1}{2}} \end{pmatrix}, i)^{n+2} = h(\begin{pmatrix} y^{\frac{1}{2}} & \frac{x}{y^{\frac{1}{2}}} \\ 0 & y^{-\frac{1}{2}} \end{pmatrix})y^{-\frac{n}{2}-1}$$

and hence our Fourier expansion (4.139) becomes FouH

$$h^{\Phi}(z) = y^{-\frac{n}{2}-1}\sum_{a=1}^{\infty} \tilde{\phi}_{n+2}(\begin{pmatrix} ay & ax \\ 0 & 1 \end{pmatrix})h^{(0)}_{\pi_f}(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}) \tag{4.140}$$

We have the formula (4.125) for $\tilde{\phi}_{n+2}$ and then this becomes

$$h^{\Phi}(z) = \sum_{a=1}^{\infty} a^{\frac{n}{2}+1}h^{(0)}_{\pi_f}(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix})e^{2\pi iza} \tag{4.141}$$

This is now the classical Fourier expansion of a holomorphic cusp eigenform of weight $k = n+2$, ([51]). The numbers $c(\pi_f, a) = a^{\frac{n}{2}+1}h^{(0)}_{\pi_f}(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix})$ are the Fourier coefficients and they also the eigenvalues of the operator T_a -defined in by Hecke in [51]- on h^{Φ}. If we apply the Eichler-Shimura isomorphism and interpret h^{Φ} as a cohomology class then it is an eigenclass in $H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{C})$ and for any prime p the number $c(\pi_f, p)$ is the eigenvalue of the operator T_p defined in **??**.

We briefly come back to the question of convergence. Hecke proves in [51] that Estone

$$|c(\pi_f, a)| \leq Ca^{n+1+\epsilon} \tag{4.142}$$

and with this estimate the convergence becomes obvious.

Actually there is a much better estimate, which will be discussed in the "probably removed" section. Lfu

4.1.10 The L-functions

We still assume that $K_f = \mathrm{Gl}_2(\hat{\mathbb{Z}})$ or what amounts to the same that $\Gamma = \mathrm{Sl}_2(\mathbb{Z})$. We start from an eigenspace $H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_\lambda \otimes F)(\pi_f)$, now π_f is simply a homomorphism $\pi_f : \mathcal{H}_{K_f} \to \mathcal{O}_F$. To this homomorphism we attach the cohomological L-function

$$L^{\mathrm{coh}}(\pi_f, s) = \prod_p \frac{1}{1 - \pi_p(T_p)p^{-s} + p^{1+n-2s}} \tag{4.143}$$

here T_p is the Hecke operator defined in **??**, it differs from the Hecke operator defined by convolution by a factor $p^{\frac{n}{2}}$ in front. If we expand this product over all primes we get

$$L^{\mathrm{coh}}(\pi_f, s) = \sum_{a=1}^{\infty} \frac{c(\pi_f, a)}{a^s} \tag{4.144}$$

and this is exactly the L-function Hecke attaches to the cusp form provided by π_f. But we want to stress that this cohomological L-function is defined in purely combinatorial terms (See section 3.2.1, and Chapter 7).

At this moment this L function is a formal expression, it is a formal Dirichlet series with coefficients in our field F, which is simply a finite extension of $\mathbb{Q}$. If we assume that $F \subset \mathbb{C}$. then we may interpret s as a complex variable and the above estimate of the size of the coefficients implies that this series converges absolutely and locally uniformly for $\Re(s) > n+2$ and hence gives a holomorphic function in this halfspace. But something much better is true. We define the completed L function

$$\Lambda^{\text{coh}}(\pi_f, s) = \frac{\Gamma(s)}{(2\pi)^s} L^{\text{coh}}(\pi_f, s), \tag{4.145}$$

for this completed L-function Hecke proved HFu

Theorem 4.1.7. *The function $\Lambda^{\text{coh}}(\pi_f, s)$ has holomorphic continuation into the entire complex plane and satisfies the functional equation*

$$\Lambda^{\text{coh}}(\pi_f, s) = (-1)^{\frac{n}{2}+1} \Lambda^{\text{coh}}(\pi_f, n+2-s)$$

Proof. We could refer to Hecke, but for some reason we give an outline of the argument. We have the integral representation (Mellin-transform)

$$\Lambda^{\text{coh}}(\pi_f, s) = \int_0^\infty \sum_{a=1}^\infty c(\pi_f, a) e^{-2\pi a y} y^s \frac{dy}{y} = \int_0^\infty h^\Phi(iy) y^s \frac{dy}{y}$$

of course here we have to be courageous (or stupid) enough to exchange integration and summation. But since $e^{-2\pi a y}$ goes rapidly to zero if $y \to \infty$ there is no problem with the upper integration limit ∞. If $\Re(s) >> 0$ the y^s also tends to zero fast enough, so that we do not have a problem with the lower integration limit. But now we can split the integration into two parts

$$\int_0^\infty \sum_a^\infty c(\pi_f, a) e^{-2\pi a y} y^s \tfrac{dy}{y} =$$

$$\int_0^1 \sum_a^\infty c(\pi_f, a) e^{-2\pi a y} y^s \tfrac{dy}{y} + \int_1^\infty \sum_a^\infty c(\pi_f, a) e^{-2\pi a y} y^s \tfrac{dy}{y}$$

the second integration is converging for all values of s. To handle the first integral we observe that $h^\Phi(-\frac{1}{z}) = z^{n+2} h^\Phi(z)$, Hence we can substitute $y \to \frac{1}{y}$ in the first integral and get

$$\Lambda^{\text{coh}}(\pi_f, s) =$$

$$\sum_a^\infty \Big(\frac{1}{(2\pi)^s} \frac{c(\pi_f, a)}{a^s} \Gamma(s, 2\pi a) + \frac{(-1)^{\frac{n}{2}+1}}{(2\pi)^{n+2-s}} \frac{c(\pi_f, a)}{a^{n+2-s}} \Gamma(n+2-s, 2\pi a) \Big). \tag{4.146}$$

Here $\Gamma(,)$ is the incomplete Γ function, which defined by $\Gamma(s, A) = \int_A^\infty e^{-y} y^s \frac{dy}{y}$, it has the virtue that for any given value of s it decays rapidly if A goes to infinity.

Therefore we see that $\Lambda^{\text{coh}}(\pi_f, s)$ can be written as a sum of two infinite series which are convergent very rapidly, hence it follows that $\Lambda^{\text{coh}}(\pi_f, s)$ is holomorphic in the entire s plane and the functional equation also becomes obvious. □

We included the proof of the above theorem, because the above formula also gives us a very effective procedure to compute the numerical value of $\Lambda^{\text{coh}}(\pi_f, s_0)$ with high accuracy. We will come back to this issue in section 5.6.

Here reference to general L-functions and especially symmetric square

periods

4.1.11 The Periods

Together with the map $\mathcal{F}$ comes the map

$$\tilde{\mathcal{F}} = \mathrm{Id}\otimes\mathcal{F}\otimes\mathrm{Id}:\ \mathrm{Hom}_{\tilde{K}_\infty}(\Lambda(\mathfrak{g}/\tilde{\mathfrak{k}}),\mathcal{W}(\pi)\otimes\tilde{\mathcal{M}}_\lambda)\to$$
$$\mathrm{Hom}_{\tilde{K}_\infty}(\Lambda^\bullet(\mathfrak{g}/\tilde{\mathfrak{k}}),\mathcal{C}_\infty(\mathrm{Gl}_2(\mathbb{Q})\backslash(\mathrm{Gl}_2(\mathbb{R})\times\mathrm{Gl}_2(\mathbb{A}_f)/K_f)\otimes\tilde{\mathcal{M}}_\lambda)$$

The purpose of the following computations is to fix a specific choice of basis elements $\omega^\dagger_\pm\in\ \mathrm{Hom}_{\tilde{K}_\infty}(\Lambda^1(\mathfrak{g}/\mathfrak{k}),\mathcal{D}_{\lambda^\vee}{}^\dagger\otimes\tilde{\mathcal{M}}_\lambda)$ (in case A) $\omega^\dagger_{1,2}\in\ \mathrm{Hom}_{\tilde{K}_\infty}(\Lambda^{1,2}(\mathfrak{g}/\mathfrak{k}),\mathcal{D}^\dagger_{\lambda^\vee}\otimes\tilde{\mathcal{M}}_\lambda)$ (in case B)) These "canonical" generators serve us to define the periods.

In case A) we have

$$\mathfrak{g}/\tilde{\mathfrak{k}}\xrightarrow{\sim}\mathbb{Q}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\oplus\mathbb{Q}\begin{pmatrix}0&1\\1&0\end{pmatrix}=\mathbb{Q}H\oplus\mathbb{Q}V=\mathfrak{p}\qquad(4.147)$$

If we put $P=H+V\otimes i,\bar{P}=H-V\otimes i\in\mathfrak{g}/\tilde{\mathfrak{k}}\otimes\mathbb{Q}(i)$

Notation abklaeren $V=E_+$ auf S. 123 ?) then

$$\mathfrak{g}/\tilde{\mathfrak{k}}\otimes\mathbb{Q}(i)=\mathbb{Q}(i)P\oplus\mathbb{Q}(i)\bar{P}\ \text{ and }\ e(\phi)Pe(-\phi)=e^{22\pi i\varphi}P;e(\phi)\bar{P}e(-\phi)=e^{-22\pi i\varphi}\bar{P}\qquad(4.148)$$

Let $P^\vee,\bar{P}^\vee\in\ \mathrm{Hom}(\mathfrak{g}/\tilde{\mathfrak{k}},\mathbb{Q}(i))$ be the dual basis. Then we check easily that Pvee

$$P^\vee(H)=\bar{P}^\vee(H)=\frac{1}{2}\ \text{ and }\ P^\vee(V)=-\frac{i}{2},\bar{P}^\vee(V)=\frac{i}{2}\qquad(4.149)$$

The module $\tilde{\mathcal{M}}_\lambda\otimes\mathbb{Q}(i)$ decomposes under the action of $\tilde{K}_\infty$ into eigenspaces under $\tilde{K}_\infty$

$$\mathcal{M}_{\lambda^\vee}\otimes\mathbb{Q}(i)=\bigoplus_\nu^n\mathbb{Q}(i)(X+Y\otimes i)^{n-\nu}(X-Y\otimes i)^\nu\qquad(4.150)$$

where

$$e(\phi)((X+Y\otimes i)^{n-\nu}(X-Y\otimes i)^\nu)=e^{\pi i(n-2\nu)\phi}\cdot(X+Y\otimes i)^{n-\nu}(X-Y\otimes i)^\nu.$$

Then we define the basis elements

$$\omega^\dagger=P^\vee\otimes\tilde{\psi}_{n+2}\otimes(X-Y\otimes i)^n\ ;\ \bar{\omega}^\dagger=\bar{P}^\vee\otimes\tilde{\psi}_{-n-2}\otimes(X+Y\otimes i)^n\qquad(4.151)$$

We still have our involution $\mathbf{c}\in\tilde{K}^*_\infty$(See (4.25)) and clearly we have $\mathbf{c}\omega^\dagger=i^n\bar{\omega}^\dagger$ (Remember $n\equiv 0 \mod 2$.)

Now we put OPM

$$\omega^\dagger_+=\frac{1}{2}(\omega^\dagger+i^n\bar{\omega}^\dagger)\ ;\ \omega^\dagger_-=\frac{1}{2}(\omega^\dagger-i^n\bar{\omega}^\dagger)\qquad(4.152)$$

then these elements

$$\omega^\dagger_\pm=\frac{1}{2}(\omega^\dagger\pm i^n\bar{\omega}^\dagger)\ \in\ \mathrm{Hom}_{K_\infty}(\Lambda^1(\mathfrak{g}/\mathfrak{k}),\tilde{\mathcal{D}}_\lambda\otimes\mathcal{M}_\lambda)_\pm$$

and they are generators of these one dimensional spaces. The choice of these generators seems to be somewhat arbitrary, in [**?**] we give some motivation for this choice.

There is an alternative way to select $\omega^\dagger_\pm$. If we evaluate $\omega^\dagger_\pm$ on the element $H \in \mathfrak{g}/\mathfrak{k} = \mathfrak{p}$ then

$$\omega^\dagger_\pm(H) = \frac{1}{4}(\psi^\dagger_{n+2} \otimes (X - Y \otimes i)^n \pm i^n(\psi^\dagger_{-n-2} \otimes (X + Y \otimes i)^n) \in \mathcal{D}^\dagger_\lambda \otimes \mathcal{M}_\lambda$$

These are functions on $\mathrm{Gl}_2(\mathbb{R})$ with values in $\mathcal{M}_\lambda$. We pair these functions with an $\mathcal{M}_\lambda \otimes \mathbb{C}$ valued function, more precisely we consider the function $g \mapsto <\omega^\dagger_\pm(\mathrm{Ad}(g)H)(g), \rho_\lambda(g) X^\nu Y^\nu>$.

We restrict these scalar valued functions to the real points of the split torus

$$< \omega^\dagger_\pm(H)(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}), \rho_\lambda(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) X^\nu Y^{n-\nu} >=$$

$$< \tfrac{1}{4}(\psi^\dagger_{n+2}(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) \otimes (X - Y \otimes i)^n \pm i^n \psi^\dagger_{-n-2}(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) \otimes (X + Y \otimes i)^n), X^\nu Y^{n-\nu} > t^{-\frac{n}{2}+\nu}$$

Now let ϵ be a variable which can take the values $+, -$, then $\epsilon = +1, -1$. Our formula (4.9) gives us $< (X - \epsilon Y \otimes i)^n, X^\nu Y^{n-\nu} >= (-\epsilon i)^{n-\nu}$ and combing this with the explicit formula (**??**) for the values of $\psi^\dagger_{\epsilon(n+2)}(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix})$ we get

$$< \omega^\dagger_\epsilon(H)(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}), \rho_\lambda(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) X^\nu Y^{n-\nu} >= \begin{cases} (-i)^{n-\nu} t^{\frac{n}{2}+1} e^{-2\pi t} t^{-\frac{n}{2}+\nu} & \text{for } t > 0 \\ \epsilon i^{n-\nu} (-t)^{\frac{n}{2}+1} e^{2\pi t} (-t)^{-\frac{n}{2}+\nu} & \text{for } t < 0 \end{cases}.$$

(Here we use that n is even, but with suitable minor modifications we can also treat the case n odd.) Then a straight forward computation yields Mellinone

$$\begin{array}{l} \int_{T^{\mathrm{ad}}(\mathbb{R})} < \omega^\dagger_\epsilon(H)(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}), \rho_\lambda(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) X^\nu Y^{n-\nu} > \frac{dt}{t} = \\ \frac{1}{2} \begin{cases} \frac{\Gamma(n+1-\nu)}{(2\pi)^{n+1-\nu}} & \text{if } (-1)^{\frac{n}{2}-\nu} = \mathrm{sg}(\epsilon) \\ 0 & \text{else} \end{cases} \end{array} \tag{4.153}$$

For each choice of the sign $\epsilon = \pm 1$ one of these equation determines the generator $\omega^\dagger_\pm$. This formula will be of importance when we discuss the special values of L-functions.

In case B) we do basically the same, in some sense it is even simpler because K_∞ is maximal compact in this case, i.e. $K_\infty = K^*_\infty$. But on the other hand we need some very explicit information about the theory of irreducible representations of K_∞ and also about the decomposition of tensor products of these representations. We will also use some explicit formulas for Bessel functions.

Probably removed paragraph

The quotient $\mathfrak{g}/\tilde{\mathfrak{k}}$ is a three-dimensional vector space over $\mathbb{Q}$ the group K_∞ acts by the adjoint representation and this gives us the standard three dimensional representation of $K_\infty = U(2)$, which in addition is trivial on the center. (See 4.1.2). This module is given by the highest weight $2\gamma_c$. We must have $\lambda = n(\gamma + \bar{\gamma}) + ..$, if we want $\mathcal{E}^{(2)}(\lambda, w, \Gamma) \neq 0$, and then the formulae 4.1.6 and 4.54 imply that for $\bullet = 1, 2$

$$\dim_{\mathbb{C}} \ \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\tilde{\mathfrak{k}}), \mathcal{D}^\dagger_{\lambda^\vee} \otimes \mathcal{M}_{\lambda^\vee}) = 1 \tag{4.154}$$

Now we recall that we have defined a structure of a $R = \mathbb{Z}[\frac{1}{2}]$ module on all the modules on the stage, hence we see that

$$\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\tilde{\mathfrak{k}}), \mathcal{D}_{\lambda^\vee} \otimes \mathcal{M}_{\lambda^\vee}) = \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\tilde{\mathfrak{k}})_R, \mathcal{D}_{\lambda^\vee R} \otimes \mathcal{M}_{\lambda^\vee R}) \otimes \mathbb{C}, \tag{4.155}$$

here we are a little bit sloppy: The first subscript K_∞ is the compact group and the second subscript is a smooth groups scheme over R. For both choices of $\bullet$ the second term in the above equation is a free R module of rank 1. We choose generators

$$\omega^{\dagger,\bullet} \in \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\tilde{\mathfrak{k}})_R, \mathcal{D}_{\lambda^\vee R} \otimes \mathcal{M}_{\lambda^\vee R}).$$

These generators $\omega^{\dagger,1}, \omega^{\dagger,2}$ are well defined up to an element in $R^\times$.

End of removed paragraph

The quotient $\mathfrak{g}/\tilde{\mathfrak{k}}$ is a three-dimensional vector space over $\mathbb{Q}$ the group K_∞ acts by the adjoint representation and this gives us the standard three dimensional representation of $K_\infty = U(2)$, which in addition is trivial on the center. (See 4.1.2). This module is given by the highest weight $2\gamma_c$. We must have $\lambda = n(\gamma + \bar{\gamma}) + ..$, if we want $\mathcal{E}^{(2)}(\lambda, w, \Gamma) \neq 0$, and then the formulae 4.1.6 and 4.54 imply that for $\bullet = 1, 2$

$$\dim_{\mathbb{C}} \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\tilde{\mathfrak{k}}), \mathcal{D}^\dagger_{\lambda^\vee} \otimes \mathcal{M}_{\lambda^\vee}) = 1 \tag{4.156}$$

We fix these generators by prescribing values of certain Mellin transforms. To do this we need a little bit of representation theory. Of course we may replace K_∞ by SU(2) because the action of the center on the different modules cancels out. The modules $\mathfrak{g}/\mathfrak{k} \otimes \mathbb{C}, \mathcal{D}^\dagger_{\lambda^\vee}$ and $\mathcal{M}_{\lambda^\vee} \otimes \mathbb{C}$ extend naturally to $\mathrm{Sl}_2(\mathbb{C})$ modules and hence we have to find an explicit generator in

$$\mathrm{Hom}_{\mathrm{Sl}_2(\mathbb{C})}(\mathfrak{g}/\mathfrak{k} \otimes \mathbb{C}, \mathcal{D}^\dagger_{\lambda^\vee} \otimes \mathcal{M}_{n\gamma} \otimes \mathcal{M}_{n\bar{\gamma}}).$$

We have an explicit basis for $\mathfrak{g}/\mathfrak{k} \otimes \mathbb{C}$ (See (4.18), our module $\mathcal{M}_{\lambda^\vee} = \mathcal{M}^\flat_{n\gamma} \otimes \mathcal{M}^\flat_{n\gamma} \otimes_{\mathcal{O}} \mathbb{C}$ is given explicitly to us.

Our module $\mathcal{D}^\dagger_{\lambda^\vee} \subset \mathfrak{I}_B^G \lambda_{\mathbb{R}} \rho^2_{\mathbb{R}}$, and this last module decomposes into SU(2) -types (See(4.31). These SU(2) modules canonically extend to $\mathrm{Sl}_2(\mathbb{C})$-modules, we have

$$\mathfrak{I}_B^G \lambda_{\mathbb{R}} \rho^2_{\mathbb{R}} = \bigoplus_{\nu=0}^{\infty} \mathcal{M}_{2\nu\gamma} = \bigoplus_{\nu=0}^{\infty} \mathfrak{I}_B^G \lambda_{\mathbb{R}} \rho^2_{\mathbb{R}}(2\nu)$$

and

$$(\mathfrak{I}_B^G \lambda_{\mathbb{R}} \rho^2_{\mathbb{R}}(2(n+1)))^\dagger = \mathcal{D}^\dagger_{\lambda^\vee}(2(n+1))$$

Now it is clear that we have the problem to select a specific generator in

$$\mathrm{Hom}_{\mathrm{Sl}_2(\mathbb{C})}(\mathfrak{g}/\mathfrak{k} \otimes \mathbb{C}, \mathcal{D}^\dagger_{\lambda^\vee}(2(n+1)) \otimes \mathcal{M}^\flat_{n\gamma} \otimes \mathcal{M}^\flat_{n\bar{\gamma}} \otimes \mathbb{C}).$$

The modules $\mathfrak{g}/\mathfrak{k} \otimes \mathbb{C}, \mathcal{M}^\flat_{n\gamma}, \mathcal{M}^\flat_{n\bar{\gamma}}$ come with an explicit basis (See ???), if we want to write down a specific generator $\omega^{\dagger,\bullet}$ we have to write down a basis of $\mathcal{D}^\dagger_{\lambda^\vee}(2(n+1))$.

Again we start from our exact sequence

$$0 \to \mathcal{D}_{\lambda^\vee} \to \mathfrak{I}_B^G \to \mathcal{M}_\lambda \to 0 \tag{4.157}$$

we apply the map $\mathcal{F}$ to it and get an exact sequence of Whittaker modules

$$0 \to \mathcal{D}^{\dagger}_{\lambda^\vee} \to \mathfrak{I}_B^{G,\dagger} \to \mathcal{M}_\lambda \to 0 \tag{4.158}$$

To get such a basis we start from a basis element $\Phi_\lambda \in \tilde{\mathfrak{I}_B^G}\lambda_{\mathbb{R}}\rho^2_{\mathbb{R}}(0)$. We recall the definition of $\mathfrak{I}_B^G\lambda_{\mathbb{R}}\rho^2_{\mathbb{R}}$ as an induced representation, the space of K_∞ invariant vectors is spanned by the spherical function

$$\psi_{\lambda,0}(bk) = \psi_{\lambda,0}(bk)((\begin{pmatrix} t_1 & u \\ 0 & t_2 \end{pmatrix}\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}) = \lambda_{\mathbb{R}}\rho^2_{\mathbb{R}}(b).$$

We map the induced representation to its Whittaker model by

$$\mathcal{F} : \psi \mapsto \{g \mapsto \int \psi(w\begin{pmatrix} 1 & x+iy \\ 0 & 1 \end{pmatrix} g)e^{2\pi i x}dxdy\} \tag{4.159}$$

our basis element will be $\phi^\dagger_{\lambda,0} = \mathcal{F}(\psi_{\lambda,0})$. A straightforward computation yields

$$\phi^\dagger_{\lambda,0}(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) = \mathcal{F}(\psi_{\lambda,0})(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) = \int_\infty^\infty \frac{t^{n+2}}{(t^2+x^2+y^2)^{n+2}} e^{2\pi i x}dxdy$$

The educated reader knows that this function in the variable t is well known, we have

$$\Phi_\lambda(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) = \frac{2\pi^{n+2}}{\Gamma(n+2)} t K_{n+1}(2\pi t)$$

where $K_n(2\pi t)$ is the modified Bessel function. Of course Φ_λ is a function on $G(\mathbb{R}) = \mathrm{Gl}_2(\mathbb{C})$, it is right invariant under K_∞ and of course

$$\phi^\dagger_{\lambda,0}(\begin{pmatrix} 1 & x+iy \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) = e^{2\pi i x}\phi^\dagger_{\lambda,0}(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix})$$

hence it is defined by its restriction to $T^{\mathrm{ad}}(\mathbb{R})_{>0}$.

Starting from this function we construct the desired basis of $\mathcal{D}^\dagger_{\lambda^\vee}(2(n+1))$. The Lie-algebra $\mathfrak{g}$ acts on $\mathfrak{I}_B^G\lambda_{\mathbb{R}}\rho^2_{\mathbb{R}}$, we restrict this action to $\mathfrak{p}$ and it is clear that under this action

$$\mathfrak{p} \otimes \mathfrak{I}_B^G\lambda_{\mathbb{R}}\rho^2_{\mathbb{R}}(2\nu) \to \mathfrak{I}_B^G\lambda_{\mathbb{R}}\rho^2_{\mathbb{R}}(2\nu+2) \oplus \mathfrak{I}_B^G\lambda_{\mathbb{R}}\rho^2_{\mathbb{R}}(2\nu) \oplus \mathfrak{I}_B^G\lambda_{\mathbb{R}}\rho^2_{\mathbb{R}}(2\nu-2)$$

and if we extend this action to the tensor algebra we get a map

$$\mathfrak{U}_{n+1} : \mathfrak{p}^{\otimes(n+1)} \otimes \mathfrak{I}_B^G\lambda_{\mathbb{R}}\rho^2_{\mathbb{R}}(0) \to \bigoplus_{\nu=0}^{n+1} \mathfrak{I}_B^G\lambda_{\mathbb{R}}\rho^2_{\mathbb{R}}(2\nu). \tag{4.160}$$

here we may replace $n+1$ by any positive integer k.

The group K_∞ acts on $\mathfrak{p}^{\otimes(n+1)}$ by the adjoint action and the above map is of course a K_∞ homomorphism. On the right hand side we can project to the highest K_∞ type $\mathfrak{I}_B^G\lambda_{\mathbb{R}}\rho^2_{\mathbb{R}}(2n+2) = \mathcal{D}^\dagger_{\lambda^\vee}(2(n+1))$, i.e. we get a surjective homomorphism

$$\Pi_{n+1} : \mathfrak{p}^{\otimes(n+1)} \otimes \mathfrak{I}_B^G\lambda_{\mathbb{R}}\rho^2_{\mathbb{R}}(0) \to \mathcal{D}^\dagger_{\lambda^\vee}(2(n+1)), \tag{4.161}$$

again we may replace $n+1$ by any positive integer k.

We have the standard surjective homomorphism $\mathfrak{p}^{\otimes(n+1)} \to \mathrm{Sym}^{n+1}(\mathfrak{p})$, let us denote its kernel by I_{n+1}. For any $f \in \mathfrak{I}_B^G \lambda_{\mathbb{R}} \rho_{\mathbb{R}}^2$ and $X', X'' \in \mathfrak{p}$ we have

$$(X'X'' - X''X',)f = [X', X'']f.$$

Since the Lie bracket $[X_1, X_2] \in \mathfrak{k}$ it follows easily that Π_{n+1} vanishes on the kernel I_{n+1}. Hence our homomorphism Π_{n+1} factors over the quotient, i.e.

$$\Pi_{n+1} : \mathrm{Sym}^{n+1}(\mathfrak{p}) \to \mathcal{D}^{\dagger}_{\lambda^\vee}(2(n+1)).$$

We change our notation for the basis of $\mathfrak{p} \otimes \mathbb{C}$ (see 4.18) and put

$$\begin{gathered} X_1 = \tfrac{1}{2}(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - i \otimes \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}); X_0 = \tfrac{1}{\sqrt{2}}(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}) \\ X_{-1} = \tfrac{1}{2}(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + i \otimes \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}) \end{gathered} \tag{4.162}$$

We have the following proposition

Proposition 4.1.7. *The $2n+3$ elements*

$$\{X_1^{n+1}, X_0 X_1^n, \ldots, X_0^{n+1}, X_0^n X_{-1}, \ldots, X_{-1}^{n+1}\}$$

form a basis of a K_∞ invariant subspace of Sym$^{n+1}(\mathfrak{p}) \otimes \mathbb{C}$. This subspace is irreducible, it is isomorphic to $\mathcal{M}_{2n+2}$. These basis elements are the weight eigenvectors for the action of T_c.

Proof. The representation of the algebraic group K_∞ on $\mathfrak{p}$ extends to a representation of the algebraic group $\mathrm{Sl}_2/\mathbb{C}$ on $\mathfrak{p} \otimes \mathbb{C}$. As such it is isomorphic to the symmetric square $\mathrm{Sym}^2(\mathbb{C}^2)$ of the tautological representation, i.e. to the module $\mathcal{M}_2$ of polynomials $aU^2 + bUV + cV^2$. We get an isomorphism $\mathcal{M}_2 \xrightarrow{\sim} \mathfrak{p} \otimes \mathbb{C}$ by sending $U^2 \mapsto X_1, UV \mapsto X_0, V^2 \mapsto X_1$.. Now $\mathrm{Sym}^{2n+2}(\mathcal{M}_2) \subset \mathrm{Sym}^{n+1}(\mathrm{Sym}^2(\mathbb{C}^2)) = \mathrm{Sym}^{n+1}(\mathfrak{p} \otimes \mathbb{C})$ is an invariant submodule. It has the basis $U^{2n+2-\nu}V^\nu$ and clearly

$$U^{2n+2-\nu}V^\nu = X_1^{n+1-\nu} X_0^\nu \text{ if } \nu \leq n+1 \text{ and } X_0^{n+1\nu} X_1$$

and this implies the assertion. □

This implies that the elements

$$\begin{gathered} \{\Pi_{n+1}(X_1^{n+1}\phi^\dagger_{\lambda,0}), \Pi_{n+1}(X_0 X_1^n \phi^\dagger_{\lambda,0}), \ldots, \Pi_{n+1}(X_0^n X_1 \phi^\dagger_{\lambda,0}), \\ \Pi_{n+1}(X_0^{n+1}, \phi^\dagger_{\lambda,0}), \Pi_{n+1}(X_0^n X_{-1} \phi^\dagger_{\lambda,0}), \ldots, \Pi_{n+1}(X_{-1}^{n+1}\phi^\dagger_{\lambda,0})\} \end{gathered} \tag{4.163}$$

form a basis of $\mathcal{D}^\dagger_{\lambda^\vee}(2(n+1))$.

We change our notation slightly. For $m < 0$ we put $X_1^m := X_{-1}^{-m}$ and for $0 \leq \nu \leq 2n+2$ we put $[\nu] = \nu$ if $\nu \leq n+1$ and $[\nu] = 2n+2-\nu$ if $\nu \geq n+1$. Then our above basis can be written as

$$\{\ldots, \Pi_{n+1}(X_0^{[\nu]} X_1^{n+1-\nu} \phi^\dagger_{\lambda,0}), \ldots\}_{\nu=0,\ldots,\nu=2n+2}, \tag{4.164}$$

these are the weight vectors of weight $2(n+1-\nu)\gamma$. We introduce the notation

$$\phi^\dagger_{\lambda,n+1-\nu} := \Pi_{n+1}(X_0^{[\nu]}X_1^{n+1-\nu}\phi^\dagger_{\lambda,0})$$

These functions $\phi^\dagger_{\lambda,\mu}$ are Whittaker functions they satisfy

$$\phi^\dagger_{\lambda,\mu}(\begin{pmatrix}1 & x+iy\\ 0 & 1\end{pmatrix}g) = e^{2\pi i x}\phi^\dagger_{\lambda,\mu}(g).$$

They are not K_∞ invariant, but they are weight vectors for the torus, we have

$$\phi^\dagger_{\lambda,\mu}(g\begin{pmatrix}e^{2\pi i\varphi} & 0\\ 0 & 1\end{pmatrix}) = e^{4\mu\pi i\varphi}\phi^\dagger(g) \tag{4.165}$$

and more generally $\phi^\dagger_{\lambda,\nu}(gk) = \sum_\mu a_{\nu,\mu}(k)\phi^\dagger_{\lambda,\mu}(g)$ where the $a_{\nu,\mu}(k)$ are the matrix coefficients of $\mathcal{M}_{2n+2}$. (above proposition).

We consider the restriction of the functions $\phi^\dagger$ to the maximal torus $T(\mathbb{R})$. Since $\mathfrak{I}_B^G\lambda_\mathbb{R}\rho_\mathbb{R}^2(2\nu)$ has a central character, it suffices to consider the restriction

$$\phi^\dagger \to \{z \mapsto \phi^\dagger(\begin{pmatrix}z & 0\\ 0 & 1\end{pmatrix}),\}$$

we write $z = te^{2\pi i\varphi}$. This means that we map the module $\mathfrak{I}_B^G\lambda_\mathbb{R}\rho_\mathbb{R}^2$ to its Kirillow realisation $\mathfrak{I}_B^{G,\kappa}\lambda_\mathbb{R}\rho_\mathbb{R}^2 \subset \mathcal{C}_\infty(\mathbb{C}^\times)$. (See [28] , §2 5.), especially this map is injective.

We express the restriction of these functions $\phi^\dagger_{\lambda,\nu}$ to the torus $T^{\text{ad}}(\mathbb{R})_{>0}$ in terms of Bessel functions. We introduce the notation

$$\mathfrak{I}_B^G[2k] := \bigoplus_{\nu=0}^{k}\mathfrak{I}_B^G\lambda_\mathbb{R}\rho_\mathbb{R}^2(2\nu) \tag{4.166}$$

For any Whittaker function $\phi^\dagger \in \mathfrak{I}_B^G[2k]^\dagger$ we have

$$\Pi_{k+1}(X_1\phi^\dagger)(\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix}) = \frac{\phi^\dagger(\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix}\exp(\epsilon X_1)) - \phi^\dagger(\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix})}{\epsilon}$$

We write $X_1 = \frac{1}{2}(\begin{pmatrix}0 & 2\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 2i\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} + \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix})$ the last two matrices are in $\mathfrak{k}$ so they preserve the K_∞ type and

$$\phi^\dagger(\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix}\exp(\epsilon(\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & i\\ 0 & 0\end{pmatrix}))) =$$

$$\phi^\dagger(\exp(\epsilon(\begin{pmatrix}0 & t\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & it\\ 0 & 0\end{pmatrix})\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix}) = \phi^\dagger(((\begin{pmatrix}1 & \epsilon(t+it)\\ 0 & 1\end{pmatrix}\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix}) =$$
$$e^{2\pi i\epsilon t}\phi^\dagger(\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix}) = \phi^\dagger(\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix})(1+it\epsilon))$$

and hence

$$\Pi_{k+1}(X_1\phi^\dagger)(\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix}) = 2it\phi^\dagger(\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix})$$

If $\phi^\dagger$ is a weight vector, i.e. $\phi^\dagger(\begin{pmatrix} te^{2\pi i\varphi} & 0 \\ 0 & 1\end{pmatrix}) = e^{2\pi i\mu\varphi}\phi^\dagger(\begin{pmatrix} t & 0 \\ 0 & 1\end{pmatrix})$ then $X_1\phi^\dagger$ is also a weight vector with weight $e^{2\pi i(\mu+2)\varphi}$.

This gives us

$$\phi^\dagger_{\lambda,n+1}(\begin{pmatrix} t & 0 \\ 0 & 1\end{pmatrix}) = (X_1^{n+1}\phi^\dagger_{\lambda,0})(\begin{pmatrix} t & 0 \\ 0 & 1\end{pmatrix}) = \frac{2^{2n+3}\pi^{2n+3}}{\Gamma(n+2)} t^{n+2}K_{n+1}(2\pi t) \tag{4.167}$$

Since this function is of weight $2n+2$ we can forget the projection Π_{n+1}.

We have recursion formulas for the Bessel functions

$$\begin{aligned} \tfrac{d}{dt}K_n(t) &= -\tfrac{1}{2}(K_{n-1}(t)+K_{n+1}(t)) \\ K_{n+1}(t) &= K_{n-1}(t) + \tfrac{2n}{t}K_n(t) \end{aligned} \tag{4.168}$$

A straightforward calculation yields

$$t\frac{d}{dt}t^\mu K_\nu(2\pi t) = (\mu-\nu)t^\mu K_\nu(2\pi t) - 2\pi t^{\mu+1}K_{\nu-1}(2\pi t) \tag{4.169}$$

Then $\phi^\dagger_{\lambda,n+1-\nu}(\begin{pmatrix} t & 0 \\ 0 & 1\end{pmatrix}) = \Pi_{n+1}(X_0^{[\nu]}X_1^{n+1-\nu}\phi^\dagger_{\lambda,n+1}(\begin{pmatrix} t & 0 \\ 0 & 1\end{pmatrix}))$. We get

$$X_1^{n+1-\nu}\phi^\dagger_{\lambda,n+1}(\begin{pmatrix} t & 0 \\ 0 & 1\end{pmatrix})) = \frac{(2\pi)^{n+2+|n+1-\nu|}}{\Gamma(n+2)} t^{1+|n+1-\nu|}K_{n+1}(2\pi t).$$

To this we apply $X_0^{[\nu]}$. The operator X_0 is $t\frac{d}{dt}$, then the above formula gives

$$\Pi_{n+1}(X_0^{[\nu]}X_1^{n+1-\nu}\phi^\dagger_\lambda(\begin{pmatrix} t & 0 \\ 0 & 1\end{pmatrix})) = \Pi_{n+1}(\cdots + \frac{2^{2n+3}\pi^{2n+3}}{\Gamma(n+2)} t^{n+2}K_{n+1-\nu}(2\pi t)) \tag{4.170}$$

where the dots $\cdots$ are a sum of those terms which are in the image of $\mathfrak{I}_B^{G,\kappa}\lambda_{\mathbb{R}}\rho_{\mathbb{R}}^2[2n]$ hence they vanish under $\Pi_{n+1.}$ and consequently

$$\phi^\dagger_{\lambda,\mu}(\begin{pmatrix} t & 0 \\ 0 & 1\end{pmatrix}) = \frac{2^{2n+3}\pi^{2n+3}}{\Gamma(n+2)} t^{n+2}K_\mu(\begin{pmatrix} t & 0 \\ 0 & 1\end{pmatrix}) \tag{4.171}$$

where μ runs from $n+1$ to $-n-1$ and of course $K_\mu = K_{-\mu}$.

Decompositions of tensor products

If $\lambda_1 = n_1\gamma, \lambda_2 = n_2\gamma$ are two highest weights and if we consider the highest weight modules $\mathcal{M}_{\lambda_1,\mathbb{Q}}, \mathcal{M}_{\lambda_2,\mathbb{Q}}$ then it is a classical theorem that

$$\mathcal{M}_{\lambda_1,\mathbb{Q}} \otimes \mathcal{M}_{\lambda_2,\mathbb{Q}} = \mathcal{M}_{(n_1+n_2)\gamma,\mathbb{Q}} \oplus \mathcal{M}_{(n_1+n_2-2)\gamma,\mathbb{Q}} \oplus \cdots \oplus \mathcal{M}_{(n_1-n_2)\gamma,\mathbb{Q}} \cdots$$

where we assume $n_1 \geq n_2$, we put $n = n_1+n_2$. Our next aim is to give an explicit homomorphism

$$j_{n_1,n_2} : \mathcal{M}^\flat_{(n_1+n_2)\gamma} \hookrightarrow \mathcal{M}^\flat_{n_1\gamma} \otimes \mathcal{M}^\flat_{n_2\gamma} \tag{4.172}$$

in other words we want to write explicit tensors for the images of $e^\flat_\mu, \mu = n_1+n_2, n_1+n_2-2, \dots, -n_1-n_2$. Of course we send thehighest weight vector $e^\flat_{n_1+n_2} \mapsto {}'e^\flat_{n_1} \otimes {}''e^\flat_{n_2}$, this vector is the highest weight vector in the direct summand $\mathcal{M}^\flat_{(n_1+n_2)\gamma,\mathbb{Q}} \subset \mathcal{M}_{(n_1+n_2)\gamma,\mathbb{Q}} \oplus \cdots \oplus \mathcal{M}_{(n_1-n_2)\gamma,\mathbb{Q}}$. In terms of the explicit realisation of these modules we can say

$$X^{n_1+n_2} \mapsto {}'X^{n_1} \otimes {}''X^{n_2} \tag{4.173}$$

Now we apply the matrix $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$ to it, here we may think of t as an in determinant. Then we see

$$(X+tY)^{n_1+n_2} \mapsto ({}'X + t\,{}'Y)^{n_1} \otimes ({}''X + t\,{}''Y)^{n_2} \tag{4.174}$$

We expand on both sides and find

$$\textstyle\sum_{\mu=0}^{n_1+n_2} \binom{n_1+n_2}{\mu} t^\mu X^{n_1+n_2-\mu} Y^\mu \mapsto$$

$$\textstyle\sum_{\mu=0}^{n_1+n_2} t^\mu (\sum_{\mu_1,\mu_2:\mu_1+\mu_2=\mu} \binom{n_1}{\mu_1} {}'X^{n_1-\mu_1}{}'Y^{\mu_1} \otimes \binom{n_2}{\mu_2} {}''X^{n_2-\mu_2} \otimes {}''X^{n_2-\mu_2}{}''Y_{\mu_2}) \tag{4.175}$$

We remember the definition of the basis elements $e^\flat_\mu$, the formula above gives us

$$j_{n_1,n_2} : e^\flat_\mu \mapsto \sum_{\mu_1+\mu_2=\mu} {}'e^\flat_{\mu_1} \otimes {}''e^\flat_{\mu_2} \tag{4.176}$$

We apply this to the SU(2) -module

$$(\mathfrak{g}/\mathfrak{k})^\vee_\mathbb{F} \otimes \mathcal{M}_{n\gamma} \otimes_F \mathcal{M}_{n\bar\gamma},$$

this module contains a unique copy of $\mathcal{M}^\flat_{2n+2}$. We write

$$\mathfrak{g}/\mathfrak{k}^\vee_\mathbb{F} = F\,{}^0e^\flat_2 \oplus F\,{}^0e^\flat_0 \oplus F\,{}^0e^\flat_{-2},\ \mathcal{M}_{n_1\gamma,F} = \bigoplus_{\mu_1} Fe^\flat_{\mu_1},\ \mathcal{M}_{n_2\gamma,F} = \bigoplus_{\mu_2} F\bar{e}^\flat_{\mu_2} \tag{4.177}$$

where of course μ_i run from n_i to $-n_i$ and $\mu_i \equiv n_i \mod 2$. Then our copy of $\mathcal{M}^\flat_{2n+2}S$ comes with the basis

$$\tilde{e}^\flat_\mu = \sum_{\mu_0+\mu_1+\mu_2=\mu} {}^0e^\flat_{\mu_0} \otimes e^\flat_{\mu_1} \otimes \bar{e}^\flat_{\mu_2}$$

We have the invariant pairing (4.9) and this tells us that we can choose as our generator cangen

$$\omega^{\dagger,\bullet} = \sum_{\mu=0}^{n_1+n_2+2} \phi^\dagger_{\lambda,\mu} \otimes (\sum_{\mu_0+\mu_1+\mu_2=n+1-\mu} {}^0e^\flat_{\mu_0} \otimes e^\flat_{\mu_1} \otimes \bar{e}^\flat_{\mu_2}) \tag{4.178}$$

This generator is only determined up to a scalar, it is fixed once we choose a generator $c_n\phi^\dagger_{\lambda,n+1}$.

The "canonical" choice of the generator

Again we can fix the generator by requiring that certain Mellin transforms have a prescribed value at certain prescribed arguments.

We do essentially the same as in the case A). We can interpret $\omega^{\dagger,1}$ as a differential 1- form on $G(\mathbb{R})$ with values in $\mathcal{M}_\lambda^\flat \otimes \mathbb{C}$. We can restrict this 1-form to the torus $T^{\text{ad}}(\mathbb{R})_{>0} = \{\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} | t > 0\}$. We have the "cycles" $e_{\mu_1} \otimes e_{\mu_2} \in \mathcal{M}_\lambda^\vee)$. We evaluate $\omega^{\dagger,1}(X_0)$ on these "cycles" and get

$$< \omega^{\dagger,\bullet}(X_0), e_{\mu_1} \otimes e_{\mu_2} > (\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) = \phi^\dagger_{\lambda,n-\mu_1-\mu_2}(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) t^{\mu_1+\mu_2} = \\ c_n' t^{n+2+\mu_1+\mu_2} K_{n-\mu_1-\mu_2}(2\pi t) \quad (4.179)$$

Later -when we study the special values of L-functions- we need to know the value

$$\int_0^\infty < \omega^{\dagger,\bullet}(X_0), e_{\mu_1} \otimes e_{\mu_2} > (\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) \frac{dt}{t} = c_n' \int_0^\infty t^{n+2+\mu_1+\mu_2} K_{n-\mu_1-\mu_2}(2\pi t) \frac{dt}{t} \quad (4.180)$$

We also will need formulas for the Mellin transforms of these Bessel functions. Here we quote [1] .p.331,334 and recall two of them (the second one for later use)

$$\int_0^\infty K_\nu(2\pi t) t^s \tfrac{dt}{t} = 2^{s-2}(2\pi)^{-s}\Gamma(\tfrac{s-\nu}{2})\Gamma(\tfrac{s+\nu}{2})$$

$$\int_0^\infty K_\mu(2\pi t) K_\nu(2\pi t) t^s \tfrac{dt}{t} = 2^{s-3}(2\pi)^{-s}\Gamma(\tfrac{s-\mu-\nu}{2})\Gamma(\tfrac{s-\mu+\nu}{2})\Gamma(\tfrac{s+\mu-\nu}{2})\Gamma(\tfrac{s+\nu+\mu}{2}) \quad (4.181)$$

the first one gives us

$$\int_0^\infty t^{n+2+\mu_1+\mu_2} K_{n-\mu_1-\mu_2}(2\pi t) \frac{dt}{t} = \frac{\Gamma(n+1)}{4\pi} \frac{\Gamma(\mu+1)}{\pi^{\mu+1}} \quad (4.182)$$

We observe that the first factor in front does not depend on μ_1, μ_2. So we renormalise our generator and for $\mu = -n-1, -n, \dots, n+1$ we now put Phi

$$\phi^\dagger_{\lambda,\mu}(t) = \frac{4\pi}{\Gamma(n+1)} t^{n+2} K_{n+1-\mu}(t) \quad (4.183)$$

and with this choice of $\phi^\dagger_{\lambda,\nu}$ the $\omega^{\dagger,1}$ (4.178) is now our canonical generator. Now our formula (4.179) becomes

$$< \omega^{\dagger,\bullet}(X_0), e_{\mu_1} \otimes e_{\mu_2} > (\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) = \frac{\Gamma(\mu+1)}{\pi^{\mu+1}} \quad (4.184)$$

Hence we may just choose $\mu_1 = \mu_2 = 0$ to nail down $\omega^{\dagger,\bullet}$, it is not clear to me whether or not it is a "miracle" that the above relation holds for all values of μ_1, μ_2.

The definition of the periods

The inner cohomology with rational coefficients is a semi-simple module under the action of the Hecke algebra (See Theorem **??**). We find a finite Galois-extension $F/\mathbb{Q}$ such that we get a decomposition into absolutely irreducible modules

$$H^\bullet_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_\lambda \otimes F) = \bigoplus_{\pi_f} H^\bullet_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_\lambda \otimes F)(\pi_f) \tag{4.185}$$

Since we assume that $\Gamma = \mathrm{Gl}_2(\mathbb{Z})$, hence the π_f are homomorphisms $\pi_f : \mathcal{H} \to \mathcal{O}_F$. (See ???) In the case A) such an isotypical piece is a direct sum

$$H^\bullet_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_\lambda \otimes F)(\pi_f) = H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_\lambda \otimes F)(\pi_f)_+ \oplus H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_\lambda \otimes F)(\pi_f)_- \tag{4.186}$$

where both summands are of dimension one over F.

In case B) we get

$$H^\bullet_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_\lambda \otimes F)(\pi_f) = H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_\lambda \otimes F)(\pi_f) \oplus H^2_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_\lambda \otimes F)(\pi_f) \tag{4.187}$$

and again the summands are one dimensional.

We have defined the module of integral classes $H^1_{!,\,\mathrm{int}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}^\flat_\lambda \otimes \mathcal{O}_F) \subset H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}^\flat_\lambda \otimes F)$ (See 2.66) and we consider the intersection

$$H^\bullet_{!,\,\mathrm{int}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}^\flat_\lambda \otimes \mathcal{O}_F)(\pi_f)_\epsilon = H^\bullet_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_\lambda \otimes F)(\pi_f)_\epsilon \cap H^1_{!,\,\mathrm{int}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}^\flat_\lambda \otimes \mathcal{O}_F)$$

is a locally free $\mathcal{O}_F$ -module of rank 1, here $\epsilon = \pm, \bullet = 1$(resp. $\epsilon = 1, \bullet \in \{1,2\}$). We assume for simplicity that it is actually free, otherwise the formulation of the following becomes slightly more complicated. (See below). On the set of π_f which occur in this decomposition we have an action of the Galois group (See (6.62)) and the Galois action yields canonical isomorphisms

$$\Phi_{\sigma,\tau} : H^\bullet_{!,\,\mathrm{int}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}^\flat_\lambda \otimes \mathcal{O}_F)({}^\sigma\pi_f)_\epsilon \xrightarrow{\sim} H^\bullet_{!,\,\mathrm{int}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}^\flat_\lambda \otimes \mathcal{O}_F)({}^\tau\pi_f)_\epsilon \tag{4.188}$$

We choose generators ${}^\sigma e^\bullet_\epsilon(\pi_f)$ and a simple argument using Hilbert theorem 90 shows that we can assume the consistency condition H90

$$\Phi_{\sigma,\tau}(e^\bullet_\epsilon({}^\sigma\pi_f)) = e^\bullet_\epsilon({}^\tau\pi_f) \tag{4.189}$$

We get isomorphisms

$$\mathcal{F}^\bullet(\omega^\dagger_\epsilon) : \mathcal{W}({}^\sigma\pi_f) \otimes_F \mathbb{C} \xrightarrow{\sim} H^\bullet_\epsilon(\Gamma\backslash\mathbb{H}, \mathcal{M}_{\lambda^\vee})({}^\sigma\pi_f) \otimes_F \mathbb{C} \tag{4.190}$$

which are defined by Armand1

$$\mathcal{F}^\bullet(\omega^\dagger_\epsilon) : h_{{}^\sigma\pi_f} \mapsto [\mathcal{F}(\omega^\dagger_\epsilon \times h_{{}^\sigma\pi_f})], \tag{4.191}$$

here $\mathcal{F}(\omega_\epsilon^\dagger \times h_{^\sigma\pi_f})$ is viewed as a closed $\mathcal{M}_\lambda \otimes \mathbb{C}$ valued differential via theidentification 4.96, and $[\dots]$ is its class in cohomology.

Since we assume that π_f is unramified everywhere $\mathcal{W}(\pi_f)$ we have the canonical basis element $h_f^{(0)} = \prod_p h^{(0)}_{^\sigma\pi_p}$ where $h^{(0)}_{\sigma\pi_p}$ is defined by the equality 4.119. Then we have obviously $\sigma(h^{(0)}_{\pi_p}) = h^{(0)}_{^\sigma\pi_p}$.

Then we define the *periods* by the relation

$$\mathcal{F}(\omega_\epsilon^\dagger)(h^{(\dagger,0)}_{^\sigma\pi_f}) = \Omega^\bullet(\epsilon \times \pi_f,))e^\bullet(\epsilon \times \pi_f) \tag{4.192}$$

These periods depend of course on our choice of the "canonical" generator $\omega_\epsilon^\dagger$. We see that the numbers $\Omega^\bullet(^\sigma\pi_f, \epsilon)$ are well defined up to an element in $\mathcal{O}_F^\times$.

If $H^\bullet_{!,\ \mathrm{int}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}^\flat_\lambda \otimes \mathcal{O}_F)(\pi_f)_\epsilon$ is not a free $\mathcal{O}_F$ module, then we can find a covering by two open subsets U_1, U_2 of $\mathrm{Spec}(\mathcal{O}_F)$ such that $H^\bullet_{!,\ \mathrm{int}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}^\flat_\lambda \otimes$ $\otimes\mathcal{O}_F(U_i))(\epsilon \times \pi_f)$ is free. We can apply the above procedure and we get periods $\Omega_1(\epsilon \times \pi_f,), \Omega_2(\epsilon \times \pi_f)$, they are well defined up to an element in $\mathcal{O}_F(U_1)^\times, \mathcal{O}_F(U_2)^\times$ respectively. The ratio of these periods is an element in $\mathcal{O}_F(U_1 \cap U_2)^\times$.

Perhaps at this point we should introduce the sheaf $\mathcal{P}$ of periods over F. For any open subset $U \subset \mathrm{Spec}(\mathcal{O}_F)$ we put $\mathcal{P}^*_F(U) := \mathbb{C}^\times/\mathcal{O}_F(U)^\times$, this is a Zariski preasheaf on $\mathrm{Spec}(\mathcal{O}_F)$, the associated sheaf is our sheaf of periods $\mathcal{P}_F$.

Now we can interpret the generators $e^\bullet(\epsilon \times \pi_f)$ as (the unique) section in the sheaf of generators modulo $\mathcal{O}_F^\times$ and then the equation (4.192) makes sense without the assumption on the class number.

These considerations will play a role in the following chapter.

Some little subtleties

We should notice that these periods are defined with respect to the "small" sheaves $\tilde{\mathcal{M}}^\flat_\lambda$. We have $\tilde{\mathcal{M}}^\flat_\lambda \subset \tilde{\mathcal{M}}_\lambda$ and therefore the map

$$H^\bullet_{!,\ \mathrm{int}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}^\flat_\lambda \otimes \mathcal{O}_F)(\pi_f)_\epsilon \to H^\bullet_{!,\ \mathrm{int}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_\lambda \otimes \mathcal{O}_F)(\pi_f)_\epsilon \tag{4.193}$$

may not be surjective. (The reader should not be puzzled by the fact that $\tilde{\mathcal{M}}^\flat_\lambda \otimes F = \tilde{\mathcal{M}}_\lambda \otimes F$.) Therefore, if we would work with $\tilde{\mathcal{M}}_\lambda$ instead and define the periods $\Omega^{\bullet,\#}(^\sigma\pi_f, \epsilon)$ by the same procedure. Then we will get a relation

$$\Omega^{\bullet,\#}(^\sigma\pi_f, \epsilon) = d(\pi_f, \epsilon)\Omega^\bullet(^\sigma\pi_f, \epsilon)$$

where $d(\pi_,, \epsilon)$ is a non zero factor in $\mathcal{O}_F$. The primes in these factors are the divisors of the binomial coefficients.

But we could also with the module $H^\bullet(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}^\flat_\lambda \otimes \mathcal{O}_F)_{\ \mathrm{int},!}(\pi_f)_\epsilon$ and define the periods with respect to this module. Again these periods will integral multiples of the periods $\Omega^\bullet(\pi_f, \epsilon)$.

In the following Chapter 5 we will discuss the rationality results (Manin and Shimura) which relate these periods to special values of the $L-$ function (see section 5.6). But we also want to discuss this method not only for cuspidal classes but also for the Eisenstein cohomology classes, therefore we close this Chapter with a brief account of these Eisenstein classes.

4.1.12 The Eisenstein cohomology class

In section 3.3.6 we claimed the existence of the specific cohomology class $\text{Eis}_n \in H^1(\Gamma\backslash\mathbb{H}, \mathcal{M}_n)$. In this section we give s construction of this class on transcendental level, i.e. we construct a cohomology class $\text{Eis}(\omega_n) \in H^1(\Gamma\backslash\mathbb{H}, \mathcal{M}_n \otimes \mathbb{C})$ whose restriction to the boundary $H^1(\partial(\Gamma\backslash\mathbb{H}), \mathcal{M}_n \otimes \mathbb{C})$ is a given class ω_n. For the general theory of Eisenstein cohomology we refer to Chapter 9.

We start from our highest weight module $\mathcal{M}_\lambda$ and we observe that by definition we have an inclusion

$$i_0 : \mathfrak{I}_B^G \lambda_{\mathbb{R}}^{w_0} \hookrightarrow \mathcal{C}_\infty(\Gamma_\infty^+\backslash G^+(\mathbb{R}))$$

where

$$\Gamma_\infty^+ = \{\begin{pmatrix} t_1 & m \\ 0 & t_1 \end{pmatrix} |m \in \mathbb{Z} \,; t_1 = \pm 1\}.$$

Therefore we get an isomorphism

$$H^1(\mathfrak{g}, K_\infty, \mathfrak{I}_B^G \lambda_{\mathbb{R}}^{w_0} \otimes \mathcal{M}_\lambda \otimes \mathbb{C}) \xrightarrow{\sim} H^1(\Gamma_\infty^+\backslash \mathcal{M}_\lambda \otimes \mathbb{C}) = H^1(\partial(\Gamma\backslash\mathbb{H}), \mathcal{M}_\lambda \otimes \mathbb{C})$$

The inclusion i_0 sends the module $\mathfrak{I}_B^G \lambda_{\mathbb{R}}^{w_0}$ into a space of functions which are Γ_∞^+ invariant under left translations. Therefore we get a homomorphism

$$\text{Eis} : \mathfrak{I}_B^G \lambda_{\mathbb{R}}^{w_0} \to \mathcal{C}_\infty(\Gamma\backslash \text{Sl}_2(\mathbb{R}))$$

if we make it invariant by summation, i.e. for $f \in \mathfrak{I}_B^G \lambda_{\mathbb{R}}^{w_0}$ we define ESeries

$$\text{Eis}(f)(x) = \sum_{\Gamma_\infty^+\backslash \text{Sl}_2(\mathbb{Z})} f(\gamma x) \tag{4.194}$$

Of course we have to discuss the convergence of this infinite series. We could quote H. Jacquet: "Let us speak about convergence later", but here is a short interlude discussing this issue.

Interlude: Here is the point: We twist our module, for any complex number $z \in \mathbb{C}$ we consider the induced module

$$\mathfrak{I}_B^G \lambda_{\mathbb{R}}^{w_0} |\rho|^z \subset \mathcal{C}_\infty(\Gamma_\infty^+\backslash \text{Sl}_2(\mathbb{R}))$$

and again we write down the Eisenstein series. Now it an elementary exercise to show that the map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (c, d)$$

provides a bijection

$$\Gamma_\infty^+\backslash \text{Sl}_2(\mathbb{Z}) \xrightarrow{\sim} \{(c, d) \in \mathbb{Z} \times \mathbb{Z} \mid (c, d) \text{ coprime }\}/\{\pm 1\} = \mathbb{P}^1(\mathbb{Q}).$$

An element $x \in \text{Sl}_2(\mathbb{R})$ can be written as $x = \begin{pmatrix} t & u \\ 0 & t^{-1} \end{pmatrix} k$ with $k \in K_\infty$. Then for $f \in \mathfrak{I}_B^G \lambda_{\mathbb{R}}^{w_0} |\rho|^z$

$$f(\gamma x, z) =$$

$$f(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} t & u \\ 0 & t^{-1} \end{pmatrix} k, z) =$$

$$f(\begin{pmatrix} (c^2t^2 + (cv + dt^{-1})^2)^{-1/2} & * \\ 0 & (c^2t^2 + (cv + dt^{-1})^2)^{1/2} \end{pmatrix})f(k(\gamma g)k, z) =$$

$$(c^2t^2 + (cv + dt^{-1})^2)^{-n-2-z} f(k(\gamma g)k).$$

Since $|f(k(\gamma)k)|$ is bounded the series

$$\text{Eis}(f, z)(x) = \sum_{\Gamma_\infty^+ \backslash \text{Sl}_2(\mathbb{Z})} f(\gamma x, z)$$

is converging if $\Re(z) >> 0$ and then it is also holomorphic in z. Selberg and others showed that it can be extended to a meromorphic function in the entire complex plane, it is now a special case of a theorem of Langlands [69]. If now the function $x \mapsto \text{Eis}(f, z)(x)$ is holomorphic at $z = 0$ then we do not care about convergence and we simply define

$$\text{Eis}(f)(x) = \sum_{\Gamma_\infty^+ \backslash \text{Sl}_2(\mathbb{Z})} f(\gamma x) = \text{Eis}(f, 0)(x).$$

In our special case it is easy to see that the series is convergent at $z = 0$ provided we have $n > 0$ and this is the only case where we will apply this construction. *End interlude*

This provides a homomorphism

$$\text{Eis}^\bullet : H^1(\mathfrak{g}, K_\infty, \mathfrak{I}_B^G \lambda_{\mathbb{R}}^{w_0} \otimes \mathcal{M}_\lambda \otimes \mathbb{C})) \to H^1(\Gamma \backslash \mathbb{H}, \mathcal{M}_\lambda \otimes \mathbb{C}) \tag{4.195}$$

In ??? we wrote down a distinguished generator $\omega_n \in H^1(\mathfrak{g}, K_\infty, \mathfrak{I}_B^G \lambda_{\mathbb{R}}^{w_0} \otimes \mathcal{M}_\lambda \otimes \mathbb{C})$ and we define

$$\text{Eis}_n = \text{Eis}(\omega_n)$$

Proposition 4.1.8. *The restriction of* Eis_n *to* $H^1(\partial(\Gamma \backslash \mathbb{H}), \mathcal{M}_\lambda \otimes \mathbb{C})$ *is the class* $[Y^n]$

discuss the Biancji case?

Chapter 5

Application to Number Theory

5.1 Modular symbols, L-values and denominators of Eisenstein classes.

In this chapter we want to restrict to the case $\Gamma = \mathrm{Sl}_2(\mathbb{Z})$ or $\Gamma = \mathrm{Sl}_2(\mathcal{O})$ where $\mathcal{O}$ is the ring of integers of an imaginary quadratic extension. We refer to section 4.1.1 then this means that $\Gamma = \mathcal{G}(\mathbb{Z})$. Our coefficient systems will be obtained from the modules $\mathcal{M}_\lambda$. We assume that we have $d = 0$ and hence $n \equiv 0 \mod 2$ in case A), and $d_1 = d_2 = 0,\ n_1 = n_1$ in case B). This has the effect that $\lambda^\vee = \lambda$.

We want to study the pairing

$$H^1_c(\Gamma\backslash X, \tilde{\mathcal{M}}^\flat_\lambda) \times H_1(\Gamma\backslash X, \partial(\Gamma\backslash X), \underline{\mathcal{M}}_\lambda) \to \mathbb{Z}, \tag{5.1}$$

5.1.1 Modular symbols attached to a torus in Gl_2.

In a first step we construct (relative) cycles in $C_1(\Gamma\backslash X, \underline{\mathcal{M}}_\lambda), C_1(\Gamma\backslash X, \partial(\Gamma\backslash X), \underline{\mathcal{M}}_\lambda)$. Our starting point is a maximal torus $T/\mathbb{Q} \subset G/\mathbb{Q}$ and we assume that it is split over a real quadratic extension $F/\mathbb{Q}$. Then the group of real points

$$T(\mathbb{R}) = \mathbb{R}^\times \times \mathbb{R}^\times$$

act on $\mathbb{H}$ and $\bar{\mathbb{H}}$ and it has two fixed points $r, s \in \mathbb{P}^1(F)$. There is a unique geodesic (half) circle $\bar{C}_{r,s} \subset \bar{\mathbb{H}}$ joining these two points. Then $T(\mathbb{R})$ acts transitively on $C_{r,s} = \bar{C}_{r,s} \setminus \{r, s\}$. We have two cases:

a) The torus $T/\mathbb{Q}$ is split. Then the two points $r, s \in \mathbb{P}^1(\mathbb{Q})$. Here for instance we can take $r = 0, s = \infty$, then the geodesic circle is the line $\{iy, y > 0\}$ and the torus is the standard diagonal split torus.

b) Here $\{r, s\} \in \mathbb{P}^1(F) \setminus \mathbb{P}^1(\mathbb{Q})$, then r, s are Galois-conjugates of each other. Our torus $T/\mathbb{Q}$ is given by a suitable embedding

$$j : R_{F/\mathbb{Q}}(\mathbb{G}_m/F) = T \hookrightarrow \mathrm{Gl}_2/\mathbb{Q}.$$

In case a) we can choose any reasonable homeomorphism $[0,1] \xrightarrow{\sim} [0,\infty]$ - for instance $x \mapsto x/(1-x)-$ and then we get a one chain

$$\sigma : [0,1] \xrightarrow{\sim} \bar{C}_{r,s} = \mathbb{R}_{>0} \cup \{0\} \cup \{\infty\}, \sigma(0) = r, \sigma(1) = s \in \partial(\bar{\mathbb{H}}),$$

and for any $m \in \mathcal{M}$ we can consider the image of $\sigma \otimes m \in C_1(\bar{\mathbb{H}}) \otimes \mathcal{M}$ in $C_1(\Gamma\backslash\bar{\mathbb{H}}, \partial(\Gamma\backslash\bar{\mathbb{H}}), \underline{\mathcal{M}})$. By definition this is a cycle and hence we get a (relative) homology class

$$[\bar{C}_{r,s} \otimes m] \in H_1(\Gamma\backslash\bar{\mathbb{H}}, \partial(\Gamma\backslash\bar{\mathbb{H}}), \underline{\mathcal{M}}_\lambda), \tag{5.2}$$

it is easy to see that it does not depend on the choice of σ.

In case b) we have $T(\mathbb{Q}) \xrightarrow{\sim} F^\times$. Then the group $T(\mathbb{Q}) \cap \Gamma$ is a subgroup of finite index in the group of units $\mathcal{O}_F^\times = \{\epsilon_0\} \times \{\pm 1\}$, where ϵ_0 is a fundamental unit. Hence

$$\Gamma_T = T(\mathbb{Q}) \cap \Gamma = \{\epsilon_T\} \times \mu_T \tag{5.3}$$

where ϵ_T is an element of infinite order and μ_T is trivial or $\{\pm 1\}$. This element ϵ_T induces a translation on $C_{r,s}$. The quotient $C_{r,s}/\Gamma_T$ is a circle. If we pick any point $x \in C_{r,s}$ then $[x, \epsilon_T x] \subset C_{r,s}$ is an interval and as above we can find a $\sigma : [0,1] \xrightarrow{\sim} [x, \epsilon_T x], \sigma(0) = x, \sigma(1) = \epsilon_T x$, As before we can consider the 1-chain $\sigma \otimes m \in C_1(\mathbb{H}) \otimes \mathcal{M}$. Its boundary boundary is the zero chain $\{x\} \otimes m - \{\epsilon_T x\} \otimes m$. If we look at the images in $C_\bullet(\Gamma\backslash\mathbb{H}, \underline{\mathcal{M}}_\lambda)$ then

$$\partial_1(\sigma \otimes m) = \sigma(0) \otimes (m - \epsilon_T m) = r \otimes (m - \epsilon_T m) \tag{5.4}$$

Hence we see that $\sigma \otimes m$ is a 1 -cycle if and only if $m = \epsilon_T m$ and hence $m \in \mathcal{M}^T$. We have constructed homology classes

$$[C_{r,s} \otimes m] \in H_1(\Gamma\backslash\mathbb{H}, \underline{\mathcal{M}}_\lambda) \text{ for all } m \in \mathcal{M}_\lambda^{<\epsilon_T>} = \mathcal{M}_\lambda^T \tag{5.5}$$

PDualsec

5.1.2 Evaluation of cuspidal classes on modular symbols

The following issue will also be discussed in greater generality and more systematically in chapter 8.2.1.

We start from a highest weight $\lambda = n\gamma$ for simplicity we assume n to be even and $d = 0$. Then $\lambda = \lambda^\vee$, we consider the two modules $\mathcal{M}_\lambda$ and $\mathcal{M}_{\underline{\lambda}}^\flat$. Then we have the pairings

$$\begin{gathered} H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{\underline{\lambda}}^\flat) \times H_1(\Gamma\backslash\mathbb{H}, \underline{\mathcal{M}}_\lambda) \to \mathbb{Z} \\ H_c^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{\underline{\lambda}}^\flat) \times H_1(\Gamma\backslash\mathbb{H}, \partial(\Gamma\backslash\mathbb{H}), \underline{\mathcal{M}}_\lambda) \to \mathbb{Z} \end{gathered} \tag{5.6}$$

These two pairings are non degenerate if we invert 6 and divide by the torsion on both sides. (See [book]).

We have the surjective homomorphism $H^1_c(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}^\flat_{\underline{\lambda}})\to H^1_!((\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}^\flat_{\underline{\lambda}})$ and over a suitably large finite extension $F/\mathbb{Q}$ we have the isotypical decomposition

$$H^1_!((\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}^\flat_{\underline{\lambda}}\otimes F)=\bigoplus_{\pi_f} H^1_!((\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}^\flat_{\underline{\lambda}}\otimes F)(\pi_f) \tag{5.7}$$

where the π_f are absolutely irreducible. (See Theorem 5.7, of course here it does not matter whether we work with $\mathcal{M}_\lambda$ or $\mathcal{M}^\flat_{\underline{\lambda}}$) . We choose an embedding $\iota: F\hookrightarrow\mathbb{C}$, in section 4.1.11 we constructed the isomorphism

$$\mathcal{F}^1_1(\omega^\dagger_\epsilon):\mathcal{W}(\pi_f)\otimes_{F,\iota}\mathbb{C}\to H^1_{\epsilon,!}(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}^\flat_{\underline{\lambda}}\otimes F)({}^\iota\pi_f) \tag{5.8}$$

The space $\mathcal{W}(\pi_f)$ is a very explicit space. Since we want to stick to the case $K_f=K_f^{(0)}$ it is of dimension one and is generated by the element

$$h^{\dagger,0}_{\pi_f}=\prod_p h^{\dagger,0}_p\in\prod_p\mathcal{W}(\pi_p)\text{ where }h^{\dagger,0}_p(e)=1 \tag{5.9}$$

Now we want to compute the value

$$<\mathcal{F}^1_1(\omega^\dagger_\epsilon)(h^{\dagger,0}_{\pi_f})),\bar{C}_{r,s}\otimes m)>. \tag{5.10}$$

here we assume that the torus is split, i.e. $r,s\in\mathbb{P}^1(\mathbb{Q})$. Then this expression is problematic. The argument $C_{r,s}$ on the left lives in the relative homology group, hence the argument on the right should be in $H^1_c(\Gamma\backslash\mathbb{H},\mathcal{M}_n\otimes\mathbb{C})$. Of course we can lift the class $\mathcal{F}^1_1(\omega^\dagger_\epsilon)(h^{\dagger,0}_{\pi_f})$ to a class

$$\widetilde{\mathcal{F}^1_1(\omega^\dagger_\epsilon)}(h^{\dagger,0}_{\pi_f})\in H^1_c(\Gamma\backslash\mathbb{H},\mathcal{M}_n\otimes\mathbb{C}).$$

Then

$$<\mathcal{F}^1_1(\widetilde{\omega^\dagger_\epsilon\times h^{\dagger,0}_{\pi_f}}),C_{r,s}\otimes m>$$

makes sense, but the result may depend on the lift. We have paircusp

Proposition 5.1.1. *If $\partial(C_{r,s}\otimes m)$ gives the trivial class in $H_0(\partial(\Gamma\backslash\bar{\mathbb{H}}),\tilde{\mathcal{M}}_\lambda\otimes\mathbb{C})$ then $<\widetilde{\mathcal{F}^1_1(\omega^\dagger_\epsilon)}(h^{\dagger,0}_{\pi_f}),C_{r,s}\otimes m>$ does not depend on the lift, i.e. the value $<\mathcal{F}^1_1(\omega^\dagger_\epsilon)(h^{\dagger,0}_{\pi_f}),C_{r,s}\otimes m>$ is well defined.*

Proof. This is rather clear, we refer to the systematic discussion in 6.3.11. □

Now we compute the value of the pairing. We realised the relative homology class by a $\mathcal{M}_\lambda$ valued 1-chain $\sigma\otimes m$. The cohomology class $\mathcal{F}^1_1(\omega^\dagger_\epsilon)(h^{\dagger,0}_{\pi_f})$ is represented by $\mathcal{F}^1(\widetilde{\omega^\dagger_\epsilon\times h^{\dagger,0}_{\pi_f}})$. (See 4.96 ,8.4). We consider the pullback $\sigma^*(\mathcal{F}^1(\widetilde{\omega^\dagger_\epsilon}\times h^{\dagger,0}_{\pi_f}))$, since $\mathcal{F}^1(\omega^\dagger_\epsilon\times h^{\dagger,0}_{\pi_f})$ is rapidly decaying if $x\to 0$ or $x\to 1$ this gives us a 1-form with values in $\mathcal{M}_\lambda\otimes\mathbb{C}$ on the closed interval $[0,1]$.

We claim - under the assumption $[\partial(C_{r,s}\otimes m)]=0$-that

$$<\mathcal{F}^1_1(\omega^\dagger_\epsilon)(h^{\dagger,0}_{\pi_f}),C_{r,s}\otimes m>=\int_0^1<\sigma^*(\mathcal{F}^1(\widetilde{\omega^\dagger_\epsilon\times h^{\dagger,0}_{\pi_f}}),m>. \tag{5.11}$$

We have to be a little bit careful at this point. Of course our assumption implies that the integral class $[\partial(C_{r,s}\otimes m)] \in H_0(\partial(\Gamma\backslash\bar{\mathbb{H}}),\tilde{\mathcal{M}}_\lambda)$ is a torsion class, Let $\delta_{r,s}(m)$ be the order of this torsion class, hence we can write

$$\delta_{r,s}(m)\partial C_{r,s}\otimes m = \partial c_{r,s} \text{ with } c_{r,s}\in C_1(\partial(\Gamma\backslash\bar{\mathbb{H}},\underline{\mathcal{M}_\lambda}). \tag{5.12}$$

This 1-chain lies in the boundary of the Borel-Serre compactification (see section 1.2.7). We consider the special case that T is the standard split diagonal torus, this means that $\{r,s\}=\{0,\infty\}$. We can pull the cycle $\delta_{r,s}(m)C_{r,s}\otimes m - c_{r,s}$ into the interior $\Gamma\backslash\mathbb{H}$ by a simple homotopy, this means we replace it by $\delta_{r,s}(m)[iy_0^{-1},iy_0]\otimes m-\partial\delta_{r,s}(m)(y_0)$ where $y_0>>1$ and $\delta_{r,s}(m)(y_0)$ is the 1-chain $c_{r,s}$ on the level y_0. Then

$$\delta_{r,s}(m)<\mathcal{F}_1^1(\omega_\epsilon^\dagger)(h_{\pi_f}^{\dagger,0}),C_{r,s}\otimes m>=<\mathcal{F}_1^1(\omega_\epsilon^\dagger)(h_{\pi_f}^{\dagger,0}),\delta_{r,s}(m)[iy_0^{-1},iy_0]\otimes m-c_{r,s}(y_0)>. \tag{5.13}$$

where now the value on the right hand side is an integral over the truncated cycle. Since the differential form $\mathcal{F}_1^1(\omega_\epsilon^\dagger)(h_{\pi_f}^{\dagger,0})$ is rapidly decreasing if $y_0\to\infty$ we get $\delta_{r,s}(m)<\mathcal{F}_1^1(\omega_\epsilon^\dagger)(h_{\pi_f}^{\dagger,0}),C_{r,s}\otimes m>=\lim_{y_0\to\infty}<\mathcal{F}_1^1(\omega_\epsilon^\dagger)(h_{\pi_f}^{\dagger,0}),\delta_{r,s}(m)[iy_0^{-1},iy_0]\otimes m>$.

We use the above identification $[0,1]=[0,\infty]$ and our 1- chain is given by the map

$$\sigma:[0,\infty]\to\bar{\mathbb{H}}:t\mapsto\begin{pmatrix}t&0\\0&1\end{pmatrix}i=ti\in\bar{\mathbb{H}}, \tag{5.14}$$

especially $\sigma(0)=0$ and $\sigma(\infty)=i\infty$. The group $T(\mathbb{R})$ acts transitively on the open part $C_{0,i\infty}$. This action can be used to trivialize the tangent bundle. The tangent space at $i\in\mathbb{H}$ is identified to the subspace $\mathfrak{p}\subset\mathfrak{g}$ (see 4.1.11) and $\frac{H}{2}$ is a generator of the tangent space of $C_{0,i\infty}$ at one. Using the translations by $T(\mathbb{R})$ we get an invariant vector field on $C_{0,i\infty}$. If we identify $C_{0,i\infty}=\mathbb{R}_{>0}$, an easy calculation shows that this vector field is $t\frac{d}{dt}=D^*$.

Now an easy calculation (See 8.4) shows that (here e_f is the identity element in $G(\mathbb{A}_f)$)

$$\mathcal{F}^1(\widetilde{\omega_\epsilon^\dagger\times h_{\pi_f}^{\dagger,0}})(D^*)((\begin{pmatrix}t&0\\0&1\end{pmatrix}e_f)=\rho_\lambda(\begin{pmatrix}t^{-1}&0\\0&1\end{pmatrix})\mathcal{F}^1(\omega_\epsilon^\dagger\times h_{\pi_f}^{\dagger,0})(\frac{H}{2})((\begin{pmatrix}t&0\\0&1\end{pmatrix},e_f))$$

and our integral in the formula above becomes

$$\int_0^\infty<\rho_\lambda(\begin{pmatrix}t^{-1}&0\\0&1\end{pmatrix})\mathcal{F}^1(\omega_\epsilon^\dagger(\frac{H}{2})\times h_{\pi_f}^{\dagger,0})(\begin{pmatrix}t&0\\0&1\end{pmatrix},e_f),m>\frac{dt}{t}. \tag{5.15}$$

Our formulas in 4.1.11 give

$$\omega_\pm^\dagger(\frac{H}{2})=\frac{1}{8}(\tilde{\psi}_{n+2}\otimes(X-Y\otimes i)^n\pm\tilde{\psi}_{-n-2}\otimes(X+Y\otimes i)^n \tag{5.16}$$

this is an element in $\tilde{\mathcal{D}}_\lambda^\pm\otimes\mathcal{M}_\lambda$. We apply $\mathcal{F}^1$ to $\omega_\pm^\dagger(\frac{H}{2})\times h_{\pi_f}^{\dagger,0})$ and evaluate at $(\begin{pmatrix}t&0\\0&1\end{pmatrix},e_f)$. Applying $\mathcal{F}^1$ means that we have to sum over $a\in\mathbb{Q}^\times$ but since

$h^{\dagger,0}_{\pi_f}$) is the Whittaker function attached to the unramified spherical function only the terms with $a \in \mathbb{Z}$ can be non zero. Hence get

$$\mathcal{F}^1(\omega^\dagger_\pm(\tfrac{H}{2}) \times h^{\dagger,0}_{\pi_f})(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}, e_f) =$$

$$\frac{1}{8}\sum_{a\in\mathbb{Z};a\neq 0}(\tilde{\psi}_{n+2}(\begin{pmatrix} at & 0 \\ 0 & 1 \end{pmatrix}) \otimes (X - Y\otimes i)^n \pm \tilde{\psi}_{-n-2}(\begin{pmatrix} at & 0 \\ 0 & 1 \end{pmatrix}) \otimes (X + Y\otimes i)^n h^{\dagger,0}_{\pi_f}(a) \tag{5.17}$$

We have seen that $\tilde{\psi}_{n+2}(\begin{pmatrix} at & 0 \\ 0 & 1 \end{pmatrix}) = 0$ if $at < 0$ and $\tilde{\psi}_{n+2}(\begin{pmatrix} -at & 0 \\ 0 & 1 \end{pmatrix}) = \tilde{\psi}_{-n-2}(\begin{pmatrix} at & 0 \\ 0 & 1 \end{pmatrix})$ and therefore our Fourier expansion becomes

$$\frac{1}{8}\sum_{a=1}^{\infty}\tilde{\psi}_{n+2}(\begin{pmatrix} at & 0 \\ 0 & 1 \end{pmatrix}) \otimes ((X - Y\otimes i)^n \pm i^n(X + Y\otimes i)^n)h^{\dagger,0}_{\pi_f}(a) \tag{5.18}$$

We have

$$\begin{gathered}\rho_\lambda(\begin{pmatrix} t^{-1} & 0 \\ 0 & 1 \end{pmatrix})((X - Y\otimes i)^n \pm i^n(X + Y\otimes i)^n) = \\ \sum_{\nu=0}^{n}\binom{n}{\nu} t^{\frac{n}{2}-\nu}X^\nu Y^{n-\nu}(i^{n+\nu} \pm i^{-\nu}),\end{gathered} \tag{5.19}$$

we remember that n is even, then the last factor is equal to $i^{-\nu}((-1)^{\frac{n}{2}+\nu} \pm 1)$. and this is $i^{-\nu}$ times 2 or 0 or -2, depending on the choices of signs and the parity of $\frac{n}{2}$ and ν. The elements $e_\nu = X^\nu Y^{n-\nu}$ form the dual basis to the basis $\binom{n}{n-\nu}X^{n-\nu}Y^\nu$ of $\mathcal{M}^\flat_\lambda$, this implies: If we choose $m = e_{n-\nu}$ in our expression above then pairinf

$$< \rho_\lambda(\begin{pmatrix} t^{-1} & 0 \\ 0 & 1 \end{pmatrix})((X - Y\otimes i)^n \pm i^n(X + Y\otimes i)^n), m >= t^{\frac{n}{2}-\nu}(i^{n-\nu} \pm i^{-\nu}) \tag{5.20}$$

and hence we have to compute

$$\frac{i^{n+\nu} \pm i^{-\nu}}{8}\int_0^\infty \sum_{a=1}^{\infty}\tilde{\psi}_{n+2}(\begin{pmatrix} at & 0 \\ 0 & 1 \end{pmatrix})t^{\frac{n}{2}-\nu}h^{\dagger,0}_{\pi_f}(a)\frac{dt}{t}. \tag{5.21}$$

We remember $\tilde{\psi}_{n+2}(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) = t^{\frac{n}{2}+1}e^{-2\pi t}$, we exchange summation and integration and after some innocent substitutions we get

$$\frac{i^{n+\nu} \pm i^{-\nu}}{8}\int_0^\infty \frac{t^{n-\nu+1}}{(2\pi)^{n-\nu+1}}e^{-t}\frac{dt}{t}\sum_{a=1}^{\infty}\frac{h^\dagger_{\pi_f}(a)a^{\frac{n}{2}}}{a^\nu} \tag{5.22}$$

We refer to the discussion of the L -function attached to π_f and get

$$\int_0^\infty \frac{t^{n-\nu+1}}{(2\pi)^{n-\nu+1}} e^{-t} \frac{dt}{t} \sum_{a=1}^{\infty} \frac{h^\dagger_{\pi_f}(a) a^{\frac{n}{2}}}{a^\nu} = \Lambda^{\mathrm{coh}}(\pi, n+1-\nu) \tag{5.23}$$

Of course some question concerning convergence have to be discussed, for this we refer to the proof of Theorem 4.1.7.

In the case that $\nu \neq 0, n$ we know that $\partial(C_{0,\infty} \otimes X^{n-\nu} Y^\nu)$ is a torsion element in $H^0(\partial(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})$ and therefore the value of the integral is also the evaluation of the cohomology class $\mathcal{F}_1^1(\omega_\epsilon^\dagger)(h^{\dagger,0}_{\pi_f})$ on a integral homology class. We get

$$< \mathcal{F}_1^1(\omega_\epsilon^\dagger)(h^{\dagger,0}_{\pi_f}, C_{0,\infty} \otimes X^\nu Y^{n-\nu}) >= \frac{i^{n+\nu} \pm i^{-\nu}}{8} \Lambda^{\mathrm{coh}}(\pi, n+1-\nu) \tag{5.24}$$

In the factor in front on the right side we have $\epsilon = \pm 1$, this factor is zero unless we have $\epsilon = (-1)^{\frac{n}{2}-\nu}$ (see 4.153) and then it is simply $\pm\frac{1}{4}$.

If the class number of $\mathcal{O}_F$ is one we defined the periods $\Omega(\epsilon \times \pi_f)$, (see 4.1.11) we then know that

$$\frac{1}{\Omega(\epsilon \times \pi_f)} \mathcal{F}_1^1(\omega_\epsilon^\dagger)(h^{\dagger,0}_{\pi_f}) \in H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathcal{O}_F) \tag{5.25}$$

and hence we can conclude for $\nu \neq 0, n$ ratint

$$\frac{\delta_{0,\infty}(e_\nu)}{\Omega(\epsilon \times \pi_f)} \Lambda^{\mathrm{coh}}(\pi, n+1-\nu) \in \mathcal{O}_F \tag{5.26}$$

If the class number is not one we have to interpret $\Omega(\epsilon \times \pi_f)$ as section in the sheaf of periods and $\mathcal{O}_F$ has to be replaced by the monoid of integral ideals in $\mathcal{O}_F$. Notice that the term $\delta_{0,\infty}(e_\nu)$ has only prime factors $< n$. We will improve this term after the following discussion of the cases $\nu = 0, \nu = n$.

This argument fails for $\nu = 0, n$ because $\partial(C_{0,\infty} \otimes X^n) = \infty \otimes (X^n - Y^n)$ is not a torsion class in $H_0(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_\lambda)$ (See section 3.2.1). We apply the Manin-Drinfeld principle to show that the rationality statement also holds for $\nu = 0, n$ but we will get a denominator.

We pick a prime p then we know that the class $[\partial(C_{0,\infty} \otimes X^n)]$ is an eigenclass modulo torsion for T_p, i.e.

$$T_p([\partial(C_{0,\infty} \otimes X^n]) = (p^{n+1}+1)[\partial(C_{0,\infty} \otimes X^n)] \tag{5.27}$$

This implies that $\partial(T_p([\mathbb{C}_{0,\infty} \otimes X^n]) - (p^{n+1}+1)[(C_{0,\infty} \otimes X^n]))$ is a torsion class, hence we can apply proposition 5.1.1 and get that the value of the pairing is equal to the integral against the modular symbol. If we exploit the adjointness formula for the Hecke operator then we get

$$< T_p([\mathbb{C}_{0,\infty} \otimes X^n]) - (p^{n+1}+1)[(C_{0,\infty} \otimes X^n]), \mathcal{F}_1^1(\omega_\epsilon^\dagger \otimes h^{\dagger,0}_{\pi_f}) >$$

$$= \textstyle\int_0^\infty (< C_{0,\infty} \otimes X^n, \mathcal{F}_1^1(\omega_\epsilon^\dagger \otimes T_p(h_{\pi_f})^{\dagger,0}) > \tag{5.28}$$

$$-(p^{n+1}+1) < C_{0,\infty} \otimes X^n, \mathcal{F}_1^1(\omega_\epsilon^\dagger) \otimes ((h^{\dagger,0}_{\pi_f}) >)) \tfrac{dt}{t}$$

We have $T_p(h^{\dagger,0}_{\pi_f}) = a_p h^{\dagger,0}_{\pi_f}$ where $a_p \in \mathcal{O}_F$ and hence we get

$$\begin{aligned} &< T_p([\mathbb{C}_{0,\infty} \otimes X^n]) - (p^{n+1}+1)[(C_{0,\infty} \otimes X^n]), \mathcal{F}_1^1(\omega_\epsilon^\dagger \otimes h^{\dagger,0}_{\pi_f}) > \\ &= (a_p - (p^{n+1}+1))\Lambda^{\text{coh}}(\pi_f, n+1) \end{aligned} \tag{5.29}$$

It is again the Manin-Drinfeld principle that tells us that for almost all primes p the number $a_p - (p^{n+1}+1) \neq 0$. Let $(Z(n))$ be the ideal in $\mathcal{O}_F$ generated by these numbers. of these numbers. We will see (Theorem 5.1.2) that

$$(\text{numerator}(\zeta(-1-n))) \subset (Z(n)) \tag{5.30}$$

Ribet gives an argument in [87] that yields even equality.

Now we can conclude: *For $\nu = 0, n+1$* $\boxed{ratintE}$

$$\frac{Z(n)}{\Omega(\epsilon \times \pi_f)}\Lambda^{\text{coh}}(\pi, n+1-\nu) \in \mathcal{O}_F \tag{5.31}$$

We want to have an estimate of the denominator ideal of

$$\frac{\Lambda^{\text{coh}}(\pi, n+1-\nu)}{\Omega(\epsilon \times \pi_f)}$$

for all values of ν. For $\nu = 0, \nu = n$ we have the estimate $Z(n)$. For the other values of ν we have the $\delta_{0,\infty}(e_\nu)$, but we can do much better. Notice that this denominator ideal is an ideal in $\mathcal{O}_F$. We pick a prime $p < n$ which then may divide $\delta_{0,\infty}(e_\nu)$. We work locally at p and replace $\mathbb{Z}$ by $\mathbb{Z}_{(p)}$, the local ring at p. It follows from proposition 3.3.1 that for $0 < \nu < n$ the torsion element $[\partial(C_{0,\infty} \otimes e_\nu^\vee))]$ is annihilated by a sufficiently high power of the Hecke operator T_p^m. Hence we see that $T_p^m(c)$ can be lifted to an element $\widetilde{T_p^m(c)} \in H_1(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}^\flat_{\underline{\lambda}} \otimes \mathbb{Z}_{(p)})$. Hence we can lift $T_p^m(C_{0,\infty} \otimes e_\nu^\vee))$ to an element $T_p^m(\widetilde{C_{0,\infty}} \otimes e_\nu^\vee)) \in H_1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}^\flat_{\underline{\lambda}} \otimes \mathbb{Z}_{(p)})$. We know that

$$< \mathcal{F}_1^1(\omega_\epsilon^\dagger)(h^{\dagger,0}_{\pi_f}, T_p^m(\widetilde{C_{0,\infty} \otimes e_\nu^\vee})) \in \mathcal{O}_F \otimes \mathbb{Z}_{(p)}. \tag{5.32}$$

Again we can use the adjointness property of T_p and we get

$$\pi_f(T_p)^m < F_1^1(\omega_\epsilon^\dagger)(h^{\dagger,0}_{\pi_f}, (C_{0,\infty} \otimes e_\nu^\vee)) >= \frac{\pi_f(T_p)^m}{\Omega(\epsilon \times \pi_f)}\Lambda^{\text{coh}}(\pi, n+1-\nu) \in \mathcal{O}_F \otimes \mathbb{Z}_{(p)} \tag{5.33}$$

We consider the ideal $\mathfrak{n}(p, \nu, \pi_f) = (\delta_{0,\infty}(e_\nu), \pi_f(T_p)^m) \subset \mathcal{O}_F \otimes \mathbb{Z}_{(p)}$. This ideal may be much larger than $(\delta_{0,\infty}(e_\nu)$. We put $\mathfrak{n}(\nu, \pi_f) = \prod_p \mathfrak{n}(p, \nu, \pi_f)$ for $\nu \neq 0, n$ and for convenience $\mathfrak{n}(n) = \mathfrak{n}(0) = Z(n)$

Then we get the final result: **Noch mal genauer diskutieren und vorher sagen was $\delta_{0,\infty}(e_\nu)$ ist**

Theorem 5.1.1. *For any π_f which occurs in (5.7) and any $\nu = 0 \ldots n$ the ideal*

$$\frac{\mathfrak{n}(\nu, \pi_f)}{\Omega(\epsilon \times \pi_f)}\Lambda^{\text{coh}}(\pi, n+1-\nu)) \tag{5.34}$$

is an integral ideal in $\mathcal{O}_F$. The primes $\mathfrak{p}$ dividing $\mathfrak{n}(\nu, \pi_f)$ a lie over primes $p < n$. Furthermore these primes are not ordinary for π_f, i.e if $\mathfrak{p}$ divides $\mathfrak{n}(\nu, \pi_f)$ then $\pi_f(T_p) \equiv 0 \mod \mathfrak{p}$.

These rationality results go back to Manin and Shimura, In principle we may say that also the integrality assertion goes back to these authors, but here we have to take into account the fine tuning of the periods. (Deligne conjecture? Later if we speak about motives)

It is clear that this compatible with the action of the Galois group Gal$(F/\mathbb{Q})$, for $\sigma \in$ Gal$(F/\mathbb{Q})$ we have

$$\sigma(\frac{1}{\Omega(\epsilon \times \pi_f)}\Lambda^{\mathrm{coh}}(\pi, n+1-\nu)) = \frac{1}{\Omega(\epsilon \times \pi_f)}\Lambda^{\mathrm{coh}}({}^{\sigma}\pi, n+1-\nu)) \qquad (5.35)$$

There is still a slightly different way to look at the theorem above. For each choice of $\epsilon = \pm$ we can look at the array of numbers

$$\{\dot{\Lambda}^{\mathrm{coh}}(\pi, n+1-\nu)), \dots\}_{\nu=0,\dots n;(-1)^{\frac{n}{2}-\nu}=\epsilon} \qquad (5.36)$$

Since we may assume that $n \geq 10$ it is easy to see that not all of the entries entries can be zero, hence we can project the arrays to a point $\mathbf{\Lambda}(\epsilon, \pi_f)$ in the projective space $\mathbb{P}^{d(\epsilon,n)}(\mathbb{C})$. Then a slightly weakened form of our results asserts

$$\mathbf{\Lambda}(\epsilon, \pi_f) \in \mathbb{P}^{d(\epsilon,n)}(F) = \mathbb{P}^{d(\epsilon,n}(\mathcal{O}_F) \text{ and } \sigma(\mathbf{\Lambda}(\epsilon, \pi_f)) = \mathbf{\Lambda}(\sigma(\epsilon, \pi_f))) \qquad (5.37)$$

In this formulation we do not see the period. But now we can fix the period as a section in the period sheaf: We require that the arrays of ideals

$$\{\dots, \frac{\mathfrak{n}(\nu, \pi_f)}{\Omega(\epsilon \times \pi_f)}\Lambda^{\mathrm{coh}}(\pi, n+1-\nu), \dots\}_{\nu=0,\dots n;(-1)^{\frac{n}{2}-\nu}=\epsilon} \qquad (5.38)$$

is an ideal of integral and coprime ideals. This period is not necessarily equal to our period we defined earlier, but they may only differ at primes $\mathfrak{p}$ dividing $\mathfrak{n}(\nu, \pi_f)$.

We pay so much attention to the careful choice of the periods because we conjecture that the factorisation of the numbers $\frac{\mathfrak{n}(\nu,\pi_f)}{\Omega(\epsilon\times\pi_f)}\Lambda^{\mathrm{coh}}(\pi, n+1-\nu))$ has influence on the structure of the integral cohomology of some other groups. We expect that prime ideals $\mathfrak{p} \subset \mathcal{O}_f$ which divide an ideal $\frac{\mathfrak{n}(\nu,\pi_f)}{\Omega(\epsilon\times\pi_f)}\Lambda^{\mathrm{coh}}(\pi, n+1-\nu))$ will also divide the denominator of an Eisenstein class on the symplectic group. A prototype of such an assertion has been discussed in [43]. We will resume this discussion in section 8.3.5.

In the following section we discuss another (simpler) example,, where we see the relationship between divisibility of certain L-values and denominators of Eisenstein classes.

5.1.3 Evaluation of Eisenstein classes on capped modular symbols

In the following we consider cohomology with coefficients in $\mathcal{M}_n$. We have seen that MDEis

$$H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}^{\flat}_{\underline{\lambda}} \otimes \mathbb{Q}) = H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}^{\flat}_{\underline{\lambda}} \otimes \mathbb{Q}) \oplus \mathbb{Q}\mathrm{Eis_n} \qquad (5.39)$$

where $\mathrm{Eis_n}$ is defined by the two conditions

$$r(\mathrm{Eis_n}) = [Y^n] \text{ and } T_p(\mathrm{Eis_n}) = (p^{n+1}+1)\mathrm{Eis_n}, \tag{5.40}$$

for all Hecke operators T_p, in our special situation it suffices to check the second condition for $p = 2$. In section **??** we raised the question to determine the denominator of the class $\mathrm{Eis_n}$, i.e. we want to determine the smallest integer $\Delta(n) > 0$ such that $\Delta(n)\mathrm{Eis_n}$ becomes an integral class.

To achieve this goal we compute the evaluation of $\mathrm{Eis_n}$ on the first homology group, i.e we compute the value $< c, \mathrm{Eis_n} >$ for $c \in H_1(\Gamma\backslash\mathbb{H}, \underline{\mathcal{M}}_\lambda)$. We have the exact sequence

$$H_1(\partial(\Gamma\backslash\mathbb{H}), \underline{\mathcal{M}}_\lambda) \xrightarrow{j} H_1(\Gamma\backslash\mathbb{H}, \underline{\mathcal{M}}_\lambda) \to H_1(\Gamma\backslash\mathbb{H}, \partial(\Gamma\backslash\mathbb{H}), \underline{\mathcal{M}}_\lambda) \xrightarrow{\delta} H_0(\partial(\Gamma\backslash\mathbb{H}), \underline{\mathcal{M}}_\lambda) \tag{5.41}$$

It follows from the construction of $\mathrm{Eis_n}$ that $< c, \mathrm{Eis_n} > \in \mathbb{Z}$ for all the elements the image of j. Therefore we only have to compute the values $< \tilde{c}_\nu, \mathrm{Eis_n} >$, where $\tilde{c}_\mu$ are lifts of a system of generators $\{c_\mu\}$ of $\ker(\delta)$.

In our special case the elements $C_{0,\infty} \otimes e_\nu$, where $\nu = 0, 1 \dots, n$ form a set of generators of $H_1(\Gamma\backslash\mathbb{H}, \partial(\Gamma\backslash\mathbb{H}), \underline{\mathcal{M}}_\lambda)$. (Diploma thesis Gebertz). We observe:

The boundary of the element $C_{0,\infty} \otimes e_n^\vee (= \pm C_{0,\infty} \otimes e_0^\vee)$ *is an element of infinite order in* $H^0(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}^\flat_{\underline{\lambda}})$,

The boundary of an elements $C_{0,\infty} \otimes e_\nu^\vee$ with $0 < \nu < n$ are torsion elements in $H^0(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}^\flat_{\underline{\lambda}})$, This implies

Proposition 5.1.2. *The elements* $C_{0,\infty} \otimes m \in H_1(\Gamma\backslash\mathbb{H}, \partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}^\flat_{\underline{\lambda}})$ *with* $\partial(C_{0,\infty} \otimes m) = 0$ *are of the form*

$$c = C_{0,\infty} \otimes (\sum_{\nu=1}^{\nu=n-1} a_\nu e_\nu^\vee); \quad \textit{with } a_\nu \in \mathbb{Z}$$

Now it seems to be tempting to choose for our generators above the $C_{0,\infty} \otimes e_\nu^\vee$, but this is not possible because for $\delta(C_{0,\infty} \otimes e_\nu^\vee)$ is not necessarily zero, it is only a torsion element. So we see that it is not clear how to find a suitable system of generators.

To overcome this difficulty we use the Hecke operators. If we want to determine the denominator $\Delta(n)$ we can localize, i.e. for each prime p we have to determine the highest power $p^{d(n,p)}$ which divides $\Delta(n)$. As usual we write $d(n,p) = \mathrm{ord}_p(\Delta(n))$. We replace the ring $\mathbb{Z}$ by its localization $\mathbb{Z}_{(p)}$ and replace all our cohomology and homology groups by he localized groups. In other words we have to check we have to find a set of generators $\{\dots, \tilde{c}_\nu \dots\}_\nu \subset H_1(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}^\flat_{\underline{\lambda}} \otimes \mathbb{Z}_{(p)})$ and compute the denominator $< \tilde{c}_\nu, \ \mathrm{Eis}_n > \in \mathbb{Z}_{(p)}$.

It follows from proposition 3.3.1 that for $0 < \nu < n$ the torsion element $\partial(c) = \partial(C_{0,\infty} \otimes (\sum_{\nu=1}^{\nu=n-1} a_\nu e_\nu^\vee))$ is annihilated by a sufficiently high power of the Hecke operator T_p^m and hence we see that $T_p^m(c)$ can be lifted to an element $\widetilde{T_p^m(c)} \in H_1(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}^\flat_{\underline{\lambda}} \otimes \mathbb{Z}_{(p)})$. Now

$$< \widetilde{T_p^m(c)}, \ \mathrm{Eis}_n > = < c, T_p^m(\ \mathrm{Eis}_n) > = (p^{n+1}+1)^m < c, \ \mathrm{Eis}_n > \tag{5.42}$$

and hence $\mathrm{ord}_p(<\widetilde{T_p^m(c)},\ \mathrm{Eis}_n>) = \mathrm{ord}_p(<c,\ \mathrm{Eis}_n>)$. Hence we get

Proposition 5.1.3. *If ν runs from 1 to $n-1$ and if $T_p^m(\widetilde{C_{0,\infty}} \otimes e_\nu^\vee)$ is any lift of $T_p^m(e_\nu^\vee)$ then*

$$d(n,p) = -\min\Big(\min_\nu(\ ord_p(<T_p^m(\widetilde{C_{0,\infty}} \otimes e_\nu^\vee),\ Eis_n>)), 0\Big)$$

Proof. This is now obvious. □

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5.1.4 The capped modular symbol

Therefore we have to compute $<T_p^m(\widetilde{C_{0,\infty}} \otimes e_\nu),\ \mathrm{Eis}_n>)$. At this point some meditation is in order. Our cohomology class Eis_n is represented by a closed differential form $\mathrm{Eis}(\omega_n)$ (See (???)) and this differential form lives on $\Gamma\backslash\mathbb{H}$ a hence provides a cohomology class in $\Gamma\backslash\mathbb{H}$. But we know that the inclusion provides an isomorphism

$$H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}^\flat_{\underline{\lambda}}) \xrightarrow{\sim} H^1(\Gamma\backslash\bar{\mathbb{H}}, \tilde{\mathcal{M}}^\flat_{\underline{\lambda}})$$

and since $T_p^m(\widetilde{C_{0,\infty}} \otimes e_\nu) \in H_1(\Gamma\backslash\bar{\mathbb{H}}, \underline{\mathcal{M}}_\lambda)$ we can evaluate the cohomology class $\mathrm{Eis}(\omega_n)$ on the cycle. But we want get this value $<T_p^m(\widetilde{C_{0,\infty}} \otimes e_\nu),\ \mathrm{Eis}_n>$ by integration of the differential form against the cycle. This is problematic because the cycle has non trivial support in $\partial(\Gamma\backslash\mathbb{H})$, and on this circle at infinity the differential form is not really defined.

There are certainly several ways out of this dilemma. The Borel-Serre boundary is a circle $\Gamma_\infty\backslash\mathbb{R}$ where $\Gamma_\infty = \{\pm\mathrm{Id}\} \times \{\mathcal{T}_\infty^m\}$ and $T_\infty = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The cycle is the sum of two 1-chains:

$$T_p^m(\widetilde{C_{0,\infty}} \otimes e_\nu) = C_{0,\infty} \otimes m_\nu + [i\infty, T_\infty i\infty] \otimes P_\nu$$

(recall definition of Borel-Serre construction from earlier chapters) where

$$\partial(C_{0,\infty} \otimes m_\nu) = \infty \otimes (m_\nu - w m_\nu) + \infty \otimes (1 - T_\infty)P_\nu = 0$$

One possibility is to deform the cycle $T_p^m(\widetilde{C_{0,\infty}} \otimes e_\nu)$ and "pull" it into the interior $\Gamma\backslash\mathbb{H}$. Recall that $C_{0,\infty}$ is the continuous extension of $t \mapsto \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} i$ from $\mathbb{R}^\times_{>0}$ to $\mathbb{H}$ to a map from $[0,\infty] \to \bar{\mathbb{H}}$. We choose a sufficiently large $t_0 \in \mathbb{R}^\times_{>0}$ and restrict $C_{0,\infty}$ to $[t_0^{-1}, t_0]$ we get the one chain $C_{0,\infty}(t_0) \otimes m_\nu$. The boundary of this 1-chain is $\partial(C_{0,\infty}(t_0) \otimes m_\nu) = t_0 \otimes (m_\nu - w m_\nu)$. Now we can do at this level the same thing as what we do at infinity we get a 1-cycle

$$C_{0,\infty}\widetilde{(t_0)} \otimes m_\nu = C_{0,\infty}(t_0) \otimes m_\nu + [t_0, T_\infty t_0] \otimes P_\nu$$

This 1-cycle clearly defines the same class as $T_p^m(\widetilde{C_{0,\infty}} \otimes e_\nu)$ and since it is a cycle in $C_1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})$ we get

$$<T_p^m(\widetilde{C_{0,\infty}} \otimes e_\nu),\ \mathrm{Eis}_n> = \int_{C_{0,\infty}(t_0)\otimes m_\nu + [t_0, T t_0]\otimes P_\nu} \mathrm{Eis}_n \tag{5.43}$$

The value of this integral does not depend on t_0 and we check easily that for both summands the limit for $t_0 \to \infty$ exists. We find that Nenner1

$$< T_p^m(\widetilde{C_{0,\infty}} \otimes e_\nu^\vee),\ \mathrm{Eis}(\omega_n) >=$$

$$\int_0^\infty < T_p^m(C_{0,\infty} \otimes e_\nu^\vee),\ \mathrm{Eis}_n > \tfrac{dt}{t} + \lim_{t_0 \to \infty} \int_0^1 < [it_0, it_0 + x] \otimes P_\nu,\ \mathrm{Eis}_n > dx \tag{5.44}$$

For the first integral we have

$$\int_0^\infty < T_p^m(C_{0,\infty} \otimes e_\nu^\vee),\ \mathrm{Eis}_n > \frac{dt}{t} = (1+p^{n+1})^m \int_0^\infty < C_{0,\infty} \otimes e_\nu^\vee,\ \mathrm{Eis}_n > \frac{dt}{t}$$

and (handwritten notes page 49)

$$\int_0^\infty < C_{0,\infty} \otimes e_\nu^\vee,\ \mathrm{Eis}_n > \frac{dt}{t} = \frac{\zeta(-\nu)\zeta(\nu-n)}{\zeta(-1-n)} \tag{5.45}$$

remember this holds for $0 < \nu < n$.

For the second term we have to observe that it depends on the choice of P_ν. We can replace P_ν by $P_\nu + V$ where $V^T = V$. (This means of course that $V = aX^n$) Then $[V] \in H^0(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}_\lambda)$ and

$$\lim_{t_0\to\infty} \int_0^1 < [it_0, it_0+x] \otimes (P_\nu+V),\ \mathrm{Eis}_n > dx = \lim_{t_0\to\infty} \int_0^1 < [it_0, it_0+x] \otimes P_\nu,\ \mathrm{Eis}_n > dx + < V, \omega_n > .$$

Therefore the second term is only defined up to a number in $\mathbb{Z}_{(p)}$ but this is ok because we are interested in the p-denominator in (5.44).

We have to evaluate the expression $< [it_0, it_0+x] \otimes (P_\nu+V),\ \mathrm{Eis}_n >$. Using the formula (8.4) we find

$$< [it_0, it_0 + x] \otimes (P_\nu + V),\ \mathrm{Eis}_n >=< \begin{pmatrix} t_0 & x \\ 0 & 1 \end{pmatrix} P_\nu,\ \mathrm{Eis}(\omega_n)(E_+)(\begin{pmatrix} t_0 & x \\ 0 & 1 \end{pmatrix} > \tag{5.46}$$

We know that for $t_0 >> 1$ the Eisenstein series is approximated by its constant term, i.e.

$$\mathrm{Eis}(\omega_n)(E_+)(\begin{pmatrix} t_0 & x \\ 0 & 1 \end{pmatrix}) = t_0^{-n} Y^n + O(e^{-t_0}) \tag{5.47}$$

On the other hand we can write $P_\nu(X,Y) = \sum p_\mu^{(\nu)} X^{n-\mu} Y^\mu$ with $p_\mu^{(\nu)} \in \mathbb{Z}$. Then

$$\begin{pmatrix} t_0 & x \\ 0 & 1 \end{pmatrix} P_\nu = t_0^n p_0^{(\nu)} X^n + \dots \tag{5.48}$$

and

$$< \begin{pmatrix} t_0 & x \\ 0 & 1 \end{pmatrix} P_\nu,\ \mathrm{Eis}(\omega_n)(E_+)(\begin{pmatrix} t_0 & x \\ 0 & 1 \end{pmatrix}) >= p_0^{(\nu)} + O(e^{-t_0}) \tag{5.49}$$

and hence we see that the limit exists and we get

$$\lim_{t_0\to\infty}\int_0^1 < [it_0, it_0+x]\otimes(P_\nu+V),\ \mathrm{Eis}_n > dx = p_0^{(\nu)} = P_\nu(1,0) \tag{5.50}$$

and hence we have the final formula

$$< T_p^m(\widetilde{C_{0,\infty}\otimes e_\nu}),\ \mathrm{Eis}_n >= \frac{\zeta(-\nu)\zeta(\nu-n)}{\zeta(-1-n)} + P_\nu(1,0) \mod \mathbb{Z}_{(p)}. \tag{5.51}$$

Therefore we have to compute $P_\nu(1,0) \mod \mathbb{Z}_{(p)}$. Recall that for any $\nu, \nu \neq 0, n$ we have to choose a very large $m > 0$ such that the zero chain $T_p^m(e_\nu)$ is homologous to

$$T_p^m(e_\nu) \sim \{\infty\}\otimes L_\nu = \{\infty\}\otimes(1-T)Q_\nu \tag{5.52}$$

with $Q_\nu \in \mathcal{M}_n$. Then we find $P_\nu = Q_\nu \pm Q_{n+1-\nu}$.

Hence we have to compute $T_p^m(e_\nu)$. A straightforward but lengthy computation yields

$$Q_\nu(1,0) \in \begin{cases} \mathbb{Z}_{(p)} & \text{if } (p-1)\not| \nu+1 \\ \frac{1}{p^{\frac{\nu+1}{p-1}}} + \mathbb{Z}_{(p)} & \text{else} \end{cases} \tag{5.53}$$

Now we are ready to compute $d(n,p)$, it is the maximum over all ν denomest

$$d(n,p,\nu) = -\ \mathrm{ord}_p\Big(\frac{\zeta(-\nu)\zeta(\nu-n)}{\zeta(-1-n)} + (Q_\nu(1,0) + Q_{n-\nu}(1,0)) \mod \mathbb{Z}_{(p)}\Big). \tag{5.54}$$

We analyse this expression. We exploit the old theorems of Kummer and of von Staudt-Clausen. For an odd positive integer m the number $\zeta(-m)$ is a rational number. The theorem of von Staudt-Clausen asserts

$$\begin{cases} \zeta(-m) \in \mathbb{Z}_{(p)} & \text{if } p-1 \not| m+1 \\ \zeta(-m) + \frac{1}{p^{\frac{m+1}{p-1}}} \in \mathbb{Z}_{(p)} & \text{if } p-1 | m+1 \end{cases} \tag{5.55}$$

We distinguish cases.

I) We have $(p-1) \not| n+2$, then $\mathrm{ord}_p(\zeta(-1-n)) = \mathrm{ord}_p(\mathrm{Numerator}\zeta(-1-n))$, and $p-1$ can divide at most one of the two numbers $\nu+1$ or $n+1-\nu$.

Ia) Let us assume it divides neither of them. Then in (5.54)

$$d(n,p,\nu) = -\ \mathrm{ord}_p((\zeta(-\nu)\zeta(\nu-n)) +\ \mathrm{ord}_p(\zeta(-1-n)) \tag{5.56}$$

Ib) Alternatively we assume that $p-1|\nu+1$ we write $\nu+1 = p^{\alpha-1}\nu_0$, with $p^{\alpha-1}||\nu+1$. Then the p-denominator of $\zeta(-\nu)$ is p^α and $\nu - n \equiv -n-1 \mod (p-1)p^{\alpha-1}$. The Kummer congruences imply

$$\zeta(\nu-n) = \zeta(-n-1) + p^\alpha Z(\nu,n)\ ;\ \text{with } Z(\nu,n)\in\mathbb{Z}_{(p)} \tag{5.57}$$

and then $\mod \mathbb{Z}_{(p)}$

$$\begin{gathered}\tfrac{\zeta(-\nu)\zeta(\nu-n)}{\zeta(-1-n)} + (Q_\nu(1,0)+Q_{n-\nu}(1,0)) = \\ \zeta(-\nu)(1+p^\alpha\tfrac{Z(\nu,n)}{\zeta(-1-n)}) + Q_\nu(1,0) = \zeta(-\nu)p^\alpha\tfrac{Z(\nu,n)}{\zeta(-1-n)}\end{gathered} \tag{5.58}$$

This implies that

$$d(n,p,\nu) = \text{ ord}_p(\text{Numerator}(\zeta(-1-n)) - \text{ ord}_p(Z(\nu,n),$$

the factor in front is a unit.

II) We have $p-1|n+2$. Then p does not divide Numerator$(\zeta(-1-n))$ and hence we have to prove $d(n,p,\nu)=0$ for all ν. This is obvious if $p-1$ does not divide $\nu+1$ and hence also does not divide $n+1-\nu$.

Therefore assume $p-1|\nu+1$. We write $\nu+1=(p-1)xp^{a-1}, n+1-\nu=(p-1)yp^{b-1}$ with $a>0, b>0$ and x,y prime to p. We assume $a\le b$ and compute

$$\frac{\zeta(1-(p-1)xp^{a-1})\zeta(1-(p-1)yp^{b-1})}{\zeta(1-(p-1)p^{a-1}(x+yp^{b-a}))} \mod \mathbb{Z}_{(p)} \tag{5.59}$$

For a value $\zeta(1-m)$ with $p-1|m$ we write $m=(p-1)xp^{k-1}$ with $(x,p)=1$. We apply again the von Staudt-Clausen theorem

$$\zeta(1-m) = \zeta(1-(p-1)xp^{k-1} = -\frac{1}{xp^k} + Z(x) \text{ where } Z(x)\in\mathbb{Z}_{(p)}$$

In our case this gives -let us assume $a<b$ - for our expression above

$$\frac{-\frac{1}{(xp^a}+Z(x))(-\frac{1}{(yp^b}+Z(y))}{-\frac{1}{(x+yp^{b-a})p^a}+Z(x+yp^{b-a}))} = -\frac{(x+yp^{b-a})(\frac{1}{x}+p^aZ(x))(\frac{1}{yp^b}+Z(y))}{1+p^a(x+yp^{b-a})Z(x+p^{b-a}y)} \tag{5.60}$$

The denominator is a unit, we need to know it modulo p^b, the numerator is a sum of eight terms we can forget all the terms in $\mathbb{Z}_{(p)}$. Then the above expression simplifies

$$\frac{\frac{1}{yp^b}+\frac{1}{xp^a}+\frac{p^{a-b}xZ(x)}{y}}{1+p^axZ(x+yp^{b-a})} \tag{5.61}$$

We want this to be equal to $\frac{1}{yp^b}+\frac{1}{xp^a}$. Hence we have to verify the equality

$$\frac{1}{yp^b}+\frac{1}{xp^a}+\frac{p^{a-b}xZ(x)}{y} = (\frac{1}{yp^b}+\frac{1}{xp^a})(1+p^axZ(x+yp^{b-a})) \tag{5.62}$$

and this comes down to

$$p^{a-b}\frac{xZ(x)}{y} \equiv p^{a-b}\frac{xZ(x+yp^{b-a})}{y} \mod \mathbb{Z}_{(p)} \tag{5.63}$$

and this means

$$Z(x) \equiv Z(x + yp^{b-a}) \mod p^{b-a}$$

and this congruence is easy to verify.

Basically the same argument works if $a = b$. Then it can happen that $x+y \equiv 0 \mod p$. Then we have to write $x + y = p^c z$. Then (5.60) changes into

$$\frac{(-\frac{1}{xp^a} + Z(x))(-\frac{1}{yp^a} + Z(y))}{-\frac{1}{zp^{a+c}} + Z(z))} = -\frac{zp^c(\frac{1}{x} + p^a Z(x))(\frac{1}{yp^a} + Z(y))}{1 + p^{a+c} z Z(z)}. \tag{5.64}$$

We ignore the denominator then the only non integral term is

$$(x + y)\frac{1}{x}\frac{1}{yp^a} = \frac{1}{xp^a} + \frac{1}{yp^a}$$

We see that in case $p - 1 \mid n + 2$ the prime p does not divide the numerator of $\zeta(-1 - n)$ and that the prime p does not divide the denominator $\Delta(n)$.

If $p - 1 \not| \, n + 2$ then p must be an irregular prime. We look at the maximal value of $d(n, p, \nu)$ in (5.54), this means we look for the minimum value of $\mathrm{ord}_p((\zeta(-\nu)\zeta(\nu - n))$ for $\nu = 1, 3, \ldots \frac{n}{2}$. We claim that this minimum value is actually equal to zero. Now it is extremely likely that this is true, because simply too many random integers have to be divisible by p. But as always it is not easy to prove.

For our given prime p the index of irregularity of p is the number of even numbers k with $2 \leq k \leq p - 3$ such that $p|\zeta(1 - k) = \frac{B_k}{k}$, it is denoted by $i(p)$. Probabilistic considerations suggest that $i(p) = O(\log(p)/\log\log(p))$, but this can not be proved at the present time. (Again a Wieferich dilemma). Therefore it seems to be very plausible that always $i(p) < \frac{n}{4}$. Then not all of the $\frac{n}{4}$ numbers $\zeta(-\nu)\zeta(\nu - n)$ can be divisible by p. The above assertion that $i(p) < \frac{n}{4}$ is certainly true for all primes $p \leq 163577833$. (See [15]). In the same paper the authors assert that for the above set of primes the largest the index of irregularity $i(p) \leq 7$ and $i(32012327) = 7$.

There is a way out of this dilemma. In his paper [16] L. Carlitz proves a very crude estimate for the index of irregularity. This estimate says that

$$i(p) < \frac{p+3}{4} - \frac{\log(2)}{\log(p)}\frac{p-1}{4} \tag{5.65}$$

and this implies that $i(p) < \frac{p-3}{4} - 2$ provided $p > 100$.

If we now assume assume $n > p$ then we see that not all the $\frac{p-3}{2}$ numbers $\zeta(-\nu)\zeta(\nu - n)$ can be divisible by p and hence we proved $d(n, p) = \mathrm{ord}_p(\zeta(-1 - n)$ and hence the theorem below under this assumption.

denomEis

Theorem 5.1.2. *If* $\Gamma = Sl_2(\mathbb{Z})$ *then the denominator of the Eisenstein class in* $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}^\flat_{\underline{\lambda}})$ *is the numerator of* $\zeta(-1 - n)$.

Proof. We have to remove the assumption $p < n$. We use Hida's method of p-adic interpolation, we refer to the approach in [42]. In section 3.3.12 we explain how the fact $p^\delta || \Delta(n)$ is reflected in the structure of the Hecke-module $H^1_{\text{ord}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}^\flat_{\underline{\lambda}} \otimes \mathbb{Z}/p^\delta\mathbb{Z})$. In [42] we prove that we have an isomorphism of Hecke modules

$$H^1_{\text{ord}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}^\flat_{\underline{\lambda}} \otimes \mathbb{Z}/p^\delta\mathbb{Z}) \xrightarrow{\sim} H^1_{\text{ord}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}^\flat_{\lambda'} \otimes \mathbb{Z}/p^\delta\mathbb{Z})$$

provided we have $\lambda \equiv \lambda' \mod p^\delta$ i.e. $n \equiv n' \mod p^\delta$. Hence we can replace n by an $n' > p$ and apply the previous argument. □

I slightly weaker version of this theorem has been proved by Haberland in [32]. Somewhat later C. Kaiser proved a more general version in his Diploma thesis and in about the same time the theorem was proved in my class.

This theorem is a paradigm for a much larger assemble of statements, which are still mostly conjectural. Roughly my general expectation is that there is a connection between the prime factorisation of certain special values of L-functions and denominators of Eisenstein classes. Examples for these conjectural statements will be discussed in Chapter 9.

Of course we can generalise the above theorem if we pass to congruence subgroups of $\mathrm{Gl}_2(\mathbb{Z})$, then the special values of the $\zeta-$ function have to be replaced by special values of Dirichlet L functions.

Another generalisation where the above method might lead to some success is the case of Hilbert modular varieties, i.e. $G/\mathbb{Q} = R_{F/\mathbb{Q}}(\mathrm{Gl}_2/F)$ and $F/\mathbb{Q}$ a totally real field.

5.1.5 The Deligne-Eichler-Shimura theorem

In this section the material is not presented in a satisfactory form. One reason is that it this point we should start using the language of adeles, but there are also other drawbacks. So in a final version of these notes this section will probably be removed.

Begin of probably removed section

In this section I try to explain very briefly some results which are specific for Gl_2 and a few other low dimensional algebraic groups. These results concern representations of the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ which can be attached to irreducible constituents Π_f in the cohomology. These results are very deep and reaching a better understanding and more general versions of these results is a fundamental task of the subject treated in these notes. The first cases have been tackled by Eichler and Shimura, then Ihara made some contributions and finally Deligne proved a general result for $\mathrm{Gl}_2/\mathbb{Q}$.

We start from the group $G = \mathrm{Gl}_2/\mathbb{Q}$, this is now only a reductive group and its centre is isomorphic to $\mathbb{G}_m/\mathbb{Q}$. Its group of real points is $\mathrm{Gl}_2(\mathbb{R})$ and the centre $\mathbb{G}_m(\mathbb{R})$ considered as a topological group has two components, the connected component of the identity is $\mathbb{G}_m(\mathbb{R})^{(0)} = \mathbb{R}^\times_{>0}$. Now we enlarge the maximal compact connected subgroup $\mathrm{SO}(2) \subset \mathrm{Gl}_2(\mathbb{R})$ to the group $K_\infty = SO(2) \cdot \mathbb{G}_m(\mathbb{R})^{(0)}$. The resulting symmetric space $X = \mathrm{Gl}_2(\mathbb{R})/K_\infty$ is now a union of a upper and a lower half plane: We write $X = \mathbb{H}_+ \cup H_-$.

We choose a positive integer $N > 2$ and consider the congruence subgroup $\Gamma(N) \subset \mathrm{Gl}_2(\mathbb{Q}))$. We modify our symmetric space: This modification may look a little bit artificial at this point, it will be justified in the next chapter and is in fact very natural (see section **??** .At this point I want to avoid to use the language of adeles.)

We replace the symmetric space by

$$X = (\mathbb{H}_+ \cup \mathbb{H}_-) \times \mathrm{Gl}_2(\mathbb{Z}/N\mathbb{Z}).$$

On this space we have an action of $\Gamma = \mathrm{Gl}_2(\mathbb{Z})$, on the second factor it acts via the homomorphism $\mathrm{Gl}_2(\mathbb{Z}) \to \mathrm{Gl}_2(\mathbb{Z}/N\mathbb{Z})$ by translations from the left. Again we look at the quotient of this space by the action of $\mathrm{Gl}_2(\mathbb{Z})$. This quotient space will have several connected components. The group $\mathrm{Gl}_2(\mathbb{Z})$ contains the group $\mathrm{Sl}_2(\mathbb{Z})$ as a subgroup of index two, because the determinant of an element is ± 1. The element $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ interchanges the upper and the lower half plane and hence we see

$$\mathrm{Gl}_2(\mathbb{Z})\backslash X = \mathrm{Gl}_2(\mathbb{Z})\backslash((\mathbb{H}_+ \cup \mathbb{H}_-) \times \mathrm{Gl}_2(\mathbb{Z}/N\mathbb{Z})) = \mathrm{Sl}_2(\mathbb{Z})\backslash(\mathbb{H}_+ \times \mathrm{Gl}_2(\mathbb{Z}/N\mathbb{Z})),$$

the connected components of $(\mathbb{H}_+ \times \mathrm{Gl}_2(\mathbb{Z}/N\mathbb{Z}))$ are indexed by elements $g \in \mathrm{Gl}_2(\mathbb{Z}/N\mathbb{Z})$. The stabilizer of such a component is the full congruence subgroup

$$\Gamma(N) = \{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a, d \equiv 1 \mod N, b, c \equiv 0 \mod N\}$$

this group is torsion free because we assumed $N > 2$.

The image of the natural homomorphism $\mathrm{Sl}_2(\mathbb{Z}) \to \mathrm{Gl}_2(\mathbb{Z}/N\mathbb{Z})$ is the subgroup $\mathrm{Sl}_2(\mathbb{Z}/N\mathbb{Z})$ (strong approximation), therefore the quotient is by this subgroup is $(\mathbb{Z}/N\mathbb{Z})^\times$.

We choose as system of representatives for the determinant the matrices $t_a = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, a \in (\mathbb{Z}/N\mathbb{Z})^\times$. The stabiliser of then we get an isomorphism

$$\mathcal{S}_N = \mathrm{Gl}_2(\mathbb{Z})\backslash(\mathbb{H} \times \mathrm{Gl}_2(\mathbb{Z}/N\mathbb{Z})) \xrightarrow{\sim} (\Gamma(N)\backslash\mathbb{H}) \times (\mathbb{Z}/N\mathbb{Z})^\times.$$

We consider the cohomology groups $H_c^\bullet(\mathcal{S}_N, \tilde{\mathcal{M}}_n), H^\bullet(\mathcal{S}_N, \tilde{\mathcal{M}}_n), H^\bullet(\partial\mathcal{S}_N, \tilde{\mathcal{M}}_n)$, again we have the fundamental long exact sequence and we define $H_!^\bullet(\mathcal{S}_N, \tilde{\mathcal{M}}_n)$ as before.

To any prime p, which does not divide N we can again attach Hecke operators. Again we can attach Hecke operators

$$T_{p^r} = T\left(\begin{pmatrix} p^r & 0 \\ 0 & 1 \end{pmatrix}, u_{\begin{pmatrix} p^r & 0 \\ 0 & 1 \end{pmatrix}}\right)$$

these to the double cosets and using strong approximation we can prove the recursion formulae (for this and the following see the next chapter 6). We define $\mathcal{H}_p := \mathbb{Z}[T_p]$. We also have a Hecke algebra $\mathcal{H}_p$ for the primes $p|N$, but this will not be commutative.anymore. We get an action of a larger Hecke algebra

$$\mathcal{H}_N^{\mathrm{large}} = \bigotimes_p{}' \mathcal{H}_p.$$

We apply 3.1.1 and find a finite normal extension $F/\mathbb{Q}$ such that we get an isotypical decomposition

$$H^\bullet_!(\mathcal{S}_N, \tilde{\mathcal{M}}_n \otimes F) = \bigoplus H^\bullet_!(\mathcal{S}_N, \tilde{\mathcal{M}}_n \otimes F)(\pi_f) \tag{5.66}$$

where $\pi_f = \otimes' \pi_p$ and the π_p are isomorphism types of absolutely irreducible $\mathcal{H}_p$ modules. For $p \not| N$ this $\mathcal{H}_p$-module is a one dimensional F-vector space $H_{\pi_p} = F$ and π_p is simply a homomorphism $\pi_p : \mathcal{H}_p \to \mathcal{O}_F$. If $p|N$ then the $\mathcal{H}_p$ module is $F^{d(\pi_p)}$ with $d(\pi_p) \geq 1$ and the theory of semi-simple algebras tells us that the map $\mathcal{H}_p \to \mathrm{End}_F(H_{\pi_p})$ is surjective. Hence we know the isomorphism type π_p once we know the two sided ideal $I(\pi_p)$ of this map.

Now we have some input from the theory of automorphic forms

Theorem 5.1.3. *The isomorphism type π_f is determined by its restriction to the central subalgebra $\otimes_{p \not| N} \mathcal{H}_p$. Under the action of the group $\pi_0(\mathrm{Gl}_2(\mathbb{R})) = \{\pm 1\}$ decomposes into two eigenspaces*

$$H^\bullet_!(\mathcal{S}_N, \tilde{\mathcal{M}}_n \otimes F)(\pi_f) = H^\bullet_!(\mathcal{S}_N, \tilde{\mathcal{M}}_n \otimes F)_+(\pi_f) \oplus H^\bullet_!(\mathcal{S}_N, \tilde{\mathcal{M}}_n \otimes F)_-(\pi_f) \tag{5.67}$$

and these two eigenspaces are absolutely irreducible of type π_f. (These assertions are summarised under "strong multiplicity one")

Of course we have the action of the Galoisgroup $\mathrm{Gal}(F/\mathbb{Q})$ on the cohomology groups $H^\bullet_!(\mathcal{S}_N, \tilde{\mathcal{M}}_n \otimes F)$ and it is clear that this induces an action on the isomorphism types π_f. For $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$ we have

$$\sigma(H^\bullet_!(\mathcal{S}_N, \tilde{\mathcal{M}}_n \otimes F)(\pi_f)) = H^\bullet_!(\mathcal{S}_N, \tilde{\mathcal{M}}_n \otimes F)(\sigma(\pi_f)). \tag{5.68}$$

I want to discuss some applications.

A) To any isotypical component π_f we can attach an (so called automorphic) L function

$$L(\pi_f, s) = \prod_p L(\pi_p, s)$$

where for $p \not| N$ we define

$$L(\pi_p, s) = \frac{1}{1 - \lambda(\pi_p)p^{-s} + p^{n+1}\omega(\pi_f)(p)p^{-2s}}$$

and for $p|N$ we have

$$L(\pi_p, s) = \begin{cases} \frac{1}{1-p^{n+1}\omega(\Pi_f)(p)p^{-s}} & \text{if } \pi_p \text{ is a Steinberg module} \\ 1 & \text{else} \end{cases}$$

This L-function, which is defined as an infinite product is holomorphic for $\Re(s) >> 0$ it can written as the Mellin transform of a holomorphic cusp form F of weight $n+2$ and this implies that

$$\Lambda(\pi, s) = \frac{\Gamma(s)}{2\pi^s} L(\pi_f, s)$$

has a holomorphic continuation into the entire complex plane and satisfies a funtional equation

$$\Lambda(\pi_f, s) = W(\pi_f)(N(\pi_f))^{s-1-n/2}\Lambda(\pi_f, n+2-s)$$

Here $W(\Pi_f)$ is the so called root number, it can be computed from the π_p where $p|N$, its value is ± 1, the number $N(\pi_f)$ is the conductor of π_f it is a positive integer, whose prime factors are contained in the set of prime divisors of N.

Now we exploit the fact, that the disjoint union of Riemann surfaces $\Gamma(N)\backslash X$ is in fact the space of complex points of the moduli scheme $M_N \to \mathrm{Spec}(\mathbb{Z}[1/N])$. This has been explained at several places in the literature. I refer to the second edition of my book [40] section 5.2.5 where I try to explain that the functor schemes $S \to \mathrm{Spec}(\mathbb{Z}[1/N])$ to elliptic curves over S with N-level structure is representable, provided $N \geq 3$, More precisely we have a smooth quasiprojective scheme $M_N \to \mathrm{Spec}(\mathbb{Z}[1/N])$ with one dimensional fibers and we have the universal elliptic curve with N level structure

$$\begin{array}{ll} \mathcal{E}; & \{e_1, e_2\} \\ \downarrow \pi & \\ M_N & \end{array} \tag{5.69}$$

where $e_i : M_N \to \mathcal{E}$ are sections which yield a pair of generators of the group of N-division points. The group $\mathrm{Gl}_2(\mathbb{Z}/NZ)$ acts on the group of N-division points, this gives an action of $\mathrm{Gl}_2(\mathbb{Z}/NZ)$ on M_N. We can define the moduli stack $M_1 \to \mathrm{Spec}(\mathbb{Z})$ of elliptic curves without level structure. For any $N \geq 3$ we have $M_1 \times \mathrm{Spec}(Z[\frac{1}{N}]) = M_N/\mathrm{Gl}_2(\mathbb{Z}/\mathbb{N}\mathbb{Z})$.

On $\mathcal{E}$ we have the constant ℓ-adic sheaf $\mathbb{Z}_\ell$. For $i = 0, 1, 2$ we can consider the ℓ- adic sheaves $R^i\pi_*(\mathbb{Z}_\ell)$ on M_N. We have the spectral sequence

$$H^p(M_N \times \bar{\mathbb{Q}}, R^q\pi_*(\mathbb{Z}_\ell)) \Rightarrow H^n(\mathcal{E} \times \bar{\mathbb{Q}}, \mathbb{Z}_\ell).$$

We can take the fibered product of the universal elliptic curve

$$\mathcal{E}^{(n)} = \mathcal{E} \times_{M_N} \mathcal{E} \times \cdots \times_{M_N} \mathcal{E} \xrightarrow{\pi_N} M_N$$

where n is the number of factors. This gives us a more general spectral sequence

$$H^p(M_N \times \bar{\mathbb{Q}}, R^q\pi_{N,*}(\mathbb{Z}_\ell)) \Rightarrow H^n(\mathcal{E}^{(n)} \times \bar{\mathbb{Q}}, \mathbb{Z}_\ell).$$

The stalk $R^q\pi_{N,*}(\mathbb{Z}_\ell)_y$) of the sheaf $R^q\pi_{N,*}(\mathbb{Z}_\ell)$ in a geometric point y of M_N is the q-th cohomology $H^q(\mathcal{E}_y^{(n)}, \mathbb{Z}_\ell)$ and this can be computed using the Kuenneth formula

$$H^q(\mathcal{E}_y^{(n)}, \mathbb{Z}_\ell) \xrightarrow{\sim} \bigoplus_{a_1, a_2 \dots, a_n} H^{a_1}(\mathcal{E}_y, \mathbb{Z}_\ell) \otimes H^{a_2}(\mathcal{E}_y, \mathbb{Z}_\ell) \cdots \otimes H^{a_n}(\mathcal{E}_y, \mathbb{Z}_\ell),$$

where the $a_i = 0, 1, 2$ and sum up to q. We have $H^0(\mathcal{E}_y, \mathbb{Z}_\ell) = \mathbb{Z}_\ell(0), H^2(\mathcal{E}_y, \mathbb{Z}_\ell) = \mathbb{Z}_\ell(-1)$ and the most interesting factor is $H^1(\mathcal{E}_y, \mathbb{Z}_\ell)$ which is a free $\mathbb{Z}_\ell$ module af rank 2.

This tells us that the sheaf decomposes into a direct sum according to the type of Kuenneth summands. We also have an action of the symmetric group S_n which is obtained from the permutations of the factors in $\mathcal{E}^{(n)}$ which also permutes the Kuenneth summands. We are mainly interested in the case $q = n$ and then we have the special summand where $a_1 = a_2 \cdots = a_n = 1$. This summand is invariant under S_n and contains a summand on which S_n acts by the signature character $\sigma : S_n \to \{\pm 1\}$. This defines a unique subsheaf $R^n\pi_{*,n}(\mathbb{Z}_\ell)(\sigma) \subset R^n\pi_{*,n}(\mathbb{Z}_\ell)$ and hence we get an inclusion

$$H^1(M_N \times \bar{\mathbb{Q}}, R^n\pi_{*,n}(\mathbb{Z}_\ell)(\sigma) \hookrightarrow H^{n+1}(\mathcal{E}^{(n)} \times \bar{\mathbb{Q}}, \mathbb{Z}_\ell) \tag{5.70}$$

and we can do the same thing for the cohomology with compact supports.

Now I claim that

A) The restriction of the etale sheaf $R^n\pi_{*,n}(\mathbb{Z}_\ell)(\sigma)$ on $M_N \times \mathbb{C}$ to the topological space $\mathcal{S}_N = M_N(\mathbb{C})$ is isomorphic to $\mathcal{M}_n \otimes \mathbb{Z}_\ell$. Then the comparison theorem gives us

$$H^1(M_N(\mathbb{C}), \mathcal{M}_n \otimes \mathbb{Z}_\ell) \xrightarrow{\sim} H^1(M_N \times \bar{\mathbb{Q}}, R^n\pi_{*,n}(\mathbb{Z}_\ell)(\sigma))$$

B) The Hecke operators T_p for $p \not| N$ are coming from algebraic correspondences $T_p \subset M_N \times M_N$ and induce endomorphisms $T_p : H^1(M_N \otimes \bar{\mathbb{Q}}, R^n\pi_{*,n}(\mathbb{Z}_\ell)(\sigma)) \to H^1(M_N \otimes \bar{\mathbb{Q}}, R^n\pi_{*,n}(\mathbb{Z}_\ell)(\sigma))$ which commute with the action of $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ on the cohomology.

This gives us the structure of a $Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \times \mathcal{H}_\Gamma$ on $H^1(M_N(\mathbb{C}), \mathcal{M}_n \otimes \mathbb{Z}_\ell)$.

C) The operation of the Galois group on $H^1(M_N(\mathbb{C}), \mathcal{M}_n \otimes \mathbb{Z}_\ell)$ is unramified outside N and ℓ therefore we have the conjugacy class Φ_p^{-1} for all $p \not| N$ as endomorphism of $H^1(M_N(\mathbb{C}), \mathcal{M}_n \otimes \mathbb{Q}_\ell)$.

We choose our normal extension $F/\mathbb{Q}$ and a prime $\mathfrak{l}$ above ℓ. Then an isotypical component $H^1(M_N(\mathbb{C}), \mathcal{M}_n \otimes F_\mathfrak{l})(\pi_f)$ is a Galois module. Let H_{π_f} be a vector space over F which is an irreducible $\mathcal{H}_\Gamma$ module which is of isomorphism type π_f. Then $W(\pi_f) = \mathrm{Hom}_{\mathcal{H}_\gamma}(H_{\pi_f} \otimes E_\mathfrak{l}, H^1_!(M_N(\mathbb{C}), \mathcal{M}_n \otimes F_\mathfrak{l}9$ is a Galois module which is unramified outside N and ℓ

We now apply our theorem 2 to the cohomology $H^1_!(M_N(\mathbb{C}), \mathcal{M}_n \otimes Z_\ell)$, as a module under this large Hecke algebra. Then the isotypical summands will be invariant under the Galois group.

Theorem 5.1.4. *(Deligne) For all primes $p \not| N, p \neq \ell$*

$$tr(\Phi_p^{-1}|W(\pi_f)) = \lambda(\pi_p), \det(\Phi_p^{-1}|W(\pi_f)) = p^{n+1}\omega(\pi_f)(p)$$

This theorem is much deeper than the previous ones. The assertion a) follows from the theory of automorphic forms on Gl_2 and b) requires some tools from algebraic geometry. We have to consider the reduction $M_N \times \mathrm{Spec}(\mathbb{F}_p)$ and to look at the reduction of the Hecke operator T_p modulo p. I will resume this discussion in Chap. V.

We conclude by giving a few applications.

A) is a function on the upper half plane $\mathbb{H} = \{z | \Im(z) > 0\}$ and it satisfies

$$\Delta(\frac{az+b}{cz+d}) = (cz+d)^{12}\Delta(z)$$

and this means that it is a modular form of weight 12. Since it goes to zero if $z = iy \to \infty$ it is even a modular cusp form.

For such a modular cusp form we can define the Hecke L-function

$$L(\Delta, s) = \int_0^\infty \Delta(iy) y^s \frac{dy}{y} = \frac{\Gamma(s)}{(2\pi)^s} \sum_{n=1}^\infty \frac{\tau(n)}{n^s} = \frac{\Gamma(s)}{(2\pi)^s} \prod_p \frac{1}{1 - \tau(p)p^{-s} + p^{11-2s}}$$

the product expansion has been discovered by Ramanujan and has been proved by Mordell and Hecke.

Now it is in any textbook on modular forms that the transformation rule

$$\Delta(-\frac{1}{z}) = z^{12}\Delta(z)$$

implies that $L(\Delta, s)$ defines a holomorphic function in the entire s plane and satisfies the functional equation

$$L(\Delta, s) = (-1)^{12/2} L(\Delta, 12 - s) = L(\Delta, 12 - s).$$

This function $L(\Delta, s)$ is the prototype of an automorphic L-function. The above theorem shows that it is equal to a "motivic" L-function. We gave some vague explanations of what this possibly means: We can interpret the projective system $(\mathcal{M}/\ell^n\tilde{\mathcal{M}})_{et}$ as the $\ell-$adic realization of a motive:

$$\mathcal{M} = \mathrm{Sym}^{10}(R^1(\pi : \mathcal{E} \to S))$$

(All this is a translation of Deligne‘s reasoning into a more sophisticated language.)

It is a general hope that "motivic" L-functions $L(M, s)$ have nice properties as functions in the variable s (meromorphicity, control of the poles, functional equation). So far the only cases, in which one could prove such nice properties are cases where one could identify the "motivic" L-function to an automorphic L function. The greatest success of this strategy is Wiles‘ proof of the Shimura-Taniyama-Weil conjecture, but also the Riemann ζ-function is a motivic $L-$function and Riemann‘s proof of the functional equation follows exactly this strategy.

B) But we also have a flow of information in the opposite direction. In 1973 Deligne proved the Weil conjectures, which in this case say that the two roots of the quadratic equation

$$x^2 - \tau(p)x + p^{11} = 0$$

have absolute value $p^{11/2}$, i.e. they have the same absolute value. This implies the famous Ramanujan- conjecture

$$\tau(p) \leq 2p^{11/2}$$

and for more than 50 years this has been a brain-teaser for mathematicians working in the field of modular forms.

End of probably removed section

The ℓ-adic Galois representation in the first non trivial case

Again we consider the module $\mathcal{M} = \mathcal{M}_{10}[-10]$. We choose a prime ℓ and for some reason let us assume $\ell > 7$. Then we can consider the cohomology groups

$$H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}/\ell^n\tilde{\mathcal{M}})$$

and the projective limit

$$H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}\otimes\mathbb{Z}_\ell) = \lim_{\leftarrow} H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}/\ell^n\tilde{\mathcal{M}}).$$

We can define etale torsion sheaves $(\mathcal{M}/\ell^n\tilde{\mathcal{M}})_{et}$ on the stack M_1 and we know that

$$H^1_{et}(M_1 \times_{\operatorname{Spec}(\mathbb{Z})} \bar{\mathbb{Q}}, (\mathcal{M}/\ell^n\tilde{\mathcal{M}})_{et}) \xrightarrow{\sim} H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}/\ell^n\tilde{\mathcal{M}}).$$

On the etale cohomology groups we have an action of the Galois group hence we get an action

$$\rho_\ell^{(n)} : \operatorname{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \operatorname{Gl}(H^1(\Gamma\backslash\mathbb{H}, (\mathcal{M}/\ell^n\tilde{\mathcal{M}})_{et})). \tag{5.71}$$

From Galois theory we get a finite normal extension $K_\ell^{(n)}/\mathbb{Q}$ which is defined by $\operatorname{Gal}(\bar{\mathbb{Q}}/K_\ell^{(n)}) = \ker(\rho^{(n)})$. The representation $\rho_\ell^{(n)}$ is unramified outside ℓ, and this means that the finite extension $K_\ell^{(n)}/\mathbb{Q}$ is unramified outside ℓ.

From the fundamental exact sequence we get a diagram

$$\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \downarrow & & & & & & \\
 & & H^0(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}\otimes\mathbb{Z}_\ell) & & & & & & \\
 & & \downarrow & & & & & & \\
0 & \to & H^1_c(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}\otimes\mathbb{Z}_\ell) & & & & & & \\
 & & \downarrow & & & & & & \\
0 & \to & H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}\otimes\mathbb{Z}_\ell) & \to & H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}\otimes\mathbb{Z}_\ell) & \to & H^1(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}\otimes\mathbb{Z}_\ell) & \to & 0 \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array} \tag{5.72}$$

the vertical and the horizontal sequence are exact sequences of Hecke$\times$ Galois modules. Here we may replace $\mathbb{Z}_\ell$ by $\mathbb{Z}/\ell^n\mathbb{Z}$. We computed these Hecke modules in section 3.3.4, the cohomology $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}\otimes\mathbb{Z}_\ell)$ is free of rank 3 and $H^1(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}\otimes\mathbb{Z}_\ell)$ is free of rank one. We get the two Galois modules

$$\rho_! : \operatorname{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \operatorname{Gl}(H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}\otimes\mathbb{Z}_\ell)), \text{ and } \rho_\partial : \operatorname{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \operatorname{Gl}(\mathbb{Z}_\ell^\times). \tag{5.73}$$

The ℓ-adic Tate character $\alpha_\ell:\ \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_\ell^\times$ is defined by the rule: For all $\sigma \in\ \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and all ℓ^n-th roots of unity $\zeta \in \bar{\mathbb{Q}}$ we have $\sigma(\zeta) = \zeta^{\alpha_\ell(\sigma)}$. Then it is not difficult to see (or well known) that $\rho_\partial = \alpha_\ell^{11}$. The representation ρ_∂ is the $\ell-$ adic realisation of the Tate-motive $\mathbb{Z}(-11)$. (For a slightly more precise explanation I refer to MixMot.pdf on my home-page). On $\mathbb{Z}_\ell(-1) = H^2(\mathbb{P}^1 \times \bar{\mathbb{Q}}, \mathbb{Z}_\ell)$ the Galois group acts by the Tate-character α_ℓ

For the representation $\rho_!$ the above theorem of Deligne gives

$$\det(\mathrm{Id} - \rho(\Phi_p^{-1})t | H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{Z}_\ell)) = 1 - \tau(p)t + p^{11}t^2 \tag{5.74}$$

We also have $\det(\rho(\sigma)) = \alpha_\ell^{11}(\sigma)$ and we can ask what is the image of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ in $\mathrm{Gl}(H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{Z}_\ell) = \mathrm{Gl}_2(\mathbb{Z}_\ell)$. This question is discussed in [105]. If $\ell \neq 691$ then the Hecke algebra induces a splitting (Manin-Drinfeld principle)

$$H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{Z}_\ell) = H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{Z}_\ell)) \oplus \mathbb{Z}_\ell \tag{5.75}$$

where T_p acts by multiplication by $p^{11}+1$ on the second summand.

Now Swinnerton-Dyer shows in [105] that for $\ell \neq 23, 691$ the image of the Galois group under $\rho_!$ is as large possible, it is the inverse image of $(\mathbb{F}_\ell^\times)^{11}$

From now on we choose $\ell = 691$ and our coefficient system $\tilde{\mathcal{M}}_{10}$. Then we get a diagram of Hecke modules

$$\begin{array}{ccccccccc} & & 0 & & & & & & \\ & & \downarrow & & & & & & \\ & & H^0(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}} \otimes \mathbb{Z}/\ell\mathbb{Z}) & & & & & & \\ & & \downarrow & & & & & & \\ 0 & \to & H^1_c(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{Z}/\ell\mathbb{Z}) & & & & & & \\ & & \downarrow & & & & & & \\ 0 & \to & H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{Z}/\ell\mathbb{Z}) & \to & H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{Z}/\ell\mathbb{Z}) & \to & H^1(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}} \otimes \mathbb{Z}/\ell\mathbb{Z}) & \to & 0 \\ & & \downarrow & & & & & & \\ & & 0 & & & & & & \end{array} \tag{5.76}$$

We learned in the probably removed section that we have an action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on this diagram and this action of the Galois group commutes with the action of the Hecke algebra. The two modules $H^0(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}} \otimes \mathbb{Z}/\ell\mathbb{Z}), H^1(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}} \otimes \mathbb{Z}/\ell\mathbb{Z})$ are isomorphic to $\mathbb{Z}/\ell\mathbb{Z}$ and a Hecke operator T_p acts by the eigenvalue $p^{11}+1 \mod \ell$. The module $H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{Z}/\ell\mathbb{Z}) = \mathbb{Z}/\ell\mathbb{Z} \oplus \mathbb{Z}/\ell\mathbb{Z}$ and the Hecke operator acts by the eigenvalue $\tau(p)$.

The Galois group acts on $H^0(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}} \otimes \mathbb{Z}/\ell\mathbb{Z})$, resp.$H^1(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}} \otimes \mathbb{Z}/\ell\mathbb{Z})$ by α_ℓ^0 resp. α_ℓ^{-11}, here α_ℓ is the reduction of the Tate character $\mod \ell$. We also know that we have the inclusion of Galois modules

$$j : \mathbb{Z}/\ell\mathbb{Z}(-11) \hookrightarrow H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{Z}/\ell\mathbb{Z}), \tag{5.77}$$

We want to understand the two Galois modules $H^1_c(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{Z}/\ell\mathbb{Z})$ and $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{Z}/\ell\mathbb{Z})$, There is perfect pairing with values in $\mathbb{Z}/\ell\mathbb{Z}(-11)$ between them, hence we have to study only one of them say $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{Z}/\ell\mathbb{Z})$, From the above considerations it follows that we have ia basis e_1, e_0, e_{-1} of this module such that a $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts by the matrix

$$\rho(\sigma) = \begin{pmatrix} \alpha_\ell(\sigma)^{-11} & u_{12}(\sigma) & u_{13}(\sigma) \\ 0 & 1 & u_{23}(\sigma) \\ 0 & 0 & \alpha_\ell(\sigma)^{-11} \end{pmatrix} \in B(\mathbb{Z}/\ell\mathbb{Z}) \tag{5.78}$$

We want to describe the image of the Galois group in $B(\mathbb{Z}/\ell\mathbb{Z})$. Let $T^{(1)}(\mathbb{Z}/\ell\mathbb{Z})$ be the torus

$$\begin{pmatrix} t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t \end{pmatrix}; t \in \mathbb{Z}/\ell\mathbb{Z}^\times \tag{5.79}$$

and let $U(\mathbb{Z}/\ell\mathbb{Z})$ be the unipotent radical in $B(\mathbb{Z}/\ell\mathbb{Z})$. Then I claim

Theorem 5.1.5. *The image of the Galois group is $T^{(1)}(\mathbb{Z}/\ell\mathbb{Z}) \ltimes U(\mathbb{Z}/\ell\mathbb{Z})$*

Here are arguments why this must be the case.

The quotient $B(\mathbb{Z}/\ell\mathbb{Z})/U(\mathbb{Z}/\ell\mathbb{Z}) = T^{(1)}(\mathbb{Z}/\ell\mathbb{Z})$ and the resulting map $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to T^{(1)}(\mathbb{Z}/\ell\mathbb{Z})$ is surjective. We have to show that the restriction map $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\zeta_\ell)) \to U(\mathbb{Z}/\ell\mathbb{Z})$ is surjective. Then it becomes clear that it suffices to show that for any pair $<$ of indices we have to find a $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ such that $\alpha(\sigma)^{11} \equiv 1 \mod 691$ and $u_{i,j}(\sigma) \neq 0$. Now we apply the congruence relation which says that for any p we have

$$\rho(\Phi_p)^2 - T_p\rho(\Phi_p) + p^{11} Id = 0. \tag{5.80}$$

and if we are courageous enough to compute with $3 \times 3 >$ matrices for $\sigma = \phi_p$ we find

$$\begin{pmatrix} 0 & 0 & u_{12}(\Phi_p)u_{23}(\Phi_p) + (-1+p^{11})u_{13}(\Phi_p) - p^{11}t^{(p)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0. \tag{5.81}$$

Hence the right upper entry must be zero. If $p \equiv 1 \mod 691$ this means that $u_{12}(\Phi_p)u_{23}(\Phi_p) - t^{(p)} = 0$ and this implies: If $t^{(p)} \neq 0$ then $u_{12}(\Phi_p)$ and $u_{23}(\Phi_p) \neq 0$. But we have seen that $t^{(p)} = 0$ implies the stronger congruence $\tau(p) \equiv p^{11} + 1 \mod 691^2$. (See section 3.3.4) But now the prime $p_1 = 6911$ is congruent to 1 mod 691 but $\tau(p_1)$ is not congruent to $691^{11} + 1$ modulo 691^2. Hence $u_{12}(\Phi_{p_1}) \neq 0, u_{23}(\Phi_{p_1}) \neq 0$. But then also $u_{13}(\Phi_{p_1})$ or $u_{13}(\Phi_{p_1}^2)$ is not zero. The claim follows.

By definition $K_\ell^{(1)}/\mathbb{Q}$ is the normal extension of $\mathbb{Q}$ such that

$$\mathrm{Gal}(K_\ell^{(1)}/\mathbb{Q}) = T^{(1)}(\mathbb{Z}/\ell\mathbb{Z}) \ltimes U(\mathbb{Z}/\ell\mathbb{Z}) := B^{(1)}(\mathbb{Z}/\ell\mathbb{Z}), \tag{5.82}$$

this extension is unramified outside ℓ. It contains the field of ℓ-th roots of unity, i.e. $\mathbb{Q}(\zeta_\ell) \subset K_\ell^{(1)}$. The Galois group $\mathrm{Gal}(K_\ell^{(1)}/\mathbb{Q}(\zeta_\ell)) = U(\mathbb{Z}/\ell\mathbb{Z})$. This group has a center $U_{13}(\mathbb{Z}/\ell\mathbb{Z}) = \mathbb{Z}/\ell\mathbb{Z}$, this is also the center of the larger group $\mathrm{Gal}(K_\ell^{(1)})/\mathbb{Q})$. We define the subfield $K_\ell^{(1,0)}$ by requiring that $\mathrm{Gal}(K_\ell^{(1,0)}/\mathbb{Q}) = \mathrm{Gal}(K_\ell^{(1)}/\mathbb{Q}(\zeta_\ell))/U_{13}(\mathbb{Z}/\ell\mathbb{Z})$. Then $K_\ell^{(1,0)}/\mathbb{Q})$ is the composite of two cyclic extensions $K_\ell^{(1,!)}/\mathbb{Q}(\zeta_\ell)$ and $K_\ell^{(1,\partial)}/\mathbb{Q}(\zeta_\ell)$. These two extensions have the faithful two dimensional representations

$$\begin{aligned} \rho_! :\ & \mathrm{Gal}(K_\ell^{(1,!)}/\mathbb{Q}) \to \mathrm{Gl}(H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}\otimes \mathbb{Z}/\ell\mathbb{Z})) \\ & \sigma \mapsto \rho_!(\sigma) = \begin{pmatrix} 1 & u_{23}(\sigma) \\ 0 & \alpha_\ell(\sigma)^{-11} \end{pmatrix} \\ \rho_\partial :\ & \mathrm{Gal}(K_\ell^{(1,\partial)}/\mathbb{Q}) \to \mathrm{Gl}(H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}\otimes \mathbb{Z}/\ell\mathbb{Z})/\mathbb{Z}/\ell\mathbb{Z}e_1) \\ & \sigma \mapsto \rho_\partial(\sigma) = \begin{pmatrix} \alpha_\ell(\sigma)^{-11} & u_{12}(\sigma) \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{5.83}$$

The extension $K_\ell^{(1,!)}/\mathbb{Q}(\zeta_\ell)$ is unramified, it is the extension which has been constructed by Ribet in [87].

This unramified extension extension is also discussed in [47]. At the end of that paper we raise the question for a decomposition law. This means that for any prime p we want to find a rule to determine the conjugacy class of $\rho(\Phi_p), \rho_!(\Phi_p)\dots$. This clear if $p \not\equiv 1 \mod \ell$. in this case the two conjugacy classes $\rho_!(\Phi_p), \rho_\partial(\Phi_p)$ are semi simple and determined by their eigenvalues. But if $p \equiv 1 \mod \ell$ then $\rho(\Phi_p)$ is unipotent and here are several possibilities for the conjugacy class.

Theorem 5.1.6. *If $p \equiv 1 \mod \ell$ and if the horizontal long exact sequence of Hecke modules (5.76) splits then p splits completely either in the field $K_\ell^{(1,!)}$ or in the field $K_\ell^{(1,\partial)}$.*

If $p \equiv 1 \mod \ell$ and if the horizontal long exact sequence of Hecke modules (5.76) does not split then both fields $K_\ell^{(1,!)}/\mathbb{Q}(\zeta_\ell)$ and the field $K_\ell^{(1,\partial)}/\mathbb{Q}(\zeta_\ell)$ are inert at the primes above p.

The density of primes which satisfy $p \equiv 1 \mod 691$ and $\tau(p) \equiv p^{11}+1 \mod 691^2$ is equal to $\frac{1}{238395}$

For the curios reader: The first such prime is $p = 3178601$. We leave it as an exercise for the reader to find out whether it splits completely in $K_\ell^{(1,!)}$ or in $K_\ell^{(1,\partial)}$. It is the 228759-th prime.

Finally we have a brief look at the action of the Galois group on $H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}\otimes \mathbb{Z}_\ell)$. Again we choose a basis e_1, e_0, e_{-1} the element e_1 maps to a generator in the boundary cohomology and e_0, e_{-1} form a basis of $H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}\otimes\mathbb{Z}_\ell)$ we assume that this basis reduces $\mod \ell$ to the basis which we denoted by the same letters. Then

$$\rho(\sigma) = \begin{pmatrix} \alpha_\ell(\sigma)^{-11} & u_{12}(\sigma) & u_{13}(\sigma) \\ 0 & a(\sigma) & b(\sigma) \\ 0 & \ell c(\sigma) & d(\sigma) \end{pmatrix} \in \mathrm{Gl}_3(\mathbb{Z}_\ell), \tag{5.84}$$

where $a(\sigma) \equiv 1 \mod \ell, d(\sigma) \equiv \alpha^{-11}(\sigma) \mod \ell$.

We claim that there is a σ with $c(\sigma) \not\equiv 0 \mod \ell$

For a prime p and the Frobenius Φ_p we get $a(\Phi_p)d(\Phi_p) - \ell b(\Phi_p)c(\Phi_p) = p^{11}$ and $\tau(p) = a(\Phi_p) + d(\Phi_p)$. Now an straightforward calculation shows that for a prime $p \equiv 1 \mod \ell$ which in addition satisfies $c(\Phi_p) \equiv 0 \mod \ell$ we must have $\tau(p) \equiv p^{11}+1 \mod \ell^2$. But $p = 1+10*691 = 6911$ does not satisfy this congruence, hence $c(\Phi_{6911}) \not\equiv 0 \mod \ell$.

The cohomology $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}\otimes\mathbb{Z}_\ell)$ has the submodule $H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}\otimes\mathbb{Z}_\ell)\oplus \mathbb{Z}_\ell e_1^\dagger$, where $e_1^\dagger = \ell f_{10}^\dagger(see(3.60))$ this submodule is determined by T_2. Therefore it is invariant under the action of the Galois group, with respect to the basis $e_1^\dagger, e_0, e_{-1}$ the Galois action is given by

$$\rho^\dagger(\sigma) = \begin{pmatrix} \alpha_\ell(\sigma)^{-11} & 0 & 0 \\ 0 & a(\sigma) & b(\sigma) \\ 0 & \ell c(\sigma) & d(\sigma) \end{pmatrix} \in \mathrm{Gl}_3(\mathbb{Z}_\ell), \tag{5.85}$$

where we still have $\alpha^{11}(\sigma) = \det(\begin{pmatrix} a(\sigma) & b(\sigma) \\ \ell c(\sigma) & d(\sigma)\end{pmatrix})$. It is clear from the above considerations that the image of the Galois group is given by those matrices in $\mathrm{Gl}_3(\mathbb{Z}_\ell)$ which satisfy the conditions above.

But we want to know the image of the Galois group with respect to our basis e_1, e_0, e_{-1}. For this we write $e_1 = \frac{x_0 e_{-1}+e_1^\dagger}{\ell}$ and then clearly

$$\rho(\sigma) = \begin{pmatrix} \alpha_\ell(\sigma)^{-11} & c(\sigma)x_0 & \frac{(d(\sigma)-\alpha_\ell(\sigma)^{-11})}{\ell}x_0 \\ 0 & a(\sigma) & b(\sigma) \\ 0 & \ell c(\sigma) & d(\sigma) \end{pmatrix} \in \mathrm{Gl}_3(\mathbb{Z}_\ell), \tag{5.86}$$

We put

$$a(x_0,\sigma) = \alpha_\ell^{11}(\sigma)(u_{12}(x_0,\sigma), u_{13}(x_0,\sigma))) \tag{5.87}$$

then $\sigma \mapsto a(x_0,\sigma)$ is a one-cocycle with values in $H^1_!(11) := H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}\otimes\mathbb{Z}_\ell)\otimes\mathbb{Z}_\ell(11)$. We compute its cohomology class $[\underline{v}] \in H^1(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}, H^1_!(11))$. We start from the exact sequence of Galois-modules

$$0 \to H^1_!(11) \to \frac{1}{\ell}H^1_!(11) \to \frac{1}{\ell}H^1_!(11)/H^1_!(11) \to 0 \tag{5.88}$$

where of course $\frac{1}{\ell}H^1_!(11)/H^1_!(11) = H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}\otimes\mathbb{Z}/\ell\mathbb{Z})$. This yields a long exact sequence in Galois cohomology. The element v provides a well defined element in $\tilde{v} \in H^0(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}, H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}\otimes\mathbb{Z}/\ell\mathbb{Z})$ and clearly $\delta(\tilde{v}) = \underline{v}$.

Now we can say that the image of the Galois group under ρ consists of the matrices

$$\{\begin{pmatrix} x & u_{12} & u_{13} \\ 0 & a & b \\ 0 & c & d \end{pmatrix} | ad-bc = x; \ell u_{12} = c; \ell u_{13} = d-x\} \subset \mathrm{Gl}_3(\mathbb{Z}_\ell) \tag{5.89}$$

It is the cocycle condition which makes this to a subgroup. Now it is not difficult to see that this is the group of $\mathbb{Z}_\ell-$ valued points of a smooth groups scheme $\mathcal{I}^{(1)}/\mathbb{Z}_\ell \subset \mathrm{Gl}_2/\mathbb{Z}$.

We can say that we constructed a Galois-extension $K_\ell^{(\infty)}/Q$ which is unramified outside ℓ and we have an isomorphism

$$\rho_\ell \ \mathrm{Gal}(K_\ell^{(\infty)}/\mathbb{Q}) \xrightarrow{\sim} \mathcal{I}^{(1)}(\mathbb{Z}_\ell) \tag{5.90}$$

We also consider the finite extensions $K_\ell^{(r)}(\mathbb{Z}/\ell^r\mathbb{Z}) \xrightarrow{\sim} \mathcal{I}^{(1)}(\mathbb{Z}/\ell^r\mathbb{Z})$ and for $r = 1$ we have $\mathcal{I}^{(1)}(\mathbb{Z}/\ell\mathbb{Z}) = B^{(1)}(\mathbb{Z}/\ell\mathbb{Z})$.

Of course we expect this to be a special case of a general ensemble of theorems. We may start from any irregular prime ℓ and an even integer $n > 0$ such that $\ell \mid \zeta(-1-n)$. The general problem will be to understand the Galois-modules $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{Z}_\ell)\{\bar\pi_f^{\mathrm{Eis}}\}$ or $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{Z}/(\ell^r))\{\bar\pi_f^{\mathrm{Eis}}\}$. In our example we are lucky and $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{Z}_\ell)\{\bar\pi_f^{\mathrm{Eis}}\} = H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n \otimes \mathbb{Z}_\ell)$, but I think this is only seemingly of relevance. More important is the fact that in our example the multiplicity $m(\bar\pi_f) = 1$ and we for the moment we restrict our attention to this case.

In our example we needed that there is a prime $p \equiv 1 \mod \ell$ which does not satisfy $\tau(p) \equiv 2^{11}+1 \mod \ell^2$. we say that the Hecke-module *is not degenerated*. We have to assume that an analogous fact is true in the general case. Of course we expect that this is always the case. But this may not so easy or even impossible to verify that a given Hecke module is non degenerate in a concrete case. In our baby example we relied on the tables for the values $\tau(p)$ provided by Mathematica, we can always verify it in principle but not in practice. It is just another case of a Wieferich dilemma.

It is also conceivable that the above condition may be replaced by another one which is easier to verify in a given case. For instance it is conceivable that an algorithm which computes some small Hecke operators $T_3, T_5, \dots$ could be helpful to show that a Hecke module is non degenerate.

Of course we may also interpolate ℓ -adically , and look at cases where we have a higher power dividing of ℓ dividing $\zeta(-1-n)$. It may be worth to investigate the structure of the Galois module in the situation of Theorem 3.3.7.

It remains the case of higher multiplicity. Fortunately we know a prime -namely $\ell = 547$ for which we have multiplicity 2, but we still have weak multiplicity one. For this case we made some experimental computations in section 3.3.11 and we made some predictions about the structure of the Hecke modules $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{484+\alpha(\ell-1)} \otimes \mathbb{Z}/(\ell^2))$. In this case it is an interesting question to consider that case $\alpha = 100$ and find out what the structure of the Galois module is, especially describe the image of the Galois group.

Chapter 6

Cohomology in the adelic language

6.1 The spaces

6.1.1 The (generalized) symmetric spaces

Our basic datum is a connected reductive group $G/\mathbb{Q}$. Let $G^{(1)}/\mathbb{Q}$ be its derived group and let $C/\mathbb{Q}$ the connected component of the identity of its centre. Then $G^{(1)}/\mathbb{Q}$ is semi simple and $C/\mathbb{Q}$ is a torus. The multiplication provides a canonical map

$$m : G^{(1)} \times C \to G, \tag{6.1}$$

it is is an isogeny, this means that the kernel $\mu_C = C \cap G^{(1)}$ of this map is a finite group scheme of multiplicative type. (A finite group scheme of multiplicative type is simply a finite abelian group together with an action of the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on it.) If we have such an isogeny as in (6.1) we write $G = C \cdot G^{(1)}$.

Let $S/\mathbb{Q}$ be the maximal $\mathbb{Q}$ -split torus in $C/\mathbb{Q}$. Up to isogeny we have $C = C_1 \cdot S$ where C_1 is the maximal anisotropic subtorus of $C/\mathbb{Q}$. We have an exact sequence

$$1 \to G^{(1)} \to G \xrightarrow{d_C} C' \to 1,$$

the quotient C' is a torus. The restriction of d_C to C is an isogeny. It is also called $d_C : C \to C'$.

If $\tilde{G}^{(1)}/\mathbb{Q}$ is the simply connected covering of $G^{(1)}$ (see section (1.1.5), then we get an isogeny

$$m_1 : \tilde{G} = \tilde{G}^{(1)} \times C \to G \tag{6.2}$$

Let $\mathfrak{g}, \mathfrak{g}^{(1)}, \mathfrak{c}, \mathfrak{c}_1, \mathfrak{z}$ be the Lie algebras of $G/\mathbb{Q}, G^{(1)}/\mathbb{Q}, C/\mathbb{Q}, C_1/\mathbb{Q}, S/\mathbb{Q}$, then the differential of m_1 induces an isomorphism

$$D_{m_1} : \mathfrak{g} \to \mathfrak{g}^{(1)} \oplus \mathfrak{c}_1 \oplus \mathfrak{z} \tag{6.3}$$

On $\mathfrak{g}$ we have the Killing form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{Q}$ it is defined by the rule (See 1.18). For $Y_1, Y_2 \in \mathfrak{g}$ we have

$$(Y_1, Y_2) \mapsto \mathrm{trace}(\mathrm{ad}(Y_1) \circ \mathrm{ad}(Y_2)) \tag{6.4}$$

Actually the Killing form is a bilinear form on $\mathfrak{g}^{(1)} = \mathfrak{g}/(\mathfrak{c}_1 \oplus \mathfrak{z})$ and the restriction $B : \mathfrak{g}^{(1)} \times \mathfrak{g}^{(1)} \to \mathbb{Q}$ is nondegenerate (see chap2 and chap4).

An automorphism $\Theta : \tilde{G}^{(1)} \times_{\mathbb{Q}} \mathbb{R} \to \tilde{G}^{(1)} \times_{\mathbb{Q}} \mathbb{R}$ is called a Cartan involution if $\Theta^2 = \mathrm{Id}$ and if the bilinear form

$$B_\Theta(Y_1, Y_2) = B(Y_1, \Theta(Y_2)) \tag{6.5}$$

on $\mathfrak{g} \otimes \mathbb{R}$ is negative definite.

If Θ is a Cartan involution then it induces an automorphism -also called Θ- on the Lie algebra $\mathfrak{g}_{\mathbb{R}} = \mathfrak{g} \otimes \mathbb{R}$ and decomposes it into a $+$ and a $-$ eigenspace

$$\mathfrak{g}_{\mathbb{R}} = \mathfrak{k} \oplus \mathfrak{p} \tag{6.6}$$

and then clearly the $+$ eigenspace $\mathfrak{k}$ is a Lie subalgebra and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. The Killing form is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$. This explains the above assertion on B_Θ.

If $\tilde{G}^{(1)}$ is split and if $T^{(1)}$ is a split maximal torus (see section 1.1.5) then find a Cartan involution Θ_0 which induces the map $t \mapsto t^{-1}$ and which induces on the little subgroups $i_\alpha : H_\alpha \xrightarrow{\sim} \mathrm{Sl}_2$ (See 1.1.5) the involution $g \mapsto^t g^{-1}$ (provided we chose the obvious identification i_α)

The topological group of real points $\tilde{G}^{(1)}(\mathbb{R})$ is connected, if $\tilde{G}^{(1)}$ is split this follows from the fact that $\tilde{G}^{(1)}(\mathbb{R})$ is generated by the groups $U_\alpha(\mathbb{R})$. (*see* sction 1.1.5) We have the classical theorem

Theorem 6.1.1. *The fixed group $K_\infty^{(1)} = \tilde{G}^{(1)}(\mathbb{R})^\Theta$ is a maximal compact subgroup and it is also connected. The Cartan involutions are conjugate under the action of $\tilde{G}^{(1)}(\mathbb{R})$, and therefore the maximal compact subgroups of $\tilde{G}^{(1)}(\mathbb{R})$ are conjugate.*

The group $K_\infty^{(1)}$ is obviously the group of real points of a reductive group, which is also called $K_\infty^{(1)}/\mathbb{R}$, so at this point we do distinguish between the group of $\mathbb{R}$-valued points and the algebraic group.

We introduce the space $\tilde{X}^{(1)}$ of Cartan involutions on $\tilde{G}^{(1)} \times_{\mathbb{Q}} \mathbb{R}$, it is a homogenous space under the action of $\tilde{G}^{(1)}(\mathbb{R})$ by conjugation and if we choose a Θ or $K_\infty^{(1)}$ then

$$\tilde{X}^{(1)} = \tilde{G}^{(1)}(\mathbb{R})/K_\infty^{(1)} \tag{6.7}$$

This is the symmetric space attached to $\tilde{G}^{(1)} \times_{\mathbb{Q}} \mathbb{R}$. The action of $G^{(1)}(\mathbb{R})$ by left translations is denoted by $L_{g_\infty} : x \mapsto g_\infty x$

Proposition 6.1.1. *The symmetric space $\tilde{X}^{(1)} = \tilde{G}^{(1)}(\mathbb{R})/K_\infty^{(1)}$ is diffeomorphic to $\mathbb{R}^d$, where $d = \dim \mathfrak{p}$, it carries a Riemannian metric which is $\tilde{G}^{(1)}(\mathbb{R})$ invariant.*

At this point it seems to be appropriate to introduce the *compact dual group* of $G^{(1)} \times \mathbb{R}$. Our Cartan involution Θ provides a homomorphism $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to \mathrm{Aut}(G^{(1)} \times \mathbb{R})$ simply by sending the complex conjugation $\mathbf{c}$ to Θ. This is also a 1-cocycle and hence this gives us a class $[\Theta] \in H^1(\mathbb{C}/\mathbb{R}, \mathrm{Aut}(G^{(1)} \times \mathbb{R}))$. This class gives us a $\mathbb{R}$-form $G_c^{(1)}/\mathbb{R}$ (see section 1.1.6). The group

$$G_c^{(1)}(\mathbb{R}) = \{g \in G^{(1)}(\mathbb{C}) \mid \Theta \circ \mathbf{c}(g) = g\} \tag{6.8}$$

We consider the group $R_{\mathbb{C}/\mathbb{R}}(G^{(1)} \times \mathbb{C})$ its group of real points is $G^{(1)}(\mathbb{C})$, on this group we have two involutions namely $\mathbf{c}$(resp $\mathbf{c}\circ\Theta$) and $G^{(1)}(\mathbb{R})$(resp. $G_c^{(1)}(\mathbb{R})) \subset R_{\mathbb{C}/\mathbb{R}}(G^{(1)} \times \mathbb{C})(\mathbb{R})$. The Lie-algebra of $R_{\mathbb{C}/\mathbb{R}}(G^{(1)} \times \mathbb{C})$ is the real vector space $\mathfrak{g} \otimes \mathbb{C}$ and the Lie-algebras of our two groups

$$\mathfrak{g} \otimes \mathbb{R} = \mathfrak{k} + \mathfrak{p} \text{ resp. } \mathrm{Lie}(G_c^{(1)}/\mathbb{R}) = \mathfrak{k} \oplus i \otimes \mathfrak{p}. \tag{6.9}$$

Since the Killiing-form is negative definite on $\mathfrak{g} \oplus i \otimes \mathfrak{p}$ it becomes clear that $G_c^{(1)}(\mathbb{R})$ is indeed compact.

We consider the special case that $G^{(1)} \times \mathbb{R}$ is split and $T^{(1)}/\mathbb{R}$ is a split maximal torus. Let Θ_0 be a Cartan involution which induces $t \mapsto t^{-1}$ on $T^{(1)}$. Then this torus provides the maximal torus $T_c^{(1)} \subset G_c^{(1)}$. This torus is compact, i.e. $T_c^{(1)}(\mathbb{R}) = (S^1)^r$ where $r = \dim(T^{(1)})$. In a certain sense we can say that $G^{(1)}/\mathbb{R}$ and $G_c^{(1)}/\mathbb{R}$ share a maximal torus. This observation turns out to be useful if we want to compute the factor $\kappa_\infty(G)$ in (6.108). We also see that the little subgroups H_α are invariant under Θ_0, hence we also have the little $H_{c,\alpha} \subset G_c^{(1)}$, the group $H_{c,\alpha}(\mathbb{R}) = \mathrm{O}(3)$ or $\mathrm{SU}(2)$.

Of course $G^{(1)}(\mathbb{R})$ can be compact, in this case $\Theta = \mathrm{Id}$ is the identity. Then $\tilde{G}^{(1)}(\mathbb{R}) = K_\infty^{(1)}$ and our symmetric space is a point.

We return to our reductive group $G/\mathbb{Q}$. We compare it to $\tilde{G}$ via the homomorphism m_1 in (6.2). Let K_∞^C be the connected component of the identity of the maximal compact subgroup in $C_1(\mathbb{R})$ and let $Z'(\mathbb{R})^{(0)}$ be the connected component of the identity of the group of real points a subtorus $Z' \subset S$. Then we put

$$K_\infty = m_1(K_\infty^{(1)} \times K_\infty^C \times Z'(\mathbb{R})^{(0)})$$

This group K_∞ is connected and if we divide by $Z'(\mathbb{R})^{(0)}$ it is compact, more precisely we can say that $K_\infty/Z'(\mathbb{R})^{(0)}$ is the connected component of a maximal compact subgroup in $G(\mathbb{R})/Z'(\mathbb{R})^{(0)}$. The choice of the subtorus Z' is arbitrary and in a certain sense irrelevant. We could choose $Z' = S$ then we call K_∞ *saturated* , this choice is very convenient but it certain situations it is better to make a different choice, for instance we may choose $Z' = 1$.

To such a pair (G, K_∞) we attach the *(generalized) symmetric space*

$$X = G(\mathbb{R})/K_\infty.$$

Here are a few comments concerning the structure of this space. (see also Chap II. 1.3) We observe that by construction K_∞ is connected, hence we have that $K_\infty \subset G(\mathbb{R})^0$. So if as usual $\pi_0(G(\mathbb{R}))$ denotes the set of connected components, then we see that

$$\pi_0(X) = \pi_0(G(\mathbb{R})).$$

The connected component of the identity of $\tilde{G}(\mathbb{R})$ maps under m_1 to the connected component of he identity of $G(\mathbb{R})$, i.e.

$$\tilde{G}(\mathbb{R}) = \tilde{G}^{(1)}(\mathbb{R}) \times C_1(\mathbb{R})^0 \times S(\mathbb{R})^0 \to G(\mathbb{R})^0$$

and if we divide by $K_\infty^{(1)} \times K_\infty^C \times Z'(\mathbb{R})^{(0)}$, resp. K_∞ we get a diffeomorphism with the connected component corresponding to the identity

$$\tilde{G}^{(1)}(\mathbb{R})/K_\infty^{(1)} \times C_1(\mathbb{R})^0/K_\infty^C \times S(\mathbb{R})^0/Z'(\mathbb{R}) \xrightarrow{\sim} X_1 \subset X.$$

We want to describe the other connected components of X. It is well known that we can find a maximal split torus $\tilde{S}_1 \subset \tilde{G}^{(1)} \times \mathbb{R}$ which is invariant under our given Cartan involution Θ. The homomorphism m_1 maps $\tilde{G}^{(1)}(\mathbb{R}) \to G^{(1)}(\mathbb{R})$. The fixed group $G^{(1)}(\mathbb{R})^\Theta$ is a compact subgroup whose connected component of the identity is the image of $K_\infty^{(1)}$ under m_1. Our torus $\tilde{S}_1$ sits as the first component in the maximal split torus

$$\tilde{S}_2 = \tilde{S}_1 \times C_1^{\text{split}} \times S$$

Then it is clear that Θ induces the involution $t \mapsto t^{-1}$ on $\tilde{S}_1$. Let S_2 be the image of $\tilde{S}_2$ under m_1. We have the following proposition

Proposition 6.1.2. *a)The group of 2-division points $S_2[2]$ normalizes K_∞.*

b) We have an exact sequence

$$\to \tilde{S}_2[2] \to S_2[2] \xrightarrow{r} \pi_0(G(\mathbb{R})) \to 0$$

c) If K_∞^0 is the image of $K_\infty^{(1)} \times K_\infty^C$ then $K_\infty^0 \cdot S_2[2]$ is a maximal compact subgroup of $G(\mathbb{R})$.

Proof. Rather obvious, the surjectivity of r requires an argument in Galois cohomology. (Details later) □

Now we can write down all the connected components. We choose a system Ξ of representatives for $S_2[2]/\tilde{S}_2[2]$ and for any $\xi \in \Xi$ we get a diffeomorphism

$$\begin{gathered}\tilde{G}^{(1)}(\mathbb{R})/K_\infty^{(1)} \times C_1(\mathbb{R})^0/K_\infty^C \times S(\mathbb{R})^0/Z'(\mathbb{R}) \to X_\xi \subset X \\ g \mapsto g\xi\end{gathered} \tag{6.10}$$

We may formulate this differently

Proposition 6.1.3. *The multiplication from the left by $S_2[2]$ on $G(\mathbb{R})$ induces an action of $S_2[2]/\tilde{S}_2[2]$ on X and this action is simple transitive on the set of connected components.*

Let $x_0 = K_\infty \in X$. For any other point $x \in X$ we find an element $g \in X$ which translates x_0 to x. Then the derivative of the translation provides an isomorphism between the tangent spaces

$$D_g : T_{x_0} = \mathfrak{p} \xrightarrow{\sim} T_x.$$

This isomorphism depends of course on the choice of g. (This will play a role in section (8.1)). But we apply this to the highest exterior power and get an isomorphism

$$D_g : \Lambda^d(\mathfrak{p}) \xrightarrow{\sim} \Lambda^d(T_x)$$

which does not depend on the choice of g because the connected group K_∞ acts trivially on $\Lambda^d(\mathfrak{p})$. Hence we can say that we can find a *consistent* orientation on X : We chose a generator in $\Lambda^d(\mathfrak{p})$ the D_g yields a generator in $\Lambda^d(T_x)$.

As a standard example we can take $G/\mathbb{Q} = \mathrm{Gl}_2/\mathbb{Q}$, then the connected component of the real points of the centre is $\mathbb{R}^\times_{>0}$ and in this case we can take $K_\infty = SO(2) \cdot \mathbb{R}^\times_{>0} \subset \mathrm{Gl}_2(\mathbb{R}))$. In this case the symmetric space is the union of an upper and a lower half plane. It we choose for our split torus $S_1/\mathbb{R}$ the standard diagonal torus, then $S_1[2]$ is the group of diagonal matrices with entries ± 1 and this normalizes K_∞.

6.1.2 The locally symmetric spaces

Let $\mathbb{A}$ be the ring of adeles, we decompose it into its finite and its infinite part: $\mathbb{A} = \mathbb{R} \times \mathbb{A}_f$. We have the group of adeles $G(\mathbb{A}) = G(\mathbb{R}) \times G(\mathbb{A}_f)$. We denote elements in the adele group by underlined letters $\underline{g}, \underline{h} \dots$ and so on. If we decompose an element $\underline{g}$ into its finite and its infinite part then we denote this by $g_\infty \times \underline{g}_f$. Let K_f be a (variable) open compact subgroup of $G(\mathbb{A}_f)$. We always assume that this group is a product of local groups $K_f = \prod_p K_p$.

To get such subgroups we choose an integral structure (explain at some other place) $\mathcal{G}/\operatorname{Spec}(\mathbb{Z})$. Then we know that we have $K_p = \mathcal{G}(\mathbb{Z}_p)$ for almost all p. Furthermore we know that $\mathcal{G}\times \operatorname{Spec}(\mathbb{Z}_p)/\operatorname{Spec}(\mathbb{Z}_p)$ is a reductive group scheme for almost all primes p.

If $\mathcal{G}/\operatorname{Spec}(\mathbb{Z})$ and K_f are given, then we select a finite set Σ of finite primes which contains the primes p where $\mathcal{G}/\mathbb{Z}_p$ is not reductive and those where K_p is not equal to $\mathcal{G}(\mathbb{Z}_p)$. This set Σ will be called the set of *ramified* primes.

The general agreement will be that we use letters $\mathcal{G}, \mathcal{T}, \mathcal{U}, \dots$ for group schemes over the integers, or over $\mathbb{Z}_p$ and then their general fiber will be $G, T, U, \dots$.

Readers who are not so familiar with this language may think of the simple example where $G/\mathbb{Q} = GSp_n/\mathbb{Q}$ is the group of symplectic similitudes on $V = \mathbb{Q}^{2n} = \mathbb{Q}e_1 \oplus \cdots \oplus \mathbb{Q}e_n \oplus \mathbb{Q}f_1 \oplus \cdots \oplus \mathbb{Q}f_n$ with the standard symplectic form which is given by $< e_i, f_i >= 1$ for all i and where all other products zero. The vector space contains the lattice $L = \mathbb{Z}^{2n} = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n \oplus \mathbb{Z}f_n \oplus \cdots \oplus \mathbb{Z}f_1$. This lattice defines a unique integral structure $\mathcal{G}/\mathbb{Z}$ on $G/\mathbb{Q}$ for which $\mathcal{G}(\mathbb{Z}_p) = \{g \in G(\mathbb{Q}_p) | g(L \otimes \mathbb{Z}_p) = (L \otimes \mathbb{Z}_p)\}$. In this case the group scheme is reductive over $\operatorname{Spec}(\mathbb{Z})$. This integral structure gives us a privileged choice of an open maximal compact subgroup: Within the ring $\mathbb{A}_f$ of finite adeles we have the ring $\hat{\mathbb{Z}} = \lim\limits_{\leftarrow} \mathbb{Z}/m\mathbb{Z}$ of integral finite adeles and we can consider $K_f^0 = \mathcal{G}(\hat{\mathbb{Z}}) = \prod_p \mathcal{G}(\mathbb{Z}_p)$. This is a very specific choice. In this case the set $\Sigma = \emptyset$, we say that $K_f = K_f^0$ is unramified.

Starting from there we can define new subgroups K_f by imposing some congruence conditions at a finite set Σ of primes. These congruence conditions then define congruence subgroups $K_p \subset K_p^0$. This set Σ of places where we impose congruence condition will then be the set of ramified primes.(See the example further down.) Then we define the level subgroup

$$K_f = \prod_{p \in \Sigma} K_p \times \prod_{p \notin \Sigma} \mathcal{G}(\mathbb{Z}_p). \tag{6.11}$$

The space $(G(\mathbb{R})/K_\infty) \times (G(\mathbb{A}_f)/K_f)$ can be seen as a product of the symmetric space and an infinite discrete set, on this space $G(\mathbb{Q})$ acts properly discontinuously (see below) and the quotients

$$\mathcal{S}^G_{K_f} = G(\mathbb{Q}) \setminus (G(\mathbb{R})/K_\infty \times G(\mathbb{A}_f)/K_f)$$

are the locally symmetric spaces whose topological properties we want to study. We denote by

$$\pi : G(\mathbb{R})/K_\infty \times G(\mathbb{A}_f)/K_f \to \mathcal{S}^G_{K_f} = G(\mathbb{Q}) \setminus (G(\mathbb{R})/K_\infty \times G(\mathbb{A}_f)/K_f)\,,$$

the projection map.

To get an idea of how this space looks like we consider the action of $G(\mathbb{Q})$ on the discrete space $G(\mathbb{A}_f)/K_f$. It follows from classical finiteness results that this quotient is finite, let us pick representatives $\{\underline{g}_f^{(i)}\}_{i=1..m}$. We look at the stabilizer of the coset $\underline{g}_f^{(i)}K_f/K_f$ in $G(\mathbb{Q})$. This stabilizer is obviously equal to $\Gamma^{\underline{g}_f^{(i)}} = G(\mathbb{Q}) \cap \underline{g}_f^{(i)}K_f(\underline{g}_f^{(i)})^{-1}$ which is an arithmetic subgroup of $G(\mathbb{Q})$. This subgroup acts properly discontinuously on X (See Chap. II, 1.6).

Now we call the level subgroup K_f neat, if all the subgroups $\Gamma^{\underline{g}_f^{(i)}}$ are torsion free. It is not hard to see, that for any choice of K_f we can pass to a subgroup of finite index K'_f, which is neat. Then we have

conncomp

Proposition 6.1.4. *For any subgroup K_f the space $\mathcal{S}^G_{K_f}$ is a finite union of quotient spaces $\Gamma^{\underline{g}_f^{(i)}} \setminus X$ where $X = G(\mathbb{R})/K_\infty$ and the $\Gamma_i = \Gamma^{\underline{g}_f^{(i)}}$ are varying arithmetic congruence subgroups. If K_f is neat, these spaces are locally symmetric spaces. If K_f is not neat then we may pass to a neat subgroup K'_f which is even normal in K_f: We get a covering $\mathcal{S}^G_{K'_f} \to \mathcal{S}^G_{K_f}$ which induces coverings $\Gamma'_j\backslash X \to \Gamma_i\backslash X$, where the Γ'_j are torsion free and normal in Γ_i. So we see that in general the quotients are orbifold locally symmetric spaces. For any point $y \in \mathcal{S}^G_{K_f}$ we can find a neighbourhood V_y such that $\pi^{-1}(V_y)$ is the disjoint union of connected components $W_{\underline{x}}, \underline{x} = (x_\infty, \underline{g}_f) \in \pi^{-1}(y)$, and $V_y = \Gamma_{x_\infty}\backslash W_{\underline{g}_f}$, where Γ_{x_∞} is the stabilizer of x_∞ intersected with $\Gamma^{\underline{g}_f}$.*

We will consider the special case where $G/\mathbb{Q}$ is the generic fibre of a split reductive scheme $\mathcal{G}/\mathbb{Z}$. In that case we can choose $K_f = \prod_p \mathcal{G}(\mathbb{Z}_p)$, this is then a maximal compact subgroup in $G(\mathbb{A}_f)$. Then K_f is unramified we will also say that the space $\mathcal{S}^G_{K_f}$ is unramified. If in addition the derived group $G^{(1)}/\mathbb{Q}$ is simply connected, then it is not difficult to see, that $G(\mathbb{Q})$ acts transitively on $G(\mathbb{A}_f)/K_f$ and hence we get

$$\mathcal{S}^G_{K_f} \xrightarrow{\sim} \mathcal{G}(\mathbb{Z})\backslash X.$$

The homomorphism $\mathcal{G}(\mathbb{Z}) \to \pi_0(C'(\mathbb{R}))$ is surjective we can conclude that $\mathcal{G}(\mathbb{Z})$ acts transitively on $\pi_0(X)$ and if Γ_0 is the stabilizer of a connected component X^0 of X then we find

$$\mathcal{S}^G_{K_f} \xrightarrow{\sim} \Gamma_0\backslash X^0$$

especially we see that the quotient is connected. We discuss an example.

We start from the group $\mathcal{G}/\operatorname{Spec}(\mathbb{Z}) = \mathrm{Gl}_n/\operatorname{Spec}(\mathbb{Z})$ then we may choose $K_\infty = \mathrm{SO}(n) \times \mathbb{R}^\times_{>0} \subset \mathrm{Gl}_n(\mathbb{R})$. and $X = \mathrm{Gl}_n(\mathbb{R})/K_\infty$ is the disjoint union of two copies of the space X of positive definite symmetric $(n \times n)$ matrices up to homothetie by a positive scalar (or what amounts to the same with determinant one). If we choose K_f as above then we find

$$\mathcal{S}^G_{K_f} = \mathrm{Sl}_n(\mathbb{Z})\backslash X.$$

We have another special case. Let us assume that $G/\mathbb{Q}$ is semi simple and simply connected. The group $G \times \mathbb{R}$ is a product of simple groups over $\mathbb{R}$ and we assume in addition that there is at least one non compact factor. Then we have the strong approximation theorem ([65],[82])) which says that for any choice of K_f the map from $G(\mathbb{Q})$ to $G(\mathbb{A}_f)/K_f$ is surjective, i.e. any $\underline{g}_f \in G(\mathbb{A}_f)$ can be written as $\underline{g}_f = a\underline{k}_f, a \in G(\mathbb{Q}), \underline{k}_f \in K_f$. This clearly implies that then

$$\mathcal{S}^G_{K_f} = \Gamma\backslash G(\mathbb{R})/K_\infty \tag{6.12}$$

where $\Gamma = K_f \cap G(\mathbb{Q})$.

There is a contrasting case, this is the case when $G/\mathbb{Q}$ is still semi simple and simply connected, but where $G(\mathbb{R})$ is compact. In this case our symmetric space X is simply a point $*$ and

$$\mathcal{S}^G_{K_f} = G(\mathbb{Q})\backslash(* \times G(\mathbb{A}_f)/K_f).$$

This means that our topological space is simply a finite set of points, hence it looks as if this is an entirely uninteresting and trivial case. But this is not so. To determine the finite set and the stabilizers is a highly non trivial task. Later we will construct sheaves and discuss the action of the Hecke algebra on the cohomology of these sheaves. Then it turns out that it is not only the set of points and the stabilizers that is of interest but also the "interaction" among these points is of interest. Then it turns out that this case is as difficult as the case where $\Gamma\backslash X$ becomes an honest space.

We give a few examples of such spaces

In the choice of our group K_∞ a subtorus $Z' \subset S$ enters. The choice of this subtorus has very little influence on the structure of our locally symmetric space $\mathcal{S}^G_{K_f}$. Remember that the isogeny m in (6.1) induces an isogeny $C \to C'$ and this isogeny yields an isogeny from S to the maximal split subtorus $S' \subset C'$. This homomorphism induces an isomorphism $S(\mathbb{R})^0 \to S'(\mathbb{R})^0$. If $G_1(\mathbb{R})$ is the inverse image of the group of 2-division points $S'[2]$ then we get from this isomorphism that $G(\mathbb{R}) = G_1(\mathbb{R}) \times S(\mathbb{R})^0$. If we now consider the two spaces $\mathcal{S}^G_{K_f}$ and $(\mathcal{S}^G_{K_f})^\dagger$, the first one defined with an arbitrary torus Z' the second one with $Z' = S$ then the arguments above imply that

$$\mathcal{S}^G_{K_f} = (\mathcal{S}^G_{K_f})^\dagger \times (S(\mathbb{R})^0/Z'(\mathbb{R})^{(0)}) \tag{6.13}$$

the second factor on the right hand side is isomorphic to $\mathbb{R}^b$ and since we are interested in the cohomology group of this space, ihe second factor is irrelevant.

In certain situations we encounter cases where it is natural to choose a subgroup K_∞ which is slightly larger and not connected. If this is the case we denote the connected component $K_\infty^{(1)}$ and we get two locally symmetric spaces and a finite map

$$G(\mathbb{Q})\setminus\left(G(\mathbb{R})/K_\infty^{(1)}\times G(\mathbb{A}_f)/K_f\right)\to G(\mathbb{Q})\setminus(G(\mathbb{R})/K_\infty\times G(\mathbb{A}_f)/K_f) \tag{6.14}$$

This map is a covering if K_f is neat and the space on the right is a quotient of the space on the left by an action of the finite elementary abelian [2]-group $K_\infty/K_\infty^{(1)}$.

In accordance with the terminology in number theory we call the space $\mathcal{S}^G_{K_f}$ *narrow* if $K_\infty^{(1)}=K_\infty$ and in general we call the space on the left the *narrow cover* of $G(\mathbb{R})/K_\infty\times G(\mathbb{A}_f)/K_f$.

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6.1.3 The group of connected components, the structure of $\pi_0(\mathcal{S}^G_{K_f})$.

If we keep our assumptions that $G/\mathbb{Q}$ is reductive and $G^{(1)}/\mathbb{Q}$ is simply connected and satisfies strong approximation. We choose a level subgroup $K_f\subset G(\mathbb{A}_f)$ and we put $d_{C'}(K_\infty\times K_f)=K_\infty^{C'}\times K_f^{C'}$. Then we claim that under these conditions pinull

$$\pi_0(\mathcal{S}^G_{K_f})\xrightarrow{\sim}\pi_0(\mathcal{S}^{C'}_{K_\infty^{C'}\times K_f^{C'}}). \tag{6.15}$$

To see this we need a theorem of Tate which says that the map $C'(\mathbb{Q})\to\pi_0(C'(\mathbb{R})$ is surjective. This implies that $\pi_0(\mathcal{S}^{C'}_{K_\infty^{C'}\times K_f^{C'}})=C'(\mathbb{Q})^{(0)}\backslash C'(\mathbb{A}_f)/K_f^{C'}$, where $C'(\mathbb{Q})^{(0)}\subset C'(\mathbb{Q})$ are the elements whose image lies in $C'(\mathbb{R})^{(0)}$. Now we need a little argument from Galois cohomology. The map $G(\mathbb{A}_f)\to C'(\mathbb{A}_f)$ is surjective because for all primes p $H^1(\mathbb{Q}_p,G^{(1)})$ consist of the trivial class only. (Kneser and Bruhat-Tits ([14]).) This implies the surjectivity: For the injectivity assume $\underline{x},\underline{y}\in C'(\mathbb{A}_f)$ and there is an element $a\in C(\mathbb{Q})^{(0)}$ with $a\underline{x}=\underline{y}$. Then we need to find a lift of a to an element $b\in G(\mathbb{Q})$. Again we invoke the standard argument from Galois cohomology. We have the exact sequence

$$G(\mathbb{Q})\to C'(\mathbb{Q})\xrightarrow{\delta}H^1(\mathbb{Q},G^{(1)})$$

the obstruction to find b is an element $\delta(a)\in H^1(\mathbb{Q},G^{(1)})$. We have the Hasse principle $H^1(\mathbb{Q},G^{(1)})\xrightarrow{\sim}H^1(\mathbb{R},G^{(1)})$ ([**?**]) but since $a\in C'(\mathbb{Q})^{(0)}$ it follows that the image of $\delta(a)\in H^1(\mathbb{R},G^{(1)})$ is trivial, hence $\delta(a)$ is trivial.

We have seen in the previous section that we can choose a consistent orientation on $X=G(\mathbb{R})/K_\infty$ provided K_∞ is narrow. Then it clear this induces also a consistent orientation on $\mathcal{S}^G_{K_f}$.

BSC1

6.1.4 The Borel-Serre compactification

In general the space $\mathcal{S}^G_{K_f}$ is not compact. Recall that in the definition of this quotient the choice of a subtorus $Z'/\mathbb{Q}$ of $S/\mathbb{Q}$ enters. This If $Z' \neq S$ then the quotient will never be compact. But this kind of non compactness is "uninteresting". In the following we assume that $Z' = S$.

In this case we have the criterion of Borel - Harish-Chandra which says

The quotient space $\mathcal{S}^G_{K_f}$ is compact if and only if the group $G/\mathbb{Q}$ has no proper parabolic subgroup over $\mathbb{Q}$.

If we have a non trivial parabolic subgroup $P/\mathbb{Q}$ then we add a boundary part $\partial_P \mathcal{S}^G_{K_f}$ to $\mathcal{S}^G_{K_f}$ it will depend only the $G(\mathbb{Q})$-conjugacy class of P. We will describe this boundary piece later. We define the Borel-Serre boundary

$$\partial(\mathcal{S}^G_{K_f}) = \bigcup_P \partial_P \mathcal{S}^G_{K_f},$$

where P runs over the set of $G(\mathbb{Q})$ conjugacy classes of parabolic subgroups. We will put a topology on this space and if $Q \subset P$ then $\partial_Q \mathcal{S}^G_{K_f}$ will be in the closure of $\partial_P \mathcal{S}^G_{K_f}$. Then

$$\mathcal{S}^{\bar{G}}_{K_f} = \mathcal{S}^G_{K_f} \cup \partial(\mathcal{S}^G_{K_f})$$

will be a compact Hausdorff-space.

We describe the construction of this compactification in more detail, more precisely we describe a tubular neighbourhood $\overset{\bullet}{\mathcal{N}}(\partial \mathcal{S}^G_{K_f})$ of the boundary. To achieve this we simply translate the considerations in section **??** into the adelic language. Let $P/\mathbb{Q}$ be a parabolic subgroup and let S_P be a maximal split torus of P then $\mathrm{Hom}(P, \mathbb{G}_m) \otimes \mathbb{Q} = \mathrm{Hom}(S_P, \mathbb{G}_m) \otimes \mathbb{Q}$. For any character $\gamma \in \mathrm{Hom}(P, \mathbb{G}_m)$ we get a homomorphism $\gamma_A : P(\mathbb{A}) \to \mathbb{G}_m(\mathbb{A}) = I_{\mathbb{Q}}$, the group of ideles. We have the idele norm $|\;| : \underline{x} \mapsto |\underline{x}|$ from the idele group to $\mathbb{R}^\times_{>0}$ and then we get by composing $|\gamma| : P(\mathbb{A}) \xrightarrow{|\;|\circ\gamma} \mathbb{R}^\times_{>0}$. It is obvious that we can extend this definition to characters $\gamma \in \mathrm{Hom}(P, \mathbb{G}_m) \otimes \mathbb{Q}$, for such a γ we find a positive non zero integer m such that $m\gamma \in \mathrm{Hom}(P, \mathbb{G}_m)$ and then we define

$$|\gamma| = (|m\gamma|)^{\frac{1}{m}}.$$

Later we will even extend this to a homomorphism $\mathrm{Hom}(P, \mathbb{G}_m) \otimes \mathbb{C} \to \mathbb{C}^\times$ by the rule $\boxed{\text{XtimesC}}$

$$\gamma \otimes z \mapsto |\gamma|^z \tag{6.16}$$

If we have a parabolic subgroup $P/\mathbb{Q}$ and a point $(x, \underline{g}_f) \in X \times G(\mathbb{A}_f)/K_f$ then we attach to it a (strictly positive) number

$$p(P, (x, \underline{g}_f)) = \mathrm{vol}_{d_x u}(U(\mathbb{Q}) \cap \underline{g}_f K_f \underline{g}_f^{-1} \backslash U(\mathbb{R})). \tag{6.17}$$

This needs explanation. The group $U(\mathbb{Q}) \cap \underline{g}_f K_f \underline{g}_f^{-1} = \Gamma_{U, \underline{g}_f}$ is a cocompact discrete lattice in $U(\mathbb{R})$, we can describe it as the group of elements $\gamma \in U(\mathbb{Q})$ which fix $\underline{g}_f K_f$, so it can be viewed as a lattice of integral elements where integrality is determined by $\underline{g}_f$. The component x defines a positive definite bilinear form B_{Θ_x} on the Lie algebra $\mathfrak{g} \otimes \mathbb{R}$, and this bilinear form can be restricted to the Lie-algebra $\mathfrak{u}_P \otimes \mathbb{R}$ and this provides a volume form $d_x u$ on $U(\mathbb{R})$. Then the above number is the volume of the nilmanifold $\Gamma_{U,\underline{g}_f} \backslash U(\mathbb{R})$ with respect to this measure.

These numbers have some obvious properties

a) They are invariant under conjugation by an element $a \in G(\mathbb{Q})$, this means we have

$$p(a^{-1}Pa, (x, \underline{g}_f)) = p(P, a(x, \underline{g}_f)) \tag{6.18}$$

b) If $\underline{p} \in P(\mathbb{A})$ then we have

$$p(P, \underline{p}(x, \underline{g}_f)) = p(P, (x, \underline{g}_f))|\rho_P|^2 \tag{6.19}$$

Here ρ_p is of course again the half sum of positive roots in the unipotent radical U_P.

If we are in the special case that $G = \mathrm{Sl}_2/\mathbb{Q}$ and $K_f = \mathrm{Sl}_2(\hat{\mathbb{Z}})$ then a parabolic subgroup P is a point $r = \frac{p}{q} \in \mathbb{P}^1(\mathbb{Q})$ (or ∞). Then $p(P, (z, 1))$ is small if z lies in a small Farey circle $D(c, \frac{p}{q})$ (see section 1.2.6). If $r = \infty$ and $z = iy$ then $p(\infty, (iy, 1)) = \frac{1}{y}$.

The $G(\mathbb{Q})$ conjugacy classes of parabolic are in one to one correspondence with the subsets π of the set relative simple roots π_G :The minimal parabolic corresponds to the empty set, the non proper parabolic subgroup $G/\mathbb{Q}$ corresponds to $\pi = \pi_G$ itself. In general $\pi = \pi_M$ is the set of relative simple roots of the semi simple part M of the reductive quotient of the parabolic subgroup $P/\mathbb{Q}$. For a parabolic subgroup P corresponding to π we put $d(P') = \#(\pi_G \backslash \pi)$.

As in section 1.2.8 we want to define the sets $X^P(c_P, r(c_P)) := \{(x, \underline{g}_f) \subset X \times G(\mathbb{A}_f)/K_f$, again these will be the sets of point which are very close to the boundary stratum $\partial_P(\mathcal{S}^G_{K_f})$ but keep a certain distance to lower dimensional strata.

For any $i \in \pi_G \backslash \pi$ we denote the maxima parabolic group attached to $\pi_G \backslash \{i\}$ by $P_i/\mathbb{Q}$. Then $\mathrm{Hom}(P_i, \mathbb{G}_m)$ is of rank one and generated by the fundamental character $\gamma_i : P_i \to \mathbb{G}_m$. The $2\rho_{P_i} = f_i\gamma_i$ with some integer $f_i > 0$ and for any parabolic subgroup $P \subset P_i$ we put

$$n_{\gamma_i}(P, (x, \underline{g}_f)) = p(P_i, (x, \underline{g}_f))^{1/f_i} \tag{6.20}$$

Now we extend this definition to any character $\gamma = \sum_i r_i\gamma_i \in \mathrm{Hom}(P_i, \mathbb{G}_m) \otimes \mathbb{Q}$ by

$$n_\gamma(P, (x, \underline{g}_f)) := \prod_i n_{\gamma_i}(P, (x, \underline{g}_f))^{r_i}. \tag{6.21}$$

We have the relative roots $\alpha_i^P \in \mathrm{Hom}(P_i, \mathbb{G}_m) \otimes \mathbb{Q}$ (see 1.93)and hence we have defined the numbers $n_{\alpha_i^P}(P, (x, \underline{g}_f))$ for $\alpha_i \in \pi_G \setminus \pi$.

We have a finite coset decomposition

$$G(\mathbb{A}_f) = \bigcup_{\underline{\xi}_f} P(\mathbb{A}_f)\xi_f K_f,$$

for any $\underline{\xi_f}$ put $K_f^P(\underline{\xi}_f) = P(\mathbb{A})_f \cap \underline{\xi}_f K_f \underline{\xi}_f^{-1}$. Then we have a disjoint union

$$P(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K_f = \bigcup_{\underline{\xi}_f} P(\mathbb{Q})\backslash X \times P(\mathbb{A}_f)/K_f^P(\underline{\xi}_f)\underline{\xi}_f,$$

If $U_P \subset P$ is the unipotent radical, then $M = P/U_P$ is a reductive group. For any open compact subgroup $K_f \subset G(\mathbb{A}_f)$(resp. for $K_\infty \subset G_\infty$) we define $K_f^M(\xi_f) \subset M(\mathbb{A}_f)$(resp. $K_\infty^M \subset M_\infty$) to be the image of $K^P(\underline{\xi}_f)$ in $M(\mathbb{A}_f)$ (resp. M_∞). We put

$$\mathcal{S}^M_{K_f^M(\xi_f)} = M(\mathbb{Q})\backslash M(\mathbb{A})/K_\infty^M K_f^M(\xi_f). \tag{6.22}$$

and get a fibration

$$\pi_{P,\underline{\xi}_f} : P(\mathbb{Q})\backslash X \times P(\mathbb{A}_f)/K_f^P(\underline{\xi}_f) \to \mathcal{S}^M_{K_f(\xi_f)}. \tag{6.23}$$

Now we apply reduction theory to the group $M/\mathbb{Q}$. For any real number $0 < r$ which is smaller than a suitable number $r_0 < 1$ we can define open, relatively compact subsets $\mathcal{S}^M_{K_f(\xi_f)}(r) \subset \mathcal{S}^M_{K_f(\xi_f)}$ such that the inclusion is a homotopy equivalence and $\bigcup_{r>0} \mathcal{S}^M_{K_f(\xi_f)}(r) = \mathcal{S}^M_{K_f(\xi_f)}$. (The points in $\mathcal{S}^M_{K_f(\xi_f)}(r)$ are those which have a certain distance -controlled by r - to $\partial(\mathcal{S}^M_{K_f(\xi_f)})$.) (See [**?**]) Now we define as in section 1.2.8 the sets

$$\begin{gathered} X^P(c_P, r(c_P)) := \{(x, \underline{g}_f) \in X \times G(\mathbb{A}_f)/K_f \mid \\ n_{\alpha_i^P}(P,(x,\underline{g}_f)) < c_P, \text{ for } \alpha_i \in \pi_G \setminus \pi \text{ and } \pi_{P,\underline{\xi}_f}(x,\underline{g}_f) \in \mathcal{S}^M_{K_f(\xi_f)}(r).\} \end{gathered} \tag{6.24}$$

Now proceed as in section 1.2.8: We assume that the number (c_P, r_P) are well chosen, then $\Gamma_P\backslash X^P(c_P, r(c_P))$ is an open subset of $\mathcal{S}^G_{K_f}$. (If $(x, \underline{g}_f)$ and $\gamma(x, \underline{g}_f)) \in X^P(c_P, r(c_P))$ then $\gamma \in P(\mathbb{Q})$.)

We define the punctured tubular neighbourhood:

$$\overset{\bullet}{\mathcal{N}}(\mathcal{S}^G_{K_f}) = \bigcup_{P \mod G(\mathbb{Q}): P \neq G} P(\mathbb{Q})\backslash X^P(c_P, r(c_P)) \subset \mathcal{S}^G_{K_f}. \tag{6.25}$$

Here we do not have the parameters $(c_p, r(c_p))$ on the left hand side, the reason is that their actual value rather irrelevant, we should also think of $\overset{\bullet}{\mathcal{N}}$ $(\mathcal{S}^G_{K_f})$ as a family of punctured neighbourhoods of $\partial(\mathcal{S}^G_{K_f})$.

Again we we have a finite double coset decompositions $M(\mathbb{A}_f) = \bigcup_{\eta_f} M(\mathbb{Q})\eta_f K_f^M$ and $P(\mathbb{A}_f) = P(\mathbb{Q})\eta_f^+ K_f^P$ where the η_f^* are lifts of the η_f (the unipotent radical satisfies strong approximation). Then we get again

$$P(\mathbb{Q})\backslash X \times P(\mathbb{A}_f)/K_f^P = \bigcup_{\eta_f^*} \Gamma^{\eta_f^*}\backslash X \;;\; \mathcal{S}^M_{K_f} = \bigcup_{\eta_f} \Gamma_M^{\eta_f}\backslash X^M \tag{6.26}$$

and our fibration can be restricted to the open subsets

$$\Gamma_P^{(\eta_f^*)}\backslash X \to \Gamma_M^{(\eta_f)}\backslash X^M \tag{6.27}$$

We have a closer look at this fibration, We are in the same situation as in section 1.2.8 for the convenience of the reader we basically repeat the reasoning here. The symmetric space is $X = G(\mathbb{R})/K_\infty$, the Iwasawa decomposition gives us $G(\mathbb{R}) = P(\mathbb{R})\cdot K_\infty$ hence we get

$$X = P(\mathbb{R})/K_\infty \cap P(\mathbb{R}) = P(\mathbb{R})/K_\infty^P \tag{6.28}$$

If U_P is the unipotent radical of P, then the reductive quotient $M = U_P\backslash P$, then $X^M = U_P(\mathbb{R})\backslash X$. The group K_∞^P maps isomorphically to a compact subgroup $K_\infty^M \subset M(\mathbb{R})$. The group K_∞^M is not necessarily connected but its connected component of the identity is a maximal compact connected subgroup, hence $X^M = M(\mathbb{R})/K_\infty^M$ is a symmetric space attached to M. We introduce the homomorphism

$$\underline{\alpha}^P : M(\mathbb{R}) \to (\mathbb{R}_{>0}^\times)^{d(P)} \,; m_\infty \mapsto (\dots, \alpha_i^P(m_\infty), \dots\}_{\alpha_i\in\pi_G\backslash\pi} \tag{6.29}$$

Let $M^{(1)}(\mathbb{R}), P^{(1)}(\mathbb{R})$ be the kernel of $\underline{\alpha}^P$. Then clearly $\Gamma_P^{(\eta_f^*)} \subset P^{(1)}(\mathbb{R})$ and $\Gamma_M^{(\eta_f)} \subset M^{(1)}(\mathbb{R})$. Then

$$\Gamma_M\backslash X^M = \Gamma_M^{(\eta_f)}\backslash M^{(1)}(\mathbb{R})/K_\infty^M \times (\mathbb{R}_{>0}^\times)^{d(P)} \tag{6.30}$$

We choose a section $s : M/\mathbb{Q} \to P/\mathbb{Q}$ we identify M with its image . Then $P = U_P \rtimes M$ and $X = U_P(\mathbb{R}) \rtimes (M^{(1)}(\mathbb{R})/K_\infty^M) \times (\mathbb{R}_{>0}^\times)^{d(P)}$. Our fibration (6.27) becomes

$$\begin{gathered}\Gamma_P\backslash(U_P(\mathbb{R}) \rtimes (M^{(1)}(\mathbb{R})/K_\infty^M)) \times (\mathbb{R}_{>0}^\times)^{d(P)} \\ \downarrow r_1 \times \mathrm{Id} \\ \Gamma_M\backslash((M^{(1)}(\mathbb{R})/K_\infty^M)) \times (\mathbb{R}_{>0}^\times)^{d(P)}\end{gathered} \tag{6.31}$$

If Γ_M is torsion free then r_1 is a locally trivial fibration, if not it is only an orbi-fibration: For any point $x \in \Gamma_M\backslash((M^{(1)}(\mathbb{R})/K_\infty^M))$ we consider its inverse image $p_M^{-1}(x) \subset M^{(1)}(\mathbb{R})/K_\infty^M$ and then

$$r_1^{-1}(x) = \{f : p_M^{-1}(x) \to \Gamma_{U_P}\backslash U_P(\mathbb{R}) | \, f(\gamma_M x') = \mathrm{Ad}(\gamma_M) f(x')\} \tag{6.32}$$

We restrict this fibration to $\overset{\bullet}{\mathcal{N}}\,(\mathcal{S}_{K_f}^G) \supset \Gamma_P\backslash X^P(c_P, r(c_P)) \subset \mathcal{S}_{K_f}^G$ then we get the neighbourhoods at infinity then our fibration looks like

$$\begin{gathered}\Gamma_P\backslash(U_P(\mathbb{R}) \rtimes (M^{(1)}(\mathbb{R})/K_\infty^M))(r_P) \times (c_P, 0))^{d(P)} \\ \downarrow r_1 \times \mathrm{Id} \\ \Gamma_M\backslash((M^{(1)}(\mathbb{R})/K_\infty^M))(r_P) \times (c_P, 0)^{d(P)}\end{gathered} \tag{6.33}$$

where ther r_P describes a relatively compact open subset in $(M^{(1)}(\mathbb{R})/K_\infty^M))$

which is homotopy equivalent to $\Gamma_M\backslash(M^{(1)}(\mathbb{R})/K_\infty^M)$. We may take the closure of this open subset in $\partial(\mathcal{S}_{K_f}^G)$ and this means we allow the numbers $\alpha_i^P(m_\infty)$ to go to zero. our fibration extends to

$$\begin{array}{cc} \Gamma_P\backslash(U_P(\mathbb{R})\rtimes(M^{(1)}(\mathbb{R})/K_\infty^M))(r_P)\times(c_P,0])^{d(P)} & \subset \bar{\mathcal{S}}_{K_f}^G \\ \downarrow r_1\times\mathrm{Id} & \\ \Gamma_M\backslash((M^{(1)}(\mathbb{R})/K_\infty^M))(r_P)\times(c_P,0]^{d(P)} & \end{array} \tag{6.34}$$

The intersection $\Gamma_P\backslash(U_P(\mathbb{R})\rtimes(M^{(1)}(\mathbb{R})/K_\infty^M))(r_P)\times(c_P,0])^{d(P)}\cap\partial(\bar{\mathcal{S}}_{K_f}^G$ consists of those points where some of the coordinates in $(c_P,0])^{d(P)}$ are equal to zero. More precisely we can say that for a parabolic $P_1\supset P$ a point $\bar{x}\in\Gamma_P\backslash(U_P(\mathbb{R})\rtimes(M^{(1)}(\mathbb{R})/K_\infty^M))(r_P)\times(c_P,0]^{d(P)}$ lies in $\partial_{P_1}(\bar{\mathcal{S}}_{K_f}^G)$ if and only if exactly those coordinates in $(c_P,0]^{d(P)}$ are zero for which $\alpha_i\in\pi_G\setminus\pi_1$.

Of course eventually we put $\mathcal{N}(\mathcal{S}_{K_f}^G)=\overset{\bullet}{\mathcal{N}}(\mathcal{S}_{K_f}^G)\cup\partial(\mathcal{S}_{K_f}^G)$

6.1.5 The easiest but very important example

If we take for instance $\mathcal{G}/\mathbb{Z}=\mathrm{Gl}_2/\mathbb{Z}$ and if we pick an integer N then we can define the congruence subgroup $K_f(N)=\prod_p K_p(N)\subset\mathcal{G}(\hat{\mathbb{Z}})$. It is defined by the condition that at all primes p dividing N the subgroup

$$K_p(N)=\{\gamma\in\mathcal{G}(\hat{\mathbb{Z}})|\gamma\equiv\mathrm{Id}\mod p^{n_p}\}$$

where of course p^{n_p} is the exact power of p dividing N. At the other primes we take the full group of integral points. For the discussion of the example we put $K_f(N)=K_f$. As usual let $G/\mathbb{Q}$ be the generic fiber.

If we consider the action of $G(\mathbb{Q})$ on $G(\mathbb{A}_f)/K_f$ then the determinant gives us a map

$$G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f\to\mathbb{G}_m(\mathbb{A}_f)/\mathbb{Q}^*\mathfrak{U}_N$$

where $\mathfrak{U}_N$ is the group of unit ideles in $I_{\mathbb{Q},f}=\mathbb{G}_m(\mathbb{A}_f)$ which satisfy $u_p\equiv 1\mod p^{n_p}$. This map is a bijection as one can easily see from strong approximation in Sl_2, and the right hand side is equal to $(\mathbb{Z}/N\mathbb{Z})^*/\{\pm1\}$. At the infinite place we have that our symmetric space has two connected components, we have

$$X=\mathrm{Gl}_2(\mathbb{R})/SO(2)=\mathbb{C}\setminus\mathbb{R}=\mathbb{H}_+\cup\mathbb{H}_-$$

where $\mathbb{H}_\pm$ are the upper and lower half plane, respectively. We have a complex structure on X which is invariant under the action of $G(\mathbb{R})$ and we consider the space $\mathcal{S}_{K_f}^G=G(\mathbb{Q})\backslash X\times G(/A_f)/K_f$. The fibers of the map

$$\mathcal{S}_{K_f}^G\to\pi_0(X)\times\mathbb{G}_m(\mathbb{A}_f)/\mathbb{Q}^*\mathfrak{U}_N \tag{6.35}$$

are exactly the connected components of $\mathcal{S}_{K_f}^G$ and these components are

$$\Gamma(N)\backslash\begin{pmatrix}t_\infty & 0\\ 0 & 1\end{pmatrix}H_+\times\begin{pmatrix}t_f & 0\\ 0 & 1\end{pmatrix}K_f(N) \tag{6.36}$$

where $\underline{t}$ runs through a set of representatives of $I_{\mathbb{Q}}/\mathbb{Q}^*\mathbb{R}^*_{>0}\mathfrak{U}_N = (\mathbb{Z}/N\mathbb{Z})^*$. and where of course $\Gamma(N) \subset \mathrm{Sl}_2(\mathbb{Z})$ is the full congrunce subgroup modulo N.

These connected components are Riemann surfaces which are not compact. They can be compactified by adding a finite number of points, the so called *cusps*. These are in one to one correspondence with the orbits of $\Gamma(N)$ on $\mathbb{P}^1(\mathbb{Q})$.

If now P is the parabolic subgroup corresponding to $r = \frac{p}{q} \in \mathbb{P}^1(\mathbb{Q})$ then the $X^P(c_P, r(c_P))$ are small Farey circles touching the real axis in the points r. (See section 1.2.6)

6.2 The sheaves and their cohomology

6.2.1 Basic data and simple properties

Let $\mathcal{M}_{\mathbb{Q}}$ be a finite dimensional $\mathbb{Q}$-vector space, let

$$r : G/\mathbb{Q} \to \mathrm{Gl}(\mathcal{M}_{\mathbb{Q}})$$

a rational representation. This representation r provides a sheaf $\tilde{\mathcal{M}}$ on $\mathcal{S}^G_{K_f}$ whose sections on an open subset $V \subset \mathcal{S}^G_{K_f}$ are given by

$$\tilde{\mathcal{M}}_{\mathbb{Q}}(V) = \{s : \pi^{-1}(V) \to \tilde{\mathcal{M}}_{\mathbb{Q}} | s \text{ locally constant and } s(\gamma v) = r(\gamma)s(v), \gamma \in G(\mathbb{Q})\}.$$

We call this the *right module description* of $\tilde{\mathcal{M}}_{\mathbb{Q}}$.

We can describe the stalk of the sheaf in a point $y \in \mathcal{S}^G_{K_f}$, we choose a point $\underline{x} = (x_\infty, \underline{g}_f)$ in $\pi^{-1}(y)$ and we choose a neighbourhood V_y as in 1.2.1. Then we can evaluate an element $s \in \tilde{\mathcal{M}}_{\mathbb{Q}}(V_y)$ at $\underline{x}$ and this must be an element in $\mathcal{M}^{\Gamma_{x_\infty}}$, this means we get an isomorphism

$$e_{\underline{x}} : (\tilde{\mathcal{M}}_{\mathbb{Q}})_y \xrightarrow{\sim} \mathcal{M}_{\mathbb{Q}}^{\Gamma_{x_\infty}}.$$

By definition we have $e_{\gamma\underline{x}} = \gamma e_{\underline{x}}$.

In our previous example such a representation r is of the following form: We take the homogeneous polynomials $P(X,Y)$ of degree n in two variables and with coefficients in $\mathbb{Q}$. This is a $\mathbb{Q}$-vector space of dimension $n+1$, we choose another integer m and now we define an action of $\mathrm{Gl}_2/\mathbb{Q}$ on this vector space

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} P(X,Y) = P(aX + cY, bX + dY)\det(\begin{pmatrix} a & b \\ c & d \end{pmatrix})^m.$$

This Gl_2 module will be called $\mathcal{M}_n[m]_{\mathbb{Q}}$ and it yields sheaves $\tilde{\mathcal{M}}_n[m]_{\mathbb{Q}}$ on our space $\mathcal{S}^G_{K_f}$.

Integral coefficient systems

We assume again that we have a rational representation of our group $G/\mathbb{Q}$, the following considerations easily generalize to the case of an arbitrary number field as base field. We want to define a subsheaf $\tilde{\mathcal{M}}_{\mathbb{Z}} \subset \tilde{\mathcal{M}}_{\mathbb{Q}}$. To do this we embed the field $\mathbb{Q} \hookrightarrow \mathbb{A}_f$ and we consider the resulting sheaf of $\mathbb{A}_f$-modules $\tilde{\mathcal{M}} \otimes \mathbb{A}_f$. We consider the diagram

$$\begin{array}{ccccc} & & G(\mathbb{R})/K_\infty \times (G(\mathbb{A}_f)/K_f) & & \\ & \overset{\pi'}{\nearrow} & & \overset{\pi}{\searrow} & \\ G(\mathbb{R})/K_\infty \times G(\mathbb{A}_f) & & \xrightarrow{\qquad\qquad\Pi\qquad\qquad} & & \mathcal{S}^G_{K_f} \\ & \underset{\Pi_1}{\searrow} & & \underset{\Pi_2}{\nearrow} & \\ & & G(\mathbb{Q})\backslash G(\mathbb{R})/K_\infty \times G(\mathbb{A}_f) & & \end{array} \tag{6.37}$$

this means that the division by the action by K_f on the right and by $G(\mathbb{Q})$ on the left (this gives Π) is divided into two steps: In the lower diagram the projection Π_1 is division by the action of $G(\mathbb{Q})$ and then Π_2 gives the division by the action of K_f on the right.

The sheaf $\tilde{\mathcal{M}}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{A}_f$ can be rewritten. For any open subset $V \subset \mathcal{S}^G_{K_f}$ we consider $W = \Pi^{-1}(V)$ and by definition

$$\tilde{\mathcal{M}}_{\mathbb{Q}} \otimes \mathbb{A}_f(V) = \{s : \Pi^{-1}(W) \to \mathcal{M}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{A}_f | s(\gamma(x_\infty, \underline{g}_f \underline{k}_f)) = \gamma(s(x_\infty, \underline{g}_f)),$$

where these sections s are locally constant in the variable x_∞. For any $s \in \mathcal{M} \otimes \mathbb{A}_f(V)$ we define a map $\tilde{s} : W \to \mathcal{M} \otimes \mathbb{A}_f$ by the formula

$$\tilde{s}(x_\infty, \underline{g}_f) = \underline{g}_f^{-1} s(x_\infty, \underline{g}_f \underline{K}_f),$$

this makes sense because $\mathcal{M} \otimes \mathbb{A}_f$ is a $G(\mathbb{A}_f)-$ module. For $\gamma \in G(\mathbb{Q})$ we have $\tilde{s}(\gamma(x_\infty, \underline{g}_f)) = \tilde{s}((x_\infty, \underline{g}_f))$ hence we can view $\tilde{s}$ as a map

$$\tilde{s} : G(\mathbb{Q})\backslash G(\mathbb{R})/K_\infty \times G(\mathbb{A}_f) \to \mathcal{M} \otimes_{\mathbb{Q}} \mathbb{A}_f.$$

We consider the projection

$$\Pi_2 : G(\mathbb{Q})\backslash G(\mathbb{R})/K_\infty \times G(\mathbb{A}_f) \to G(\mathbb{Q})\backslash G(\mathbb{R})/K_\infty \times G(\mathbb{A}_f)/K_f = \mathcal{S}^G_{K_f}$$

and then it becomes clear that $\tilde{\mathcal{M}} \otimes \mathbb{A}_f$ can be described as

$$\widetilde{\mathcal{M} \otimes \mathbb{A}}_f(V) = \{\tilde{s} : (\Pi_1^{-1}(V) \to \mathcal{M} \otimes_{\mathbb{Q}} \mathbb{A}_f |$$
$$\tilde{s} \text{ locally constant in } x_\infty \text{ and } \tilde{s}((x_\infty, \underline{g}_f \underline{k}_f)) = \underline{k}_f^{-1} \tilde{s}((x_\infty, \underline{g}_f))\}.$$

Hence we have identified the sheaf $\tilde{\mathcal{M}} \otimes_{\mathbb{Q}} \mathbb{A}_f$ which is defined in terms of the action of $G(\mathbb{Q})$ on $\mathcal{M}$ to the sheaf $\widetilde{\mathcal{M} \otimes_{\mathbb{Q}} \mathbb{A}_f}$ which is defined in terms of the action of K_f on $\mathcal{M} \otimes_{\mathbb{Q}} \mathbb{A}_f$.

Now we assume that our group scheme $G/\mathbb{Q}$ is the generic fiber of a flat group scheme $\mathcal{G}/\operatorname{Spec}(\mathbb{Z})$ (See 1.2). We choose our maximal compact subgroup $K_f = \prod_p K_p$ such that $K_p \subset \mathbb{G}(\mathbb{Z}_p)$ and with equality for all primes outside a finite set Σ. We can extend the vector space $\mathcal{M}$ to a free $\mathbb{Z}$ module $\tilde{\mathcal{M}}_{\mathbb{Z}}$ of the same rank which provides a representation $\mathcal{G}/\operatorname{Spec}(\mathbb{Z}) \to \mathrm{Gl}(\mathcal{M}_{\mathbb{Z}})$.

As usual $\hat{\mathbb{Z}}$ will be the ring of integral adeles. Then it is clear that $\mathcal{M}_{\mathbb{Z}} \otimes \hat{\mathbb{Z}} \subset \mathcal{M} \otimes_{\mathbb{Q}} \mathbb{A}_f$ is invariant under K_f and hence we can define the sub sheaf

$$\widetilde{\mathcal{M}_{\mathbb{Z}} \otimes \hat{\mathbb{Z}}} \subset \widetilde{\mathcal{M} \otimes_{\mathbb{Q}} \mathbb{A}_f},$$

this is the sheave where the sections $\tilde{s}$ have values in $\mathcal{M}_{\mathbb{Z}} \otimes \hat{\mathbb{Z}}$. We put

$$\tilde{\mathcal{M}}_{\mathbb{Z}} = \widetilde{\mathcal{M}_{\mathbb{Z}} \otimes \hat{\mathbb{Z}}} \cap \tilde{\mathcal{M}},$$

of course it depends on our choice of $\mathcal{M}_{\mathbb{Z}} \subset \mathcal{M}$. We get two exact sequences of sheaves

$$\begin{array}{ccccccc} 0 & \to & \tilde{\mathcal{M}}_{\mathbb{Z}} & \to & \tilde{\mathcal{M}} & \to & \widetilde{\mathcal{M} \otimes (\mathbb{Q}/\mathbb{Z})} \to 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & \widetilde{\mathcal{M} \otimes \hat{\mathbb{Z}}} & \to & \widetilde{\mathcal{M} \otimes_{\mathbb{Q}} \mathbb{A}_f} & \to & \widetilde{\mathcal{M} \otimes (\mathbb{A}_f/\hat{\mathbb{Z}})} \to 0 \end{array}$$

The far most vertical arrow to the right is an isomorphism, the inclusions $\mathbb{Z} \hookrightarrow \hat{\mathbb{Z}}$ and $\mathbb{Q} \hookrightarrow \mathbb{A}_f$ are flat. Writing down the resulting long exact sequences provides a diagram

$$\begin{array}{ccccc} \to & H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathbb{Z}}) & \xrightarrow{j_{\mathbb{Q}}} & H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}) & \to \\ & \downarrow i_{\mathbb{Z}} & & \downarrow i_{\mathbb{Q}} & . \\ \to & H^\bullet(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M} \otimes \hat{\mathbb{Z}}}) & \xrightarrow{j_{\mathbb{A}}} & H^\bullet(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M} \otimes_{\mathbb{Q}} \mathbb{A}_f}) & \to \end{array}$$

The above remarks imply that the vertical arrows are injective, the horizontal arrows in the middle have the same kernel and kokernel. This implies

Proposition 6.2.1. *The integral cohomology*

$$H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathbb{Z}})$$

consists of those elements in $H^\bullet(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M} \otimes \hat{\mathbb{Z}}})$ *which under* $j_{\mathbb{A}}$ *go to an element in the image under* $i_{\mathbb{Q}}$ *or in brief*

$$H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathbb{Z}}) = j_{\mathbb{A}}^{-1}(\mathrm{im}(i_{\mathbb{Q}}))$$

Highest weight modules

We assume that $G/\mathbb{Q}$ is a quasisplit group over $\mathbb{Q}$. This generalizes to the case where we have a representation $r : G \times F \to \mathrm{Gl}(\mathcal{M})$ where $\mathcal{M}$ is a vector space over F. If our group scheme is an extension of a flat group scheme $\mathcal{G}/\operatorname{Spec}(\mathcal{O}_F)$ then can find a lattice $\mathcal{M}_{\mathcal{O}_F}$ which yields a representation of $\mathcal{G} \to \mathrm{Gl}(\mathcal{M}_{\mathcal{O}_F})$. Then we can define the sheaf $\tilde{\mathcal{M}}_{\mathcal{O}_F}$ and define the cohomology groups

$$H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathcal{O}_F})$$

Sheaves with support conditions

We can extend the sheaves to the Borel-Serre compactification. We have the inclusion

$$i : \mathcal{S}^G_{K_f} \to \bar{\mathcal{S}}^G_{K_f}$$

and we can extend the sheaf by the direct image functor $i_*(\tilde{\mathcal{M}})$. It follows easily from the description of the neighbourhood of a point in the boundary (see **??**) that $R^q i_*(\mathcal{M}) = 0$ for $q = 0$ and hence we get that the restriction map

$$H^\bullet(\bar{\mathcal{S}}^G_{K_f}, i_*(\tilde{\mathcal{M}})) \to H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})$$

is an isomorphism.

We may also extend the sheaf by zero (See [Vol I], 4.7.1), this yields the sheaf $i_!(\tilde{\mathcal{M}})$ whose stalk at $x \in \mathcal{S}^G_{K_f}$ is equal to $\tilde{\mathcal{M}}_x$ and whose stalk ist zero in points $x \in \partial\mathcal{S}^G_{K_f}$. Then we have by definition

$$H^\bullet_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}) = H^\bullet(\bar{\mathcal{S}}^G_{K_f}, i_!(\tilde{\mathcal{M}}))$$

this is the cohomology with compact supports.

We are interested in the *integral* cohomology modules $H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\mathbb{Z}), H^\bullet_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\mathbb{Z})$. We introduced the boundary $\partial\mathcal{S}^G_{K_f}$ of the Borel-Serre compactification then we have a first general theorem, which is due to Raghunathan.

Raghunathan

Theorem 6.2.1. *(i) The cohomology groups* $H^i(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\mathbb{Z}), H^i(\partial\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\mathbb{Z})$ *and* $H^i_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\mathbb{Z})$ *are finitely generated.*

(ii) We have the well known **fundamental long exact sequence** *in cohomology*

$$\longrightarrow H^{i-1}(\partial\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\mathbb{Z}) \longrightarrow H^i_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\mathbb{Z}) \longrightarrow H^i(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\mathbb{Z}) \xrightarrow{r} H^i(\partial\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\mathbb{Z}) \longrightarrow .$$

We introduce the notation $H_?(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\mathbb{Z})$ meaning that for ? = blank we take the cohomology without support, for ? = c we take the cohomology with compact support and for ? = ∂ we take cohomology of the boundary of the Borel-Serre compactification. Later on we will also allow ? =! this denotes the inner cohomology. The above proposition 6.2.1 holds for all choices of ?.

Let $\Sigma = \{P_1, \dots, P_s\}$ be a finite set of parabolic subgroups, we assume that none of them is a subgroup of another parabolic subgroup in this set. The union of the closures of the strata

$$\bigcup_i \bigcup_{Q \subset P_i} \partial_Q(\mathcal{S}^G_{K_f}) = \partial_\Sigma(\mathcal{S}^G_{K_f})$$

is closed . We have the inclusions

$$j_\Sigma : \mathcal{S}^G_{K_f} \to \bar{\mathcal{S}}^G_{K_f} \setminus \partial_\Sigma(\bar{\mathcal{S}}^G_{K_f}), j^\Sigma : \bar{\mathcal{S}}^G_{K_f} \setminus \partial_\Sigma(\bar{\mathcal{S}}^G_{K_f}) \to \bar{\mathcal{S}}^G_{K_f}.$$

The inclusion $i : \mathcal{S}^G_{K_f} \to \bar{\mathcal{S}}^G_{K_f}$ is the composition $i = j^\Sigma \circ j_\Sigma$ we define the intermediate extension suppcond

$$i_{\Sigma,*,!}(\tilde{\mathcal{M}}) = j^\Sigma_! \circ j_{\Sigma,*}(\tilde{\mathcal{M}}), \tag{6.38}$$

this means that the stalk $i_{\Sigma,*,!}(\tilde{\mathcal{M}})_y$ at a point $y \in \partial_\Sigma(\bar{\mathcal{S}}^G_{K_f})$ is zero. Now we can define the cohomology with supports $H^\bullet(\mathcal{S}^G_{K_f}, i_{\Sigma,*,!}(\tilde{\mathcal{M}}))$. If $\Sigma = \emptyset$ then $H^\bullet(\Sigma,*,!(\tilde{\mathcal{M}})_y) = H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})$ and if Σ is the set of all maximal parabolic subgroups then $H^\bullet(\Sigma,*,!(\tilde{\mathcal{M}})_y) = H^\bullet({}_c\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})$.

For these cohomology groups coefficients in sheaves with intermediate support conditions we can also formulate assertion like the one in the above theorem. Hence we get filtrations on the cohomology

$$W_0H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}) = H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}) \subset W_1H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}) \subset \cdots \subset H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}) \tag{6.39}$$

on the cohomology, the bottom of this filtration will be the inner cohomology and the filtration steps will the images cohomology with intermediate supports.

Functorial properties

The groups have some functorial properties if we vary the level subgroup K_f. If we pass to a smaller open subgroup $K'_f \subset K_f$ then we get a surjective map

$$\pi_{K_f,K'_f} : \mathcal{S}^G_{K'_f} \to \mathcal{S}^G_{K_f},$$

whose fibers are finite. This induces maps between cohomology groups

$$\pi_{K'_f,K_f}{}^\bullet_? : H^\bullet_?(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\mathbb{Z}) \to H^\bullet_?(\mathcal{S}^G_{K'_f}, \tilde{\mathcal{M}}_\mathbb{Z}),$$

for $? = c$ we exploit the fact that the fibers are finite.

We construct homomorphisms in the opposite direction. We exploit the finiteness a second time and find that the direct image functor $(\pi_{K'_f,K_f})_*$ is exact and hence

$$H^\bullet_?(\mathcal{S}^G_{K'_f}, \tilde{\mathcal{M}}_\mathbb{Z}) = H^\bullet_?(\mathcal{S}^G_{K_f}, (\pi_{K'_f,K_f})_*(\tilde{\mathcal{M}}_\mathbb{Z})).$$

We define a trace homomorphism $(\pi_{K'_f,K_f})_*(\tilde{\mathcal{M}}_\mathbb{Z}) \to \tilde{\mathcal{M}}_\mathbb{Z}$: A section $s \in (\pi_{K'_f,K_f})_*(\tilde{\mathcal{M}}_\mathbb{Z})(V)$ is a map $\tilde{s} : \Pi^{-1}(V) \to \tilde{\mathcal{M}}_\lambda \otimes \hat{\mathbb{Z}}$ such that

$$\tilde{s}(\gamma(x_\infty, \underline{g}_f \underline{k}'_f)) = (k'_f)^{-1}\tilde{s}((x_\infty, \underline{g}_f)) \text{ for all } k'_f \in K'_f.$$

This is a section of $\tilde{\mathcal{M}}_\mathbb{Z}$ if and only if the corresponding section s takes values in $\mathcal{M}$. Then we define

$$\operatorname{tr}(\tilde{s})(x_\infty, \underline{g}_f) = \sum_{\underline{\xi}_f \in K_f/K'_f} \underline{\xi}_f^{-1}\tilde{s}(x_\infty, \underline{g}_f)$$

and this now satisfies

$$\operatorname{tr}(\tilde{s})(\gamma(x_\infty, \underline{g}_f \underline{k}_f)) = k_f^{-1}\tilde{s}((x_\infty, \underline{g}_f)) \text{ for all } k_f \in K_f.$$

and since the corresponding section $\operatorname{tr}(s)$ takes values in $\mathcal{M}$ we see that $\operatorname{tr}(\tilde{s}) \in \tilde{\mathcal{M}}_\mathbb{Z}(V)$.

Remark: It may happen that this trace map is not the optimal choice, it can be the integral multiple of another homomorphism between these two sheaves. This happens the intersection $C(\mathbb{Q}) \cap K_f$ is non trivial.

Then the homomorphism between the sheaves induces

$$H^\bullet_?(\mathcal{S}^G_{K'_f}, \tilde{\mathcal{M}}_\mathbb{Z}) = H^\bullet_?(\mathcal{S}^G_{K_f}, (\pi_{K'_f,K_f})_*(\tilde{\mathcal{M}}_\mathbb{Z})) \stackrel{\pi_{K'_f,K_f}\bullet}{\longrightarrow} H^\bullet_?(\mathcal{S}^G_{K_f}, (\tilde{\mathcal{M}}_\mathbb{Z})).$$

Later on our maps between the spaces will be denoted $\pi, \pi_1, \dots$ and the notation simplifies accordingly.

6.2.2 Rational systems of coefficients

We will decompose the cohomology into smaller pieces under the action of the Hecke-algebra. For this we have to pass to finite normal extension $F/\mathbb{Q}$. Then we should require that the G-modules $\mathcal{M}$ should be absolutely irreducible., but we also want them to be $\mathbb{Q}$-vector spaces. There is no problem to construct such modules if the semi simple component $G^{(1)}/\mathbb{Q}$ is split and the central torus satisfies a very mild condition. But we will show that we may also work with absolutely irreducible modules $\mathcal{M}$ which are defined over $F/\mathbb{Q}$, and if we keep track of the Galois- conjugate modules ${}^\sigma\mathcal{M}_\lambda$ then we still can formulate rationality statements over $\mathbb{Q}$.

We start again from a quasisplit reductive group scheme $G/\mathbb{Q}$, let $B/\mathbb{Q}$ a Borel subgroup and $T/\mathbb{Q} \subset B/\mathbb{Q}$ a maximal torus. Then we find a normal extension $F_0/\mathbb{Q}$ such that $T \times_\mathbb{Q} F_0$ is split If we choose F_0 minimal then $\mathrm{Gal}(F_0/\mathbb{Q})$ acts faithfully on $X^*(T \times_\mathbb{Q} F_0)$, it acts by permutations on the set of positive roots. To any dominant $\lambda \in X^*(T \times_\mathbb{Q} F_0)$ we can define the absolutely irreducible highest weight representation $\mathcal{M}_\lambda$. This representation is defined over the field $F_0[\lambda]$, where $\mathrm{Gal}(F_0/F_0[\lambda])$ is the stabiliser of λ. This means that $\mathcal{M}_\lambda$ is an F_0-vector space and the representation is a representation of $G \times_\mathbb{Q} F_0[\lambda]$. For any $\sigma \in \mathrm{Gal}(F_0/\mathbb{Q})$ we can consider this highest weight ${}^\sigma\lambda$ and the resulting highest weight module ${}^\sigma\mathcal{M}_\lambda = \mathcal{M}_{{}^\sigma\lambda}$. It follows from the construction of these modules that there is an obvious σ-linear map $\Phi_\sigma : \mathcal{M}_\lambda \to^\sigma \mathcal{M}_\lambda$. The map Φ_σ is a σ linear isomorphism between the modules, we have

$$\text{For } g \in G(F_0[\lambda]), m \in \mathcal{M}_\lambda \text{ we have } \Phi_\sigma(gm) = \sigma(g)\Phi_\sigma(m). \tag{6.40}$$

These semi-linear maps satisfy the cocycle relation

$${}^\sigma\Phi_\tau \circ \Phi_\sigma = \Phi_{\sigma\tau}. \tag{6.41}$$

Then we will call the collection

$$\{\dots, \mathcal{M}_{{}^\sigma\lambda}, \Phi_\sigma, \dots\}_{\sigma \in \mathrm{Gal}(F_0/\mathbb{Q})}$$

a *rational system of representations* (see also [41], 6.2.8).

From this rational system of representations we also get a rational system of sheaves $\{\dots, \tilde{\mathcal{M}}_{{}^\sigma\lambda}, \tilde{\Phi}_\sigma, \dots\}_{\sigma \in \mathrm{Gal}(F_0/\mathbb{Q})}$ and from this we get a rational system of cohomology groups

$$\{\dots, H^\bullet_?(\mathcal{S}^G_{K_f}, \mathcal{M}_{{}^\sigma\lambda}), \tilde{\Phi}^\bullet_\sigma, \dots\}_{\sigma \in \mathrm{Gal}(F_0/\mathbb{Q})}.$$

We can construct a flat extension $\mathcal{G}/\mathbb{Z}$, which is semi simple outside the set of primes which ramify in F_0. For any $\lambda \in X^*(T \times_{\mathbb{Q}} F_0)$ we can construct a locally free, finitely generated $\mathcal{G} \times_{\mathbb{Z}} \mathcal{O}_{F_0[\lambda]}$ module $\mathcal{M}_{\lambda,\mathcal{O}_{F_0}}$, such that after tensoring with $F_0[\lambda]$ we get $\mathcal{M}_\lambda$. This module is unique if we invert some finitely many primes. Then we can arrange these data such that the maps Φ_σ induce isomorphism $\Phi_\sigma : \mathcal{M}_{\lambda,\mathcal{O}_{F_0}} \xrightarrow{\sim} \mathcal{M}_{^\sigma\lambda,\mathcal{O}_{F_0}}$. The we may call the collection

$$\{\dots, \mathcal{M}_{^\sigma\lambda,\mathcal{O}_{F_0}}, \Phi_\sigma, \dots\}_{\sigma\in \mathrm{Gal}(F_0/\mathbb{Q})}$$

an *integral rational system of representations* .

Heckalg

6.3 The action of the Hecke-algebra

6.3.1 The action on rational cohomology

In this section we assume that our coefficient systems are obtained from rational representations of a reductive group scheme $G/\mathbb{Q}$ hence they are $\mathbb{Q}$ vector spaces. We consider the *rational* cohomology groups

$$H^i_?(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathbb{Q}}) = H^i_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathbb{Q}}), H^i(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathbb{Q}}), H^i(\partial(\mathcal{S}^G_{K_f}), \tilde{\mathcal{M}}_{\mathbb{Q}}),$$

These cohomology groups are finite dimensional $\mathbb{Q}$-vector spaces and they are related exact fundamental sequence. We can pass to the direct limit

$$H^i_?(\mathcal{S}^G, \tilde{\mathcal{M}}_{\mathbb{Q}}) = \lim_{K_f} H^i_?(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathbb{Q}}).$$

GAF

Proposition 6.3.1. *On these limits we have an action of the group $\pi_0(G(\mathbb{R}))\times G(\mathbb{A}_f)$. We recover the cohomology with fixed level K_f by taking the invariants ,under this action, i.e. we have*

$$H^i_?(\mathcal{S}^G, \tilde{\mathcal{M}}_{\mathbb{Q}})^{K_f} = H^i_?(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathbb{Q}})$$

To define this action we represent an element in $\pi_0(G(\mathbb{R}))$ by an element k_∞ in the in the normalizer of K_∞ in $G(\mathbb{R})$. An element $\underline{x} = (k_\infty, \underline{x}_f) \in G(\mathbb{R})\times G(\mathbb{A}_f))$ defines by multiplication from the right an isomorphism of spaces

$$m_{\underline{x}} : G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K_f \xrightarrow{\sim} G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/\underline{x}_f^{-1}K_f\underline{x}_f.$$

It is clear from the definition that $m_{\underline{x}}$ yields a bijection between the fibers $\pi^{-1}(\underline{\bar{g}}), \underline{\bar{g}} \in G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K_f$ and $\pi^{-1}(m_{\underline{x}})(\underline{\bar{g}})$ and since the sheaf is described in terms of the left action by $G(\mathbb{Q})$ we get $m_{\underline{x},*}(\tilde{\mathcal{M}}) = \tilde{\mathcal{M}}$. Passing to the limit gives us the action on $H^i_?(\mathcal{S}^G, \tilde{\mathcal{M}}_{\mathbb{Q}})$. The second assertion is obvious, but here we need that our coefficients are $\mathbb{Q}$ vector spaces, we need to take averages.

We introduce the notation $\tilde{G}(\mathbb{A}) := \pi_0(G(\mathbb{R})) \times G(\mathbb{A}_f)$ and then we denote this action by

$$\rho_{\tilde{\mathcal{M}}_{\mathbb{Q}}} : \tilde{G}(\mathbb{A}) \times H^i_?(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathbb{Q}}) \to H^i_?(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathbb{Q}}).$$

The interesting component of this representation is of course the action of the finite component $G(\mathbb{A}_f)$, it is simply the action which is induced by the right translation action of $G(\mathbb{A}_f)$ on $\mathcal{S}^G$.

Now we fix a level $K_f \subset G(\mathbb{A}_f)$ The Hecke algebra $\mathcal{H}_{K_f}$ consists of the compactly supported functions $h : G(\mathbb{A}_f) \to \mathbb{Q}$, which are biinvariant under the action of K_f, we also write $\mathcal{H}_{K_f} = \mathcal{C}_c(G(\mathbb{A}_f)//K_f, \mathbb{Q})$. An element $h \in \mathcal{H}_{K_f}$ is simply a finite linear combination of characteristic functions $h = \sum c_{\underline{a}_f} \chi_{K_f \underline{a}_f K_f}$ with rational coefficients $c_{\underline{a}_f}$. The algebra structure is given by convolution with respect to the Haar measure on $G(\mathbb{A}_f)$ which gives volume 1 to K_f. This convolution is given by

$$h_1 * h_2(\underline{g}_f) = \int_{G(\mathbb{A}_f)} h_1(\underline{x}_f) h_2(\underline{x}_f{}^{-1} \underline{g}_f) d\underline{x}_f.$$

With this choice of the measure it is clear that the characteristic function of K_f is the identity element of this algebra.

The action of the group $G(\mathbb{A}_f)$ induces an action of $\mathcal{H}_{K_f}$ on the cohomology with fixed level $H^i_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathbb{Q}}), H^i(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathbb{Q}}), \cdots$: For an element $v \in H^i_?(\mathcal{S}^G, \tilde{\mathcal{M}}_{\mathbb{Q}})$ we define

$$T_h(v) = \int_{G(\mathbb{A}_f)} h(\underline{x}_f) \underline{x}_f v d\underline{x}_f,$$

where the measure is still the one that gives volume 1 to K_f. Clearly we have $T_{h_1 * h_2} = T_{h_1} T_{h_2}$.

(Actually the integral is a finite sum: We find an open subgroup $K'_f \subset K_f$ such that v is fixed by K'_f and then it is clear that

$$T_h(v) = \int_{G(\mathbb{A}_f)} h(\underline{x}_f) \underline{x}_f v d\underline{x}_f = \frac{1}{[K_f : K'_f]} \sum_{\underline{a}_f} \sum_{\underline{\xi}_f \in G(\mathbb{A}_f)/K'_f} c_{\underline{a}_f} \chi_{K_f \underline{a}_f K_f)}(\underline{\xi}_f) \underline{\xi}_f v.$$

This makes it clear why we need rational coefficients .)

It is clear that $T_h(v) \in H^i_?(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})$ and hence T_h gives us an endomorphism of $H^i_?(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})$. We will show later that we also get endomorphisms on the cohomology of the boundary and therefore $\mathcal{H}$ also acts on the fundamental long exact sequence (Seq) .

If our function h is the characteristic function of a double coset $K_f \underline{x}_f K_f$ then we change notation and write $T_h = \mathbf{ch}(\underline{x}_f)$. We give another definition of the Hecke operator $\mathbf{ch}(\underline{x}_f)$ in terms of sheaf cohomology. We imitate the construction of the Hecke operators in Chapter 3, 3.1. We put $K_f^{(\underline{x}_f)} = K_f \cap \underline{x}_f K_f \underline{x}_f^{-1}$ and consider the diagram

$$\begin{array}{ccc} S^G_{K_f^{(\underline{x}_f)}} & \xrightarrow{m_{\underline{x}_f}} & S^G_{K_f^{(\underline{x}_f^{-1})}} \\ \searrow \pi_1 & & \swarrow \pi_2 \\ & \mathcal{S}^G_{K_f} & \end{array} \tag{6.42}$$

where $m_{\underline{x}_f}$ is induced by the multiplication by $\underline{x}_f$ from the right. This yields in cohomology

$$H^\bullet(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\mathbb{Q}) \xrightarrow{\pi_1^\bullet} H^\bullet(S^G_{K_f^{(\underline{x}_f)}},\tilde{\mathcal{M}}_\mathbb{Q}) \xrightarrow{m_{\underline{x}_f,*}} H^\bullet(S^G_{K_f^{(\underline{x}_f^{-1})}}, m_{\underline{x}_f,*}(\tilde{\mathcal{M}}_\mathbb{Q})) \quad (6.43)$$

Since we described the sheaf by the action of $G(\mathbb{Q})$ from the left and the map $m_{\underline{x}_f}$ by multiplication from the right we have $m_{\underline{x}_f,*}(\tilde{\mathcal{M}}_\mathbb{Q}) = \tilde{\mathcal{M}}_\mathbb{Q}$, this yields an isomorphism $i_{\underline{x}_f}$. Since π_2 is finite we have the trace homomorphism

$$\pi_{2,\bullet} : H^\bullet(S^G_{K_f^{(\underline{x}_f)^{-1}}},\tilde{\mathcal{M}}_\mathbb{Q}) \to H^\bullet(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\mathbb{Q})$$

and the composition is our Hecke operator

$$\pi_{2,\bullet}\circ i_{\underline{x}_f}\circ m_{\underline{x}_f,*}\circ \pi_1^\bullet = \mathbf{ch}(\underline{x}_f) : H^\bullet(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\mathbb{Q}) \to H^\bullet(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\mathbb{Q}).$$

This is simpler than the construction Chap.II 2.2. because we do not need the intermediate homomorphism u_α. But we do not get Hecke operators on the integral cohomology.

6.3.2 The integral cohomology as a module under the Hecke algebra

We resume the discussion of the integral Hecke algebra acting on $H^\bullet_?(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\mathbb{Z})$ from Chapter 3. Inside the Hecke algebra we may also look at the sub algebra of $\mathbb{Z}$ -valued functions. This is in principle the algebra which is generated by the characteristic functions $\mathbf{ch}(\underline{x}_f)$ of double cosets $K_f\underline{x}_fK_f$. These characteristic functions act by convolution on the cohomology $H^\bullet(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\mathbb{Q})$ but this does not induce an action on the integral cohomology. Our next aim is to define a fractional ideal $\mathfrak{n}(\underline{x}_f)\subset\mathbb{Q}$ or more generally $\mathfrak{n}(\underline{x}_f)\subset F$ such that for any $a\in\mathfrak{n}(\underline{x}_f)$ we can define an endomorphism

$$a\cdot\mathbf{ch}(\underline{x}_f) : H^\bullet(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\mathbb{Z}) \to H^\bullet(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\mathbb{Z}\ )$$

and if we send this to the rational cohomology then on $H^\bullet(\mathcal{S}^G_{K_f},\mathcal{M}_\mathbb{Q})$ this will be the convolution endomorphism induced by $\mathbf{ch}(\underline{x}_f)$ multiplied by a.

This ideal will depend on $\underline{x}_f$ and on λ and further down we compute it in special cases.

(iv) These endomorphisms $a\cdot\mathbf{ch}(\underline{x}_f)$ generate an algebra $\mathcal{H}^{(\lambda)}_\mathbb{Z}$ acting on the integral cohomology and the arrows in the fundamental exact sequence above commute with this action.

v) Moreover, we have an action of $\pi_0(G(\mathbb{R}))$ on the above sequence and this action also commutes with the action of the Hecke algebra. Hence we know that our above sequence is long exact sequence of $\pi_0(G(\mathbb{R}))\times\mathcal{H}^{(\lambda).}_\mathbb{Z}$

We come to the definition of the ideal.

If we are in the special case that our group has strong approximation then we have

$$\Gamma\backslash X \xrightarrow{\sim} G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K_f$$

(See (6.12)). We pick an element $\alpha \in G(\mathbb{Q})$. In Chap. 3 , 3.1. we defined the Hecke operator $T(\alpha, u_\alpha)$ where $u_\alpha : \mathcal{M}^{(\alpha)} \to \mathcal{M}$ is the canonical choice. Let us denote the image of α in $G(\mathbb{A}_f)$ by $\underline{\alpha}_f$. We just attached a Hecke operator to the double coset $K_f\underline{\alpha}_f.K_f$. We have the diagram of spaces

$$\begin{array}{ccc} \Gamma(\alpha)\backslash X & \longrightarrow & G(\mathbb{Q})\backslash G(\mathbb{R})/K_\infty \times G(\mathbb{A}_f)/K_f^{\underline{\alpha}_f} \\ \Big\downarrow l(\alpha^{-1}) & & \Big\downarrow r(\underline{\alpha}_f) \\ \Gamma(\alpha^{-1})\backslash X & \longrightarrow & G(\mathbb{Q})\backslash G(\mathbb{R})/K_\infty \times G(\mathbb{A}_f)/K_f^{\underline{\alpha}_f^{-1}} \end{array} \qquad (6.44)$$

Here the horizontal arrows are the isomorphisms provided by strong approximation, the arrow $l(\alpha^{-1})$ is the isomorphism induced by left multiplication by α^{-1} and $r(\underline{\alpha}_f)$ by right multiplication by $\underline{\alpha}_f$. These two maps enter in the definition of the Hecke operators $T(\alpha^{-1}, u_{\alpha^{-1}})$ and $\mathbf{ch}(\underline{\alpha}_f)$ and a straightforward inspection of the sheaves yields

$$\mathbf{ch}(\underline{\alpha}_f) = T(\alpha^{-1}, u_{\alpha^{-1}}).$$

Hence we can conclude that under this assumption our newly defined Hecke operators coincide with the Hecke operators defined in Chap.3. This also tells us what we have to do if we want to define Hecke operators on integral cohomology.

Here we want to point out that we can interpret the above diagram as a correspondence, it defines a "finite valued" map from $T(\alpha_f) : "\Gamma\backslash X \to \Gamma\backslash X$.

To define the action of the Hecke algebra on the integral cohomology without the assumption of simple connectedness we have to translate their definition into the right module description. Then our sheaf $\widetilde{\mathcal{M} \otimes \mathbb{A}_f}$ is described by the action of K_f on $\mathcal{M} \otimes \mathbb{A}_f$ and this allows us to define the sub sheaf $\mathcal{M}_\mathbb{Z} \otimes \hat{\mathbb{Z}}$. We look at the same diagram. But now the sheaf $m_{\underline{x}_f,*}(\widetilde{\mathcal{M} \otimes \mathbb{A}_f})$ is the sheaf described by the $K_f^{(\underline{x}_f)^{-1}}$ module $(\mathcal{M} \otimes \mathbb{A}_f)^{(\underline{x}_f)}$. This module is $\mathcal{M} \otimes \mathbb{A}_f$ as abelian group, but $\underline{g}_f \in K_f^{(\underline{x}_f)^{-1}}$ acts by $\underline{m}_f \mapsto \underline{x}_f \underline{g} \underline{x}_f^{-1} \underline{m}_f$. The map $\underline{m}_f \to \underline{x}_f \underline{m}_f$ induces an isomorphism $[\underline{x}_f]$ between the two $K_f^{(\underline{x}_f)^{-1}}$ modules $(\mathcal{M} \otimes \mathbb{A}_f)^{(\underline{x}_f)}$ and $(\mathcal{M} \otimes \mathbb{A}_f)$. We now consider the diagram (6.42) and replace in the sequence of maps the homomorphism $i_{\underline{x}_f}$ by the map $[\underline{x}_f^\bullet]$ induced by the isomorphism $[\underline{x}_f]$ between the sheaves. Then we can proceed as before and get an operator

$$p_{1,*} \circ [\underline{x}_f]^\bullet \circ m_{\underline{x}_f,*} \circ p_2^* = \mathbf{ch}(\underline{x}_f).$$

It is straightforward to check that this operator is an extension $\pi_{2,\bullet} \circ i_{\underline{x}_f} \circ m_{\underline{x}_f,*} \circ \pi_1^\bullet$ to $H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}} \otimes \mathbb{A}_f)$.

Our right module sheaf contains the submodule sheaf $\tilde{\mathcal{M}}_\lambda \otimes \hat{\mathbb{Z}}$, we can write the same diagram but now it can happen that $[\underline{x}_f]$ does not map $\mathcal{M}_\mathbb{Z} \otimes \hat{\mathbb{Z}}$ into itself. This forces us to make the following definition

$$\mathfrak{n}(\underline{x}_f) = \{a \in \mathbb{Q} |\ [a\underline{x}_f] : \mathcal{M}_\mathbb{Z} \otimes \hat{\mathbb{Z}} \subset \mathcal{M}_\mathbb{Z} \otimes \hat{\mathbb{Z}}\}$$

Then we can again go back to our above diagram and it becomes clear that we can define Hecke operators

$$a \cdot \mathbf{ch}(\underline{x}_f) : H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\mathbb{Z}) \to H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\mathbb{Z}) \text{ for all } a \in \mathfrak{n}(\underline{x}_f). \tag{6.45}$$

It is clear that the same construction also works for the cohomology with compact supports. But of course we also can define the action of the Hecke algebra on the cohomology of the boundary. We recall that for a suitable punctured tubular neighbourhood we have $H^i(\partial(\mathcal{S}^G_{K_f}), \tilde{\mathcal{M}}_\mathbb{Z}) = H^i(\overset{\bullet}{\mathcal{N}}\ (\mathcal{S}^G_{K_f}), \tilde{\mathcal{M}}_\mathbb{Z})$. If we take this tubular neighbourhood sufficiently small then we can find another slightly larger tubular neighbourhood $\overset{\bullet}{\mathcal{N}}_1\ (\mathcal{S}^G_{K_f})$, such that the "multivalued map" $T(\alpha_f)$ restricted to $\overset{\bullet}{\mathcal{N}}\ (\mathcal{S}^G_{K_f})$ values in $\overset{\bullet}{\mathcal{N}}_1\ (\mathcal{S}^G_{K_f})$. Since now $H^i(\overset{\bullet}{\mathcal{N}}_1\ \mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\mathbb{Z})$ is also isomorphic to the cohomology of the boundary we have defined an action of the Hecke algebra on the cohomology of the boundary. It is clear that the fundamental long exact sequence is an exact sequence of Hecke modules.

The case of a split group

We want to discuss this in the special case that $\mathcal{G}/\operatorname{Spec}(\mathbb{Z})$ is split reductive, we assume that the derived group $\mathcal{G}^{(1)}/\operatorname{Spec}(\mathbb{Z})$ is simply connected, we assume that the center $\mathcal{C}/\operatorname{Spec}(\mathbb{Z})$ is a (split)-torus and that $\mathcal{C} \cap \mathcal{G}^{(1)}$ is equal to the center $Z^{(1)}$ of $\mathcal{G}^{(1)}$. This center is a finite multiplicative group scheme (See 6.1.1).

Accordingly we get decompositions up to isogeny of the character and cocharacter modules of the torus

$$X^*(\mathcal{T}) \hookrightarrow X^*(\mathcal{T}^{(1)}) \oplus X^*(\mathcal{C})\ X_*(\mathcal{T}^{(1)}) \oplus X_*(\mathcal{C}) \hookrightarrow X_*(\mathcal{T}) \tag{6.46}$$

they become isomorphisms after taking the tensor product by $\mathbb{Q}$. We numerate the simple positive roots $I = \{1, 2, \dots, r\} = \{\alpha_1, \alpha_2, \dots, \alpha_r\} \subset X^*(\mathcal{T})$ and we define dominant fundamental weights $\gamma_i \in X^*(\mathcal{T})_\mathbb{Q}$ which restricted to $\mathcal{T}^{(1)}$ are the usual fundamental dominant weights and restricted to $\mathcal{C}$ are trivial. Then a dominant weight can be written as

$$\lambda = \sum_{i \in I} a_i \gamma_i + \delta = \lambda^{(1)} + \delta, \tag{6.47}$$

where $\delta \in X^*(\mathcal{C})$ and we must have the congruence condition

$$(\lambda^{(1)} + \delta)|Z^{(1)} = 1 \tag{6.48}$$

We can construct a highest weight module $\mathcal{M}_{\lambda,\mathbb{Z}}$. We pick a prime p, we assume that is unramified (with respect to K_f), this means that $K_p = \mathcal{G}(\mathbb{Z}_p)$.

Any element $t_p \in T(\mathbb{Q}_p)$ defines a double coset $K_p t_p K_p$. Of course only the image of t_p in $T(\mathbb{Q}_p)/\mathcal{T}(\mathbb{Z}_p)$ matters and

$$T(\mathbb{Q}_p)/\mathcal{T}(\mathbb{Z}_p) = X_*(T)$$

we find $\chi \in X_*(T)$ such that $\chi(p) = t_p$. We take a χ in the positive chamber, i.e. we assume $<\chi, \alpha> \geq 0$ for all α. We can produce the element

$$\underline{\chi}_p = (1 \dots, 1, \dots, \chi(p), 1 \dots, 1, \dots) \in T(\mathbb{A}_f)$$

and the Hecke operator

$$\mathcal{H}(\underline{\chi}_p) : H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}} \otimes \mathbb{Q}) \to H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}} \otimes \mathbb{Q})$$

We have to look at the ideal of those integers a for which

$$a\, \mathcal{H}(\underline{\chi}_p)(\mathcal{M}_{\lambda,\mathbb{Z}} \otimes \mathbb{Z}_p) \subset (\mathcal{M}_{\lambda,\mathbb{Z}} \otimes \mathbb{Z}_p).$$

This is easy: We have the decomposition into weight spaces

$$\mathcal{M}_{\lambda,\mathbb{Z}} = \oplus_\mu \mathcal{M}_{\lambda,\mathbb{Z}}(\mu)$$

and on a weight space the torus element $\mathcal{H}(\underline{\chi}_p)$ acts by

$$\mathcal{H}(\underline{\chi}_p) x_\mu = p^{<\chi,\mu>} x_\mu.$$

We get the smallest exponent if we choose for μ, the lowest weight vector. We denote by w_0 the longest element in the Weyl group, which sends all the positive roots into negative roots. The element $-w_0$ induces an involution $i \to i'$ on the set of simple roots. Then this lowest weight vector is

$$\lambda_- = w_0(\lambda) = -\sum a_{i'} \gamma_i + \delta. \tag{6.49}$$

We say that our weight is (essentially) *self dual* if we have $a_i = a_{i'}$. In this case $\lambda_- = -\lambda^{(1)} + \delta$.

Hence we see that our ideal is the principal ideal is given by

$$(p^{-<\chi, w_0\lambda^{(1)}> - <\chi,\delta>}) \text{ or if } \lambda \text{ self dual } (p^{<\chi,\lambda^{(1)}> - <\chi,\delta>}). \tag{6.50}$$

Hence we have defined the Hecke operator

$$T^{\mathrm{coh},\lambda}_{p,\chi} = p^{-<\chi,w_0\lambda> - <\chi,\delta>} \cdot \mathcal{H}(\underline{\chi}_p) : H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,\mathbb{Z}}) \to H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,\mathbb{Z}}) \tag{6.51}$$

We introduce the notation $c(w,\lambda) :=< -\chi, w_0\lambda - <\chi,\delta>>$, the number $-<\chi, w_0\lambda^{(1)}>$ is the relevant contribution in the exponent (let us call this the semi-simple term), the second term $-<\chi,\delta>$ is a correction term (the abelian contribution) and it takes care of the central character. It only serves to fulfill a parity condition. We come back to this in section 7.1.3.

Modules of congruence origin and Hecke operators

We also can define an action of the Hecke algebra if the coefficient system is of congruence origin. Our assumptlions are as above and we consider a finitely generated $\mathcal{G}(\mathbb{Z}/N\mathbb{Z})-$ module $\mathcal{V}$. The finite group $\mathcal{G}(\mathbb{Z}/N\mathbb{Z})-$ is of course a quotient of $\mathcal{G}(\hat{\mathbb{Z}}) = K_f$ and hence we can define the sheaf $\tilde{\mathcal{V}}$ by the action of K_f. This is now a sheaf of congruence origin in the adelic context.

We consider the subalgebra $\mathcal{H}^{(N)}_{K_f} = \mathcal{C}_c(G(\mathbb{A}^{(N)}_f)//K^{(N)}_f, \mathbb{Z})$ where $\mathbb{A}^{(N)}_f$ is the partial adele ring where we take the restricted product over all primes not dividing N. If $\mathbb{Z}_{(N)} \subset \mathbb{Q}$ is the semi local ring of rational numbers which are integral at $p|N$ then we have a surjective homomorphism $\mathbb{Z}_{(N)} \to \mathbb{Z}/N\mathbb{Z}$. Hence we can view $\mathcal{V}$ as a $\mathcal{G}(\mathbb{Z}_{(N)})$ module.

We apply the usual procedure to construct a sheaf $\tilde{\mathcal{V}}$ on $\mathcal{S}^G_{K_f}$, but here $\mathcal{V}$ is not a $G(\mathbb{Q})$ module but a $K_f = \mathcal{G}(\hat{\mathbb{Z}})$ module. If we want to attach a Hecke operator T_h to the double coset $K_f \underline{x}_f K_f$ with $\underline{x}_f \in G(\mathbb{A}^{(N)})$ we have to define a map $\alpha_{\underline{x}_f} : m_{\underline{x}_f,*}(\tilde{\mathcal{V}}) \to \tilde{\mathcal{V}}$. Let $K_f(N)$ be the kernel of $\mathcal{G}(\hat{\mathbb{Z}}) \to \mathcal{G}(\mathbb{Z}/N\mathbb{Z})$. We have to make an assumption

The map $\pi_N : G(\mathbb{Z}_{(N)}) \to G(\mathbb{A}_f)/K_f(N)$ *is surjective*

(This assumption is certainly true if the group $G/\mathbb{Q}$ is semi simple and simply connected.) Our assumption says that we can find an $u_{\underline{x}_f} \in G(\mathbb{Z}_{(N)})$ with $\pi_N(u_{\underline{x}_f}) = \underline{x}_f$.

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6.3.3 Excursion: Finite dimensional $\mathcal{H}-$modules and representations.

In the following we start from a a flat group scheme $\mathcal{G}/\mathbb{Z}$, we assume that the generic fiber $G/\mathbb{Q}$ is reductive. Let $K_f = \prod_p K_p$. be an open compact subgroup in $\otimes' G(\mathbb{Q}_p)$ this means that for almost all primes p we have $K_p = \mathcal{G}(\mathbb{Z}_p)$ and $K_p \subset G(\mathbb{Q}_p)$ is open for all primes p. For any prime p let $\mathcal{C}_c(G(\mathbb{Q}_p)//K_p)$ the space of $\mathbb{Q}$ valued functions h on $G(\mathbb{Q}_p)$ which have compact support and which are biinvariant under K_p, i.e. $h(k_1 g k_2) = h(g)$. These functions form an algebra under convolution (See 6.3) and the characteristic function e_p of K_p is the identity element.

The Hecke algebra is the restricted tensor product

$$\mathcal{H} = \bigotimes_p{}' \mathcal{H}_p = \bigotimes_p{}' \mathcal{C}_c(G(\mathbb{Q}_p)//K_p)$$

As the notation indicates we take the tensor product over all finite primes. This tensor product has to be taken in a restricted sense: for an element of the form $h_f = \otimes h_p$ the local factor h_p is equal to the identity element e_p for almost all primes p (here e_p is the characteristic function of K_p). All other elements are finite linear combinations of elements of the form above. We have the obvious embedding

$$\mathcal{H}_p \hookrightarrow \mathcal{H} \text{ we simply send } h_p \mapsto \otimes \dots e_{p'} \otimes h_p \otimes e_{p'\prime}, \dots . \tag{6.52}$$

The subalgebras $\mathcal{H}_p$ commute with each other.

We say that prime p is unramified (with respect to K_f) if $\mathcal{G}\times_{\mathbb{Z}}\mathbb{Z}_p$ is reductive and $K_p = \mathcal{G}(\mathbb{Z}_p)$, At unramified primes the local factor $\mathcal{H}_p$ is commutative, its exact structure is given by the Satake isomorphism (See 6.3.5).

HWe define the ideal $I^!_{K_f}$ to be the kernel of the action on $H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,\mathbb{Q}})$,

then $\mathcal{H}/I^!_{K_f} = \mathcal{A}$ is a finite dimensional algebra. It is known- and will be proved later (8.1.8)- that $H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,\mathbb{Q}})$ is a semi simple module.

A central subalgebra

Let Σ be the set of ramified primes. For $p \notin \Sigma$ the algebra $\mathcal{H}_p$ is finitely generated, integral and commutative.

The subalgebra

$$\mathcal{H}^{(\Sigma)} = {\bigotimes}'_{p\notin\Sigma}\mathcal{H}_p \tag{6.53}$$

is commutative and lies in the centre, and therefore its image $\mathcal{H}^{(\Sigma)} \subset \mathcal{H}$ lies in the center of $\mathcal{H}$. Since $H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,\mathbb{Q}})$ is semi simple $\mathcal{H}^{(\Sigma)}$ is a direct sum of fields and hence we have orthogonal system of idempotents $\{e_i\}$ such that

$$\mathcal{H}^{(\Sigma)} = \bigoplus_i \mathcal{H}^{(\Sigma)} e_i$$

gives a decomposition of $\mathcal{H}^{(\Sigma)}$ into a direct sum of fields. Hence we get a decomposition into isotypical modules

$$H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,\mathbb{Q}}) = \bigoplus_i e_i H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,\mathbb{Q}}) \tag{6.54}$$

We can decompose further, if $F \subset \bar{\mathbb{Q}}$ is a finite normal extension which "contains " the field $\mathcal{H}^{(\Sigma)} e_i$ then decoFabs

$$e_i H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,F}) = \bigoplus_{\sigma:\mathcal{H}^{(\Sigma)}e_i\to F} e_i H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,F})[\sigma]. \tag{6.55}$$

The composition $\pi^{(\Sigma)}(e_i,\sigma) : \mathcal{H}^{(\Sigma)} \to e_i\mathcal{H}^{(\Sigma)} \xrightarrow{\sigma} F$ is a homomorphism $\mathcal{H}^{(\Sigma)} \xrightarrow{\pi^{(\Sigma)}} F$ and

$$e_i H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,F})[\sigma] = H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,F})(\pi^{(\Sigma)}(e_i,\sigma))$$

is the eigenspace with eigenvalue $\pi^{(\Sigma)} = \pi^{(\Sigma)}(e_i,\sigma)$.. Hence we can rewrite the decomposition (6.55) as

$$H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,F}) = \bigoplus_{\pi^{(\Sigma)}} H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,F})(\pi^{(\Sigma)}) \tag{6.56}$$

where now the set of $\{\pi^{(\Sigma)}\} = \{\pi(e_i,\sigma)\}$. We have seen earlier that $\pi^{(\Sigma)} = \prod_{p\notin\Sigma}\pi_p$, where $\pi_p : \mathcal{H}_p \to F$.

We change our notation slightly instead of $\mathrm{Spec}_{\mathcal{H}^{(\Sigma)}}(H^\bullet_!(\mathcal{S}^G_{K_f},\mathcal{M}_{\lambda,\mathbb{Q}}))$ we define the set

$$\mathrm{Coh}^{(\Sigma)}_!(\mathcal{S}^G_{K_f},\mathcal{M}_{\lambda,\mathbb{Q}})=\{\dots,e_i,\dots\}, \tag{6.57}$$

this is the set of isomorphism classes of irreducible $\mathcal{A}^{(\Sigma)}$ modules which occur non trivially in $H^\bullet_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\lambda,\bar{\mathbb{Q}}})$, and

$$\mathrm{Coh}^{(\Sigma)}_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\lambda,\bar{\mathbb{Q}}})=\{\dots,\pi^{(\Sigma)},\dots\} \tag{6.58}$$

this is the set of isomorphism classes of absolutely irreducible $\mathcal{A}^{(\Sigma)}$ modules which occur non trivially in $H^\bullet_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\lambda,\bar{\mathbb{Q}}})$. We have the projection map

$$\mathrm{Coh}^{(\Sigma)}_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\lambda,\bar{\mathbb{Q}}})\to\mathrm{Coh}^{(\Sigma)}_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\lambda,\mathbb{Q}});(e_i,\sigma)\mapsto e_i, \tag{6.59}$$

the fibers of this map are the orbits of the action of the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on $\mathrm{Coh}^{(\Sigma)}_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\lambda,\bar{\mathbb{Q}}})$.

We have a canonical way to realise an isomorphism type $\pi^{(\Sigma)}$, resp. π_p. We consider the subfield F resp.$F[p]\bar{\mathbb{Q}}$ which is generated by the values of $\pi^{(\Sigma)}$ resp. π_p. Then $H_{\pi^{(\Sigma)}}=F$ and $\mathcal{A}^{(\Sigma)}$ acts on F via $\pi^{(\Sigma)}$. Since F is a field we have a canonical generator, this is given by the element $1\in F$. We may do the same for π_p and define H_{π_p} this by definition is a one dimension vector space over $F[p]$. But if we remember that π_p is the local components of $\pi^{(\Sigma)}$ then we modify our definition and define $H_{\pi_p}=H_{\pi_p}\otimes_{F[p]}F$.

Then we can say that $H_{\pi^{(\Sigma)}}$ is the restricted tensor product

$$H_{\pi^{(\Sigma)}}=\bigotimes_{p\notin\Sigma}{}' H_{\pi_p}. \tag{6.60}$$

Here on the right hand side we only allow tensors $\otimes a_p\otimes\cdots\otimes a_q\otimes\dots$ where for almost all p^* the local factor $a_{p^*}=1$.

Most of the time we are only interested in the unramified part of the action of the Hecke algebra. But of course we may also consider the action of the entire Hecke-algebra $\mathcal{H}$. We define $\mathcal{H}_{(\Sigma)}=\prod_{p\in\Sigma}\mathcal{H}_p$, this algebra acts on $H^\bullet_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\lambda,F})$ and respects the decomposition (6.56). Hence we have to look at the action of $\mathcal{H}_{(\Sigma)}$ on $H^\bullet_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\lambda,F})(\pi^{(\Sigma)})$. We denote the analogous of the ideals $I^!_{K_f}$ by $J_p\subset\mathcal{H}_p$ and put $\mathcal{A}_p=\mathcal{H}_p/I_p$. Then an absolutely irreducible module for $\mathcal{H}_{(\Sigma)}$ is of the form $\bigotimes_{p\in\Sigma}V_{\pi_p}$ where V_{π_p} is an absolutely irreducible $\mathcal{A}_p$- module. The structure of these modules has been described in the previous section, they are standard irreducible modules over full matrix algebras with entries in an extension $L_1/\mathbb{Q}$. These matrix algebras are quotients of $\mathcal{A}_p\otimes L_1$ by a two sided ideal.

The finite dimensional $\mathcal{H}\otimes L_1$ modules form a category $\mathbf{Mod}_{\mathcal{H}\otimes L_1}$, this is not a set. We can define the set $[\mathbf{Mod}_{\mathcal{H}\otimes L_1}]$ of isomorphism classes. The elements in this set will be denoted by π_f. We introduce the same notation for the elements π_p in the set $[\mathbf{Mod}_{\mathcal{H}_p\otimes L_1}]$. If $\mathcal{H}_p$ is commutative and π_p is absolutely irreducible then π_p is a homomorphism $\pi_p:\mathcal{H}_p\to L_1$. In general

π_p is a quotient $\mathcal{H}_p \otimes L_1/J(\pi_p)$, where $J(\pi_p)$ is a two sided ideal such that $\mathcal{H}_p/J(\pi_p) = M_r(L_1)$, and π_p is the standard absolutely irreducible module over $\mathcal{H}_p/J(\pi_p)$. If we denote an (absolutely irreducible) $\mathcal{H}_p \otimes L_1$ -module by H_{π_p} then this means that the isomorphism class of this module is π_p. If we have an absolutely irreducible $\mathcal{H} \otimes L_1$ module which occurs in $H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,F})$ then its isomorphism type is

$$\pi_f = \prod_{p\in\Sigma} \pi_p \times \pi^{(\Sigma)} = \prod_p \pi_p$$

On the cohomology $H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})$ we still have the action of the group $\pi_0(G(\mathbb{R}))$, this action commutes with the action of the Hecke algebra. (See (6.3.9) This is an elementary abelian 2- group and we may decompose further according to characters $\epsilon : \pi_0(G(\mathbb{R})) \to \{\pm 1\}$. Hence we get finally that after choosing a suitable finite (normal) extension $F/\mathbb{Q}$ we have an isotypical decomposition

$$H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,F}) = \bigoplus_{\epsilon\times\pi_f} H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,F})(\epsilon \times \pi_f) \tag{6.61}$$

As before we denote by $\mathrm{Coh}_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,\bar{\mathbb{Q}}})$ the set of isomorphism classes of absolutely irreducible $\mathcal{H}$ modules. On this set we have an action of the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}, /\mathbb{Q})$, this action factors over $\mathrm{Gal}(F/\mathbb{Q})$. On the other hand $\mathrm{Gal}(F/\mathbb{Q})$ acts upom $H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,F})$ via the action on F and clearly for $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$ we have

$$\sigma(H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,F})(\epsilon \times \pi_f)) = H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,F})(\sigma(\epsilon) \times \sigma(\pi_f)) \tag{6.62}$$

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6.3.4 Representations and Hecke modules

For $p \in \Sigma$ the category of finite dimensional modules is complicated, since the Hecke algebra will not be commutative in general.

Let F be a field of characteristic zero, let V be an F-vector space. An admissible representation of the group $G(\mathbb{Q}_p)$ is an action of $G(\mathbb{Q}_p)$ on V which has the following two properties

(i) For any open compact subgroup $K_p \subset G(\mathbb{Q}_p)$ the space V^{K_p} of K_p invariant vectors is finite dimensional.

(ii) For any vector $v \in V$ we can find an open compact subgroup K_p so that $v \in V^{K_p}$ in other words $V = \lim_{K_p} V^{K_p}$.

An admissible $G(\mathbb{Q}_p)$ -module V is irreducible if it does not contain an invariant proper submodule.

It is clear that the vector spaces V^{K_p} are modules for the Hecke algebra $\mathcal{H}_{K_p}$. VKirr

Proposition 6.3.2. *If $V \neq (0)$ is a irreducible $G(\mathbb{Q}_p)$ modules, and if K_p is an open compact subgroup with $V^{K_p} \neq (0)$. Then V^{K_p} is an irreducible $\mathcal{H}_{K_p}$-module.*

Proof. To see this we take the identity element e_{K_p} in our Hecke algebra, it induces a projector on V and a decomposition

$$V = V^{K_p} \oplus V' = e_{K_p}V \oplus (1 - e_{K_p})V.$$

Let assume we have a proper $\mathcal{H}_{K_p}$-invariant submodule $W \subset V^{K_p}$ Now we convince ourselves that the $G(\mathbb{Q}_p)$-invariant subspace $\tilde{W}$ generated by the elements gw is a proper subspace. We compute the integral

$$\int_{K_p} kgwdk = \int_{K_p \times K_p} k_1 g k_2 w dk_2 dk_1.$$

The first integral gives us the projection to V^{K_p}, the second integral is the Hecke operator, hence the result is in W. We conclude that $e_{K_p}\tilde{W} \subset W$ and tis shows that $(0) \neq \tilde{W} \neq V$. □

Now it is not hard to see, that the assignment

$$V \to V^{K_p}$$

from irreducible admissble $G(\mathbb{Q}_p)$-modules with $V^{K_p} \neq (0)$ to finite dimensional irreducible $\mathcal{H}_{K_p}$-modules induces an bijection between the isomorphism classes of the respective types of modules. If we start from V^{K_p} we can reconstruct V by an appropriate form of induction.**Refenrenz Godement ??**

The dual module

Let us assume that V is a finite dimensional F-vector space with an action of the Hecke algebra $\mathcal{H}$ (we fix the level). We have an involution on the Hecke algebra which is defined by

$${}^t h(\underline{x}_f) = h(\underline{x}_f^{-1})$$

a simple calculation shows that ${}^t h_1 * {}^t h_2 ={}^t (h_2 * h_1)$.

This allows us to introduce a Hecke-module structure on $V^\vee = \mathrm{Hom}_F(V, F)$ we for $\phi \in V^\vee$we simply put

$$T_h(\phi)(v) = \phi(T_{{}^t h}(v))$$

for all $v \in V$.

Unitary and essentially unitary representations

Here it seems to be a good moment to recall the notion of unitary Hecke modules and unitary representations. In this book we make the convention that a character is a continuous homomorphism from a topological group $H \to \mathbb{C}^\times$, we do not require that its values have absolute value one. If this is the case we call the character unitary. Our ground field will now be $F = \mathbb{C}$, let V be a $\mathbb{C}$ vector space. We pick a prime p. We call a representation $\rho : G(\mathbb{Q}_p) \to \mathrm{Gl}(V)$ unitary if there is given a positive definite hermitian scalar product $< \, , \, > V \times V \to \mathbb{C}$ which is invariant under the action of $G(\mathbb{Q}_p)$.

If our representation is irreducible then it has a central character $\zeta_\rho : C(\mathbb{Q}_p) \to \mathbb{C}^\times$. In this case the scalar product is unique up to a scalar. A necessary condition for the existence of such a scalar product is that $|\zeta_\rho| = 1$, in other words ζ_ρ is unitary.

If this is not the case then our representation may still be *essentially unitary*: We have a unique homomorphism $|\zeta_\rho^*| : C'(\mathbb{Q}_p) \to \mathbb{R}_{>0}^\times$ whose restriction to $C(\mathbb{Q}_p)$ under d_C (see 1.1) is equal to $|\zeta_\rho|$. Then we may form the twisted representation $\rho^* = \rho \otimes |\zeta_\rho^*|^{-1}$. Then the central character of ρ^* is unitary. We say that σ is called essentially unitary if ρ^* is unitary.

If our representation is not irreducible we still can define the notion of being essential unitary. This means that there exists a homomorphism $|\zeta_\rho^*| : C'(\mathbb{Q}_p) \to \mathbb{R}_{>0}^\times$, such that the twisted representation $\rho^* = \rho \otimes |\zeta_\rho^*|^{-1}$ is unitary.

The same notions apply to modules for the Hecke algebra. A (finite dimensional) $\mathbb{C}$ vector space V with an action $\pi_p : \mathcal{H}_p \to \mathrm{End}(V)$ is called unitary, if there is given a positive definite scalar product $< \, , \, >: V \times V \to \mathbb{C}$ such that

$$< T_h(v), w >=< v, T_{{}^t h}(w) > \tag{6.63}$$

Recall that we always assume that our functions $h \in \mathcal{H}_p$ take values in $\mathbb{Q}$, hence we do not need a complex conjugation bar in the expression on the right.

The restriction of π_p to $C(\mathbb{Q}_p)$ in induces a homomorphism $\zeta_{\pi_p} : C(\mathbb{Q}_p) \to \mathbb{C}^\times$. We call π_p isobaric if this action of the center is semi simple - and therefore a direct sum of characters $\zeta_{\pi_p} = \sum \zeta_{\pi_p}^\nu$ - and if all these characters have the same absolute values $|\zeta_{\pi_p}^\nu| = |\zeta_{\pi_p}|$. This means that we can find $|\zeta_{\pi_p}^*|$ as above. Then we call π_p essentially unitary if the Hecke module $\pi_p^* = \pi_p \otimes |\zeta_{\pi_p}^*|^{-1}$ is unitary.

These boring considerations will be needed later, we will see that for an irreducible coefficient system $\mathcal{M}$ the $H_!^\bullet(\mathcal{S}_{K_f}^G, \tilde{\mathcal{M}}) \otimes \mathbb{C}$ is essentially unitary (see 8.1.7).

Abelian representations

Let us assume that the derived group $G^{(1)}(\mathbb{Q}_p)$ has non proper normal subgroup of finite index (this is true in most of the cases, for instance for $G = Gl_n/\mathbb{Q}_p$, then it is easy to see that a finite dimensional, admissible and absolutely irreducible representation is one dimensional and given by a character $\chi_p : G(\mathbb{Q}_p)/G^{(1)}(\mathbb{Q}_p) \to \mathbb{C}^\times$.

6.3.5 The Satake isomorphism

In the formulation of this theorem I will use the language of group schemes, the reader not so familiar with this language may think of Gl_n or the group of symplectic similitudes GSp_n. Since we assumed that for $p \notin \Sigma$ the integral structure $\mathcal{G}/\mathrm{Spec}(\mathbb{Z}_p)$ is reductive it is also quasisplit. We can find a Borel subgroup $\mathcal{B}/\mathrm{Spec}(\mathbb{Z}_p) \subset \mathcal{G}/\mathrm{Spec}(\mathbb{Z}_p)$ and a maximal torus $\mathcal{T}/\mathrm{Spec}(\mathbb{Z}_p) \subset \mathcal{B}/\mathrm{Spec}(\mathbb{Z}_p)$. Then our torus $\mathcal{T}/\mathrm{Spec}(\mathbb{Z}_p)$ splits over an unramified extension $E_p/\mathbb{Q}_p$ and the Galois group $\mathrm{Gal}(E_p/\mathbb{Q}_p)$ acts on the character module $X^*(\mathcal{T} \times E_p) = \mathrm{Hom}(\mathcal{T} \times E_p, \mathbb{G}_m)$. Let $\{\alpha_1, \alpha_2, \ldots, \alpha_r\} \subset X^*(\mathcal{T} \times E_p)$ be the set of positive simple roots, it is invariant under the action of the Galois group. Let $W(\mathbb{Z}_p)$ be the centraliser of the Galois action in the absolute Weyl group W. We introduce the module of unramified characters on the torus this is

$$\mathrm{Hom}_{\mathrm{unram}}(\mathcal{T}(\mathbb{Q}_p), \mathbb{C}^\times) = \mathrm{Hom}(\mathcal{T}(\mathbb{Q}_p)/\mathcal{T}(\mathbb{Z}_p), \mathbb{C}^\times) = \Lambda(\mathcal{T}). \tag{6.64}$$

Since we have $T(\mathbb{Q}_p) = B(\mathbb{Q}_p)/U(\mathbb{Q}_p)$ the character $\eta_p \in \Lambda(\mathcal{T})$ yields a character $\eta_p : B(\mathbb{Q}_p) \to \mathbb{C}^\times$. We write the module structure additively, i.e. $(\eta_{1,p} + \eta_{2,p})(x) = \eta_{1,p}(x)\eta_{2,p}(x)$.

The group of (rational) characters $\mathrm{Hom}(\mathcal{T}, \mathbb{G}_m) = X^*(T)^{\mathrm{Gal}(E_p/\mathbb{Q}_p)}$ is a subgroup of $\Lambda(T)$: An element $\gamma \in X^*(\mathcal{T})^{\mathrm{Gal}(E_p/\mathbb{Q}_p)}$ defines a homomorphism $\mathcal{T}(\mathbb{Q}_p) \to \mathbb{Q}_p^\times$ and this gives us the following element $x \mapsto |\gamma(x)|_p \in \Lambda(\mathcal{T})$ which we denote by $|\gamma|_p$. Here of course $|a|_p$ is the usual p-adic absolute value of $a \in \mathbb{Q}_p$. We can even do this for elements $\gamma \otimes \frac{1}{n} \in X^*(T) \otimes \mathbb{Q}$, then $\gamma \otimes \frac{1}{n}(x) = |\gamma(x)|_p^{1/n} \in \mathbb{R}_{>0}^\times$.

Our open compact subgroup will be $K_p = \mathcal{G}(\mathbb{Z}_p)$. Since we have the Iwasawa decomposition $G(\mathbb{Q}_p) = B(\mathbb{Q}_p)\mathcal{G}(\mathbb{Z}_p) = B(\mathbb{Q}_p)K_p$ we can attach to any $\eta_p \in \Lambda(\mathcal{T})$ a *spherical function*

$$\phi_{\eta_p}(g) = \phi_{\eta_p}(b_p k_p) = \eta_p(b_p) \tag{6.65}$$

We introduce the induced representation

$$\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\eta_p = \{f : G(\mathbb{Q}_p) \to \mathbb{C} | f(bg) = \eta_p(b)f(g)\} \tag{6.66}$$

(see 6.70) It is clear that $(\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\eta_p)^{K_p} = \mathbb{C}\phi_{\eta_p}$ We call these representations *spherical representations.*

Since $\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\eta_p$ is also a module for $\mathcal{H}_p$ it follows that spherical function is of course an eigenfunction for $\mathcal{H}_p$ for $h_p \in \mathcal{H}_p$ $\boxed{\text{FT1}}$

$$\int_{G(\mathbb{Q}_p)} \phi_{\eta_p}(gx)h_p(x)dx = \hat{h}_p(\eta_p)\phi_{\eta_p}(g) \tag{6.67}$$

and $\mathfrak{S}(\eta_p) : h_p \mapsto \hat{h}_p(\eta_p)$ is an algebra homomorphism from $\mathcal{H}_p$ to $\mathbb{C}$, hence $\mathfrak{S}(\eta_p) \in Hom_{\mathrm{alg}}(\mathcal{H}_p, \mathbb{C})$. Of course the measure dx gives volume 1 to $\mathcal{G}(\mathbb{Z}_p) = K_p$.

The subgroup $W(\mathbb{Z}_p)$ of the absolute Weyl group acts on $X^*(T)$ and hence on $\Lambda(T)$, we denote this action by $(w, \eta_p) \mapsto s\eta_p$. We also define the twisted action by

$$(w, \eta_p) \mapsto w \cdot \eta := (w\eta_p)(|w\rho - \rho|_p, \tag{6.68}$$

here $\rho \in X^*(T)$ is the half sum of positive roots.

The theorem of Satake asserts:

Theorem 6.3.1. *The map $\mathfrak{S}$ is invariant under the twisted action, i.e we have $\mathfrak{S}(w \cdot \eta_p)) = \mathfrak{S}(\eta_p)$ and*

$$\Lambda(\mathcal{T})/W(\mathbb{Z}_p) \xrightarrow{\mathfrak{S}} Hom_{\mathrm{alg}}(\mathcal{H}_p, \mathbb{C})$$

is an isomorphism.

The Hecke algebra is generated by the characteristic functions of double cosets $K_p t_p K_p$ where $t_p \in T(\mathbb{Q}_p)$ and where for all simple roots $\alpha \in \pi$ we have $|\alpha(t_p)|_p \leq 1$, i.e. $t_p \in T_+(\mathbb{Q}_p)$. Then the evaluation in (6.67) comes down to the computation the integrals

$$\int_{K_p t_p K_p} \phi_{\eta_p}(gx)dx = \hat{t}_p(\eta_p)\phi_{\eta_p}(g) \tag{6.69}$$

We discuss this evaluation in (7.1.2)

Admissible basis +ramified induced rpps

6.3.6 Spherical representations

Now we assume that Let $F \subset \mathbb{C}$ be a finite extension of $\mathbb{Q}$ and let V/F be a vector space. We choose $K_p = \mathcal{G}(\mathbb{Z}_p)$, i.e. p is unramified. An admissible representation (i.e. for any open subgroup K_p' the space $V^{K_p'}$ of invariants is finite dimensional and $V = \cup V^{K_p'}$)

$$\tilde{\pi}_p : G(\mathbb{Q}_p) \to \mathrm{Gl}(V)$$

is called *spherical* if $V^{K_p} \neq 0$, and this space is a module for the Hecke algebra. If the representation is absolutely irreducible, then it is well known (**Reference**) that $\dim_F V^{K_p} = 1$, this is a one dimensional module for $\mathcal{H}_{K_p}$, i.e. a homomorphism $\pi_p : \mathcal{H}_{K_p} \to F$. The $G(\mathbb{Q}_p)-$ module V is determined by the $\mathcal{H}_{K_p}$- module V^{K_p}. Then it is well known that we can find a finite normal extension F_1/F and an $\eta_p \in \mathrm{Hom}(\Lambda(\mathcal{T}), F_1^\times)$ such that $V \otimes F_1$ is isomorphic to a subquotient of the induced representation

indprinc

$$\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \eta_p = \{f : G(\mathbb{Q}_p) \to F_1 | f(bg) = \eta_p(b) f(g)\} \tag{6.70}$$

where f satisfies the (obvious) condition that there exists a finite index subgroup $K_p' \subset K_p$ such that f is invariant under right translations by elements $k' \in K_p'$. In general the induced representation will be irreducible and then it is isomorphic to the representation $V \otimes_F F_1$.

6.3.7 Intertwining operators

The theorem of Satake implies that the two Hecke modules ($\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \eta_p)^{K_p}$ and ($\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} w \cdot \eta_p)^{K_p}$ are isomorphic. We give a proof of this fact, since we need it later in Chapter 9 when we discuss the Eisenstein cohomology.

We only discuss the case that $G/\mathbb{Z}_p$ is split, at the end we say something how to modify the argument for quasisplit groups, We change our standpoint slightly, we introduce the field $F_1[[q]]$ of Laurent power series

$$F_1[[q]] := \{P(q) = \sum_{\nu \geq N}^{\infty} a_\nu q^\nu | a_\nu \in F_1\}.$$

For any character $\gamma \in X^*(T)$ we define a homomorphism $\gamma' \to Z$. It defined by the requirement that for any cocharacter $\chi \in X_*(T)$ we have the relation $\gamma(\chi(t)) = t^{<\chi,\gamma>}$ (See 1.1.2). Now we consider characters $\eta_p \otimes \gamma : T(\mathbb{Q}_p)/T(\mathbb{Z}_p) \to F_1[[q]]^\times$ which are given by $x \mapsto \eta_p(x)q^{\gamma'(x)}$. :

As before (6.70) we define the induced representation

$$\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\eta_p \otimes \gamma = \{f : G(\mathbb{Q}_p) \to F_1[[q]] \mid f(bg) = \eta_p(b)q^{\gamma'(b)}f(g)\} \tag{6.71}$$

The vector space $\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\eta_p \otimes \gamma$ can be identified to the vector space of $F_1[[q]]$ valued functions f on $G(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p) = \mathcal{B}(\mathbb{Z}_p)\backslash\mathcal{G}(\mathbb{Z}_p)$, which are invariant under right translations by elements of a suitable open sub group K'_f. (depending on f). Then it is clear that this space has a countable basis $f_0, f_1, \ldots$ consisting of F_1 valued functions which are invariant under smaller and smaller open compact subgroups. If $g \in G(\mathbb{Q}_p)$ we have $R_g(f_i) = \sum a_{i,j}f_f$ where only finitely many of the matrix coefficients $a_{i,j}$ are zero.

We analyse how an element $g \in G(\mathbb{Q}_p)$ acts on $\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\eta_p \otimes \gamma$. We know that $\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\eta_p \otimes \gamma = \cup_{K'_p}(\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\eta_p \otimes \gamma)^{K'_p}$ where K'_p runs over all open compact subgroups. Then the right translation by g maps

$$R_g : (\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\eta_p \otimes \gamma)^{K'_p} \to (\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\eta_p \otimes \gamma)^{g^{-1}K'_pg} \tag{6.72}$$

The functions in these induced modules are determined by their restriction to $K_p = \mathcal{G}(\mathbb{Z}_p)$. We can find a basis $\{f_0, f_1, f_2 \ldots, f_t\}$ given by functions $f_i : \mathcal{B}(\mathbb{Z}_p)\backslash\mathcal{G}(\mathbb{Z}_p)/K'_p \to F_1$ Then the right translation by an element $g \in G(\mathbb{Q}_p)$ is given by

$$R_g : f = \{x \mapsto f(x)\} \mapsto \{x \mapsto f(xg)\};\ \text{here } x \in \mathcal{G}(\mathbb{Z}_p) \tag{6.73}$$

then we use the Iwasawa decomposition and write

$$xg = b(xg)k(xg) = \eta_p(b(xg))q^{\gamma'(bx)}f(k(xg)).$$

We have $\mathcal{B}(\mathbb{Q}_p)\backslash\mathbb{G}(\mathbb{Q}_p) = \mathcal{B}(\mathbb{Z}_p)\backslash\mathcal{G}(\mathbb{Z}_p)$ and the right multiplication by g is given by $x \mapsto k(xg)$. If now choose a basis $f_0^*, f_1^*, f_2^*, \ldots, f_t^*$ as above for $(\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\eta_p \otimes \gamma)^{g^{-1}K'_pg}$ then it becomes clear that

$$R_g(f_i) = \sum a_{i,j}f_j^*$$

where the matrix coefficients $a_{i,j} \in F_1[q, q^{-1}]$.

Now we describe the well known process to write an explicit intertwining operator. This operator is discussed at many places in the literature, but there the basic field for the vector spaces is always $\mathbb{C}$. Here we are in an arithmetic context and our representations are defined over a number field F or over $F[[q]]$ and this requires some algebraic arguments. But in principles there is no essential change.

Under the assumption that γ is in the positive chamber (see below) our intertwining operator is given by an integral

$$T(w,\eta_p\otimes\gamma):\ \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\eta_p\otimes\gamma \overset{T(w,\eta_p\otimes\gamma)}{\longrightarrow} \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}w\cdot\eta_p\otimes w\cdot\gamma$$
$$f(g)\mapsto \int_{U^{(w)}(\mathbb{Q}_p)} f(wug)du \tag{6.74}$$

This needs some explanation. Here $U^{(w)}$ is the product of all the one parameter subgroups $U_\beta\subset U$ for which $w^{-1}\beta<0$. Then it is clear that for any $u_0\in U(\mathbb{Q}_p)$

$$\int_{U^{(w)}(\mathbb{Q}_p)} f(wuu_0g)du = \int_{U^{(w)}(\mathbb{Q}_p)} f(wug)du.$$

Moreover we see that for an element $t\in T(\mathbb{Q}_p)$

$$\int_{U^{(w)}(\mathbb{Q}_p)} f(wutg)du = \int_{U^{(w)}(\mathbb{Q}_p)} f(wtww^{-1}t^{-1}utg)du = w\eta_p\otimes\zeta(t) =$$

$$= \prod_{\beta\in\Delta^{(w)}}|\beta|_p(t)\int_{U^{(w)}(\mathbb{Q}_p)} f(wug)du$$

and it is rather obvious that the factor in front is $w\cdot(\eta_p\otimes\gamma)$. Hence we see that indeed the image of $T(w,\eta_p\otimes\gamma)$ lands in the right space.

We have to discuss the "convergence" of the integral. For this we consider the special case that $w=s_1$ the reflection at a positive simple root α_i Then the two unipotent groups $U_{\alpha_i}/\mathbb{Z}$ and $U_{-\alpha_i}/\mathbb{Z}$ generate a three dimensional semi-simple subgroup H_{α_i}. We have a surjective homomorphism $h_{\alpha_i}:\mathrm{Sl}_2/\mathbb{Z}\to H_{\alpha_i}$ which induces isomorphisms of $U_\pm/\mathbb{Z}\xrightarrow{\sim}U_{\pm\alpha_i}/\mathbb{Z}$. Then we can say

$$\int_{U_{\alpha_i}(\mathbb{Q}_p)} f(s_iu_{\alpha_i}g)du_{\alpha_i} = \int_{\mathbb{Q}_p} f(h_{\alpha_i}(\begin{pmatrix}0&1\\-1&1\end{pmatrix}\begin{pmatrix}1&u\\0&1\end{pmatrix})g)du \tag{6.75}$$

Since we assumed that f is right invariant under some open compact subgroup K_p' , i.e. $f(gk_p)=f(g)$ for $k_p\in K_p'$ we can an integer $m_0\geq 0$ (depending on g) such that $f(h_{\alpha_i}(\begin{pmatrix}0&1\\-1&1\end{pmatrix}\begin{pmatrix}1&u+v\\0&1\end{pmatrix})g)=f(h_{\alpha_i}(\begin{pmatrix}0&1\\-1&1\end{pmatrix}\begin{pmatrix}1&u\\0&1\end{pmatrix})g)$ for $v\in p^{m_0}$. Hence our integral becomes

$$\int_{\mathbb{Q}_p/p^{m_0}\mathbb{Z}_p} f(h_{\alpha_i}(\begin{pmatrix}0&1\\-1&1\end{pmatrix}\begin{pmatrix}1&u\\0&1\end{pmatrix})g)du =$$
$$\sum_{n>m_0}\int_{(p^{-n}\mathbb{Z}_p\setminus p^{-n+1}\mathbb{Z}_p)/p^{m_0}\mathbb{Z}_p} f(h_{\alpha_i}(\begin{pmatrix}0&1\\-1&1\end{pmatrix}\begin{pmatrix}1&u\\0&1\end{pmatrix})g)du \tag{6.76}$$

For each summand the integral is a finite sum. For $n>0$ we write $u=p^{-n}\varepsilon$ then

$$\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}1&u\\0&1\end{pmatrix}=\begin{pmatrix}p^n&v\\0&p^{-n}\end{pmatrix})k_p \text{ with } k_p\in\mathrm{Sl}_2(\mathbb{Z}_p).$$

We introduce the cocharacter $\alpha_i^\vee:t\mapsto h_{\alpha_i}(\begin{pmatrix}t&0\\0&t^{-1}\end{pmatrix})$ then

$$f(\alpha_i^\vee(p^n)k_p(\varepsilon)g)=\eta_p(\alpha_i^\vee(p))^n q^{n<\alpha_i^\vee,\gamma>}f(k_p(\varepsilon)g)$$

We assume that γ is in the positive chamber, i.e $< \alpha_i^\vee, \gamma >> 0$ then it becomes clear that our integral in (9.4.2) yields an honest Laurent power series in the variable q, and hence we see that the integral provides an intertwining operator in the case $w = s_{\alpha_i}$

We have a closer look at the case $f_0 = \phi_{\eta_p}$ In (9.4.2), we choose $g = 1$ and $m_0 = 0$. Then the integral simplifies to (the right hand side becomes)

$$1 + (1 - \frac{1}{p}) \sum_{n=1}^{\infty} p^n \eta_p(\alpha_i^\vee(p))^n q^{n<\alpha_i^\vee,\gamma>} = \frac{1 - \eta_p(\alpha_i^\vee(p)) q^{<\alpha_i^\vee,\gamma>}}{1 - \eta_p(\alpha_i^\vee(p)) p q^{<\alpha_i^\vee,\gamma>}}$$

We observe that this last expression is a rational function in the variable q. This has simple consequences for the intertwining operator on the entire induced representation.

It is well known that the induced module $\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \eta_p \otimes \gamma$ is irreducible, therefore the translates $R_g(f_0)$ generate this module. Then we can conclude that

$$T(s_{\alpha_i}, \eta_p \otimes \gamma)(R_g\ f_0) = \frac{1 - \eta_p(\alpha_i^\vee(p)) q^{<\alpha_i^\vee,\gamma>}}{1 - \eta_p(\alpha_i^\vee(p)) p q^{<\alpha_i^\vee,\gamma>}} (R_g\ f_0)$$

and this means that the matrix coefficients of the intertwining operator are ratios of Laurent polynomials in q, q^{-1} with coefficients in F_1 divided by the polynomial $1 - \eta_p(\alpha_i^\vee(p)) p q^{<\alpha_i^\vee,\gamma>}$.

Then it is clear that we can replace the assumption $< \alpha_i^\vee, \gamma >> 0$ by $< \alpha_i^\vee, \gamma > \neq 0$, we also constructed the operator $T(s_{\alpha_i}, s_{\alpha_i}, \cdot(\eta_p \otimes \gamma)$.

Now it is easy to understand the general intertwining operator $T(w, \eta_p \otimes \gamma)$. We denote by $\Delta_+^{(w)}$ the set of positive roots β for which $w^{-1}\beta < 0$,(then our subgroup $U^{(w)} = \prod_{\beta \in \Delta_+^{(w)}} U_\beta$). If it is not empty (i.e. if $w \neq e$), then it contains a simple root $\beta_1 = \alpha_{i_1}$. Then $w = s_{\alpha_{i_1}} w_1'$ and $\Delta_+^{(w_1')} = s_{\alpha_{i_1}} \Delta_+^{(w)} \setminus \{-\alpha_{i_1}\}$. This set $\Delta_+^{(w_1')}$ again contains a simple root $\alpha_{i_2} = s_{\alpha_{i_1}} \beta_2$. This way we get an expression as a product of reflections $w = s_{\alpha_{i_1}} \dots s_{\alpha_{i_{l(w)}}}$, this expression is of shortest length. This also gives us an ordered listing $\Delta_+^{(w)} = \{\beta_1, \beta_2, \dots, \beta_{l(w)}\}$.. Then we get for our intertwining operator

$$T(w, \eta_p \otimes \gamma) = T(s_{\alpha_{i_{l(w)}}}, w'' \cdot (\eta_p \otimes \gamma)) \dots \circ T(s_{\alpha_{i_2}}, s_{\alpha_{i_1}} \cdot (\eta_p \otimes \gamma)) \circ T(s_{\alpha_{i_1}}, (\eta_p \otimes \gamma)) \tag{6.77}$$

We look at the intermediate expressions $w = s_{\alpha_{i_1}} \dots s_{\alpha_{i_\nu}} w_\nu' = w_\nu w_\nu'$ and consider $T(s_{\alpha_{i_{\nu+1}}}, w_\nu \cdot (\eta_p \otimes \gamma))$, we look at its effect on the spherical function

$$T(s_{\alpha_{i_\nu}}, w_{\nu-1}^{-1} \cdot (\eta_p \otimes \gamma)) \phi_{w_{\nu-1}^{-1} \cdot (\eta_p \otimes \gamma))} = \frac{1 - w_{\nu-1}^{-1} \cdot (\eta_p \otimes \gamma)(\alpha_{i_\nu}^\vee(p))}{1 - p w_{\nu-1}^{-1} \cdot (\eta_p \otimes \gamma)(\alpha_{i_\nu}^\vee(p))} \phi_{w_\nu^{-1} \cdot (\eta_p \otimes \gamma)} \tag{6.78}$$

Now it is easy to see that

$$w_{\nu-1}^{-1} \cdot (\eta_p \otimes \gamma)(\alpha_{i_\nu}^\vee(p)) = w_{\nu-1}^{-1}(\eta_p \otimes \gamma)((\alpha_{i_\nu}^\vee(p)) | w_{\nu-1}^{-1}\rho - \rho|_p = (\eta_p \otimes \gamma)(\beta_\nu^\vee(p)) p^{-s(\beta_\nu)+1}$$

where $s(\beta_\nu)$ is the sum of the coefficients if we write β as the sum of simple roots. Hence we finally get

Hence we get

$$T(w,\eta_p\otimes\gamma)\phi_{\eta_p\otimes\gamma}=\prod_{\nu=0}^{l(w)-1}\frac{1-(\eta_p\otimes\gamma)(\beta_\nu^\vee(p))p^{-s(\beta_\nu)+1}}{1-p(\eta_p\otimes\gamma)(\beta_\nu^\vee(p))p^{-s(\beta_\nu)+1}}\phi_{w\cdot\eta_p\otimes\gamma} \tag{6.79}$$

We get the following

Proposition 6.3.3. *The matrix coefficients of the intertwining operator are rational functions in q. More precisely the become polynomials in q, q^{-1} if we multiply them by $\prod_{\nu=0}^{l(w)-1}(1-p(\eta_p\otimes\gamma)(\beta_\nu^\vee(p))p^{-s(\beta_\nu)+1})$.*

Now we can specialise $q\to 1$ provided $\prod_{\nu=0}^{l(w)-1}(1-p\eta_p(\beta_\nu^\vee(p))p^{-s(\beta_\nu)+1})\neq 0$. Hence we get an intertwining operator

$$T(w,\eta_p):\ \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\eta_p\to\ \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}w\cdot\eta_p \tag{6.80}$$

This implies that we get an intertwining operator between the one dimensional Hecke modules

$$(T(w,\eta_p):(\ \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\eta_p)^{K_p}\to(\ \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}w\cdot\eta_p)^{K_p} \tag{6.81}$$

which is an isomorphism provided the numerator evaluated at $q=1$ in (6.79) is non zero. This shows the that the Satake map $\mathfrak{S}$ is invariant under the twisted action of the Weyl group, i.e we have $\hat{h}(w\cdot\eta_p)=\hat{h}(\eta_p)$ in (6.67) , but still under the proviso that the numerator and the denominator in (6.79) are non zero for the given η_p and $q=1$. But it is easy to see that we can drop this assumption. To see this we look at an individual factor $T(s_{\alpha_{i_\nu}},w_{\nu-1}^{-1}\cdot(\eta_p\otimes\gamma))$. If the denominator evaluated at $q=1$ is zero we normalise by multiplying by the denominator and then clearly the normalised operator evaluated at $q=1$ yields an isomorphism

$$1-pw_{\nu-1}^{-1}\cdot(\eta_p\otimes\gamma)(\alpha_{i_\nu}^\vee(p))T(s_{\alpha_{i_\nu}},w_{\nu-1}^{-1}\cdot(\eta_p\otimes\gamma))|_{q=1}:$$

$$(\ \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}w_{\nu-1}^{-1}\eta_p)^{K_p}\xrightarrow{\sim}(\ \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}w_\nu^{-1}\cdot\eta_p)^{K_p}$$

It the denominator is non zero but the numerator $1-(\eta_p)(\beta_\nu^\vee(p))p^{-s(\beta_\nu)+1}=0$, then we apply the same considerations to the operator in the opposite direction

$$T(s_{\alpha_{i_\nu}},w_\nu^{-1}\cdot(\eta_p\otimes\gamma)):(\ \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}w_\nu^{-1}\cdot\eta_p\otimes\gamma)\to(\ \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}w_{\nu-1}^{-1}\eta_p\otimes\gamma) \tag{6.82}$$

and now an easy calculation shows that we get an isomorphism of Hecke modules $(\ \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}w_\nu^{-1}\cdot\eta_p)^{K_p}\xrightarrow{\sim}(\ \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}w_{\nu-1}^{-1}\eta_p)^{K_p}$. This shows that the Hecke algebra modules $(\ \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}w\cdot\eta_p)^{K_p}$ are all isomorphic, so we almost proved Theorem 6.3.1.

The orbit

$$\{\eta_p,\dots,w\cdot\eta_p,\dots\}_{w\in W}=\omega(\eta_p)$$

will be called the *Satake parameter* of the representation $\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\eta_p$. (If it is irreducble). We come back to this in the next chapter.

If $\tilde{\pi}_p^\vee$ is the spherical representation attached to the Satake parameter η_p^{-1} then we have a pairing dualSat

$$\begin{aligned} H_{\tilde{\pi}_p} \times H_{\tilde{\pi}_p^\vee} &\to \mathbb{C} \\ f_1 \times f_2 &\mapsto \int_{K_p} f_1(k_p) f_2(k_p) dk_p \end{aligned} \tag{6.83}$$

This tells us that the dual module to $H_{\pi_p} = H_{\tilde{\pi}_p}^{K_p}$ has the Satake parameter η_p^{-1}. The representations $H_{\tilde{\pi}_p}$ are called the representations of the unramified principal series.

We may consider the case that η_p is a unitary character, this means that $\eta_p : \mathcal{T}(\mathbb{Q}_p)/\mathcal{T}(\mathbb{Z}_p) \to \mathbb{S}^1$. Then we have $\eta_p^{-1}(t) = \overline{\eta_p}(t)$ and our above pairing defines a positive definite hermitian scalar product

$$< \, , \, >: H_{\tilde{\pi}_p} \times H_{\tilde{\pi}_p} \to \mathbb{C} \tag{6.84}$$

which is given by

$$< f_1, f_2 >= \int_{K_p} f_1(k_p) \overline{f_2(k_p)} dk_p \tag{6.85}$$

If we allow for $f \in H_{\tilde{\pi}_p}$ all the functions whose restriction to K_p lies in $L^2(K_p)$ then $H_{\tilde{\pi}_p}$ becomes a Hilbert space and the representation of $G(\mathbb{Q}_p)$ on $H_{\tilde{\pi}_p}$ is a unitary representation.

These representations are called the unitary principal series representations. It is not the case that these representations are the only unramified principal series representations which carry an invariant positive definite scalar product. (See [Sat]).

6.3.8 Back to cohomology

The case of a torus and the central character

We consider the case that our group is a torus $T/\mathbb{Q}$. This case is already discussed in [35]. Our torus splits over a finite normal extension $F/\mathbb{Q}$, here we choose $F \subset \mathbb{C}$. Our absolutely irreducible representation is simply a character $\gamma : T \times_{\mathbb{Q}} F \to \mathbb{G}_m$, it defines a one dimensional $T \times_{\mathbb{Q}} F-$ module $F[\gamma]$. where $F[\gamma]$ is simply the one dimensional vector space F over F with $T \times_{\mathbb{Q}} F$ acting by the character γ. We now choose an open compact subgroup $K_f^T \subset T(\mathbb{A}_f)$ and consider the cohomology $H^\bullet(\mathcal{S}^T_{K_f^T}, \widetilde{F[\gamma]})$. Since in this case the group $T(\mathbb{A})$ is abelian these cohomology groups are modules under the group $\tilde{T}(\mathbb{A}) = \pi_0(T(\mathbb{R})) \times T(\mathbb{A}_f)$ and we want to understand the cohomology groups as such. For the following see also [35] 2.6.

We consider the map $p : \mathcal{S}^T_{K_f^T} \to \pi_0(\mathcal{S}^T_{K_f^T})$ and we see easily that $\pi_0(\mathcal{S}^T_{K_f^T}) = T(\mathbb{Q})\backslash\pi_0(T(\mathbb{R})) \times T(\mathbb{A}_f)/K_f^T$ is a finite abelian group, it is a generalised ideal class group. The fibre of this map is $E(T)\backslash(T(\mathbb{R})^{(0)}/K_\infty^T)$ where $E(T) = T(\mathbb{Q}) \cap \{e\} \times K_f$, this is some kind of group of units. Our torus contains a maximal anisotropic sub torus $T^{(an)}/\mathbb{Q}$ and $E(T) \subset T^{(an)}(\mathbb{R})^{(0)}$. The torus $T^{(an)} \times \mathbb{R}$ again contains a maximal anisotropic sub torus $T^{(an_\mathbb{R})}$ and by construction we have $K_\infty^T = T^{(an_\mathbb{R})}(\mathbb{R})$, this is a product of circles. The quotient $T^{(an)} \times \mathbb{R}/T^{(an_\mathbb{R})}$ is split over $\mathbb{R}$. This implies that $T^{(an)} \times \mathbb{R}/T^{(an_\mathbb{R})}(\mathbb{R})^{(0)} = (\mathbb{R}_{>0}\times)^t = \mathbb{R}^t$ where $t = \dim(T^{(an)}) - \dim(T^{(an_\mathbb{R})})$. The group $E(T)$ is a lattice in $\mathbb{R}^t$ and the quotient $E(T)\backslash T^{(an)} \times \mathbb{R}/T^{(an_\mathbb{R})}(\mathbb{R})^{(0)} = (S^1)^t$ is a product of circles. If $T^{(sp)}/\mathbb{Q}$ is the split component of $T/\mathbb{Q}$ then $T^{(sp)}(\mathbb{R})^{(0)} = \mathbb{R}^s$ and hence we see for the cohomology of the fibre (the connected component)

$$H^\bullet(E(T)\backslash T(\mathbb{R})^{(0)}/K_\infty^T, \tilde{F[\gamma]}) = \Lambda^\bullet(E(T)) \otimes F[\gamma]$$

We recall the notion of an algebraic Hecke character of type γ. We assumed $F \subset \mathbb{C}$, then γ induces a homomorphisms $T(\mathbb{C}) \to \mathbb{C}^\times$. The restriction of this homomorphism to $T(\mathbb{R})$ is called $\gamma_\infty : T(\mathbb{R}) \to \mathbb{C}^\times$.

A continuos homomorphism

$$\phi = \phi_\infty \times \Pi_p \phi_p = \phi_\infty \times \phi_f \ : T(\mathbb{Q})\backslash T(\mathbb{A}) \to \mathbb{C}^\times$$

is called an *algebraic Hecke character of type* γ if the restrictions to the connected component of the identity satisfy

$$\phi_\infty|_{T^{(0)}(\mathbb{R})} = \gamma_\infty^{-1}|_{T^{(0)}(\mathbb{R})}.$$

The finite part $\phi_f : T(\mathbb{A}_f) \to \bar{\mathbb{Q}}^\times$ is trivial on some open compact subgroup $K_f^T \subset T(\mathbb{A}_f)$. We also say that a homomorphism $\psi_f : T(\mathbb{A}_f)/K_f^T \to \bar{\mathbb{Q}}^\times$ is an algebraic Hecke-character, if it is the finite part of an algebraic Hecke character $\psi = \psi_\infty \times \psi_f$ which then is uniquely defined.

If this character ψ is of type γ then clearly $\psi_\infty|E(T) = 1$ and we can say that our character γ with type$(\psi) = \gamma$. must be trivial on the Zariski closure of $E(T)$. This Zariski closure does not depend on the level K_f^T and it is is a sub torus $T^{(\mathrm{CM})} \subset T/\mathbb{Q}$. We also define the CM (complex multiplication) component $T^{(\mathrm{CM})} := T/T^{(\mathrm{nCM})}$.

We summarise:

Proposition 6.3.4. *A character $\gamma \in X^*(T \times E)$ is the type of a Hecke character ϕ if and only if $\gamma \in X^*(T^{(\mathrm{CM})} \times E)$. If K_f^T is given then the algebraic Hecke characters of type γ form a torsor under the group of Dirichlet characters*

$$\chi : T(\mathbb{Q})\backslash(\pi_0(T(\mathbb{R})) \times T(\mathbb{A}_f)/K_f^T \to \mathbb{C}^\times,$$

i.e.. if ϕ_0 is a Hecke character of type γ then the others are the characters $\phi_0\chi$

In [35], 2.5.5 we explain that the cohomology $H^\bullet(S^T_{K_f^T}, F[\gamma]$ vanishes (for any choice of K_f^T) if γ is not the type of an algebraic Hecke character. Hence we assume that $\gamma \in X^*(T^{(\mathrm{CM})} \times E)$, then it is easy to see that there is a finite normal extension $F_1 \subset F$ - depending on the level K_f^T- such that for all characters $\phi_f : T(\mathbb{A}_f)/K_f^T \to \mathbb{C}^\times$ the values are in $F_1^\times$. for Now we give the complete description of the cohomology in [35], 2.6:

$$H^\bullet(S^T_{K_f^T}, F[\gamma]) \otimes F_1 = \bigoplus_{\phi_f : T(\mathbb{A}_f)/K_f^T \to F_1 : \mathrm{type}(\phi_f) = \gamma} \Lambda^\bullet(E(T)) \otimes F_1[\phi_f]. \quad (6.86)$$

If we return to our group $G/\mathbb{Q}$ and if we start from an absolutely irreducible representation $G\times_{\mathbb{Q}} F\to \mathrm{Gl}(\mathcal{M})$ then its restriction to the center $C/\mathbb{Q}$ is a character $\zeta_{\mathcal{M}}$. Our remark above implies that this character must be the type of an algebraic Hecke character if we want the cohomology $H^\bullet_?(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}})$ to be non trivial. (Look at a suitable spectral sequence).

In any case we can consider the sub algebra $C_{K_f}\subset \mathcal{H}_{K_f}$ generated by central double cosets $K_f\underline{z}_f K_f = K_f\underline{z}_f$. with $\underline{z}_f\in C(\mathbb{A}_f)$ This provides an action of the group $C(\mathbb{A}_f)/K_f^C$ on the cohomology $H^\bullet_?(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}})$. Then the following proposition is obvious

Proposition 6.3.5. *Let H_{π_f} be an absolutely irreducible subquotient in the Jordan Hölder series in any of our cohomology groups. Then $C(\mathbb{A}_f)/K_f^C$ acts by a character ζ_{π_f} on H_{π_f} and ζ_{π_f} is an algebraic Hecke character of type $\zeta_{\mathcal{M}}$.*

Note that $\zeta_{\mathcal{M}}$ is the restriction of the abelian component δ in $\lambda=\lambda^{(1)}+\delta$ to the center.

The cohomology in degree zero

Let us start from an absolutely irreducible representation $r: G\times F\to \mathrm{Gl}(\mathcal{M})$, we want to understand $H^0(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}})$: To do this we have to understand the connected components of the space and the spaces of invariants in $\tilde{\mathcal{M}}$ under the discrete subgroups $\Gamma^{\underline{g}_f}$ in 1.2.1. We assume that the groups $\Gamma^{\underline{g}_f}\cap G^{(1)}(\mathbb{Q})$ are Zariski dense in $G^{(1)}$. Then it is clear that we can have non trivial cohomology in degree zero if $\mathcal{M}$ is one dimensional and $G^{(1)}$ acts trivially. Hence $\mathcal{M}$ is given by a character $\delta: C'\times F\to \mathbb{G}_m\times F$.

To simplify the situation we assume that the assumptions in (6.1.3) are fulfilled and we have a bijection

$$\pi_0(\mathcal{S}^G_{K_f}) \xrightarrow{\sim} \pi_0(\mathcal{S}^{C'}_{K^{C'}_\infty\times K^{C'}_f}) \tag{6.87}$$

where $K^{C'}_\infty$ and $K^{C'}_f$ are the images of the chosen compact subgroups respectively. With these data we define $\mathcal{S}^{C'}_{K^{C'}_f}$ and we can view $\mathcal{M}$ as a sheaf on $\mathcal{S}^{C'}_{K^{C'}_f}$, in our previous notation it is the sheaf $\tilde{F}[\delta]$.

Then we get for an absolutely irreducible $G\times F$ module $\mathcal{M}$ -and under the assumption that the $\Gamma^{\underline{g}_f}\cap G^{(1)}(\mathbb{Q})$ are Zariski dense in $G^{(1)}$- that (See **??**)

$$H^0(\mathcal{S}^G_{K_f},\mathcal{M}\otimes F_1)=\begin{cases}0 & \text{if } \dim(\mathcal{M})>1\\ \bigoplus_{\phi_f:\mathrm{type}(\phi_f)=\delta} F_1\phi & \text{if } \mathcal{M}=F[\delta]\end{cases} \tag{6.88}$$

The density assumption is fulfilled if $G^{(1)}/\mathbb{Q}$ is quasisplit. We also observe that we have the isogeny $d_C: C\to C'$ (See (1.1). Then it is clear that the composition $d_C\circ\delta$ is the character $\zeta_{\mathcal{M}}$ in section **??**. **Remark on Poincare duality**

6.3.9 The Manin-Drinfeld principle

We return to the general situation. We start from a rational (preferably absolutely irreducible) representation $\rho: G\times_{\mathbb{Q}} F_0\to \mathrm{Gl}(\mathcal{M}_{F_0})$ where $\mathcal{M}_{F_0}$ is a finite dimensional F_0 vector spaces. We have an action of $\mathcal{H}$ on our cohomology groups $H^\bullet_?(\mathcal{S}^G_{K_f},\mathcal{M}_{F_0})$. Most of the time we will consider the restriction of this action to the central sub algebra $\mathcal{H}^{(\Sigma)}$.We choose a finite normal extension $F/\mathbb{Q}, F\supset F_0$ such that all irreducible subquotients are absolutely irreducible. We introduced the sets $\mathrm{Coh}(H^\bullet_?(\mathcal{S}^G_{K_f},\mathcal{M})), \mathrm{Coh}^{(\Sigma)}(H^\bullet_?(\mathcal{S}^G_{K_f},\mathcal{M}))$.

We say that for a cohomology groups $H^i(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F)$ (resp. $H^\bullet_c(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F)$ satisfy the (strong) *Manin-Drinfeld principle*, if

$$\mathrm{Coh}^{(\Sigma)}(H^i_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F)\cap \mathrm{Coh}^{(\Sigma)}(H^i(\partial(\mathcal{S}^G_{K_f}),\tilde{\mathcal{M}}_F)=\emptyset$$

(resp

$$\mathrm{Coh}^{(\Sigma)}(H^i_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F)\cap \mathrm{Coh}^{(\Sigma)}(H^{i-1}(\partial(\mathcal{S}^G_{K_f}),\tilde{\mathcal{M}}_F)=\emptyset.$$

An equivalent formulation is: The $\mathcal{H}^{(\Sigma)}$ module $H^i_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F)$ is complete in $H^i(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F)$

If the Manin-Drinfeld principle is valid we get canonical decompositions

$$H^i(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F) = \ \mathrm{Im}(H^i(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F)\longrightarrow H^i(\partial(\mathcal{S}^G_{K_f}),\tilde{\mathcal{M}}_F))\oplus H^i_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F)$$

$$H^i_c(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F) = \ \mathrm{Im}(H^{i-1}(\partial\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F)\longrightarrow H^i_c(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F))\oplus H^i_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F). \tag{6.89}$$

which is invariant under the action of the Hecke algebra.

In the first case we can consider the module $H^i_{\mathrm{Eis}}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F)\subset \ \mathrm{Im}(H^i(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F)\longrightarrow H^i(\partial(\mathcal{S}^G_{K_f}),\tilde{\mathcal{M}}_F))$ as a submodule in $H^i(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F)$ and this submodule is called the Eisenstein cohomology. In the second case we will call the above image of the boundary cohomology the Eisenstein subspace or compactly supported Eisenstein cohomology and denote it by

$$\mathrm{Im}(H^{i-1}(\partial\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F)\longrightarrow H^i_c(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F)) = H^i_{c,\ \mathrm{Eis}}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F).$$

Therefore we get the decompositions

$$\begin{aligned} H^i(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F) &= H^i_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F)\oplus H^i_{\mathrm{Eis}}\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F) \\ H^i(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F) &= H^i_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F)\oplus H^i_{c,\mathrm{Eis}}\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F) \end{aligned} \tag{6.90}$$

We could also speak of the *weak* Manin-Drinfeld principle where we replace $\mathcal{H}^{(\Sigma)}$ by the full Hecke algebra.

If we know the Manin-Drinfeld principle we can ask new questions. We return to the integral cohomology $H^i_?(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathcal{O}_F})$ and map it into the rational cohomology, then the image is called $H^\bullet_?(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathcal{O}_F})_{\ \mathrm{int}}\subset H^\bullet_?(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_F)$ this is also the module which we get if we divide $H^\bullet_?(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathcal{O}_F})))$ by the torsion. (This may be not true for ? =!.(*see*??))

We introduce some terminology. Let R be any Dedekind ring, let K be its quotient field. We consider finitely generated modules over R. If X is a finitely generated R-module then we have the map $X \to X \otimes_R K$. The kernel of this map is the module X_{tors} of torsion elements, the image is called X_{int} it is a locally free R-module and equal to X/X_{tors}. If we have a decomposition submodules $X \otimes K = U \oplus V$ then we consider $U_{\text{int}}, V_{\text{int}} \subset X_{\text{int}}$ and we get a decomposition *up to isogeny*

$$X_{\text{int}} \supset U_{\text{int}} \oplus V_{\text{int}} \text{ with } X_{\text{int}}/(U_{\text{int}} \oplus V_{\text{int}}) \text{ finite}$$

where the term *up to isogeny* is a synonym for the finiteness of the quotient on the right. At this point we notice that the quotients $X_{\text{int}}/U_{\text{int}}, X_{\text{int}}/V_{\text{int}}$ are torsion free. We call a submodule $Y \subset X_{\text{int}}$ *saturated* , if X_{int}/Y is torsion free. Therefore we will call the above decomposition up to isogeny also a *decomposition into saturated submodules.*

For instance the Manin-Drinfeld decomposition above yields (a decomposition up to isogeny

$$H^i_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathcal{O}_F})_{\text{int}} \oplus H^i_{\text{Eis}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathcal{O}_F})_{\text{int}} \subset H^i(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathcal{O}_F})_{\text{int}},$$

It is one of the central questions discussed in this book to understand the quotient

$$H^i(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathcal{O}_F})_{\text{int}}/(H^i_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathcal{O}_F})_{\text{int}} \oplus H^i_{\text{Eis}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathcal{O}_F})_{\text{int}}) \tag{6.91}$$

In the earlier chapters 3-5 we discuss this problem in a very specific case. Our group is $G/\mathbb{Z} = \mathrm{Gl}_2/\mathbb{Z}$, the open compact subgroup is $K_f = \prod_p \mathrm{Gl}_2(\mathbb{Z}_p)$. Then $\mathcal{S}^G_{K_f} = \mathrm{Sl}_2(\mathbb{Z})\backslash\mathbb{H}$, our coefficient system is the module $\mathcal{M}^\flat_n$ (See section [**?**]) and we give an answer to the above question.

I am convinced that there are many more cases in which the above question is interesting and has an interesting answer. The structure of the quotient should be related to the arithmetic of special values of L-functions which are attached to Hecke eigenclasses in $H^\bullet(\partial\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_F)$ (See Chapter 9) This is highly conjectural but the experimental data are very convincing.

The same applies to the decomposition of $H^i_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})_{\text{int}}$ in isotypical summands. We put

$$H^i_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})(\pi_f) \cap H^i_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathcal{O}_F})_{\text{int}} = H^i_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathcal{O}_F})_{\text{int}}(\pi_f).$$

Then we get an decomposition up to isogeny

$$\bigoplus_{\pi_f} H^i_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathcal{O}_F})_{\text{int}}(\pi_f) \subset H^i_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathcal{O}_F})_{\text{int}}. \tag{6.92}$$

It is a very interesting question to learn something about the structure of the quotient of the right hand side by the left hand side. The structure of this quotient should be related to the arithmetic of special values of L-functions. (See [53]).

The action of $\pi_0(G(\mathbb{R}))$

We have seen that we can choose a maximal torus $T/\mathbb{Q}$ such that $T(\mathbb{R})[2]$ normalizes K_∞. We know that $T(\mathbb{R})[2] \to \pi_0(G(\mathbb{R}))$ is surjective and that

$T(\mathbb{R})[2] \cap G^{(1)}(\mathbb{R}) \subset K_\infty$. This allows us to define an action of $\pi_0(G(\mathbb{R}))$ on the various cohomology groups and this action commutes with the action of the Hecke-algebra. Therefore we can decompose any isotypical subspace in a cohomology group into eigenspaces under this action

$$H^\bullet_?(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_F)(\pi_f) = \bigoplus_{\epsilon_\infty} H^\bullet_?(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_F)(\pi_f \times \epsilon_\infty) \tag{6.93}$$

and for the integral lattices we get a decomposition up to isogeny

$$\bigoplus_{\pi_f \times \epsilon_\infty} H^i_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathcal{O}_F})_{\text{ int}}(\pi_f \times \epsilon_\infty) \subset H^i_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathcal{O}_F})_{\text{ int}} \tag{6.94}$$

6.3.10 Some questions and some general facts

Homology

We may also define homology groups $H_i(\mathcal{S}^G_{K_f}, \underline{\mathcal{M}}_\lambda)$ and $H_i(\mathcal{S}^G_{K_f}, \partial\mathcal{S}^G_{K_f}, \underline{\mathcal{M}}_\lambda)$, here $\mathcal{M}_\lambda$ is a "cosheaf". The "costalk" $\underline{\mathcal{M}}_{\mathbb{Z},x}$ is obtained as follows: We consider $\pi^{-1}(x)$ and

$$\bigoplus_{\underline{y}=y\times \underline{g}_f K_f/K_f} \underline{g}_f \mathcal{M}_\lambda,$$

and the action of $G(\mathbb{Q})$ on this direct sum. Then $\underline{\mathcal{M}}_{\lambda,x}$ is the module of coinvariants. If we pick a point $\underline{y} = y \times \underline{g}_f K_f/K_f$, which maps to $x \in \mathcal{S}^G_{K_f}$ then we get an isomorphism

$$\underline{\mathcal{M}}_{\lambda,x} \simeq (g_f \mathcal{M}_\lambda)_{\Gamma^{(\underline{g}_f)}_y}.$$

We define the chain complex

$$C_i(\mathcal{S}^G_{K_f}, \underline{\mathcal{M}}_\lambda)$$

and the above homology groups are given by the homology of this complex.

If we assume that $\mathcal{S}^G_{K_f}$ is oriented (ref. to prop 1.3) then we know (Chap. II 2. 1. 5) that we have isomorphisms which are compatible with the fundamental exact sequence

$$\begin{array}{ccc}
\downarrow & & \downarrow \\
H^{i-1}(\partial\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) & \xrightarrow{\sim} & H_{d-i}(\partial\mathcal{S}^G_{K_f}, \underline{\mathcal{M}}_\lambda) \\
\downarrow & & \downarrow \\
H^i_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) & \xrightarrow{\sim} & H_{d-i}(\mathcal{S}^G_{K_f}, \underline{\mathcal{M}}_\lambda) \\
\downarrow & & \downarrow \\
H^i(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) & \xrightarrow{\sim} & H_{d-i}(\mathcal{S}^G_{K_f}, \partial\mathcal{S}^G_{K_f}, \underline{\mathcal{M}}_\lambda) \\
\downarrow & & \downarrow \\
H^i(\partial\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) & \xrightarrow{\sim} & H_{d-i-1}(\partial\mathcal{S}^G_{K_f}, \underline{\mathcal{M}}_\lambda) \\
\downarrow & & \downarrow
\end{array}$$

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6.3.11 Poincaré duality

We assume that $\mathcal{S}^G_{K_f}$ is connected. If we denote the dual representation by $\mathcal{M}^\vee_\lambda = \mathcal{M}_{w_0(\lambda)}$ (we choose the right lattice $\mathcal{M}^\vee_\mathbb{Z} \subset \mathcal{M}^\vee_\mathbb{Q}$) we have the canonical homomorphism $\Phi_\lambda : \mathcal{M}_\lambda \otimes \mathcal{M}^\vee_\lambda \to \mathbb{Z}$ and the standard pairing between the homology and the cohomology groups yields pairings

$$\begin{array}{ccccc} H^i_c(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda)\times H_i(\mathcal{S}^G_{K_f},\partial\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}^\vee_\lambda) & \to & H^0(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda\otimes\tilde{\mathcal{M}}^\vee_\lambda) & \to & H^0(\mathcal{S}^G_{K_f},\mathbb{Z}) \\ \downarrow \qquad\qquad \uparrow & & \downarrow & & \downarrow \\ H^i(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda)\times H_i(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\lambda^\vee}) & \to & H^0(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda\otimes\tilde{\mathcal{M}}^\vee_\lambda) & \to & H^0(\mathcal{S}^G_{K_f},\mathbb{Z}) \end{array}$$

This pairing is of course compatible with the isomorphism between homology and cohomology and then the pairing becomes the cup product. We get the diagram

$$\begin{array}{ccccc} H^i_c(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda)\times H^{d-i}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}^\vee_\lambda) & \to & H^d_c(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda\otimes\tilde{\mathcal{M}}^\vee_\lambda) & \to & H^d_c(\mathcal{S}^G_{K_f},\mathbb{Z}) \\ \downarrow \qquad\qquad \uparrow & & \downarrow & & \downarrow \\ H^i(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda)\times H^{d-i}_c(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\lambda^\vee}) & \to & H^d_c(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda\otimes\tilde{\mathcal{M}}^\vee_\lambda) & \to & H^d_c(\mathcal{S}^G_{K_f},\mathbb{Z}) \end{array}$$

We know that the manifold with corners $\partial\mathcal{S}^G_{K_f}$ "smoothable" it can be approximated by a $\mathcal{C}-$ manifold and therefore we also have a pairing $< , >_\partial$ on the cohomology of the boundary. This pairing is consistent with the fundamental long exact sequence (Thm. 6.2.1). We write this sequence twice but the second time in the opposite direction and the pairing $< , >$ in vertical direction:

$$\begin{array}{ccccccc} \to & H^p(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda) & \xrightarrow{r} & H^p(\partial\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda) & \xrightarrow{\delta} & \\ & \times & & \times & & \\ \leftarrow & H^{d-p}_c(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\lambda^\vee}) & \xleftarrow{\delta} & H^{d-p-1}(\partial\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\lambda^\vee}) & \leftarrow \\ & \downarrow < , > & & \downarrow < , >_\partial & & \\ & H^d_c(\mathcal{S}^G_{K_f},\mathbb{Z}) & \xleftarrow{\delta_d} & H^{d-1}_c(\partial\mathcal{S}^G_{K_f},\mathbb{Z}) & \end{array} \tag{6.95}$$

then we have: For classes $\xi \in H^p(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda), \eta \in H^{d-p-1}(\partial\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\lambda^\vee})$ we have the equality

$$< \xi, \delta(\eta) >= \delta_d(< r(\xi), \eta >_\partial) \tag{6.96}$$

Non degeneration of the pairing

The spaces $\mathcal{S}^G_{K_f}$ and $\partial\mathcal{S}^G_{K_f}$ are not connected in general. Let us assume that we have a consistent orientation on $\mathcal{S}^G_{K_f}$. Then each connected component M of $\mathcal{S}^G_{K_f}$ is an oriented manifold which is natural embedded into its compactification $\bar{M}$. It is obvious that the cohomology groups are the direct sums of the cohomology groups of the connected components and that we may restrict the pairing to the components

$$H^p(M,\tilde{\mathcal{M}}_\lambda) \times H^{d-p}_c(M,\tilde{\mathcal{M}}_{\lambda^\vee}) \to H^d_c(M,\mathbb{Z}) = \mathbb{Z}. \tag{6.97}$$

We recall the results which are explained in Vol. I 4.8.4. The fundamental group $\pi_1(M)$ is an arithmetic subgroup $\Gamma_M \subset G(\mathbb{Q})$ and $\mathcal{M}_\lambda, \mathcal{M}_{\lambda^\vee}$ are Γ_M

modules. For any commutative ring with identity $\mathbb{Z} \to R$ the Γ_M modules $\mathcal{M}_\lambda \otimes R, \mathcal{M}_{\lambda^\vee} \otimes R$ provide local systems $\widetilde{\mathcal{M}_\lambda \otimes R}, \widetilde{\mathcal{M}_{\lambda^\vee} \otimes R}$, and we have the extension of the cup product pairing

$$H^p(M, \widetilde{\mathcal{M}_\lambda \otimes R}) \times H_c^{d-p}(M, \widetilde{\mathcal{M}_{\lambda^\vee} \otimes R}) \to H_c^d(M, R) = R$$

Proposition 6.3.6. *If $R = k$ is a field then the pairing is non degenerate. .*

If R is a Dedekind ring then the pairing then the cohomology may contain some torsion submodules and

$$H^p(M, \widetilde{\mathcal{M}_\lambda \otimes R})/\mathrm{Tors} \times H_c^{d-p}(M, \widetilde{\mathcal{M}_{\lambda^\vee} \otimes R})/\mathrm{Tors} \to H_c^d(M, R) = R$$

is non degenerate.

(See Vol. I 4.8.9)

We want to discuss the consequences of this result for the cohomology of $H^\bullet_?(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$. Before we do this we want to recall some simple facts concerning the representations of the algebraic group $G/\mathbb{Q}$. We consider two highest weights $\lambda, \lambda_1 \in X^*(T \times F)$ which are dual modulo the center. By this we mean that we have (See 6.46)

$$\lambda = \lambda^{(1)} + \delta, \lambda_1 = -w_0(\lambda^{(1)}) + \delta_1 \tag{6.98}$$

Then $\delta + \delta_1$ is a character on $X^*(C' \times F)$ and yields a one dimensional module

(??????) $H_c^i(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) \times H^{d-i}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}^\vee_\lambda) \to H_c^d(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \tilde{\mathcal{M}}^\vee_\lambda) \to H_c^d(\mathcal{S}^G_{K_f}, \mathbb{Z})$
for $G \times F$, of course the action of $G^{(1)}$ on this module is trivial. Then we get a G invariant non trivial pairing

$$\mathcal{M}_{\lambda,F} \times \mathcal{M}_{\lambda_1,F} \to \mathcal{N}_{\lambda\circ\lambda_1}$$

which induces a pairing

$$H_c^i(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,F}) \times H^{d-i}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda_1,F}) \to H_c^d(\mathcal{S}^G_{K_f}, \mathcal{N}_{\lambda\circ\lambda_1}),$$

this only a slight generalization of the previous pairing.

Now we recall that (under certain assumptions) we have the inclusion $\pi_0(\mathcal{S}^G_{K_f}) \hookrightarrow \pi_0(\mathcal{S}^{C'}_{K^{C'}_\infty \times K^{C'}_f})$ and then we get

$$H_c^d(\mathcal{S}^G_{K_f}, \mathcal{N}_{\lambda\circ\lambda_1}) \subset H^0(\mathcal{S}^{C'}_{K^{C'}_\infty \times K^{C'}_f}, \mathcal{N}_{\lambda\circ\lambda_1}) = \bigoplus_{\chi':\mathrm{type}(\chi')=\lambda\circ\lambda_1} F\chi'$$

The character χ' has a restriction to $C(\mathbb{A})$ let us call this restriction χ.

The group $C(\mathbb{A}_f)$ acts on the cohomology groups and this action has an open kernel K_f^C. Hence we can decompose the cohomology groups on the left hand side according to characters

$$H_c^i(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,F}) = \bigoplus_{\zeta_f:\mathrm{type}(\zeta_f)=\delta} H_c^i(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,F})(\zeta_f) \tag{6.99}$$

$$H^{d-i}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda_1,F}) = \bigoplus_{\zeta_{1,f}:\mathrm{type}(\zeta_{1,f})=\delta_1} H^{d-i}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda_1,F})(\zeta_{1,f}). \tag{6.100}$$

With these notations we get another formulation of Poincaré duality.

Proposition 6.3.7. *If we have three algebraic Hecke characters $\zeta_f, \zeta_{1,f}, \chi'_f$ of the correct type and if we have the relation $\zeta_f \cdot \zeta_{1,f} = \chi_f$ then the cup product induces a non degenerate pairing*

$$H^i_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,F})(\zeta_f) \times H^{d-i}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda_1,F})(\zeta_{1,f}) \to F\chi'$$

This is an obvious consequence of our considerations above. Fixing the central characters has the advantage that the target space of the pairing becomes one dimensional over F, The field F should contain the values of the characters.

We return to the diagram (6.95) and consider the images $\mathrm{Im}(r^q)(\zeta_f) = \mathrm{Im}(H^q_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,F})(\zeta_f) \to H^{d-q-1}_c(\partial\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}^\vee_{\lambda,F})(\zeta_f)$ and $\mathrm{Im}(r^{\vee,d-q-1})$. Then the following proposition is an obvious consequence of the non degeneration of the pairing and (6.96) PDinner

Proposition 6.3.8. *The images $Im(r^p(\zeta_f))$ and $Im(r^{\vee,d-p-1})(\zeta_{1,f})$ are mutual orthogonal complements of each other with respect to $< \, , \, >_\partial$.*

The pairing in proposition 6.3.7 induces a non degenerate pairing

$$\cup : H^i_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,F})(\zeta_f) \times H^{d-i}_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda_1,F})(\zeta_{1,f}) \to F\chi'.$$

Proof. Let $\eta \in H^{d-p-1}(\zeta_{1,f})$ Then we know from the exactness of the sequence that $\eta \in \mathrm{Im}(r^{\vee,d-p-1})(\zeta_{1,f}) \iff \delta(\eta) = 0 \iff <\delta(\eta), \xi>= 0$ for all $\xi \in H^p(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)(\zeta_f) \iff <\eta, r(\xi)>= 0$ for all $\xi \in H^p(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)(\zeta_f) \iff <\eta, \xi'>_\partial= 0$ for all $\xi' \in \mathrm{Im}(r^q)(\zeta_f)$.

The second assertion is rather obvious. If we have $\xi \in H^p_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)(\zeta_f), \xi_1 \in H^{d-p}_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda^\vee})(\zeta_f)$ then we can lift either of these classes - say ξ_1- to a class $\tilde{\xi_1} \in H^p_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)(\zeta_f)$ and then $<\xi_1, \xi_2>=<\tilde{\xi_1}, \xi_2>$. It is clear that the result does not depend on the choice of class which we lift. It is also obvious that the pairing is non degenerate. □

Of course we also have a version of proposition 6.3.8 for the integral cohomology. Since we fixed the level we have only a finite number of possible central characters $\zeta_f, \zeta_{1,f}$ of the required type. The values of these characters evaluated on $C(\mathbb{A}_f)$ lie in a finite extension $F/\mathbb{Q}$ and of of course they are integral. If we now invert a few small primes and pass to a quotient ring $R = \mathcal{O}_F[1/N]$ then we get the decomposition (6.99) but with coefficient systems which are R-modules:

$$H^i_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,R}) = \bigoplus_{\zeta_f:\mathrm{type}(\zeta_f)=\delta} H^i_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,R})(\zeta_f) \qquad (6.101)$$

$$H^{d-i}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda_1,R}) = \bigoplus_{\zeta_{1,f}:\mathrm{type}(\zeta_{1,f})=\delta_1} H^{d-i}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda_1,R})(\zeta_{1,f}) \qquad (6.102)$$

Then it becomes clear that we get an integral version of proposition 6.3.7 where replace the F-vector space coefficient systems $\tilde{\mathcal{M}}_{\lambda,F}$ by R -module coefficient systems. We get a pairing (See [40] 4.8.4)

$$H^i_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,R})(\zeta_f)/\mathrm{Tors} \times H^{d-i}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda_1,R})(\zeta_{1,f})/\mathrm{Tors} \to R\chi' \qquad (6.103)$$

and this pairing is non degenerate. (See [40] Thm. 4.8.9. The finiteness assumptions are easy consequences of reduction theory)

We recall the notion of non degenerate. Our ring R is a Dedekind ring and all our cohomology groups are finitely generated R modules. If we divide any finitely generated R-module by the subgroups of torsion elements then the result is a projective R-module and it is locally free for Zariski topology. An element $\xi \in H^i_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,R})(\zeta_f)/\mathrm{Tors}$ is called *primitive* if the submodule $R\xi$ is -locally for the Zariski topology- a direct summand or what amounts to the same if $H^i_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,R})(\zeta_f)/\mathrm{Tors}/R\xi$ is torsion free. Then the assertion that the above pairing is non degenerate means:

For any primitive element $\eta \in H^i_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,R})(\zeta_f)/\mathrm{Tors}$ *we find elements* $\xi_1, \xi_2, \dots, \xi_r \in H^{d-i}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda_1,R})(\zeta_{1,f})/\mathrm{Tors}$ *such that the ideal generated by* $< \xi_1, \eta >, < \xi_2, \eta >, \dots, < \xi_r, \eta >$ *is equal to* R.

We want to formulate an integral version of (6.96). Here the statement is not quite symmetric. It is clear from ??? that we get a pairing intint

$$H^i_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,R})(\zeta_f)_{\text{ int}} \times H^{d-i}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda_1,R})(\zeta_{1,f})_{\text{ int},!} \to R\chi'. \tag{6.104}$$

It is also clear from proposition (6.3.6)

Proposition 6.3.9. *This pairing is partially non degenerate. For any primitive element* $\eta \in H^{d-i}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda_1,R})(\zeta_{1,f})_{\text{ int},!}$ *we find elements*

$$\xi_1, \xi_2, \dots, \xi_r \in H^i_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,R})(\zeta_f)_{\text{ int}}$$

such that the ideal generated by $< \eta, \xi_1 >, < \eta, \xi_2 >, \dots, < \eta, \xi_r >$ *is equal to* R.

Here we see that the possibility that

$$H^i(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda_1,R})(\zeta_{1,f})_{\text{ int},!}/H^i_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,R})(\zeta_f)_{\text{ int}} \neq (0)$$

plays a role.

Inner Congruences

We choose a highest weight $\lambda = \lambda^{(1)} + d\delta$ and the dual weight $\lambda^\vee = -w_0(\lambda) - d\delta$. Let us also fix a central character ζ_f whose type is equal to the restriction of $d\delta$ to the central torus C.

We look at the pairing in prop. 6.3.8 where we assume in addition that $\zeta_{1,f} = \zeta_f^{-1}$ and we take the action of the Hecke algebra into account, i.e we look at the decomposition into eigenspaces (see(**??**)). Then we get a non degenerate pairing between isotypical subspaces

$$H^i_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,F})(\pi_f) \times H^{d-i}_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda^\vee,F})(\pi_f^\vee) \to F$$

where we assume that the central characters of the summands are ζ_f, ζ_f^{-1}.

If we try to extend this to the integral cohomology. In this case the above decomposition yields decompositions up to isogeny

$$\begin{aligned} H^i_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,R})/\mathrm{Tors} &\supset \textstyle\bigoplus_{\pi_f} H^i_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,R})/\mathrm{Tors}(\pi_f) \\ H^{d-i}_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda^\vee,R})/\mathrm{Tors} &\supset \textstyle\bigoplus_{\pi_f} H^{d-i}_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda^\vee,R})/\mathrm{Tors}(\pi_f^\vee) \end{aligned} \tag{6.105}$$

where we should fix the central characters as above. We choose a pair $\pi_f, \pi_f^\vee$. Then our non degenerate pairing from the above proposition induces a pairing

$$H^i_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,R})/\mathrm{Tors}(\pi_f) \times H^{d-i}_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda^\vee,R})/\mathrm{Tors}(\pi_f^\vee) \to R \qquad (6.106)$$

and this pairing is non degenerate if and only if both modules are direct summands in the above decomposition up to isogeny.

But it may happen that the values of the pairing generate a proper ideal $\Delta(\pi_f) \subset R$, and in this case the above submodules will not be direct summands and this implies that we will have congruences between the Hecke-module π_f and some other module in the decomposition up to isogeny. This yields the *inner congruences.*

The ideal $\Delta(\pi_f)$ should be expressed in terms of L-values, in the classical case this has been done by Hida [Hi].

6.3.12 The Gauss-Bonnet formula

Of course we can be more modest we may only ask for the dimension of the cohomology groups $H^i(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\mathbb{Q})$. This question can be answered in some cases, for instance we gave the answer $\mathrm{Sl}_2(\mathbb{Z})$ in section 2.1.4, and we will give the answer in some more cases further down.

If we are even more modest we can ask for the Euler characteristic

$$\chi(H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\mathbb{Q})) = \sum_i (-1)^i \dim(H^i(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\mathbb{Q}))$$

This question has an answer. We assume for the beginning that the subgroup K_f is neat (See 1.1.2.1) and we also assume that K_∞ is narrow. Then $\mathcal{S}^G_{K_f}$ is a disjoint union of locally symmetric spaces on we can choose- in a consistent way- an orientation on $\mathcal{S}^G_{K_f}$. On these spaces exists a differential form of highest degree, which is obtained from differential geometric data, this is the Gauss-Bonnet form ω^{GB}. Then the Gauss-Bonnet theorem yields that

$$\chi(H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\mathbb{Q})) = \dim(\mathcal{M}_\mathbb{Q}) \int_{\mathcal{S}^G_{K_f}} \omega^{GB} \qquad (6.107)$$

(See [33], [102],[90]).

We have a closer look at this formula. We can compute the differential form ω^{GB} explicitly.. The connected components of $\mathcal{S}^G_{K_f}$ are of the form $\Gamma_i\backslash X$ where $X = G(\mathbb{R})/K_\infty$. Then the top degree form ω^{GB} is a $G(\mathbb{R})$ invariant form on the symmetric space X. Since this form is $G(\mathbb{R})$ invariant it is determined by its value on $\Lambda^d(\mathfrak{p})$. On $\mathfrak{p}$ we have the euclidian metric given by the Killing form and we chose an orientation. These two data provide a second top degree form ω^{Kill} on $\Lambda^d(\mathfrak{p})$, and hence an invariant form also called ω^{Kill} on X. These two forms are proportional, i.e. we have kappa

$$\omega^{GB} = \kappa_\infty(G)\omega^{Kill}, \qquad (6.108)$$

the proportionality factor can be computed from the curvature tensor (See [33], 2.2 Kobayashi-Nomizu) We have the following

Proposition 6.3.10. *The factor $\kappa_\infty(G)$ is non zero if and only if $G\times_{\mathbb{Q}}\mathbb{R}$ is an inner form of its compact dual $G_c/\mathbb{R}$ or in other words $G\times_{\mathbb{Q}}\mathbb{R}$ has a compact maximal torus. If it is non zero then it is a real number and its sign is $(-1)^{\frac{d}{2}}$*

We remember that $G/\mathbb{R} = G\times_{\mathbb{Q}}\mathbb{R}$, the Lie-algebra $\mathfrak{g}$ is a $\mathbb{Q}$-vector space. The Killing form $\mathrm{B}:\mathfrak{g}\times\mathfrak{g}\to\mathbb{Q}$ is non degenerate and hence defines a top degree form ω_{B} on $\Lambda^{d_G}\mathfrak{g}$. If we use the decomposition $\mathfrak{g}\otimes\mathbb{R} = \mathrm{Lie}(K_\infty)\oplus\mathfrak{p}$ then we find $\omega_{\mathrm{B}} = \omega_{\mathrm{B}}^{K_\infty}\wedge\omega^{Kill}$ and hence

$$\int_{K_\infty}\omega_{\mathrm{B}} = \mathrm{vol}_{\omega_{\mathrm{B}}^{K_\infty}}(K_\infty)\omega^{Kill} \tag{6.109}$$

The top degree form ω_{B} which is defined over $\mathbb{Q}$ also provides invariant measures $\omega_{\mathrm{B},p}$ on all the groups $G(\mathbb{Q}_p)$ and also an invariant measure $\omega_{\mathrm{B},\infty}$ on $G(\mathbb{R})$. We can multiply these measures and get the Tamagawa measure

$$\omega_G^{Tam} = \omega_{\mathrm{B},\infty}\times\prod_p\omega_{\mathrm{B},p} = \omega_{\mathrm{B},\infty}\times\omega_{G,f}^{Tam} \tag{6.110}$$

this product is absolutely convergent and provides an $G(\mathbb{A})$-invariant measure on $G(\mathbb{A})$. This is the *Tamagawa measure* on $G(\mathbb{A})$. It is an important fact, that this measure does not depend on the choice of the top degree form ω_{B}. If we multiply ω_{B} by a non zero number $a\in\mathbb{Q}^\times$ then the local measures get multiplied by $|a|_\infty$ at infinity and $|a|_p$ at the finite places. Hence we see that ω_{B} and $a\omega_{\mathrm{B}}$ yield the same Tamagawa measure. The number

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})}\omega_G^{Tam} = \tau(G)$$

is called the Tamagawa number.

If we have written the Tamagawa measure as a product as in (6.110) we say that the Tamagawa measure is represented by ω_{B}. The remark above tells us that we may replace ω_{B} by any non zero invariant top degree form.

Now a miracle occurs

Theorem 6.3.2. *If $G/\mathbb{Q}$ is simply connected then*

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})}\omega_G^{Tam} = \tau(G) = 1$$

This theorem was conjectured by Weil, he ..... For a general semi simple $G/\mathbb{Q}$ we can consider the universal covering by a simply connected $\pi: G^{sc}/\mathbb{Q}\to G/\mathbb{Q}$, then $\tau(G)$ is a rational number which can be expressed in terms of Galois cohomology data of the finite kernel of π.

This gives us a way to compute the integral in (6.107). We recall that $\mathcal{S}^G_{K_f} = \bigcup_i\Gamma_i\backslash G(\mathbb{R})/K_\infty = \bigcup_i\Gamma_i\backslash G(\mathbb{R})/K_\infty\times\underline{x}_iK_f/K_f$ (prop. 6.1.4) and hence

$$\int_{\mathcal{S}^G_{K_f}}\omega^{GB} = \sum_i\int_{\Gamma_i\backslash X}\omega^{GB} = \kappa_\infty(G)\sum_i\int_{\Gamma_i\backslash X}\omega^{Kill} =$$

$$\frac{\kappa_\infty(G)}{\mathrm{vol}_{\omega_B^{K_\infty}}(K_\infty)}\sum_i\int_{\Gamma_i\backslash\mathcal{X}}\omega_B = \frac{\kappa_\infty(G)}{\mathrm{vol}_{\omega_B^{K_\infty}}(K_\infty)\,\mathrm{vol}_{\omega_{G,f}^{Tam}}(K_f)}\int_{G(\mathbb{Q})\backslash G(\mathbb{A})}\omega_G^{Tam} \tag{6.111}$$

Hence we see that for a neat K_f

$$\chi(H^\bullet(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\mathbb{Q})) = \frac{\kappa_\infty(G)}{\mathrm{vol}_{\omega_B^{K_\infty}}(K_\infty)\ \mathrm{vol}_{\omega^{Tam}_{G,f}}(K_f)}\dim(\mathcal{M}_\mathbb{Q})\tau(G) \tag{6.112}$$

We study the factor in front. We assume that the level group K_f is a product over local factors. i.e. we assume $K_f = \prod_{p:p \text{ prime}} K_p$. Then clearly

$$\frac{1}{\mathrm{vol}_{\omega^{Tam}_{G,f}}(K_f)} = \prod_p \frac{1}{\mathrm{vol}_{\omega_{B,p}}(K_p)},$$

the message is that the factor in front is a product over local contributions at the places of $\mathbb{Q}$, i.e. a local contribution at infinity and a local contribution at each prime.

We study the local factors $\mathrm{vol}_{\omega_{B,p}}(K_p)$. Of course we have to tell what K_p should be. To define such a subgroup K_f we choose a flat integral structure $\mathcal{G}/\mathbb{Z}$ (See section 1.2.1) of $G/\mathbb{Q}$ and define K_p a congruence subgroup of $\mathcal{G}(\mathbb{Z}_p)$. We know that there is a finite set Σ of primes such that for $p \notin \Sigma$ the following is true

a) The group scheme $\mathcal{G} \times \mathbb{Z}_p/\mathbb{Z}_p$ is semi simple and $K_p = \mathcal{G}(\mathbb{Z}_p)$

b) The top degree form $\omega_{B,p}$ on $\Lambda^{d_G}(\mathfrak{g} \otimes \mathbb{Z}_p)$ is non zero $\mod p$.

We think that it was Tamagawa who pointed out that under these conditions, i.e. $p \notin \Sigma$ we have

$$\mathrm{vol}_{\omega_{B,p}}(K_p) = p^{-d_G}\#\mathcal{G}(\mathbb{F}_p) \tag{6.113}$$

For the following we refer to the lectures of R. Steinberg [104]. We recall the well known formula for $\#\mathcal{G}(\mathbb{F}_p)$. For the moment we assume that $G/\mathbb{Q}$ is an inner form of the split $\mathbb{Q}$ -form, hence $\mathcal{G} \times \mathbb{F}_p$ is a split Chevalley group. Then it is well known that

$$p^{-d_G}\#\mathcal{G}(\mathbb{F}_p) = (1-\frac{1}{p})^r\prod_{i=1}^r((1+\frac{1}{p}+\dots+\frac{1}{p^{m_i}}) = \prod_{i=1}^r(1-\frac{1}{p^{m_i+1}}) \tag{6.114}$$

where r is the rank of $G/\mathbb{Q}$ (the dimension of a maximal torus) and the m_i are so called exponents (See [?], [?] ,,,). The expression $(1-\frac{1}{p^{m_i+1}}) = \zeta_p(m_i+1)^{-1}$ where $\zeta_p(s)$ is the local Euler factor of the Riemann-zeta function at p.

Hence we get

$$\frac{1}{\mathrm{vol}_{\omega^{Tam}_{G,f}}(K_f)} = (\prod_{p\in\Sigma}\frac{\prod_{i=1}^r\zeta_p(m_i+1)^{-1}}{\mathrm{vol}_{\omega_{B,p}}(K_p)})\prod_{i=1}^r\zeta(m_i+1) \tag{6.115}$$

Hence we are left with the computation of $\mathrm{vol}_{\omega_{B,p}}(K_p)$ at the finitely many ramified places $p \in \Sigma$. Of course $\mathrm{vol}_{\omega_{B,p}}(K_p) = \frac{\mathrm{vol}_{\omega_{B,p}}(\mathcal{G}(\mathbb{Z}_p))}{[\mathcal{G}(\mathbb{Z}_p):K_p]}.$ The computation of $\mathrm{vol}_{\omega_{B,p}}(\mathcal{G}(\mathbb{Z}_p))$ may become tedious depending on how badly ramified the group scheme $\mathcal{G} \times \mathbb{Z}_p$ at $p \in \Sigma$ wilfre know that for $r >> 0$

$$\mathrm{vol}_{\omega_{B,p}}(\mathcal{G}(\mathbb{Z}_p)) = \#\mathcal{G}(\mathbb{Z}_p/p^r)p^{a(\omega_{B,p})-\dim(G)r} \tag{6.116}$$

where $a(\omega_{B,p})$ is an integer depending on the choice of ω_G^{Tam}. Therefore it is clear that $\mathrm{vol}_{\omega_{B,p}}(K_p)$ is a non zero rational number. For a very special case we discuss this computation further down

Therefore we can sum up and get for a split and simply connected $G/\mathbb{Q}$ the final formula

$$\chi(H^\bullet(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\mathbb{Q}})) = \frac{\kappa_\infty(G)}{\mathrm{vol}_{\omega_B^{K_\infty}}(K_\infty)}(\prod_{p\in\Sigma}\frac{\prod_{i=1}^r \zeta_p(m_i+1)^{-1}}{\mathrm{vol}_{\omega_{B,p}}(K_p)})\dim(\mathcal{M}_{\mathbb{Q}})\prod_{i=1}^r\zeta(m_i+1) \tag{6.117}$$

Formulas of this kind have been proved by C. L. Siegel, I. Satake ([102] [90]) and others. For the case of general semi simple groups this is in [33]. The numbers m_i in [33] are the numbers m_i+1 here.

We also discuss the case that $G/\mathbb{Q}$ is not an inner form of the split form $G_0/\mathbb{Q}$, we assume that $G_0/\mathbb{Q}$ is simple, let Φ be its Dynkin diagrams. It is one of the form $A_n, n\geq 2, D_n, n\geq 4$ or E_6. In this case there is a unique normal extension L/Q and a faithful action of $\mathrm{Gal}(L/\mathbb{Q})$ on Φ, in other words an inclusion $j:\ \mathrm{Gal}(L/\mathbb{Q})\hookrightarrow \mathrm{Aut}(\Phi)$. For these above diagrams the group of automorphisms is of order 2 except we are in the case D_4, in this cases it is the symmetric group in three letters (See for instance [104] , Chap. 10, p.85).

Again we find a finite set Σ of primes such that for $p\notin\Sigma$ the group $\mathcal{G}\times\mathbb{F}_p$ is semi simple, but possibly only quasisplit,.assume we are in the case that $[L:\mathbb{Q}]=2$, this is certainly the case if $G_0/\mathbb{Q}$ is not of type D_4. If now $p\notin\Sigma$ and p splits, then the formula (6.114) is still valid. If p is inert in L, i.e. $\mathcal{O}_L/p\mathcal{O}_L=\mathbb{F}_{p^2}$ then we get from [104] the recipe how to modify the right hand side in (6.114).

a) In case $G_0/\mathbb{Q}$ is of type A_n, D_n and n odd or E_6 then we have to replace the factor $(1-\frac{1}{p^{m_i+1}})$ by $(1+\frac{1}{p^{m_i+1}})$ in case m_i is even.

b) In the case D_n and n even then $m_i=n-1$ occurs twice and we have to replace the factor

$$(1-\frac{1}{p^n})^2 \text{ by } (1-\frac{1}{p^n})(1+\frac{1}{p^n}).$$

Finally we come to the case D_4 and $\mathrm{Gal}(L/\mathbb{Q})$ is cyclic of order 3 or the symmetric group in three letters. Let $\mathfrak{P}$ be a prime ideal in $\mathcal{O}_L$ which lies over p. Then $\mathcal{O}_L/\mathfrak{P}=\mathbb{F}_p,\mathbb{F}_{p^2}$ or $\mathbb{F}_{p^3}$. The first two cases are handled by b) . In the third case we have to replace (See[104], Table on p. 105)

$$(1-\frac{1}{p^4})^2 \text{ by } 1+\frac{1}{p^4}+\frac{1}{p^8}=(1-\zeta_3p^{-4})(1-\zeta_3^2p^{-4})$$

where ζ_3 is a third root of unity ($\neq 1$.)

Now it is clear how we have to modify the formula (6.117). If $[L:\mathbb{Q}]=2$ we have the character $\chi_{L/\mathbb{Q}}$ corresponding to this extension For $p\notin\Sigma$ we have $\chi_{L/\mathbb{Q}}(p)=-1$ if and only if p does not split in L. Attached to this character we have the Dirichlet L- function

$$L(\chi_{L/\mathbb{Q}},s)=\prod_p\frac{1}{1-\chi_{L/\mathbb{Q}}(p)p^{-s}}.$$

This means that in formula (6.117) we have to replace the factors $\zeta(m_1+1)$ by $L(\chi_{L/\mathbb{Q}}^{m_i+1}, m_i+1)$. If we agree that for m_i+1 even $L(\chi_{L/\mathbb{Q}}^{m_i+1}, s) = \zeta(s)$ then (6.117) in case $[L:\mathbb{Q}] = 2$ becomes

$$\chi(H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\mathbb{Q})) =$$

$$\frac{\kappa_\infty(G)}{\mathrm{vol}_{\omega_B^{K_\infty}}(K_\infty)}\Big(\prod_{p\in\Sigma}\frac{\prod_{i=1}^r L_p(\chi_{L/\mathbb{Q}}^{\epsilon(m_i)}, m_i+1)^{-1}}{\mathrm{vol}_{\omega_{B,p}}(K_p)}\Big)\dim(\mathcal{M}_\mathbb{Q})\prod_{i=1}^r L(\chi_{L/\mathbb{Q}}^{\epsilon(m_i)}, m_i+1) \tag{6.118}$$

here we have to say what $\epsilon(m_i)$ is.

a) If $G_0/\mathbb{Q}$ is of type A_n, E_6 or D_n with n odd then $\epsilon(m_i) = 0$ if m_i is odd and $\epsilon(m_i) = 1$ if m_i is even.

b) If $G_0/\mathbb{Q}$ is of type D_n with n even then the exponent $n-1 = m_{n/2} = m_{n/2+1}$ occurs twice. In this case $\epsilon(m_i) = 0$ for $i \neq \frac{n}{2}$ or $\frac{n}{2}+1$ and $\epsilon(m_{n/2}) = 0, \epsilon(m_{n/2+1}) = 1$ (Hence we see that in the case D_n we have exactly one genuine Dirichlet L- function in the product.)

c) Finally we look at the case D_4 and assume $\mathrm{Gal}(L/\mathbb{Q})$ is the symmetric group in three letters or cyclic of order three. In this case we have an irreducible representation $\rho_2 :\ \mathrm{Gal}(L/\mathbb{Q}) \to \mathrm{Gl}_2(\mathbb{Q})$. to this representation we attach the Artin-L- function. Then it is clear that we have to replace the factor $\zeta(4)^2$ by the factor $L(\rho_2, 4)$.

This implies of course, that for a covering $\mathcal{S}^G_{K'_f} \to \mathcal{S}^G_{K_f}$, where $K'_f \subset K_f$ and both groups are neat,we get

$$\chi(\mathcal{S}^G_{K'_f}, \tilde{\mathcal{M}}) = \chi(H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})[K_f : K'_f],$$

a fact which also follows easily from topological considerations.

This leads us-following C.T.C. Wall- to introduce the orbifold Euler characteristic for a not necessarily neat K_f by

$$\chi_{\mathrm{orb}}(H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})) = \frac{1}{[K'_f : K_f]}\chi(\mathcal{S}^G_{K'_f}, \tilde{\mathcal{M}}) \tag{6.119}$$

where $K'_f \subset K_f$ is a neat subgroup of finite index. The orbifold Euler characteristic may differ from the Euler characteristic $\chi(H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}))$ by a sum of contributions coming from the set of fixed points of the Γ_i on X (See 1.1.2.1). Hence the formulae (6.117) and (6.118) remain valid without the assumption K_f neat once we replace $\chi(\mathcal{S}^G_{K'_f}, \tilde{\mathcal{M}})$ by $\chi_{\mathrm{orb}}(H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}))$ on the left hand side.

The Gauss-Bonnet formula implies that the orbifold Euler characteristic is linear in $\dim(\mathcal{M}_\mathbb{Q})$. But this is an obvious consequence of our considerations in section 2.1.3. We compute the cohomology from the Čech complex given by an orbiconvex covering. If our group K_f is neat then all the terms $\tilde{\mathcal{M}}(U_{\underline{i}})$ in (**??** are of the form $\tilde{\mathbb{Q}}(U_{\underline{i}}) \otimes \mathcal{M}$ where of course $\tilde{\mathbb{Q}}$ is the sheaf obtained from the trivial one-dimensional representation. The differentials in the complex are only acting on the first factor and hence

$$C^\bullet(\mathfrak{U}, \tilde{\mathcal{M}}) \xrightarrow{\sim} C^\bullet(\mathfrak{U}, \tilde{\mathbb{Q}}) \otimes \mathcal{M}.$$

Since $\chi(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}) = \chi(C^\bullet(\mathfrak{U}, \tilde{\mathcal{M}})) = \chi(C^\bullet(\mathfrak{U}, \tilde{\mathbb{Q}})) \times \dim \mathcal{M}$ the linearity follows.

Gauss-Bonnet and the special values

We discuss some arithmetic consequences of the Gauss-Bonnet formula. By definition $\chi_{\mathrm{orb}}(H^\bullet(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}))\in\mathbb{Q}$, hence we can conclude that the right side must a rational number. This argument gives us non trivial consequences for the special values $\zeta(m_i+1)$, but only if the curvature factor $\kappa_\infty(G)\neq 0$. We analyse this condition.

Remember that we want to assume that $G/\mathbb{Q}$ is absolutely simple, We consider the base extension $G\times_{\mathbb{Q}}\mathbb{R}$, then the complex conjugation $\mathbf{c}$ induces an involution on Φ. Now it is known

Proposition 6.3.11. *The Dynkin diagrams is of the form $A_1, B_n, C_n, E_7, E_8, F_4, G_2$ have trivial automorphism groups and $G\times_{\mathbb{Q}}\mathbb{R}$ is an inner form of its compact dual. The Dynkin diagram D_n has non trivial automorphisms. In this case $G\times_{\mathbb{Q}}\mathbb{R}$ is an inner form of its compact dual if*

a) the complex conjugation $\mathbf{c}$ acts trivially on Φ if n is even.

b) the complex conjugation $\mathbf{c}$ acts non trivially on Φ if n is odd.

In the remaining cases $G\times_{\mathbb{Q}}\mathbb{R}$ is an inner form of its compact dual if $\mathbf{c}$ acts non trivially.

In the cases where $\mathbf{c}$ acts trivially all the m_i are odd.

Now we have a look at the numbers $\zeta(m_i+1)$ and $L(\chi_{L/\mathbb{Q}}^{\epsilon(m_i)}, m_i+1)$ on the right hand side. Euler has shown that

$$\text{For even integers } m\geq 2 \text{ the numbers } \frac{\zeta(m)}{\pi^m}\in\mathbb{Q}. \tag{6.120}$$

We also know that the matching answer for $L(\chi_{L/\mathbb{Q}}, m)$ depends on the parity $p(\chi_{L/\mathbb{Q}})$ of $\chi_{L/\mathbb{Q}}$, where $p(\chi_{L/\mathbb{Q}})=1$ if $L_\infty=\mathbb{C}$ and 0 else. If we write $L=\mathbb{Q}[\sqrt{d}]$ we have $p(\chi_{L/\mathbb{Q}})=1$ if and only if $d<0$. Then we have the more general result

$$\text{For integers } m>0 \text{ and } m+p(\chi_{L/\mathbb{Q}}) \text{ even the numbers } \frac{L(\chi_{L/\mathbb{Q}}, m)}{\pi^m\sqrt{|d|}}\in\mathbb{Q} \tag{6.121}$$

The values $\zeta(m)$ for $m=2,4,\dots$ or $L(\chi_{L/\mathbb{Q}}, m)$ for $m>0$; $m+p(\chi_{L/\mathbb{Q}})\equiv 0 \mod 2$ are the so called *special values* of the Riemann ζ function or more generally Dirichlet -L function.

For the following we refer to [78] Chapter VII. We still have the functional equation. We introduce the Euler-factor at infinity

$$L_\infty(\chi_{L/\mathbb{Q}}, s) := (\frac{|d|}{\pi})^{\frac{s}{2}}\Gamma(\frac{s+p(\chi_{L/\mathbb{Q}})}{2})$$

where Γ is the Gamma-function and here we also assume that d is squarefree. We define the completed L-function

$$\Lambda(\chi_{L/\mathbb{Q}}, s) = L_\infty(\chi_{L/\mathbb{Q}}, s)L(\chi_{L/\mathbb{Q}}, s) \tag{6.122}$$

If $\chi_{L/\mathbb{Q}}$ is not trivial then $\Lambda(\chi_{L/\mathbb{Q}}, s)$ is holomorphic in the entire complex plane, if $\chi_{L/\mathbb{Q}}$ is the trivial character then we get the completed Riemann ζ function $\Lambda(s)=\frac{\Gamma(s/2)}{(2\pi)^{s/2}}\zeta(s)$. It is meromorphic function in the entire complex plane and has two simple poles at $s = 1$ and $s = 0$. For this completed L function we have the functional equation (See [78], Chap. VII, Theorem 2.8)

$$\Lambda(\chi_{L/\mathbb{Q}}, s) = W(\chi)\Lambda(\chi_{L/\mathbb{Q}}, 1-s) \tag{6.123}$$

where in this special case $W(\chi)$ is an integral power of $i = \sqrt{-1}$ and we observe that $\chi_{L/\mathbb{Q}}$ is real.

This tells us something about the special values at negative integers: For $m > 0; m + p(\chi_{L/\mathbb{Q}}) \equiv 0 \mod 2$ we get

$$L(\chi_{L/\mathbb{Q}}, 1-m) = \frac{L_\infty(\chi_{L/\mathbb{Q}}, m)}{L_\infty(\chi_{L/\mathbb{Q}}, 1-m)} L(\chi_{L/\mathbb{Q}}, m). \tag{6.124}$$

Using the functional equation for the Γ-function and $\Gamma(\frac{1}{2})) = \sqrt{\pi}$ we get for ratio of the two Euler factors at infinity

$$|\frac{d}{\pi}|^{m-\frac{1}{2}} \frac{\Gamma(\frac{m+p(\chi_{L/\mathbb{Q}}))}{2})}{\Gamma(\frac{1-m-p(\chi_{L/\mathbb{Q}})}{2})} = (\frac{|d|}{\pi})^m \frac{\sqrt{\pi}}{\sqrt{|d|}} \frac{\Gamma(m)}{\sqrt{\pi}2^{m-1}}$$

and if we insert this in (6.124) we get

$$L(\chi_{L/\mathbb{Q}}, 1-m) = |d|^m \frac{\Gamma(m)}{2^{m-1}} \frac{L(\chi_{L/\mathbb{Q}}, m).}{\sqrt{|d|}\pi^m} \in \mathbb{Q}^\times \tag{6.125}$$

Finally we have understand the contribution from the infinite place, i.e. the term $\frac{\kappa_\infty(G)}{\mathrm{vol}_{\omega_B^{K_\infty}}(K_\infty)}$. This term has been computed in [33] in the case of split groups G/F, where F is a totally real number field. In our case $F = \mathbb{Q}$ this gives

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$$\frac{\kappa_\infty(G)}{\mathrm{vol}_{\omega_B^{K_\infty}}(K_\infty)} = (-1)^{\frac{d}{2}} \frac{\prod_{i=1}^r (m_i+1)!}{\#W_{K_\infty} \pi^{r+\sum_{i=1}^r m_i}} = \frac{a_\infty(G))}{\pi^{r+\sum_{i=1}^r m_i}} \tag{6.126}$$

here W_{K_∞} is the Weyl group of K_∞ and $a_\infty(G))$ is an explicitly computable rational number.Hence equation (6.117) becomes

$$\chi_{\mathrm{orb}}(H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})) = a_\infty(G)a(G, K_f)\dim(\mathcal{M}_\mathbb{Q}) \prod_{i=1}^r \frac{\zeta(m_i+1)}{\pi^{m_i+1}} \text{ with } a(G, K_f) \in \mathbb{Q}^\times, \tag{6.127}$$

the non zero rational number $a(G, K_f)$ can be explicitly computed.

The formula becomes much nicer if we apply the functional equation for the ζ-function and look at the special values at the negative arguments.

If $\mathcal{G}_0/\mathbb{Z}$ is split semi simple and simply connected and if we choose $K_f = \prod_p \mathcal{G}_0(\mathbb{Z}_p)$ then the computation in [33] p. 452-453 gives

$$\chi_{\mathrm{orb}}(H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})) = \frac{\#W_G}{2^r \#W_{K_\infty}} \dim(\mathcal{M}_\mathbb{Q}) \prod_{i=1}^r \zeta(-m_i) \tag{6.128}$$

Now we see that the Gaus-Bonnet formula reproves Euler's rationality results. We consider the split groups schemes of type A_1 and B_n or C_n. In these cases the exponents are the odd numbers $1, 3, \dots 2n-1$. (See [13] , Planche II, III) If we apply the formula for A_1 we get $\zeta(2)/\pi^2 \in \mathbb{Q}^\times$. Applying it for B_2 or C_2 gives $\zeta(2)/\pi^2 \times \zeta(4)/\pi^4 \in \mathbb{Q}^\times$ hence $\zeta(4)/\pi^4 \in \mathbb{Q}^\times$. Clearly we get Euler's result by induction.

We also get the corresponding rationality results for the Dirichlet-L functions attached to quadratic characters $\chi_{L/\mathbb{Q}}$. We consider absolutely simple groups $G/\mathbb{Q}$ with non trivial action of $\mathrm{Gal}(L/\mathbb{Q})$ on their Dynkin diagram Φ. If $L/\mathbb{Q}$ is real quadratic, i.e. $p(\chi_{L//Q}) = 0$ then the situation is the essentially same as in the case of a trivial character.

If $L = \mathbb{Q}[\sqrt{d}]$ is imaginary quadratic we choose a group $G/\mathbb{Q}$ of type A_n with $n \geq 2$, or D_n with n odd, or E_6. In these cases the Gauss-Bonnet formula becomes

$$\chi_{\mathrm{orb}}(H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})) = a_\infty(G) a(G, K_f) \dim(\mathcal{M}_\mathbb{Q}) \prod_{i=1}^{r} L(\chi_{L/\mathbb{Q}}^{\epsilon(m_i)}, -m_i) \quad (6.129)$$

Hence see that for an imaginary quadratic extension $L/\mathbb{Q}$ we can start from groups $G/\mathbb{Q}$ of type A_n for $n = 2, 3, 4 \dots$ to prove (**??**).

Of course we can also consider the case that $G(\mathbb{R})$ is compact. (See ??) in this case the Gauss-Bonnet theorem is a tautology, the quotient

$$\mathcal{S}^G_{K_f} = G(\mathbb{Q}) \backslash * \times G(\mathbb{A}_f)/K_f$$

is a finite set. If K_f is neat then

$$\chi(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}) = \dim H^0(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}) = \#(G(\mathbb{Q}) \backslash * \times G(\mathbb{A}_f)/K_f) \dim(\mathcal{M}_\mathbb{Q}).$$

If $K_f^0 \subset G(\mathbb{A}_f)$ is not then we choose a normal neat subgroup $K_f \subset K_f^0$, and we consider the diagram

$$\begin{array}{ccc} G(\mathbb{A}_f)/K_f & \xrightarrow{\pi_{K_f}} & G(\mathbb{Q}) \backslash G(\mathbb{A}_f)/K_f = \mathcal{S}^G_{K_f} \\ q_0 \downarrow & & q_1 \downarrow \\ G(\mathbb{A}_f)/K_f^0 & \xrightarrow{\pi_{K_f^0}} & G(\mathbb{Q}) \backslash G(\mathbb{A}_f)/K_f^0 = \mathcal{S}^G_{K_f^0} \end{array} \quad (6.130)$$

We want to compute the orbifold Euler characteristic

$$\chi_{\mathrm{orb}}(\mathcal{S}^G_{K_f^0}, \tilde{\mathcal{M}}) = \frac{1}{[K_f^0 : K_f]} \chi(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})$$

to do this we have to understand the fibers of q_1. For a point $\underline{x} \in \mathcal{S}^G_{K_f^0}$ we pick a point $\underline{y} \in \pi^{-1}_{K_f^0}(\underline{x})$. Then we choose a point $\underline{y}_1 \in q_0^{-1}(\underline{y})$, now we can identify $q_0^{-1}(\underline{y}_1) = K_f^0/K_f$ by $\underline{k}_f \mapsto \underline{y}_1 \underline{k}_f$. If we apply π_{K_f} to the fiber $q_0^{-1}(\underline{y}_1)$ we get the fiber $q_1^{-1}(\underline{y})$. Now two points $\underline{y}_1 \underline{k}_f, \underline{y}_1 \underline{k}'_f$ map to the same point in $q_1^{-1}(\underline{y})$ if there is a $\gamma \in G(\mathbb{Q})$ such that $\gamma \underline{y}_1 \underline{k}_f K_f^0 = \underline{y}_1 \underline{k}'_f K_f^0$. Since K_f was a normal subgroup this means that $\gamma \underline{y}_1 \underline{k}'_f \underline{k}_f^{-1} \in \underline{y}_1 K_f^0$ and hence $\gamma \in \underline{y}_1 K_f^0 \underline{y}_1^{-1}$. Since K_f is neat we get an injection

$$\Gamma_{\underline{y}_1} := G(\mathbb{Q}) \cap \underline{y}_1 K_f^0 \underline{y}_1^{-1} / \underline{y}_1 K_f \underline{y}_1^{-1} \hookrightarrow \underline{y}_1 K_f^0 \underline{y}_1^{-1} / \underline{y}_1 K_f \underline{y}_1^{-1} \xrightarrow{\sim} K_f^0 / K.f$$

Now it is easy to see that the conjugacy class of the finite subgroup $\Gamma_{\underline{y}_1} \subset K_f^0/K.f$ only depends on $\underline{x} \in \mathcal{S}^G_{K_f^0}$ and not on the choices $\underline{y}$ or $\underline{y}_1$. Therefore all the $\Gamma_{\underline{y}_1}$ are isomorphic and we put $\#\Gamma_{\underline{x}} := \#\Gamma_{\underline{y}_1}$.

Now we see

$$\chi_{\text{orb}}(\mathcal{S}^G_{K_f^0}, \tilde{\mathcal{M}}) = \frac{1}{[K_f^0 : K_f]} \Big(\sum_{\underline{x} \in \mathcal{S}^G_{K_f^0}} \sum_{\underline{x}_1 \in q_1^{-1}(\underline{x})} \dim_{\mathbb{Q}}(\mathcal{M}) \Big) = \Big(\sum_{\underline{x} \in \mathcal{S}^G_{K_f^0}} \frac{1}{\#\Gamma_{\underline{x}}} \Big) \dim(\mathcal{M}_{\mathbb{Q}}) \tag{6.131}$$

We see that the orbilocal system $\tilde{\mathcal{M}}$ enters only by its dimension. This changes if we look at the Euler characteristic itself. Then we get obviously

$$\chi(\mathcal{S}^G_{K_f^0}, \tilde{\mathcal{M}}) = \sum_{\underline{x} \in \mathcal{S}^G_{K_f^0}} \dim(\mathcal{M}_{\mathbb{Q}}^{\Gamma_{\underline{x}}}) \tag{6.132}$$

and we notice that G-module structure of $\mathcal{M}$ matters. Since in general $\frac{\dim(\mathcal{M})_{\mathbb{Q}})}{\Gamma_{\underline{x}}} \neq \dim(\mathcal{M}^{\Gamma_{\underline{x}}})$ we see that we should expect $\chi_{\text{orb}}(\mathcal{S}^G_{K_f^0}, \tilde{\mathcal{M}}) \neq \chi(\mathcal{S}^G_{K_f^0}, \tilde{\mathcal{M}})$ in general, for instance if $\mathcal{M} = \mathbb{Q}$. Of course the formulas (6.117), (6.118) also apply in this situation. But here we have the advantage that the curvature factor $\kappa_\infty(G) = 1$.

There are cases where $\mathcal{G}/\mathbb{Z}$ is semi-simple and simply connected and $\mathcal{G}(\mathbb{R})$ is compact. In this case the computation in [33] gives us a variant of equation (6.128)

$$\chi_{orb}(\mathcal{S}^G_{K_f}, \mathbb{Q}) = \frac{1}{2^r} \prod_{i=1}^{r} \zeta(-m_i) \tag{6.133}$$

To give an example we consider n-dimensional unimodular lattices. An unimodular lattice L is a free $\mathbb{Z}$ - module, which is equipped with a symmetric bilinear form $F : L \times L \to \mathbb{Z}$ which has the following properties

a) It is positive definite, i.e. $F(x,x) > 0$ for all $x \neq 0$

b) For any saturated (See (???)) $x \in L$ we find a $y \in L$ such that $F(x,y) = 1$.

c) The values $F(x,x)$ are even.

Then we know that $n \equiv 0 \mod 8$ (See ???) and the group $\mathrm{SO}(F)/\mathbb{Z}$ is indeed a semi simple. (If we consider the base extension $\mathrm{SO}(F) \times_{\mathbb{Z}} \mathbb{Z}_p$ for any prime p the group scheme is isomorphic to $\mathrm{SO}(\frac{n}{2}, \frac{n}{2})/\mathbb{Z}_p$). This group scheme is not simply connected we have a degree 2 covering $\mathcal{G}/\mathbb{Z} = \mathrm{Spin}(F)/\mathbb{Z} \to \mathrm{SO}(F)/\mathbb{Z}$. Then equation (6.128) yields

$$\frac{1}{2^{\frac{n}{2}}} \prod_i \zeta(-m_i) = \sum_{\underline{x} \in \mathcal{S}^G_{K_f}} \frac{1}{\#\Gamma_{\underline{x}}} \tag{6.134}$$

We do this for $n = 8$. In this case the root lattice (L_8, F_8) of E_8 (See [**?**], or infinitely many other references) satisfies the conditions a), b), c) above and we get

$$\frac{1}{2^4}\zeta(-1)\zeta(-3)\zeta(-3)\zeta(-5) = \frac{1}{696729600} = \sum_{\underline{x}\in\mathcal{S}^G_{K_f}} \frac{1}{\#\Gamma_{\underline{x}}} \tag{6.135}$$

This implies that $\mathcal{S}^G_{K_f} = \mathcal{G}(\mathbb{Q})\backslash * \times\mathcal{G}(\mathbb{A}_f)/\mathcal{G}(\hat{\mathbb{Z}})$ consists of one element, namely the identity $\underline{x}_1$ and

$$\Gamma_{\underline{x}_1} = \mathcal{G}(\mathbb{Z}) = \mathcal{G}(\mathbb{Q}) \cap \mathcal{G}(\hat{\mathbb{Z}}).$$

We have the homomorphism $\mathcal{G}(\mathbb{Z}) \to \mathrm{SO}(F)(\mathbb{Z})$ its kernel is the group $\mu_2 = \{\pm 1\}$, and a simple calculation shows that the cokernel is also $\{\pm 1\}$, Since (L_8, F_8) is the root lattice of E_8 we get that $\mathrm{SO}(F_8)(\mathbb{Z})$ is the Weyl $W(E_8)$ group of E_8. The order of the Weyl group of E_8 is 696729600, hence we have verified equation (6.128) in this particular case.

If we play the same game for $n = 16$ then we start from the lattice $L_8 \oplus L_8$. The automorphism group of this lattice is $\Gamma_{\underline{x}_1} \times \Gamma_{\underline{x}_1} \times \mathbb{Z}/2\mathbb{Z}$, we may flip the two summands. Then

$$\frac{1}{2^8}\zeta(-1)\zeta(-3)\zeta(-5)\zeta(-7)^2\zeta(-9)\zeta(-11)\zeta(-13) = \sum_{\underline{x}} \frac{1}{\#\Gamma_{\underline{x}}} \tag{6.136}$$

On the right hand side we have the summand $\frac{1}{2*696729600^2}$ we subtract it and get

$$\frac{1}{2^8}\zeta(-1)\dots\zeta(-13) - \frac{1}{2*696729600^2} = \frac{1}{685597979049984000}$$

Hence we see that $\mathcal{S}^G_{K_f}$ consists of exactly two elements, we have the lattice $L_8 \oplus L_8$ and still another one. This has been discovered by E. Witt in [115]. In the same paper Witt mentions that he has found more then 10 different lattices for $n = 24$.

The case $n = 24$ was solved by Niemeier in [79], he showed that there exactly 24 different lattices, one of them is the famous Leech lattice (See editors note to the paper [115]).

Unimodular lattices are studies intensively in the book of G. Chenevier-J. Lannes [18].

We get also get (semi)- simple group schemes $\mathcal{G}/\mathbb{Z}$ with $\mathcal{G}(\mathbb{R})$ compact if start from a Dynkin diagram for which the simply connected group has trivial center. This follows from the Hasse principle. Hence we can find such a $\mathcal{G}/\mathbb{Z}$ of type E_8. Then we get

$$\frac{1}{2^8}\zeta(-1)\zeta(-7)\zeta(-11)\zeta(-13)\zeta(-17)\zeta(-19)\zeta(-23)\zeta(-29) =$$

$$\frac{2155741910416889170788798426985697}{1547054925088594115690496000000} = \sum_{\underline{x}\in\mathcal{S}^G_{K_f}} \frac{1}{\#\Gamma_{\underline{x}}} \tag{6.137}$$

The comparison

If we have two semi simple groups $G/\mathbb{Q}, G'/\mathbb{Q}$ which are inner forms of each other, then for both groups the product of $L-$ values in (6.129) is the same. Hence we see that the ratio $\chi_{\rm orb}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}})/\chi_{\rm orb}(\mathcal{S}^{G'}_{K'_f},\tilde{\mathcal{M}}')$ of the Euler characteristics is a number which can be computed from comparing local data at a finite number of primes, this is sometimes called the Hirzebruch proportionality principle.

We want to be a little more precise. We need to find a way to compare the groups K_f and K'_f. To make such a comparison possible, we use the ideas of Bruhat-Tits.

We start from any extension of $G/\mathbb{Q}, G'/\mathbb{Q}$ to a smooth group schemes over $\mathbb{Z}$. (See section 1.2.1). For all primes p outside a finite set Σ these two extensions will be semi simple at p. Then we get semi simple extensions $\mathcal{G}^*/(\text{ Spec}(\mathbb{Z})\setminus\Sigma)$ and ${}^*\mathcal{G}'/(\text{ Spec}(\mathbb{Z})\setminus\Sigma)$. At the primes $p\in\Sigma$ we choose extensions $G/\mathbb{Q}, G'/\mathbb{Q}$ to flat, smooth Bruhat-Tits group schemes $\mathcal{G}\times_{\mathbb{Z}}\mathbb{Z}_p$ and $\mathcal{G}'\times_{\mathbb{Z}}\mathbb{Z}_p$. We require that the two group schemes $\mathcal{G}\times_{\mathbb{Z}}\mathbb{Z}_p$ and $\mathcal{G}'\times_{\mathbb{Z}}\mathbb{Z}_p$ are locally isomorphic for the etale topology. In less educated language this means that we can find a finite unramified extension $F_{\mathfrak{p}}/\mathbb{Q}_p$ such that $\mathcal{G}\times_{\mathbb{Z}_p}\mathcal{O}_{F,\mathfrak{p}}$ and $\mathcal{G}'\times_{\mathbb{Z}_p}\mathcal{O}_{F,\mathfrak{p}}$ become isomorphic. Then we can use these extensions to extend ${}^*\mathcal{G},{}^*\mathcal{G}'$ to flat, smooth group scheme $\mathcal{G}/\mathbb{Z},\mathcal{G}'/\mathbb{Z}$ (We will give an example further down).

Now we may choose $K_p=\mathcal{G}(\mathbb{Z}_p), K'_p=\mathcal{G}'(\mathbb{Z}_p)$ and $K_f=\prod_p K_p, K'_f=\prod_p K'_p$. At a finite set of primes we may modify our choice and take full congruence subgroups $K_p=\mathcal{G}(\mathbb{Z}_p)(p^r), K'_p=\mathcal{G}'(\mathbb{Z}_p)(p^r)$. This makes it clear that -up to a power of p -the ratio

$$\frac{\mathrm{vol}_{\omega_{B,p}}(K_p)}{\mathrm{vol}_{\omega_{B,p}}(K'_p)}=p^{\delta_p(G,G')}\frac{\#\mathcal{G}(\mathbb{F}_p)}{\#\mathcal{G}'(\mathbb{F}_p)}, \tag{6.138}$$

the exponent $\delta(G,G')$ is explicitly computable, and the orders of the finite groups follow from Bruhat-Tits. The computation is basically straightforward but not completely trivial.

We discuss an example. Let $G/\mathbb{Q}=\mathrm{Sl}_2/\mathbb{Q}$, we choose a prime $p\equiv 3 \mod 4$ and we consider the division algebra $D(-p,-1)$ (see section 1.1.6) and put $G'/\mathbb{Q}=D^{(1)}(-p,-1)$ the norm one group of this division algebra. We put $L=\mathbb{Q}[\sqrt{-1}]$ then we get (see (1.40))

$$G'(\mathbb{Q})=\{x\in\mathrm{Sl}_2(L)\mid x=\begin{pmatrix}0&-p\\1&0\end{pmatrix}\sigma(x)\begin{pmatrix}0&-p\\1&0\end{pmatrix}^{-1}\} \tag{6.139}$$

and with $x=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ this means

$$G'(\mathbb{Q})=\{\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}\sigma(c)&-p\sigma(c)\\p^{-1}\sigma(b)&\sigma(a)\end{pmatrix}\} \tag{6.140}$$

and hence

$$G'(\mathbb{Q})=\{\begin{pmatrix}a&pb\\-\sigma(b)&\sigma(a)\end{pmatrix}\mid a\sigma(a)+pb\sigma(b)=1;\ a,b\in L\} \tag{6.141}$$

We choose extensions $\mathcal{G}'/\mathbb{Z}$ and $\mathcal{G}/\mathbb{Z}$ of of our groups, For any prime ℓ we require that

$$\mathcal{G}'(\mathbb{Z}_\ell) = \{\begin{pmatrix} a & pb \\ -\sigma(b) & \sigma(a) \end{pmatrix} \mid a\sigma(a) + pb\sigma(b) = 1;\ a, b \in \mathbb{Z}_\ell[i]\}$$

For $\ell \neq 2$ or $\ell \neq p$ it is easy to see that $\mathcal{G}/\mathbb{Z}_\ell \xrightarrow{\sim} \mathrm{Sl}_2/\mathbb{Z}_\ell$ and hence semi simple. For $\ell = 2$ we have to use $p \equiv 3 \mod 4$ and hence $-p \in N_{L/\mathbb{Q}}(\mathbb{Z}_2[i]^\times)$. Then it is again easy to see that $\mathcal{G}'/\mathbb{Z}_\ell$ must be semi-simple. It remains the case $\ell = p$. In this case p does not split in $\mathbb{Z}[i]$ and hence $\mathbb{Z}[i]/(p) = \mathbb{F}_{p^2}$. Hence we see that the reduction $\mod p$ gives us

$$\mathcal{G}'(\mathbb{F}_p) = \{\begin{pmatrix} a & 0 \\ -\sigma(b) & \sigma(a) \end{pmatrix} \mid a\sigma(a)) = 1\ a, b \in \mathbb{F}_{p^2}\}$$

Now we see that $\mathcal{G}' \times_{\mathbb{Z}} \mathbb{F}_p$ is not semi-simple, it has a non trivial unipotent radical, which is isomorphic to $R_{\mathbb{F}_{p^2}/\mathbb{F}_p}(\mathbb{G}_a)$. We get $\#\mathcal{G}'(\mathbb{F}_p) = (p+1)p^2$.

It is clear that this extension of $G'/\mathbb{Q}$ to $\mathcal{G}'/\mathbb{Z}$ is "optimal".

Now we extend $\mathrm{Sl}_2/\mathbb{Z}$ to $\mathcal{G}/\mathbb{Z}$, for any prime $\ell \neq p$ we choose the obvious extension $\mathrm{Sl}_2/\mathbb{Z}_\ell$. For p we choose an Iwahori -Bruhat-Tits group scheme $\mathcal{G}/\mathbb{Z}_p$, it is smooth and flat and

$$\mathcal{G}(\mathbb{Z}_p) = \{\begin{pmatrix} a & b \\ pc & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}_p, ad - pbc = 1\}.$$

The reduction $\mod p$ gives $\mathcal{G} \times_{\mathbb{Z}} \mathbb{F}_p = \mathbb{G}_m \ltimes (\mathbb{G}_a \times \mathbb{G}_a)$. It is clear that $\mathcal{G} \times \mathbb{Z}_p$ and $\mathcal{G}' \times \mathbb{Z}_p$ are locally isomorphic in the etale topology.

Hence we see that we can choose for our set $\Sigma = \{p\}$ and we get

$$\frac{\#\mathcal{G}(\mathbb{F}_p)}{\#\mathcal{G}'(\mathbb{F}_p)} = \frac{p-1}{p+1}$$

We still have to discuss the factor $p^{\delta_p(G,G')}$ and the contribution from the infinite place. We must go back to the definition of the Tamagawa measure. Since $\mathcal{G}/\mathbb{Z}$ and $\mathcal{G}'/\mathbb{Z}$ are smooth the Lie algebras of these group schemes are free $\mathbb{Z}$- modules, they are given by

$$\mathrm{Lie}(\mathcal{G}) = \mathbb{Z}H \oplus \mathbb{Z}pE_+ \oplus \mathbb{Z}E_-\ ;\ \mathrm{Lie}(\mathcal{G}') = \mathbb{Z}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \oplus \mathbb{Z}\begin{pmatrix} 0 & p \\ -1 & 0 \end{pmatrix} \oplus \mathbb{Z}\begin{pmatrix} 0 & pi \\ i & 0 \end{pmatrix}. \tag{6.142}$$

To define the Tamagawa measure we have to choose top degree non zero invariant differential forms, in this situation we gauge them by requiring

$$\omega_{\mathcal{G}}(H \wedge pE_+ \wedge E_-) = 1;\ \omega_{\mathcal{G}'}(\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \wedge \begin{pmatrix} 0 & p \\ -1 & 0 \end{pmatrix} \wedge \begin{pmatrix} 0 & pi \\ i & 0 \end{pmatrix}) = 1.$$

For any prime ℓ these linear forms provide invariant measures $\omega_{\mathcal{G},\ell}, \omega_{\mathcal{G}',\ell}$ on $G(\mathbb{Q}_\ell), G'(\mathbb{Q}_\ell)$ and $\omega_{\mathcal{G},\infty}, \omega_{\mathcal{G}',\infty}$ at the infinite place, the product of these local measures gives the Tamagawa -measures $\omega_G^{\mathrm{Tam}}, \omega_{G'}^{\mathrm{Tam}}$. We recall that these two measures do not depend on the choice of $\omega_{\mathcal{G}}, \omega'_{\mathcal{G}}$, but the local factors do. In a sense the choice of these two forms is optimal with respect to the " arithmetic aspects".

With this product realisation of the Tamagawa measure we get for the local factor at p in formula (See 6.117)

$$\frac{\zeta_p(2)}{\mathrm{vol}_{\omega_G^{\mathrm{Tam,f}}}(K_p)} = \frac{1}{p}(p+1); \ \frac{\zeta_p(2)}{\mathrm{vol}_{\omega_{G'}^{\mathrm{Tam,f}}}(K'_p)} = \frac{1}{p}(p-1)$$

We have to consider the contributions at the infinite place. We have to compare the measures $\omega_{\mathcal{G},\infty}, \omega_{\mathcal{G}',\infty}$ to the measures defined by the Killing form, and we have to compute the factor $\kappa_\infty(G)$.

We consider the cases $G = \mathrm{Sl}_2/\mathbb{Q}$ first. Then we have the decomposition

$$\mathfrak{g} = \mathbb{Q}Y \oplus \mathbb{Q}H \oplus \mathbb{Q}V = \mathfrak{k} \oplus \mathfrak{p}$$

this is an orthonormal decomposition for the Killingform B and $\mathrm{B}(Y,Y) = -8, \mathrm{B}(H,H) = 8, \mathrm{B}(Y,Y) = 8.$ and now we replace B by $\frac{1}{8}\mathrm{B}$. This normalised Killing form defines a top differential ω_{B} which satisfies $\omega_{\mathrm{B}}(Y,H,V) = 1$ this differential is of the form $\omega_{\mathrm{B}}^{K_\infty} \wedge \omega_{|\mathfrak{p}\mathrm{B}}$, here $\omega_{|\mathfrak{p}\mathrm{B}}$ is a 2-form on $\mathfrak{p}$, it is normalised by $\omega_{|\mathfrak{p}\mathrm{B}}(H,V) = 1$.

We have to compare ω_{B} and ω_G. We must express Y, H, V as linear combination of H, pE_+, E_- and then we get easily

$$\omega_{\mathrm{B}} = \pm\frac{p}{2}\omega_{\mathcal{G}}$$

Now it is well known that in this case $\omega^{GB} = -\frac{1}{2\pi}\omega_{|\mathfrak{p},\mathrm{B}}$ (i.e. $\kappa_\infty(G) = -\frac{1}{2\pi}$.) (See ???) and hence finally

$$\omega^{GB} = \pm\frac{p}{4\pi}\omega_{\mathcal{G}} \tag{6.143}$$

Now we represent the Tamagawa number by $\omega_{\mathcal{G}}$. We get

$$1 = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \omega^{\mathrm{Tam}} = \int_{\mathcal{G}(\mathbb{Z})\backslash G(\mathbb{R})} \omega_{\mathrm{B},\infty} \times \int_{K_f} \omega_{\mathrm{B},f} = \frac{p}{p+1}\zeta(2)^{-1} \times \int_{\mathcal{G}(\mathbb{Z})\backslash \mathbb{G}(\mathbb{R})} \omega_{\mathcal{G},\infty} \tag{6.144}$$

For the last factor we have

$$\int_{\mathcal{G}(\mathbb{Z})\backslash \mathbb{G}(\mathbb{R})} \omega_{\mathcal{G},\infty} = \pm\frac{2}{p}\int_{\mathcal{G}(\mathbb{Z})\backslash \mathbb{G}(\mathbb{R})} \omega_{\mathrm{B},\infty} \tag{6.145}$$

Now $\int_{K_\infty} \omega_{\mathrm{B}}^{K_\infty} = 2\pi$ and hence

$$\int_{\mathcal{G}(\mathbb{Z})\backslash \mathbb{G}(\mathbb{R})} \omega_{\mathrm{B},\infty} = \pm\frac{4\pi}{p}\int_{\mathcal{G}(\mathbb{Z})\backslash X} \omega_{|\mathfrak{p},\mathrm{B}} = \pm\frac{8\pi^2}{p}\int_{\mathcal{G}(\mathbb{Z})\backslash X} \omega^{GB} \tag{6.146}$$

and this finally results in

$$\chi_{\mathrm{orb}}(\mathcal{G}(\mathbb{Z})\backslash X) = \pm\int_{\mathcal{G}(\mathbb{Z})\backslash X} \omega^{GB} = -\frac{p+1}{8}\frac{\zeta(2)}{\pi^2} = -\frac{p+1}{48?} \tag{6.147}$$

Here we recollect that $\mathcal{G}(\mathbb{Z}) = \Gamma_0(p) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}_2(\mathbb{Z}) | c \equiv 0 \mod p\}$ and $X = \mathbb{H}$ the upper half plane.

We do the same calculation for $\mathcal{G}'$. The group

$$G'(\mathbb{R}) = \{\begin{pmatrix} x & py \\ -\sigma(y) & \sigma(x) \end{pmatrix} \mid x\sigma(x) + py\sigma(y) = 1;\ x, y \in \mathbb{C}\}$$

and we get the well known isomorphism

$$\Phi : G'(\mathbb{R}) \xrightarrow{\sim} S^3 \subset \mathbb{C}^2; \Phi : \begin{pmatrix} x & py \\ -\sigma(y) & \sigma(x) \end{pmatrix} \mapsto (x, \sqrt{p}\, y) \tag{6.148}$$

The identity element $e_{Gp} \in G'(\mathbb{R})$ is mapped to $(1,0) \in \mathbb{C}^2$(" = "$(1,0,0,0) \in \mathbb{R}^4$). The tangent space to the sphere at this point is $i\mathbb{R} \oplus \mathbb{C}$. The Lie algebra $\mathfrak{g}' \otimes \mathbb{R}$ is the tangent space of $G'(\mathbb{R})$ at $e_{G'}$ and the derivative D_Φ maps

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \mapsto (i,0)\text{" = "}(0,1,0,0); \begin{pmatrix} 0 & p \\ -1 & 0 \end{pmatrix} \mapsto (0, \sqrt{p}); \begin{pmatrix} 0 & p\, i \\ i & 0 \end{pmatrix} \mapsto (0, \sqrt{p}\, i),$$

we see that the images of our three basis vectors are orthogonal to each other. Hence the euclidian volume form evaluated at the triple of these three vectors gives

$$\omega^{eucl}((i,0), (0,\sqrt{p}), (0, \sqrt{p}\, i) = p$$

The volume of the 3-sphere with respect to the euclidian volume form is $4\pi^2$. Hence we get $\mathrm{vol}_{\omega_{\mathcal{G}',\infty}}(K'_\infty) = \frac{4\pi^2}{p}$. If we perform the same computation as in the first case we end up with

$$\chi_{\mathrm{orb}}(\mathcal{S}^{\mathcal{G}'}_{K'_f}, \tilde{\mathcal{M}}') = \frac{p-1}{4\pi^2}\zeta(2) = \frac{p-1}{48} \tag{6.149}$$

6.3.13 Some (philosophical) remarks

Of course the Gauss-Bonnet theorem only gives an alternating sum of dimensions of cohomology groups, we do not have any control of the possible cancellation. This is especially bad in the case when we have $\kappa_\infty(G) = 0$. We have seen that in the case where $G(\mathbb{R})$ is compact the space $\mathcal{S}^G_{K_f}$ is of dimension zero and hence all the cohomology sits in degree zero and there is no cancellation. In this case the differential geometric subleties also disappear, i.e. we have $\kappa_\infty(G) = 1$.

We still may ask: How do the cohomology groups behave once we vary the level K_f or the coefficient system $\mathcal{M}$. What happens if K_f gets smaller and smaller? If our coefficient system is a highest weight module $\mathcal{M}_\lambda$ where $\lambda = \sum n_i\gamma_i$, what happens if all the $n_i \to \infty$?

Let us fix a neat reference level $K_f^{(0)}$, and the reference coefficient system $\mathbb{Q}$. Then we have seen that for $K_f \subset K_f^{(0)}$ and any $\mathcal{M}_\lambda$ we have

$$\chi(\mathcal{S}^G_{K_f}, \mathcal{M}) = [K_f^{(0)} : K_f] \times \chi(\mathcal{S}^G_{K_f^{(0)}}, \mathbb{Q}) \times \dim(\mathcal{M})$$

We consider the case $\kappa_\infty(G) \neq 0$, then the dimension $d = \dim X$ is even. In this case one might expect that in a certain sense

$$\chi(\mathcal{S}^G_{K_f}, \mathcal{M}) \simeq (-1)^{\frac{d}{2}} \dim H^{\frac{d}{2}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}). \tag{6.150}$$

This expectation can be verified and also made precise in a few cases and it is also supported by experimental data.

If the highest weight λ is regular, i.e. if $n_i > 0 \ \forall i$ then it has been shown by J. S. Li -J. Schwermer in [?] and L. Saper in [?] that $H^\nu(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) = 0$ for $\nu < \frac{d}{2}$. Moreover it can be shown that all the cohomology $H^\nu(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$ with $\nu > \frac{d}{2}$ is Eisenstein cohomology (see section 9.2), and this gives us some confirmation of the statement above.

We drop the assumption on λ and vary K_f, then we have results by Lück and ...that

$$\lim_{K_f \to e} \frac{1}{[K_f^{(0)} : K_f]} H^\nu(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}) = \begin{cases} 0 & \text{if } \nu \neq \frac{d}{2} \\ \chi_{orb}(\mathcal{S}^G_{K_f}, \mathcal{M}) & \text{if } \nu = \frac{d}{2} \end{cases} \tag{6.151}$$

This is another piece of evidence for the above principle.

This last formula makes also sense if $\kappa_\infty(G) = 0$.

6.3.14 The topological trace formula

The Gauss-Bonnet formula is a special case of the topological trace formula (See [5],[29],[46]). The topological trace formula is a tool to compute the trace

$$\mathrm{tr}(T_h | H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}) = \sum_\nu (-1)^\nu \ \mathrm{tr}(T_h | H^\nu(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})) \tag{6.152}$$

of a Hecke operator T_h (See section 6.3). If we choose for h the characteristic function of K_f then this trace is equal to $\chi_{\mathrm{orb}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})$. The topological trace formula gives a formula for the traces of Hecke operators on the cohomology in terms of orbifold Euler characteristics of fixed point sets. These fixed point sets are again locally symmetric spaces and hence again can be computed using the Gauss-Bonnet theorem.

We come back to our two groups $G/\mathbb{Q}, G'/\mathbb{Q}$ and we assume that we have chosen compatible extensions $\mathcal{G}/\mathbb{Z}$ and $\mathcal{G}'/\mathbb{Z}$ as above. Again Σ will be the set of places where $\mathcal{G}, \mathcal{G}'$ are not semi simple. Then we can identify the central sub algebras $\mathcal{H}^{(\Sigma)} = \bigotimes_{p \notin \Sigma} \mathcal{H}_p = \mathcal{H}^{(\prime,\Sigma)} = \bigotimes_{p \notin \Sigma} \mathcal{H}'_p$. This means that we can compare the $\mathcal{H}^{(\Sigma)}$ - modules $H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})$ and $H^\bullet(\mathcal{S}^{G'}_{K'_f}, \mathcal{M}')$.

To get such a comparison we can invoke the topological trace formula, We have to make some clever choices of Hecke operators $h = h_{(\Sigma)} \times h^{(\Sigma)} \in \prod_{p \in \Sigma} \mathcal{H}_p \times \mathcal{H}^{(\Sigma)}$ and $h' = h'_{(\Sigma)} \times h^{(\Sigma)} \in \prod_{p \in \Sigma} \mathcal{H}'_p \times \mathcal{H}^{(\Sigma)}$.

Now we compute the traces on the two cohomology groups using the topological trace formula

$$\begin{aligned} \mathrm{tr}(T_h | H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}) &= \textstyle\sum_{\underline{x}:\text{fixed point}} T_h \underline{x} \\ \mathrm{tr}(T_{h'} | H^\bullet(\mathcal{S}^{G'}_{K'_f}, \mathcal{M}')) &= \textstyle\sum_{\underline{x}':\text{fixed point}} T_{h'} \underline{x}' \end{aligned} \tag{6.153}$$

where the numbers $T_h\underline{x}, T_{h'}\underline{x}'$ are local contributions, they are Euler characteristics of fixed point sets times so called orbital integrals.

Now we try to establish a correspondence between the two sets of fixed points such that for two corresponding points $\underline{x} \leftrightarrow \underline{x}'$ we have adapted our Hecke operators such that $T_h\underline{x} = T_{h'}\underline{x}'$ In case we do not find a corresponding point $\underline{x}'$ to a given $\underline{x}$ we must have $T_h\underline{x} = 0$. If we are lucky-and in fact we are in a few cases- we can show that the right hand sides in (6.153) are equal.

This comparison between the cohomology groups attached to two different groups has been executed in some detail in [46] for a pair $G = \mathrm{Gl}_2/\mathbb{Q}$ and the multiplicative group $G'/\mathbb{Q}$ of a division algebra. We also discuss the much more subtle of comparing $\mathrm{Sl}_2/\mathbb{Q}$ and where $G'/\mathbb{Q}$ is the norm one group of a division algebra. (See also [67],)

Arthur-Selberg trace formula vs. topological trace formula

In Chapter 8 we will discuss the description of the cohomology $H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\mathbb{C})$ in terms of automorphic forms. In the theory of automorphic forms we also can compare the spaces of automorphic forms for a pair of groups which are inner forms of each other. This occurs the first time in the fundamental book of Jacquet and Langlands [57] for the two groups Gl_2 and the multiplicative group of a division algebra. The result is the Jacquet- Langlands correspondence, which plays a predominant role in the theory of modular forms, The Jacquet -Langlands correspondence implies the above results on cohomology.

The main tool to prove the Jacquet-Langlands correspondence is the Arthur-Selberg trace formula, J. Arthur and many other people have developed this instrument to the case of general reductive groups. As an application they get results which allow a comparison of spaces of automorphic forms (See Arthur's papers) on different groups. The formulation and the proof of the Arthur-Selberg trace formula are peppered with enormous analytical difficulties, which make it difficult to apply it. The problem is the non compactness of $\mathcal{S}^G_{K_f}$, one encounters situations in which certain infinite sums or certain integrals are divergent and one has to renormalise them.

These subtle analytical problems disappear if we use the topological trace formula instead. In this context we encounter the problem how to treat the "fixed points at infinity", this is discussed and solved in [5] in the rank one case and in [29] in greater generality. It should be possible to prove many of the relevant consequences of the Arthur-Selberg trace formula by using the topological trace formula, provided we restrict our attention to the "cohomological part" of the space of automorphic forms. This applies especially to the comparison of the cohomology of to different groups and to questions of endoscopy. The proofs would dramatically simplify. On the other hand the problems with the stabilisation of the trace formula remain the same.

In section 3.2.1 we gave a general strategy how to write an algorithm to compute -at least in principle- cohomology groups $H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})$ and in addition to compute the action of a Hecke operator $T^{\mathrm{coh},\lambda}_{p,\chi}$ (**??**) on it. Together with H. Gangl we wrote such an algorithm in a baby case (3.3), and we used this to verify an assertion about denominators of Eisenstein classes experimentally.

We come back to this algorithm and discuss it in the case that $G(\mathbb{R})$ is compact. In this case $\mathcal{S}^G_{K_f} = *\times G(\mathbb{A}_f)/K_f$ is a finite set, let $\mathcal{A}(*\times G(\mathbb{A}_f)/K_f,\mathbb{Z})$ be the module of $\mathbb{Z}-$ valued functions on this finite set, it is of course equal to $H^0(\mathcal{S}^G_{K_f},\mathbb{Z})$. The space $\mathcal{A}(*\times G(\mathbb{A}_f)/K_f,\mathbb{C})$ is also called the space of algebraic modular forms. We have a basis given by the delta functions $\delta_{\underline{x}}$, where $\underline{x}$ runs through the points in $\mathcal{A}(*\times G(\mathbb{A}_f)/K_f,\mathbb{C})$.

We recall the definition of Hecke operators in this special situation. We pick an element $x_p \in G(\mathbb{Q}_p)$ we extend it to an adelic point $\underline{x}_p = (1,1,\dots,x_p,\dots 1,\dots)$ We consider the group $K(\underline{x}_p)_f := K_f \cap \underline{x}_p K_f \underline{x}_p^{-1}$ and the projection map $\pi_+ : G(\mathbb{Q})\backslash *\times G(\mathbb{A}_f)/K(\underline{x}_p)_f \to G(\mathbb{Q})\backslash *\times G(\mathbb{A}_f)/K_f$. If we multiply by $\underline{x}_p$ from the right we get a map

$$m(\underline{x}_p) : G(\mathbb{Q})\backslash *\times G(\mathbb{A}_f)/K(\underline{x}_p)_f \to G(\mathbb{Q})\backslash *\times G(\mathbb{A}_f)/K(\underline{x}_p^{-1})_f$$

$$m(\underline{x}_p) : \underline{y} \mapsto \underline{y}\underline{x}_p$$

Now the Hecke operator $T(\underline{x}_p) : H^0(\mathcal{S}^G_{K_f},\mathbb{Z}) \to H^0(\mathcal{S}^G_{K_f},\mathbb{Z})$ does the following: We choose a set $\underline{u}_1,\dots,\underline{u}_t$ for $K_f/K(\underline{x}_p)_f$ We pick an $\underline{x} \in G(\mathbb{Q})\backslash *\times G(\mathbb{A}_f)/K_f$ and represent by $\underline{\tilde{x}} \in G(\mathbb{A}_f)$. We consider the fiber $\pi_+^{-1}(\underline{x})$ the points in the fiber are represented by $\underline{x}\underline{u}_i, i = 1\dots t$. A point $\underline{y} \in \pi_+^{-1}(\underline{x})$ comes with a multiplicity $m(\underline{y})$, the number of times it is represented by a $\underline{\tilde{x}}\underline{u}_i$. The map $m(\underline{x}_p)$ maps the points $\underline{\tilde{x}}\underline{u}_i$ to $G(\mathbb{Q})\backslash *\times G(\mathbb{A}_f)/K(\underline{x}_p^{-1})_f$. To this set of points (with multiplicities) we apply the projection $\pi_- : G(\mathbb{Q})\backslash *\times G(\mathbb{A}_f)/K(\underline{x}_p^{-1})_f \to G(\mathbb{Q})\backslash *\times G(\mathbb{A}_f)/K_f$. We get a finite set of points $T(\underline{x},\underline{x}_p) \subset G(\mathbb{Q})\backslash *\times G(\mathbb{A}_f)/K_f$, where each point $\underline{z} \in T(\underline{x},\underline{x}_p)$ comes with a finite multiplicity $a_{\underline{x},\underline{z}}(\underline{x}_p)$, this is the number of times it is hit by a $\pi_-(m(\underline{x}_p)((\underline{\tilde{x}}\underline{u}_i)))$. Hence $a_{\underline{x},\underline{z}}(\underline{x}_p)$ is an integer > 0 for $\underline{z} \in T(\underline{x},\underline{x}_p)$, we put $a_{\underline{x},\underline{z}}(\underline{x}_p) = 0$ for $\underline{z} \notin T(\underline{x},\underline{x}_p)$. Then

$$T(\underline{x}_p)(\delta_{\underline{x}}) = \sum_{\underline{z}\in G(\mathbb{Q})\backslash *\times G(\mathbb{A}_f)/K_f} a_{\underline{x},\underline{z}}(\underline{x}_p)\delta_{\underline{z}} \tag{6.154}$$

The computation of this incidence matrix $(a_{\underline{x},\underline{z}}(\underline{x}_p))$ may become very difficult. In a slightly different context such computations are carried out in [77] and their results are presented in [18]. Even in the case of the group $\mathrm{Spin}(L_8 \oplus L_8)$ where we have only two elements in $\mathcal{S}^G_{K_f}$ the computation of the incidence matrix is by no means trivial. In [18] the authors define a Hecke operator T_2 using "Kneser Neighbors", this is essentially a $T(\underline{x}_2)$ as described above. And they give the resulting matrix

$$T_2 = \begin{pmatrix} 20025 & 18225 \\ 12870 & 14670 \end{pmatrix} \tag{6.155}$$

(One of the reasons we give this matrix here is the following observation: The difference of the two eigenvalues is divisible by 691 !) In [18] the authors also discuss T_2 for the lattice $L_8 \oplus L_8 \oplus L_8$ but they do not write the resulting $(24\times 24)-$ matrix.

On the other hand there are formulas in [18] which can not be obtained simple from a computer program, for instance **Theorem A** in section 1.2. on Kneser neighbors.

Of course we may also do this if we have a non trivial coefficient system $\mathcal{M}$. In case we need to know the incidence matrix but in addition we also have to keep track of the linear maps between the stalks of sheaves. Then even the case of the 8 dimensional lattice L_8 becomes non trivial.

It seems to be an interesting exercise to consider the two groups $\mathcal{G}'/\mathbb{Z} = \mathrm{Spin}(L_8)/\mathbb{Z}$ and $\mathcal{G}/\mathbb{Z} =$ the split Chevalley group of type D_4. Now we look at the unramified cohomology, i.e. we take $K_f = \mathcal{G}(\hat{\mathbb{Z}}), K'_f = \mathcal{G}'(\hat{\mathbb{Z}})$ and compare the Hecke modules

$$H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}) \text{ and } H^0(\mathcal{S}^{G'}_{K'_f}, \tilde{\mathcal{M}}') \tag{6.156}$$

The result should be compared to the results of Arthur.

This is perhaps the right moment, to discuss another minor technical point. When we discuss the action of the Hecke algebra $\mathcal{H}_{K_f} = \mathcal{C}_c(G(\mathbb{A}_f)//K_f, \mathbb{Q})$ on $H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})$ then we chose the same K_f for the space and for the Hecke algebra. We also normalized the measure on the group so that it gave volume 1 to K_f. But we have of course an inclusion of Hecke algebras $\mathcal{H}_{K_f} \subset \mathcal{H}_{K'_f}$. Therefore $\mathcal{H}_{K_f}$ also acts on $H^\bullet(\mathcal{S}^G_{K'_f}, \tilde{\mathcal{M}})$. This contains $H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})$ but then the inclusion is not compatible with the action of the Hecke algebra. We therefore choose a measure independently of the level, if we are in a situation where we vary the level. In such a case a measure provided by an invariant form ω_G on G (See 2.1.3) is a good choice. If we now define the action of the Hecke operators by means of this measure. With this choice of a measure the inclusion $\mathcal{H}_{K_f} \subset \mathcal{H}_{K'_f}$ is compatible with the inclusion of the cohomology groups.

Then we see the new Hecke operator $T_h^{(\omega_G)}$, and the old one are related by the formula

$$T_h = \frac{1}{\mathrm{vol}_{|\omega_G|}(K_f)} T_h^{(\omega_G)}$$

The reader might raise the question, why we work with fixed levels and why we do not pass to the limit. The reason is that for some questions we need to work with the integral cohomology, and this does not behave so well under change of level.

Chapter 7

The fundamental question

Let Σ be a finite set. Of course any product $V = \bigotimes' H_{\pi_p}$ of finite dimensional absolutely irreducible modules for the $\mathcal{H}_p$, for which $\mathcal{H}_p$ is spherical for all $p \notin \Sigma$ gives us an absolutely irreducible module for the Hecke algebra.

We may ask: Can we formulate non tautological conditions for the irreducible representation V or for the collection $\{\pi_p\}_{p:prime}$, which are necessary or (and) sufficient for the occurrence of $\otimes'_p \pi_p$ in the cohomology

This question can be formulated in the more general framework of the theory automorphic forms, but in this book we only consider "cohomological" (or certain limits of those) automorphic forms. This restricted question is difficult enough. A speculative answer is outlined in the following section

7.1 The Langlands philosophy

Let us start from a product $V = \otimes H_{\pi_p}$. For the primes outside the finite set Σ the module H_{π_p} is determined by its Satake parameter ω_p.

7.1.1 The dual group

There is another way of looking at these Satake parameters ω_p. We explain this in the case that $\mathcal{G}/\mathbb{Z}_p$ is a split reductive group. We choose a maximal split torus $\mathcal{T}$ over $\mathbb{Z}$ and a Borel subgroup $\mathcal{B}/\mathbb{Z}$. For any commutative ring with identity ring R we have a canonical isomorphism $X_*(\mathcal{T}) \otimes R^\times \xrightarrow{\sim} \mathcal{T}(R)$, which is given by $\chi \otimes a \mapsto \chi(a)$. Then $\mathcal{T}(\mathbb{Q}_p)/\mathcal{T}(\mathbb{Z}_p) = X_*(\mathcal{T}) \otimes \mathbb{Q}_p^\times/\mathbb{Z}_p^\times = X_*(\mathcal{T})$. We apply this to the maximal split torus $\mathcal{T}/\mathbb{Z}_p \subset \mathbb{G}/\mathbb{Z}_p$. Then $\Lambda(\mathcal{T}) = \mathrm{Hom}(X_*(\mathcal{T}), \mathbb{C}) = X^*(\mathcal{T}) \otimes \mathbb{C}^\times = T^\vee(\mathbb{C})$ where $T^\vee$ is the torus over $\mathbb{Q}$ whose cocharacter module is $X^*(\mathcal{T})$. This torus over $\mathbb{Q}$ is called the dual torus. There is a canonical construction of a dual group ${}^L G/\mathbb{C}$, this is a reductive group with maximal torus $T^\vee$ such that the Weyl group of $T^\vee$ in this dual group is equal to the Weyl group of $\mathcal{T} \subset \mathcal{G}$ (See also (7.1.5)). This dual torus sits in a Borel subgroup ${}^L B \subset^L G$. Recall that we have a canonical pairing

$$<,>: X_*(\mathcal{T}) \times X^*(\mathcal{T}) \to \mathbb{Z},\ \gamma \circ \chi(x) \mapsto x^{<\chi,\gamma>}. \tag{7.1}$$

The positive simple roots in $X^*(T^\vee)$ in the dual ${}^LG/\mathbb{C}$ are the cocharacters $\alpha_i^\vee \in X_*(\mathcal{T}^{(1)})$ defined by

$$< \alpha_i^\vee, \gamma_j >= \delta_{i,j}. \tag{7.2}$$

We define a coroot $\alpha^\vee$ for any root α: Let $T^{(\alpha)}$ the subtorus on which α is trivial, this torus is of codimension 1. Then the centraliser H_α is a reductive subgroup whose semi simple component $H_\alpha^{(1)}$ is the group Sl_2 or PSl_2. In any case $H_\alpha^{(1)} \cap T = T_\alpha$ is a one dimensional torus. The coroot $\alpha^\vee : \mathbb{G}_m \to T$ is the unique cocharacter which factors through T_α and satisfies $< \alpha^\vee, \alpha >= 2$.

The fundamental weights in ${}^LG$ are the cocharacters χ_i defined by

$$< \chi_i, \alpha_j >= \delta_{i,j.}$$

The dominant weights in $X^*(T^\vee)$ are the linear combinations

$$X^*(T^\vee)^+ = \{\chi = \sum n_i \chi_i | n_i \in \mathbb{Z}, n_i \geq 0\}.$$

To any $\chi \in X^*(T^\vee)^+$ we attach a highest weight module $\mathcal{E}_\chi$, the representation is denoted by r_χ.

We can interpret $\omega_p \in \Lambda(T) = X^*(\mathcal{T}) \otimes \mathbb{C}^\times = T^\vee(\mathbb{C})$ as a semi simple conjugacy class in ${}^LG(\mathbb{C})$. Remember that ω_p is only determined by the local component π_p up to an element in the Weyl group, hence we only get a conjugacy class.

Let $\pi_f \in \mathrm{Coh}_!(G, K_f, \lambda)$ be absolutely irreducible and defined over a finite extension $E/\mathbb{Q}$. Hence we see that our absolutely irreducible π_f provides a collection of conjugacy classes $\{\omega(\pi_p) = \omega_p\}_{p \notin \Sigma}$ in the dual group ${}^LG(E)$.

A rather vague formulation of the general very bold Langlands philosophy predicts:

The isotypical components under the action of the Hecke algebra, namely the $H^i_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})(\pi_f)$, should correspond to a collection $\{\mathbb{M}(\pi_f, r_\chi)\}_{r_\chi}$ of motives (with coefficients in E). The correspondence should be defined via the equality of certain automorphic and motivic L-functions.

This formulation is definitely somewhat cryptic, we will try to make it a little bit more precise in the following sections.

One may think of such a motive could in principle be a "direct summand" in the cohomology $H^i(X)$ of a smooth projective scheme $X/\mathbb{Q}$, which in a certain sense is cut out by a projector (see also Mix-Mot.pdf). In some cases, where the space $\mathcal{S}^G_{K_f}$ "is a Shimura variety", these motives have been constructed, we will discuss this issue in Chap. V. ??????????????

The cyclotomic case

We consider the special case that $G = \mathbb{G}_m/\mathbb{Q}$, we choose $K_\infty = \mathbb{R}^\times_{>0}$ and $K_f = \prod_p K_p \subset \hat{\mathbb{Z}}^\times = \prod_p \mathbb{Z}_p^\times$ an open compact subgroup. The space $\mathcal{S}^G_{K_f}$ is a finite set, actually it is a finite abelian group. It is a generalised ideal class group. The representation $\mathbb{G}_m \to \mathrm{Gl}_1(\mathbb{Q})$ providing the coefficient system is given by the character $[n] : x \mapsto x^n$, the $\mathbb{G}_m-$ module is denoted by $\mathbb{Q}(n)$. We study the cohomology

$$H^0(\mathcal{S}^G_{K_f}, \tilde{\mathbb{Q}}(n)).$$

On these cohomology groups we have an action of the Hecke algebra. In this case this Hecke algebra is simply the group ring $\mathbb{Z}[\mathbb{G}_m(\mathbb{A}_f)/K_f]$. If we include the real place the cohomology $H^0(\mathcal{S}^G_{K_f}, \tilde{\mathbb{Q}}(n))$ becomes a $\pi_0(\mathbb{R}^\times) \times \mathbb{G}_m(\mathbb{A}_f)/K_f$ module. We decompose into irreducible modules (see 6.3.8)

$$H^0(\mathcal{S}^G_{K_f}, \tilde{\mathbb{Q}}(n)) = \bigoplus_\Phi \mathbb{Q}(\Phi) \tag{7.3}$$

Here $\mathbb{Q}(\Phi)$ is a finite extension of $\mathbb{Q}$, and $\Phi : \mathbb{G}_m(\mathbb{Q})\backslash\mathbb{G}_m(\mathbb{A})/K_f \to \mathbb{Q}(\Phi)^\times$ is a homomorphism. The field $\mathbb{Q}(\Phi)$ is generated by the image of Φ and it is even generated by the image of Φ_f. Hence it is a cyclotomic extension. If we extend our field to $\bar{\mathbb{Q}}$ we get a decomposition

$$H^0(\mathcal{S}^G_{K_f}, \bar{\mathbb{Q}}(n))) = \bigoplus_{\phi:type(\phi)=[n]} \bar{\mathbb{Q}}(\phi).$$

The Tate character $\alpha : \mathbb{G}_m(\mathbb{Q})\mathbb{G}_m(\mathbb{A}) \to \mathbb{R}^\times_{>0}, \alpha : \underline{x} \to |\underline{x}|$ is an algebraic Hecke character of type $[-1]$, therefore the general algebraic Hecke character of type $[-n]$ is of the form $\Phi = \alpha^n \cdot \chi$ where $\chi : \mathcal{S}^G_{K_f} \to \bar{\mathbb{Q}}^\times$ is a Dirichlet character.

We can attach two different kinds of L-functions to our isotypical component Φ_f namely an "automorphic" L-function and a "motivic" L-function. It turns out that these two L-functions are the same, this is the Artin-reciprocity law for cyclotomic extensions.

Actually we get a collection of such L-functions which are labelled by the embeddings $\iota : \mathbb{Q}(\Phi) \to \bar{\mathbb{Q}} \subset \mathbb{C}$. Such an embedding yields an algebraic Hecke character

$$\phi_f^{(\iota)} = \iota \circ \Phi_f : G(\mathbb{A}_f) = I_{\mathbb{Q},f} \to \bar{\mathbb{Q}}^\times$$

and

$$\phi^{(\iota)} = \iota \circ \Phi : G(\mathbb{Q})\backslash G(\mathbb{A}) = \mathbb{Q}^\times\backslash I_\mathbb{Q} \to \mathbb{C}^\times$$

and to any of these Hecke characters we attach the (the automorphic L-function) namely

$$L(\phi^{(\iota)}, s) = \prod_p (1 - \phi^{(\iota)}(p)p^{-s})^{-1} \tag{7.4}$$

where $\phi^{\ (\iota)}(p) = \phi^{\ (\iota)}(1, 1 \dots, p, \dots)$ at unramified primes and zero at ramified places , i.e. those places where $K_p \neq \mathbb{Z}_p^\times$. The $\phi^{(\iota)}$ are Hecke characters of type $[n]$.

Now we can attach a motive $\mathbb{M}(\Phi)$ to our isotypical component. To do this we assume first $K_f = \hat{\mathbb{Z}}$, i.e. we are in the unramified case. Then $\mathbb{Q}(\Phi) = \mathbb{Q}$, Then we have

$$\Phi_0(\underline{x}) = \phi(\underline{x}) = |\ \underline{x}\ |^{-n}.$$

This is an algebraic Hecke character of type $[n] : x \mapsto x^n$.

We attach the motive

$$\mathbb{Z}(-n) = H^{2n}(\mathbb{P}^n, \mathbb{Z})$$

to this Hecke character. At this moment we do not need to know what a motive is. The only thing we need to know is that it provides a compatible system of ℓ -adic representations

$$\rho_{n,\ell} : \ \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gl}(H^{2n}(\mathbb{P}^n \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Z}_\ell) = \mathbb{Z}_\ell^\times$$

of the Galois group (see **??**) In this case this representations are easy to describe. For a given ℓ we consider the field $\mathbb{Q}(\zeta_{\ell^\infty})$, this is the cyclotomic extension which is obtained from adjoining all ℓ^k-th roots of unity. Then we have the homomorphism $\pi_\ell : \ \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \ \mathrm{Gal}(\mathbb{Q}(\zeta_{\ell^\infty})/\mathbb{Q}) = \mathbb{Z}_\ell^\times$. Now $\pi_\ell(\sigma) = x \in \mathbb{Z}_\ell$ acts on $\mathbb{Z}_\ell^\times(-n)$ by the rule

$$\rho_n(\sigma) = \{y \mapsto x^{-n}y\}$$

is a module for the Galois-group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. This Galois -module $\mathbb{Z}_\ell(-n)$ is unramified outside ℓ and for any prime $p \neq \ell$ we have an action of the Frobenius element $\mathbf{F}_p$ and this action is given by the multiplication by p^n.

We define the local Euler factor

$$L_p(\mathbb{Z}(-n), s) = \frac{1}{\det(1 - \mathbf{F}_p^{-1} | \mathbb{Z}_\ell(-n) p^{-s})} = \frac{1}{1 - p^n p^{-s}}.$$

The local Euler-factor does not depend on ℓ as long we avoid the case $\ell = p$. (This is the compatibly of the system of Galois modules). This leads us to define

$$L(\mathbb{Z}(-n), s) = \prod_p \frac{1}{\det(1 - \mathbf{F}_p^{-1} | \mathbb{Z}_\ell(-n) p^{-s})} = \zeta(s - n), \tag{7.5}$$

this is now the motivic L - function attached to $\mathbb{Z}(-n)$. But it is also equal to the automorphic L function in 7.4, where $\phi^{(\iota)} = \alpha^n$.

In the general case we observe that a Hecke character ϕ of type $[n]$ is of the form $\alpha^n \cdot \chi$ where χ is a Dirichlet character, i.e. $\chi : \mathcal{S}^G_{K_f} \to \bar{\mathbb{Q}}^\times$. The reciprocity law implies that there is a finite abelian extension $F_{K_f}/\mathbb{Q}$ such that we have the reciprocity isomorphism

$$\mathrm{Art} : \mathcal{S}^G_{K_f} \xrightarrow{\sim} \ \mathrm{Gal}(F_{K_f}/\mathbb{Q}).$$

Hence we see that our Dirichlet character χ can be viewed as a character $\mathrm{Gal}(F_{K_f}/\mathbb{Q}) \to \bar{\mathbb{Q}}^\times$. The finite extension $\mathbb{Q}(\chi)$ of $\mathbb{Q}$ which is generated by the values of χ is contained in F_{K_f}. Hence we get for any prime $\mathfrak{l}$ in $\mathcal{O}_{F_{K_f}}$ a representation

$$\rho_n \otimes \chi : \ \mathrm{Gal}(\mathbb{Q}(\zeta_{\ell^\infty}) F_{K_f}/\mathbb{Q}) \to \mathcal{O}^\times_{F_{K_f}\,\mathfrak{l}}$$

and this is a compatible system of $\mathfrak{l}$ -adic representations.

We attach a motivic L-function to this system of representations. It is a product over local Euler-factors and for a prime p ($l \neq p$) which is unramified in $\mathbb{Q}(\chi)$ this local Euler-factor is

$$L_p(\rho_n \otimes \chi, s) = \frac{1}{\det(1 - \chi(\mathbf{F}_p)\mathbf{F}_p^{-1} | \mathcal{O}_{F_{K_f}}(-n)p^{-s})}$$

and this is exactly the local Euler-factor for the automorphic L-function.

At a prime where χ is ramified the local Euler-factor of the automorphic L-function is 1. But this is also the case fro the motivic L function. In this case the action of the inertia group $I_p \subset \operatorname{Gal}(F_{K_f}/\mathbb{Q})$ on $\mathcal{O}_{F_{K_f}}$ is non trivial and hence the module of invariants is zero. Hence the local Euler-factor is 1. Now it is clear that the two L functions are the same.

The situation becomes more complicated once we replace $\mathbb{G}_m/\mathbb{Q}$ by an arbitrary torus $T/\mathbb{Q}$. The process to attach a motive to an algebraic Hecke character is intricate, we illustrate this in a specific example. We consider the field $K = \mathbb{Q}(\sqrt{-1})$ and our torus $T = R_{K/\mathbb{Q}}(\mathbb{G}_m)$. In this case $X^*(T \times_{\mathbb{Q}} K) = \mathbb{Z} \oplus \mathbb{Z}$. We construct algebraic Hecke characters $\phi, \bar{\phi}$ of type $(1,0),(0,1)$ and hence of any type (a,b). To do this we choose a level subgroup $K_f \subset \prod_p T(\mathbb{Z}_p)$: For $p \neq 2$ we choose $K_p = \mathbb{Z}_p[i]^\times$ and for $p = 2$ we choose $K_2 = \{x \in \mathbb{Z}_2[i] \mid x \equiv 1 \mod (1+i)^3\}$. We define $\phi : T(\mathbb{Q})\backslash T(\mathbb{A})/K_f \to \mathbb{C}^\times$: Any $\underline{x} \in T(\mathbb{A})$ can be written uniquely as $\underline{x} = \gamma(x_\infty, \underline{k}_f)$ with $\underline{k}_f \in K_f$. (For this we use that $\mathbb{Z}[i]$ is a principal ideal domain and $\{i^\nu\} = \mathbb{Z}[i]^\times$.) Then we define

$$\phi(\underline{x}) = x_\infty^{-1};\ \bar{\phi}(\underline{x}) = \bar{x}_\infty^{-1}.$$

We can check easily that this gives us Hecke-characters of type $(1,0),(0,1)$. Let $I_{,K}(2)$ be the group of divisors prime to 2. If $\mathfrak{p} \subset \mathbb{Z}[i]$; $\mathfrak{p} \neq (1+i)$ and if $\mathfrak{p} = (\pi_\mathfrak{p})$ and

$$\underline{\pi}_\mathfrak{p} = (1, 1 \dots, \pi_\mathfrak{p}, 1, \dots) \in G(\mathbb{A})$$

then we put $\phi(\mathfrak{p}) := \phi(\underline{\pi}_\mathfrak{p})$. We see that ϕ also yields a homomorphism $I_K(2) \xrightarrow{\phi} K^\times$.

Now we attach motives to these Hecke characters. This is 19-th century mathematics. We consider the elliptic curve $\mathcal{E} = \{(x,y,z) | zy^2 - z^2x + x^3 = 0\}$. This is a smooth elliptic curve over $\mathbb{Q}$. We get a compatible system of Galois-representations

$$\rho_\mathcal{E} :\ \operatorname{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \operatorname{Gl}(H^1(\mathcal{E} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Z}_\ell)) \tag{7.6}$$

We know that $H^1(\mathcal{E} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Z}_\ell) = \mathbb{Z}_\ell^2$. Moreover we know that the curve $\mathcal{E} \times_{\mathbb{Q}} K$ has an automorphism of order 4, namely $\mathbf{i} : (x,y) \mapsto (-x, iy)$. (The curve has complex multiplication by $\mathbb{Z}[i]$). This automorphism induces an automorphism of order 4 on $H^1(\mathcal{E} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Z}_\ell)$ and if we extend the coefficients from $\mathbb{Z}_\ell$ to $\mathbb{Z}_\ell[i]$ we get two eigenspaces

$$H^1_+ = \{\xi \in H^1(\mathcal{E} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Z}_\ell[i]) |\ \mathbf{i}\xi = i\xi\}; H^1_- = \{\xi \in H^1(\mathcal{E} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Z}_\ell[i]) |\ \mathbf{i}\xi = -i\xi\}.$$

The restriction of the Galois action to $\mathrm{Gal}(\bar{\mathbb{Q}}, K)$ commutes with $\mathbf{i}$ and hence this restriction respects this decomposition, Finally we see easily that this elliptic curve has good reduction at all primes $p > 2.$, hence for any prime $p > 2$ we get a Frobenius $\rho_{\mathcal{E}}(\mathbf{F}_p) \in \mathrm{End}(\mathbb{Z}_\ell^2)$. We distinguish the cases p is inert ,i.e. $p \equiv 3 \mod 4$ and p split, i.e. $p = \mathfrak{p}\bar{\mathfrak{p}}$. Then the classical results on complex multiplication (Weber , Kronecker,..) imply

Theorem 7.1.1. *For any prime* $\mathfrak{p} \subset \mathbb{Z}[i], \mathfrak{p} \neq (1+i)$ *the invers Frobenius* $\rho_{\mathcal{E}}(\mathbf{F}_{\mathfrak{p}}^{-1})$ *acts by multiplication with* $\phi(\pi_{\mathfrak{p}})$(*resp* $\bar{\phi}(\pi_{\mathfrak{p}})$ *on* H^1_+ *resp.* H^1_-.

The characteristic polynomial of $\rho_{\mathcal{E}}(\mathbf{F}_p^{-1})$ *is*

$$\det(Id - t\rho_{\mathcal{E}}(\mathbf{F}_p^{-1})|H^1(\mathcal{E}\times_{\mathbb{Q}}\bar{\mathbb{Q}}, \mathbb{Z}_\ell) = \begin{cases} 1 - (\phi(\mathfrak{p}) + \bar{\phi}(\mathfrak{p}))t + N(\mathfrak{p})t^2 & \text{if } N(\mathfrak{p}) = p \\ 1 + pt^2 & \text{if } \mathfrak{p} = (p) \end{cases}$$

This can be made very explicit:

If p is inert, i.e. if $p \equiv 3 \mod 4$ then the first half of the theorem says that $\rho_\ell(\mathbf{F}_{p^2}$ acts by multiplication with $-p$ hence the eigenvalues of $\rho_{\mathcal{E}}(\mathbf{F}_p)$ should be $\sqrt{-p}, -\sqrt{-p}$.

If p splits then we solve $p = a^2 + b^2 = (a+bi)(a-bi)$ and $\pi_{\mathfrak{p}} = a + bi$. Now the condition $\pi_{\mathfrak{p}} \equiv 1 \mod (1+i)^3$ means $b \equiv 0 \mod 2$ and $a \equiv b + 1 \mod 4$. Under this condition the solution is unique and we the characteristic polynomial becomes

$$\det(\mathrm{Id} - t\rho_{\mathcal{E}}(\mathbf{F}_p^{-1})|H^1(\mathcal{E} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Z}_\ell) = 1 - 2at + pt^2$$

We know of course (see []) that the value at $t = 1$ gives us the number of points of th e curve over $\mathbb{F}_p$, hence we get for any odd prime p

$$\#\mathcal{E}(\mathbb{F}_p) = \begin{cases} p + 1 & \text{if } p \equiv 3 \mod 4 \\ 1 - 2a + p & \text{if } p \equiv 1 \mod 4 \end{cases} \tag{7.7}$$

and in this formulation this was known to Gauss (see []).

Now we can say that H^1_+ and H^1_- are motives attached to the two Hecke characters ϕ and $\bar{\phi}$. Actually these motives are only defined over K because we restricted the Galois group action to $\mathrm{Gal}(\bar{\mathbb{Q}}/K)$ We should also say that these motives have coefficients in K, because the Galois modules are $\mathbb{Z}_\ell \otimes \mathcal{O}_K$ modules.

Actually it should be possible to verify the above prediction of the Langlands philosophy for any torus $T/\mathbb{Q}$. Here we can only give a very general reference to the work of Shimura and Langlands own contribution [70].

7.1.2 The (non abelian) L-functions

We return to the general case, we consider primes p for which π_p is unramified and define local Euler factors.

Let us choose a cocharacter $\chi : \mathbb{G}_m \to T$, we assume that it is in the positive chamber, i.e. we have $< \chi, \alpha_i > \geq 0$ for all positive simple roots. It yields an element $\chi(p) \in T(\mathbb{Q}_p)$, and let $\mathcal{E}_\chi$ the highest weight module for $G^\vee$ provided by χ. For $\omega_p \in \Lambda(T)$ we consider the two expressions EVTchi

$$\begin{aligned} S_\chi(\omega_p) &= p^{<\chi,\rho>} \textstyle\sum_{w \in W/W_\chi} \omega_p(w(\chi(p)) \\ Ch_\chi(\omega_p) &= p^{<\chi,\rho>} \operatorname{tr}(r_\chi(\omega_p)|\mathcal{E}_\chi) \end{aligned} . \tag{7.8}$$

These two terms are related by

$$Ch_\chi(\omega_p) = S_\chi(\omega_p) + \sum_{\chi' \in X^*(T^\vee)^+ : \chi' < \chi} a(\chi,\chi') S_{\chi'}(\omega_p) \tag{7.9}$$

where $\chi' < \chi$ means that $\chi - \chi' \in X^*(T^\vee)^+$ and the coefficients $a(\chi,\chi')$ are positive integers. Here we have to use that $< \chi - \chi', \rho > \geq 0$. We get a formula

$$\int_{\mathcal{H}(\chi(p))} \phi_{\omega_p}(xg)dg = (S_\chi(\omega_p) + \sum_{\chi' < \chi} b(\chi,\chi') S_{\chi',\omega_p}) \phi_{\omega_p}(x) \tag{7.10}$$

where the coefficients $a(\chi,\chi') \in \mathbb{Z}$. The expression on the right hand side is invariant under W and hence only depends on ω_p modulo W (see [?])

The number $< \chi, \rho >$ is a half integer, hence $p^{<\chi,\rho>}$ may not lie in a fixed number field if p varies. But for those χ' which may occur in the summation we have $< \chi - \chi', \rho > \in \mathbb{Z}$. The theorem of Satake yields that we can define a Hecke operator $S_\chi \in \mathcal{H}_p$ such that $S_\chi * \phi_{\omega_p} = S_\chi(\omega_p)\phi_{\omega_p}$ and the formula (7.10) tells us that we get another recursion

$$S_\chi = \mathcal{H}(\chi) + \sum_{\chi' < \chi} b(\chi,\chi') \mathcal{H}(\chi') \tag{7.11}$$

where again $b(\chi,\chi') \in \mathbb{Z}$.

Since we assume that our absolutely irreducible module $V_{\pi_f}, \pi_f = \otimes' \pi_p$ occurs in $\mathrm{Coh}_!(G, K_f, \lambda)$, the Hecke module is a vector space over a finite extension $F/\mathbb{Q}$. We can conclude that the eigenvalue of the convolution operator $\mathcal{H}(\chi)$ is in F and it follows that

$$S_\chi(\omega_p) \in F$$

for any cocharacter χ.

The local Euler factor at unramified primes p is given by

$$L_p(\pi_f, r_\chi, s) = \frac{1}{\det(\mathrm{Id} - r_\chi(\omega_p)p^{-s} | \mathcal{E}_\chi)} \tag{7.12}$$

In this book we do not discuss the local Euler-factors at ramified primes. This is a very subtle issue, and in the general case the definition is only conjectural. Here we assume that we have a consistent consistent definition for $L_p(\pi_f, r_\chi, s)$ at the ramified primes too. We may also assume that our π_f are unramified everywhere, or have only some very mild ramification. The problems which we discuss in Chapter 9 do not become easier if we make this assumption on ramification.

Then we define the (*automorphic*) L-function: We put

$$L(\pi_f, r_\chi, s) = \prod_{p \in \Sigma} L_p(\pi_f, r_\chi, s) (\prod_{p \notin \Sigma} \frac{1}{\det(\mathrm{Id} - r_\chi(\omega_p)p^{-s} | \mathcal{E}_\chi)}) = \prod_p L_p(\pi_f, r_\chi, s) \tag{7.13}$$

The restriction of π_f to the center is the (central) character $\zeta_{\pi_f} : C(\mathbb{A}_f) \to \mathbb{C}^\times$. We say that π_f is unitary if this central character takes values in the unit circle $\mathbb{S}^1$. In this case it follows from theorem 8.1.1 that we can construct a positive definite Hermitian scalar product $< , >: V_{\pi_f} \times V_{\pi_f} \to \mathbb{C}$, for which the algebra of Hecke operators is selfadjoint. We also have some estimates for the eigenvalues of $r_\chi(\omega_p)$ which imply that the infinite product is absolutely and locally uniformly converging in a half plane $\Re(z) > T_0$. Therefore $L(\pi_f, r_\chi, s)$ is a holomorphic function in that half plane. It is a deep conjecture that these L-functions have meromorphic continuation into the entire complex plane. If $\bar{\pi}_f$ is the complex conjugate module then it is also conjectured that we have a functional equation

$$L_\infty(z)L(\pi_f, r_\chi, z) = \epsilon(\pi_f, z)L_\infty(1-z)L(\bar{\pi}_f, r_\chi, 1-z) \tag{7.14}$$

where $\epsilon(\pi_f, z)$ is an exponential factor and $L_\infty(z)$ is the local Euler factor at infinity, it is essentially a product of Γ factors.

This conjecture is proved in very few cases, we come back to this later. We discussed it for the group $\mathrm{Gl}_2/\mathbb{Q}$ in section (4.1.10).

The isogeny d_C induces a homomorphism $d' : C(\mathbb{Q})\backslash C(\mathbb{A}) \to C'(\mathbb{Q})\backslash C'(\mathbb{A})$ and it is well known that this map has a compact kernel. We compose ζ_π with the norm $| \; | : \mathbb{C}^\times \to \mathbb{R}^\times_{>0}$, this composition is trivial on the kernel of d'. Therefore we find a homomorphism $|\zeta_\pi|^* : C'(\mathbb{A}_f) \to \mathbb{R}^\times_{>0}$ which satisfies $| \; | \circ \zeta_\pi = |\zeta_\pi|^* \circ d'$. We look at the finite components of these characters and put as in (6.3.4)

$$\pi_f^* = \pi_f \otimes (|\zeta_\pi|^*)^{-1}. \tag{7.15}$$

This module has a unitary central character. It is easy to see how the Satake parameter changes under the twisting. We have the homomorphism $T(\mathbb{A}) \to C'(\mathbb{A})$ and therefore $(|\zeta_\pi|^*)^{-1}$ induces also a homomorphism from $T(\mathbb{A}_f)$ to $\mathbb{R}^\times_{>0}$. Then it is clear that we get for the Satake parameters the equality

$$\omega(\pi_p \otimes (|\zeta_\pi|_p^*)^{-1}) = \omega(\pi_p)(|\zeta_\pi|_p^*)^{-1} \tag{7.16}$$

Let us assume that π_f occurs as an isotypical subspace in some $H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{C})$, where $\lambda = \lambda^{(1)} + \delta$. The element δ is an element in $X^*(C') \otimes \mathbb{Q}$. To an element $\eta \in X^*(C') \otimes \mathbb{R}$ we have attached an element $|\eta$ and since ζ_{π_f} is of type $\delta \circ d_C$ we have

$$(|\zeta_\pi|^*)^{-1} = |\delta.$$

We also have the cocharacter $\chi : \mathbb{G}_m \to T$ then it is clear that the composition $(|\zeta_\pi|^*)^{-1} \circ \chi$ induces a homomorphism $\mathbb{G}_m(\mathbb{Q})\backslash\mathbb{G}_m(\mathbb{A}) \to \mathbb{R}^\times_{>0}$ which is of the form

$$((|\zeta_\pi|^*)^{-1} \circ \chi)_{\mathbb{A}} : \underline{x} \mapsto |\underline{x}|^{<\chi, \delta>}. \tag{7.17}$$

Then we have

$$L(\pi_f^*, r_\chi, s) = L(\pi_f, r_\chi, s+ < \chi, \delta >) \tag{7.18}$$

The cohomological L-function

We still have another L function which is attached to a Hecke module π_f which occurs in the cohomology, this is the *cohomological* L function. Let us decompose the representation $\mathcal{E}_\lambda$ into weight spaces

$$\mathcal{E}_\chi = \bigoplus_\nu \mathcal{E}_{\chi,\nu} = \bigoplus_{\nu \in X_{*,+}(T)} \bigoplus_{w \in W/W_\nu} \mathcal{E}_{\chi, w(\nu)}$$

then we get with $m(\nu, \chi) = \dim(\mathcal{E}_{\chi,w(\nu)})$. Such a weight vector space is zero unless we have $\nu < \chi$.

$$\det(\mathrm{Id} - r_\chi(\omega_p)p^{-s}|\mathcal{E}_\chi) = \prod_{\nu \in X_{*,+}(T)} \prod_{w \in W/W_\nu} (1 - \omega_p(w(\nu))p^{-s})^{m(\nu,\chi)}$$

For a given ν we expand the inner product

$$\prod_{w \in W/W_\nu} (1 - \omega_p(w(\nu))p^{-s}) = 1 - (\sum_{w \in W/W_\nu} \omega_p(w(\nu)))p^{-s} \dots .$$

Now we recall that

$$p^{<\chi,\lambda^{(1)}>-<\chi,\delta>}\mathcal{H}(\chi) = S_\chi^{(\lambda)}$$

is an operator on the integral cohomology (See (6.51)). Then our recursion formula (7.11) implies that

$$p^{<\chi,\lambda^{(1)}>-<\chi,\delta>}S_\chi$$

is an operator on the integral cohomology, we simply have to observe that $<\chi, \lambda^{(1)}> \geq <\chi', \lambda^{(1)}>$. From this it follows directly that for $\nu \in X_{*,+}(T)$ which occurs as a weight in r_χ we have

$$p^{<\chi,\lambda^{(1)}+\rho>-<\chi,\delta>} \sum_{w \in W/W_\nu} \omega_p(w(\nu)) \in \mathcal{O}_F$$

because $<\chi, \lambda^{(1)}> > <\nu, \lambda^{(1)}>$. Then the right hand side in the above formula can be written

$$1 - p^{<\chi,\lambda^{(1)}>-<\chi,\delta>}(\sum_{w \in W/W_\nu} \omega_p(w(\nu)))p^{-s-<\chi,\lambda^{(1)}+\rho>+<\chi,\delta>} \dots$$

We introduce the new variable $s' = s+ <\chi, \lambda^{(1)}> - <\chi, \delta>$ and put autcoh

$$c(\chi, \lambda) =<\chi, \lambda^{(1)}> - <\chi, \delta> \tag{7.19}$$

$$\prod_{w \in W/W_\nu} (1 - p^{c(\chi,\lambda)}\omega_p(w(\nu))p^{-s'}) = 1 - p^{c(\chi,\lambda)}(\sum_{w \in W/W_\nu} \omega_p(w(\nu)))p^{-s'} \dots \tag{7.20}$$

Hence we define the cohomological local Euler factor at p

$$L_p^{\mathrm{coh}}(\pi_f, r_\chi, s) = \frac{1}{\det(\mathrm{Id} - p^{c(\chi,\lambda)}r_\chi(\omega_p)p^{-s})}. \tag{7.21}$$

We look at this local Euler factor from a slightly different point of view. Our π_f is an absolutely irreducible module which occurs in the cohomology

$H^{\bullet}_{?}(\mathcal{S}^G_{K_f},\mathcal{M}_\lambda\otimes F)$, where $F/\mathbb{Q}$ is an abstract (normal) finite extension of $\mathbb{Q}$. For an unramified prime p the local factor is simply a homomorphism $\pi_p:\mathcal{H}_p\to E$. The previous computations show that the denominator is equal to a polynomial in the "variable" p^{-s} and with coefficients in $\mathcal{O}_F$, i.e.

$$\det(\mathrm{Id}-p^{c(\chi,\lambda)}r_\chi(\omega_p)p^{-s}) = 1-A_1(p,\lambda,\chi)(\pi_p)p^{-s}+A_2(p,\lambda,\chi)(\pi_p)p^{-2s}\dots\in\mathcal{O}_F[p^{-s}] \tag{7.22}$$

where the $A_i(p,\lambda,\chi)$ are certain explicitly computable elements in $\mathcal{H}^{(\lambda)}_{\mathbb{Z}}$. (We showed this only for $A_1(p,\lambda,\chi)$ but the same kind of reasoning gives it for the other $A_i(p,\lambda,\chi)$.) In the expression of the right hand side the Satake parameter does not enter.

The cohomological L function is defined as

$$L^{\mathrm{coh}}(\pi_f,r_\chi,s)=\prod_{p\in\Sigma}L^{\mathrm{coh}}_p(\pi_p,r_\chi,s)\prod_{p\notin\Sigma}\frac{1}{1-A_1(p,\lambda,\chi)(\pi_p)p^{-s}+A_2(p,\lambda,\chi)(\pi_p)p^{-2s}\dots}. \tag{7.23}$$

Again we do not discuss the factors at the primes in Σ.

In the definition of the automorphic L function the Satake parameter is an element in ${}^LG(\mathbb{C})$ or in other words $\omega_p(\nu)\in\mathbb{C}^\times$ and $L^{\mathrm{aut}}_p(\pi_f,r_\chi,s)$ is an honest analytic function in the complex variable s for $\Re(s)>>0$.

If we want to compare the cohomological L-function to the automorphic L -function we have to pick an element $\iota\in I(F,\mathbb{C})$, then $\iota\circ\pi_f$ is an absolutely irreducible Hecke module over $\mathbb{C}$. To $\iota\circ\pi_p$ belongs a Satake parameter ω_p and then

$$\det(\mathrm{Id}-r_\chi(\omega_p)p^{-s+c(\chi,\lambda)})=1-\iota(A_1(p,\lambda,\chi))(\pi_p))p^{-s}+\iota(A_2(p,\lambda,\chi))(\pi_p)p^{-2s}\dots$$

and this tells us that we have

$$L^{\mathrm{coh}}(\iota\circ\pi_f,r_\chi,s)=L(\iota\circ\pi_f,r_\chi,s-c(\chi,\lambda)) \tag{7.24}$$

7.1.3 Invariance under twisting

We remember that we introduced the quotient $\mathcal{C}'=\mathcal{T}/\mathcal{T}^{(1)}$ and the isogeny $d_C:\mathcal{C}\to\mathcal{C}'$. (See 6.1.1). The map d_C in 1.1 induces a map from our locally symmetric space

$$\mathcal{S}^G_{K_f}\xrightarrow{d_{\mathcal{C}'}}\mathcal{S}^{\mathcal{C}'}_{K^{\mathcal{C}'}_\infty\times K^{\mathcal{C}'}_f}$$

We assume that K_∞ is connected and then $K^{\mathcal{C}'}_\infty$ is also connected.

We can modify our system of coefficients if we replace λ by $\lambda+\delta_1$ with $\delta_1\in X^*(\mathcal{C}')$. Then δ_1 provides a local coefficient system $\mathbb{Z}[\delta_1]$ on $\mathcal{S}^{\mathcal{C}'}_{K^{\mathcal{C}'}_\infty\times K^{\mathcal{C}'}_f}$ and since $K^{\mathcal{C}'}_\infty$ is connected we get a canonical class

$$e_{\delta_1}\in H^0(\mathcal{S}^{\mathcal{C}'}_{K^{\mathcal{C}'}_\infty\times K^{\mathcal{C}'}_f},\mathbb{Z}[\delta_1])$$

which generates the rank one submodule of type $|\delta_f|^{-1}$ in the decomposition (6.86). We pull this back by $d'_{\mathcal{C}}$ and we get a class in

$$e_{\delta_1} \in H^0(\mathcal{S}^G_{K_f}, \mathbb{Z}[\delta_1]) \tag{7.25}$$

(see section (6.3.8)). We have the isomorphism $\mathcal{M}_{\lambda,\mathbb{Z}} \otimes \mathbb{Z}[\delta_1] \xrightarrow{\sim} \mathcal{M}_{\lambda+\delta_1,\mathbb{Z}}$ and then the cup product with e_{δ_1} yields an isomorphism

$$H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,\mathbb{Z}}) \xrightarrow{\cup e_{\delta_1}} \xrightarrow{\sim} H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda+\delta_1,\mathbb{Z}}) \tag{7.26}$$

This isomorphism is compatible with the action of the integral Hecke algebra provided we choose the right identification

$$\mathcal{H}^{(\lambda)}_{\mathbb{Z}} \to \mathcal{H}^{(\lambda+\delta_1)}_{\mathbb{Z}}$$

which is given by $a \cdot \mathbf{ch}(\underline{x}_f) \mapsto p^{<\mathbf{ch}(\underline{x}_f),\delta_1>} a \cdot \mathbf{ch}(\underline{x}_f)$.

If we extend the coefficients to F then this cup product yields an isomorphism

$$H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,F})(\pi_f) \xrightarrow{\sim} H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda+\delta_1,F})(\pi_f \otimes |\delta_{1,f}|^{-1}) \tag{7.27}$$

Then our cohomological L-function has the property cup

$$L^{\mathrm{coh}}(\pi_f \otimes |\delta_{1,f}|^{-1}, r_\chi, s) = L^{\mathrm{coh}}(\pi_f, r_\chi, s) \tag{7.28}$$

This invariance under twists is of course also a consequence of the definition in terms of the automorphic L-function.

We may interpret this differently. Our λ is a sum of a semi-simple component $\lambda^{(1)}$ plus an abelian part δ We can use the isomorphisms in (7.27) to define a vector space

$$H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda^{(1)}+?,F})\{\pi_f\}, \tag{7.29}$$

this vector space has a distinguished isomorphism to any of the $H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda+\delta_1,F})(\pi_f \otimes |\delta_{1,f}|^{-1})$, we could say that it the direct limit of all these spaces. By $\{\pi_f\}$ we understand the array

$$\{\pi_f\} = \{\dots, \pi_f \otimes |\delta_{1,f}|^{-1}, \}_{\delta_1 \in X^*(\mathcal{C}')}.$$

Using (7.28) we have now defined $L^{\mathrm{coh}}(\{\pi_f\}, r_\chi, s)$.

For any pair $\chi \in X_*(T), \lambda \in X^*(T)$, where χ is in the positive chamber and λ a dominant weight we define the weight

$$\mathbf{w}(\chi, \lambda) = 2 < \chi, \lambda^{(1)} + \rho > . \tag{7.30}$$

Here we observe that χ provides a highest weight representation $r = r_\chi$ of ${}^LG$ and λ a highest weight representation of G so we could also write

$$\mathbf{w}(\chi, \lambda) = \mathbf{w}(r_\chi, \mathcal{M}_\lambda) = \mathbf{w}(r, \mathcal{M}). \tag{7.31}$$

This means that we may consider the weight as a number attached to a pair of irreducible rational representations of ${}^LG$ and G. It also depends only on the semi simple part of λ.

Functional equation (7.32)

A different look

We could look at the previous discussion from another point of view. Given our coefficient system $\mathcal{M}_\lambda$ where $\lambda = \lambda^{(1)} + \delta$ and an absolutely irreducible module $\pi_f \in \mathrm{Coh}_!(G, \lambda, K_f)$. As explained above we get $X^*(C')$ torsor $(\lambda+\delta', \pi_f \otimes |\delta'_f|)$ of such objects. If we choose a $\iota : F \hookrightarrow \mathbb{C}$ then we can think of $\iota \circ \pi_f$ as the finite part of an automorphic representation π. Then we get a second torsor for the above group $\Xi = X^*(C') \otimes \mathbb{R}$. The inclusion $X^*(C') \hookrightarrow \Xi$ yields an interpolation of the first torsor into the second one. To any element $\pi \otimes \xi$ we defined the automorphic L function $L^{\mathrm{aut}}(\iota \circ \pi_f \otimes \xi_f, r_\chi, s)$. Now the unitary and the cohomological L-function are defined as the automorphic L function of a specific point in the torsor, i.e. a specific trivialization.

To define the unitary L function we choose the specific point for which the central character is unitary, for the cohomological L -function we choose the "optimal" point $\pi_f \otimes |\delta'_f|$ for which we have

$$L_p^{\mathrm{coh}}(\pi_f \otimes |\delta'_f|, r_\chi, s)^{-1} \in \mathcal{O}_F[p^{-s}]. \tag{7.33}$$

If we are investigating analytic questions concerning automorphic forms the unitary L is the right object, but if we want to capture the integral structure of the cohomology we prefer to work with the cohomological L function.

7.1.4 The motives

We consider an isotypical submodule $H_!^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda;F})(\pi_f)$ in the inner cohomology. The Langlands philosophy predicts the existence of a collection of pure motives over $\mathbb{Q}$ with coefficients in F.

$$\{\mathbb{M}(\pi_f, r_\chi)\}_{r_\chi}$$

which has certain properties. We will not be absolutely precise in the following but we list certain properties this motive should have. We should assume that π_f is not some kind of exceptional Hecke module (for instance it should not be endoscopic), and I can not give a precise definition what that means. We will make it more precise later when we discuss the case that our group is Gl_n.

This motive should be invariant under twists, i.e. we want that

$$\mathbb{M}(\pi_f \otimes |\delta_f|, r_\chi) = \mathbb{M}(\pi_f, r_\chi)$$

First of all this motive has a Betti-realization $\mathbb{M}(\pi_f, r_\chi)_B$, which is simply an F vector space of dimension $\dim(r_\chi)$. Such a motive has a de-Rham realization $\mathbb{M}(\pi_f, r_\chi)_{dRh}$, this is another F-vector space of the same dimension. It has a descending filtration

$$\mathbb{M}(\pi_f, r_\chi)_{dRh} = F^0(\mathbb{M}(\pi_f, r_\chi)_{de-Rh}) \supset F^1(\mathbb{M}(\pi_f, r_\chi)_{de-Rh}) \supset \dots$$

$$\dots \supset F^{\mathbf{w}}(F^0(\mathbb{M}(\pi_f, r_\chi)_{dRh}) \supset F^{\mathbf{w}+1}(F^0(\mathbb{M}(\pi_f, r_\chi)_{dRh}) = 0.$$

The number $\mathbf{w} = \mathbf{w}(\pi_f, \chi)$ is the weight of the motive it is equal to $\mathbf{w}(\chi, \lambda)$.

Furthermore we have a comparison isomorphism

$$I_{B-dRh} : \mathbb{M}(\pi_f, r_\chi)_B \otimes \mathbb{C} \xrightarrow{\sim} \mathbb{M}(\pi_f, r_\chi)_{dRh} \otimes \mathbb{C},$$

this yields periods and these periods should be related to π_f, this is rather mysterious.

For any prime ℓ and any prime $\mathfrak{l}|\ell$ in F we get a Galois representation

$$\rho(\pi_f,\chi): \ \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL(\mathbb{M}(\pi_f, r_\chi)_B \otimes F_{\mathfrak{l}})$$

which is unramified outside $\Sigma \cup \{l\}$ and for any such prime we have

$$\det(\mathrm{Id} - \rho(\pi_f,\chi)(\Phi_p^{-1})p^{-s}, \mathbb{M}(\pi_f, r_\chi)_B \otimes F_{\mathfrak{l}}) = L_p^{\mathrm{coh}}(\pi_f, r_\chi, s)^{-1},$$

or in other words we expect that the semi-simple conjugacy classes

$$\rho(\pi_f,\chi)(\Phi_p^{-1}) \sim p^{c(\chi,\lambda)} r_\chi(\omega_p) \tag{7.34}$$

and hence we want

$$L^{\mathrm{coh}}(\pi_f, r_\chi, s) = L(\mathbb{M}(\pi_f, r_\chi), s)$$

The existence of these hypothetical motives has a lot of consequences. Once we have established such a relation

$$L^{\mathrm{coh}}(\pi_f, r_\chi, s) = L(\mathbb{M}(\pi_f, r_\chi), s)$$

then we can exploit this in both directions. We have a certain chance to prove the conjectural analytic properties and the conjectural functional equation for the L-function of the motive $\mathbb{M}(\pi_f, r_\chi)$, provided we can prove this for $L^{\mathrm{coh}}(\pi_f, r_\chi, s)$. On the automorphic side we know many cases in which we can prove these properties of the L-function using the theory of automorphic forms.

In the other direction we have Deligne's theorem concerning the absolute values of the Frobenius. This implies Ramanujan (more details later)

We seem to be very far away from proving these conjectures, but there are many instances where some parts of this program have been established and there are also some very interesting cases where this correspondence has been verified experimentally.

Deligne's Conjectures on special values

7.1.5 The case $G = \mathrm{Gl}_n$

Notations for the dual group ${}^LG$

We want to verify formula (7.10) in the special case $G = \mathrm{Gl}_n/\mathbb{Z}$. In this case t we have the cocharacters χ_i which send t to the diagonal matrix $t \mapsto \mathrm{diag}(t,\dots,t,1\dots,1)$ where t is placed to the first i dots. They satisfy $<\chi_i, \alpha_j>= \delta_{i,j}$ for $1 \le i \le n, 1 \le j \le n-1$. They are uniquely determined by this condition modulo the cocharacter χ_n which identifies $\mathbb{G}_m$ with the center. For $1 \le i \le n-1$ the cocharacter χ_i determines a maximal parabolic subgroup $P_i \supset T$ whose roots $\Delta_{P_i} = \{\alpha| <\chi_i, \alpha>\ge 0\}$. The parabolic subgroup P_i^- will be the opposite parabolic subgroup.

Let $\epsilon_i : \mathbb{G}_m \to T$ be the cocharacter which sends t to t on the $i-$ th spot on the diagonal and to 1 at all others. If we identify the module of cocharacters with the character group of the dual torus $T^\vee \subset^L G = \mathrm{Gl}_n$ then the differences $\epsilon_i - \epsilon_j$ will be the roots, the simple roots are $\epsilon_i - \epsilon_{i+1}$ and the fundamental dominant weights are the semi simple components $(\sum_{i=1}^i \epsilon_i^{(1)})$.

Formulas for the Hecke operators

We consider the homomorphism $r : K_p = \mathrm{Gl}_n(\mathbb{Z}_p) \to \mathrm{Gl}_n(\mathbb{F}_p)$ then we check easily that the intersection $K_p \cap \chi_i(p) K_p \chi_i(p)^{-1} = K_p^{(\chi_i(p))}$ is the inverse image of the parabolic subgroup $P_i^-(\mathbb{F}_p)$ under r.

We want to evaluate the integral

$$\int_{K_p \chi_i(p) K_p} \phi_{\omega_p}(x) dx$$

We write choose representatives ξ for the cosets of $K_p / K_p^{(\chi_i(p))}$ and write $K_p = \cup_\xi \xi K_p^{(\chi_i(p))}$. We observe that ϕ_{ω_p} is constant on the cosets $\xi K_p^{(\chi_i(p))}$. Hence we see that

$$\int_{K_p \chi_i(p) K_p} \phi_{\omega_p}(x) dx = \sum_\xi \phi_{\omega_p}(\xi \chi_i(p))) \tag{7.35}$$

The Bruhat decomposition gives us a nice system of representatives for $K_p / K_p^{(\chi_i(p))} = \mathrm{Gl}_n(\mathbb{F}_p)/P_i^-(\mathbb{F}_p)$. Let W_{M_i} be the Weyl group of the standard Levi subgroup $M_i = P_i \cap P_i^-$ and we choose a system of representatives W^{P_i} for W/W_{M_i} Then we get a disjoint decomposition

$$\mathrm{Gl}_n(\mathbb{F}_p) = \bigcup_{w \in W^{P_i}} U_B(\mathbb{F}_p) w P_i^-(\mathbb{F}_p),$$

here U_B is the unipotent radical of the standard Borel subgroup. The function ϕ_{ω_p} is constant on the double cosets. If we write a representative in the form $\xi = uw$ then the factor w is determined by ξ but the factor u is not. This factor is only unique up to multiplication from the right by a factor $u \in U_B^{(w,-)}(\mathbb{F}_p) = U_B(\mathbb{F}_p) \cap w P_i^- w^{-1}(\mathbb{F}_p)$. Hence we may choose our u in the subgroup

$$U_B^{(w,+)}(\mathbb{F}_p) = \prod_{\alpha \in \Delta^+ | <\chi_i, w^{-1}\alpha>>0} U_\alpha(\mathbb{F}_p) \tag{7.36}$$

and our sum in (7.35) becomes

$$\sum_{w \in W^{P_i}} \sum_{u \in U_B^{(w,+)}(\mathbb{F}_p)} \phi_{\omega_p}(uw\chi_i(p))) = \sum_{w \in W^{P_i}} p^{l(w)} \phi_{\omega_p}(w \chi_i(p) w^{-1})) \tag{7.37}$$

where $l(w)$ is the cardinality of the set $\{\alpha \in \Delta^+ | < \chi_i, w^{-1}\alpha >> 0\}$. We recall

the definition of the spherical function and get for our integral

$$\sum_{w \in W/W_{M_i}} p^{l(w)} \omega_p(w\chi_i(p)w^{-1}))|\rho|_p(w\chi_i(p)w^{-1})) = \sum_{w \in W/W_{M_i}} p^{l(w) - <\chi_i, w^{-1}\rho>} \omega_p((w\chi_i)(p)) \tag{7.38}$$

Now one checks easily that $p^{l(w) - <\chi_i, w^{-1}\rho>} = p^{<\chi_i, \rho>}$ and hence we get the desired formula

$$\int_{K_p \chi_i(p) K_p} \phi_{\omega_p}(x) dx = p^{<\chi_i, \rho>} \sum_{w \in W/W_{M_i}} \omega_p((w\chi_i)(p)) \tag{7.39}$$

This is the formula (7.10) for the group Gl_n and the special choice of the cocharacters $\chi = \chi_i$. The only cocharacter $\chi' < \chi_i$ is the trivial cocharacter, in our situation its contribution to (7.10) is zero.

Let us have a brief look at an arbitrary reductive (split or may be only quasisplit) group $G/\mathbb{Q}$, let us assume that the center is a connected torus $C/\mathbb{Q}$. We choose a maximal torus $T/\mathbb{Q}$ which is contained in a Borel subgroup $B/\mathbb{Q}$. We have the homomorphism to the adjoint group $G \to G_{\mathrm{ad}}$ it maps T to $T_{\mathrm{ad}} = T/C$. Again we may also define the fundamental cocharacters $\chi_i : \mathbb{G}_m \to T$ which satisfy $< \chi_i, \alpha_j >= \delta_{i,j}$. They are only well defined modulo cocharacters $\chi : \mathbb{G}_m \to C$ but this does not matter so much. Our above method to compute the eigenvalue of $\mathcal{H}(\chi_i)$ still works if the cocharacter χ_i is ”minuscule” which means that $< \chi_i, \alpha_j >\in \{-1, 0, 1\}$. In this case the formula (7.39) is still valid, again there is no contribution from the trivial character.

We return to $G = \mathrm{Gl}_n$ and to our speculations about motives. We choose a weight module $\mathcal{M}_\lambda$ where $\lambda = \sum_i a_i\gamma_i + d\delta$, where the γ_i are the fundamental weights and δ is the determinant. The a_i are integers and we have the consistency condition $\sum ia_i \equiv nd \mod n$. Let us pick an isotypical submodule $H^\bullet(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda \otimes F)(\pi_f)$. In section 6.3.2 we define the Hecke operators

$$T^{\mathrm{coh},\lambda}_\chi : H^\bullet_?(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda) \to H^\bullet_?(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda)$$

and these endomorphisms induce endomorphisms

$$T^{\mathrm{coh},\lambda}_\chi : H^\bullet_{?,\,\mathrm{int}}(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda \otimes F)(\pi_f) \to H^\bullet_{?,\,\mathrm{int}}(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda \otimes F)(\pi_f)$$

Let $\pi_f = \otimes\pi_p$ be an irreducible Hecke module and at an unramified place p let ω_p be the Satake parameter. Our Satake parameter is determined by the n-tuple of numbers

$$\omega_p(\eta_i(p)) = \omega_{i,p} \text{ for } i = 1, \dots, n$$

The cocharacter $\chi_n : \mathbb{G}_m \to T$ identifies $\mathbb{G}_m$ with the center of Gl_n. Our Hecke-module π_f has a central character and this provides a Hecke character

$$\pi_f \circ \chi_n : \mathbb{G}_m(\mathbb{A}_f) = I_{\mathbb{Q},f} \to F^\times$$

The restriction of $\mathcal{M}_\lambda$ to $\mathbb{G}_m$ is the character $\omega_\lambda : t \mapsto t^{nd}$ and the type of $\pi_f \circ \chi_n$ is of course ω_λ.

Our cocharacters χ_i define representations of the dual group which is again Gl_n and in fact χ_1 yields the tautological representation $r_1 : \mathrm{Gl}_n \xrightarrow{\sim} \mathrm{Gl}(V)$. Then χ_i yields the representation $r_i = \Lambda^i(r_1) : \mathrm{Gl}_n \to \mathrm{Gl}(\Lambda^i(V))$. For any subset $I \subset \{1, 2, \dots, n\}$ we define

$$\omega_{I,p} = \prod_{i \in I} \omega_{i,p}$$

and then our formula (7.39) in combination with the formula (6.51) in section 6.3.2 and the observation that $< \chi_i, \delta >= i$ yields

$$T^{\mathrm{coh},\lambda}_{\chi_i}(\pi_p) = p^{<\chi_i, \lambda^{(1)} + \rho> - id} \sum_{I : \#I = i} \omega_{I,p} \tag{7.40}$$

and by the same token we get for the cohomological L-function

$$L^{\text{coh}}(\pi_f, r_\nu, s) = \prod_{p\in S} L_p^{\text{coh}}(\pi_f, r_i, s) \prod_{p\notin S}\Big(\prod_{I:\#I=i} \frac{1}{(1-p^{<\chi_i,\lambda^{(1)}+\rho>-id}\omega_{I,p}p^{-s})}\Big) \tag{7.41}$$

Here we see in a very transparent way the independence of the twist: If we modify λ to $\lambda + r\delta$ then we have to modify π_f to $\pi_f \otimes |\delta_f|^{-r}$. This means that the $\omega_{I,p}$ get multiplied by p^{ir} and the modifications cancel out.

We assume that $\pi_f \in \text{Coh}(H_!^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda))$, then we will see in section 8.1.6 that π_f is essentially unitary. The central character of $\mathcal{M}_\lambda$ is $x \mapsto x^{nd}$ and hence we get that $\pi_f^* = \pi_f \otimes |\delta_f|^d$ is unitary. Then the Satake parameter of π_f^* is given by

$$\omega_{i,p}^* = \omega_{i,p}p^{-d} \text{ for } i = 1, \dots, n \tag{7.42}$$

where the factor $p^{-d} = |p|_p^d$ and we observe that these numbers are also invariant under twists by a power of $|\delta_f|$.

Since the operators $T_{\chi_i}^{\text{coh},\lambda}$ operate on the integral cohomology it follows that the numbers $T_{\chi_i}^{\text{coh},\lambda}(\pi_f)$ are algebraic integers. We easily check that for all $i \le n$

$$i(<\chi_1, \lambda^{(1)} + \rho > -d) \geq <\chi_i, \lambda^{(1)} + \rho > -id$$

and this implies that the numbers

$$\sum_{I:\#I=i} \prod_{\nu\in I} p^{<\chi_1,\lambda^{(1)}+\rho>-d}\omega_{\nu,p}$$

are algebraic integers and hence we can conclude

The numbers

$$\tilde{\omega}_{i,p} = p^{<\chi_1,\lambda^{(1)}+\rho>-d}\omega_{i,p} = p^{<\chi_1,\lambda^{(1)}+\rho>}\omega_{i,p}^* \tag{7.43}$$

are algebraic integers

Observe that these numbers are invariant under twists by a power of $|\delta_f|$.

We want t make few remarks about the relationship between the automorphic and the cohomological L-functions, especially we comment the shift in the variable s.

For the automorphic L -function we assume that we are over $\mathbb{C}$, we have chosen an embedding $\iota : F \hookrightarrow \mathbb{C}$. If our isotypical Hecke module π_f is cuspidal (see Thm. 8.1.1) then the considerations around this theorem show that π_f is essentially unitary. The center $C = \mathbb{G}_m$, the quotient $C' = \mathbb{G}_m$ and the isogeny $d_C : x \mapsto x^n$.

We come back to the Langlands philosophy. It predicts that for our a "cuspidal" π_f and the cocharacter χ_1 we should be able to attach a motive $\mathbb{M}(\pi_f, r_1) = \mathbb{M}(\pi_f, \chi_1)$ with coefficients in F. This motive provides a compatible system of $\mathfrak{l}$- adic Galois representations

$$\rho_{\mathfrak{l}}(\pi_f, \chi_1) : \ \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \text{Gl}_n(F_{\mathfrak{l}}) = \text{Gl}(\mathbb{M}(\pi_f, \chi_1)_{\text{ét},\mathfrak{l}}) \tag{7.44}$$

which are unramified outside $\{l\} \cup S$ and for $p \notin S \cup \{l\}$ we should have

$$\det(\mathrm{Id} - \rho_{\mathfrak{l}}(\pi_f, \chi_1)(\Phi_p^{-1})p^{-s}) = \prod_i (1 - p^{<\chi_1, \lambda^{(1)}+\rho>-d}\omega_{i,p}p^{-s}) \tag{7.45}$$

and this means that up to the local factors at the bad primes we should have

$$L^{\mathrm{mot}}(\mathbb{M}(\pi_f, \chi_1), s) = L^{\mathrm{coh}}(\pi_f, \chi_1, s) \tag{7.46}$$

The existence of the compatible system of Galois representation has been shown by Harris - Kai-Wen Lan -Taylor and Thorne and by P. Scholze.

Once we have the motive for the cocharacter χ_1 we easily get it the other χ_i we simply have to look at the exterior powers $\Lambda^i(\mathbb{M}(\pi_f, \chi_1))$.

Now we see that numbers $\tilde{\omega}_{\nu,p}$ can be interpreted as the eigenvalues of the Frobenius on $\mathbb{M}_{\text{ét},\mathfrak{l}}(\pi_f, \chi_1)$. Under the assumption that π_f is "cuspidal" we expect that the motive $\mathbb{M}(\pi_f, \chi_1)$ is pure of weight $\mathbf{w}(\chi_1, \lambda)$ we get

$$|\tilde{\omega}_{\nu,p}| = p^{\frac{\mathbf{w}(\chi_1,\lambda)}{2}}$$

and this is the Ramanujan conjecture. We will explain in the section on analytic aspects, that for cuspidal π_f the Ramanujan conjecture says that for any embedding $\iota : F \hookrightarrow \mathbb{C}$ we have

$$|\iota \circ \omega^*_{\nu,p}| = 1$$

This suggests that we call the array $\tilde{\omega}_p = \{\tilde{\omega}_{1,p}, \ldots, \tilde{\omega}_{n,p}\}$ the *motivic* Satake parameter (with respect to the tautological representation r_1 .) Of course it can always be defined, independently of the existence of the motive.

We will see in the next section that the inner cohomology is trivial unless our highest weight is essentially self dual, this means that $\lambda^{(1)} = -w_0(\lambda^{(1)})$. Let us assume that this is the case. If $r_1^\vee$ is the dual of the tautological representation then the eigenvalues of $r_1^\vee(\omega_p)$ are by

$$r_1^\vee(\omega_p) = \{\omega_{1,p}^{-1}, \ldots, \omega_{n,p}^{-1}\}.$$

The highest weight of $r_1^\vee$ is the cocharacter $-\eta_n = \sum_{i=1}^{n-1} \eta_i - \det$ (This has to be read in $X^*(T^\vee)$) Then

$$c(-\eta_n, \lambda) =< \chi_1, -w_0(\lambda^{(1)}) > +d$$

and under our assumption that λ is essentially self dual we know

$$< \chi_1, -w_0(\lambda^{(1)}) >=< \chi_1, \lambda^{(1)} >= \frac{\mathbf{w}(\chi_1, \lambda)}{2}.$$

This implies that the motivic Satake parameters with respect to the dual representation $r_1^\vee$ are the numbers

$$\{p^{<\chi_1,\lambda^{(1)}>+d\delta}\omega_{1,p}^{-1}, \ldots, p^{<\chi_1,\lambda^{(1)}>+d\delta}\omega_{n,p}^{-1}\} \tag{7.47}$$

In the following section on Poincaré duality we will see that for any isotypical module $H^i_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,F})(\pi_f)$ the dual module $\pi_f^\vee$ appears in $H^{d-i}_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda^\vee,F})$. Then we get an equality of local Euler factors

$$L^{\mathrm{coh}}(\pi_p, r_1^\vee, s) = L^{\mathrm{coh}}(\pi_p^\vee, r_1, s) \tag{7.48}$$

The concept of motives allows us to define thedual motive. If our motive has weight $\mathbf{w}(M)$ then Poincaré duality suggests that we define the motive

$$\mathbb{M}^\vee = \mathrm{Hom}(\mathbb{M}, \mathbb{Z}(-\mathbf{w}(M)) \tag{7.49}$$

The $\mathfrak{l}$ adic realization as Galoismodule gives us

$$\mathbb{M}^\vee_{\text{ét},\mathfrak{l}} = \mathrm{Hom}(\mathbb{M}_{\text{ét},\mathfrak{l}}, \mathbb{Z}_\mathfrak{l}(-\mathbf{w}(M))$$

If $\{\alpha_1, \dots, \alpha_m\}$ are the eigenvalues of Φ_p^{-1} on $\mathbb{M}_{\text{ét},\mathfrak{l}}$ then $\{\alpha_1^{-1}p^{\mathbf{w}(M)}, \dots, \alpha_m^{-1}p^{\mathbf{w}(M)}\}$ are the eigenvalues of Φ_p^{-1} on $\mathbb{M}^\vee_{\text{ét},\mathfrak{l}}$.

Therefore we can say: If we find a motive $\mathbb{M}(\pi_f, \chi_1)$ for π_f the we also find the motive for $\pi_f^\vee$ and we have

$$\mathbb{M}(\pi_f^\vee, \chi_1) = \mathbb{M}(\pi_f, \chi_1)^\vee$$

Chapter 8

Analytic methods

8.1 The representation theoretic de-Rham complex

8.1.1 Rational representations

We start from a reductive group $G/\mathbb{Q}$ for simplicity we assume that the semi simple component $G^{(1)}/\mathbb{Q}$ is quasisplit. There is a unique finite normal extension $F/\mathbb{Q}, F \subset \mathbb{C}$ such that $G^{(1)} \times_{\mathbb{Q}} F$ becomes split. If $T^{(1)}/\mathbb{Q}$ is a maximal torus which is contained in a Borel subgroup $B/\mathbb{Q}$ then the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on $X^*(T^{(1)} \times_{\mathbb{Q}} F)$. It acts by permutations on the set of positive roots $\pi_G \subset X^*(T^{(1)} \times_{\mathbb{Q}} F)$ corresponding to $B/\mathbb{Q}$. This action factors over the quotient $\mathrm{Gal}(F/\mathbb{Q})$. Then it also acts on the set of highest weights. Since our group is quasi split we find for any highest weight an absolutely irreducible $G \times_{\mathbb{Q}} F$-module $\mathcal{M}_\lambda$.

$$r : G \times_{\mathbb{Q}} F \to \mathrm{Gl}(\mathcal{M}_\lambda)$$

whose highest weight is λ. Since we assumed that $\mathbb{Q} \subset F \subset \bar{\mathbb{Q}} \subset \mathbb{C}$ we get the extension

$$r_{\mathbb{C}} : (G \times_{\mathbb{Q}} F) \times_F \mathbb{C} \to \mathrm{Gl}(\mathcal{M}_\lambda \otimes_F \mathbb{C}).$$

Given such an absolutely irreducible rational representation, we can construct two new representations. At first we can form the dual $\mathcal{M}^\vee_{\lambda,\mathbb{C}} = \mathrm{Hom}_{\mathbb{C}}(\mathcal{M}_\lambda, \mathbb{C})$ and the complex conjugate $\overline{\mathcal{M}}_{\mathbb{C}}$ of our module $\mathcal{M}_\lambda$. On the dual module we have the contragredient representation $r^\vee$, which is defined by $\phi(r_{\mathbb{C}}(g)(v)) = r^\vee_{\mathbb{C}}(g^{-1})(\phi)(v)$.

To get the rational representation on the conjugate module $\bar{\mathcal{M}} \otimes_F \mathbb{C}$, we recall its definition: As abelian groups we have $\mathcal{M} \otimes_F \mathbb{C} = \bar{\mathcal{M}} \otimes_F \mathbb{C}$ but the action of the scalars is conjugated, we write this as $z \cdot_c m = \bar{z}m$. Then the identity gives us an identification

$$\mathrm{End}_{\mathbb{C}}(\mathcal{M} \otimes_F \mathbb{C}) = \mathrm{End}_{\mathbb{C}}(\bar{\mathcal{M}}_\lambda \otimes_F \mathbb{C}).$$

Now we define an action $\bar{r}_{\mathbb{C}}$ on $\bar{\mathcal{M}}_\lambda \otimes_F \mathbb{C}$: For $g \in G(\mathbb{C})$ we put

$$\bar{r}_{\mathbb{C}}(g)m = r_{\mathbb{C}}(g) \cdot_c m.$$

This defines an action of the abstract group $G(\mathbb{C})$, but this is in fact obtained from a rational representation. Therefore $\mathcal{M}_{\mathbb{C}}^{\vee}$ and $\overline{\mathcal{M}}_C$ both are given by a highest weight.

The highest weight of $\mathcal{M}_{\lambda}^{\vee}$ is $-w_0(\lambda)$. Here w_0 is the unique element $w_0 \in W$, which sends the system of positive roots Δ^+ into the system $\Delta^- = -\Delta^+$.

The highest weight of $\bar{\mathcal{M}}_{\lambda} \otimes_F \mathbb{C}$ is $c(\lambda)$ where $c \in \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \subset \mathrm{Gal}(F/\mathbb{Q})$ is the complex conjugation acting on $X^*(T \times_{\mathbb{Q}} F)$. So we may say: $\overline{\mathcal{M}_{\lambda C}} = \mathcal{M}_{\bar{\lambda}}$.

We will call the module $\mathcal{M}_{\lambda}$- *conjugate-autodual* or simply *c-autodual* if

$$c(\lambda) = -w_0(\lambda) \tag{8.1}$$

If our group $G/\mathbb{Q}$ is split then c acts trivially on the character module and the condition becomes $\lambda = -w_0(\lambda)$. If now in addition the element w_0 acts by -1 on the character module, the every λ is conjugate-autodual.

In the following few sections (until 8.1.6 we will always assume that our local system (resp. the corresponding representation) are local systems in $\mathbb{C}$-vector spaces (resp. $\mathbb{C}$-vector spaces $\tilde{\mathcal{M}}_{\lambda}$). Therefore we will suppress the factor $\otimes\mathbb{C}$.

HCmod

8.1.2 Harish-Chandra modules and $(\mathfrak{g}, K_\infty)$-cohomology.

Now we consider the group of real points $G(\mathbb{R})$, it has the Lie algebra $\mathfrak{g}$, inside this Lie algebra we have the Lie algebra $\mathfrak{k}$ of the group K_∞. We have the notion of a $(\mathfrak{g}, K_\infty)$ module: This is a $\mathbb{C}$-vector space V together with an action of $\mathfrak{g}$ and an action of the group K_∞. We have certain assumptions of consistency:

i) The action of K_∞ is differentiable, this means it induces an action of $\mathfrak{k}$, the derivative of the group action.

ii) The action of $\mathfrak{g}$ restricted to $\mathfrak{k}$ is the derivative of the action of K_∞.

iii) For $k \in K_\infty, X \in \mathfrak{g}$ and $v \in V$ we have

$$(\mathrm{Ad}(k)X)v = k(X(k^{-1}v)).$$

Inside V we have have the subspace of K_∞ finite vectors, a vector v is called K_∞ finite if the $\mathbb{C}$- subspace generated by all translates kv is finite dimensional, i.e. v lies in a finite dimensional K_∞ invariant subspace. The K_∞ finite vectors form a subspace $V^{(K_\infty)}$ and it is obvious that $V^{(K_\infty)}$ is invariant under the action of $\mathfrak{g}$, hence it is a $(\mathfrak{g}, K_\infty)$ sub module of V. We call a $(\mathfrak{g}, K_\infty)$ module a Harish-Chandra module if $V = V^{(K_\infty)}$.

For such a $(\mathfrak{g}, K_\infty)$-module we can write down a complex

$$\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), V) = \{0 \to V \to \mathrm{Hom}_{K_\infty}(\Lambda^1(\mathfrak{g}/\mathfrak{k}), V) \to \mathrm{Hom}_{K_\infty}(\Lambda^2(\mathfrak{g}/\mathfrak{k}), V) \to \dots\}$$

where the differential is given by liealgc

$$\begin{aligned} d\omega(X_0, X_1, \dots, X_p) = \textstyle\sum_{i=0}^{p}(-1)^i X_i \omega(X_0, \dots, \hat{X}_i, \dots, X_p) + \\ \textstyle\sum_{0 \le i < j \le p}(-1)^{i+j}\omega([X_i, X_j], X_0, \dots, \hat{X}_i, , \dots, \hat{X}_j, \dots, \end{aligned} \tag{8.2}$$

A few comments are in order. We have inclusions

$$\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), V) \subset \mathrm{Hom}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), V) \subset \mathrm{Hom}(\Lambda^\bullet(\mathfrak{g}), V).$$

The above differential defines the structure of a complex for the rightmost term, we have to verify that the leftmost term is a subcomplex, this is not so difficult.

We define the $(\mathfrak{g}, K_\infty)$ cohomology as the cohomology of this complex, i.e.

$$H^\bullet(\mathfrak{g}, K_\infty, V) = H^\bullet(\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), V)) \tag{8.3}$$

It is clear that the map

$$H^\bullet(\mathfrak{g}, K_\infty, V^{(K_\infty)}) \to H^\bullet(\mathfrak{g}, K_\infty, V)$$

is an isomorphism.

If we have two $(\mathfrak{g}, K_\infty)$ modules V_1, V_2 and form the algebraic tensor product $W = V_1 \otimes V_2$ the we have a natural structure of a $(\mathfrak{g}, K_\infty)$ -module on W : The group K_∞ acts via the diagonal and $U \in \mathfrak{g}$ acts by the Leibniz-rule $U(v_1 \otimes v_2) = Uv_1 \otimes v_2 + v_1 \otimes Uv_2$. If both modules are Harish-Chandra modules, then the tensor product is also a Harish-Chandra module. Of course any finite dimensional rational representation of the algebraic group also yields a Harish-Chandra module.

deRhamiso

8.1.3 The representation theoretic de-Rham isomorphism

For us the $(\mathfrak{g}, K_\infty)$ module $\mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f)$,- this is the space of functions which are $\mathcal{C}_\infty$ in the variable g_∞- is one of the most important $(\mathfrak{g}, K_\infty)$ -modules. We may also consider the limit over smaller and smaller levels K_f we get the space $\mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A}))$, which consists of those functions on $G(\mathbb{A})$, which are left invariant under $G(\mathbb{Q})$, right invariant under a suitably small open subgroup $K_f \subset G(\mathbb{A}_f)$ and which are $\mathcal{C}_\infty$ in the variable g_∞. On these functions the group $G(\mathbb{A})$ acts by translations from the right, since our functions are $\mathcal{C}_\infty$ we also get an action of the Lie algebra $\mathfrak{g}$. Hence this is also a $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$-module.

If we fix the level see that $\mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f))$ is a $(\mathfrak{g}, K_\infty) \times \mathcal{H}_{K_f}$, the Hecke algebra acts by convolution. We choose a highest weight module $\mathcal{M}_\lambda$ and apply the previous considerations to the Harish-Chandra module

$$V = \mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f) \otimes \mathcal{M}_\lambda.$$

Notice that we can evaluate an element $f \in \mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f) \otimes \mathcal{M}_\lambda$ in a point $\underline{g} = (g_\infty, \underline{g}_f)$ and the result $f(\underline{g}) \in \mathcal{M}_\lambda$. The Hecke algebra acts via convolution on the first factor.

Let us assume that our compact subgroup $K_f \subset G(\mathbb{A}_f)$ is neat, i.e. for any $\underline{g} = (g_\infty, \underline{g}_f) \in G(\mathbb{A})$ we have $\underline{g}^{-1}(K_\infty \times K_f)\underline{g} \cap G(\mathbb{Q}) = \{e\}$. In this case we know that $\tilde{\mathcal{M}}$ is a local system and we can form the de-Rham complex $\Omega^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$.

We have an action of the Hecke algebra on this complex and we have the following fundamental fact: Borel

Proposition 8.1.1. *We have a canonical isomorphism of complexes*

$$Hom_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f)\otimes\mathcal{M}_\lambda) \xrightarrow{\sim} \Omega^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda\otimes\mathbb{C}),$$

this isomorphism is compatible with the action of the Hecke algebra on both sides

This is rather clear. We have the projection map

$$q : G(\mathbb{R})\times G(\mathbb{A}_f) \to G(\mathbb{R})/K_\infty \times G(\mathbb{A}_f)/K_f = X\times G(\mathbb{A}_f)/K_f$$

let $x_0 \in X\times G(\mathbb{A}_f)/K_f$ be the image of the identity $e\in G(\mathbb{R})$. The differential $D_q(e)$ maps the Lie algebra $\mathfrak{g} =$ tangent space of $G(\mathbb{R})$ at e to the tangent space T_{X,x_0} at $x_0\times e_f$. This provides the identification $T_{X,x_0} \xrightarrow{\sim} \mathfrak{g}/\mathfrak{k}$.

An element $\omega\in$ $\mathrm{Hom}_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}), \mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f)\otimes\mathcal{M}_\lambda)$ can be evaluated on a p-tuple $(X_0, X_1, \dots, X_{p-1})$ and the result

$$\omega(X_0, X_1,\dots, X_{p-1}) \in \mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f)\otimes\mathcal{M}_\lambda.$$

We want to produce an element $\tilde{\omega}$ in the de-Rham complex $\Omega^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$. Pick a point $x\times \underline{g}_f \in X\times G(\mathbb{A}_f)/K_f$, we find an element $(g_\infty, \underline{g}_f)\in G(\mathbb{R})\times G(\mathbb{A}_f)$ such that $g_\infty x_0 = x$. Our still to be defined form $\tilde{\omega}$ can be evaluated at a p-tuple $(Y_0,\dots, Y_{p-1})$ of tangent vectors in $x\times\underline{g}_f$ and the result has to be an element in $\mathcal{M}_{\mathbb{C},x}$. We find a p-tuple $(X_0, X_1,\dots, X_{p-1})$ of tangent vectors at x_0 which are mapped to $(Y_0,\dots, Y_{p-1})$ under the differential D_{g_∞} of the left translation by L_{g_∞}. We put Armand

$$\tilde{\omega}(Y_0,\dots,Y_{p-1})(x\times\underline{g}_f) = g_\infty^{-1}(\omega(X_0,\dots,X_{p-1})(g_\infty,\underline{g}_f)). \tag{8.4}$$

At this point I leave it as an exercise to the reader that this gives the isomorphism we want.(Ref ???)

We recall that the de-Rham complex (Reference Book Vol. !) computes the cohomology and therefore we can rewrite the de-Rham isomorphism BodeRh

$$H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) \xrightarrow{\sim} H^\bullet(\ \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f)\otimes\mathcal{M}_\lambda) \tag{8.5}$$

From now on the complex $\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f)\otimes\mathcal{M}_\lambda)$ will also be called the de-Rham complex.

By the same token we can compute the cohomology with compact supports BodeRhcs

$$H^\bullet_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) \xrightarrow{\sim} H^\bullet(\ \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathcal{C}_{c,\infty}(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f)\otimes\mathcal{M}_\lambda) \tag{8.6}$$

where $\mathcal{C}_{c,\infty}(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f)$ are the $\mathcal{C}_\infty$ function with compact support. These isomorphisms are also valid if we drop the assumption that K_f is neat.

The Poincaré duality on the cohomology is induced by the pairing on the de-Rham complexes: PD

Proposition 8.1.2. *If* $\omega_1 \in$ $Hom_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f)\otimes\tilde{\mathcal{M}})$ *is a closed form and* $\omega_2\in$ $Hom_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathcal{C}_{\infty,c}(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f)\otimes\tilde{\mathcal{M}}^\vee)$ *a closed form with compact support in complementary degree then the value of the cup* *product pairing of the classes* $[\omega_1] \in H^p(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda), [\omega_2] \in H_c^{d-p}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}^\vee_\lambda)$ *is given by*

$$< [\omega_1] \cup [\omega_2] >= \int_{\mathcal{S}^G_{K_f}} < \omega_1 \wedge \omega_2 >$$

(Reference Book Vol. !)

8.1.4 Input from representation theory of real reductive groups.

Let us consider an arbitrary irreducible $(\mathfrak{g}, K_\infty)$- module V. We also assume that for any $\vartheta \in \hat{K_\infty}$ the multiplicity of ϑ in V is finite (we say that V is admissible). Then we can extend the action of the Lie-algebra $\mathfrak{g}$ to an action of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ on V and we can restrict this action to an action of the centre $\mathfrak{Z}(\mathfrak{g})$. The structure of this centre is well known by a theorem of Harish-Chandra, it is a polynomial algebra in $r =$ rank(G) variables, here the rank is the absolute rank, i.e. the dimension of a maximal torus in $G/\mathbb{Q}$. (See Chap. 4 sect. 4)

Clearly this centre respects the decomposition into K_∞ types, since these K_∞ types come with finite multiplicity we can apply the standard argument, which proves the Lemma of Schur. Hence $\mathfrak{Z}(\mathfrak{g})$ has to act on V by scalars, we get a homomorphism $\chi_V : \mathfrak{Z}(\mathfrak{g}) \to \mathbb{C}$, which is defined by

$$zv = \chi_V(z)v.$$

This homomorphism is called the *central character* of V.

A fundamental theorem of Harish-Chandra asserts that for a given central character there exist only finitely many isomorphism classes of irreducible, admissible $(\mathfrak{g}, K_\infty)$-modules with this central character.

Of course for any rational finite dimensional representation $r : G/\mathbb{Q} \to \mathrm{Gl}(\mathcal{M}_\lambda)$ we can consider $\mathcal{M}_\lambda \otimes \mathbb{C}$ as $(\mathfrak{g}, K_\infty)$-module. If $\mathcal{M}_\lambda$ is absolutely irreducible with highest weight λ (See chap. IV) then it also has a central character $\chi_\mathcal{M} = \chi_\lambda$.

Wigner's lemma: *Let V be an irreducible, admissible $(\mathfrak{g}, K_\infty)$-module, let $\mathcal{M} = \mathcal{M}_\lambda$, a finite dimensional, absolutely irreducible rational representation. Then $H^\bullet(\mathfrak{g}, K_\infty, V \otimes \mathcal{M}_\mathbb{C}) = 0$ unless we have*

$$\chi_V(z) = \chi_{\mathcal{M}^\vee}(z) = \chi_{\mathcal{M}_{\lambda^\vee}}(z) \textit{ for all } z \in \mathfrak{Z}(\mathfrak{g})$$

Since we also know that the number of isomorphism classes of irreducible, admissible $(\mathfrak{g}, K_\infty)$-modules with a given central character is finite, we can conclude that for a given absolutely irreducible rational module $\mathcal{M}_\lambda$ the number of isomorphism classes of irreducible, admissible $(\mathfrak{g}, K_\infty)$-modules V with $H^\bullet(\mathfrak{g}, K_\infty, V \otimes \mathcal{M}_\mathbb{C}) \neq 0$ is finite.

The proof of Wigner's lemma is very elegant. We have $\mathcal{M} \otimes V = \mathcal{M}^\vee \otimes V$ and hence we have $H^0(\mathfrak{g}, K_\infty, \mathcal{M} \otimes V) =$ $\mathrm{Hom}(\mathcal{M}^\vee, V)^{(\mathfrak{g}, K_\infty)} =$ $\mathrm{Hom}_{\mathfrak{g}, K_\infty}(\mathcal{M}^\vee, V)$. In [11] , Chap.I 2.4 it is shown, that the category of $\mathfrak{g}, K_\infty$ -modules has enough injective and projective elements (See [11], I. 2.5) . If I is an injective $(\mathfrak{g}, K_\infty)$-module then $\mathcal{M} \otimes I$ is also injective because for any $\mathfrak{g}, K_\infty$-module A we have $\mathrm{Hom}(A, \mathcal{M} \otimes I) = \mathrm{Hom}(\mathcal{M}^\vee, I)$. Hence an injective resolution $0 \to V \to I^0 \to I^1 \dots$ yields an injective resolution $0 \to \mathcal{M} \to \mathcal{M} \otimes I^0 \to \mathcal{M} \otimes I^1 \dots$ and from this we get

$$H^q(\mathfrak{g}, K_\infty, \mathcal{M} \otimes V) = \mathrm{Ext}^q_{(\mathfrak{g}, K_\infty)}(\mathcal{M}^\vee, V).$$

Any $z \in \mathfrak{Z}(\mathfrak{g})$ induces an endomorphism of $\mathcal{M}_\lambda$ and V. Since $\mathrm{Ext}^\bullet$ is functorial in both variables, we see that z induces endomorphisms z_1 (via the action on $\mathcal{M}_\lambda$) and z_2 (via the action on V) on $\mathrm{Ext}^q_{\mathfrak{g}, K_\infty}(\mathcal{M}^\vee, V)$. We show that $z_1 = z_2$. This is clear by definition for $\mathrm{Ext}^0_{\mathfrak{g}, K_\infty}(\mathcal{M}^\vee, V) = \mathrm{Hom}_{\mathfrak{g}, K_\infty}(\mathcal{M}^\vee, V)$: For $z \in \mathfrak{Z}(\mathfrak{g})$ and $\phi \in \mathrm{Hom}_{\mathfrak{g}, K_\infty}(\mathcal{M}^\vee, V), m \in \mathcal{M}_\lambda$ we have $z_1\phi(m) = \phi(zm) = z_2(\phi(m))$. To prove it for an arbitrary q we use devissage and induction. We embed V into an injective $\mathfrak{g}, K_\infty$ module I and get an exact sequence

$$0 \to V \to I \to I/V \to 0$$

and from this and $\mathrm{Ext}^q_{\mathfrak{g}, K_\infty}(\mathcal{M}_\lambda, I)$ for $q > 0$ we get

$$\mathrm{Ext}^{q-1}(\mathfrak{g}, K_\infty, \mathcal{M}_\lambda, I/V) = \mathrm{Ext}^q(\mathfrak{g}, K_\infty, \mathcal{M}_\lambda, V) \text{ for } q > 0.$$

Now by induction we know $z_1 = z_2$ on the left hand side, so it also holds on the right hand side.

If now $\chi_V \neq \chi_{\mathcal{M}^\vee}$ then we can find a $z \in \mathfrak{Z}(\mathfrak{g})$ such that $\chi_{\mathcal{M}^\vee}(z) = 0, \chi_V(z) = 1$. This implies that $z_1 = 0$ and $z_2 = 1$ on all $\mathrm{Ext}^q(\mathfrak{g}, K_\infty(\mathcal{M}_\lambda, V)$. Since we know that $z_1 = z_2$ we see that the identity on $\mathrm{Ext}^q(\mathfrak{g}, K_\infty(\mathcal{M}_\lambda, V)$ is equal to zero and this implies the assertion.

On the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ we have an antiautomorphism $u \mapsto^t u$ which is induced by the antiautomorphism $X \mapsto -X$ on the Lie algebra $\mathfrak{g}$. If V is an admissible $(\mathfrak{g}, K_\infty)$-module, then we can form the dual module $V^\vee$ and if we denote the pairing between $V, V^\vee$ by $< , >_V$ then

$$< Uv, \phi >_V = < v, ^t U\phi >_V \text{ for all } U \in \mathfrak{U}(\mathfrak{g}), v \in V, \phi \in V^\vee.$$

If V is irreducible, then it has a central character and we get

$$\chi_{V^\vee}(z) = \chi_V(^t z).$$

This applies to finite dimensional and to infinite dimensional $(\mathfrak{g}, K_\infty)$-modules.

8.1.5 Representation theoretic Hodge-theory.

We consider irreducible unitary representations $G(\mathbb{R}) \to U(H)$. We know from the work of Harish-Chandra:

1) If we fix an isomorphism class ϑ irreducible representations of K_∞ then the isotypical subspace $\dim_{\mathbb{C}} H(\vartheta) \leq \dim(\vartheta)^2$, i.e. ϑ occurs at most with multiplicity $\dim(\vartheta)$.

2) The direct sum $\sum_{\vartheta \subset \hat{K_\infty}} H(\vartheta) = H^{(K)} \subset H$ is dense in H and it is an admissible irreducible Harish-Chandra -module.

We call an irreducible $(\mathfrak{g}, K_\infty)$-module unitary, if it is isomorphic to such an $H^{(K_\infty)}$.

For a given $G/\mathbb{R}$ and any rational irreducible module $\mathcal{M}_\lambda$ Vogan and Zuckerman give a finite list of certain irreducible, admissible $(\mathfrak{g}, K_\infty)-$ modules $A_\mathfrak{q}(\lambda)$, for which $H^\bullet(\mathfrak{g}, K_\infty, A_\mathfrak{q}(\lambda) \otimes \mathcal{M}_\lambda) \neq 0$ they compute these cohomology group. This list contains all unitary, irreducible $(\mathfrak{g}, K_\infty)-$modules, which have non trivial cohomology with coefficients in $\mathcal{M}_\lambda$.

For the following we refer to [11] Chap. II , §1-2 . We want to apply the methods of Hodge-theory to compute the cohomology groups $H^\bullet(\mathfrak{g}, K_\infty, V \otimes \mathcal{M}_\lambda)$ for an unitary $(\mathfrak{g}, K_\infty)$-module V. This means have a positive definite scalar product $< , >_V$ on V, for which the action of K_∞ is unitary and for $U \in \mathfrak{g}$ and $v_1, v_2 \in V$ we have $< Uv_1, v_2 >_V + < v_1, Uv_2 >_V = 0$.

We assume that $\mathcal{M}_\lambda$ is conjugate-autodual. In the next step we introduce for all p a hermitian form on $\mathrm{Hom}_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}), V \otimes \mathcal{M}_\lambda)$. To do this we construct a hermitian form on $\mathcal{M}_\lambda$.

(The following considerations are only true modulo the centre). We consider the Lie algebra and its complexification $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \otimes \mathbb{C}$. On this complex vector space we have the complex conjugation $^- : U \mapsto \bar{U}$. We rediscover $\mathfrak{g}$ as the set of fixed points under $^-$. We also have the Cartan involution Θ which is the involution which has $\mathfrak{k}$ as its fixed point set. Then we get the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \text{ where } \mathfrak{p} \text{ is the -1 eigenspace of } \Theta.$$

The Killing form is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$, we have for the Lie bracket $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. We consider the invariants under $^- \circ \Theta$, this is the Lie algebra $\mathfrak{g}_c = \mathfrak{k} \oplus \sqrt{-1} \otimes \mathfrak{p}$. On this real Lie algebra the Killing form is negative definite and $\mathfrak{g}_c$ is the Lie algebra of an algebraic group $G_c/\mathbb{R}$ whose base extension $G_c \otimes_\mathbb{R} \mathbb{C} \xrightarrow{\sim} G \otimes_\mathbb{R} \mathbb{C}$ and whose group $G_c(\mathbb{R})$ of real points is compact (this is the so called compact form of G). We still have the representation $G_c/\mathbb{R} \to \mathrm{Gl}(\mathcal{M}_\lambda)$ which is irreducible and hence we find a hermitian form $< , >_\lambda$ on $\mathcal{M}_\lambda$, which is invariant under $G_c(\mathbb{R})$ and which is unique up to a scalar.

This form satisfies the equations

$$< Um_1, m_2 >_\mathcal{M} + < m_1, Um_2 >_\lambda = 0 \text{ for all } m_1, m_2 \in \mathcal{M}_\lambda, U \in \mathfrak{k}$$

this is the invariance under K_∞ and

$$< Um_1, m_2 >_\mathcal{M} = < m_1, Um_2 >_\lambda \text{ for all } m_1, m_2 \in \mathcal{M}_\lambda, U \in \mathfrak{p}$$

this is the invariance under $\sqrt{-1} \otimes \mathfrak{p}$.

Now we define a hermitian metric on $V \otimes \mathcal{M}_\lambda$, we simply take the tensor product $< , >_V \otimes < , >_\lambda = < , >_{V \otimes \lambda}$. Finally we define the (hermitian) scalar product on $\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), V \otimes \mathcal{M}_\lambda)$. We choose and orthonormal (with respect to the Killing form) basis $E_1, E_2, \ldots, E_d$ on $\mathfrak{p}$, we identify $\mathfrak{g}/\mathfrak{k} \xrightarrow{\sim} \mathfrak{p}$. Then a form $\omega \in \mathrm{Hom}_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}), V \otimes \mathcal{M}_\lambda)$ is given by its values $\omega(E_I) \in V \otimes \mathcal{M}_\lambda$, where $I = \{i_1, i_2, \ldots, i_p\}$ runs through the ordered subsets of $\{1, 2, \ldots, d\}$ with p elements. For $\omega_1, \omega_2 \in \mathrm{Hom}_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}), V \otimes \mathcal{M}_\lambda)$ we put

$$< \omega_1, \omega_2 >= \sum_{I,|I|=p} < \omega_1(E_I), \omega_2(E_I) >_{V\otimes\lambda} \tag{8.7}$$

Now we can define an adjoint operator

$$\delta : \ \mathrm{Hom}_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}), V \otimes \mathcal{M}_\lambda) \to \ \mathrm{Hom}_{K_\infty}(\Lambda^{p-1}(\mathfrak{g}/\mathfrak{k}), V \otimes \mathcal{M}_\lambda), \tag{8.8}$$

which can be defined by a straightforward calculation. We simply write a formula for δ: For an element E_i we define $E_i^*(v \otimes m) = -E_i v \otimes m + v \otimes E_i m$. Then we can define δ by the following formula:

We have to evaluate $\delta(\omega)$ on $E_J = (E_{i_1}, \ldots, E_{i_{p-1}})$ where $J = \{i_1, \ldots, i_{p-1}\}$. We put

$$\delta(\omega)(E_J) = \sum_{i \notin J} (-1)^{p(i,J\cup\{i\})} E_i^* \omega_{J\cup\{i\}},$$

where $p(i, J \cup \{i\})$ denotes the position of i in the ordered set $J \cup \{i\}$. With this definition we get for a pair of forms $\omega_1 \in \ \mathrm{Hom}_{K_\infty}(\Lambda^{p-1}(\mathfrak{g}/\mathfrak{k}), V \otimes \mathcal{M}_\lambda)$ and $\omega_2 \in \ \mathrm{Hom}_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}), V \otimes \mathcal{M}_\lambda)$ (See [11], II, prop. 2.3)

$$< d\omega_1, \omega_2 >=< \omega_1, \delta\omega_2 > \tag{8.9}$$

We define the Laplacian $\Delta = \delta d + d\delta$. Then we have ([11] , II ,Thm.2.5)

$$< \Delta\omega, \omega >\geq 0 \text{ and we have equality if and only if } d\omega = 0, \delta\omega = 0 \tag{8.10}$$

Inside $\mathfrak{Z}(\mathfrak{g})$ we have the Casimir operator C (See Chap. 4). An element $z \in \mathfrak{Z}(\mathfrak{g})$ acts on $V \otimes \mathcal{M}_\lambda$ by $z \otimes \mathrm{Id}$ via the action on the first factor and by the scalar $\chi_\lambda(z)$ via the action on the second factor. Then we have

Kuga's lemma : *The action of the Casimir operator and the Laplace operator on* $Hom_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}), V \otimes \mathcal{M}_\lambda)$ *are related by the identity*

$$\Delta = C \otimes Id - \chi_\lambda(C).$$

If the $(\mathfrak{g}, K_\infty)$ *module is irreducible, then* Δ *acts by multiplication by the scalar* $\chi_V(C) - \chi_\lambda(C)$

This has the following consequence

If V *is an irreducible unitary* $\mathfrak{g}, K_\infty$*- module and if* $\mathcal{M}_\lambda$ *is an irreducible representation with highest weight* λ *then*

$$H^\bullet(\mathfrak{g}, K_\infty, V \otimes \mathcal{M}_\mathbb{C}) = \begin{cases} 0 & \text{if } \chi_V(C) - \chi_\lambda(C) \neq 0 \\ Hom_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), V \otimes \mathcal{M}_\lambda) & \text{if } \chi_V(C) - \chi_\lambda(C) = 0 \end{cases}.$$

This only applies for unitary $\mathfrak{g}, K_\infty$-modules, but for these it is much stronger: It says that under the assumption $\chi_V(C) = \chi_\lambda(C)$ we have $\chi_V = \chi_\lambda$ (we only have to test the Casimir operator) and it says that all the differentials in the complex are zero.

8.1.6 Input from the theory of automorphic forms

We apply this to the spaces of square integrable functions on $G(\mathbb{Q})\backslash G(\mathbb{A})/K_f$. Because of the presence of a non trivial center, we have to consider functions which transform in a certain way under the action of the center. We may assume that coefficient system $\mathcal{M}_\lambda$ has a central character and this central character defines a character ζ_λ on the maximal $\mathbb{Q}$-split torus $S \subset C$. This character can be evaluated on $S^0(\mathbb{R})$ this is the connected component of the identity of the real valued points of S. The map $z_\infty \mapsto (z_\infty, 1, \dots, 1, \dots) \in S(\mathbb{A})$ is an embedding of $S^0(\mathbb{R})$ into $G(\mathbb{A})$. It follows from [9] that the quotient $G(\mathbb{Q})S^0(\mathbb{R})\backslash G(\mathbb{A})/K_f$ has finite volume. We define the space of functions

$$\mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f, \zeta_\infty^{-1}) \tag{8.11}$$

to be the subspace of those $\mathcal{C}_\infty$ functions which satisfy $f(z_\infty \underline{g}) = \zeta_\infty^{-1}(z_\infty) f(\underline{g})$ for all $z_\infty \in S^0(\mathbb{R})$. The isogeny $d_C : C \to C'$ (see 6.1.1) induces an isomorphism $S^0(\mathbb{R}) \xrightarrow{\sim} S'^{,0}(\mathbb{R})$, where S' is the maximal $\mathbb{Q}$ split torus in C'. Therefore we get a character $\zeta'_\infty : S'^{,0}(\mathbb{R}) \to \mathbb{R}^\times_{>0}$ and this is also a character $\zeta'_\infty : G(\mathbb{R}) \to \mathbb{R}^\times_{>0}$. Its restriction to $S^0(\mathbb{R})$ is ζ_∞. If now $f \in \mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f, \zeta_\infty^{-1})$ then

$$f(\underline{g})\zeta'_\infty(\underline{g}) \in \mathcal{C}_\infty(G(\mathbb{Q})S^0(\mathbb{R})\backslash G(\mathbb{A})/K_f) \tag{8.12}$$

We say that $f \in \mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f, \zeta_\infty^{-1})$ is square integrable if sqint

$$\int_{(G(\mathbb{Q})S^0(\mathbb{R})\backslash G(\mathbb{A})/K_f)} |f(\underline{g})\zeta'_\infty(\underline{g})|^2 d\underline{g} < \infty \tag{8.13}$$

and this allows us to define the Hilbert space $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f, \zeta_\infty^{-1})$. Since the space $(G(\mathbb{Q})S^0(\mathbb{R})\backslash G(\mathbb{A})/K_f)$ has finite volume we know that

$$\zeta'_\infty \in L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f, \zeta_\infty^{-1}).$$

The group $G(\mathbb{R})$ acts on $\mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f, \zeta_\infty^{-1})$ by right translations and hence we get by differentiating an action of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ on it. We define by $\mathcal{C}^{(2)}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f, \zeta_\infty^{-1})$ the subspace of functions f for which Uf is square integrable for all $U \in \mathfrak{U}(\mathfrak{g})$.

This allows us to define a sub complex of the de-Rham complex Ltwo

$$\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathcal{C}^{(2)}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f, \zeta_\infty^{-1}) \otimes \mathcal{M}_\lambda). \tag{8.14}$$

We will not work with this complex because its cohomology may show some bad behavior. (See remark below).

We do something less sophisticated, we simply define $H^\bullet_{(2)}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) \subset H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$ to be the image of the cohomology of the complex (8.14) in the cohomology. Hence $H^\bullet_{(2)}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$ is the space of cohomology classes which can be represented by square integrable forms.

Remark: Some authors also define L^2- de-Rham complexes, using the above complex (8.14) and then they take suitable completions to get complexes of

Hilbert spaces. These complexes also give cohomology groups which run under the name of L^2-cohomology. These L^2-cohomology groups are related but not necessarily equal to our $H^\bullet_{(2)}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$. They can be infinite dimensional.

The Hilbert space $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f, \zeta_\infty^{-1})$ is a module for $G(\mathbb{R})\times\mathcal{H}_{K_f}$ the group $G(\mathbb{R})$ acts by unitary transformations and the algebra $\mathcal{H}_{K_f}$ is selfadjoint.

Let us assume that $H = H_{\pi_\infty\times\pi_f}$ is an irreducible unitary module for $G(\mathbb{R})\times \mathcal{H} = \bigotimes'_p \mathcal{H}_p$ and assume that we have an inclusion of this $G(\mathbb{R})\times\mathcal{H}$-module

$$j : H \hookrightarrow L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f, \zeta_\infty^{-1}).$$

It follows from the finiteness results in 8.1.5 that induces an inclusion into the space of square integrable $\mathcal{C}_\infty$ functions

$$H^{(K_\infty)} \hookrightarrow \mathcal{C}^{(2)}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f, \zeta_\infty^{-1})^{(K_\infty)}.$$

We consider the $(\mathfrak{g}, K_\infty)-$ cohomology of this module with coefficients in our irreducible module $\mathcal{M}_\lambda$, we assume $\chi_V(C) = \chi_\lambda(C)$. We have $H^\bullet(\mathfrak{g}, K_\infty, H\otimes \mathcal{M}_\lambda) = \mathrm{Hom}_{K_\infty}(\mathfrak{g}, K_\infty, H^{(K_\infty)}\otimes\mathcal{M}_\lambda)$ and get

$$H^\bullet(\mathfrak{g}, K_\infty, H^{(K_\infty)}\otimes\mathcal{M}_\mathbb{C}) \xrightarrow{j^\bullet} H^\bullet(\mathfrak{g}, K_\infty, \mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f, \zeta_\infty^{-1})^{(K_\infty)}\otimes\mathcal{M}_\lambda).$$

This suggests that we try to "decompose" $\mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f, \zeta_\infty^{-1})^{(K_\infty)}$ into irreducibles and then investigate the contributions of the irreducible summands to the cohomology. Essentially we follow the strategy of [Bo-Ga] and [7] but instead of working with complexes of Hilbert spaces we work with complexes of $\mathcal{C}_\infty$ forms and modify the arguments accordingly.

It has been shown by Langlands, that we have a decomposition into a discrete and a continous spectrum

$$L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f) = L^2_{\text{disc}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f) \oplus L^2_{\text{cont}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f),$$

where $L^2_{\text{disc}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f)$ is the closure of the sum of all irreducible closed subspaces occuring in $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f)$ and where $L^2_{\text{cont}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f)$ is the complement.

The discrete spectrum $L^2_{\text{disc}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f)$ contains as a subspace the *cuspidal spectrum* $L^2_{\text{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f)$:

A function $f\in L^2(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f)$ is called a *cusp form* if for all proper parabolic subgroups $P/\mathbb{Q}\subset G/\mathbb{Q}$, with unipotent radical $U_P/\mathbb{Q}$ the integral

$$\mathcal{F}^P(f)(g) = \int_{U_P(\mathbb{Q})\backslash U_P(\mathbb{A})} f(\underline{u}\underline{g})d\underline{u} = 0,$$

this means that the integral is defined for almost all $\underline{g}$ and zero for almost all $\underline{g}$. The function $\mathcal{F}^P(f)(\underline{g})$, which is an almost everywhere defined function on $P(\mathbb{Q})\backslash G(\mathbb{A})/K_f$ is called the constant Fourier coefficient of f along $P/\mathbb{Q}$. The cuspidal spectrum the intersection of all the kernels of the $\mathcal{F}^P$.

If our group is anisotropic, then it does not have any proper parabolic subgroup and in this case we have $L^2_{\text{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f) = L^2_{\text{disc}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f) = L^2(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f)$.

For any unitary $G(\mathbb{R}) \times \mathcal{H}$- module $H_\pi = H_{\pi_\infty} \otimes H_{\pi_f}$ we put

$$W_{\text{cusp}}(\pi) := \operatorname{Hom}_{G(\mathbb{R})\times\mathcal{H}}(H_\pi, L^2_{\text{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f)). \tag{8.15}$$

We can ignore the $\mathcal{H}$-module structure and define

$$W_{\text{cusp}}(\pi_\infty) = \operatorname{Hom}_{G(\mathbb{R})}(H_{\pi_\infty}, L^2_{\text{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f)).$$

Then then dimension of $W_{\text{cusp}}(\pi_\infty)$ is the multiplicity $m_{\text{cusp}}(\pi_\infty)$. It has been shown by Gelfand-Graev and Langlands that

$$m_{\text{cusp}}(\pi_\infty) = \sum_{\pi_f} \dim(W_{\pi,\text{cusp}}) < \infty.$$

We get a decomposition into isotypical subspaces

$$L^2_{\text{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f) = \overline{\bigoplus_{\pi_\infty\otimes\pi_f} (L^2_{\text{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f)(\pi_\infty \times \pi_f)},$$

where $(L^2_{\text{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f)(\pi_\infty \times \pi_f)$ is the image of $W_{\pi,\text{cusp}} \otimes H_\pi$ in $L^2_{\text{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f)$.

The cuspidal spectrum has a complement in the discrete spectrum, this is the *residual spectrum* $L^2_{\text{res}}((G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f)$. It is called residual spectrum, because the irreducible subspaces contained in it are obtained by residues of Eisenstein classes.

Again we define $W_{\text{res}}(\pi) = \operatorname{Hom}_{G(\mathbb{R})\times\mathcal{H}}(H_\pi, L^2_{\text{res}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f))$, (resp. $W_{\text{res}}(\pi_\infty) = \operatorname{Hom}_{G(\mathbb{R})}(H_{\pi_\infty}, L^2_{\text{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f))$, and it is a deep theorem of Langlands that $m_{\text{res}}(\pi_\infty) = \dim(W_{\text{res}}(\pi_\infty) < \infty$. Hence we get a decomposition

$$L^2_{\text{res}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f) = \overline{\bigoplus_{\pi_\infty\otimes\pi_f} (L^2_{\text{res}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f)(\pi_\infty \times \pi_f)}.$$

If our group $G/\mathbb{Q}$ is isotropic, then the one dimensional space of constants is in the residual (discrete) spectrum but not in the cuspidal spectrum.

Langlands has given a description of the continuos spectrum using the theory of Eisenstein series, we have a decomposition decomp-cont

$$L^2_{\text{cont}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f) = \overline{\bigoplus_{\Sigma} \tilde{H}^+_P(\pi_\Sigma)}, \tag{8.16}$$

we briefly explain this decomposition following [Bo-Ga]. The Σ are so called cuspidal data, this are pairs (P, π_Σ) where P is a proper parabolic subgroup and π_Σ is a representation of $M(\mathbb{A}) = P(\mathbb{A})/U(\mathbb{A})$ occurring in the discrete spectrum $L^2_{\text{cusp}}(M(\mathbb{Q})\backslash M(\mathbb{A}))$.

Let $M^{(1)}/\mathbb{Q}$ be the semi simple part of M and recall that C/Q was the center of $G/\mathbb{Q}$. We consider the character module $Y^*(P) = \operatorname{Hom}(C \cdot M^{(1)}, \mathbb{G}_m)$. The elements $Y^*(P)\otimes\mathbb{C}$ provide homomorphisms $\gamma\otimes z : M(\mathbb{A})/C(\mathbb{A})M^{(1)}(\mathbb{A}) \to \mathbb{C}^\times$.

(See (6.16)). The module $Y^*(P)\otimes\mathbb{Q}$ comes with a canonical basis which is given by the dominant fundamental weights γ_μ which are trivial on $M^{(1)}$. We define

$$\Lambda_\Sigma = Y^*(P)\otimes i\mathbb{R} = \{\sum_\mu \gamma_\mu\otimes it_\mu | t_\mu\in\mathbb{R}\}$$

this is a group of unitary characters. For $\sigma\in\Lambda_\Sigma$ we define the unitarily induced representation

$$\begin{gathered}\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}\pi_\Sigma\otimes(\sigma+\rho_P) = I_P^G\pi_\Sigma\otimes\sigma \\ \{f : G(\mathbb{A})\to L^2_{\mathrm{res}}(M(\mathbb{Q})\backslash M(\mathbb{A}))(\pi_\Sigma) | f(\underline{p}\underline{g}) = (\sigma+|\rho_P|)(\underline{p})\pi_\Sigma(\underline{p})f(\underline{g})\}\end{gathered} \tag{8.17}$$

where of course $\underline{p}\in P(\mathbb{A}), \underline{g}\in G(\mathbb{A})$ and $\rho_P\in Y^*(P)\otimes\mathbb{Q}$ is the half sum of the roots in the unipotent radical of P. This gives us a unitary representation of $G(\mathbb{A})$. Let d_Σ be the Lebesgue measure on Λ_Σ then we can form the direct integral unitary representations

$$H_P(\pi_\Sigma) = \int_{\Lambda_\Sigma} I_P^G\pi_\Sigma\otimes\sigma\, d_\Sigma\sigma \tag{8.18}$$

The theory of Eisenstein series gives us a homomorphism of $G(\mathbb{R})\times\mathcal{H}$ -modules

$$\mathrm{Eis}_P(\pi_\Sigma) : H_P(\pi_\Sigma)\to L^2_{\mathrm{cont}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f). \tag{8.19}$$

Let us put

$$\Lambda_\Sigma^+ = \{\sum_\mu \gamma_\mu\otimes it_\mu | t_\mu\geq 0\}$$

then the restriction

$$\mathrm{Eis}_P(\pi_\Sigma) : H_P^+(\pi_\Sigma) = \int_{\Lambda_\Sigma^+} I_P^G\pi_\Sigma\otimes\sigma\, d_\Sigma\sigma \to L^2_{\mathrm{cont}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f). \tag{8.20}$$

is an isometric embedding. The image will be denoted by $\tilde{H}_P^+(\pi_\Sigma)$ these spaces are the elementary subspaces in [B-G]. Two such elementary subspaces $\tilde{H}_P^+(\pi_\Sigma), \tilde{H}_{P_1}^+(\pi_{\Sigma_1})$ are either orthogonal to each other or they are equal. We get the above decomposition if we sum over a suitable set of representatives of cuspidal data.

Now we are ready to discuss the contribution of the continuous spectrum to the cohomology. If we have a closed square integrable form

$$\omega\in\ \mathrm{Hom}_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}), \mathcal{C}_\infty^{(2)}(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f)\otimes\mathcal{M}_\lambda),$$

then we can decompose it

$$\omega = \omega_{\mathrm{res}} + \omega_{\mathrm{cont}},$$

both summands are $\mathcal{C}^2_\infty$ and closed.

Proposition 8.1.3. *The cohomology class* $[\omega_{\mathrm{cont}}]$ *is trivial.*

Proof. This now the standard argument in Hodge theory, but this time we apply it to a continuous spectrum instead of a discrete one. We follow Borel-Casselman and prove their Lemma 5.5 (See[B-C]) in our context. We may assume that ω_∞

lies in one of the summands, i.e. $\omega_{\text{cont}} = \text{Eis}(\int_{\Lambda_\Sigma} \omega^\vee(\sigma) d_\Sigma\sigma)$ where $\omega^\vee(\sigma) \in \text{Hom}_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}), I_P^G\pi_\Sigma \otimes \sigma \otimes \mathcal{M}_\lambda))$ is the Fourier transform of ω_∞ in the L^2., (theorem of Plancherel). As it stands the expression $\int_{\Lambda_\Sigma} \omega^\vee(\sigma) d_\Sigma\sigma)$ does not make sense because the integrand is in L^2 and not necessarily in L^1. If we choose a symmetric positive definite quadratic form $h(\sigma) = \sum_{\nu,\mu} b_{\nu,\mu} t_\nu t_\mu$ and a positive real number τ then the function

$$h_\tau(\sigma) = (1 + \tau h(\sigma)^m)^{-1} \in L^2(\Lambda_\Sigma)$$

and then $\omega^\vee(\sigma) h_\tau(\sigma)$ is in L^1 and by definition

$$\lim_{\tau \to 0} \int_{\Lambda_\Sigma} \omega^\vee(\sigma) h_\tau(\sigma) d_\Sigma\sigma) = \int_{\Lambda_\Sigma} \omega^\vee(\sigma) d_\Sigma\sigma \tag{8.21}$$

where the convergence is in the L^2 sense. Since $\omega_\infty \in \text{Hom}_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}), I_P^G\pi_\Sigma \otimes \sigma \otimes \mathcal{M}_\lambda)$ we get get that $\omega^\vee(\sigma)$ has the following property

For any polynomial $P(\sigma) = \sum a_{\underline{\mu}} t^{\underline{\mu}}$ in the variables t_μ and with real coefficients the section $\boxed{\text{diffmult}}$

$$\omega^\vee(\sigma) P(\sigma) \text{ is square integrable} \tag{8.22}$$

this follows from the well known rules that differentiating a function provides multiplication by the variables for the Fourier transform.

The Lemma of Kuga implies

$$\Delta(\omega^\vee(\sigma)) = (\chi_\sigma(C) - \chi_\lambda(C)) \omega^\vee(\sigma)$$

and if $\sigma = \sum \gamma_\mu \otimes it\_\mu$ the eigenvalue is

$$\chi_\sigma(C) - \chi_\lambda(C) = \sum a_{\nu,\mu} t_\nu t_\mu + \sum b_\mu t_\mu + c_{\pi_\Sigma} - c_\lambda. \tag{8.23}$$

where c_{π_Σ} is the eigenvalue of the Casimir operator of $M^{(1)}$ on π_Σ If the $t_\mu \in \mathbb{R}$ then this expression is always ≤ 0 especially we see that the quadratic form on the right hand side is negative definite. This implies that for $\sigma \in \Lambda_F$ the expression $\chi_\sigma(C) - \chi_\lambda(C)$ assumes a finite number of maximal values all of them ≤ 0 and hence

$$V_\Sigma = \{\sigma | \chi_\sigma(C) - \chi_\lambda(C) = 0\} \tag{8.24}$$

is a finite set of point. This set has measure zero, since we assumed that P was a proper parabolic subgroup. The of σ for which $H^\bullet(\mathfrak{g}, K_\infty, H_{\Lambda_\Sigma}(\sigma) \otimes \mathcal{M}_\mathbb{C}) \neq 0$ is finite. We choose a $\mathcal{C}_\infty$ function $h_\Sigma(\sigma)$ which is positive, which takes value 1 in a small neighbourhood of V_Σ, which takes values ≤ 1 in a slightly larger neighbourhood and which is zero outside this second neighbourhood. Then we write

$$\omega_\infty = \text{Eis}(\int_{\Lambda_\Sigma^+} h_\Sigma(\sigma) \omega^\vee(\sigma) d_\Sigma\sigma) + \text{Eis}(\int_{\Lambda_\Sigma^+} (1 - h_\Sigma(\sigma)) \omega^\vee(\sigma) d_\Sigma\sigma)$$

We have $d\omega^\vee(\sigma) = 0$ and hence we get

$$\Delta((1 - h_\Sigma(\sigma)) \omega^\vee(\sigma) = d\big((\chi_\sigma(C) - \chi_\lambda(C))(1 - h_{\Sigma(\sigma)}) \delta\omega^\vee(\sigma)\big)$$

and this implies that

$$\mathrm{Eis}(\int_{\Lambda_\Sigma^+}(1-h_\Sigma(\sigma))\omega^\vee(\sigma)d_\Sigma\sigma) = d\ \mathrm{Eis}(\int_{\Lambda_\Sigma^+}(1-h_\Sigma(\sigma))(\chi_\sigma(C)-\chi_\lambda(C))^{-1}\delta\omega^\vee(\sigma)d_\Sigma\sigma)$$

It is clear that the integrand in the second term- $\int_{\Lambda_\Sigma^+}(1-h_\Sigma(\sigma))(\chi_\sigma(C)-\chi_\lambda(C))^{-1}\delta\omega^\vee(\sigma)$ still satisfies (8.22) and then our well known rules above imply that $\psi = \mathrm{Eis}(\int_{\Lambda_\Sigma^+}(1-h_\Sigma(\sigma))(\chi_\sigma(C)-\chi_\lambda(C))^{-1}\delta\omega^\vee(\sigma)d_\Sigma\sigma)$ is $\mathcal{C}^2_\infty$. Therefore the second term in our above formula is a boundary.

$$\omega_{\mathrm{cont}} = \int_{\Lambda_\Sigma} h_\Sigma(\sigma)\omega(\sigma)d_\Sigma\sigma + d\psi.$$

This is true for any choice of h_Σ. Hence the scalar product $< \omega - d\psi, \omega - d\psi >$ can be made arbitrarily small. Then we claim that the cohomology class $[\omega] \in H^\bullet(\mathrm{Hom}_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}), \mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f) \otimes \mathcal{M}_\lambda)$ must be zero. This needs a tiny final step.

We invoke Poincaré duality: A cohomology class in $[\omega] \in H^p(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$ is zero if and only the value of the pairing with any class $[\omega_2] \in H^{d-p}_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}^\vee_\lambda)$ is zero. But the (absolute) value $[\omega] \cup [\omega_2]$ of the cup product can be given by an integral (See Prop.8.1.2). Therefore it can be estimated by the norm $< \omega - d\psi, \omega - d\psi >$ (Cauchy-Schwarz inequality) and hence must be zero. □

As usual we denote by $\widehat{G(\mathbb{R})}$ the unitary spectrum, for us it is simply the set of unitary irreducible representations of $G(\mathbb{R})$. Given $\tilde{\mathcal{M}}_\lambda$, we define

$$\mathrm{Coh}_2(\lambda) = \{\pi_\infty \in \widehat{G(\mathbb{R})} |\ H^\bullet(\mathfrak{g}, K_\infty, H_{\pi_\infty} \otimes \tilde{\mathcal{M}}_\lambda) \neq 0\} \tag{8.25}$$

The theorem of Harish-Chandra says that this set is finite.

Let

$$H_{\mathrm{Coh}_2(\lambda)} = \bigoplus_{\pi:\pi_\infty \in \mathrm{Coh}_2(\lambda)} L^2_{\mathrm{disc}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f)(\pi_\infty \times \pi_f) = \bigoplus_{\pi:\pi_\infty \in \mathrm{Coh}_2(\lambda)} H_{\pi_\infty}(\pi_f) \tag{8.26}$$

the theorem of Gelfand-Graev and Langlands assert that this is a finite sum of irreducible modules. This space decomposes again into $H^{\mathrm{cusp}}_{\mathrm{Coh}_2(\lambda)} \oplus H^{\mathrm{res}}_{\mathrm{Coh}_2(\lambda)}$

Then we get the following theorem which is due to Borel, Garland, Matsushima and Murakami [Bo-Ga-Mu]

Theorem 8.1.1. *a)The map*

$$H^\bullet(\mathfrak{g}, K_\infty, H^{(K_\infty)}_{\mathrm{Coh}_2(\lambda)} \otimes \mathcal{M}_\lambda) = \ Hom_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), H^{(K_\infty)}_{\mathrm{Coh}_2(\lambda)} \otimes \mathcal{M}_\lambda) \to H^\bullet_{(2)}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$$

surjective. Especially the image contains $H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$.

b) (Borel) The homomorphism

$$H^\bullet(\mathfrak{g}, K_\infty, H^{(\mathrm{cusp},\mathrm{K}_\infty)}_{\mathrm{Coh}_2(\lambda)} \otimes \mathcal{M}_\lambda) \to H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$$

is injective.

In [8] Prop.5.6, they do not consider the above space $H^\bullet_{(2)}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda)$ we added an $\epsilon > 0$ to this proposition by claiming that this space is the image.

In general the homomorphism

$$H^\bullet(\mathfrak{g},K_\infty,H^{\mathrm{res}}_{\mathrm{res}(\lambda)}\otimes\mathcal{M}_\lambda)\to H^\bullet(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda)$$

is not injective. We come back to this issue in the next section.

If we denote by $H^\bullet_{\mathrm{cusp}}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda)$ the image of the homomorphism in b), then we get a filtration of the cohomology by four subspaces four

$$H^\bullet_{\mathrm{cusp}}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda)\subset H^\bullet_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda)\subset H^\bullet_{(2)}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda)\subset H^\bullet(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda). \quad (8.27)$$

We get the representation theoretic Hodge decomposition

$$\bigoplus_{\pi_\infty} W_{\mathrm{cusp}}(\pi_\infty)\otimes H^\bullet_{\mathrm{cusp}}(\mathfrak{g},K_\infty,H_{\pi_\infty}\otimes\mathcal{M}_\lambda)\xrightarrow{\sim} H^\bullet_{\mathrm{cusp}}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda) \qquad (8.28)$$

If we replace the subscript $_{\mathrm{cusp}}$ by $_!$ the corresponding map is still surjective but may be not injective.

We want to point out that our space $H^\bullet_{(2)}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda)$ is not the space denoted by the same symbol in the paper [7]. They define L^2 cohomology as the complex of square integrable forms, i.e. ω and $d\omega$ have to be square integrable. But then a closed form ω which is in L^2 gives the trivial class in their cohomology if we can write $\omega = d\psi$ where ψ must also be square integrable. In our definition we do not have that restriction on ψ.

The semi-simplicity of the inner cohomology

Now we assume again that our representation $\tilde{\mathcal{M}}_\lambda$ is defined over some number field F we consider it as a subfield of $\mathbb{C}$. In other word we have a representation $r: G\times F\to \mathrm{Gl}(\mathcal{M}_\lambda)$. We have defined $H^\bullet_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}})$, this is a finite dimensional F-vector space and Theorem 3.1.1 in Chapter 3 asserts that this is a semi simple module under the Hecke algebra. The following argument shows that this is an easy consequence of our results above.

The module $H_1\subset L^2_{\mathrm{disc}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f)$ can also be decomposed into a finite direct sum of irreducible $G(\mathbb{R})\times\mathcal{H}_{K_f}$ modules

$$H_1=\bigoplus_{\pi_\infty\otimes\pi_f\in\hat{H}_1}(H_{\pi_\infty}\otimes H_{\pi_f})^{m_1(\pi_\infty\times\pi_f)},$$

this module is clearly semi-simple. Of course it is not a $(\mathfrak{g},K_\infty)$-module, but we can restrict to the K_∞-finite vectors and get

$$H^\bullet(\mathfrak{g},K_\infty,H_1^{(K_\infty)}\otimes\mathcal{M}_\lambda\otimes\mathbb{C})=\bigoplus_{\pi_\infty\otimes\pi_f\in\hat{H}_1}(\ \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}),H_{\pi_\infty}\otimes\mathcal{M}_{\mathbb{C}})\otimes H_{\pi_f})^{m_1(\pi_\infty\times\pi_f)}$$

This is a decomposition of the left hand side into irreducible $\mathcal{H}_{K_f}$ modules. Now we have the surjective map

$$H^\bullet(\mathfrak{g}, K_\infty, H_1^{(K_\infty)} \otimes \mathcal{M}_\lambda \otimes \mathbb{C}) \to H^\bullet_{(2)}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{C})$$

hence it follows that $H^\bullet_{(2)}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{C}))$ is a semi simple $\mathcal{H}_{K_f}$ module and hence also $H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$ is a semi simple $\mathcal{H}_{K_f}$ module.

Friendship

We touch upon a question which comes up naturally in this context. Assume we have a non zero isotypical submodule $H^\bullet_!(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda)(\pi_f)$. Then we know that there is a unitary $(\mathfrak{g}, K_\infty)$ module H_{π_∞} with $\pi_\infty \in \mathrm{Coh}(\lambda)$ such that we can embed $H_{\pi_\infty} \times H_{\pi_f}$ into $L^2_{\mathrm{disc}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f)$. The interesting question is:

Given π_f, what are the possible choices for π_∞?.

We can formulate this differently. We recall that

$$W_?(\pi_\infty \otimes \pi_f) = \mathrm{Hom}_{G(\mathbb{R})\times\mathcal{H}_{K_f}}(H_{\pi_\infty} \otimes H_{\pi_f}, L^2_?(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f)$$

where ? = cusp or = (2) resp. disc then we get the surjective map

$$\bigoplus_{\pi_\infty} W_?(\pi_\infty \times \pi_f) \otimes H_{\pi_\infty} \otimes H_{\pi_f} \to H^\bullet_?(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)(\pi_f) \qquad (8.29)$$

which is an isomorphism if ? = cusp. The friends of π_f are those π_∞ where $W_?(\pi_\infty \times \pi_f) \neq 0$.

This question may become very delicate and we will not discuss it profoundly. (As J. Arthur puts it : π_f looks around and asks "Who is my friend?") In principle we give a complete answer to these questions in the low dimensional cases discussed in section (4.1.5), i.e $G/\mathbb{R} = \mathrm{Gl}_2/\mathbb{Q}$ and $G/\mathbb{R} = R_{\mathbb{C}(\mathbb{R}}(\mathrm{Gl}_2/\mathbb{C})$.

In section (8.1.5) we mentioned the Vogan-Zuckerman classification of unitary representations with non trivial cohomology. More precisely Vogan and Zuckerman construct a family of $(\mathfrak{g}, K_\infty)$ irreducible modules $A_\mathfrak{q}(\lambda)$ for which they show $H^\bullet(\mathfrak{g}, K_\infty, A_\mathfrak{q}(\lambda) \otimes \mathcal{M}_\lambda) \neq 0$, and they compute $H^\bullet(\mathfrak{g}, K_\infty, A_\mathfrak{q} \otimes \mathcal{M}_\lambda)$ explicitly. Moreover they show that any irreducible unitary module V with $H^\bullet(\mathfrak{g}, K_\infty, V \otimes \mathcal{M}_\lambda) \neq 0$, is isomorphic to an $A_\mathfrak{q}(\lambda)$.

We give some very cursory description of their construction. Let $T_1^c/\mathbb{R}$ be a maximal torus in $K_\infty^{(1)}/\mathbb{R}$. Then it is clear that the centraliser $T/\mathbb{R}$ is a maximal torus in $G/\mathbb{R}$. In section 9.4.3 we introduce the one dimensional torus $\mathbb{S}^1/\mathbb{R}$ and we choose an isomorphism $i_0 : \mathbb{S}^1 \times_\mathbb{R} \mathbb{C} \xrightarrow{\sim} \mathbb{G}_m/\mathbb{C}$. We consider cocharacters $\chi : \mathbb{S}^1/\mathbb{R} \to T_1^c/\mathbb{R}$. Such a cocharacter defines a centraliser $Z_\chi \subset G/\mathbb{R}$ and a parabolic $P_\chi/\mathbb{C} \subset G \times_\mathbb{R} \mathbb{C}$, this parabolic subgroup also depends on i_0. (See section 9.4.3) The Lie-algebra $\mathfrak{q} = \mathrm{Lie}(P_\chi/\mathbb{C})$ is the $\mathfrak{q}$ in $A_\mathfrak{q}(\lambda)$. We will denote these modules also by $A_\chi(\lambda)$, i.e. $A_\chi(\lambda) := A_\mathfrak{q}(\lambda)$ if χ and $\mathfrak{q}$ are related as above.

The second datum is a highest weight $\lambda \in X^*(T \times_\mathbb{R} \mathbb{C})$, it has to satisfy two conditions

a) The weight λ is c-autodual (see (8.1)) i.e. $c(\lambda) = -w_0\lambda)$.

b) The highest weight λ is trivial on the semi simple part $Z_\chi^{(1)}$ or what amounts to the same λ extends to a character $\lambda : P_\chi \to \mathbb{G}_m/\mathbb{C}$.

We have two extreme cases. In the first case the cocharacter χ is trivial, then the centraliser is the entire group $G \times_{\mathbb{R}} \mathbb{C}/\mathbb{C}$ and then the condition b) implies that $\lambda = 0$. This implies that $\mathcal{M}_\lambda$ is one dimensional, $\mathfrak{q} = \mathfrak{g}$ and $A_{\mathfrak{q}}(0)$ is this trivial one dimensional $(\mathfrak{g}, K_\infty)$-module.

But on the other hand for $\lambda = 0$ we do not have any constraint on the χ, i.e. we get a non trivial irreducible module $A_\chi(0)$ for any χ. But it is not known in general which of these modules are unitary.

In the second case χ is regular, this means that $Z_\chi = T$ and $P_\chi = B_\chi$ is a Borel subgroup, we have no constraint on λ. In this case the $A_{\mathfrak{q}}(\lambda) = A_{\mathfrak{b}}(\lambda)$ are the so called *tempered representations* (see [11], IV, 3.6).

The regular cocharacters $\chi \in X_*(T_1^c) \otimes \mathbb{R}$ lie in the complement of finitely many hyperplanes, hence the set $(X_*(T_1^{(c)}) \otimes \mathbb{R}))^{(0)}$ of regular characters is a finite union of connected components. It is clear from the description that the module $A_{\mathfrak{q}}(\lambda)$ does not change, if χ moves inside a connected component. Finally we have the action of the real Weyl group $W(\mathbb{R}) = N(T)(\mathbb{R})/T(\mathbb{R})$ on $X_*(T_1^c) \otimes \mathbb{R}$ and again it is clear that the isomorphism type does not change if we conjugate χ by an element in $W(\mathbb{R})$. Hence we can say that the tempered $A_{\mathfrak{p}}(\lambda)$ are parametrised by $\pi_0((X_*(T_1^{(c)}) \otimes \mathbb{R}))^{(0)})/W(\mathbb{R})$.

We have a brief look at the case that $G^{(1)}/\mathbb{R}$ has a compact maximal torus T_1^c,i.e. $T = T$. This case played an important role in the section on the Gauss-Bonnet formula. Then

$$T_1^c \times_{\mathbb{R}} \mathbb{C} \subset K_\infty^{(1)} \times_{\mathbb{R}} \mathbb{C} \subset G^{(1)} \times_{\mathbb{R}} \mathbb{C},$$

hence T_1^c is a maximal torus in both reductive groups. We have the two (absolute) Weyl groups $W_{K_\infty} = W(\mathbb{R}) = N_{K_\infty}(T)(\mathbb{C})/T(\mathbb{C})$ and $W_G = N_G(T)(\mathbb{C})/T(\mathbb{C})$ The big Weyl group W_G acts simply transitively on the set of connected components of $(X_*(T_1^{(c)}) \otimes \mathbb{R}))^{(0)}$. Hence we have $W_G = \pi_0(X_*(T_1^{(c)}) \otimes \mathbb{R}))^{(0)})$ once we choose a base point $[\chi_0] \in \pi_0(X_*(T_1^{(c)}) \otimes \mathbb{R}))^{(0)})$ and therefore we get a family

$$\{A_{w\chi_0}(\lambda)\}_{w \in W_{K_\infty} \backslash W_G}, \tag{8.30}$$

and the results of Vogan and Zuckerman assert:

These repesentations are unitary, they are pairwise non isomorphic, and they are the Harish-Chandra modules attached to the discrete series representations of $G(\mathbb{R})$.

The cohomology groups are given by

$$H^q(\mathfrak{g}, K_\infty, A_{w\chi_0}(\lambda) \otimes \mathcal{M}_\lambda) = \begin{cases} \mathbb{C} & \textit{if } q = \frac{d}{2} \\ 0 & \textit{else} \end{cases} \tag{8.31}$$

Now it is clear that for a regular highest weight λ regular the condition b) forces the cocharacter χ to be regular.

We come back to the question raised above. Assume λ is regular and we have an isotypical component $H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)(\pi_f)$. Then the possible "friends" are the $A_\chi(\lambda)$ with χ regular. Hence we get

$$H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)(\pi_f) = \bigoplus_{w\in W_K\backslash W_G} H^{\frac{d}{2}}(\mathfrak{g}, K_\infty, A_{w\chi_0}(\lambda)\otimes \mathcal{M}_\lambda)^{m(w\chi_0\times\pi_f)} = \bigoplus_{w\in W_K\backslash W_G} \mathbb{C}^{m(w\chi_0\times\pi_f)} \tag{8.32}$$

where $m(w\chi_0\times\pi_f)$ is the multiplicity of $A_{w\chi_0}(\lambda)\times\pi_f$. in $L^2_{\mathrm{disc}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f)$. (In Arthurs' words : If) λ is regular then the only friends of a $\pi_f \in \mathrm{Coh}(H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$ are the $w\chi_0$.)

If we refrain from decomposing into isotypical subspaces then we get a simpler formula

$$H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) = \bigoplus_{w\in W_K\backslash W_G} \mathbb{C}^{m(w\chi_0)} \tag{8.33}$$

where of course $m(w\chi_0)$ is the multiplicity of $A_{w\chi_0}(\lambda)$ in $L^2_{\mathrm{disc}}(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f)$. Actually we know that $A_{w\chi_0}(\lambda)$ must even lie in the cuspidal spectrum (see [110]). In principle we already used this fact because we tacitly used the theorem of Borel (see Thm 8.1.1, b).

8.1.7 Cuspidal vs. inner

Now we remember that in the previous sections we made the convention (See end of (8.1.1)) that our coefficient systems $\mathcal{M}_\lambda$ are $\mathbb{C}$ vector spaces. We now revoke this convention and recall that the coefficient systems $\mathcal{M}_\lambda$ should be replaced by $\mathcal{M}_\lambda\otimes_F\mathbb{C}$, where F is some number field over which $\mathcal{M}_\lambda$ is defined. Then in the above list (8.27) of four subspaces in the cohomology the second and the fourth subspace have a natural structure of F-vector spaces and they have a combinatorial definition, whereas the first and third subspace need some input from analysis in their definition. In other words if we replace $\mathcal{M}_\lambda$ in (8.27) by $\mathcal{M}_\lambda\otimes_F\mathbb{C}$ then (8.27) can be written as

$$H^\bullet_{\mathrm{cusp}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda\otimes_F\mathbb{C}) \subset H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)\otimes_F\mathbb{C} \subset H^\bullet_{(2)}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda\otimes_F\mathbb{C}) \subset H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)\otimes_F\mathbb{C} \tag{8.34}$$

It is a very important question to understand the discrepancy between the first two steps. If λ is regular then it follow from the results of [73] that in fact

$$H^\bullet_{\mathrm{cusp}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda\otimes_F\mathbb{C}) = H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)\otimes_F\mathbb{C} \tag{8.35}$$

but without the assumption λ regular this is not true for interesting reasons.

Of course we should also take the action of the Hecke algebra into account. If π_f is the isomorphism type of an absolutely irreducible Hecke module which is defined over F. Then we can consider

$$H^\bullet_{\mathrm{cusp}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda\otimes_F\mathbb{C})(\pi_f) \subset H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)\otimes_F\mathbb{C}(\pi_f) \tag{8.36}$$

and compare these two modules. We will say that π_f is *strongly inner* if we have equality.

We come back to this issue in Chapter 9 after stating proposition 9.2.1.

A formula for the Poincaré duality pairing

We assume that $-w_0(\lambda) = c(\lambda)$. We have the positive definite hermitian scalar product on $\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), H^{(K_\infty)}_{\mathrm{Coh}(\lambda)} \otimes \mathcal{M}_\lambda)$ (See(8.7)). On the other hand we have the Poincaré duality pairing

$$H^i_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)(\omega_f) \times H^{d-i}_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda^\vee})(\omega_{1,f}) \to \mathbb{C} \tag{8.37}$$

where $\omega_f \cdot \omega_{1,f} = 1$. To relate these two products we recall the Hodge $*$-operator. (See for instance Vol. I. 4.11) This operator yields an isomorphism

$$\begin{aligned} * : \ &\mathrm{Hom}_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}), \mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f) \otimes \mathcal{M}_\lambda) \xrightarrow{\sim} \\ &\mathrm{Hom}_{K_\infty}(\Lambda^{d-p}(\mathfrak{g}/\mathfrak{k}), \mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f) \otimes \mathcal{M}_{c\lambda}) \end{aligned} \tag{8.38}$$

We can use the $*$ operator to define the adjoint $\delta = (-1)^{d(p+1)+1} * d*$ and hence the Laplacian Δ (See (8.8). Especially the $*$ operator yields an identification between the $\mathcal{C}_\infty$-functions and the $\mathcal{C}_\infty$ differential forms in top degree.

We consider two differential forms

$$\omega_1, \omega_2 \in \ \mathrm{Hom}_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}), \mathcal{C}^{(2)}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f) \otimes \mathcal{M}_\lambda)$$

which are square integrable, then we defined the scalar product (See(8.7)) $< \omega_1, \omega_2 >$ of these two forms. This scalar product is an integral

$$< \omega_1, \omega_2 >= \int_{\mathcal{S}^G_{K_f}} \omega_1 \wedge *\omega_2.$$

If we have two closed forms $\omega_1 \in \ \mathrm{Hom}_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}), \mathcal{C}^{(2)}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f) \otimes \mathcal{M}_\lambda), \omega_2 \in \ \mathrm{Hom}_{K_\infty}(\Lambda^{d-p}(\mathfrak{g}/\mathfrak{k}), \mathcal{C}^{(2)}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f) \otimes \mathcal{M}_{\lambda^\vee})$ and if one of these forms has compact support -say ω_2-then they define cohomology classes $[\omega_1] \in H^p(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda), [\omega_2] \in H^{d-p}_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda^\vee})$ and the cup product $[\omega_1] \cup [\omega_2]$ is defined and given by an integral (See proposition 8.1.2) over a form in top degree. Now we check easily - and this is the way how the $*$ operator is designed that for $\omega_1, \omega_2 \in \ \mathrm{Hom}_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}), \mathcal{C}^{(2)}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f) \otimes \mathcal{M}_\lambda)$ the integrand

$$[\omega_1 \cup [\omega_2] = \int_{\mathcal{S}^G_{K_f}} \omega_1 \wedge *\omega_2$$

Now we look at the same question, but we only assume that $[\omega_1], [\omega_2] \in H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$. Then the cup product $[\omega_1 \cup [\omega_2]$ is defined and we want to express this as an integral. We have

Proposition 8.1.4. *If* $\omega_1, \omega_2 \in \ Hom_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}), H^{(K_\infty)}_{\mathrm{Coh}_2(\lambda)} \otimes \mathcal{M}_\lambda)$ *and if both classes* $[\omega_1], [*\omega_2]$ *are inner classes, then*

$$[\omega_1] \cup [*\omega_2] =< \omega_1, \omega_2 >$$

Proof. Here we give only a sketch of the proof, for some details we refer to section 8.1.10. Of course we have to recall that the right hand side is defined since we have proposition 6.3.8, here we need that both classes are inner classes. We write $\omega_1 = \tilde{\omega}_1 + d\psi_0$ where $\tilde{\omega}_1$ has compact support, this means that the restriction of $\omega_1 - d\psi_0$ to some punctured tubular neighbourhood is zero. Then the value of this cup product is equal to cupl2

$$[\omega_1] \cup [*\omega_2] = \int_{\mathcal{S}^G_{K_f}} \tilde{\omega}_1 \wedge *\omega_2 = \int_{\mathcal{S}^G_{K_f}} \omega_1 \wedge *\omega_2 - \int_{\mathcal{S}^G_{K_f}} d\psi_0 \wedge *\omega_2 \tag{8.39}$$

The rightmost term does not depend on the choice of ψ_0, we want to show that this term is zero. We can assume that ψ_0 is square integrable (The proof of this fact is a little bit tedious: See proposition 8.1.8). Then we construct a $\mathcal{C}_\infty$ function h with values in $[0,1]$, which has value 1 in a still smaller punctured tubular neighbourhood $\overset{\bullet}{\mathcal{N}}(\mathbf{c}')$ and has a bounded derivative. We replace ψ_0 by $h\psi_0$. Then the rightmost term in (8.39) becomes

$$\int_{\mathcal{S}^G_{K_f}} dh\psi_0 \wedge *\omega_2 = \int_{\mathcal{S}^G_{K_f}} dh \wedge \psi_0 \wedge *\omega_2 + \int_{\mathcal{S}^G_{K_f}} hd\psi_0 \wedge *\omega_2 \tag{8.40}$$

We can arrange our function h_t in such a way, that its support is contained in a very small tubular neighbourhood $\mathcal{N}(\mathbf{c}'')$. Then the absolute value the second term on the right hand side can be estimated by

$$|\int_{\mathcal{S}^G_{K_f}} hd\psi_0 \wedge *\omega_2| \leq |\int_{\overset{\bullet}{\mathcal{N}}(\mathbf{c})} \omega_1 \wedge \omega_2| \tag{8.41}$$

and this becomes arbitrarily small. But a similar argument applies to the first term, it can be estimated by (Cauchy-Schwarz inequality)

$$|\int_{\mathcal{S}^G_{K_f}} \psi_0 \wedge dh \wedge *\omega_2| \leq C \int_{\overset{\bullet}{\mathcal{N}}(\mathbf{c}'')} \psi_0$$

and since ψ_0 is square integrable this also becomes arbitrarily small, therefore it must be zero. □

. If the quotient $\mathcal{S}^G_{K_f}$ is compact, then it is of course a consequence of Hodge theory. But if this is not the case we really need that both classes are inner. In fact we have the standard example which shows that this assumption is needed. If take $\omega_1 = \omega_2$ to be the form in degree zero given by the constant function 1. Then the left hand side is non zero but the class $*1$ is the volume form which yields the trivial class if $\mathcal{S}^G_{K_f}$ is not compact, and therefore the right hand side is not zero.

The proposition has the following nice corollary

Corollary 8.1.1. *If* $\omega \in \ Hom_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}), H^{(K_\infty)}_{\mathrm{Coh}_2(\lambda)} \otimes \mathcal{M}_\lambda)$ *is non zero and if the restrictions of* $[\omega]$ *and* $[*\omega]$ *to the boundary are zero then* $[\omega] \neq 0$.

This last Corollary could be useful if we want to understand the kernel of the map

$$\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), H^{(K_\infty)}_{\mathrm{Coh}_2(\lambda)} \otimes \mathcal{M}_\lambda) \to H^\bullet(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda), \tag{8.42}$$

but a closer look tells us that this may not be so easy, because the restriction the cohomology to the boundary is not so easy to understand.

At this point it seems that a commentary is in order. If we have two differential forms $\omega_1, \omega_2 \in \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), H^{(K_\infty)}_{\mathrm{Coh}_2(\lambda)} \otimes \mathcal{M}_\lambda)$ sitting complementary degrees $p, d-p$ then we can define their scalar product

$$< \omega_1, \omega_2 >= \int_{\mathcal{S}^G_{K_f}} \omega_1 \wedge \omega_2 \tag{8.43}$$

the integral is convergent because the forms are square integrable. These forms represent cohomology classes $[\omega_1], [\omega_2]$, if $\mathcal{S}^G_{K_f}$ is not compact, it does not make sense to take their cup product. Hence there is no way to interpret the above scalar product as a cup product. But if these two classes are in $H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$ then the cup product is defined see (6.3.8). If we want to express the cup product as an integral we need that one of the forms has compact support. But our forms ω_i never have compact support, if they are non zero. Our above arguments show that nevertheless

$$[\omega_1] \cup [\omega_2] =< \omega_1, \omega_2 >$$

Will not appear in the final version of this book At this very moment the above proposition is controversial, because it contradicts a result in the paper by Lee and Sczcarba [72] In this paper the authors claim that for $G = \mathrm{Sl}_4/\mathbb{Z}$ and $K_f = \mathrm{Sl}_4(\hat{\mathbb{Z}})$ (the standard maximal compact in $G(\mathbb{A}_f)$) the cohomology

$$H^4(\mathcal{S}^G_{K_f}, \mathbb{Q}) = H^5(\mathcal{S}^G_{K_f}, \mathbb{Q}) = 0$$

of course here $\mathcal{S}^G_{K_f} = \mathrm{Sl}_4(\mathbb{Z})\backslash X$ and $X = \mathrm{Sl}_4(\mathbb{R})/\mathrm{SO}(4)$. Now we know from various sources that the ring of invariant differential forms $\Omega^\bullet(X)^{G(\mathbb{R})}$ has non trivial elements ω_4, ω_5 in degree 4 and 5. If one of this classes restricts non trivially to the boundary, then we get non trivial cohomology in this degree. If both classes restricted to the boundary are zero (what I think happens) then the corollary says that we have non zero cohomology in both degrees.

My argument may contains a fundamental error, . but on the other hand the reasoning is crystal clear to me. For the special case that $G = R_{F/\mathbb{Q}}(\mathrm{Gl}_2)$ (for any number field $F/\mathbb{Q}$) I have proved a version of this result in [35].

Continue Conseq

8.1.8 Consequences

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Vanishing theorems

If V is unitary and irreducible, then we have that $\bar{V} \xrightarrow{\sim} V^\vee$ and this implies for the central character

$$\overline{\chi_V(z)} = \chi_{V^\vee}(z) \text{ for all } z \in \mathfrak{Z}(\mathfrak{g}).$$

Combining this with Wigner's lemma we can conclude

If V is an irreducible unitary $(\mathfrak{g}, K_\infty)$-module, $\mathcal{M}_\lambda$ is an irreducible rational representation, and if

$$H^\bullet(\mathfrak{g}, K_\infty, V \otimes \mathcal{M}_\lambda) \neq 0$$

then $\chi_{\mathcal{M}_\lambda^\vee}(z) = \chi_{\mathcal{M}_\lambda}({}^t z) = \chi_{\bar{\mathcal{M}}_\lambda}(z)$

In other words: For an unitary irreducible $(\mathfrak{g}, K_\infty)$-module V the cohomology with coefficients in an irreducible rational representation $\mathcal{M}$ vanishes, unless we have $\mathcal{M}_\lambda^\vee \xrightarrow{\sim} \bar{\mathcal{M}}_\lambda$, or in terms of highest weights unless $-w_0(\lambda) = c(\lambda)$. (See 3.1.1)

If we combine this with the considerations following Wigner's lemma we get

Corollary 8.1.2. *If $\mathcal{M}$ is an absolutely irreducible rational representation and if $\mathcal{M}_\lambda^\vee$ is not isomorphic to $\bar{\mathcal{M}}_\lambda$ then*

$$H^\bullet_{(2)}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) = 0.$$

Hence also

$$H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) = 0.$$

We will discuss examples for this in section 8.1.8

The group $G/\mathbb{Q} = \mathbf{Sl}_2/\mathbb{Q}$

Let us consider the group $G/\mathbb{Q} = \mathrm{Sl}_2/\mathbb{Q}$. We have tautological representation $\mathrm{Sl}_2 \hookrightarrow \mathrm{Gl}(\mathbb{Q}^2) = \mathrm{Gl}(V)$ and we get all irreducible representations of we take the symmetric powers $\mathcal{M}_n = \mathrm{Sym}^n(V)$ of V. (See 2, these are the $\mathcal{M}_n[m]$ restricted to Sl_2, then the m drops out.)

In this case the Vogan-Zuckerman list is very short. It is discussed in [Slzwei] for the groups $\mathrm{Sl}_2(\mathbb{R})$ and $\mathrm{Sl}_2(\mathbb{C})$, where both groups are considered as real Lie-groups.

In the case $\mathrm{Sl}_2(\mathbb{R})$ we have the trivial module $\mathbb{C}$ and for any integer $k \geq 2$ we have two irreducible unitarizable $(\mathfrak{g}, K_\infty)$-modules $\mathcal{D}_k^\pm$ (the discrete series representations) (See [Slzwei], 4.1.5). These are the only $(\mathfrak{g}, K_\infty)$-modules which have non trivial cohomology with coefficients in a rational representation. If we now pick one of our rational representation $\mathcal{M}_n$, then the non vanishing cohomology groups are

$$H^q(\mathfrak{g}, K_\infty, \mathcal{M}_n \otimes \mathbb{C}) = \mathbb{C} \text{ for } l = 0, q = 0, 2$$
$$H^q(\mathfrak{g}, K_\infty, \mathcal{D}_k^\pm \otimes \mathcal{M}_n \otimes \mathbb{C}) = \mathbb{C} \text{ for } l = k-2, q = 1$$

The trivial $(\mathfrak{g}, K_\infty)$-module $\mathbb{C}$ occurs with multiplicity one in $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f)$ hence we get for the trivial coefficient system a contribution

$$H^\bullet(\mathfrak{g}, K_\infty, \mathbb{C}\otimes\mathcal{M}_n\otimes\mathbb{C}) = H^0(\mathfrak{g}, K_\infty, \mathbb{C})\oplus H^2(\mathfrak{g}, K_\infty, \mathbb{C}) = \mathbb{C}\oplus\mathbb{C} \to H^\bullet_{(2)}(\mathcal{S}^G_{K_f}, \mathbb{C}).$$

This map is injective in degree 0 and zero in degree 2.

For the modules $\mathcal{D}^\pm_k$ we have to determine the multiplicities $m^\pm(k)$ of these modules in the discrete spectrum of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f)$. A simple argument using complex conjugation tells us $m^+(k) = m^-(k)$. Now we have the fundamental observation made by Gelfand and Graev, which links representation theory to automorphic forms:

We have an isomorphism

$$Hom_{\mathfrak{g},K_\infty}(\mathcal{D}^+_k, L^2_{\mathrm{disc}}(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f) \xrightarrow{\sim} S_k(G(\mathbb{Q})\backslash \mathbb{H}\times G(\mathbb{A}_f)/K_f) =$$

space of holomorphic cusp forms of weight k and level K_f

This is also explained in [Slzwei] on the pages following 23. We explain how we get starting from a holomorphic cusp form f of weight k an inclusion

$$\Phi_f : \mathcal{D}^+_k \hookrightarrow L^2_{\mathrm{disc}}(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f)$$

and that this map $f \mapsto \Phi_f$ establishes the above isomorphim. This gives us the famous Eichler-Shimura isomorphism

$$S_k(G(\mathbb{Q})\backslash \mathbb{H}\times G(\mathbb{A}_f)/K_f) \oplus \overline{S_k(G(\mathbb{Q})\backslash \mathbb{H}\times G(\mathbb{A}_f)/K_f)} \xrightarrow{\sim} H^1_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{k-2}).$$

The group $G/\mathbb{Q} = R_{F/\mathbb{Q}}(\mathbf{Sl}_2/F)$.

For any finite extension $F/\mathbb{Q}$ we may consider the base restriction $G/\mathbb{Q} = R_{F/\mathbb{Q}}(Sl_2/F)$. (See Chap-II. 1.1.1). Here we want to consider the special case the $F/\mathbb{Q}$ is imaginary quadratic. In this case we have $G\otimes\mathbb{C} = \mathrm{Sl}_2\times\mathrm{Sl}_2/\mathbb{C}$ the factors correspond to the two embeddings of F into $\mathbb{C}$. The rational irreducible representations are tensor products of irreducible representations of the two factors $\mathcal{M}_\lambda = \mathcal{M}_{k_1}\otimes\mathcal{M}_{k_2}$ where again $\mathcal{M}_k = \mathrm{Sym}^k(\mathbb{C}^2)$. These representations are defined over F.

In this case we discuss the Vogan-Zuckerman list in [Slzwei], here we want to discuss a particular aspect. We observe that

$$\mathcal{M}^\vee_\lambda \xrightarrow{\sim} \mathcal{M}_{k_1}\otimes\mathcal{M}_{k_2}, \bar{\mathcal{M}}_\lambda = \mathcal{M}_{k_2}\otimes\mathcal{M}_{k_1}$$

and hence our corollary above yields for any choice of K_f

$$H^\bullet_!(\mathcal{S}^G_{K_f}, \mathcal{M}) = 0 \text{ if } k_1 \neq k_2.$$

In Chapter II we discuss the special examples in low dimensions. We take $F = \mathbb{Q}[i]$ and $\Gamma = \mathrm{Sl}_2[\mathbb{Z}[i]]$ this amounts to taking the standard maximal compact subgroup $K_f = \mathrm{Sl}_2[\hat{\mathcal{O}}_F]$. If now for instance $k_1 > 0$ and $k_2 = 0$, then we get $H^\bullet_!(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda) = 0$. Hence we have by definition $H^\bullet_!(\mathcal{S}^G_{K_f}, \mathcal{M}) = H^\bullet_{\mathrm{Eis}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})$ and we have complete control over the Eisenstein- cohomology in this case. Hence we know the cohomology in this case if we apply the analytic methods.

On the other hand in Chapter 2 we have written an explicit complex of finite dimensional vector spaces, which computes the cohomology. It is not clear to me how we can read off this complex the structure of the cohomology groups.

We get another example where this phenomenon happens, if we consider the group $\mathrm{Sl}_n/\mathbb{Q}$ if $n > 2$. In Chap.2 1.2 we describe the simple roots $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$, accordingly we have the fundamental highest weights $\omega_1, \ldots, \omega_{n-1}$. The element w_0 (See 8.1.1) has the effect of reversing the order of the weights. Hence we see that for $\lambda = \sum n_i \omega_i$ we have

$$H^\bullet_!(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda) = 0$$

unless we have $-w_0(\lambda) = \lambda$ and this means $n_i = n_{n-1-i}$.

The algebraic K-theory of number fields

I briefly recall the definition of the K-groups of an algebraic number field $F/\mathbb{Q}$. We consider the group $\mathrm{Gl}_n(\mathcal{O}_F)$, it has a classifying space BG_n. We can pass to the limit $\lim\limits_{n\to\infty} \mathrm{Gl}_n(\mathcal{O}_F) = \mathrm{Gl}(\mathcal{O}_F) = G$ and let BG its classifying space. Quillen invented a procedure to modify this space to another space BG^+, whose fundamental group is now abelian, but which has the same homology and cohomology as BG. Then he defines the algebraic K-groups as

$$K_i(\mathcal{O}_F) = \pi_i(\mathrm{BG}^+).$$

The space is an H-space, this means that we have a multiplication $m : \mathrm{BG}^+ \times \mathrm{BG}^+ \to \mathrm{BG}^+$ which has a two sided identity element. Then we get a homomorphism $m^\bullet : H^\bullet(\mathrm{BG}^+, \mathbb{Z}) \to H^\bullet(\mathrm{BG}^+ \times \mathrm{BG}^+, \mathbb{Z})$ and if we tensorize by $\mathbb{Q}$ and apply the Künneth-formula then we get the structure of a Hopf algebra on the Cohomology

$$m^\bullet : H^\bullet(\mathrm{BG}^+, \mathbb{Q}) \to H^\bullet(\mathrm{BG}^+, \mathbb{Q}) \otimes H^\bullet(\mathrm{BG}^+, \mathbb{Q})$$

Then a theorem of Milnor asserts that the rational homotopy groups

$$\pi_i(\mathrm{BG}^+) \otimes \mathbb{Q} = \mathrm{prim}(H^i(\mathrm{BG}, \mathbb{Q}),$$

where prim are the primitive elements, i.e. those elements $x \in H^i(\mathrm{BG}, \mathbb{Q})$ for which

I sketch a second application. We discuss the group $G = R_{F/\mathbb{Q}}(\mathrm{Gl}_n/F)$, where $F/\mathbb{Q}$ is an algebraic number field. the coefficient system $\tilde{\mathcal{M}}_\lambda = \mathbb{C}$ is trivial. In this case Borel, Garland and Hsiang have shown hat in low degrees $q \leq n/4$

$$H^q(\mathcal{S}^G_{K_f}, \mathbb{C}) = H^q_{(2)}\mathcal{S}^G_{K_f}, \mathbb{C}).$$

On the other hand it follows from the Vogan-Zuckerman classification ([108], that the only irreducible unitary $(\mathfrak{g}, K_\infty)$ modules V, for which $H^q(\mathfrak{g}, K_\infty, V) \neq 0$ and $q \leq n/4$ are one dimensional.

Hence we see that in low degrees

$$H^q(\mathfrak{g}, K_\infty, \mathbb{C}) \to H^q(\mathcal{S}^G_{K_f}, \mathbb{C})$$

is an isomorphism (Injectivity requires some additional reasoning.)

On the other hand we have $H^q(\mathfrak{g}, K_\infty, \mathbb{C}) = \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathbb{C})$ and obviously this last complex is isomorphic to the complex $\Omega^\bullet(X)^{G(\mathbb{R})}$ of $G(\mathbb{R})$-invariant forms on the symmetric space $G(\mathbb{R})/K_\infty$. Our field has different embeddings $\tau : F \hookrightarrow \mathbb{C}$, the real embeddings factor through $\mathbb{R}$, they form the set S_∞^{real} and the pairs of may conjugate embeddings into $\mathbb{C}$ form the set S_∞^{comp} Then

$$X = \prod_{v \in S_\infty^{\mathrm{real}}} \mathrm{Sl}_n(\mathbb{R})/SO(n) \times \prod_{S_\infty^{\mathrm{comp}}} \mathrm{Sl}_n(\mathbb{C})/SU(n).$$

Now the complex $\Omega^\bullet(X)^{G(\mathbb{R})}$ of invariant differential forms (all differentials are zero) does not change if we replace the group

$$G(\mathbb{R}) = \prod_{v \in S_\infty^{\mathrm{real}}} \mathrm{Sl}_n(\mathbb{R}) \times \prod_{S_\infty^{\mathrm{comp}}} \mathrm{Sl}_n(\mathbb{C})$$

by its compact form $G_c(\mathbb{R})$ and then we get the complex of invariant forms on the compact twin of our symmetric space

$$X_c = \prod_{v \in S_\infty^{\mathrm{real}}} SU_n(\mathbb{R})/SO(n) \times \prod_{S_\infty^{\mathrm{comp}}} (SU(n) \times SU(n))/SU(n),$$

but then

$$\Omega(X_c)^{G_c(\mathbb{R})} = H^\bullet(X_c, \mathbb{C}).$$

The cohomology of the topological spaces like the one on the right hand side has been computed by Borel in the early days of his career. **Referenz**

If we let n tend to infinity, we can consider the limit of these cohomology groups, then the limit becomes a Hopf algebra and we can consider the primitive elements

At this point we encounter an interesting problem. We have the three subspaces (See end of 3.2)

$$H^\bullet_{\mathrm{cusp}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{C}) \subset H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) \otimes \mathbb{C} \subset H^\bullet_{(2)}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{C}) \subset H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) \otimes \mathbb{C},$$

note the positions of the tensor symbol $\otimes$. The first and the third space are only defined after we tensorize the coefficient system by $\mathbb{C}$, whereas the second and the fourth cohomology groups by definition F vector spaces tensorized by $\mathbb{C}$.

Now the question is whether the first and the third space also have a natural F -vector space structure. Of course we get a positive answer, if the Manin-Drinfeld principle holds. All the vector spaces are of course modules under the Hecke algebra and we and we can look at their spectra

$$\begin{array}{cc} \Sigma(H^\bullet_{\mathrm{cusp}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{C})) = \Sigma_{\mathrm{cusp}} & \Sigma(H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{C})) = \Sigma_! \\ \Sigma(H^\bullet_{(2)}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{C})) = \Sigma_{(2)} & \Sigma(H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{C})) = \Sigma \end{array}.$$

If now for instance $\Sigma_{\mathrm{cusp}} \cap (\Sigma_! \setminus \Sigma_{\mathrm{cusp}}) = \emptyset$ then we can define $H^\bullet_{\mathrm{cusp}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) \subset H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$ as the subspace which is the sum of the isotypical components in Σ_{cusp}.

If this is the case we say that the cuspidal cohomology is *intrinsically definable* and we get a canonical decomposition

$$H^\bullet_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda) = H^\bullet_{\text{cusp}}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda)\oplus H^\bullet_{!,\text{noncusp}}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda).$$

The classical Manin-Drinfeld principle refers to the two spectra $\Sigma_! \subset \Sigma$, if it is true in this case we get a decomposition

$$H^\bullet(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda) = H^\bullet_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda)\oplus H^\bullet_{\text{Eis}}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda)$$

the canonical complement is called the Eisenstein cohomology. (See Chap. II 2.2.3 and Chap III 5.)

8.1.9 Growth of cohomology classes

The fundamental exact sequence yields a short sequence fil0

$$0 \to H^\bullet_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}) \to H^\bullet(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}) \to H^\bullet(\overset{\bullet}{\mathcal{N}}(\mathcal{S}^G_{K_f}),\tilde{\mathcal{M}}) \tag{8.44}$$

and we gained some understanding of $H^\bullet_!(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}})$ using analytic methods. We have seen that classes in $[\omega]\in H^\bullet_{(2)}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}})$ can be represented by harmonic forms ω. Of course the condition that ω is square integrable implies some restriction on the growth of ω. It is our goal in this section to find criteria which imply that a closed form ω or a class $[\omega]$ is square integrable.

To attack this kind of question we study the "asymptotic behavior" of the cohomology at infinity, this means that we have to study $H^q(\overset{\bullet}{\mathcal{N}}(\mathcal{S}^G_{K_f}),\tilde{\mathcal{M}})$. We apply reduction theory (See section 1.2.8) and start from the covering (See 1.119)

$$\overset{\bullet}{\mathcal{N}}(\mathcal{S}^G_{K_f}) = \bigcup_{P:P\text{proper}} \Gamma_P\backslash X^P(c_{\pi'},r(\pi')) \tag{8.45}$$

Of course we know: The form ω is square integrable if and only if its restriction to the open sets in this covering are square integrable.

We start by describing the cohomology of the open sets in the covering, i.e. we consider the cohomology $H^\bullet(X^P(c_{\pi'},r(\pi')),\tilde{\mathcal{M}})$ we recall that we have the spectral sequence (2.48)

$$H^p(\Gamma_M\backslash X^M(r), H^q((\widetilde{\Gamma_{U_P}\backslash U_P(\mathbb{R})}),\tilde{\mathcal{M}})) \Rightarrow H^{p+q}(\Gamma_P\backslash X^P(C(\underline{\tilde{c}})),\tilde{\mathcal{M}})$$

and the first step is to get more information on the M- module $H^\bullet(\Gamma_{U_P}\backslash U_P(\mathbb{R}),\tilde{\mathcal{M}})$.

The cohomology of unipotent groups

We drop the subscript $_P$, we know that the group scheme $U/\mathbb{Q}$ is a unipotent group scheme, let $A = A(U)$ be its affine algebra (see section 1.1.1).. Then $U/\mathbb{Q}$ has a filtration by subschemes $U_0 = \{e\}\subset U_1\subset U_2\subset \dots U_{m-1}\subset U_m$ such that $U_i/U_{i-1}\xrightarrow{\sim}\mathbb{G}_a$. The subgroup $\Gamma_U\subset U(\mathbb{Q})$ is Zariski dense, more precisely we know the following: If $\Gamma_i = U_i(\mathbb{Q})\cap\Gamma$ then $\Gamma_i/\Gamma_{i-1}\xrightarrow{\sim}\mathbb{Z}\subset U_i/U_{i-1}(\mathbb{R})\xrightarrow{\sim}\mathbb{R}$.

We consider the category of $U/\mathbb{Q}$ modules $\mathrm{Mod_U}$ Then it is clear that the functor $\mathcal{M}\to\mathcal{M}^U$ is equal to $\mathcal{M}\to\mathcal{M}^{\Gamma_U}$. (Our $\mathbb{Z}-$ module $\mathcal{M}$ above is now a $\mathbb{Q}-$ vector space, i.e. we consider coefficient systems with rational coefficients.)

We choose the action of U on A by left translations on A. It follows from Frobenius reciprocity that the $U/\mathbb{Q}$ module A is an injective module in $\mathrm{Mod}_{\mathrm{U}}$. (See ???) This implies that we get an injective resolution of the $U/\mathbb{Q}$ -module $\mathbb{Q}$ by

$$0 \to \mathbb{Q} \to A \to (A/\mathbb{Q}) \otimes A \to \dots = 0 \to \mathbb{Q} \to I^0 \to I^1 \to \tag{8.46}$$

and hence

$$H^q(U, \mathcal{M}) = H^q(\Gamma_U \backslash U(\mathbb{R}), \mathcal{M}) = H^q(0 \to (I^1 \otimes \mathcal{M})^U \to (I^2 \otimes \mathcal{M})^U \to \dots) =$$
$$H^q((I^\bullet \otimes \mathcal{M})^U) \tag{8.47}$$

Since $U/\mathbb{Q}$ is the unipotent radical of the parabolic group $P/\mathbb{Q}$, the parabolic group $P/\mathbb{Q}$ acts via the adjoint action on the modules I^m. This action respects the submodules $(I^m)^U$ and $U/\mathbb{Q}$ acts trivially on $(I^m)^U$, this implies that the modules $(I^m)^U$ are $M/\mathbb{Q} = (P/U)/\mathbb{Q}$ modules. The group $M/\mathbb{Q}$ is reductive and we know that the category of $M/\mathbb{Q}$ modules is semi simple (???). This implies that we can decompose

$$(I^\bullet)^U = \mathbb{H}^\bullet(U, \mathcal{M}) \oplus ACI(I^\bullet)^U \tag{8.48}$$

where the first summand is a complex of $M/\mathbb{Q}$ -modules in which all the differentials are zero and the second is an acyclic complex of $M/\mathbb{Q}$- modules. Hence

$$H^\bullet(U, \mathcal{M}) = H^\bullet(\Gamma_U, \mathcal{M}) \xrightarrow{\sim} \mathbb{H}^\bullet(U, \mathcal{M}) \tag{8.49}$$

We get a "smaller" resolution from the (algebraic) de-Rham complex of differential forms. On the smooth affine scheme $U/\mathbb{Q}$ we have the sheaves of differential forms $\Omega^p_U = \Lambda^p \Omega^1_U$ ([41] ,7.5) and we have the de-Rham complex

$$\Omega(U)^\bullet = 0 \to \mathbb{Q} \to A \to \Omega^1(U) \to \Omega^2(U) \to \dots \tag{8.50}$$

where $\Omega^p(U) = \Omega^p_U(U)$ is the module of global sections and $A = \Omega^0(U)$. These modules of differentials are free A modules, hence they are injective. Since our unipotent group scheme $U/\mathbb{Q}$ is isomorphic to the affine space $\mathbb{A}^d$ (as affine scheme) we see easily that this complex is exact, hence it provides an acyclic resolution. As before we get the cohomology by taking the complex $(\Omega^p(U) \otimes \mathcal{M})^U$ of invariants under the action of $U/\mathbb{Q}$. Since an $U/\mathbb{Q}$- invariant differential form with values in $\mathcal{M}$ is determined by its value at the identity e the complex of invariants under $U/\mathbb{Q}$ becomes

$$0 \to \mathcal{M} \to \mathrm{Hom}(\mathfrak{u}, \mathcal{M}) \to \mathrm{Hom}(\Lambda^2\mathfrak{u}, \mathcal{M}) \to \dots = 0 \to \mathrm{Hom}(\Lambda^\bullet\mathfrak{u}, \mathcal{M}) \tag{8.51}$$

and the cohomology of this complex is the cohomology $H^\bullet(\mathfrak{u}, \mathcal{M})$. We still have the action of $P/\mathbb{Q}$ on $\mathfrak{u}$ by the adjoint action, hence we get an action of P on $\mathrm{Hom}(\Lambda^\bullet\mathfrak{u}, \mathcal{M})$ and we have

Theorem 8.1.2. *(van Est [?])*

$$H^\bullet(\mathfrak{u}, \mathcal{M}) \xrightarrow{\sim} \mathbb{H}^\bullet(\mathfrak{u}, \mathcal{M}) = (\mathrm{Hom}(\Lambda^\bullet\mathfrak{u}, \mathcal{M}))^U,$$

and therefore $H^\bullet(\mathfrak{u}, \mathcal{M})$ is a $M/\mathbb{Q}$ module.

Proof. later □

A theorem of Kostant yields a description of the $M/\mathbb{Q}$ module $(\mathrm{Hom}(\Lambda^\bullet\mathfrak{u},\mathcal{M}))^U$, it gives us the decomposition into highest modules. Let $\lambda \in X^*(T)$ be the highest weight of $\mathcal{M}$, i.e. we have $\mathcal{M} = \mathcal{M}_\lambda$. The set

$$W^P = \{w \in W \mid w^{-1}(\alpha) \in \Delta^+\} \tag{8.52}$$

is the set of Kostant representatives for $W^M\backslash W$. For any $w \in W^P$ we define the element

$$\omega_w = \Lambda_{\alpha\in\Delta_U;w^{-1}\alpha<0}\; u_\alpha^\vee \otimes e_{w\lambda} \tag{8.53}$$

Proposition 8.1.5. *This element ω_w lies in $\mathbb{H}^\bullet(\mathfrak{u},\mathcal{M})$ and it is a highest weight vector for the action of $M/\mathbb{Q}$, the weight is $w\cdot\lambda = w\lambda + w\rho - \rho = w(\lambda+\rho)-\rho$.*

Proof. This is an easy computation. □

This highest weight vector provides an irreducible highest weight module $\mathcal{M}_{w\cdot\lambda}$ for $M/\mathbb{Q}$ and we have the famous theorem of Kostant

Theorem 8.1.3.

$$\mathbb{H}^\bullet(\mathfrak{u},\mathcal{M}) = \bigoplus_{w\in W^P} \mathcal{M}_{w\cdot\lambda}[l(w)]$$

where the summand $\mathcal{M}_{w\cdot\lambda}$ sits in degree $l(w) =?$.

Proof. Rather clear after the preparation. □

We change the notation slightly and put $\mathbb{H}^{l(w)}(\mathfrak{u},\mathcal{M})[w\cdot\lambda] := \mathcal{M}_{w\cdot\lambda}[l(w)]$. Since the differentials in the complex $\mathbb{H}^\bullet(\mathfrak{u},\mathcal{M})$ are zero, the spectral sequence degenerates and we get [cohboundstrat]

$$H^n(\Gamma_P\backslash X^P(C(\underline{\tilde{c}})),\tilde{\mathcal{M}}) \xrightarrow{\sim} \bigoplus_{w\in W^P} H^{n-l(w)}(\Gamma_M\backslash X^M(c_P),\mathbb{H}^{l(w)}\widetilde{(\mathfrak{u},\mathcal{M})}[w\cdot\lambda]), \tag{8.54}$$

this is the decomposition of the cohomology of the boundary stratum into weight spaces.

The cohomology groups $H^\bullet(\Gamma_P\backslash X^P(C(\underline{\tilde{c}})),\tilde{\mathcal{M}}\otimes\mathbb{C})$ can be computed as the cohomology groups of the de-Rham complex

$$H^p(\Gamma_P\backslash X^P(C(\underline{\tilde{c}})),\tilde{\mathcal{M}}\otimes\mathbb{C}) \xrightarrow{\sim} H^p(\Omega^\bullet(\Gamma_P\backslash X^P(C(\underline{\tilde{c}})),\tilde{\mathcal{M}}\otimes\mathbb{C}) \tag{8.55}$$

here $\Omega^\bullet(\Gamma_P\backslash X^P(C(\underline{\tilde{c}})$ is the complex of those $\mathcal{C}_\infty$ differential forms which extend to a $\mathcal{C}_\infty$ form into a small open neighbourhood of $X^P(C(\underline{\tilde{c}})$. We want to use the decomposition of the cohomology into weight spaces to establish a "much smaller" sub-complex

$$\Omega^\bullet_{\log}(\Gamma_P\backslash X^P(C(\underline{\tilde{c}})),\tilde{\mathcal{M}}\otimes\mathbb{C}) \hookrightarrow \Omega^\bullet(\Gamma_P\backslash X^P(C(\underline{\tilde{c}})),\tilde{\mathcal{M}}\otimes\mathbb{C})$$

such that the inclusion induces an isomorphism in cohomology. We recall the map

$$q_{P,M} : \Gamma_P\backslash X^P(c_{\pi'}, r(\underline{c}_{\pi'})) \to \Gamma_M\backslash X^M(r(\underline{c}_P)) \times \prod_{\alpha\in\pi'}(0, c_\alpha] \tag{8.56}$$

it provides a map

$$q^\bullet_{P,M} : \Omega^\bullet(\Gamma_M\backslash X^M(r(\underline{c}_P)) \times \prod_{\alpha\in\pi'}(0, c_\alpha]) \otimes \mathbb{H}^\bullet(\mathfrak{u}, \mathcal{M}) \to \Omega^\bullet(\Gamma_P\backslash X^P(C(\underline{\tilde{c}}))\otimes\mathcal{M}) \tag{8.57}$$

This map is defined as follows. Let

$$\omega^p \otimes \omega^q_U \in \Omega^p(\Gamma_M\backslash X^M(r(\underline{c}_P)) \times \prod_{\alpha\in\pi'}(0, c_\alpha]) \otimes \mathbb{H}^q(\mathfrak{u}, \mathcal{M})$$

For a point $x \in \Gamma_P\backslash X^P(c_{\pi'}, r(\underline{c}_{\pi'}))$ we have to give the value of $q^{p+q}_{P,M}(\omega^p \otimes \omega^q_U)(x)$. Hence we to determine the value of $q^{p+q}_{P,M}(\omega^p\otimes\omega^q_U)(x)$ at a $p+q$- tuple of tangent vectors. We choose p tangent vectors $t^M_1,\dots,t^M_p$ arbitrarily, they map to tangent vectors $\bar{t}_1,\dots,\bar{t}_p$ under $q_{P,M}$. Then we choose q tangent vectors $u_1,\dots,u_q$ which are tangent to the fiber. The fiber is identified to $\Gamma_U\backslash U(\mathbb{R})$ and hence $u_1,\dots,u_q \in \mathfrak{u}^\vee$. With $m = (m_1, a) = q_{P,M})$ we get

$$q^{p+q}_{P,M}(\omega^p \otimes \omega^q_U)(x) = \omega^p(\bar{t}_1,\dots,\bar{t}_p)((m_1, a))\omega^q_U(u_1,\dots,u_q) \tag{8.58}$$

and (8.54) implies that $q^\bullet_{P,M}$ induces an isomorphism in cohomology.

The image under this map is not yet what we want. Again we can consider the sub complex

$$\Omega^\bullet(\Gamma_M\backslash X^M(r(\underline{c}_P)))\otimes\Omega^\bullet(\prod_{\alpha\in\pi'}(0,c_\alpha]) \subset \Omega^\bullet(\Gamma_M\backslash X^M(r(\underline{c}_P)) \times \prod_{\alpha\in\pi'}(0, c_\alpha]).$$

this inclusion induces an isomorphism in cohomology. We pick an element $w \in W^P$ and consider the complex

$$\Omega^\bullet(\Gamma_M\backslash X^M(r(\underline{c}_P)))\otimes\Omega^\bullet(\prod_{\alpha\in\pi'}(0,c_\alpha])\otimes\mathbb{H}^{l(w)}(\mathfrak{u},\mathcal{M})[w\cdot\lambda]), \tag{8.59}$$

this complex still computes the cohomology $H^p(\Gamma_P\backslash X^P(C(\underline{\tilde{c}})), \tilde{\mathcal{M}}\otimes\mathbb{C})[w\cdot\lambda]$. We look look for a suitable small sub complex of $\Omega^\bullet(\prod_{\alpha\in\pi'}(0,c_\alpha])\otimes\mathbb{H}^{l(w)}(\mathfrak{u},\mathcal{M})[w\cdot\lambda])$. We embed $\prod_{\alpha\in\pi'}(0,c_\alpha]) \subset \prod_{\alpha\in\pi'}\mathbb{R}^\times_{>0} = A_{\pi'}$, we have the restriction

$$\Omega^\bullet(A_{\pi'})\otimes\mathbb{H}^{l(w)}(\mathfrak{u},\mathcal{M})[w\cdot\lambda]) \xrightarrow{res} \Omega^\bullet(\prod_{\alpha\in\pi'}(0,c_\alpha])\otimes\mathbb{H}^{l(w)}(\mathfrak{u},\mathcal{M})[w\cdot\lambda]).$$

Then $\mathbb{H}^{l(w)}(\mathfrak{u},\mathcal{M})[w\cdot\lambda])$ is a $A_{\pi'}$- module, the action is given by the restriction of $w\cdot\lambda$ to $A_{\pi'}$. Then this character provides a one dimensional subspace $\mathbb{C}((w\cdot\lambda) \subset \mathcal{C}_\infty(A_{\pi'})$ Let $\mathfrak{a}_{\pi'}$ be the Lie-algebra of $A_{\pi'}$ then we get a map

$$\mathrm{Hom}(\Lambda^\bullet(\mathfrak{a}_{\pi'}), \mathbb{C}(w\cdot\lambda)\otimes\mathbb{H}^{l(w)}(\mathfrak{u},\mathcal{M})[w\cdot\lambda]). \to \mathrm{Hom}(\Lambda^\bullet(\mathfrak{a}_{\pi'}), \mathcal{C}_\infty(A_{\pi'})\otimes\mathbb{H}^{l(w)}(\mathfrak{u},\mathcal{M})[w\cdot\lambda]).$$

which gives us a surjection in cohomology. But we know that the cohomology of of the right hand sides sits in degree $l(w)$ and the left hand side complex has also cohomology in higher degrees, hence the left hand side complex is too small. We introduce the space

$$P_{\log}(A_{\pi'}) = \{f \in \mathcal{C}_\infty(A_{\pi'}) | f \text{ is a polynomial in } \log(x_\alpha), \alpha \in \pi'\} \subset \mathcal{C}_\infty(A_{\pi'})$$

and define the subspace $\mathbb{C}(w\cdot\lambda)_{\log} = \mathbb{C}(w\cdot\lambda)\otimes P_{\log}(A_{\pi'}) \subset \mathcal{C}_\infty(A_{\pi'})$. Then $\Omega^\bullet_{\log}(A_{\pi'})(w\cdot\lambda) = \text{Hom}(\Lambda^\bullet\mathfrak{a}, C(w\cdot\lambda)_{\log}) \hookrightarrow \Omega^\bullet(A_{\pi'})$ and clearly

$$\Omega^\bullet_{\log}(A_{\pi'})(w\cdot\lambda)\otimes \mathbb{H}^{l(w)}(\mathfrak{u},\mathcal{M})[w\cdot\lambda]) \hookrightarrow \Omega^\bullet(A_{\pi'})\otimes \mathbb{H}^{l(w)}(\mathfrak{u},\mathcal{M})[w\cdot\lambda]$$

induces an isomorphism in the cohomology of these comlexes. If we now define $\Omega^\bullet_{\log}(\prod_{\alpha\in\pi'}(0,c_\alpha])(w\cdot\lambda)\otimes\mathbb{H}^{l(w)}(\mathfrak{u},\mathcal{M})[w\cdot\lambda])$ to be the image of $\Omega^\bullet_{\log}(A_{\pi'})(w\cdot\lambda)\otimes\mathbb{H}^{l(w)}(\mathfrak{u},\mathcal{M})[w\cdot\lambda])$ under the restriction then it is clear that

$$\begin{gathered}\Omega^\bullet(\Gamma_M\backslash X^M(r(\underline{c}_P)))\otimes\Omega^\bullet_{\log}(\textstyle\prod_{\alpha\in\pi'}(0,c_\alpha]))(w\cdot\lambda)\otimes\mathbb{H}^{l(w)}(\mathfrak{u},\mathcal{M})[w\cdot\lambda]) \hookrightarrow \\ \Omega^\bullet(\Gamma_M\backslash X^M(r(\underline{c}_P)))\otimes\Omega^\bullet(\textstyle\prod_{\alpha\in\pi'}(0,c_\alpha]))(w\cdot\lambda)\otimes\mathbb{H}^{l(w)}(\mathfrak{u},\mathcal{M})[w\cdot\lambda]) \\ \hookrightarrow \Omega^{\bullet+\bullet+l(w)}(\Gamma_P\backslash X^P(r(c_P),c_P),\tilde{\mathcal{M}}\otimes\mathbb{C})\end{gathered} \tag{8.60}$$

induces an isomorphism in cohomology.

We define a global subcomplex $\Omega^\bullet_{\log}(\Gamma\backslash X)\otimes\mathcal{M}_\mathbb{C}$, it consists of those forms whose restriction to $\Gamma_P\backslash X^P(r(c_P),c_P))$ lie asymptotically in the log sub complex, this means for a suitable choice $c'_P < c_P$ the restriction to $\Gamma_P\backslash X^P(c'_P,,r(c_P)))$ lies in $\Omega^\bullet_{\log}(\Gamma_P\backslash X^P,(c'_P,,r(c_P)))\otimes\mathcal{M}_\mathbb{C}$. Then

Proposition 8.1.6. *The inclusion*

$$\Omega^\bullet_{\log}(\Gamma\backslash X)\otimes\mathcal{M}_\mathbb{C}\hookrightarrow\Omega^\bullet(\Gamma\backslash X)\otimes\mathcal{M}_\mathbb{C}$$

induces an isomorphism in cohomology.

Proof. We pick a closed form $\omega\in\Omega^p(\Gamma\backslash X)\otimes\mathcal{M}_\mathbb{C}$, we restrict this form to the sets $\Gamma_P\backslash X^P(c_\pi,r(c_\pi))$ in 1.79 . These sets are contained in slightly larger subsets $\Gamma_P\backslash X^P(c'_\pi,r(c'_\pi))$. For the maximal parabolic subgroups we find a $\psi_P\in\Omega^{p-1}(\Gamma_P\backslash X^P(c'_\pi,r(c'_\pi))\otimes\mathcal{M}$ such that $\omega-d\psi_P$ lies in the $\Omega^p_{\log}$ subcomplex. We choose a $\mathcal{C}_\infty$ function h_P on $\Gamma\backslash X$ which takes values ≥ 0, whose support lies in $\Gamma_P X^P(c'_\pi),r(c'_\pi)$ and which has value 1 on $X^P(c_\pi,r(c_\pi))$. Then $h_P\psi_P$ is a differential form on $\Gamma\backslash X$ We define

$$\psi_1 = \sum_{P:d(P)=1} h_P\psi_P \tag{8.61}$$

if we now put $\omega_1=\omega-d\psi_1$, then for all maximal parabolic subgroups the restriction of $\omega_1|_{\Gamma_P\backslash(c_\pi,r(c_\pi))}$ lies in the $\Omega_{\log}$ subcomplex. We continue this process and choose a π_Q for all parabolic subgroups with $d(Q)=2$ and apply the same procedure we can define $\psi_2=\sum_{Q:d(Q)=2}h_Q\psi_Q$ and then then restriction of $\omega_2=\omega_1-d\psi_2$ to the $\Gamma_Q\backslash X^Q(c'_\pi,r(c'_\pi)$ with $d(Q)=2$ lies in the $\Omega_{\log}$ subcomplex. The restriction of ω_2 to the $\Gamma_P\backslash X^P(c'_\pi,r(c'_\pi)$ with $d(P)=1$ differs from ω_1 but a closer look shows that this resriction is also in the $\Omega^p_{\log}$ complex. If we continue further this process stops if we reach the minimal parabolic subgroup.

This proves at least that the map in proposition 8.1.6 induces a surjective map in cohomology. But the injectivity is also clear if we consider the spectral sequence obtained from the Čzech covering. □

We introduced this sub complex because now we can say something about the growth of cohomology classes or the asymptotic behavior. We consider this behavior on the different sets $X^P((r(c_P), c_P))$. The restriction of a form $\omega \in \Omega_{\log}(\Gamma\backslash X) \otimes \mathcal{M}$ to $\Gamma_P\backslash X^P((r(c_P), c_P))$ is asymptotically of the form $\sum_{w\in W^P} \omega_w$.

Let $\omega = \omega_w \in \Omega^p(\Gamma_M\backslash X^M(r(\underline{c}_P))) \otimes \Omega^i_{\log}(\prod_{\alpha\in\pi'}(0, c_\alpha])) \otimes \mathbb{H}^{l(w)}(\mathfrak{u}, \mathcal{M})[w \cdot \lambda])$, we study the "growth" of the value of this form. We evaluate it at points $x = (x_1, a) \in \Gamma_M\backslash X^M(r(\underline{c}_P))) \times \prod_{\alpha\in\pi'}(0, c_\alpha])$, this means we pick tangent vectors $t^M_1, \ldots, t^M_p$ at x_1 from an orthonormal basis in the tangent space. The tangent bundle on $A_{\pi'}$ is trivialized by translation invariant vector fields. An i-tuple $t^A_1, \ldots, t^A_i \in \mathrm{Lie}A_{\pi'}$ gives an i-tuple $t^A_1, \ldots, t^A_i$ of tangent vectors in the point a. Then we put $\underline{T} = (t^M_1, \ldots, t^M_p, t^A_1, \ldots, t^A_i)$ and consider the value

$$\omega_w(\underline{T})(x) := \omega^M_w(x_1)(t^M_1, \ldots, t^M_p)\omega^A_w(a)(t^A_1, \ldots, t^A_i) \in \mathbb{H}^{l(w)}(\mathfrak{u}, \mathcal{M})[w \cdot \lambda]).$$

The variable x_1 runs through a compact set and the tangent vectors are from an orthonormal basis, hence value the first factor is bounded. The second term $\omega^A_w(a)(t^A_1, \ldots, t^A_i) \in \mathbb{C}(w\cdot\lambda)P_{\log}(A_{\pi'})$. We have a hermitian scalar product $< , >$ on $\mathbb{H}^{l(w)}(\mathfrak{u}, \mathcal{M})[w \cdot \lambda])$ and we are interested in the value

$$< \omega_w(\underline{T})(x), \omega_w(\underline{T})(x) >= ||\omega^M_w(x_1)(t^M_1, \ldots, t^M_p)||^2||\omega^A_w(a)(t^A_1, \ldots, t^A_i)||^2$$

and this implies

$$< \omega_w(\underline{T})(x), \omega_w(\underline{T})(x) > < C(2w \cdot \lambda)(a)a^{-\epsilon}$$

where $\epsilon > 0$ and $a^{-\epsilon} = \prod_{\alpha\in\pi'} x_\alpha^{-\epsilon}$.

Now we can formulate a criterion to decide whether ω_w is square integrable. We have to evaluate the integral

$$\int_{\Gamma_P\backslash X^P(r(c_P), c'_P))} ||\omega_w(\underline{p})||^2 d\underline{p} \tag{8.62}$$

The measure $d\underline{p}$ is of course the restriction of the invariant measure on $\Gamma\backslash X$, it is of the form $2\rho_P(a)d\underline{u}d\underline{a}d\underline{m}$. The differential form is invariant under left translations under $U(\mathbb{R})$ and hence we have to evaluate

$$\int_{\Gamma_M\backslash X^M(r(c_P)\times(\prod_{\alpha\in\pi'}(0,c_\alpha]))} ||\omega^M_w(\underline{m})||^2||\omega^A_w(a)||^2(2\rho)(a)d\underline{m}d\underline{a} \tag{8.63}$$

The integral over $\Gamma_M\backslash X^M(r(c_P))$ is finite hence we are left with

$$\int_{\prod_{\alpha\in\pi'}(0,c_\alpha]} ||\omega^A_w(a)||^2 2\rho_P(a)d\underline{a} \tag{8.64}$$

Of course we assume that $\omega_w \neq 0$ and then we can find a constant $C > 0$ and an $\epsilon > 0$ such that

$$C(2(w(\lambda + \rho) - \rho) + 2\rho_P)(a)a^{\epsilon} \leq ||\omega^A_w(a)||^2 2\rho_P(a) \leq C(2(w(\lambda + \rho) - \rho) + 2\rho_P)(a)a^{-\epsilon} \tag{8.65}$$

Since $\rho|_{A_{\pi'}} = \rho_P|_{A_{\pi'}}$ we get

$$C(2(w(\lambda+\rho)))(a)a^{\epsilon} \leq ||\omega_w^A(a)||^2 2\rho_P(a) \leq C(2(w(\lambda+\rho)))a^{-\epsilon}. \qquad (8.66)$$

The relative roots α^P form a basis for $X^*(S)$, any character $\mu \in X^*(S)$ can be written as linear combination $\mu = \sum_{\alpha\in\pi'} r_{\mu,\alpha}\alpha^P$ with $r_{\mu,\alpha} \in \mathbb{Q}$. We say that μ is in the positive cone (with respect to the roots) if $r_{\mu,\alpha} > 0$ for all $\alpha \in \pi'$, we write $\mu >_P 0$. Then

$$w(\lambda+\rho)(a) = w(\lambda+\rho)(\{\dots, x_\alpha, \dots\}) = \prod_{\alpha\in\pi'} x_\alpha^{r_{\lambda+\rho,\alpha}}.$$

and come to the conclusion

Proposition 8.1.7. *a)The integral (8.64) is finite* $\iff w(\lambda+\rho) >_P 0$.
b) A closed differential form $\omega \in \Omega^p_{\log}(\Gamma\backslash X)\otimes\mathcal{M}$ *is square integrable if and only if for all parabolic subgroups P and the resulting decompositions*

$$\omega|X^P(r(c_P), c_P) = \sum_{w\in W^P} \omega_w$$

the components $\omega_w = 0$ *if* $w(\lambda+\rho) \not\geq_P 0$.

We are now able to show

Proposition 8.1.8. *If* $\omega \in$ *Hom*$_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}), H^{(K_\infty)}_{\mathrm{Coh}_2(\lambda)} \otimes \mathcal{M}_\lambda)$ *and if the class* $[\omega] \in H^q_!(\Gamma\backslash X, \tilde{\mathcal{M}})$ *then we can find a square integrable* $\psi \in \Omega^{p-1}(\Gamma\backslash X)\otimes\mathcal{M}\lambda)$ *such that* $\omega - d\psi$ *has compact support*

Proof. We know that we can find a form $\omega_1 \in \Omega^p_{\log}(\Gamma\backslash X)\otimes\mathcal{M}_\lambda$ which represents the same class. Our previous arguments show that ω_1 is again square integrable. □

I want to explain the basic idea to prove 8.1.8 in the [book], I look at a special case.

Let $G/\mathbb{Z}$ semi simple (simply connected) Chevalley scheme, let $T/\mathbb{Z}$ be a maximal split torus, let $B/\mathbb{Z} \supset T/\mathbb{Z}$ a Borel subgroup, let $U/\mathbb{Z}$ be its unipotent radical. Let Θ_0 be the Cartan involution which induces $t \mapsto t^{-1}$ on T and maps $U/\mathbb{Z}$ to its opposite $U_-/\mathbb{Z}$. Let $K_\infty = G(\mathbb{R})^{\Theta_0}$ be the maximal compact subgroup, and $X = G(\mathbb{R})/K_\infty$ is the symmetric space. Then $x_0 = K_\infty \in X$. We put $\Gamma := G(\mathbb{Z})$.

We consider the ring of invariant differential forms $\Omega^\bullet(X)^{G(\mathbb{R})}$ and map it to the de-Rham complex

$$\Omega^\bullet(X)^{G(\mathbb{R})} \to \Omega^\bullet(\Gamma\backslash X, \mathbb{R}) \qquad (8.67)$$

and study the resulting map in cohomology

$$J : \Omega^\bullet(X)^{G(\mathbb{R})} \to H^\bullet(\Gamma\backslash X, \mathbb{R}). \qquad (8.68)$$

In the book I claim (prop. 8.1.8.) that for an invariant differential form $\omega \in \Omega^p(X)^{G(\mathbb{R})}$, which induces an inner class $[\omega] \in H^p_!(\Gamma\backslash X, \mathbb{R})$ we can find a square-integrable $\psi \in \Omega^{p-1}(\Gamma\backslash X, \mathbb{R})$ such that $\omega - d\psi$ has compact support.

The proof given in the [book] is somewhat laconic, I claim that it follows from the considerations on the pages before.

I want to give a slightly modified and more detailed exposition of the proof of 8.1.8. We consider a tubular neighbourhood $\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X) \subset \Gamma\backslash X$ of the Borel-Serre compactification $\partial(\Gamma\backslash X)$. We look at this tubular neighbourhood near the Borel stratum $\partial_B(\Gamma\backslash X)$. To be precise: We know that $B(\mathbb{R})$ acts transitively on X. We choose a number $0 < c_0 < 1$, let $B_+(c_0)$ be the set of elements $b = tu \in B(\mathbb{R}), t \in T^+(\mathbb{R}), \alpha(t) < c_0$, for all α in the set π of simple roots.

Then our tubular neighbourhood "near" ∂_B will be

$$X^B(c_0) = \{bx_0 |\ b \in B(c_0)\} \tag{8.69}$$

Reduction theory tells us that $B(\mathbb{Z})\backslash X^B(c_0)$ is an open subset of $\Gamma\backslash X$.

Now I prove:

If $\omega \in \Omega^p(X)^{G(\mathbb{R})}$ and if the restriction of $[\omega]$ to $H^p(B(\mathbb{Z})\backslash X^B(c_0), \mathbb{R})$ is zero, then we find a square-integrable class $\psi_1 \in \Omega^{p-1}(B(\mathbb{Z})\backslash X^B(c_0))$ such that $\omega - d\psi_1|_{B(\mathbb{Z})\backslash X^B(c_0)} = 0$.

We look at the restriction

$$\Omega^\bullet(X)^{G(\mathbb{R})} = \operatorname{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathbb{R}) \xrightarrow{J_B} \Omega^\bullet(B(\mathbb{Z})\backslash X^B(c_0), \mathbb{R}) \tag{8.70}$$

On the right hand side we have the action of the semi-group $B^+(\mathbb{R}) = \{b \in B(\mathbb{R}) | 0 < \alpha_i(b) < 1\}$ and this implies that the image of this map lands in $\Omega^\bullet(B(\mathbb{Z})\backslash X^B(c_0), \mathbb{R})^{U(\mathbb{R})\cdot T^+(\mathbb{R})}$. Now $\Omega^\bullet(U(\mathbb{R}), \mathbb{R})^{U(\mathbb{R})} = \operatorname{Hom}(\Lambda^\bullet(\mathfrak{u}), \mathbb{R})$. Hence we see that our map J_B is actually a map

$$\Omega^\bullet(X)^{G(\mathbb{R})} = \operatorname{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathbb{R}) \xrightarrow{J_B} \operatorname{Hom}(\Lambda^\bullet(\mathfrak{u}), \mathbb{R}) \otimes \Lambda^\bullet(\mathfrak{t}^\vee), \tag{8.71}$$

here of course $\mathfrak{t} = \operatorname{Lie}(T), \mathfrak{u} = \operatorname{Lie}(U)$. The Lie algebra $\mathfrak{u}$ decomposed into one dimensional root subspaces

$$\mathfrak{u} = \bigoplus_{\alpha\in\Delta^+} \mathfrak{u}_\alpha$$

For any subset $\mathbf{A} \subset \Delta^+$ we define the one dimensional subspace $\mathbb{R}U_{\mathbf{A}} = \mathbb{R} \cdot \Lambda_{\alpha\in\mathbf{A}} U_\alpha \in \Lambda^p(\mathfrak{u})$, here $p = \#\mathbf{A}$. Let $\omega_{\mathbf{A}} \in \operatorname{Hom}(\Lambda^p(\mathfrak{u})$ be an element which is non zero on $U_{\mathbf{A}}$ and zero on all other $U_{\mathbf{A}'}$. Then

$$\operatorname{Hom}^\bullet(\mathfrak{u}, \mathbb{R}) = \bigoplus_p \bigoplus_{\mathbf{A}:\#\mathbf{A}=p} \mathbb{R}\cdot\omega_{\mathbf{A}} \tag{8.72}$$

This is a decomposition into eigenspaces under the action of T, the torus T acts on $\mathbb{R}\cdot\omega_{\mathbf{A}}$ by the character $\chi_{\mathbf{A}} = -\sum_{\alpha\in\mathbf{A}}\alpha$. For any character χ of the torus we put

$$\operatorname{Hom}(\Lambda^\bullet(\mathfrak{u}), \mathbb{R})(\chi) := \bigoplus_{\mathbf{A}:\chi_{\mathbf{A}}=\chi} \mathbb{R}\omega_{\mathbf{A}} \tag{8.73}$$

It follows directly from the definition of the differential that $\operatorname{Hom}(\Lambda^\bullet(\mathfrak{u}), \mathbb{R})(\chi)$ is a subcomplex. Now we have a theorem which is essentially a special case of Kostant's theorem

Theorem 8.1.4. *The complex* $Hom(\Lambda^\bullet(\mathfrak{u}),\mathbb{R})(\chi)$ *is acyclic unless there is a* $w \in W$ *such that* $\chi = w\cdot\rho = w\rho - \rho$. *If* $\chi = w\cdot\rho$ *we define* $\Delta(w) = \{\alpha \in \Delta^+ | w^{-1}\alpha < 0\}$ *and then*

$$Hom(\Lambda^\bullet(\mathfrak{u}),\mathbb{R})(w\cdot\rho) = \mathbb{R}\omega_{\Delta(w)} \tag{8.74}$$

Of course this also says that the complex $\mathrm{Hom}(\Lambda^\bullet(\mathfrak{u}),\mathbb{R})(w\cdot\rho)$ is equal to its cohomology, we write

$$H^\bullet(\ \mathrm{Hom}(\Lambda^\bullet(\mathfrak{u}),\mathbb{R})(w\cdot\rho)) = \mathbb{H}^{l(w)}(\mathfrak{u},\mathbb{R})(w\cdot\rho) \tag{8.75}$$

Hence we see that map J_B maps a form ω to the finite sum

$$J_B(\omega) = \sum_\chi \omega_\chi; \tag{8.76}$$

By construction $J_B(\omega)$ is an element in $\Omega^\bullet(B(\mathbb{Z})\backslash X^B(c_0),\mathbb{R})$. We have the fibration

$$\pi_B : X^B(c_0) \to \prod_{\alpha 1 \pi}(0,c_0) = A_\pi(c_0) \subset \prod_{\alpha 1\pi}\mathbb{R}^\times_{>0} \tag{8.77}$$

the fibers are the orbits under the action of $U(\mathbb{R})$. The tangent bundle of $A_\pi(c_0)$ is trivialised via the action of $T^+(\mathbb{R})$. Since $J_B(\omega)$ is invariant under the action of $U(\mathbb{R})$ we can evaluate at $a \in A_\pi(c_0)$, and hence

$$J_B(\omega)(a) \in \ \mathrm{Hom}(\Lambda^\bullet(\mathfrak{u}),\mathbb{R})(\chi)\otimes\Lambda^\bullet(\mathfrak{t}^\vee) \tag{8.78}$$

If we pick a ω_χ an use the invariance under $T^+(\mathbb{R})$ then we see

$$\omega_\chi(ta_0) = \chi(t)\omega_\chi(a_0). \tag{8.79}$$

In section 8.1.8 of the [book] we defined the scalar product

$$<\omega_\chi(ta_0),\omega_\chi(ta_0)>=\chi(t)^2<\omega_\chi(a_0),\omega_\chi(a_0)>$$

Of course we know that

$$\int_{B(\mathbb{Z})\backslash X^B(c_0)} < J_B(\omega), J_B(\omega) > dx < \infty \tag{8.80}$$

and here the measure $dx = t^{2\rho}dud^\times\underline{t}$ We want to make this more explicit. The variable $\underline{t} = \{\dots,t_\alpha,\dots\}_{\alpha\in\pi}$ and $t_\alpha \in (0,c_0)$. As usual 2ρ is the the sum of positive roots we write $2\rho = \sum_{\alpha\in\pi} n_\alpha\alpha$ then $\underline{t}^{2\rho}d^\times\underline{t} = \prod_{\alpha\in\pi} t_\alpha^{n_\alpha}\frac{dt_\alpha}{t_\alpha}$.

Since $J_B(\omega)$ is invariant under $U(\mathbb{R})$ the integral over $U(\mathbb{R})$ is a constant and our integral becomes up to a non zero constant factor

$$\int_{A_\pi(c_0)} < J_B(\omega)(\underline{t}), J_B(\omega)(\underline{t})\underline{t}^{2\rho}d^\times\underline{t} < \infty. \tag{8.81}$$

We look at a summand ω_χ. We have seen that the character $\chi = -\sum_{\alpha\in\pi}\chi_\alpha\alpha$ where χ_α are integers ≥ 0. Then our integral becomes

$$\int_{A_\pi(c_0)} <\omega_\chi,\omega_\chi> \underline{t}^{2\rho}d^\times\underline{t} = \prod_{\alpha\in\pi}\int_{c_0}^0 t_\alpha^{n_\alpha - 2\chi_\alpha}\frac{dt_\alpha}{t_\alpha} \tag{8.82}$$

If $\omega_\chi \neq 0$ then this integral is finite if and only if for all $\alpha \in \pi$ we have $n_\alpha - 2\chi_\alpha > .0$, i.e $\rho > -\chi$. We know that $J_B(\omega)$ square integrable, then it is easy to see that all the ω_χ must be square integrable and hence $\omega_\chi \neq 0$ implies $\rho > -\chi$.

We have $d\omega_\chi = 0$. If $\chi \notin \{w\rho - \rho\}_{w\in W}$, then $\mathrm{Hom}(\Lambda^\bullet(\mathfrak{u}), \mathbb{R})(\chi)$ is acyclic and hence $\omega_\chi = d\psi_\chi$ and ψ_χ is again square integrable.

If $\chi = w \cdot \rho$ the situation is a little more complicated. We consider the map

$$\mathbb{R}\omega_{\Delta(w)} \otimes \Lambda^\bullet(\mathfrak{t}^\vee) \xrightarrow{i_0} \Omega^\bullet(B(\mathbb{Z})\backslash X^B(c_0), \mathbb{R})^{U(\mathbb{R})} \to H^{l(w)}(B(\mathbb{Z})\backslash X^B(c_0), \mathbb{R}) = \mathbb{R}\omega_{\Delta(w)}, \tag{8.83}$$

the kernel of this composition is exactly $\mathbb{R}\omega_{\Delta(w)} \otimes (\bigoplus_{\nu>0} \Lambda^\nu(\mathfrak{t}^\vee))$. The image of i_0 in degree p consists of differential forms

$$\omega_{\Delta(w)} \otimes \chi \otimes \sum_{I\subset\pi:\#I=p} c_I \Lambda^p_{\alpha\in I} \frac{dt_\alpha}{t_\alpha} \tag{8.84}$$

where $c_I \in \mathbb{R}$

We enlarge the image slightly. Instead of taking constant coefficients c_0 we allow polynomials $P_I(, \log t_\alpha, \dots)$, i.e we allow polynomials in the $\log t_\alpha$. We call the resulting complex $\Omega^\bullet_{\log}(B(\mathbb{Z})\backslash X^B(c_0), \mathbb{R})^{U(\mathbb{R})}$ then it is elementary that

$$H^\bullet(\Omega^\bullet_{\log}(B(\mathbb{Z})\backslash X^B(c_0), \mathbb{R})^{U(\mathbb{R})}) \xrightarrow{\sim} H^{l(w)}(B(\mathbb{Z})\backslash X^B(c_0), \mathbb{R}) \tag{8.85}$$

But now we remark that

$$\chi \otimes \sum_{I\subset\pi:\#I=p} P_I(\dots, \log t_\alpha, \dots)\Lambda^p_{\alpha\in I} \frac{dt_\alpha}{t_\alpha} \tag{8.86}$$

is square integrable if $\rho > -\chi$ and hence the claim is proved.

Remark 1: We do something absolutely elementary. The de-Rham complex $\Omega^\bullet(\mathbb{R}^n)$ computes the cohomology of $\mathbb{R}^n$. The subcomplex $\Omega^\bullet(\mathbb{R}^n)^{\mathbb{R}^n}$ of translation invariant differential forms has the cohomology $\Lambda^\bullet(\mathbb{R}^n)$. But if we enlarge slightly and allow polynomial coefficients $P(\dots, x_i, \dots)dx_{i_1} \wedge \dots dx_{i_k}$ the complex computes the cohomology correctly.

Remark 2: In principle these considerations are also in the [book] before proposition 8.1.8. The special situation here is that we have the trivial representation $\mathbb{C}$ and not a more general $H_{\pi_\infty} \subset L^2_{\text{disc}}$ And we only consider the lowest dimensional stratum. But I do not see - at least in the case that $H_{\pi_\infty} = \mathbb{C}$ - any obstacle to extend these considerations to the other strata.

Remark 3: Proposition 8.1.8 fills the gap in the proof of prop.8.1.4. I make such a fuss about the proof of prop. 8.1.8 because prop. 8.1.4. contradicts the results in the Lee-Sczcarba paper, these results are widely accepted

8.1.10 Franke's Theorem

The theorem 8.1.1 tells us that we have the very small sub complex

$$\mathrm{Hom}_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}), H^{(K_\infty)}_{\mathrm{Coh}_2(\lambda)} \otimes \mathcal{M}_\lambda) \subset \mathrm{Hom}_{K_\infty}(\Lambda^p(\mathfrak{g}/\mathfrak{k}), \mathcal{C}_\infty(\Gamma\backslash G(\mathbb{R})) \otimes \mathcal{M}_\lambda)$$

such that this induces a surjective map in cohomology

$$H^\bullet(\mathfrak{g}, K_\infty, H^{(K_\infty)}_{\mathrm{Coh}_2(\lambda)} \otimes \mathcal{M}_\lambda) \xrightarrow{j_{(2)}} H^\bullet_{(2)}(\Gamma\backslash X, \mathcal{M}_\lambda).$$

If $\Gamma\backslash X$ is not compact the map j_c is not necessarily an isomorphism, the kernel can be computed in principle by using Proposition 8.1.4.

By definition

$$H^{(K_\infty)}_{\mathrm{Coh}_2(\lambda)} = \{f \in L^2_{\mathrm{disc}}(\Gamma\backslash G(\mathbb{R})) | zf = \chi_\lambda(z) f \ \forall z \in \mathfrak{Z}(\mathfrak{g}\}.$$

A. Borel proposed to replace $H^{(K_\infty)}_{\mathrm{Coh}(\lambda)}$ by a larger space

$$\mathcal{A}_\lambda(\Gamma\backslash G(\mathbb{R})) := \{f \in \mathcal{C}_\infty(\Gamma\backslash G(\mathbb{R}))| \ \exists N \text{ such that } (z - \chi_\lambda(z))^N f = 0\} \quad (8.87)$$

where f also satisfies a growth condition. Borel conjectured the following theorem which was proved by Franke

Theorem 8.1.5. *(* Franke [26]) *The inclusion* $\mathcal{A}_\lambda(\Gamma\backslash G(\mathbb{R})) \subset \mathcal{C}_\infty(\Gamma\backslash G(\mathbb{R}))$ *induces an isomorphism in cohomology*

$$H^\bullet(\mathfrak{g}, K_\infty, \mathcal{A}_\lambda(\Gamma\backslash G(\mathbb{R})) \otimes \mathcal{M}_\lambda) \xrightarrow{\sim} H^\bullet(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda)$$

The main tool for proving this theorem is again the theory of Eisenstein series. As in section 4.1.11 we start from classes of cartain induced $\mathfrak{I}^G_P \sigma \times |\rho_P|^{\underline{z}}$ where P runs over the conjugacy classes bolic subgroups, σ is a (cuspidal) cohomology class on locally symmetric space attached to the reductive quotient $M/\mathbb{Q}$ of P, and $\underline{z} \in \mathbb{C}^r$ We can write down Eisenstein series which yield an embedding

$$\mathrm{Eis}(\ ,z) : \mathfrak{I}^G_P \sigma \times |\rho_P|^z \hookrightarrow \mathcal{C}_\infty(G(\mathbb{Q})\mathbb{G}(\mathbb{A}))$$

these series are absolutely and locally uniformly converging if $\Re(z_i) >> 0$. In [69] Langlands proves that these Eisenstein series have a meromorphic continuation into the entire $\mathbb{C}^r$. If we now "evaluate at $\underline{z} = 0$" then we get functions in $\mathcal{A}_\lambda(\Gamma\backslash G(\mathbb{R}))$. But now the process of evaluation may become delicate, because the Eisenstein series may be singular at $z = 0$. So we may have to take residues and derivatives of such Eisenstein series which the will give us the space $\mathcal{A}_\lambda(\Gamma\backslash G(\mathbb{R}))$.

8.2 Modular symbols

GP

8.2.1 The general pattern

We start from a flat group scheme $\mathcal{G}/\operatorname{Spec}(\mathbb{Z})$ whose generic fiber $G/\mathbb{Q} = \mathcal{G} \times \mathbb{Q}$ is reductive. We assume that the derived group $G^{(1)}/\mathbb{Q}$ is quasi split Let $F/\mathbb{Q}$ be a finite normal extension, let $\mathcal{O}_F$ be its ring of integers. We choose a highest weight, which is defined over F and consider a representation $\rho_\lambda : \mathcal{G} \times \mathcal{O}_F \to \mathrm{Gl}(\mathcal{M}_{\mathcal{O}_F})$ which after tensorization by F becomes the highest weight representation $\mathcal{M}_{F,\lambda}$. In the following we write $\mathcal{M} = \mathcal{M}_{\mathcal{O}_F}$., if we change the ring of scalars we write $\mathcal{M}_R := \mathcal{M} \otimes_{\mathcal{O}_F} R$. Let $K_f^{(0)} = \mathcal{G}(\hat{\mathbb{Z}})$ and $K_f \subset K_f^{(0)}$ be an open subgroup.

We want to describe a general method to construct homology classes in

$$H_d(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}) \text{ resp. relative homology groups } H_d(\mathcal{S}^G_{K_f}, \partial\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}),$$

which are obtained from (reductive) subgroups $H \subset G$, these classes will be the modular symbols. In Chapter 5 we discussed the construction of such classes in a very special case and here we treat a more systematic way to construct such classes.

Let $H/\mathbb{Q}$ be a (reductive) subgroup of our ambient group $G/\mathbb{Q}$, we also consider the flat closure $\mathcal{H}/\mathbb{Z}$. We assume that its derived subgroup $H^{(1)}$ is simply connected and satisfies strong approximation. The quotient $H/H^{(1)} = C'$ is a torus. Let $K_\infty^{H,(1)}$ be the connected component of the identity of a maximal compact subgroup of $H(\mathbb{R})$ we put $X^H = H(\mathbb{R})/K_\infty^{H,(1)}$. We have the two spaces

$$\mathcal{S}^G_{K_f} = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K_f,\ \mathcal{S}^H_{K_f^H} = H(\mathbb{Q})\backslash X^H \times H(\mathbb{A}_f)/K_f.$$

and it follows from the considerations in section 6.1.3 that

$$\pi_0(\mathcal{S}^H_{K_f^H}) \xrightarrow{\sim} \pi_0(\mathcal{S}^{C'}_{K_f^{C'}}). \tag{8.88}$$

From the inclusion $i : H \to G$ we will get maps between these locally symmetric spaces

$$j(x_\infty, \underline{g}_f) : \mathcal{S}^H_{K_f^H} \to \mathcal{S}^G_{K_f}$$

which depend on the choice of "pin points" $(x_\infty, \underline{g}_f) \in X \times G(\mathbb{A}_f)$. These pin points have to be chosen with some care:

a) The point $x_\infty \in X$ can be viewed as a Cartan involution Θ_{x_∞} on $G(\mathbb{R})$ and Θ_{x_∞} should fix $H(\mathbb{R})$. Hence it is also a Cartan involution on H and we require that it is the identity on our chosen $K_\infty^{H,(1)}$. Let us denote this subset of X by

$$X^{(H, K_\infty^{H,(1)})} = \{x \in X | \Theta_x(H(\mathbb{R})) = H(\mathbb{R});\ \Theta_x = \text{ identity on } K_\infty^{H,(1)}\}.$$

Let N be the subgroup of the normalizer of $H/\mathbb{Q}$ which also normalizes $K_\infty^{H,(1)}$. Then $N(\mathbb{R})$ acts on $X^{(H, K_\infty^{H,(1)})}$. I think that this action is transitive and the orbits under the group $N(\mathbb{R})^{(1)}$ are the connected components.

b)The element $\underline{g}_f$ has to satisfy a similar condition:

$$K_f^H \underline{g}_f K_f = \underline{g}_f K_f \tag{8.89}$$

we say that $\underline{g}_f$ is adapted.

(Recall that we always have to make careful choices of the level once we deal with integral cohomology.) Such a pin point $(x_\infty, \underline{g}_f)$ provides a map pinpoint

$$j(x_\infty, \underline{g}_f) : H(\mathbb{Q})\backslash H(\mathbb{R})/K_\infty^H \times H(\mathbb{A}_f)/K_f^H \longrightarrow \mathcal{S}^G_{K_f} \tag{8.90}$$

which is defined by

$$(h_\infty, \underline{h}_f) \mapsto (h_\infty x_\infty, \underline{h}_f \underline{g}_f).$$

We restrict the representation $\mathcal{M}$ to $\mathcal{H}/\mathbb{Z}$ then we can decompose the rational module $\mathcal{M}_\lambda \otimes F = \mathcal{M}_{\lambda}{}_F^{>1} \oplus \mathcal{M}_{\lambda}{}_F^{H^{(1)}}$, where the first summand is the direct sum of irreducible modules of dimension > 1 and the second summand is the module of $H^{(1)}$ invariants. We define the module of $\mathcal{H}^{(1)}$ coinvariants

$$\mathcal{M}_{\lambda,H^{(1)}} = \mathcal{M}_\lambda / \mathcal{M}_\lambda \cap \mathcal{M}_{\lambda}{}_F^{>1}.$$

This module of coinvariants is now a module for C' we assume that our field F is large enough so that we can assume that $C' \times \mathcal{O}_F$ is a split torus. We get that $\mathcal{M}_{\lambda,H^{(1)}} = \bigoplus_{\mu \in X^*(C' \times \mathcal{O}_F)} \mathcal{M}_{\lambda,H^{(1)}}[\mu]$. Then $\mathcal{M}_{\lambda,H^{(1)}}[\mu]$ is a projective $\mathcal{O}_F$ module of finite rank on which $C' \times \mathcal{O}_F$ acts by the character μ. We assume for simplicity that $\mathcal{M}_{\lambda,H^{(1)}}[\mu]$ is actually free, hence we can write it as a direct sum of modules $\mathcal{O}_F e_{\mu,j}$ where we chose a generator for each summand (in our examples this module is always of rank one). Let $\mathcal{O}_\mu$ be the $\mathcal{O}_F-$ module $\mathcal{O}_F$ (with canonical generator 1) and with the action of C' by the character μ, then of course $\mathcal{O}_F e_{\mu,j} \xrightarrow{\sim} \mathcal{O}_\mu$. Any C' homomorphism $\phi_\mu : \mathcal{M}_{\lambda,H^{(1)}} \to \mathcal{O}_\mu$ provides a homomorphism of $\mathcal{H}$ modules

$$\phi_\mu : \mathcal{M}_\lambda \to \mathcal{O}_\mu. \tag{8.91}$$

we denote it by the same letter. This induces a homomorphism of sheaves

$$\phi_\mu^* : j(x_\infty, \underline{g}_f)^*(\tilde{\mathcal{M}}_\lambda) \to \tilde{\mathcal{O}}_\mu. \tag{8.92}$$

especially any of the $e_{\mu,j}$ gives us such a homomorphism.

Then these data provide a homomorphism for the cohomology groups

$$\phi_\mu \circ j(x_\infty, \underline{g}_f))^\bullet : H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}) \to H^\bullet(\mathcal{S}^H_{K_f^H}, \tilde{\mathcal{O}}_\mu). \tag{8.93}$$

We are interested in this homomorphism in degree $d_H = \dim \mathcal{S}^H_{K_f^H}$.

Let us assume for a moment that $\mathcal{S}^H_{K_f^H}$ is compact. We have an orientation on $\mathcal{S}^H_{K_f^H}$, because we chose the compact subgroup K_∞^H to be narrow. Therefore we see that the cohomology group $H^{d_H}(\mathcal{S}^H_{K_f^H}, \tilde{\mathcal{O}}_\mu)$ is the sum of cohomology groups over the connected components, and hence(See 6.1.4)

$$H^{d_H}(\mathcal{S}^H_{K_f^H}, \tilde{\mathcal{O}}_\mu) \supset \bigoplus_{\tilde{\mu}_f} H^{d_H}(\mathcal{S}^H_{K_f^H}, (\tilde{\mathcal{O}}_\mu)[\tilde{\mu}_f]. \tag{8.94}$$

where we sum over characters $\tilde{\tilde{\mu}}_f$ of type μ on $\tilde{C}'(\mathbb{A})/K_f^{C'}$ (See (6.3.8)). The eigenspaces are projective $\mathcal{O}$- modules of rank one let us assume that they are free and that we have chosen generators $c_{\tilde{\mu}_f}$. We will call such generators modular symbols.

We still have the variable $\underline{g}_f$, it has to satisfy the above condition b). We have to fix the level because we want to work with integral cohomology groups.

But once we tensorize our coefficient systems with F (the quotient field of $\mathcal{O}$) then we can consider the limit

$$\lim_{K_f} H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_F) = H^\bullet(\mathcal{S}^G, \tilde{\mathcal{M}}_F),$$

and this limit is now a $\tilde{G}(\mathbb{A})-$ module (Section 6.3). Doing this also with $\mathcal{S}^H_{K_f^H}$ we can forget the constraint on $\underline{g}_f$, the condition b) is certainly fulfilled for some choice of levels.

We recall the definition of an induced representation, we have

$$\mathrm{Ind}_{\tilde{H}(\mathbb{A}))}^{\tilde{G}(\mathbb{A})} H^{d_H}(\mathcal{S}^H, \tilde{F}_\mu) = \{\Phi : \tilde{G}(\mathbb{A}) \to H^{d_H}(\mathcal{S}^H, \tilde{F}_\mu))$$

where Φ satisfies $\Phi(\underline{hg}) = \rho_{F_\mu}(\underline{h})\Phi(\underline{g})$ for all $\underline{h} \in \tilde{H}(\mathbb{A})), \underline{g} \in \tilde{G}(\mathbb{A})$ and where Φ is right invariant under some open compact subgroup $K_f^\prime$. The map

$$J(\phi_\mu) : \xi \mapsto r_{\phi_\mu} \circ j(x, \underline{g}_f)(\xi) \tag{8.95}$$

yields an intertwining operator between $\tilde{G}(\mathbb{A}) = \pi_0(G(\mathbb{R})) \times G(\mathbb{A}_f)$ modules Jphi

$$J(\phi_\mu) : H^{d_H}(\mathcal{S}^G, \tilde{\mathcal{M}}_F) \to \ \mathrm{Ind}_{\tilde{H}(\mathbb{A}))}^{\tilde{G}(\mathbb{A})} H^{d_H}(\mathcal{S}^H, \tilde{F}_\mu) \tag{8.96}$$

On the right hand side we can decompose further. We have seen that decochi

$$H^{d_H}(\mathcal{S}^H, \tilde{F}_\mu) = \bigoplus_{\tilde{\mu}_f : \mathrm{type}(\tilde{\mu}_f) = \mu} \mathrm{Ind}_{\tilde{H}(\mathbb{A}))}^{\tilde{G}(\mathbb{A})} H^{d_H}(\mathcal{S}^H, \tilde{F}_\mu)[\tilde{\mu}_f] = \bigoplus_{\tilde{\mu}_f : \mathrm{type}(\tilde{\mu}_f) = \mu} F[\tilde{\mu}_f] \tag{8.97}$$

here we have to take into account that we have to enlarge our field F so that it contains the values of $\tilde{\mu}_f(C^\prime(\mathbb{A}))$.

We project to the $\tilde{\mu}_f$ component and get intertwining operators prchi

$$J(\phi_\mu, \tilde{\mu}_f) : H^{d_H}(\mathcal{S}^G, \tilde{\mathcal{M}}_F) \to \ \mathrm{Ind}_{\tilde{H}(\mathbb{A}))}^{\tilde{G}(\mathbb{A})} H^{d_H}(\mathcal{S}^H, \tilde{F}_\mu)(\tilde{\mu}_f) = \ \mathrm{Ind}_{\tilde{H}(\mathbb{A}))}^{\tilde{G}(\mathbb{A})} F \otimes \tilde{\mu}_f \tag{8.98}$$

Again the question arises to compute this intertwining operator. We have to explain what this means. At this point we only give a first approximation of what it means to compute this operator.

Assume we have an "explicitly given" absolutely irreducible $\tilde{G}(\mathbb{A})$ module $V_{\epsilon \times \pi_f}/F$, here $\epsilon \times \pi_f$ is a isomorphism type of an absolutely irreducible representation of $\tilde{G}(\mathbb{A})$ on a F-vector space. The infinite component of such a representation is simply a character $\epsilon : \pi_\infty(G(\mathbb{R})) \to \{\pm 1\}$.

Now we also assume that we have an embedding $\Phi(\pi_f) : V_{\epsilon \times \pi_f} \hookrightarrow H^{d_H}(\mathcal{S}^G, \tilde{\mathcal{M}}_F)$, i.e. $V(\pi_f)$ is a F-model space (see further down). We also choose a character $\tilde{\mu}_f$ of type μ, we assume that the values of $\tilde{\mu}_f$ are in F. Then $H^{d_H}(\mathcal{S}^H, \tilde{F}_\mu)[\tilde{\mu}_f]$

is of rank one and our intertwining operator gives us an $\tilde{G}(\mathbb{A})$ -module homomorphism

$$J(\phi_\mu, \tilde{\mu}_f) \circ \Phi(\pi_f) : V_{\epsilon \times \pi_f} \to \operatorname{Ind}_{\tilde{H}(\mathbb{A}))}^{\tilde{G}(\mathbb{A})} \tilde{\mu}_f. \tag{8.99}$$

Now we will see further down that we encounter situations where the space of these intertwining operators is of dimension one. Moreover we will be able to identify an explicit non zero such operator $I^{\text{loc}} : V(\epsilon \times \pi_f) \to \operatorname{Ind}_{\tilde{H}(\mathbb{A}))}^{\tilde{G}(\mathbb{A})} \tilde{\mu}_f$. (See (8.116)) Then we get

$$J(\phi_\mu, \tilde{\mu}_f) \circ \Phi(\pi_f) = \mathcal{L}(\pi_f, \tilde{\mu}_f) I^{\text{loc}} \tag{8.100}$$

and computing the intertwining operator means to compute the number $\mathcal{L}(\pi_f, \tilde{\mu}_f)$. Since all our vector spaces on stage are defined over F we get the rationality result

$$\mathcal{L}(\pi_f, \tilde{\mu}_f) \in F \tag{8.101}$$

We will make this more precise later.

The passage to the to the limit has the technical advantage that we are dealing with representations of $G(\mathbb{A}_f)$ instead of Hecke-modules, for the representations certain issues are easier to handle. Especially it is easier to compute dimensions of spaces of intertwining operators.

We drop the assumption that $\mathcal{S}^H_{K^H_f}$ is compact, and we go back to the case of a fixed level.

In this case we study the extension of $j(x_\infty, \underline{g}_f)$ to the compactification

$$\bar{j}(x_\infty, \underline{g}_f) : \bar{\mathcal{S}}^H_{K^H_f} \to \bar{\mathcal{S}}^G_{K_f}$$

We recall the construction of sheaves with intermediate support conditions (See(6.38)). Let us assume that we can find a Σ such that the image of $\partial(\bar{\mathcal{S}}^H_{K^H_f})$ factors through $\partial_\Sigma(\bar{\mathcal{S}}^G_{K_f})$. Then our homomorphism r together with a choice of a ϕ_μ yields a homomorphism between sheaves (see (6.38))

$$r^!_{\phi_\mu} : \bar{j}(x, \underline{g}_f)^*(i_{\Sigma,*,!}(\tilde{\mathcal{M}})) \to i_!(\tilde{\mathcal{O}}_\mu). \tag{8.102}$$

and hence we get a homomorphism in cohomology jxrone

$$(r^!_{\phi_\mu} \circ \bar{j}((x, \underline{g}_f))^{d_H} : H^{d_H}(\mathcal{S}^G_{K_f}, i_{\Sigma,*,!}(\tilde{\mathcal{M}})) \to H^{d_H}(\mathcal{S}^H_{K^H_f}, i_!(\tilde{\mathcal{O}}_\mu)) \tag{8.103}$$

and now the left hand side is again of the form the composition $J(\phi_\mu, \tilde{\mu}_f) \circ \Phi(\pi_f)$. On the right hand side we can decompose again

$$H^{d_H}(\mathcal{S}^H_{K^H_f}, i_!(\tilde{\mathcal{O}}_\mu)) = H^{d_H}_c(\mathcal{S}^H_{K^H_f}, \tilde{\mathcal{O}}_\mu) \supset \bigoplus_{\tilde{\mu}_f} H^{d_H}_c(\mathcal{S}^H_{K^H_f}, \tilde{\mathcal{O}}_\mu)[\tilde{\mu}_f], \tag{8.104}$$

and the summands are locally free $\mathcal{O}_F$ modules of rank one. If we project to the $\tilde{\mu}_f$ component we get an operator

$$J(\phi_\mu, \tilde{\mu}_f) := P_{\tilde{\mu}_f} \circ (\phi_\mu \circ \bar{j}((x, \underline{g}_f))^{d_H} : H^{d_H}(\mathcal{S}^G_{K_f}, i_{\Sigma,*,!}(\tilde{\mathcal{M}})) \to \operatorname{Ind}_{\tilde{H}(\mathbb{A}))}^{\tilde{G}(\mathbb{A})} \tilde{\mu}_f. \tag{8.105}$$

We are in the same situation as before, we have to find absolutely irreducible modules defined over F and an embedding

$$\Phi(\pi_f) : V_{\epsilon\times\pi_f} \hookrightarrow H^{d_H}(\mathcal{S}^G_{K_f}, i_{\Sigma,*,!}(\tilde{\mathcal{M}}_F)).$$

and then we can try again to investigate the operator $J(\phi_\mu, \tilde{\mu}_f) \circ \Phi(\pi_f)$.

Here we have to discuss a subtle point. Let us consider the case that Σ is the set of all maximal parabolic subgroups, then $H^{d_H}(\mathcal{S}^G_{K_f}, i_{\Sigma,*,!}(\tilde{\mathcal{M}}_F)) = H^{d_H}_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_F)$. We have the exact sequence

$$H^{d_H-1}(\partial(\mathcal{S}^G_{K_f}), \tilde{\mathcal{M}}_F) \to H^{d_H}_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_F) \to H^{d_H}_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_F) \to 0.$$

Sometimes it is easier to construct homomorphisms $\Phi : V(\epsilon\times\pi_f) \hookrightarrow H^{d_H}_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_F)$. and we would like to form again in a canonical way the composition $(r^!_{\phi_\mu} \circ \bar{j}((x, \underline{g}_f))^{d_H} \circ \Phi$.

We have two different instances, when this is possible. We look at the homology, then we have the boundary map

$$H_{d_H}(\mathcal{S}^H_{K^H_f}, \partial(\mathcal{S}^H_{K^H_f}), \underline{F_\mu}) \xrightarrow{\partial} H_{d_H-1}(\partial(\mathcal{S}^H_{K^H_f}), \underline{F_\mu}) \tag{8.106}$$

and from the target we have map

$$H_{d_H-1}(\partial(\mathcal{S}^H_{K^H_f}), \underline{F_\mu}) \xrightarrow{j} H_{d_H-1}(\partial(\mathcal{S}^G_{K_f}), \underline{\mathcal{M}}_F) \tag{8.107}$$

Hence we get a map

$$j \circ \partial_{\tilde{\mu}_f} : H_{d_H}(\mathcal{S}^H_{K^H_f}, \partial(\mathcal{S}^H_{K^H_f}), \underline{F_\mu})[\tilde{\mu}_f] \to H_{d_H-1}(\partial(\mathcal{S}^G_{K_f}), \underline{\mathcal{M}}_F) \tag{8.108}$$

Now we have the following

Proposition 8.2.1. *If the map* $j\circ\partial_{\tilde{\mu}_f} = 0$ *then the homomorphism* $J(\phi_\mu, \pi_f, \tilde{\mu}_f)$ *vanishes on* $H^{d_H-1}(\partial(\mathcal{S}^G_{K_f}), \tilde{\mathcal{M}}_F)$ *and hence it factors over the quotient*

$$J(\phi_\mu, \tilde{\mu}_f) : H^{d_H}_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_F) \to H^{d_H}_!(\mathcal{S}^H_{K^H_f}, \tilde{F}_\mu))[\tilde{\mu}_f]$$

The second instance that may be satisfied is the Manin-Drinfeld principle applies to the exact sequence above, i.e. we have an isotypical decomposition

$$H^{d_H}_{\mathrm{Eis}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_F) \oplus H^{d_H}_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_F). \tag{M_2}$$

Then we may restrict $J(\phi_\mu, \tilde{\mu}_f)$ to the second summand. We get

$$J_!(\phi_\mu, \tilde{\mu}_f) : H^{d_H}_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_F) \longrightarrow \operatorname{Ind}_{\tilde{H}(\mathbb{A}_f)}^{\tilde{G}(\mathbb{A}_f)} \tilde{\mu}_f,$$

and this intertwining operator is defined over F.

We used both arguments already in section 5.6 in a very special case.

8.2.2 Model spaces

We want to find the modules $V_{\epsilon\times\pi_f}$ and operators $\Phi(\pi_f)$. We introduce some abstract concept of the production of cohomology classes and the evaluation of these intertwining operators on these classes. For this purpose we introduce model spaces.

We assume that we have a family of smooth and admissible representations $\{V_{\pi_v}\}$ of $G(\mathbb{Q}_v)$ where v runs over all places. At this moment the V_{π_v} are $\mathbb{C}$-vector spaces. For almost all finite places p the representation $\{V_{\pi_p}\}$ should be an unramified irreducible principal series representation. We assume that V_{π_∞} is an irreducible Harish-Chandra module with non trivial cohomology $H^\bullet(\mathfrak{g}, K_\infty, V_{\pi_\infty} \otimes \mathcal{M}_\mathbb{C}) \neq 0$. We denote $\pi = \pi_\infty \times \pi_f$. Furthermore we assume that we have an intertwining operator of $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$

$$\Psi(\pi) : V_{\pi_\infty} \otimes \bigotimes_p V_{\pi_p} \longrightarrow \mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})). \tag{8.109}$$

At this point a comment is in order. We should think of the spaces V_{π_v} as very specific spaces of $\mathbb{C}$ valued functions on $G(\mathbb{Q}_v)$ on which $G(\mathbb{Q}_v)$ acts by right translations. In all cases known to the author the operator $\Psi(\pi)$ is given by an infinite summation , i.e if $f = \prod f_v \in V_{\pi_\infty} \otimes \bigotimes_p V_{\pi_p}$ then

$$\Psi(\pi)(f)(\underline{g}) = \sum_{a\in H(\mathbb{Q})} f(a\underline{g}) \tag{8.110}$$

where for instance $H/\mathbb{Q}$ is a subgroup or a quotient of a subgroup by another subgroup. In any case it is clear that the construction of these $\Psi(\pi)$ will be a transcendental process.

This induces of course an intertwining operator $\Psi(\pi)$

$$\begin{aligned} H^\bullet(\mathfrak{g}, K_\infty, V_{\pi_\infty} \otimes \mathcal{M}_\mathbb{C}) \otimes \textstyle\bigotimes_p V_{\pi_p} \quad &\xrightarrow{\Psi^\bullet(\pi)} \quad H^\bullet(\mathfrak{g}, K_\infty, \mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})) \otimes \mathcal{M}_\mathbb{C}) \\ &= \quad H^\bullet(\mathcal{S}^G, \tilde{\mathcal{M}}_\mathbb{C}) \end{aligned}$$

We introduce a subspace of $\mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A}))$. We consider the subspace of functions of moderate growth and inside this space we consider the space of functions which are cuspidal along the strata $\partial_P(\mathcal{S}^G)$ for the parabolic subgroups $P \in \Sigma$, i.e. which satisfy

$$\int_{U_P(\mathbb{Q})\backslash U_P(\mathbb{A})} f(\underline{u}\underline{g})d\underline{u} \equiv 0$$

for these parabolic subgroups. Let us call this subspace $\mathcal{C}_\infty^{(\Sigma)}(G(\mathbb{Q})\backslash G(\mathbb{A}))$. We assume that our intertwining operator factors through the subspace of Σ cuspidal functions

$$\Psi(\pi) : V_{\pi_\infty} \otimes \bigotimes_p V_{\pi_p} \longrightarrow \mathcal{C}_\infty^{(\Sigma)}(G(\mathbb{Q})\backslash G(\mathbb{A})). \tag{8.111}$$

We have an action of $\pi_0(G(\mathbb{R}))$ on $H^\bullet(\mathfrak{g}, K_\infty, V_{\pi_\infty} \otimes \mathcal{M}_\mathbb{C})$ let $\epsilon : \pi_0(G(\mathbb{R})) \to \{\pm 1\}$ be a character and let ω_ϵ be a differential form representing an eigenclass $[\omega_\epsilon]$. In [35] we explain how a Hecke character $\tilde{\mu}_f$ extends uniquely to a character $\tilde{\tilde{\mu}}_f^{-1} = \epsilon \times \tilde{\mu}_f : \pi_0(H(\mathbb{R}))H(\mathbb{A}_f) \to \{\pm 1\}$. We have the homomorphism $\pi_0(H(\mathbb{R})) \to \pi_0(G(\mathbb{R}))$ and we require that $\chi_\infty = \epsilon$.

We get a diagram

$$\begin{array}{ccc}
H^{d_H}(\mathfrak{g}, K_\infty, V_{\pi_\infty} \otimes \mathcal{M}_\mathbb{C})(\epsilon) \otimes \bigotimes_p V_{\pi_p} & & \\
\downarrow \Psi^{d_H}(\pi) & & \\
H^{d_H}(\mathfrak{g}, K_\infty, \mathcal{C}_\infty^{(\Sigma)}(G(\mathbb{Q})\backslash G(\mathbb{A})) \otimes \mathcal{M}_\mathbb{C}) & \stackrel{\mathrm{dRh}}{\longrightarrow} & H^{d_H}(\mathcal{S}^G, \tilde{\mathcal{M}} \otimes \mathbb{C}) \\
 & & \uparrow i_\Sigma^{d_H} \otimes \mathbb{C} \\
\mathrm{Ind}_{\tilde{H}(\mathbb{A})}^{\tilde{G}(\mathbb{A})} \tilde{\tilde{\mu}}_f^{-1} \otimes \mathbb{C} & \stackrel{J(\phi_\mu, \tilde{\mu}_f)}{\longleftarrow} & H^{d_H}(\mathcal{S}^G, i_{\Sigma,*,!}(\tilde{\mathcal{M}})) \otimes \mathbb{C}
\end{array}$$

Proposition 8.2.2. *The image of* dRh *is contained in the image of* $i_\Sigma^{d_H} \otimes \mathbb{C}$

Proof. We do not give the proof of this general assertion here, it is a careful analysis using reduction theory and the considerations in ???. We simply mention the case of a compact $\mathcal{S}^H_{K_f^H}$ then we may choose $\Sigma = \emptyset$ to be the set of all maximal parabolic subgroups snd $i_\Sigma^{d_H} \otimes \mathbb{C}$ is the identity and hence the proposition is obvious in this case. On the other hand if Σ is the set of all maximal parabolic subgroups. Then the image of $i_\Sigma^{d_H} \otimes \mathbb{C}$ is the inner cohomology and since in this case the functions in $\mathcal{C}_\infty^{(\Sigma)}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ are cuspidal the assertion follows from Theorem 8.1.1.

□

We put $H^{d_H}_{\Sigma,!}(\mathcal{S}^G, \tilde{\mathcal{M}}) = i_\Sigma^{d_H}(H^{d_H}(\mathcal{S}^G, i_{\Sigma,*,!}(\tilde{\mathcal{M}})))$ and we assume that one of the conditions above is satisfied, i.e. either we can apply proposition 8.2.1 or we have Manin-Drinfeld. We have the action of the $\pi_0(G(\mathbb{R}))$ on $H^\bullet(\mathfrak{g}, K_\infty, V_{\pi_\infty} \otimes \tilde{\mathcal{M}} \otimes \mathbb{C})$, we decompose into eigenspaces according to characters ϵ.

We get an arrow

$$H^\bullet(\mathfrak{g}, K_\infty, V_{\pi_\infty} \otimes \mathcal{M}_\mathbb{C})(\epsilon) \otimes \bigotimes_p V_{\pi_p} \stackrel{J(\phi_\mu, \tilde{\mu}_f) \circ \mathrm{dRh} \circ \Psi^{d_H}(\epsilon \times \pi)}{\longrightarrow} \mathrm{Ind}_{\tilde{H}(\mathbb{A})}^{\tilde{G}(\mathbb{A})} \tilde{\tilde{\mu}}_f^{-1} \otimes \mathbb{C}. \tag{8.112}$$

We choose an element $\omega_\epsilon \in \mathrm{Hom}_{K_\infty}(\Lambda^{d_H}(\mathfrak{g}/\mathfrak{k}), V_{\pi_\infty} \otimes \mathcal{M}_\mathbb{C})[\epsilon]$ then this provides a homomorphism of $G(\mathbb{A}_f)$-modules

$$J(\phi_\mu, \tilde{\mu}_f) \circ \Psi^{d_H}(\omega_\epsilon \times \pi_f)) : \bigotimes_p V_{\pi_p} \to \mathrm{Ind}_{\tilde{H}(\mathbb{A})}^{\tilde{G}(\mathbb{A})} \tilde{\tilde{\mu}}_f^{-1} \otimes \mathbb{C} \tag{8.113}$$

For an element $\psi_f \in \bigotimes_p V_{\pi_p}$ this map is given by the formula

intcyc0

$$J(\phi_\mu, \tilde{\mu}_f) \circ \Psi^{d_H}(\omega_\epsilon \times \pi_f))(\psi_f)(\underline{g}_f) = \int_{\mathcal{S}^H_{K_f^H}} \phi_\mu(j^*(x, \underline{h}_f \underline{g}_f)(\omega_\epsilon \times \psi_f)) \tilde{\mu}_f(\underline{h}_f) d\underline{h}_f, \tag{8.114}$$

here $d\underline{h}_f$ is the invariant measure on $H(\mathbb{A}_f)$ which has value one one K_f^H.

We still have the problem to compute this operator. But now the situation has changed, we can be a little bit more precise in formulating what we mean by computing this operator. The source and the target of the operator

$$J(\phi_\mu,\pi_f,\tilde{\mu}_f,\omega_\epsilon) := J(\phi_\mu,\tilde{\mu}_f)\circ \Psi^{d_H}(\omega_\epsilon\times\pi_f)) \tag{8.115}$$

are restricted tensor products of local representations. So a necessary condition for $J(\phi_\mu,\pi_f,\tilde{\mu}_f,\omega_\epsilon)\neq 0$ is that for all primes p the vector space

$$\mathrm{Hom}_{G(\mathbb{Q}_p)}(V_{\pi_p},\ \mathrm{Ind}_{H(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\tilde{\mu}_p^{-1})\neq 0. \tag{I_p}$$

Therefore we assume that this condition is fulfilled.

If the local condition (I_p) is satisfied for all primes p, we can formulate a much stronger condition

$$\dim\ \mathrm{Hom}_{G(\mathbb{Q}_p)}(V_{\pi_p},\ \mathrm{Ind}_{H(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\tilde{\mu}_p^{-1}) = 1 \tag{I_{pp}}$$

We assume that the representations V_{π_p} are somehow given to us as very concrete representations and (I_{pp}) is true for all primes p. Moreover we assume at each prime p we see some natural choice of a generator

$$I_{\tilde{\mu}_p}^{\mathrm{loc}}\in\ \mathrm{Hom}_{G(\mathbb{Q}_p)}(V_{\pi_p},\ \mathrm{Ind}_{H(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}\tilde{\mu}_p^{-1})$$

(This will be discussed in our examples.) We can define a local intertwining operator

$$I_{\tilde{\mu}_f}^{\mathrm{loc}} = \bigotimes_p I_{\tilde{\mu}_p}^{\mathrm{loc}}\in\ \mathrm{Hom}_{G(\mathbb{A}_f)}(\bigotimes_p V_{\pi_p},\ \mathrm{Ind}_{H(\mathbb{A}_f)}^{G(\mathbb{Q}_p)}\tilde{\mu}_f^{-1}) \tag{8.116}$$

and now we can formulate the following basic question:

The operator $J(\phi_\mu,\pi_f,\tilde{\mu}_f,\omega_\epsilon)$ is a multiple of $I_{\tilde{\mu}_f}^{loc}$ and the problem of computing this intertwining operator comes down to compute a number namely the proportionality factor in

$$J(\phi_\mu,\pi_f,\tilde{\mu}_f,\omega_\epsilon) = \mathcal{L}(\pi_f,\tilde{\mu})\cdot I_{\tilde{\mu}_f}^{\mathrm{loc}}. \qquad \textit{bquest}$$

The general philosophy says that this proportionality factor should be obtained from the data $\pi_f,\tilde{\mu}_f$ for instance it should be essentially a special value of an L-function attached to $\pi_f=\bigotimes_p\pi_p$, We will see in the examples in the section below that this is indeed sometimes the case.

8.2.3 Rationality and integrality results

Now go back to the situation where we fix a finite level K_f, we also assume that Σ is the set of all maximal parabolic subgroups, and we assume that proposition ??? applies (this depends on the choice of μ. see (**??**). Hence we have the map

jxr

$$\phi_\mu\circ\bar{j}((x,\underline{g}_f))^{d_H}: H_!^{d_H}\mathcal{S}_{K_f}^G,\tilde{\mathcal{M}}_F)\to H_!^{d_H}(\mathcal{S}_{K_f^H}^H,\tilde{F}_\mu)) \tag{8.117}$$

We assume that our finite extension $F/\mathbb{Q}$ is large enough so that we get an isotypical decomposition

$$H_!^{d_H}(\mathcal{S}_{K_f}^G, \tilde{\mathcal{M}}_F) = \bigoplus_{\pi_f} H_!^{d_H}(\mathcal{S}_{K_f}^G, \tilde{\mathcal{M}}_F)(\pi_f)$$

where the π_f are isomorphism types of absolutely irreducible modules for the Hecke algebra. Of course we may also require that $F/\mathbb{Q}$ is normal.

On our isotypical subspace we still have the action of $\pi_0(G(\mathbb{R}))$ which commutes with the action of the Hecke algebra. Since this group of connected components is an elementary abelian 2 group we get a decomposition

$$H_!^{d_H}(\mathcal{S}_{K_f}^G, \tilde{\mathcal{M}}_F)(\pi_f) = \bigoplus_{\epsilon} H_!^{d_H}(\mathcal{S}_{K_f}^G, \tilde{\mathcal{M}}_F)(\epsilon \times \pi_f)$$

where ϵ runs over the characters of $\epsilon : \pi_0(G(\mathbb{R})) \to \{\pm 1\}$.

We intersect $H_!^{d_H}(\mathcal{S}_{K_f}^G, \tilde{\mathcal{M}})_F(\pi_f)$ with the integral cohomology $H^{d_H}(\mathcal{S}_{K_f}^G, \tilde{\mathcal{M}}_{\mathcal{O}_F})$ and get the submodule $H_!^{d_H}(\mathcal{S}_{K_f}^G, \tilde{\mathcal{M}}_F)(\pi_f)_{\text{ int}} \subset H^{d_H}(\mathcal{S}_{K_f}^G, \tilde{\mathcal{M}})_F)_{\text{ int}}$. We have seen in ??? that we may alternatively define the submodule

$$H^{d_H}(\mathcal{S}_{K_f}^G, \tilde{\mathcal{M}}_{\mathcal{O}_F})(\pi_f)_{\text{ int},!} \subset H^{d_H}(\mathcal{S}_{K_f}^G, (\tilde{\mathcal{M}}_{\mathcal{O}_F})_{\text{ int}} \tag{8.118}$$

and we recall that the quotient

$$H^{d_H}(\mathcal{S}_{K_f}^G, \tilde{\mathcal{M}}_{\mathcal{O}_F})(\pi_f)_{\text{ int},!}/H_!^{d_H}(\mathcal{S}_{K_f}^G, \tilde{\mathcal{M}}_{\mathcal{O}_F})(\pi_f)_{\text{ int}} = \mathcal{T}(\pi_f) \tag{8.119}$$

is a torsion module which is isomorphic to a sub quotient of the torsion module of $H^{d_H}(\partial(\mathcal{S}_{K_f}^G), \tilde{\mathcal{M}}_{\mathcal{O}_F})$.

The isomorphism type $(\epsilon \times \pi_f)$ occurs with a non zero multiplicity $m(\epsilon \times \pi_f)$ in or in $H_!^{d_H}(\mathcal{S}_{K_f}^G, \tilde{\mathcal{M}}_F)$. We assume that $\tilde{\mathcal{M}}_F$ belongs to a rational system of coefficients, then ${}^\sigma(\epsilon \times \pi_f)$ occurs with the same multiplicity in $H_!^{d_H}(\mathcal{S}_{K_f}^G, {}^\sigma \tilde{\mathcal{M}}_F)$.

Rationality of the model space

We also assume that all the local components V_{π_p} of our model space are also defined over F. For a finite place p this means that V_{π_p} is a vector space over F with an action of $G(\mathbb{Q}_p)$. If we choose an open compact subgroup $K_p \subset G(\mathbb{Q}_p)$ then $V_{\pi_p}^{K_p}$ is a finite dimensional F vector space with an action of the Hecke algebra $\mathcal{H}_{K_p}$ on it. This action is absolutely irreducible if π_p is absolutely irreducible. If our underlying flat group scheme $\mathcal{G}/\mathbb{Z}$ is reductive at the prime p and $K_p = \mathcal{G}(\mathbb{Z}_p)$ (some people say that K_p is hyperspecial) then $V_{\pi_p}^{K_p}$ is of dimension one (or zero). The Hecke algebra module is given by a homomorphism $h \mapsto \pi_p(h)$ from $\mathcal{H}_{K_p} \to F$. We say that π_p is spherical if $\dim V_{\pi_p}^{K_p} = 1$ and K_p is hyperspecial. We require that π_p is spherical for almost all primes p and we also require that for all spherical $V_{\pi_p}^{K_p}$ we have chosen a generator $h_p^{(0)} \in V_{\pi_p}^{K_p}$. Then we can define

$$V_{\pi_f} = \bigotimes{}' V_{\pi_p} \tag{8.120}$$

where the restricted tensor product means that at almost all components p the factor is $h_p^{(0)}$.

Furthermore we assume that our local model spaces come as rational systems of representations, i.e. we have the families of σ-linear isomorphism $\Phi_\sigma^{(p)}: V_{\pi_p} \xrightarrow{\sim} V_{{}^\sigma\pi_p}$ satisfying the cocycle condition. Of course we also require that

$$\Phi_\sigma^{(p)}(h_p^{(0)}) = h_p^{(0)} \tag{8.121}$$

and then get the rational system of model spaces

$$\Phi_\sigma^{\mathrm{mod}} : V_{\pi_f} \xrightarrow{\sim} V_{{}^\sigma\pi_f} \tag{8.122}$$

We discuss the concept of rationality for V_{π_∞}. Our group $G/\mathbb{Q}$ is defined over $\mathbb{Q}$ we choose the Cartan-involution Θ which provides K_∞ also defined over $\mathbb{Q}$. Hence the Lie algebras $\mathfrak{g}$ and $\mathfrak{k}$ are defined over $\mathbb{Q}$, i.e. they are $\mathbb{Q}$-vector spaces. Our module $\mathcal{M}$ is obtained from an absolutely irreducible highest weight representation $\rho_\lambda : G \times_\mathbb{Q} F \to \mathrm{Gl}(\mathcal{M}_F)$, If we choose a basis $\{\dots, f_i, \dots\}$ and $a \in G(\mathbb{Q})$ then $\rho_\lambda(a) f_i = \sum a_{i,j} f_j$ with $a_{i,j} \in F$. The same applies for the action of $\mathfrak{g}$ on X_{π_∞}, this module has a countable basis $\{\dots, g_i, \dots\}$ and for $X \in \mathfrak{g}$ we get again $X f_i = \sum a_{i,j} f_j, a_{i,j} \in F$ and where the sum is finite, i.e. only finitely many $a_{i,j} \neq 0$.

We assume that $F \subset \mathbb{C}$, and assume that we have the intertwining operator

$$\Psi^{d_H}(\pi)(\omega_\epsilon) : (\bigotimes_p V_{\pi_p}) \otimes \mathbb{C}) \to H_!^{d_H}(\mathcal{S}^G, \tilde{\mathcal{M}}_F)(\epsilon \times \pi_f) \otimes \mathbb{C} \tag{8.123}$$

these are isomorphisms over $\mathbb{C}$ between absolutely irreducible $G(\mathbb{A}_f)$ modules which are defined over F. Hence we can find numbers (the periods) $\Omega(\epsilon \times \pi_f) \in \mathbb{C}^\times$

periodrat

$$\Phi^{d_H}(\pi)(\omega_\epsilon) = \frac{\Psi^{d_H}(\pi)(\omega_\epsilon)}{\Omega(\epsilon \times \pi_f)} : \bigotimes_p V_{\pi_p} \xrightarrow{\sim} H_!^{d_H}(\mathcal{S}^G, \tilde{\mathcal{M}}_F)(\epsilon \times \pi_f) \tag{8.124}$$

is an isomorphism over F.

But we can do better. We may also assume that after fixing a level we have an integral structure on our model space, i.e we have chosen lattices $V_{\pi_p, \mathcal{O}_F}^{K_p} \subset V_{\pi_p}^{K_p}$. For almost all p this lattice is of rank one and $V_{\pi_f, \mathcal{O}_F}^{K_f} = \bigotimes_p V_{\pi_p, \mathcal{O}_F}^{K_p}$ is a (locally) free module of finite rank. We require that our periods satisfy

$$\frac{\Psi^{d_H}(\pi)(\omega_\epsilon)}{\Omega(\epsilon \times \pi_f)} : (\bigotimes_p V_{\pi_p, \mathcal{O}_F}^{K_p}) \to H_!^{d_H}(\mathcal{S}_{K_f}^G, \tilde{\mathcal{M}}_{\mathcal{O}_F})(\epsilon \times \pi_f)_{\text{ int}} \tag{8.125}$$

and we require that this choice of periods is optimal, i.e.

if $a \in F$ and

$$a \frac{\Psi^{d_H}(\pi)(\omega_\epsilon)}{\Omega(\epsilon \times \pi_f)}((\bigotimes_p V_{\pi_p, \mathcal{O}_F}^{K_p}) \subset H_!^{d_H}(\mathcal{S}_{K_f}^G, \tilde{\mathcal{M}}_{\mathcal{O}_F})(\epsilon \times \pi_f)_{\text{ int}} \tag{8.126}$$

then $a \in \mathcal{O}_F$.

This pins down the periods up to an element in $\mathcal{O}_F^\times$.

If now proposition 8.2.1 applies we can then we can form the composition

$$J(\phi_\mu, \chi_f) \circ \frac{\Psi^{d_H}(\pi)(\omega_\epsilon)}{\Omega(\epsilon \times \pi_f)} : \bigotimes_p V_{\pi_p, \mathcal{O}_F}^{K_p} \to \operatorname{Ind}_{\tilde{H}(\mathbb{A})}^{\tilde{G}(\mathbb{A})} \tilde{\mu}_f^{-1}.$$

Now we assume that the condition (I_{pp}) is satisfied and we assume that our local intertwining operators I_p^{loc} are defined over F.. We define as above $I_{\tilde{\mu}_f}^{\mathrm{loc}} = \otimes I_{\tilde{\mu}_p}$ and again we get a formula

$$J(\phi_\mu, \tilde{\mu}_f) \circ \frac{\Psi^{d_H}(\pi)(\omega_\epsilon)}{\Omega(\epsilon \times \pi_f)} = \mathcal{L}(\pi \otimes \chi, \mu) I_{\tilde{\mu}_f}^{\mathrm{loc}} \tag{8.127}$$

Of course there is still the unknown quantity $\mathcal{L}(\pi \otimes \chi, \tilde{\mu})$ but we can say

Proposition 8.2.3. *If proposition 8.2.1 applies or if we have Manin-Drinfeld then* $\mathcal{L}(\pi \otimes \chi, \tilde{\mu}) \in F$

But we want to do better. On the left hand side we have the integral structure and if we evaluate at an adapted argument $\underline{g}_f$, i.e. $\underline{g}_f$ satisfies (8.89) then we get for $\psi_f \in V_{\pi_f, \mathcal{O}_F}^{K_f}$ Spv1

$$P_{\chi_f} \circ (r^!_{\phi_\mu} \circ \bar{j}((x, \underline{g}_f))^{d_H} \circ \frac{\Psi^{d_H}(\pi)(\omega_\epsilon)}{\Omega(\epsilon \times \pi_f)}(\psi_f) = \mathcal{L}(\pi \otimes \chi, \mu) I_{\tilde{\mu}_f}^{\mathrm{loc}}(\psi_f)(\underline{g}_f) \tag{8.128}$$

If we can apply proposition 8.2.1 then the left hand side is an integer in $\mathcal{O}_F$ hence we know that the right hand side $\mathcal{L}(\pi \otimes \chi, \mu) I_{\chi_f}^{\mathrm{loc}}(\psi_f)(\underline{g}_f)$ is also an integer. To get information about the denominator of $\mathcal{L}(\pi \otimes \tilde{\mu}_f, \mu)$ we have to optimize the numerator of $I_{\tilde{\mu}_f}^{\mathrm{loc}}(\psi_f)(\underline{g}_f)$.

We have to choose $\psi_f \in \bigotimes_p V_{\mathcal{O}_F}^{K_p}$, and we choose $\underline{g}_f$ such that $K_f^H \underline{g}_f K_f = \underline{g}_f K_f)$). The first choice provides an integral cohomology class in $H^{d_H}(\mathcal{S}_{K_f}^G, \tilde{\mathcal{M}}_{\mathcal{O}_F})(\pi_f)$. But this class is not necessarily the image of an integral class under $r_{\Sigma,!}$ this will be the case if we multiply it with $\Delta(\pi_f)$. Once we have done this we get that Spv2

$$j((x, \underline{g}_f), r_{\lambda,\mu})((\frac{\Phi^{d_H}(\omega_\epsilon)}{\Omega(\pi_f, \omega_\epsilon)} \times \Delta(\pi_f)\psi_f)) = \Delta(\pi_f)\mathcal{L}(\pi \otimes \tilde{\mu}) I_{\tilde{\mu}_f}^{\mathrm{loc}}(\psi_f)(\underline{g}_f) c_{\tilde{\mu}_f} \tag{8.129}$$

is a number in $\mathcal{O}_F$.

Then we have to optimise the choice of $\underline{g}_f$, this means that we have to keep the numerator of $I_{\tilde{\mu}_f}^{\mathrm{loc}}(\psi_f)(\underline{g}_f)$ small. Then we get an integrality result for the L-value.

Galois action

Then it is clear that for any $\sigma \in \operatorname{Gal}(F/\mathbb{Q})$ we can define the conjugate $(\mathfrak{g}, K_\infty)-$ module $V_{{}^\sigma\pi_\infty}$ and the conjugate $\mathcal{G}$ module ${}^\sigma\mathcal{M}$. In this sense we can say that the system $(V_{{}^\sigma\pi_\infty}, {}^\sigma\mathcal{M})$ is a rational system over $\mathbb{Q}$. We also get the system of conjugate cohomology groups

$$\{\dots, H_!^{d_H}(\mathcal{S}^G_{K_f}, {}^\sigma\tilde{\mathcal{M}}_F)({}^\sigma\epsilon \times {}^\sigma\pi_f), \dots\}_{\sigma\in \mathrm{Gal}(F/\mathbb{Q})}.$$

We made the assumption that the model space $V(\pi)$ and the embedding $|Phi(\pi)$ are somehow canonically given to us (see the next section) and we assume that these $\Phi(\pi)$ behave nicely under the action of the Galois group. More precisely we assume that the following diagram commutes

$$\begin{array}{llll} \Psi^{d_H}(\pi)(\omega_\epsilon): & (\bigotimes_p V_{\pi_p})\otimes\mathbb{C}) \to & H_!^{d_H}(\mathcal{S}^G, \tilde{\mathcal{M}}_F)(\epsilon\times\pi_f)\otimes\mathbb{C} & \\ & \downarrow \Phi_\sigma^{\mathrm{mod}}\otimes 1 & \downarrow \Phi_\sigma^\bullet\otimes 1 & (8.130) \\ \Psi^{d_H}(\pi)({}^\sigma\omega_\epsilon): & (\bigotimes_p V_{{}^\sigma\pi_p})\otimes\mathbb{C}) \to & H_!^{d_H}(\mathcal{S}^G, \tilde{\mathcal{M}}_F)({}^\sigma\epsilon\times{}^\sigma\pi_f)\otimes\mathbb{C} & \end{array}$$

If we now have chosen $\Omega(\epsilon\times\pi_f)$ then we choose $\Omega({}^\sigma\epsilon\times{}^\sigma\pi_f)$ such that the following diagram is commutative

$$\begin{array}{llll} \dfrac{\Psi^{d_H}(\pi)(\omega_\epsilon)}{\Omega(\epsilon\times\pi_f)}: & \bigotimes'_p V_{\pi_p} \to & H_!^{d_H}(\mathcal{S}^G, \tilde{\mathcal{M}}_F)(\epsilon\times\pi_f) & \\ & \downarrow \Phi_\sigma^{\mathrm{mod}} t & \downarrow \Phi_\sigma^\bullet & (8.131) \\ \dfrac{\Psi^{d_H}({}^\sigma\pi)(\omega_\epsilon)}{\Omega({}^\sigma\epsilon\times{}^\sigma\pi_f)}: & \bigotimes'_p V_{{}^\sigma\pi_p} \to & H_!^{d_H}(\mathcal{S}^G, \tilde{\mathcal{M}}_F)({}^\sigma\epsilon\times{}^\sigma\pi_f) & \end{array}$$

8.3 The special case Gl_2/F_0

Let $F_0/\mathbb{Q}$ be an algebraic number field, we consider the algebraic group $G/\mathbb{Q} = R_{F_0/\mathbb{Q}}(\mathrm{Gl}_2/F_0)$. We embed $\mathbb{G}_m\times\mathbb{G}_m = T_0$ into Gl_2/F_0 by

$$(t_1, t_2) \mapsto \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$$

the diagonal $\mathbb{G}_m \subset \mathbb{G}_m\times\mathbb{G}_m$ maps to the center C_0/F_0. Let $B_0 \supset T_0$ be the standard Borel subgroup of upper triangular matrices and U_0 its unipotent radical. Then $T/\mathbb{Q} = R_{F_0/\mathbb{Q}}(T_0), B/\mathbb{Q} = R_{F_0/\mathbb{Q}}(B_0),\ U/\mathbb{Q} = R_{F_0/\mathbb{Q}}(U_0)$ and $C/\mathbb{Q} = R_{F_0/\mathbb{Q}}(C_0)$.

We want to apply our above considerations to the following two cases

M1)

$$H = T = R_{F_0/\mathbb{Q}}(T_0) \subset R_{F_0/\mathbb{Q}}(\mathrm{Gl}_2/F_0)$$

and

M2)

$$H = R_{F_0/\mathbb{Q}}(\mathrm{Gl}_2/F_0) \subset G = R_{F_0/\mathbb{Q}}(\mathrm{Gl}_2/F_0)\times R_{F_0/\mathbb{Q}}(\mathrm{Gl}_2/F_0)$$

where the embedding is the diagonal one.

8.3.1 The spaces

Let Σ be the set of embeddings $\iota : F_0 \hookrightarrow \mathbb{C}$, on this set we have the action of complex conjugation $\mathbf{c}$. The set of embeddings $\iota : F_0 \to \mathbb{R}$ is the set of elements the fixed under conjugation, this is also the set of real places. The other embeddings come in pairs $\iota, \mathbf{c}\iota$. Let S_∞ be the set of equivalence classes under this action, then S_∞ is the set of archimedian places of F_0 and

$$F_0 \otimes \mathbb{R} = \prod_{v \in S_\infty} F_v = \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$$

of course $F_v = \mathbb{R}$ resp. $\mathbb{C}$, if v is a real (resp. complex) place, hence $G(\mathbb{R}) = \mathrm{Gl}_2(F \otimes \mathbb{R}) = \prod_{v \in S_\infty} \mathrm{Gl}_2(F_v)$. Let

$$K_v = \mathrm{SO}(2) \times C'(F_v)(\text{ resp. } U(2)) \times C^{?}{'}(F_v)^{(0)} \subset \mathrm{Gl}_2(F_v)$$

be the ess. maximal compact subgroups (See (4.1.2)). We completely neglegt the contributions from the center. Our symmetric spaces

$$\mathrm{X} = \prod_{v \in S_\infty} \mathrm{Gl}_2(K_v)/K_v = \prod_{v \in S_\infty} X_v. \tag{8.132}$$

Let $\mathcal{O}_{F_0}$ be the ring of integers in F_0, let $\hat{\mathcal{O}}_{F_0} \subset \mathbb{A}_{F_0}$ be the ring of integral adeles, we consider the group scheme $\mathcal{G}/\mathbb{Z} = R_{\mathcal{O}_{F,0}/\mathbb{Z}}(\mathrm{Gl}_2/\mathcal{O}_{F,0})$. We choose an open compact subgroup $K_f \subset \mathcal{G}(\hat{\mathcal{O}}_{F,0})$. With these choices we define again

$$\mathcal{S}^G_{K_f} = G(\mathbb{Q})\backslash \mathrm{X} \times G(\mathbb{A}_Q)/K_f \tag{8.133}$$

The Lie-algebra $\mathfrak{g} \otimes \mathbb{R}$ of $G \times_{\mathbb{Q}} \mathbb{R}$ is the direct sum $\mathfrak{g} \otimes \mathbb{R} = \oplus_{v \in S_\infty} \mathfrak{g}_v$, our standard Cartan involution is the product of the involution Θ_v. (See4.1.2). Then we get the corresponding Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \text{ where } \mathfrak{k} = \oplus_{v \in S_\infty} \mathfrak{k}_v \text{ , } \mathfrak{p} = \oplus_{v \in S_\infty} \mathfrak{p}_v. \tag{8.134}$$

8.3.2 The highest weight modules, the sheaves and their cohomology groups.

Let $\bar{\mathbb{Q}}$ the subfield of algebraic numbers of $\mathbb{C}$, then the maps $\iota \in \Sigma$ factor through $\bar{\mathbb{Q}}$. Let $F_1 \subset \bar{\mathbb{Q}}$ be the subfield generated by the $\iota(F_0)$, this is of course the normal closure of F_0 (see also section 6.2.2 the field F_1 here is the fiield F_0 there). In (4.1.1) we gave a description of the character module of T_0/F, it is

$$X^*(T \times \bar{\mathbb{Q}}) = \mathrm{Hom}(T \times \bar{\mathbb{Q}}, \mathbb{G}_m) = \prod_{\iota_0 : F_0 \to \bar{\mathbb{Q}}} X^*(T_0 \times_{F_0,\iota} \bar{\mathbb{Q}}) \tag{8.135}$$

and hence an element $\underline{\lambda} \in X^*(T \times \bar{\mathbb{Q}})$ is an array (see section 4.1.1)

$$\underline{\lambda} = \{\dots, n_\iota \gamma + d_\iota \det, \dots\}_{\iota \in \Sigma}. \tag{8.136}$$

We call $\underline{\lambda}$ dominant if $n_\iota \geq 0$ for all $\iota \in \Sigma$. The Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on $X^*(T \times \bar{\mathbb{Q}}, \mathbb{G}_m)$, this action factors over $\mathrm{Gal}(F_1/\mathbb{Q})$.Let $F[\underline{\lambda}] \subset F_1$ the be stabilizer field, i.e. $\mathrm{Gal}(\mathbb{Q}/F[\underline{\lambda}])$ is the stabilizer of $\underline{\lambda}$.

We can obviously extend our constructions in (4.1.1) to this situation and construct a $\mathcal{G}$ module $\mathcal{M}^\flat_{\underline{\lambda}}$ of highest weight $\underline{\lambda}$. This is a free $\mathcal{O}_{F_{\underline{\lambda}}}$ module, for the extension $\mathcal{M}^\flat_{\underline{\lambda}} \otimes \mathcal{O}_{\tilde{F}}$ we have a canonical isomorphism KueM

$$\mathcal{M}^\flat_{\underline{\lambda}} \otimes \mathcal{O}_{\mathcal{O}_{F[\underline{\lambda}]}} \xrightarrow{\sim} \bigotimes_{\iota\in\Sigma} \mathcal{M}^\flat_{n_\iota\gamma+d_\iota \det} \tag{8.137}$$

The tensor product $\mathcal{M}^\flat_{\underline{\lambda}} \otimes_{F[\underline{\lambda}]} F$. is of course isomorphic to the standard highest weight module $\mathcal{M}_{\underline{\lambda},F}$. We recall the explicit realization

$$\mathcal{M}^\flat_{n_\iota\gamma+d_\iota \det} = \{P(X_\iota, Y_\iota) = \sum_{m=0}^{n_\iota} a_\iota \binom{n_\iota}{m} X_\iota^{n_\iota - m} Y_\iota^m \mid a_\iota \in \mathcal{O}_F\}. \tag{8.138}$$

We apply the considerations of section (6.2) to these modules. We get get sheaves $\tilde{\mathcal{M}}^\flat_{\underline{\lambda}}$ on $\mathcal{S}^G_{K_f}$. If the field F_0 has at least one real place the sheaves $\tilde{\mathcal{M}}^\flat_{\underline{\lambda}}$ are zero unless all the coefficients d_ι are all equal, i.e. $d_\iota = d$. (See [35]). Therefore we require that this is always so. The parameter d is actually rather irrelevant it only serves to fulfill the parity condition. If the n_ι are even then we may choose $d = 0$. We also require that $\underline{\lambda}$ is unitary, this means that for all the complex embeddings we have $n_\iota = n_{\mathbf{c}\circ\iota}$ (See Thm. 4.1.2.). Of course, if F is totally real then $\underline{\lambda}$ is always unitary.

We want to investigate the cohomology groups and the fundamental exact sequence

$$\to H^\bullet_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}^\flat_{\underline{\lambda}}) \xrightarrow{j_c} H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}^\flat_{\underline{\lambda}}) \xrightarrow{r} H^\bullet(\overset{\bullet}{\mathcal{N}}\, \mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}^\flat_{\underline{\lambda}}) \to \tag{8.139}$$

as modules for the Hecke-algebra in this special case. As usual we also introduce the inner "cohomology" $H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}^\flat_{\underline{\lambda}}) = \ker(r) = \mathrm{Im}(j_c)$.

The Galois group acts on the set of dominant highest weights and is easy to see that the $\tilde{\mathcal{M}}_{\sigma\underline{\lambda}}$ form a rational system of coefficient systems, the Φ_σ simply act on the a_i in (8.138).

For the following see [35]. We have two cases - If $\lambda^{(1)} \neq 0$ In this case it is easy to see that

$$H^\bullet_{cusp}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}^\flat_{\underline{\lambda}}) = H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}^\flat_{\underline{\lambda}}) \tag{8.140}$$

If $\lambda^{(1)} = 0$ then $\mathcal{M}^\flat_{\underline{\lambda}}$ is one dimensional, all the d_ι are equal to an even number d. The restriction to $G^{(1)}$ is trivial. Each algebraic Hecke character $\chi_\infty \times \chi_f$ of

$$\text{type}\chi_f = \underline{\lambda} = \{\dots, 0 + \frac{d}{2}\det, \dots\}_{\iota\in\Sigma}$$

yields an embedding $\mathbb{C}[\chi_\infty \times \chi_f] \hookrightarrow \mathcal{C}_\infty(G(\mathbb{Q})\backslash G(\mathbb{A})K_\infty K_f)$. and hence a homomorphism

$$H^\bullet(\mathfrak{g}/\mathfrak{k}, \mathbb{C}[\chi_\infty \times \chi_f] \otimes \tilde{\mathcal{M}}^\flat_{\underline{\lambda}}) \to H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\underline{\lambda}}) \tag{8.141}$$

The direct sum of images is a complete (different eigenvalues of Hecke) submodule and hence the Manin-Drinfeld principle gives a decomposition

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$$\bigoplus_{\chi_\infty \times \chi_f} H^\bullet(\mathfrak{g}/\mathfrak{k}, \mathbb{C} \otimes \tilde{\mathcal{M}}^\flat_{\underline{\lambda}}) \oplus H^\bullet_{\mathrm{cusp}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) = H^\bullet_{(2)}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\underline{\lambda}}) \tag{8.142}$$

The classes $\bigoplus_{\chi_\infty \times \chi_f} H^\bullet(\mathfrak{g}/\mathfrak{k}, \mathbb{C} \otimes \tilde{\mathcal{M}}^\flat_{\underline{\lambda}})$ are residual Eisenstein classes, the form a submodule of total Eisenstein cohomology and hence we get a splitiing t two decompositions deco-eis-cusp

$$\begin{aligned} H^\bullet_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{Q}) &= H^\bullet_{\mathrm{cusp}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{Q}) \oplus H^\bullet_{c,\mathrm{Eis}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{Q}) \\ H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{Q}) &= H^\bullet_{\mathrm{cusp}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{Q}) \oplus H^\bullet_{\mathrm{Eis}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{Q}) \end{aligned} \tag{8.143}$$

Of course this may not give a decomposition over $\mathbb{Z}$, the reason for this are the "denominators of the Eisenstein-classes". We get a decomposition into saturated modules

$$\begin{aligned} H^\bullet_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}^\flat_{\underline{\lambda}})_{\mathrm{int}} &\supset H^\bullet_{\mathrm{cusp}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}^\flat_{\underline{\lambda}})_{\mathrm{int}} \oplus H^\bullet_{c,\mathrm{Eis}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}^\flat_{\underline{\lambda}})_{\mathrm{int}} \\ H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}^\flat_{\underline{\lambda}})_{\mathrm{int}} &\supset H^\bullet_{\mathrm{cusp}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}^\flat_{\underline{\lambda}})_{\mathrm{int}} \oplus H^\bullet_{\mathrm{Eis}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}^\flat_{\underline{\lambda}})_{\mathrm{int}} \end{aligned} \tag{8.144}$$

The quotients *left side/right side* are related to the denominators of Eisenstein classes. We have studied them in a very special case in Chapter 5, but we are convinced that theorem 5.1.2 is a special case of a much more general theorem (See also Chapter 9).

8.3.3 The modular symbols

We apply the strategy outlined in section 8.2.2 for the case M1). The reader will realise that carrying out the different steps is quite elaborate. The final result is the Theorem 8.3.1 and Corollary 8.3.2. In Chapter 5 we discuss a special case of these computation where a lot of simplifying assumptions are fulfilled. But in essence the computation is the same. For the archimedian component of our pin point we choose $x_\infty = \prod_v K_v$ (resp. $x_\infty = \prod_v (K_v \times K_v)$).

In the next step we compute the restriction of $\mathcal{M}_{\underline{\lambda}}$ to H, we do it separately for the two cases. The action of the torus $H/\mathbb{Q} = T/\mathbb{Q}$ on $\mathcal{M}_{\underline{\lambda}}$ is semi simple and it is clear from (8.137) that the restriction to $T/\mathbb{Q}$ decomposes into free rank one modules restr

$$\mathcal{M}^\flat_{\underline{\lambda}} = \bigoplus_{\underline{\mu}} \mathcal{O}_{F_{\underline{\mu}}} e^\flat_{\underline{\mu}} \tag{8.145}$$

where $\underline{\mu} = \{\dots, m_\iota \gamma_\iota + d \det, \dots\}$ where $-n_\iota \leq m_\iota \leq n_\iota, n_\iota \equiv m_\iota \mod 2$ and where

$$e^\flat_{\underline{\mu}} = \prod_\iota \binom{n_\iota}{m_\iota} X_\iota^{n_\iota - m_\iota} Y_\iota^{m_\iota}$$

is a generator. The homomorphism $\phi_{\underline{\mu}}$ will be the projection to the summand $\mathcal{O}e^\flat_{\underline{\mu}}$. The space

$$S^T_{K^T_f} = T(\mathbb{Q}) \backslash (T(\mathbb{R})/K^T_\infty \times C(\mathbb{R})^{(0)}) \times T(\mathbb{A}_f)/K^T_f,$$

it has several connected components, each of these components is isomorphic to $(S^1)^{r_1+r_2-1} \times R^{\times}_{>0}$, the dimension is $d_T = r_1 + r_2$. Our data so far provide a homomorphism evST

$$\phi^!_\mu \circ j(x_\infty, \underline{g}_f) : H^{r_1+r_2}_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}^\flat_{\underline{\lambda}}) \to H^{r_1+r_2}_c(\mathcal{S}^T_{K^T_f}, \tilde{\mathcal{O}}_{F[\underline{\lambda}]} e_\mu) \tag{8.146}$$

This homomorphism factors over the quotient $H^{r_1+r_2}_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}^\flat_{\underline{\lambda}})$ if proposition 8.2.1 applies. If this proposition does not apply, then we have Manin-Drinfeld instead then we only get a homomorphism

$$\phi^!_{\underline{\mu}} \circ j(x_\infty, \underline{g}_f) : H^{r_1+r_2}_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes F) \to H^{r_1+r_2}_c(\mathcal{S}^T_{K^T_f}, F[\underline{\lambda}] e_{\underline{\mu}}). \tag{8.147}$$

Of course we want that the cohomology $H^\bullet_c(S^T_{K^T_f}, \tilde{\mathcal{O}}_{F_\mu} e_{\underline{\mu}}) \neq 0$. A necessary condition for this to be the case, is that $\underline{\mu}$ is *pure of weight* $w(\underline{\mu})$. This means that for the real embeddings all the numbers $m_\iota = w(\underline{\mu})$ and for the pairs of complex embeddings we have $m_\iota + m_{c\iota} = 2w(\underline{\mu})$. If our field F_1 has a real embedding then we even know that always $m_\iota = m_{c\iota} = w(\underline{\mu})$. for all embeddings.

The module $H^{r_1+r_2}_{\text{cusp}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\underline{\lambda}})$ is semi simple, we can find a finite (normal over $\mathbb{Q}$) extension F/F_λ such that we get an isotypical decomposition

$$H^{r_1+r_2}_{\text{cusp}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\underline{\lambda}} \otimes F) = \bigoplus_{\underline{\epsilon}\times\pi_f} H^{r_1+r_2}_{\text{cusp}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\underline{\lambda}} \otimes F)(\underline{\epsilon} \times \pi_f)$$

where $\epsilon \times \pi_f$ is an isomorphism class of a (finite dimensional) F-vector space with an irreducible action of $\pi_0(G(\mathbb{R})) \times \mathcal{H}$ on it.

For the integral cohomology we get a decomposition up to isogeny

$$H^{r_1+r_2}_{\text{cusp}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}^\flat_{\underline{\lambda}} \otimes \mathcal{O}_F) \supset \bigoplus_{\underline{\epsilon}\times\pi_f} H^{r_1+r_2}_{\text{cusp}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}^\flat_{\underline{\lambda}} \otimes \mathcal{O}_F)(\underline{\epsilon} \times \pi_f).$$

We can also decompose the right hand side of (8.146). We have seen in (6.3.8) that we have a decomposition

$$H^{r_1+r_2}_!(\mathcal{S}^T_{K^T_f}, \tilde{F}_\lambda e_\mu) = \bigoplus_{\tilde{\mu}_f:\text{type}(\tilde{\mu}_f)=\mu} F e_{\tilde{\mu}_f}$$

and over the integers this gives us

$$H^{r_1+r_2}_!(\mathcal{S}^T_{K^T_f}, \tilde{\mathcal{O}}_{F_\lambda} e_\mu) \supset \bigoplus_{\phi:\text{type}(\phi)=\mu} \mathcal{O}_F e_{\tilde{\mu}_f}$$

and this gives us the projection map $P_{\tilde{\mu}_f} : H^{r_1+r_2}_!(\mathcal{S}^T_{K^T_f}, \tilde{\mathcal{O}}_{F_\lambda} e_\mu) \to \mathcal{O}_F \tilde{e}_{\tilde{\mu}_f}$. Hence we see: If we restrict to the $\epsilon \times \pi_f$ component on the left hand side and project to the $\tilde{\mu}_f$ component on the right hand side, then we get a homomorphism PiChi

$$P_{\tilde{\mu}_f} \phi_\mu \circ j(x_\infty, \underline{g}_f)(\epsilon \times \pi_f) : H^{r_1+r_2}_{\text{cusp}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}^\flat_{\underline{\lambda}} \otimes \mathcal{O}_F)(\underline{\epsilon} \times \pi_f) \to \mathcal{O}_F \tilde{e}_{\tilde{\mu}_f} \tag{8.148}$$

Recall that this only works if proposition 8.2.1 applies. But if we tensor our coefficient system with F then we can invoke the Manin-Drinfeld principle. We can change the level and go to the limit over smaller and smaller K_f. We get a $\pi_0(G(\mathbb{R})) \times G(\mathbb{A}_f)-$ module homomorphism

$$J(\phi_\mu, \underline{\epsilon} \times \pi_f, \tilde{\mu}_f) : H_!^{r_1+r_2}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes F)(\underline{\epsilon} \times \pi_f) \to \ \mathrm{Ind}_{T(\mathbb{A}_f)}^{G(\mathbb{A}_f)} \tilde{\mu}_f^{-1} \tag{8.149}$$

which is defined by

$$\xi \mapsto \{\underline{g}_f \mapsto P_\phi \circ \phi_\mu \circ j(x_\infty, \underline{g}_f)(\epsilon \times \pi_f)\}(\xi). \tag{8.150}$$

This is now the situation where we can try the strategy outlined in section 8.2.2. In the following section we find a model space $V_{\pi_f} = \prod_{\mathfrak{p}} V_{\pi_f}$ together with an isomorphism $\Phi_\epsilon : V_\epsilon(\pi_f) \xrightarrow{\sim} H_!^{r_1+r_2}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes F)(\underline{\epsilon} \times \pi_f)$. We get the composite

$$J(\phi_\mu, \underline{\epsilon} \times \pi_f, \tilde{\mu}_f \circ \Phi : V_\epsilon(\pi_f) \to \ \mathrm{Ind}_{T(\mathbb{A}_f)}^{G(\mathbb{A}_f)} \tilde{\mu}_f^{-1}. \tag{8.151}$$

Now the $G(\mathbb{A}_f)-$ modules are product of local $G(F_{0,\mathfrak{p}})-$ modules and we find some "natural" local operators $I^{\mathrm{loc}}_{\mathfrak{p}} : V_{\pi_{\mathfrak{p}}} \hookrightarrow \ \mathrm{Ind}_{T_0(F_{0,\mathfrak{p}})}^{\mathrm{Gl}_2(\mathbb{F}_{0,\mathfrak{p}})} \tilde{\mu}_{\mathfrak{p}}$. Our final goal is to find an explicit formula for the factor $\mathcal{L}(\underline{\epsilon} \times \pi_f, \tilde{\mu}_f)$ in the comparison

$$J(\phi_\mu, \underline{\epsilon} \times \pi_f, \tilde{\mu}_f) \circ \Phi = \mathcal{L}(\underline{\epsilon} \times \pi_f, \tilde{\mu}_f) \prod_{\mathfrak{p}} I^{\mathrm{loc}}_{\mathfrak{p}}. \tag{8.152}$$

The Whittaker models

We assume that π_f is a representation which occurs in the decomposition of $H^{r_1+r_2}_{\mathrm{cusp}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathcal{O}_F)$. Let $\pi_\infty = \otimes_{v \in S_\infty} \pi_v$ an isomorphism class of Harish-Chandra modules with $\pi_v \in \mathrm{Coh}_2(\lambda_v)$. Then the isomorphism type $\pi_\infty \otimes \pi_f$ occurs in $L^2_{\mathrm{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}))$, we have to find model spaces $V_{\pi_{\mathfrak{p}}}, V_{\pi_\infty}$. We proceed as in section 4.1.9.

The adele ring $\mathbb{A} = \mathbb{A}_{F_0}$ is the restricted product

$$\mathbb{A} = (F_0 \otimes \mathbb{R}) \times {\prod_p}' (F_0 \otimes \mathbb{Q}_p) = \prod_{v \in S_\infty} F_v \times {\prod_{\mathfrak{p}}}' F_{\mathfrak{p}}.$$

The trace map $\mathrm{tr}_{F_0/\mathbb{Q}} : F_0 \to \mathbb{Q}$ induces a homomorphism $\mathrm{tr}_{F_0/\mathbb{Q}} : \mathbb{A}/F_0 \to \mathbb{A}_{\mathbb{Q}}/\mathbb{Q}$ which is of course the product $\prod_v \mathrm{tr}_{F_v/\mathbb{Q}_v}$. We compose $\mathrm{tr}_{F_0/\mathbb{Q}}$ with the character $\psi_1 : \mathbb{A}_{\mathbb{Q}}/\mathbb{Q} \to \mathbb{C}^\times$ in (4.126) and get a character $\psi = \psi_1 \circ \mathrm{tr}_{F_0/\mathbb{Q}} : \mathbb{A} \to S^1$. This character is of course a product $\prod_{v \in S_\infty} \psi_v \times \prod_{\mathfrak{p}} \psi_{\mathfrak{p}}$.

The character $\psi_{1,p} : \mathbb{Q}_p \to S^1$ has a trivial "additive" conductor, this means that $\psi_{1,p}(\mathbb{Z}_p) = 1$ and $\psi_{1,p}(\frac{1}{p}\mathbb{Z}_p) \neq 1$, or in other words $\psi_{1,p} : \frac{1}{p}\mathbb{Z}_p \to S^1$ is a non trivial character. For any prime $\mathfrak{p}$ which lies above p there is a largest integer $d(\psi_{\mathfrak{p}})(\psi) \geq 0$ such that $\mathrm{tr}_{F_{0,\mathfrak{p}}/\mathbb{Q}_p}(\mathfrak{p}^{-d(\psi_{\mathfrak{p}})(\psi)} \mathcal{O}_{F_0,\mathfrak{p}}) \subset \mathbb{Z}_p$. Then it is clear that $d(\psi_{\mathfrak{p}})(\psi)$ is the largest integer such that $\psi_{\mathfrak{p}} : \mathfrak{p}^{-d(\psi_{\mathfrak{p}})(\psi)} \mathcal{O}_{F_0,\mathfrak{p}} \to S^1$ is the trivial character. The ideal $\mathfrak{p}^{d(\psi_{\mathfrak{p}})(\psi)}$ and sometimes also simply $d(\psi_{\mathfrak{p}})(\psi)$ is the "additive " conductor of ψ. Of course we have $d(\psi_{\mathfrak{p}})(\psi) = 0$ for almost all $\mathfrak{p}$, the ideal $\vartheta_{F_0} = \prod_{\mathfrak{p}} \mathfrak{p}^{d(\psi_{\mathfrak{p}})(\psi)}$ is the different of F_0. The character ψ is trivial on $F_0 \subset \mathbb{A}$ hence $\psi \in \mathrm{Hom}(\mathbb{A}/F_0, S^1)$. It is in some sense a distinguished element: It is obtained by a canonical construction fro ψ_1. All other characters in $\mathrm{Hom}(\mathbb{A}/F_0, S^1)$ are of the form

$$\psi^{[a]} := \underline{x} \mapsto \psi(a\underline{x}) \text{ with some } a \in F_0,$$

hence we can say $\mathrm{Hom}(\mathbb{A}/F_0, S^1) = F_0$.

Now we know that for any place v (finite or archimedian) and any -not one dimensiional- irreducible representation $\pi_{\mathfrak{p}}$ we have a Whittaker model, this means that we have an unique subspace

$$\mathcal{W}(\pi_v, \psi_v)_{\mathbb{C}} \subset \{f : \mathrm{Gl}_2(F_v) \to \mathbb{C} \mid f(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g = \psi_v(u) f(g)\} \tag{8.153}$$

which invariant under right translations and isomorphic to π_v. This Whittaker model depends of course the choice of ψ_v which in the following will always be the v-local component of the distinguished ψ. At some instances we have to use the fact that we have an isomorphism $\boxed{\text{shift}}$

$$\begin{aligned} R_{a_v} : \mathcal{W}(\pi_v, \psi_v) &\xrightarrow{\sim} \mathcal{W}(\pi_v, \psi_v^{[a_v]}) \\ f(g) &\mapsto f(\begin{pmatrix} a_v & 0 \\ 0 & 1 \end{pmatrix} g) \end{aligned} \tag{8.154}$$

At the archimedian places these are the modules $\tilde{\mathcal{D}}_{\lambda_v}$. We can form the tensor product

$$\mathcal{W}(\pi, \psi)_{\mathbb{C}} := \bigotimes_v \mathcal{W}(\pi_v, \psi_v)_{\mathbb{C}} \tag{8.155}$$

and these spaces will be our $V_{\pi_{\mathfrak{p}}}, V_{\pi_\infty}$.

Our level subgroup should be a product $K_f = \prod_{\mathfrak{p}} K_{\mathfrak{p}}$, then

$$\mathcal{W}(\pi_f^{K_f}, \psi_f)_{\mathbb{C}} = \prod_{\mathfrak{p}} \mathcal{W}(\pi_{\mathfrak{p}}^{K_{\mathfrak{p}}}, \psi_{\mathfrak{p}})_{\mathbb{C}}$$

is a module for the Hecke algebra $\mathcal{H}^G_{K_f}$.

The Fourier expansion gives us an isomorphism

$$\begin{aligned} \mathcal{F}_1 : \mathcal{W}(\pi, \psi)_{\mathbb{C}} &\to \mathcal{A}(\mathrm{Gl}_2(F)\backslash \mathrm{Gl}_2(\mathbb{A}))(\pi) \\ \underline{f}(\underline{g}) = \{\dots, f_v, \dots\} &\mapsto \textstyle\sum_{a \in T_1(\mathbb{Q})} \underline{f}(a\underline{g}) \end{aligned} \tag{8.156}$$

this will be our intertwining operator $\Psi(\pi)$ in (8.111). We get an isomorphism

$$\mathcal{F}_1^{d_T} : \bigotimes_{v \in S_\infty} H^1(\mathfrak{g}_v, K_v, \mathcal{D}_v \otimes \mathcal{M}_{\lambda_v}) \otimes \bigotimes_{\mathfrak{p}} \mathcal{W}(\pi_{\mathfrak{p}}^{K_{\mathfrak{p}}}, \psi_{\mathfrak{p}}) \xrightarrow{\sim} H^{r_1+r_2}_{\mathrm{cusp}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes F)(\pi_f) \otimes \mathbb{C} \tag{8.157}$$

A well known consequence is that $H^{r_1+r_2}_{\mathrm{cusp}}(\mathcal{S}^G, \tilde{\mathcal{M}}_\lambda \otimes F)(\underline{\epsilon} \times \pi_f) \otimes \mathbb{C}$ occurs with multiplicity one. Therefore $H^{r_1+r_2}_{\mathrm{cusp}}(\mathcal{S}^G, \tilde{\mathcal{M}}_\lambda \otimes F)(\underline{\epsilon} \times \pi_f)$ is a F- vector space with an absolutely irreducible $G(\mathbb{A}_f)$ module structure, hence we can also say that we realized π_f over F. We can write $H^{r_1+r_2}_{\mathrm{cusp}}(\mathcal{S}^G, \tilde{\mathcal{M}}_\lambda \otimes F)(\underline{\epsilon} \times \pi_f) = \bigotimes' V_{\pi_{\mathfrak{p}}}$

Rational and integral strutures on the Whittaker model.

At a finite place $\mathfrak{p}$ we can realise our local representation $V_{\pi_\mathfrak{p}}$ as a subspace $\mathcal{W}_{\overline{\mathbb{Q}}}(\pi_\mathfrak{p},\psi_\mathfrak{p}))_{\bar{\mathbb{Q}}}$ in the space of $\overline{\mathbb{Q}}$-valued functions

$$\mathcal{W}_{\overline{\mathbb{Q}}}(\psi_\mathfrak{p}) = \left\{ f : G(F_\mathfrak{p}) \to \overline{\mathbb{Q}} \mid f\left(\begin{pmatrix}1 & u_\mathfrak{p}\\ 0 & 1\end{pmatrix} g_\mathfrak{p}\right) = \psi_\mathfrak{p}(u_\mathfrak{p}) f(g_\mathfrak{p})\right\}.$$

We briefly sketch how we get this realisation. We choose a no zero linear form $L_0 : V_{\pi_\mathfrak{p}} \to F$ and then we define a second linear form $L : V_{\pi_\mathfrak{p}} \to \bar{\mathbb{Q}}$ by the integral

$$L(h) := \int_{U_0(F_\mathfrak{p})} (\rho_{\pi_\mathfrak{p}}(\begin{pmatrix}1 & u_\mathfrak{p}\\ 0 & 1\end{pmatrix})h)\overline{\psi_\mathfrak{p}(u_\mathfrak{p})} du_\mathfrak{p}$$

here is a minor issue with convergence, this will be discussed later. This linear mapy is non zero because we know that a Whittaker model exists. Then it is clear that $L((\begin{pmatrix}1 & u_\mathfrak{p}\\ 0 & 1\end{pmatrix})h) = L(h)\psi_\mathfrak{p}(u_\mathfrak{p})$ and

$$h \mapsto \{g_\mathfrak{p} \mapsto L(\rho_{\pi_\mathfrak{p}}(g_\mathfrak{p})(h)\} \tag{8.158}$$

is a $\mathrm{Gl}_2(F_\mathfrak{p})$ - isomorphism of this $V_{\pi_\mathfrak{p}}$ with a subspace in $\mathcal{W}_{\overline{\mathbb{Q}}}(\pi_\mathfrak{p},\psi_\mathfrak{p}) \subset \mathcal{W}_{\overline{\mathbb{Q}}}(\psi_\mathfrak{p})$.

On the space $\mathcal{W}_{\overline{\mathbb{Q}}}(\psi_\mathfrak{p})$ we define an action of the Galois group: The values $\psi_\mathfrak{p}(u_\mathfrak{p})$ are p^m-th roots of unity, we have the reciprocity homomorphism

$$\alpha : \ \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \ \mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q}) \xrightarrow{\sim} \mathbb{Z}_p^\times.$$

For $f \in \mathcal{W}_{\overline{\mathbb{Q}}}(\psi_\mathfrak{p}))$ and $\sigma \in \ \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ we put

$${}^\sigma f(g_\mathfrak{p}) = \sigma\left(f\left(\begin{pmatrix}\alpha(\sigma)^{-1} & 0\\ 0 & 1\end{pmatrix} g_\mathfrak{p}\right)\right),$$

If we take an element $\sigma \in \ \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ then it conjugates the representation $\pi_\mathfrak{p}$ into ${}^\sigma\pi_\mathfrak{p}$ and we get a map

$$\begin{array}{ccc} \mathcal{W}_{\overline{\mathbb{Q}}}(\pi_\mathfrak{p},\psi_\mathfrak{p})) & \xrightarrow{\tilde{\sigma}} & \mathcal{W}_{\overline{\mathbb{Q}}}({}^\sigma\pi_\mathfrak{p},\psi_\mathfrak{p})) \\ f & \mapsto & {}^\sigma f \end{array}$$

This map is a semilinear isomorphism and $\mathbb{Q}(\pi_\mathfrak{p})$ is the number field for which $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\pi_\mathfrak{p}))$ is the stabilizer of $\mathcal{W}_{\overline{\mathbb{Q}}}(\pi_\mathfrak{p},\psi_\mathfrak{p})$. The space $\mathcal{W}_{\overline{\mathbb{Q}}}(\pi_\mathfrak{p},\psi_\mathfrak{p}))$ is the union of finite dimensional $\overline{\mathbb{Q}}$ - modules $\mathcal{W}_{\overline{\mathbb{Q}}}(\pi_\mathfrak{p}^{K_\mathfrak{p}},\psi_\mathfrak{p})$ where $K_\mathfrak{p}$ runs over the open compact subgroups. The space of functions in $\mathcal{W}(\pi_\mathfrak{p}^{K_\mathfrak{p}},\psi_\mathfrak{p})$ which are invariant under $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\pi_\mathfrak{p})$ is a $\mathbb{Q}(\pi_\mathfrak{p})$ vector space $\mathcal{W}(\pi_\mathfrak{p}^{K_\mathfrak{p}},\psi_\mathfrak{p}))$ on which $\mathcal{H}(G(F_\mathfrak{p})//K_\mathfrak{p})$ acts absolutely irreducibly. Of course $\mathcal{W}(\pi_\mathfrak{p},\psi_\mathfrak{p}))$ is the union of the $\mathcal{W}(\pi_\mathfrak{p}^{K_\mathfrak{p}},\psi_\mathfrak{p})$ and clearly $\mathcal{W}(\pi_\mathfrak{p},\psi_\mathfrak{p})) \otimes_{\mathbb{Q}(\pi_\mathfrak{p})} \overline{\mathbb{Q}} = \mathcal{W}_{\overline{\mathbb{Q}}}(\pi_\mathfrak{p},\psi_\mathfrak{p})$.

Of course $\mathbb{Q}(\pi_\mathfrak{p}) \subset \mathbb{Q}(\pi_f)$, let $\mathcal{O}(\pi_f) \subset \mathbb{Q}(\pi_f)$ be the ring of integers. We have the action of $\mathcal{H}_{\mathbb{Z}}^{coh}$ (See 1.2.1.(ii)) on the cohomology and hence we get an action of the algebra $\mathcal{H}(G(F_\mathfrak{p})//K_\mathfrak{p})_{\mathbb{Z}}$ on $\mathcal{W}(\pi_\mathfrak{p},\psi_p)$ and this gives us a finitely generated $\mathcal{O}(\pi_\mathfrak{p})$- module of endomorphisms. Hence we can find invariant lattices $\mathcal{W}_{\mathcal{O}(\pi_\mathfrak{p})}(\pi_\mathfrak{p}^{K_\mathfrak{p}}, \psi_p)_{\mathcal{O}(\pi_\mathfrak{p})}$. If we invert a few more primes then we can achieve that two such choices just differ by an element $a \in \mathcal{O}(\pi_\mathfrak{p})$. We assume that such a choice of lattices has been made at all primes $\mathfrak{p}$. If we are in the unramified case then we will make a very particular choice later. We put $\mathcal{W}_{\mathcal{O}(\pi_f)}(\pi_f, \psi_\mathfrak{p})) = \bigotimes'_\mathfrak{p} \mathcal{W}_{\mathcal{O}(\pi_f)}(\pi_\mathfrak{p}, \psi_\mathfrak{p})$ (See 2.2.7).

The Newvector

For any integer $n \geq 0$ we define the congruence subgroups

$$K_{\mathfrak{p},0}(\mathfrak{p}^n)) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} | c \equiv 0 \mod \mathfrak{p}^n\} \subset \mathrm{Gl}_2(\mathcal{O}_\mathfrak{p})$$
$$K_{\mathfrak{p},1}(\mathfrak{p}^n)) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} | c \equiv 0 \mod \mathfrak{p}^n) \text{ and } a \equiv 1 \mod \mathfrak{p}^{\mathfrak{f}(\pi_\mathfrak{p})}\} \subset \mathrm{Gl}_2(\mathcal{O}_\mathfrak{p}).$$

Clearly the quotient $K_{\mathfrak{p},0}(\mathfrak{p}^n))/K_{\mathfrak{p},1}(\mathfrak{p}^n)) = \{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\}_{a \in (\mathcal{O}_\mathfrak{p}/\mathfrak{p}^n))^\times} = C(\mathcal{O}_\mathfrak{p}/\mathfrak{p}^n))$.

A theorem of Casselman and Novovorskii([?]) implies that there is smallest integer $\mathfrak{f}(\pi_\mathfrak{p}) \geq 0$ such that $\mathcal{W}_{\overline{\mathbb{Q}}}(\pi_\mathfrak{p}, \psi_\mathfrak{p}))^{K_{\mathfrak{p},1}(\mathfrak{p}^{\mathfrak{f}(\pi_\mathfrak{p})})} \neq 0$ and it also says that the dimension of this space is actually equal one. An element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_{\mathfrak{p},0}(\mathfrak{p}^{\mathfrak{f}(\pi_\mathfrak{p})})$ acts by multiplication the central character $\zeta_\mathfrak{p}(a)$ on this one dimensional space. The ideal $\mathfrak{p}^{f(\pi_\mathfrak{p})}$ is the *conductor* of $\pi_\mathfrak{p}$.

We introduce the subtorus $T_1(F_\mathfrak{p}) = \left\{\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\right\} = F_\mathfrak{p}^\times$ of $T(F_\mathfrak{p})$. We restrict the functions in $\mathcal{W}(\pi_\mathfrak{p}, \psi_\mathfrak{p})$ to $T_1(F_\mathfrak{p})$ this restriction map is injective ([28]). The space of these restrictions is the Kirillow model $\mathcal{K}(\pi_\mathfrak{p})$ hence we may consider $\mathcal{K}(\pi_\mathfrak{p})$. as a space of functions on $T_1(F_\mathfrak{p}) = F_\mathfrak{p}^\times$.

We recall the definition of the Schwartz-spaces $\mathcal{S}(F_\mathfrak{p})$- this are the locally constant $\bar{\mathbb{Q}}$ valued functions with compact support-, and $\mathcal{S}(F_\mathfrak{p}^\times)$, this is the space of those functions in $\mathcal{S}(F_\mathfrak{p})$ which vanish at 0. This is of course equal to the locally constant $\bar{\mathbb{Q}}$ valued functions on $F_\mathfrak{p}^\times$ with compact support. The Schwartz space $\mathcal{S}(F_\mathfrak{p}) \subset \mathcal{K}(\pi_\mathfrak{p})$ and it is of codimension $0, 1$ or 2(*see*[28]). Of course it is clear how $B(F_\mathfrak{p})$ acts upon the Kirillow model, it follows from the definition that for $h \in \mathcal{K}(\pi_\mathfrak{p})$

$$\pi_\mathfrak{p}(\begin{pmatrix} t_1 & u \\ 0 & t_2 \end{pmatrix})h(t) = \zeta_\mathfrak{p}(t_2)h(t_1 t_2^{-1} t)\psi_\mathfrak{p}(t_1^{-1} t). \tag{8.159}$$

The following lengthy computation serves to find an explicit formula for a generator $h_\mathfrak{p}^{(0)} \in \mathcal{K}(\pi_\mathfrak{p})^{K_{\mathfrak{p},0}(\mathfrak{p}^{\mathfrak{f}(\pi_\mathfrak{p})})}$. This is the *newvector*.

The purpose of following somewhat lengthy computations is to discuss the conditions (I_p, I_{pp}) and to find an explicit expression for a new vector $h_{\pi_\mathfrak{p}}^{(0)}$. This will help us to nail down the local intertwining operator $I_\mathfrak{p}^{\mathrm{loc}}$.

The Principal Series

To see what is going on we consider the special case that $\pi_\mathfrak{p}$ is a principal series representation. This means that

$$\chi_\mathfrak{p} : \begin{pmatrix} t_1 & * \\ 0 & t_2 \end{pmatrix} \longrightarrow \chi_{\mathfrak{p},1}(t_1) \cdot \chi_{\mathfrak{p},2}(t_2)$$

is an unramified character and $\pi_\mathfrak{p}$ is the induced representation from $\chi_\mathfrak{p}$, i.e. we consider the space of functions

$$\mathrm{Ind}_{B(F_\mathfrak{p})}^{G(F_\mathfrak{p})}\chi_\mathfrak{p} = \left\{ f : G(F_\mathfrak{p}) \to \mathbb{C} \mid f\left(\begin{pmatrix} t_1 & * \\ 0 & t_2 \end{pmatrix} g\right) = \chi_{\mathfrak{p},1}(t_1)\chi_{\mathfrak{p},2}(t_2)f(g) \right\},$$

Since we want the representation to be admissible the function f must be right invariant under some open subgroup $K'_\mathfrak{p}$.

Let us denote the restriction of $\chi_\mathfrak{p}$ to the subtorus $T^{(1)}(F_\mathfrak{p}) = \{\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} |t \in F_\mathfrak{p}^\times\} = F_\mathfrak{p}^\times\}$ by $\chi_\mathfrak{p}^{(1)}$.

Proposition 8.3.1. *i) The induced representation* $\mathit{Ind}_{B(F_\mathfrak{p})}^{G(F_\mathfrak{p})}\chi_\mathfrak{p}$ *is irreducible unless* $\chi_\mathfrak{p}^{(1)} = \mathbf{1}$*(the trivial character) or* $|\;|_\mathfrak{p}$*(the normalised* $\mathfrak{p}$*-adic absolute value)*

ii) If $\chi_\mathfrak{p}^{(1)} = \mathbf{1}$ *, theone dimensional space of functions* $g \mapsto a\ \chi_{1,\mathfrak{p}}(\det(g))$ *form an invariant subspace, the quotient by this subspace is irreducible. This quotient is called the Steinberg module* $\mathrm{St}(\chi_\mathfrak{p})$.

iii) If $\chi_\mathfrak{p}^{(1)} = |\;|_\mathfrak{p}$ *the integral*

$$f \mapsto \int_{Sl_2(\mathcal{O}_\mathfrak{p})} f(k)dk.$$

defines an invariant linear form, the kernel is irreducible and isomorphic to $\mathrm{St}(\chi_\mathfrak{p})$.

See [28]

In this case we have an obvious option for an intertwining operator to the Whittaker model:

$$R_\mathfrak{p} :\ \mathrm{Ind}_{B(F_\mathfrak{p})}^{G(F_\mathfrak{p})}\chi_\mathfrak{p} \longrightarrow \mathcal{W}(\pi_\mathfrak{p}(\chi_\mathfrak{p}), \psi_\mathfrak{p}),$$

it is given by

$$R_\mathfrak{p}(f)(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) = \int_{U(F_\mathfrak{p})} f(w\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix})\overline{\psi_\mathfrak{p}(u)}du$$

where $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and du is the additively invariant measure that gives volume one to $\mathcal{O}_{F_\mathfrak{p}}$. After a small substitution the integral becomes

$$|t|_\mathfrak{p}\chi_{2,\mathfrak{p}}(t)\int_{U(F_\mathfrak{p})} f(w\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix})\overline{\psi_\mathfrak{p}(tu)}du. \tag{8.160}$$

Our integral can be written as an infinite sum RuP

$$\int_{\mathcal{O}_{F_0,\mathfrak{p}}} f(w\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix})\overline{\psi_{\mathfrak{p}}(tu)}du+\sum_{\nu=1}^{\infty}\int_{\varpi_{\mathfrak{p}}^{-\nu}\mathcal{O}_{F_0,\mathfrak{p}}\setminus\varpi_{\mathfrak{p}}^{-\nu+1}\mathcal{O}_{F_0,\mathfrak{p}}} f(w\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix})\overline{\psi_{\mathfrak{p}}(tu)}du \tag{8.161}$$

As usual $N(\mathfrak{p}) = \#(\mathcal{O}_{F_0,\mathfrak{p}}/\mathfrak{p}$ is the number of elements in the residue field. We choose a local uniformizing element $\varpi_{\mathfrak{p}}$ and write any element $u \in F_{0,\mathfrak{p}}^{\times}$ as $u = \varpi_{\mathfrak{p}}^{-\nu}\varepsilon$ with ε a unit and of course $\nu =$ ord$(\psi_{\mathfrak{p}})(u)$. Then the ν-th summand becomes term1

$$N(\mathfrak{p})^{\nu}\int_{\mathcal{O}_{F_0,\mathfrak{p}}\setminus\varpi_{\mathfrak{p}}\mathcal{O}_{F_0,\mathfrak{p}}} f(w\begin{pmatrix}1 & \varpi_{\mathfrak{p}}^{-\nu}u\\ 0 & 1\end{pmatrix})\overline{\psi_{\mathfrak{p}}(t\varpi_{\mathfrak{p}}^{-\nu}u)}du \tag{8.162}$$

Now we apply the Iwasawa decomposition and get

$$\begin{aligned}\underline{f}(w\begin{pmatrix}1 & \varpi_{\mathfrak{p}}^{-\nu}u\\ 0 & 1\end{pmatrix}) &= f(\begin{pmatrix}\varpi_{\mathfrak{p}}^{\nu}u^{-1} & -1\\ 0 & \varpi_{\mathfrak{p}}^{-\nu}u\end{pmatrix}\begin{pmatrix}1 & 0\\ \varpi_{\mathfrak{p}}^{\nu}u^{-1} & 1\end{pmatrix})\\ &= \chi_{\mathfrak{p}}^{(1)}(\varpi_{\mathfrak{p}})^{\nu}\chi_{\mathfrak{p}}^{(1)}(u)^{-1}f(\begin{pmatrix}1 & 0\\ \varpi_{\mathfrak{p}}^{\nu}u^{-1} & 1\end{pmatrix})\end{aligned} \tag{8.163}$$

For ν large enough

$$f(\begin{pmatrix}1 & 0\\ \varpi_{\mathfrak{p}}^{\nu}u^{-1} & 1\end{pmatrix}) = f(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}),$$

hence we have to study the expression Gsum

$$G(t\varpi_{\mathfrak{p}}^{-\nu},\chi_{\mathfrak{p}}^{(1)},\psi_{\mathfrak{p}}) = \int_{\mathcal{O}_{F_0,\mathfrak{p}}\setminus\varpi_{\mathfrak{p}}\mathcal{O}_{F_0,\mathfrak{p}}}\chi_{\mathfrak{p}}^{(1)}(u)^{-1}\overline{\psi_{\mathfrak{p}}(t\varpi_{\mathfrak{p}}^{-\nu}u)}du. \tag{8.164}$$

These expressions are Gaussian sums and these Gaussian sums are computed in any textbook on algebraic number theory. we refer to [78]. We recall the results.

We also have the multiplicative conductor $\mathfrak{f}(\chi_{\mathfrak{p}}) = \mathfrak{f}(\chi_{\mathfrak{p}}^{(1)})$. This is the smallest non negative integer such that $\chi_{\mathfrak{p}}^{(1)}$ is trivial on the subgroup $\mathcal{O}_{F_0,\mathfrak{p}}^{(\mathfrak{f}(\chi_{\mathfrak{p}}))} = \{u \in \mathcal{O}_{F_0,\mathfrak{p}}^{\times} | u \equiv 1 \mod \varpi_{\mathfrak{p}}^{\mathfrak{f}(\chi_{\mathfrak{p}})}\}$. Then the Gaussian sum is zero unless the "additive" conductor of $u \mapsto \psi_{\mathfrak{p}}(t\varpi_{\mathfrak{p}}^{-\nu}u)$ is equal to the conductor $\mathfrak{f}(\chi_{\mathfrak{p}}^{(1)})$ i.e. the Gaussian sum is zero unless

$$\mathrm{ord}_{\mathfrak{p}}(t) - \nu + d(\psi_{\mathfrak{p}}) = \mathfrak{f}(\chi_{\mathfrak{p}}^{(1)}). \tag{8.165}$$

If this equality holds then a well known computation yields

$$G(t\varpi_{\mathfrak{p}}^{-\nu_0},\chi_{1,\mathfrak{p}},\psi_{\mathfrak{p}})\overline{G(t\varpi_{\mathfrak{p}}^{-\nu_0},\chi_{1,\mathfrak{p}},\psi_{\mathfrak{p}})} = N(\mathfrak{p})^{\mathfrak{f}(\chi_{1,\mathfrak{p}})}$$

From this it follows that the infinite sum in (8.161) is actually a finite sum, hence there is no problem with convergence. Furthermore it is clear that for a t with $\mathrm{ord}_{\mathfrak{p}}() << 0$ we have $R_{\mathfrak{p}}(f)(\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix}) = 0$. and this means that $R_{\mathfrak{p}}(f)(\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix})$ lies in $\mathcal{S}(F_{\mathfrak{p}})$.

Looking at these computation we easily see that the functions $t \mapsto R_{\mathfrak{p}}(f_0)(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix})$ have a simple asymptotic behaviour. For $|t| \to 0$, the computations yield

$$R_{\mathfrak{p}}(f_0)(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) \sim a|t|_{\mathfrak{p}}\chi_{2,\mathfrak{p}}(t) + b\chi_{1,\mathfrak{p}}(t) \tag{8.166}$$

We want to pin down a specific generator $f_0 \in (\mathrm{Ind}_{B(F_{\mathfrak{p}})}^{G(F_{\mathfrak{p}})}\chi_{\mathfrak{p}})^{K_{\mathfrak{p},1}(\mathfrak{p}^{f(\pi_{\mathfrak{p}})})}$(resp. $\mathrm{St}(\chi_{\mathfrak{p}})^{K_{\mathfrak{p},1}(\mathfrak{p}^{f(\pi_{\mathfrak{p}})})}$). We look at the double coset decomposition

$$\mathrm{Gl}_2(\mathcal{O}_{\mathfrak{p}}) = \bigcup_{\xi} B(\mathcal{O}_{\mathfrak{p}})\xi K_{\mathfrak{p},0}(\mathfrak{p}^{\mathfrak{f}(\pi_{\mathfrak{p}})}) \tag{8.167}$$

It is easy to see that a system of representatives for these double cosets is given by the matrices

$$\{\begin{pmatrix} 1 & 0 \\ \varpi_{\mathfrak{p}}^{\nu} & 1 \end{pmatrix}\}_{\nu=1,\dots,\mathfrak{f}(\pi_{\mathfrak{p}})} \cup \{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\} \tag{8.168}$$

The space $(\mathrm{Ind}_{B(F_{\mathfrak{p}})}^{G(F_{\mathfrak{p}})}\chi_{\mathfrak{p}})^{K_{\mathfrak{p},1}(\mathfrak{p}^{\mathfrak{f}(\pi_{\mathfrak{p}})})}$ is spanned by functions

$$f_\xi : B(\mathcal{O}_{\mathfrak{p}})\xi K_{\mathfrak{p},1}(\mathfrak{p}^{\mathfrak{f}(\pi_{\mathfrak{p}})})) \to \bar{\mathbb{Q}}$$

which are supported on $B(\mathcal{O}_{\mathfrak{p}})\xi K_{\mathfrak{p},1}(\mathfrak{p}^{\mathfrak{f}(\pi_{\mathfrak{p}})})$ and satisfy

$$f_\xi(b_1\xi k) = \chi_{\mathfrak{p}}(b_1)f(\xi)\ \forall\ b_1 \in B(\mathcal{O}_{\mathfrak{p}}), k \in K_{\mathfrak{p},1}(\mathfrak{p}^{\mathfrak{f}(\pi_{\mathfrak{p}})}.$$

This is a very restrictive condition if we want $f_\xi \neq 0$, actually it follows from the definition of $\mathfrak{f}(\pi_{\mathfrak{p}})$ and the above theorem of Casselmann and Novovorskii(that there is exactly one double coset ξ_0 for which we find a function $f_{\xi_0} = f_0 \neq 0$. We have several cases.

a) If $\mathfrak{f}(\pi_{\mathfrak{p}}) = 0$ then $K_{\mathfrak{p},1}(\mathfrak{p}^{\mathfrak{f}(\pi_{\mathfrak{p}})}) = \mathrm{Gl}_2(\mathcal{O}_{F_0,\mathfrak{p}})$ and the character $\chi_{\mathfrak{p}}$ is unramified. In this case we choose for the function f_0 the spherical function which has value one at the identity element.

b) We have $\mathfrak{f}(\pi_{\mathfrak{p}}) > 0$. Assume our double coset isi represented by $\xi_0 = \begin{pmatrix} 1 & 0 \\ \varpi_{\mathfrak{p}}^{\nu_0} & 1 \end{pmatrix}$. For any $\varepsilon \in \mathcal{O}_{F_0,\mathfrak{p}}^{\times}$ we have

$$f_{\xi_0}(\begin{pmatrix} 1 & 0 \\ \varpi_{\mathfrak{p}}^{\nu_0}\varepsilon & 1 \end{pmatrix}) = f_{\xi_0}(\begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \varpi_{\mathfrak{p}}^{\nu_0} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}) = \chi_{2,\mathfrak{p}}(\varepsilon)f_\xi(\begin{pmatrix} 1 & 0 \\ \varpi_{\mathfrak{p}}^{\nu_0} & 1 \end{pmatrix}). \tag{8.169}$$

For $\varepsilon \equiv 1 \mod \mathfrak{p}^{\mathfrak{f}(\pi_{\mathfrak{p}})-\nu_0}$ the far most left term does not depend on ε hence we can conclude that $\chi_{2,\mathfrak{p}}(\varepsilon) = 1$ or what amounts to the same

$$\mathfrak{f}(\chi_{2,\mathfrak{p}}) \geq \mathfrak{f}(\pi_{\mathfrak{p}}) - \nu_0.$$

On the other hand for any $v \in \mathcal{O}_{F_0,\mathfrak{p}}$ we have the equality

$$f_\xi(\begin{pmatrix} 1 & 0 \\ \varpi_{\mathfrak{p}}^{\nu_0} & 1 \end{pmatrix}) = f_\xi(\begin{pmatrix} 1 & 0 \\ \varpi_{\mathfrak{p}}^{\nu_0} & 1 \end{pmatrix}\begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}). \tag{8.170}$$

and

$$\begin{pmatrix} 1 & 0 \\ \varpi_{\mathfrak{p}}^{\nu_0} & 1 \end{pmatrix}\begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} (1+v\varpi_{\mathfrak{p}}^{\nu_0})^{-1} & * \\ 0 & (1+v\varpi_{\mathfrak{p}}^{\nu_0}) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{\varpi_{\mathfrak{p}}^{\{}\nu_0}{1+v\varpi_{\mathfrak{p}}^{\nu_0}} & 1 \end{pmatrix}$$

Therefore we get

$$f_\xi(\begin{pmatrix} 1 & 0 \\ \varpi_{\mathfrak{p}}^{\nu_0} & 1 \end{pmatrix}) = \chi_{1,\mathfrak{p}}^{-1}(1+v\varpi_{\mathfrak{p}}^{\nu_0}) f_\xi(\begin{pmatrix} 1 & 0 \\ \varpi_{\mathfrak{p}}^{\nu_0} & 1 \end{pmatrix}) \tag{8.171}$$

and this implies $\mathfrak{f}(\chi_{1,\mathfrak{p}}) \leq \nu_0$. Hence we see that ν_0 must satisfy

$$\mathfrak{f}(\chi_{1,\mathfrak{p}}) \geq \nu_0 \geq \mathfrak{f}(\pi_{\mathfrak{p}}) - \mathfrak{f}(\chi_{2,\mathfrak{p}})$$

If on the other hand ν_0 satisfies this inequality we can write down a non zero function f_{xi_0}. But since the ν is unique we find that actually nufone

$$\mathfrak{f}(\pi_{\mathfrak{p}}) = \mathfrak{f}(\chi_{2,\mathfrak{p}}) + \mathfrak{f}(\chi_{1,\mathfrak{p}}) \text{ and } \nu_0 = \mathfrak{f}(\chi_{1,\mathfrak{p}}). \tag{8.172}$$

Therefore we normalise the generator f_0 so that $f_0(\begin{pmatrix} 1 & 0 \\ \varpi_{\mathfrak{p}}^{\nu_0} & 1 \end{pmatrix}) = 1$

We still have the double coset $\xi_0 = w$. Then

$$f_w(w) = f_{\xi_0}(w\begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}) = f_w(\begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix} w) = \chi_{1,\mathfrak{p}}(\varepsilon) f_w(w)$$

and this implies $\chi_{1,\mathfrak{p}}(\varepsilon) = 1$. We normalise $f_w(w) = 1$. We are still in the case b) i.e. $\mathfrak{f}(\chi_{2,\mathfrak{p}}) = \mathfrak{f}(\pi_{\mathfrak{p}}) > 0$.

Now we consider the case that our induced representation is reducible, this means that $\chi_{\mathfrak{p}}^{(1)} = \mathbf{1}$(resp. $= |\ |_{\mathfrak{p}}$). In this case $\mathrm{Ind}_{B(F_{\mathfrak{p}})}^{G(F_{\mathfrak{p}})}\chi_{\mathfrak{p}}$ has the Steinberg module $\pi_{\mathfrak{p}} = \mathrm{St}(\chi_{\mathfrak{p}})$ as quotient (resp. submodule). Then $\mathfrak{f}(\pi_{\mathfrak{p}}) = 1$ and the Bruhat decomposition $\mathrm{Gl}_2(\mathcal{O}_{\mathfrak{p}}) = B(\mathcal{O}_{\mathfrak{p}}) \bigcup B(\mathcal{O}_{\mathfrak{p}}) w K_{\mathfrak{p},0}(\mathfrak{p})$ gives us

$$(\ \mathrm{Ind}_{B(F_{\mathfrak{p}})}^{G(F_{\mathfrak{p}})}\chi_{\mathfrak{p}})^{K_{\mathfrak{p},0}(\mathfrak{p})} = \bar{\mathbb{Q}} f_e + \bar{\mathbb{Q}} f_w, \tag{8.173}$$

here f_e resp. f_w are the characteristic functions of $B(\mathcal{O}_{\mathfrak{p}})$ resp. $B(\mathcal{O}_{\mathfrak{p}}) w K_{\mathfrak{p},0}(\varpi_{\mathfrak{p}})$. The element $f_e + f_w$ spans the one dimensional kernel of $\mathrm{Ind}_{B(F_{\mathfrak{p}})}^{G(F_{\mathfrak{p}})}\chi_{\mathfrak{p}} \to \mathrm{St}(\chi_{\mathfrak{p}})$, hence the image of f_e spans $\mathrm{St}(\chi_{\mathfrak{p}})^{K_{\mathfrak{p},0}(\mathfrak{p})}$. If we realise $\mathrm{St}(\chi_{\mathfrak{p}})$ as kernel of $\mathrm{Ind}_{B(F_{\mathfrak{p}})}^{G(F_{\mathfrak{p}})} |\ |_{\mathfrak{p}} \chi_{\mathfrak{p}}^{-1} \to \bar{\mathbb{Q}} |\ |_{\mathfrak{p}} \chi_{\mathfrak{p}}^{-1}$ then $f_0 := f_e - \frac{1}{q} f_w$ can be chosen as generator of $\mathrm{St}(\chi_{\mathfrak{p}})^{K_{\mathfrak{p},0}(\mathfrak{p})}$.

At this point we have chosen a specific element $f_0 \in V_{\pi_{\mathfrak{p}}}^{K_{\mathfrak{p},0}(\mathfrak{p}^{\mathfrak{f}(\pi_{\mathfrak{p}})})}$ if $\pi_{\mathfrak{p}}$ is an induced representation or a Steinberg module. It is known that these representations are exactly those representations for which $\mathcal{K}(\pi_{\mathfrak{p}})/\mathcal{S}(F_{\mathfrak{p}}) \neq (0)$. For these representations we define a distinguished newvector $h_{\pi_{\mathfrak{p}}}^{(0)} := R_{\mathfrak{p}}(f_0)$.

The remaining representations are the supercuspidal representations, for those we chose a distinguished $h^{(0)}_{\pi_f p}$ further down.

We resume the computation of the $R_{\mathfrak{p}}(f_0)$ after equation (**??**) we start with the case $\chi_{\mathfrak{p}}$ is unramified. Let f be any linear combination of these two functions, for $\nu \geq 1$ we have $f(\begin{pmatrix} 1 & 0 \\ \varpi_{\mathfrak{p}}^{\nu}\varepsilon^{-1} & 1 \end{pmatrix}) = f(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix})$ and (**??**) becomes

$$R_{\mathfrak{p}}(f)(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) = |t|_{\mathfrak{p}}\chi_{2,\mathfrak{p}}(t) \times \big[f(w)\int_{\mathcal{O}_{F_0,\mathfrak{p}}} \overline{\psi_{\mathfrak{p}}(tu)}du + f(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix})\sum_{\nu=1}^{\infty} N(\mathfrak{p})^{\nu}\chi_{\mathfrak{p}}^{(1)}(\varpi_{\mathfrak{p}}^{\nu})\int_{\mathcal{O}^{\times}_{F_0,\mathfrak{p}}} \overline{\psi_{\mathfrak{p}}(t\varpi_{\mathfrak{p}}^{-\nu}u)}du\big] \tag{8.174}$$

If we write $t = \varpi_{\mathfrak{p}}{}^{\mathrm{ord}_{\mathfrak{p}}(t)}$ then

$$\int_{\mathcal{O}^{\times}_{F_0,\mathfrak{p}}} \overline{\psi_{\mathfrak{p}}(t\varpi_{\mathfrak{p}}^{-\nu}u)}du == \begin{cases} 1 & \text{if } \mathrm{ord}_{\mathfrak{p}}(t) \geq \nu + d(\psi_{\mathfrak{p}}) \\ -\frac{1}{N(\mathfrak{p})} & \text{if } \mathrm{ord}_{\mathfrak{p}}(t) = \nu + d(\psi_{\mathfrak{p}}) - 1 \\ 0 & \text{if } \mathrm{ord}_{\mathfrak{p}}(t) < \nu + d(\psi_{\mathfrak{p}}) - 1 \end{cases} \tag{8.175}$$

and therefore

$$R_{\mathfrak{p}}(f)(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) = |t|_{\mathfrak{p}}\chi_{2,\mathfrak{p}}(\varpi_{\mathfrak{p}})^{\mathrm{ord}_{\mathfrak{p}}(t)} \times \big[f(w)\int_{\mathcal{O}_{F_0,\mathfrak{p}}} \overline{\psi_{\mathfrak{p}}(tu)}du$$

$$+f(e)(1-\tfrac{1}{N(\mathfrak{p})})(\sum_{\nu=1}^{\mathrm{ord}_{\mathfrak{p}}(t)-d(\psi_{\mathfrak{p}})} N(\mathfrak{p})^{\nu}\chi^{(1)}(\varpi_{\mathfrak{p}})^{\nu}) - N(\mathfrak{p})^{\mathrm{ord}_{\mathfrak{p}}(t)}\chi^{(1)}(\varpi_{\mathfrak{p}})^{\mathrm{ord}_{\mathfrak{p}}(t)+1}\big] \tag{8.176}$$

We consider the special case $f_0 = f_e + f_w$ then a simple manipulation gives us Wsph

$$R_{\mathfrak{p}}(f_0)(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) =$$

$$(1-\chi_{\mathfrak{p}}^{(1)}(\varpi_{\mathfrak{p}}))N(\mathfrak{p})^{-\mathrm{ord}_{\mathfrak{p}}(t)}(\textstyle\sum_{\nu=0}^{\mathrm{ord}_{\mathfrak{p}}(t)-d(\psi_{\mathfrak{p}})}(N(\mathfrak{p})\chi_{1,\mathfrak{p}}(\varpi_{\mathfrak{p}}))^{\nu}\chi_{2,\mathfrak{p}}(\varpi_{\mathfrak{p}})^{\mathrm{ord}_{\mathfrak{p}}(t)-\nu} =$$

$$(1-\chi_{\mathfrak{p}}^{(1)}(\varpi_{\mathfrak{p}}))(N(\mathfrak{p})\chi_{2,\mathfrak{p}}(\varpi_{\mathfrak{p}}))^{d(\psi_{\mathfrak{p}})}(\textstyle\sum_{\nu=0}^{\mathrm{ord}_{\mathfrak{p}}(t)-d(\psi_{\mathfrak{p}})}\chi_{1,\mathfrak{p}}(\varpi_{\mathfrak{p}})^{\nu}(N(\mathfrak{p})^{-1}\chi_{2,\mathfrak{p}}(\varpi_{\mathfrak{p}}))^{\mathrm{ord}_{\mathfrak{p}}(t)-d(\psi_{\mathfrak{p}})-\nu} \tag{8.177}$$

We look back to section 4.1.9. There $\mathfrak{p}$ was a rational prime p and we introduced the spherical Whittaker function $h^{(0)}_{\pi_p} \in \mathcal{W}(\pi_p,\psi_p)^{\mathrm{Gl}_2(\mathbb{Z}_p)}$ we normalised it so it takes value one at one, its values on the torus $T_1(\mathbb{Q}_p)$ were encoded in formula (4.119).

Essentially the same is true in this more general situation. In the above special case we had $d_p = 0$, here we may hay have $d(\psi_{\mathfrak{p}}) \neq 0$.

Assume $\chi_{\mathfrak{p}}$ is unramified and $\mathrm{Ind}_{B(F_{\mathfrak{p}})}^{G(F_{\mathfrak{p}})}\chi_{\mathfrak{p}}$ is irreducible, we define $h^{(0)}_{\pi_{\mathfrak{p}}}$ by the equation (recall that $\mathrm{Ind}_{B(F_{\mathfrak{p}})}^{G(F_{\mathfrak{p}})}\chi_{\mathfrak{p}}$ irreducible $\iff$ $\chi^{(1)}(\varpi_{\mathfrak{p}}) \neq \mathbf{1}$ or $|\ |_{\mathfrak{p}}$)

$$R_{\mathfrak{p}}(f_0)(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) := (1-\chi_{\mathfrak{p}}^{(1)}(\varpi_{\mathfrak{p}}))(N(\mathfrak{p})\chi_{2,\mathfrak{p}}(\varpi_{\mathfrak{p}}))^{d(\psi_{\mathfrak{p}})}h^{(0)}_{\pi_{\mathfrak{p}}}(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) \tag{8.178}$$

We get the following identity of formal power series in the variable q PsI

$$\sum_{\nu=0}^{\infty} h^{(0)}_{\pi_{\mathfrak{p}}}(\begin{pmatrix} \varpi_{\mathfrak{p}}^{\nu-d(\psi_{\mathfrak{p}})} & 0 \\ 0 & 1 \end{pmatrix})q^{\nu} = \frac{1}{(1-\chi_{1,\mathfrak{p}}(\varpi_{\mathfrak{p}})q)(1-N(\mathfrak{p})^{-1}\chi_{2,\mathfrak{p}}(\varpi_{\mathfrak{p}})q)} \tag{8.179}$$

If $\chi_{\mathfrak{p}}^{(1)} = \mathbf{1}$ then $\mathrm{Ind}_{B(F_{\mathfrak{p}})}^{G(F_{\mathfrak{p}})}\chi_{\mathfrak{p}}$ is reducible and $R_{\mathfrak{p}}(f_0) = 0$, the function f_0 generates the kernel of $R_{\mathfrak{p}}$ The quotient $\mathrm{Ind}_{B(F_{\mathfrak{p}})}^{G(F_{\mathfrak{p}})}\chi_{\mathfrak{p}}/Ff_0$ is the Steinberg module. Now we go back to (8.174) and we evaluate $R_{\mathfrak{p}}$ at the function f_w. Then $f_w(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) = 0$ and the terms in the summation over ν are zero. Therefore we get PsSt

$$R_{\mathfrak{p}}(f_w)(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) = N(\mathfrak{p})^{-\operatorname{ord}_{\mathfrak{p}}(t)}\chi_{2,\mathfrak{p}}(\varpi_{\mathfrak{p}})^{\operatorname{ord}_{\mathfrak{p}}(t)} \int_{\mathcal{O}_{F_0,\mathfrak{p}}} \overline{\psi_{\mathfrak{p}}(tu)}du \tag{8.180}$$

We choose as canonical generator

$$h^{(0)}_{\pi_{\mathfrak{p}}} = N(\mathfrak{p})^{d(\psi_{\mathfrak{p}})}\chi_{2,\mathfrak{p}}(\varpi_{\mathfrak{p}})^{-d(\psi_{\mathfrak{p}})}R_{\mathfrak{p}}(f_w). \tag{8.181}$$

and again we get an identity for power series

$$\sum_{\nu=0}^{\infty} h^{(0)}_{\pi_{\mathfrak{p}}}(\begin{pmatrix} \varpi_{\mathfrak{p}}^{\nu-d(\psi_{\mathfrak{p}})} & 0 \\ 0 & 1 \end{pmatrix})q^{\nu} = \frac{1}{1-N(\mathfrak{p})^{-1}\chi_{2,\mathfrak{p}}(\varpi_{\mathfrak{p}})q} \tag{8.182}$$

Now we consider the case that $\chi_{\mathfrak{p}}$ is ramified. We compute the value $R_{\mathfrak{p}}(f_\xi)$, for ξ running through the elements in (8.168). We begin with the case that $\chi_{1,\mathfrak{p}}$ is unramified, we can take $\xi = w$. Then all the terms in the summation over ν are equal to zero (**??**) , we normalise $f_w(w) = 1$ and get

$$R_{\mathfrak{p}}(f_w)(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) = |t|_{\mathfrak{p}}\chi_{2,\mathfrak{p}}(t) \int_{\mathcal{O}_{F_0,\mathfrak{p}}} \psi_{\mathfrak{p}}(ut)du \tag{8.183}$$

Consequently we define in this case

$$h^{(0)}_{\pi_{\mathfrak{p}}}(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) = N(\mathfrak{p})^{d(\psi_{\mathfrak{p}})}\chi_{2,\mathfrak{p}}(t)R_{\mathfrak{p}}(f_w)(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) \tag{8.184}$$

We see that in all cases the support of the restriction of $h^{(0)}_{\pi_{\mathfrak{p}}}$ to $T_1(F_{\mathfrak{p}}) = F_{\mathfrak{p}}^\times$ in the set $\{t \mid \operatorname{ord}_{\mathfrak{p}}(tpsi) \geq -d(\psi_{\mathfrak{p}})\}$ and the value of $h^{(0)}_{\pi_{\mathfrak{p}}}$ on $\varpi_{\mathfrak{p}}^{-d(\psi_{\mathfrak{p}})}$ is 1 (or at least a unit).

We look at the case $\xi_{\nu_0} = \begin{pmatrix} 1 & 0 \\ \varpi_{\mathfrak{p}}^{\nu_0} & 1 \end{pmatrix}$, we normalise $f_{\xi_{\nu_0}}(\xi_{\nu_0}) = 1$. In the summation in formula (8.161) the only (possibly) non zero term is $\nu = \nu_0$. Hence the value of the sum is xinu

$$R_{\mathfrak{p}}(f_{\xi_{\nu_0}})(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}) = |t|_{\mathfrak{p}}\chi_{2,\mathfrak{p}}(t)N(\mathfrak{p})^{\nu_0}\chi_{\mathfrak{p}}^{(1)}(\varpi_{\mathfrak{p}})^{\nu_0} \int_{\mathcal{O}_{F_0,\mathfrak{p}}\setminus\varpi_{\mathfrak{p}}\mathcal{O}_{F_0,\mathfrak{p}}} \chi_{1,\mathfrak{p}}(u)^{-1}\overline{\psi_{\mathfrak{p}}(t\varpi_{\mathfrak{p}}^{-\nu_0}u)}du. \tag{8.185}$$

Here a short discussion about normalisations of measures is in order. Our measure du is additively invariant on $F_{0,\mathfrak{p}}$ and $\mathrm{vol}_{du}(\mathcal{O}_{F_0,\mathfrak{p}}) = 1$ The measure $\frac{du}{|u|_\mathfrak{p}}$ is a multiplicatively invariant measure on $F_{0,\mathfrak{p}}^\times$, for the volume of the group of units we have

$$\mathrm{vol}_{\frac{du}{|u|_\mathfrak{p}}}(\mathcal{O}_{F_0,\mathfrak{p}}^\times) = (1 - \frac{1}{N(\mathfrak{p})}).$$

We define the local Tamagawa measure on $T(F_{0,\mathfrak{p}})$ by $(1 - \frac{1}{N(\mathfrak{p})})^{-1} \frac{dt_\mathfrak{p}}{|t_\mathfrak{p}|_\mathfrak{p}}$ it gives volume one to $\mathcal{O}_{F_0,\mathfrak{p}}^\times$, and we define the Tamagawa measure

$$d_{\mathrm{Tam}}\underline{t}_f = \prod_{\mathfrak{p}} (1 - \frac{1}{N(\mathfrak{p})})^{-1} \frac{dt}{|t|_\mathfrak{p}}.$$

Then we have by definition $\mathrm{vol}_{d_{\mathrm{Tam}}\underline{t}_f} T(\hat{\mathcal{O}}_{F_0}^\times)) = 1.$

Any residue class $x + \varpi_\mathfrak{p}^{\mathfrak{f}(\chi_{1,\mathfrak{p}})}$ has volume $(\frac{1}{N(\mathfrak{p})})^{\mathfrak{f}(\chi_{1,\mathfrak{p}})}$ for the measure du and therefore we can say that the Gaussian sum is equal to

$$G(t\varpi_\mathfrak{p}^{-\nu_0}, \chi_{1,\mathfrak{p}}, \psi_\mathfrak{p}) = \sum_{\varepsilon \in (\mathcal{O}_{F_0,\mathfrak{p}}/(\varpi_\mathfrak{p}^{\mathfrak{f}(\chi_{1,\mathfrak{p}})}))^\times} \chi_{1,\mathfrak{p}}(\varepsilon)^{-1} \overline{\psi_\mathfrak{p}(t\varpi_\mathfrak{p}^{-\nu_0}\varepsilon)}. \tag{8.186}$$

if (8.165) holds and zero otherwise. Then it is clear that the Gaussian sum is an algebraic integer. If we replace t by $t\eta$ with $\eta \in F_{F_0,\mathfrak{p}}^\times$ then clearly teta

$$G(t\eta\varpi_\mathfrak{p}^{-\nu_0}, \chi_{1,\mathfrak{p}}, \psi_\mathfrak{p}) = \chi_{1,\mathfrak{p}}(\eta) G(t\varpi_\mathfrak{p}^{-\nu_0}, \chi_{1,\mathfrak{p}}, \psi_\mathfrak{p}) \tag{8.187}$$

This tells us that in this case $h_{\pi_\mathfrak{p}}^{(0)}(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix})$ is supported on the annulus $\varpi_\mathfrak{p}^{\nu_0 + d(\psi_\mathfrak{p}) - \mathfrak{f}\chi_{1,\mathfrak{p}}} \mathcal{O}_{F_0,\mathfrak{p}}^\times$ and again we can normalise by requiring

$$h_{\pi_\mathfrak{p}}^{(0)}(\begin{pmatrix} \varpi_\mathfrak{p}^{\nu_0 + d(\psi_\mathfrak{p}) - \mathfrak{f}\chi_{1,\mathfrak{p}}} & 0 \\ 0 & 1 \end{pmatrix}) = 1.$$

We still have the supercuspidal representations $\pi_\mathfrak{p}$, these are the representations for which the Kirillow-model is $\mathcal{S}(F_{0,\mathfrak{p}}^\times)$, we follow the argumentation in [Cas]. Let $n_0 = \mathfrak{f}(\pi_\mathfrak{p})$ and let h_1 be a generator of the one dimensional vector space $H(\pi_\mathfrak{p})^{K_{\mathfrak{p},1}(\mathfrak{p}^{n_0})}$. We consider the element $W(\mathfrak{p}^{n_0}) = \begin{pmatrix} 0 & \varpi_\mathfrak{p}^{-n_0} \\ -1 & 0 \end{pmatrix}$ (the Atkin-Lehner involution), it conjugates $K_{\mathfrak{p},1}(\mathfrak{p}^{n_0})$ into $K'_{\mathfrak{p},1}(\mathfrak{p}^{n_0})$ where the condition $a \equiv 1 \mod \mathfrak{p}^{n_0}$ is replaced by $d \equiv 1 \mod \mathfrak{p}^{n_0}$. Therefore $\pi_\mathfrak{p}(W(\mathfrak{p}^{n_0})h_1) = h_2$ will be a generator of $H(\pi_\mathfrak{p})^{K'_{\mathfrak{p},1}(\mathfrak{p}^{n_0})}$. We look at the restriction of h_1 to the annuli $\varpi_\mathfrak{p}^\nu \mathcal{O}_{F_0,\mathfrak{p}}^\times$. It is clear that $h_1(\varpi_\mathfrak{p}^\nu \varepsilon) = h_1(\varpi_\mathfrak{p}^\nu)\zeta_\mathfrak{p}(\varepsilon)$. Let us assume that $d(\psi_\mathfrak{p}) = 0$ then it is also clear that $h_1(\varpi_\mathfrak{p}^\nu) = 0$ if $\nu < 0$. The computations in [Cas] yield

$$W(\mathfrak{p}^{n_0}) h_1(\begin{pmatrix} \varpi_\mathfrak{p}^\nu \varepsilon & 0 \\ 0 & 1 \end{pmatrix}) = C(\psi_\mathfrak{p}) h_1(\begin{pmatrix} \varpi_\mathfrak{p}^{-\nu}\varepsilon^{-1} & 0 \\ 0 & 1 \end{pmatrix})$$

The right hand side is zero if $-\nu < 0$ hence we see that the values of h_1, h_2 on an annulus $\varpi_\mathfrak{p}^\nu \mathcal{O}_{F_0,\mathfrak{p}}^\times$ is zero unless $\nu = 0$. Hence we define the generator $h_{\pi_\mathfrak{p}}^{(0)}$ by the requirement that it assumes the value 1 at the identity element. If $d(\psi_{\mathfrak{p}}) >= 0$ we use the isomorphism in equation 8.154. It tells us that

$$g \mapsto h^{(0)}_{\pi_{\mathfrak{p}}}(\begin{pmatrix} \varpi_{\mathfrak{p}}^{-d(\psi_{\mathfrak{p}})t} & 0 \\ 0 & 1 \end{pmatrix} g) \in \mathcal{W}(\pi_{\mathfrak{p}}, \psi_v^{[-d(\psi_{\mathfrak{p}})]}) \tag{8.188}$$

and since $\psi^{[-d/fp]})$ has additive character 0 we conclude that $t \mapsto h^{(0)}_{\pi_{\mathfrak{p}}}(\begin{pmatrix} \varpi_{\mathfrak{p}}^{-d(\psi_{\mathfrak{p}})}t & 0 \\ 0 & 1 \end{pmatrix} t)$ is supported on the annulus $\mathcal{O}^{\times}_{F_0,v}$. Therefore we can normalise $h^{(0)}_{\pi_{\mathfrak{p}}}$ by requiring

$$h^{(0)}_{\pi_{\mathfrak{p}}}(\begin{pmatrix} \varpi_{\mathfrak{p}}^{-d(\psi_{\mathfrak{p}})} & 0 \\ 0 & 1 \end{pmatrix}) = 1.$$

For any irreducible $\pi_{\mathfrak{p}}$ we introduce the number $\mathfrak{e}(e_{\mathfrak{p}})$, this is the smallest integer for which the value $h^{(0)}_{\pi_{\mathfrak{p}}}(\begin{pmatrix} \varpi_{\mathfrak{p}}^{\mathfrak{e}(e_{\mathfrak{p}})} & 0 \\ 0 & 1 \end{pmatrix}) \neq 0$ and the we normalise such that this value is actually equal to 1. It does not depend on the choice of the generator

Periods again

Now that we have chosen a generator $h^{(0)}_{\pi_{\mathfrak{p}}}$ at all finite places we choose generators in $\bigotimes_{v \in S_\infty} H^1(\mathfrak{g}_v, K_v, \tilde{\mathcal{D}}_{\lambda_v} \otimes \mathcal{M}_{\lambda_v})$ and of course these generators will be tensor product of the generators in the factors.

If $v \in S_\infty$ is real, then the maximal compact subgroup K_v^* containing K_v is not connected. As before we denote the Whittaker realisation of $\mathcal{D}_{\lambda_v}$ by $\tilde{\mathcal{D}}_{\lambda_v}$. The module $\tilde{\mathcal{D}}_{\lambda_v}$ is a sum of two copies which are switched under the action of K_v^*/K_v. The space

$$H^1(\mathfrak{g}_v, K_v, \tilde{\mathcal{D}}_{\lambda_v} \otimes \mathcal{M}_{\lambda_v}) = \operatorname{Hom}_{K_v}(\Lambda^1(\mathfrak{g}/\mathfrak{k}), \tilde{\mathcal{D}}_{\lambda_v} \otimes \mathcal{M}_{\lambda_v})$$

is the direct sum of a $+$ and a $-$ eigenspace, both of dimension 1. In (4.152) we wrote down generators of these one dimensional spaces

$$\omega_+^\dagger = \frac{1}{2}(\omega^\dagger + i^n \bar{\omega}^\dagger) \; ; \; \omega_-^\dagger = \frac{1}{2}(\omega^\dagger - i^n \bar{\omega}^\dagger) \tag{8.189}$$

The choice of these generators depends on the choice of several specific basis elements. To justify the selection of these basis elements we can put a $\mathbb{Z}-$ module structures on $\mathcal{D}_v, \mathfrak{g}/\mathfrak{k}$, (See [44]). The module $\mathcal{M}_{\lambda_v}$ has a $\mathbb{Z}$ -module structure by definition and the choices become natural.

We do essentially the same for a complex place v, we have seen that for $i = 1, 2$ the space $\operatorname{Hom}_{K_\infty}(\Lambda^i(\mathfrak{g}/\tilde{\mathfrak{k}})_R, \tilde{\mathcal{D}}_v \otimes \mathcal{M}_{\lambda^\vee})$ is of dimension one. For the elements e_μ in (**??**) we can choose tensor product of monomials $X^{n-\nu}Y^\nu \otimes \bar{X}^{n-\bar{\nu}}\bar{Y}^{\bar{\nu}}$. Then we require that our generator $\omega^{1,\dagger}$ in degree one satisfies

$$\int_0^\infty (< \omega_\epsilon^{1,\dagger}, H \otimes X^n \otimes \bar{X}^n >= \frac{\Gamma(2n+2)}{(2\pi)^{2n+2}} \tag{8.190}$$

In degree two we use the isomorphism $\kappa : \Lambda^{!}(\mathfrak{g}/\mathfrak{k}) \xrightarrow{\sim} \Lambda^{2}(\mathfrak{g}/\mathfrak{k})$ we define $\omega^{2,\dagger} =^{t} \kappa^{-1}(\omega^{1,\dagger})$.

Let $\nu = 1$ or 2 and $\underline{\epsilon} = \prod_{v\in S_\infty,\mathrm{real}} \epsilon_v$ be a character

$$\underline{\epsilon} : \pi_0(G(\mathbb{R})) = \prod_{v\in S_\infty,\mathrm{real}} K_v^*/K_v \to \mathbb{C}^\times$$

then

$$\omega_{\underline{\epsilon}}^{\nu,\dagger} \in \mathrm{Hom}_{K_\infty}(\Lambda^{r_1+\nu r_2}(\mathfrak{g}/\mathfrak{k}), \mathcal{D}_{\underline{\lambda}}^{\epsilon} \otimes \mathcal{M}_{\underline{\lambda}})$$

will be the product over the $v \in S_\infty$ where the local factor is $\omega_{\epsilon_v}^{\dagger}$ for a real place and $\omega^{(\nu),\dagger}$. for a complex place.

Now we have constructed an isomorphism between absolutely irreducible $G(\mathbb{A}_f)$ modules

$$\mathcal{F}_1^{(\bullet)}(\omega_{\underline{\epsilon}}^{(\nu,\dagger)}) : \bigotimes_{\mathfrak{p}} \mathcal{W}(\pi_{\mathfrak{p}}, \psi_{\mathfrak{p}}) \otimes \mathbb{C} \longrightarrow H^{r_1+\nu r_2}(\mathcal{S}^G, \mathcal{M}_{\underline{\lambda}})(\varepsilon \times \pi_f) \otimes \mathbb{C}),$$

the two vector spaces $\mathcal{W}(\pi_f, \psi_f), H^{r_1+\nu r_2}(\mathcal{S}^G, \mathcal{M}_{\underline{\lambda}})(\varepsilon \times \pi_f)$ are $\mathbb{Q}(\pi_f)$ vector spaces. and hence we can define a complex numbers $\Omega(\underline{\epsilon} \times \pi_f)$ such that

$$\frac{1}{\Omega(\underline{\epsilon} \times \pi_f)} \cdot \mathcal{F}_1^{(1)}(\omega_{\underline{\epsilon}}^{\dagger}) : \bigotimes_{\mathfrak{p}} \mathcal{W}_{\mathcal{O}(\pi_f)}(\pi_f, \psi_{\mathfrak{p}}) \xrightarrow{\sim} H^{r_1+\nu r_2}(\mathcal{S}^G, \mathcal{M})(\underline{\epsilon} \times \pi_f), \tag{8.191}$$

these numbers are well defined modulo an element in $\mathbb{Q}(\pi_f)^\times$.

But we can do better. We choose a level subgroup, actually we chose $K_f = \prod_{\mathfrak{p}} K_{\mathfrak{p},1}(\mathfrak{p}^{f(\pi_{\mathfrak{p}})})$, this is in a sense the optimal level for π_f. Then

$$H^{r_1+\nu r_2}(\mathcal{S}_{K_f}^G, \mathcal{M} \otimes \mathcal{O}_F)(\underline{\epsilon} \times \pi_f)_{\mathrm{int}} \subset H^{r_1+\nu r_2}(\mathcal{S}_{K_f}^G, \mathcal{M} \otimes \mathcal{O}_F)(\underline{\epsilon} \times \pi_f)$$

is a locally free $\mathcal{O}_F$ module of rank 1. Let us assume for the moment that it is actually free Hence

$$H^{r_1+\nu r_2}(\mathcal{S}_{K_f}^G, \mathcal{M} \otimes \mathcal{O}_F)(\pi_f) = \mathcal{O}_F e_\mu.$$

.

On the other hand we have chosen specific generators $h_{\pi_{\mathfrak{p}}}^{(0)} \in \mathcal{W}(\pi_{\mathfrak{p}}^{K_{\mathfrak{p}}}, \psi_{\mathfrak{p}})$ and hence $h_{\pi_f}^{(0)} \in \mathcal{W}(\pi_f^{K_f}, \psi_f)$. Then we can define periods by requiring

$$\frac{1}{\Omega(\underline{\epsilon} \times \pi_f)} \mathcal{F}_1(\omega_{\underline{\epsilon}} \times h_{\pi_f}^{(0)}) = e_\mu \tag{8.192}$$

these periods are now well defined up to an element of $\mathcal{O}_F^\times$, hence we may view $\Omega(\underline{\epsilon} \times \pi_f)$ as an element $\Omega(\underline{\epsilon} \times \pi_f) \in \mathbb{C}^\times/\mathcal{O}_F^\times$.

If the class number of F is not one then it is perhaps a good idea to introduce the sheaf $\mathcal{P}_F$ of periods over F. This is the Zariski sheaf of $\mathrm{Spec}(\mathcal{O}_F)$ which is obtained from the presheaf $U \to \mathbb{C}^\times/\mathcal{O}(U)^\times$. Then we can define the period as a section $\Omega(\underline{\epsilon} \times \pi_f) \in \mathcal{P}_F(\mathcal{O}_F)$ and then (8.192) holds locally in the Zariski topology.

Let us assume that we have an isotopical component $H_!^{r_1+r_2}(\mathcal{S}^G, \tilde{\mathcal{M}}^\vee_{\mathbb{Q}(\pi_f)})(\pi_f)$, then we can consider the composition

$$J(\phi_\mu, \underline{\epsilon}\times\pi_f, \tilde{\mu}_f)\circ\frac{1}{\Omega^{(\nu))}(\underline{\epsilon}\times\pi_f)}\mathcal{F}_1^{(1)}(\omega_\varepsilon) : \bigotimes_{\mathfrak{p}} \mathcal{W}_{\mathcal{O}(\pi_f)}(\pi_\mathfrak{p}, \psi_\mathfrak{p}) \longrightarrow \mathrm{Ind}_{\tilde{H}(A_f)}^{\tilde{G}(\mathbb{A}_f)} \tilde{\mu}_f^{-1}.$$

Since we will see in the following subsection that the condition (I_{pp}) is satisfied and since we have some natural choices for the local intertwining operators, this comes down to the computation of a number and this number is expressible in terms of L-values, this is our ultimate goal.

The local intertwining operators

Next issue is to investigate the space of intertwining operators, i.e. we have to check (I_p) and (I_{pp}) and to study the space

$$\mathrm{Hom}_{G(F_\mathfrak{p})}(\mathcal{W}(\pi_\mathfrak{p}, \psi_\mathfrak{p}) \ , \ \ \mathrm{Ind}_{T(F_\mathfrak{p})}^{G(F_\mathfrak{p})}\tilde{\mu}_\mathfrak{p}^{-1}).$$

Of course we need to assume that the central character $\zeta_\mathfrak{p}$ is equal to the character $\tilde{\mu}_\mathfrak{p}$ restricted to the centre. We restrict the functions in $\mathcal{W}(\pi_\mathfrak{p}, \psi_\mathfrak{p})$ to $T_1(F_\mathfrak{p})$ the space of these restrictions is the Kirillow model $\mathcal{K}(\pi_\mathfrak{p})$ this restriction map is injective ([28]) We introduce the subtorus

$$T_1(F_\mathfrak{p}) = \left\{\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\right\}$$

of $T(F_\mathfrak{p})$. Of course a function in $\mathcal{K}(\pi_\mathfrak{p})$ is determined by its restriction to $T_1(F_\mathfrak{p})$, hence we may consider $\mathcal{K}(\pi_\mathfrak{p})$. as a space of functions on $T_1(F_\mathfrak{p}) = F_\mathfrak{p}^\times$ The restriction $\tilde{\mu}_\mathfrak{p}$ to this subgroup $T_1(F_\mathfrak{p})$ is also called $\tilde{\mu}_\mathfrak{p}$, since the central character is given, this restriction determines $\tilde{\mu}_\mathfrak{p}$. For $t \in F_\mathfrak{p}^\times$, we denote by $[t]$ the matrix $[t] = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$. The space of Schwartz-functions $\mathcal{S}(F_\mathfrak{p}^\times)$ is of codimension $0, 1$ or 2 in $\mathcal{K}(\pi_\mathfrak{p})$, The action of $T(F_\mathfrak{p})$ on $\mathcal{K}(\pi_\mathfrak{p})$, is given by $\pi_\mathfrak{p}([t])(f)(x) = f(tx)$, hence it is clear that the space $\mathcal{S}(F_\mathfrak{p}^\times)$ invariant under this action. Therefore we have an intertwining operator $I_\mathfrak{p}$ from $\mathcal{S}(F_\mathfrak{p}^\times)$ to $F[\tilde{\mu}_\mathfrak{p}]$ which given by

$$I_\mathfrak{p}(f)([t_0]) = \int_{T_1(F_\mathfrak{p})} f([t_0 t]\tilde{\mu}_\mathfrak{p}([t])d^\times t,$$

this operator is unique up to a scalar. Here $d^\times t$ is (momentarily) the multiplicatively invariant measure which gives value one to $\mathcal{O}_\mathfrak{p}^\times$. We apply Frobenius and see

$$\mathrm{Hom}_{G(F_\mathfrak{p})}(\mathcal{W}(\pi_\mathfrak{p}, \psi_\mathfrak{p}) \ , \ \ \mathrm{Ind}_{T(F_\mathfrak{p})}^{G(F_\mathfrak{p})}\tilde{\mu}_\mathfrak{p}^{-1}) = \ \mathrm{Hom}_{T(F_\mathfrak{p})}(\mathcal{W}(\pi_\mathfrak{p}, \psi_\mathfrak{p}) \ , \ \tilde{\mu}_\mathfrak{p}^{-1}).$$

If $\pi_\mathfrak{p}$ is supercuspidal this is our choice of a local intertwining operator $I_\mathfrak{p}^{\mathrm{loc}}$ at $\mathfrak{p}$ up to a normalisation of the measure

If $\pi_\mathfrak{p}$ is not supercuspidal we have to discuss the question whether this operator has a (unique) extension to $\mathcal{K}(\pi_\mathfrak{p}, \psi_\mathfrak{p})$. Our representation $\pi_\mathfrak{p}$ is either a induced representation $\mathrm{Ind}_{B(F_\mathfrak{p})}^{G(F_\mathfrak{p})}\chi_\mathfrak{p}$ or it is a Steinberg representation. In both cases we can consider the quotient (See (8.166))

$$\mathcal{K}(\pi_\mathfrak{p})/\mathcal{S}(F_\mathfrak{p}) \xrightarrow{\sim} \begin{cases} F|t|\chi_{2,\mathfrak{p}}(t) \oplus F\chi_{1,\mathfrak{p}}(t) & \text{if } \pi_\mathfrak{p} \text{ is induced} \\ |t| & \text{if } \pi_\mathfrak{p} \text{ is Steinberg.} \end{cases} \tag{8.193}$$

We consider the exact sequence

$$0 \to \mathrm{Hom}_{T_1(F_\mathfrak{p})}(\mathcal{K}(\pi_\mathfrak{p})/\mathcal{S}(F_\mathfrak{p}), F[\tilde{\mu}_\mathfrak{p}^{-1}]) \to \mathrm{Hom}_{T_1(F_\mathfrak{p}^\times)}(\mathrm{Ind}_{B(F_\mathfrak{p})}^{G(F_\mathfrak{p})}F\chi_\mathfrak{p}, F[\tilde{\mu}_\mathfrak{p}^{-1}]) \to$$

$$\to \mathrm{Hom}_{T_1(F_\mathfrak{p}^\times)}(\mathcal{S}(F_\mathfrak{p}^\times),, F[\tilde{\mu}_\mathfrak{p}^{-1}]) \to \mathrm{Ext}^1_{T_1(F_\mathfrak{p}^\times)}(\mathcal{K}(\pi_\mathfrak{p})/\mathcal{S}(F_\mathfrak{p}), F[\tilde{\mu}_\mathfrak{p}^{-1}]). \tag{8.194}$$

Now it is easy to see that the abelian groups

$$\mathrm{Hom}_{T_1(F_\mathfrak{p})}(\mathcal{K}(\pi_\mathfrak{p})/\mathcal{S}(F_\mathfrak{p}), F\tilde{\mu}_\mathfrak{p}^{-1}) \text{ and } \mathrm{Ext}^1_{T_1(F_\mathfrak{p}^\times)}(\mathcal{K}(\pi_\mathfrak{p})/\mathcal{S}(F_\mathfrak{p}), \tilde{\mu}_\mathfrak{p}^{-1}) = 0$$

are both trivial unless we have

$$\chi_{1,\mathfrak{p}}|t| = \tilde{\mu}_\mathfrak{p}^{-1} \text{or } \chi_{2,\mathfrak{p}}|t|^{-1} = \tilde{\mu}_\mathfrak{p}^{-1}. \tag{\textit{pole}}$$

Hence we see: If *(pole)* is false then

$$\mathrm{Hom}_{T_1(F_\mathfrak{p}^\times)}(\mathrm{Ind}_{B(F_\mathfrak{p})}^{G(F_\mathfrak{p})}\chi_\mathfrak{p}, F[\tilde{\mu}_\mathfrak{p}^{-1}]) \to \mathrm{Hom}_{T_1(F_\mathfrak{p}^\times)}(\mathcal{S}(F_\mathfrak{p}^\times), F[\tilde{\mu}_\mathfrak{p}^{-1}]) \tag{8.195}$$

is an isomorphism, our intertwining operator has a unique extension. We want a formula for this extension and consider the expression

$$\int_{T_1(F_\mathfrak{p})} f([tt_0]\tilde{\mu}_\mathfrak{p}([t])d^\times t \tag{8.196}$$

for all functions in $f \in \mathcal{K}(\pi_\mathfrak{p})$. Of course as it stands it does not always make sense. But given f we can find an integer $N_0 > 0$ such that

$$f|\{[t] \in T_1(F_\mathfrak{p})||t|_\mathfrak{p} \le N(\mathfrak{p})^{-N_0}\} = a|t|\chi_{2,\mathfrak{p}}(t) + b\chi_{1,\mathfrak{p}}(t). \tag{8.197}$$

Let us put $c_0 = N(\mathfrak{p})^{-N_0}$ and let $T_1(F_\mathfrak{p})(\le c_0)$ be the above neighbourhood of 0. Then our above expression becomes

$$\begin{aligned}\textstyle\int_{T_1(F_\mathfrak{p})} f([tt_0])\tilde{\mu}_\mathfrak{p}([t])d^\times t = \int_{T_1(F_\mathfrak{p})(>c_0)} f([tt_0])\tilde{\mu}_\mathfrak{p}([t])d^\times t \\ + \textstyle\int_{T_1(F_\mathfrak{p})(\le c_0)} f([tt_0]\tilde{\mu}_\mathfrak{p}([t])d^\times t.\end{aligned} \tag{8.198}$$

The first term is well defined, we have to assign a value to the second term.

Let $\eta_\mathfrak{p} : F_\mathfrak{p}^\times \to F^\times$ be any (continuous) character (which is the local component of an algebraic Hecke character). We consider the formal power series (in the variable q)

$$\sum_{\nu=0}^{\infty} \Big(\int_{F_\mathfrak{p}^\times(|t|=N(\mathfrak{p})^{-\nu})} \eta_\mathfrak{p}(t)d^\times t\Big)q^\nu. \tag{8.199}$$

This power series is identically zero if $\eta_\mathfrak{p}$ is ramified and if $\eta_\mathfrak{p}$ is unramified each summand is $\eta_\mathfrak{p}(\varpi_\mathfrak{p})^\nu q^\nu$ and hence the power series sums up to $\frac{1}{1-\eta_\mathfrak{p}(\varpi_\mathfrak{p})q}$. We attach the usual local Euler factor to $\eta_\mathfrak{p}$:

$$L(\eta_\mathfrak{p}, s) := \begin{cases} \frac{1}{1-\eta_\mathfrak{p}(\varpi_\mathfrak{p})N(\mathfrak{p})^{-s}}. & \text{if } \eta_\mathfrak{p} \text{ is unramified} \\ 1 & \text{if } \eta_\mathfrak{p} \text{ is ramified} \end{cases} \tag{8.200}$$

and then

$$L(\eta_\mathfrak{p}, 0)^{-1} \int_{F_\mathfrak{p}^\times((\leq c_0)} \eta_\mathfrak{p}(t) d^\times t \tag{8.201}$$

is a finite sum and has a well defined value.

We return to our representation which is $\pi_\mathfrak{p} = \mathrm{Ind}_{B(F_\mathfrak{p})}^{G(F_\mathfrak{p})}\chi_\mathfrak{p}$ or $\pi_\mathfrak{p} = \mathrm{St}(\chi_\mathfrak{p})$. For any $\tilde{\mu}_\mathfrak{p} : T(F_\mathfrak{p}) \to E^\times$ as above we define localL

$$L(\pi_\mathfrak{p} \times \tilde{\mu}_\mathfrak{p}, s) = \begin{cases} L(| \, |_\mathfrak{p}^{-1}\chi_{1,\mathfrak{p}}\tilde{\mu}_\mathfrak{p}, s)(L(\chi_{2,\mathfrak{p}}\tilde{\mu}_\mathfrak{p}, s) & \text{if } \pi_\mathfrak{p} \text{ is induced} \\ L(| \, |_\mathfrak{p}^{-1}\chi_{1,\mathfrak{p}}\tilde{\mu}_\mathfrak{p}, s) & \text{if } \pi_\mathfrak{p} \text{ is Steinberg} \end{cases} \tag{8.202}$$

The above $\pi_\mathfrak{p}$ are those irreducible representations for which $h_{\pi_\mathfrak{p}}^{(0)}$ does not have compact support. For the other representations and especially for the cuspidal then we put $L(\pi_\mathfrak{p} \otimes \tilde{\mu}_\mathfrak{p}, s) = 1$.

Hence we define the (provisorial) local intertwining operator
Klocalintop

$$\begin{gathered} {}'I_\mathfrak{p}^{\mathrm{loc}} : \mathcal{K}(\pi_\mathfrak{p}, \psi_\mathfrak{p}) \to \mathrm{Ind}_{T(F_\mathfrak{p}}^{G(F_\mathfrak{p}}(\tilde{\mu}_\mathfrak{p}) \\ f(\cdot) \mapsto L(\pi_\mathfrak{p} \times \tilde{\mu}_\mathfrak{p}, 0)^{-1} \int_{T_1(F_\mathfrak{p})} f([t]\cdot)\tilde{\mu}_\mathfrak{p}([t])d^\times t \end{gathered} \tag{8.203}$$

We recall that we want to study the integral cohomology, therefore a level subgroup $K_\mathfrak{p}^T$ is given to us, it has to satisfy the condition b) above. In this case our function f and $\tilde{\mu}_\mathfrak{p}$ are invariant under K_f^T and then we renormalise our operator and define

$$I_\mathfrak{p}^{\mathrm{loc}}(f)(g) = \frac{[T_1(\mathcal{O}_{F_0,\mathfrak{p}}) : K_\mathfrak{p}^{T_1}]}{L(\pi_\mathfrak{p} \times \tilde{\mu}_\mathfrak{p}, 1)} \int_{T_1(F_\mathfrak{p})} f([t]g)\tilde{\mu}_\mathfrak{p}([t])d^\times t, \tag{8.204}$$

here the measure $d^\times t = d_{\mathrm{Tam}}t$. Then the right hand side of i

$$I_\mathfrak{p}^{\mathrm{loc}}(f)(g) = L(\pi_\mathfrak{p} \times \tilde{\mu}_\mathfrak{p}, 0)^{-1} \sum_{\varepsilon \in T_1(F_\mathfrak{p}^\times)/K_\mathfrak{p}^{T_1}} f(\varepsilon g)\tilde{\mu}_f(\varepsilon) \tag{8.205}$$

s actually a finite sum, if rearrange the terms.

From this we conclude that the local intertwining operator $I_\mathfrak{p}^{loc}$ is defined over $\mathbb{Q}(\pi_p, \chi_\mathfrak{p}^{(1)}, \tilde{\mu}_\mathfrak{p}.)$ If the conductur $d(\psi_\mathfrak{p}) = 0$ it transforms the spherical function

$h^{(0)}_{\pi_\mathfrak{p}}$ into the spherical function in the induced module which also takes value one at the identity element.

We come to the final computation, we choose our pin point $x_\infty \times \underline{g}_f = x_\infty \times \prod_\mathfrak{p} g_\mathfrak{p}$ and we have a level subgroup $K_f^T = \prod_\mathfrak{p} K_\mathfrak{p}^T$ which satisfies b) with respect to this pin point. We know that the Manin-Drinfeld principle is valid in this case. Hence (see also the argument further down)

$$< j((x, \underline{\underline{g}}_f), r_{\lambda,\mu})(\mathcal{F}^\bullet(\frac{[\omega_\epsilon]}{\Omega(\underline{\epsilon}\times\pi_f)} \times h^{(0)}_{\pi_f})), \tilde{\mu}_f >=$$

$$[T_1(\hat{\mathcal{O}}_{F_0}) : K_f^{T_1}] \int_{T_1(\mathbb{Q})\backslash T_1(\mathbb{A})} < \mathcal{F}(\frac{\omega_{\underline{\epsilon}}}{\Omega(\underline{\epsilon}\times\pi_f)} \times h^{(0)}_{\pi_f})(\underline{t}\underline{g}_f), \tilde{\mu}_f > d^\times_\infty t_\infty \times d_{\mathrm{Tam}}\underline{t}_f =$$

$$[T_1(\hat{\mathcal{O}}_{F_0}) : K_f^{T_1}] \int_{T_1(\mathbb{Q})\backslash T_1(\mathbb{A})} (\sum_{a\in T(\mathbb{Q})} \frac{\omega_{\underline{\epsilon}}}{\Omega(\underline{\epsilon}\times\pi_f)} \times h^{(0)}_{\pi_f}(a\underline{t}\underline{g}_f), \tilde{\mu}_f > d_{\mathrm{Tam}}\underline{t}. \tag{8.206}$$

Now we allow ourselves to interchange summation and integration (same argument as in (4.146), then the last expression becomes

$$[T_1(\hat{\mathcal{O}}_{F_0}) : K_f^{T_1}] \int_{T_1(\mathbb{A})} (< \frac{\omega_{\underline{\epsilon}}}{\Omega(\underline{\epsilon}\times\pi_f)} \times h^{(0)}_{\pi_f})[\underline{t}\underline{g}_f], \tilde{\mu}_f > d_{\mathrm{Tam}}t =$$

$$\frac{1}{\Omega(\underline{\epsilon}\times\pi_f,)} \prod_{v\in S_\infty} \int_{T_1(F_v)} < \omega_v^\dagger, e_{\mu_v} > d^\times_\infty t_v \prod_\mathfrak{p} L(\pi_\mathfrak{p}\times\tilde{\mu}_f, 0) I^{\mathrm{loc}}(h^{(0)}_\mathfrak{p})(\underline{g}_f) \tag{8.207}$$

For the archimedian places v we denoted the Whittaker model of the representation π_v by $\tilde{\mathcal{D}}^\pm_{\lambda_v}$, for real places, $\tilde{\mathcal{D}}_{\lambda_v}$ for complex places. We have the local Euler factor

$$L_v(\pi_v, s) = \frac{\Gamma(s)}{(2\pi)^s} \text{ and } L_\infty(\pi_\infty, s) = \prod_{v\in S_\infty} L_v(\pi_v, s) \tag{8.208}$$

We define the completed L- function

$$\Lambda(\pi, s) = L_\infty(\pi_\infty, s) L(\pi_f, s) \tag{8.209}$$

Now we see from the definition of the generators $\omega_v^\dagger$ we get

$$\int_{T_1(F_v)} < \omega_v^\dagger, e_{\mu_v} > d^* t_v = \frac{\Gamma(1 + w(\mu))}{(2\pi)^{1+w(\mu)}} \tag{8.210}$$

and hence we get the final formula

$$\begin{aligned} &< j((x, \underline{\underline{g}}_f), r_{\lambda,\mu})(\mathcal{F}^\bullet(\frac{[\omega_\epsilon]}{\Omega(\underline{\epsilon}\times\pi_f,)} \times h^{(0)}_{\pi_f})) = \\ &\frac{1}{\Omega(\underline{\epsilon}\times\pi_f)} \Lambda(\pi\otimes\tilde{\mu}, 0) I^{\mathrm{loc}}(h^{(0)}_{\pi_f}(g_\mathfrak{p})) \end{aligned} \tag{8.211}$$

This is now exactly the expression we want to see. We have identified the factor $\mathcal{L}(\pi\otimes\chi,\mu)$ in (8.128),(8.129) a special values of an L-function. We have to interpret this formula.

It is clear from our computations above that for all places $\mathfrak{p}$ where $\chi_\mathfrak{p},\tilde{\mu}_\mathfrak{p}$ are unramified and where $t_{0,\mathfrak{p}}\in T_1(\mathcal{O}_{F,\mathfrak{p}})$ we have $I^{\rm loc}_\mathfrak{p}(h^{(0)}_{\pi_\mathfrak{p}})(t_{0,\mathfrak{p}})=1$. Hence it is clear that in the infinite product $I^{\rm loc}(h^{(0)}_{\pi_f})([\underline{t}_{0,f}])=\prod_\mathfrak{p} I^{\rm loc}_\mathfrak{p}(h^{(0)}_{\pi_\mathfrak{p}})(t_{0,\mathfrak{p}})$ almost all factors are equal to one.

It is also clear that $I^{\rm loc}_\mathfrak{p}(h_{\pi_\mathfrak{p}}([t_{0,\mathfrak{p}}])\in F(\tilde{\mu}_f)$, this was the field extension of F which was generated by the values of $\tilde{\mu}_f$. We may evaluate the local intertwining operator at any function $\underline{h}_{\pi_f}=\prod' h_{\pi_\mathfrak{p}}$ where $h_{\pi_\mathfrak{p}}=h^{(0)}_{\pi_\mathfrak{p}}$ for almost all $\mathfrak{p}$ we can find an element $\underline{h}_{\pi_f}$ such that $I^{\rm loc}_\mathfrak{p}(h_{\pi_\mathfrak{p}}([t_{0,\mathfrak{p}}])\neq 0,?$ for all $\mathfrak{p}$. We apply our formula above to this function then we can conclude

Corollary 8.3.1. *The number* $\frac{1}{\Omega(\underline{\epsilon}\times\pi_f)}\Lambda(\pi\otimes\tilde{\mu},0)\in F[\tilde{\mu}_f]$, *For any element* $\sigma\in$ *Gal*$(F(\tilde{\mu}_f)))/\mathbb{Q})$ *we have*

$$^\sigma(\frac{1}{\Omega(\underline{\epsilon}\times\pi_f,)}\Lambda(\pi\otimes\tilde{\mu}_f,0))=\frac{1}{\Omega(^\sigma\underline{\epsilon}\times^\sigma\pi_f,)}\Lambda(^\sigma\pi\otimes^\sigma\tilde{\mu}_f,0)$$

Of course we also get an integrality statement. We are stiil working with our level subgroup $K_f=\prod_\mathfrak{p} K_{\mathfrak{p},1}(\mathfrak{p}^{f(\pi_\mathfrak{p})})$, but we choose a pin point $(x_\infty,\underline{g}_f)$ and a level subgroup $K_f^T\in T(\mathbb{A}_f)$ such that $K_f^T\underline{g}_fK_f=\underline{g}_fK_f$ -this is our condition b) above. We get the maps

$$j(x_\infty,\underline{t}_f):\mathcal{S}^T_{K_f^T}\to\mathcal{S}^G_{K_f},j(x_\infty,\underline{t}_f)^\bullet:H_c^{r_1+r_2}(\mathcal{S}^G_{K_f},\mathcal{M}^\flat\otimes\mathcal{O}_F)\to H_c^{r_1+r_2}(\mathcal{S}^T_{K_f^T},\mathcal{O}_F\otimes\tilde{\mu}_f),$$

it follows from the definition of the periods that the cohomology class

$$\mathcal{F}^\bullet(\frac{[\omega_\epsilon]}{\Omega(\underline{\epsilon}\times\pi_f)}\times h^{(0)}_{\pi_f})\in H^{r_1+r_2}(\mathcal{S}^G_{K_f},\mathcal{M}^\flat\otimes\mathcal{O}_F)(\varepsilon\times\pi_f)_{!,\rm int}.$$

Hence we can lift this class to a class $\mathcal{F}^\bullet(\frac{[\omega_\epsilon]}{\Omega(\underline{\epsilon}\times\pi_f)}\times h^{(0)}_{\pi_f})^*$ in $H_c^{r_1+r_2}(\mathcal{S}^G_{K_f},\mathcal{M}^\flat\otimes\mathcal{O}_F)$ and the restriction

$$j(x_\infty,\underline{t}_f)^\bullet(\mathcal{F}^\bullet(\frac{[\omega_\epsilon]}{\Omega(\underline{\epsilon}\times\pi_f)}\times h^{(0)}_{\pi_f})^*\in H_c^{r_1+r_2}(\mathcal{S}^T_{K_f^T},\mathcal{O}_F\otimes\tilde{\mu}_f)$$

gives us the number

$$<(j(x_\infty,\underline{t}_f)^\bullet(\mathcal{F}^\bullet(\frac{[\omega_\epsilon]}{\Omega(\underline{\epsilon}\times\pi_f)}\times h^{(0)}_{\pi_f})^*,e_{\tilde{\mu}_f}>$$

which is the number in the top line in (8.206). But this number may depend on the lift. We still have proposition 8.2.1 which says that this number does not depend on the lift if $j\circ\partial_{\chi_{\tilde{\mu}}}=0$.and hence we can say

Theorem 8.3.1. *If* $j \circ \partial_{\tilde{\mu}_f} = 0$. *then*

$$< (j(x_\infty, \underline{g}_f)^\bullet(\mathcal{F}^\bullet(\tfrac{[\omega_\epsilon]}{\Omega(\pi_f,\epsilon)} \times h^{(0)}_{\pi_f}), \tilde{\mu}_f >:=< (j(x_\infty, \underline{g}_f)^\bullet(\mathcal{F}^\bullet(\tfrac{[\omega_\epsilon]}{\Omega(\pi_f,\epsilon)} \times h^{(0)}_{\pi_f})^*, e_{\tilde{\mu}_f} >=$$

$$\int_{T_1(\mathbb{Q})\backslash T_1(\mathbb{A})} < \mathcal{F}(\frac{\omega_{\underline{\epsilon}}}{\Omega(\underline{\epsilon} \times \pi_f)} \times h^{(0)}_{\pi_f}(\underline{t}(x_\infty \times \underline{g}_f), e_{\tilde{\mu}_f} > d^*\underline{t} =$$

$$\frac{1}{\Omega(\underline{\epsilon} \times \pi_f)} \Lambda(\pi \otimes \tilde{\mu}, 0) I^{\mathrm{loc}}(h^{(0)}_{\pi_f})(\underline{g}_f)$$

where $d^*\underline{t}_f$ has volume 1 on K_f^T. As far as $j \circ \partial_{\chi_{\tilde{\mu}}} = 0$ is concerned we have

Proposition 8.3.2. *The class* $j \circ \partial_{\chi_{\tilde{\mu}}} = 0$ *unless* $\underline{\lambda}$ *is of parallel weight, i.e. all the* n_ι *are equal to the same number* m.

If $\underline{\lambda}$ *is of parallel weight we still have* $j \circ \partial_{\chi_{\tilde{\mu}}} = 0$ *unless* $\underline{\mu} = \underline{\lambda}, w_0\underline{\lambda}$.

Proof. Postponed □

If we are in the exceptional case that $j \circ \partial_{\chi_{\tilde{\mu}}} \neq 0$ we use the Manin-Drinfeld argument. We can find Hecke operators C' in (the central subalgebra) of the Hecke algebra which annihilate $j \circ \partial_{\chi_{\tilde{\mu}}} \neq 0$ and act by multiplication by a non zero algebraic integer $\pi_f(T_h) \in \mathcal{O}_F$ on $H^{r_1+r_2}_c(\mathcal{S}^G_{K_f}, \mathcal{M}^\flat \otimes \mathcal{O}_F)(\pi_f)$ (See [**?**]). We consider the ideal generated by all these numbers $\pi_f(T_h) \in \mathcal{O}_F$ and we assume for simplicity that it is a principal ideal $\Delta(\pi_f, \mu)$. Then we can apply proposition 8.2.1 and get

$$< (j(x_\infty, \underline{g}_f)^\bullet(\mathcal{F}^\bullet(\tfrac{[\omega_\epsilon]}{\Omega(\underline{\epsilon}\times\pi_f)} \times h^{(0)}_{\pi_f}), T_h\chi^{[\mu,1]} >:=< (j(x_\infty, \underline{g}_f)^\bullet(\mathcal{F}^\bullet(\tfrac{[\omega_\epsilon]}{\Omega(\underline{\epsilon}\times\pi_f)} \times h^{(0)}_{\pi_f})^*, T_h\chi^{[\mu,1]} >=$$

$$\int_{T_1(\mathbb{Q})\backslash T_1(\mathbb{A})} < \mathcal{F}(\frac{\omega_{\underline{\epsilon}}}{\Omega(\underline{\epsilon} \times \pi_f)} \times \pi_f(T_h) h^{(0)}_{\pi_f})(\underline{g}_f), \tilde{\mu}_f > d^\times\underline{t} =$$

$$\frac{\pi_f(T_h)}{\Omega(\underline{\epsilon} \times \pi_f)} \Lambda(\pi \otimes \tilde{\mu}, 0) I^{\mathrm{loc}}(h^{(0)}_{\pi_f})(\underline{g}_f) \tag{8.212}$$

This formula is a supplement to the theorem above if the proposition 8.2.1 does not apply directly. We have used this argument already before in section 5.6.

Corollary 8.3.2. *If* $j \circ \partial_{\chi_{\tilde{\mu}}} = 0$ *then*

$$\frac{\Lambda(\pi \otimes \tilde{\mu}, 0)}{\Omega(\underline{\epsilon} \times \pi_f)} I^{\mathrm{loc}}(h^{(0)}_{\pi_f})(\underline{g}_f) \in \mathcal{O}_{F[\tilde{\mu}]}$$

and if this is not the case then we get the weaker result

$$\Delta(\pi_f, \mu) \frac{\Lambda(\pi \otimes \tilde{\mu}, 0)}{\Omega(\underline{\epsilon} \times \pi_f)} I^{\mathrm{loc}}(h^{(0)}_{\pi_f})(\underline{g}_f) \in \mathcal{O}_{F[\tilde{\mu}]}$$

We look at the numbers $I^{\mathrm{loc}}(h^{(0)}_{\pi_f})(\underline{g}_f) = \prod_{\mathfrak{p}} I^{\mathrm{loc}}_{\mathfrak{p}}(h^{(0)}_{\pi_{\mathfrak{p}}})(\underline{g}_{\mathfrak{p}})$. They are algebraic integers, of course our goal must be to arrange the data such that the number $I^{\mathrm{loc}}(h^{(0)}_{\pi_{\mathfrak{p}}})(\underline{g}_{\mathfrak{p}}) \neq 0$ and to keep the (the number of) prime divisors "small" . Given our pin point $g_{\mathfrak{p}}$ we compute

$$
\begin{gathered}
[T(\mathcal{O}_{F_0,\mathfrak{p}}) : K^T_{\mathfrak{p}}] \int_{T_1(F_{0,\mathfrak{p}})} h^{(0)}_{\pi_{\mathfrak{p}}}([t_{\mathfrak{p}}]g_{\mathfrak{p}})\tilde{\mu}_{1,\mathfrak{p}}([t_{\mathfrak{p}}])d_{\mathrm{Tam}}t_{\mathfrak{p}} = \\
[T(\mathcal{O}_{F_0,\mathfrak{p}}) : K^T_{\mathfrak{p}}] \sum_{n\in\mathbb{Z}} \tilde{\mu}_{1,\mathfrak{p}}(\varpi_{\mathfrak{p}})^n \int_{\mathcal{O}^\times_{F_0,\mathfrak{p}}} h^{(0)}_{\pi_{\mathfrak{p}}}(\begin{pmatrix} \varpi^n_{\mathfrak{p}}\varepsilon & 0 \\ 0 & 1 \end{pmatrix} g_{\mathfrak{p}})\tilde{\mu}_{1,\mathfrak{p}}(\varepsilon)d_{\mathrm{Tam}}\varepsilon
\end{gathered}
\tag{8.213}
$$

We have the freedom to choose $g_{\mathfrak{p}}$, of course we always want condition b).

We start with the choice $g_{\mathfrak{p}} = [t_0] = \begin{pmatrix} \varpi^{m_0}_{\mathfrak{p}} & 0 \\ 0 & 1 \end{pmatrix}$, since $h^{(0)}_{\pi_{\mathfrak{p}}}$ is the new vector we know $h^{(0)}_{\pi_{\mathfrak{p}}}(\begin{pmatrix} \varpi^{n+m_0}_{\mathfrak{p}}\varepsilon & 0 \\ 0 & 1 \end{pmatrix}) = \zeta_{\mathfrak{p}}(\varepsilon)h^{(0)}_{\pi_{\mathfrak{p}}}(\begin{pmatrix} \varpi^{n+m_0}_{\mathfrak{p}} & 0 \\ 0 & 1 \end{pmatrix})$ and hence the right hand side in (8.213) becomes

$$
\sum_{n\in\mathbb{Z}} \tilde{\mu}_{1,\mathfrak{p}}(\varpi_{\mathfrak{p}})^n h^{(0)}_{\pi_{\mathfrak{p}}}(\begin{pmatrix} \varpi^{n+m_0}_{\mathfrak{p}} & 0 \\ 0 & 1 \end{pmatrix}) \int_{\mathcal{O}^\times_{F_0,\mathfrak{p}}} \zeta_{\mathfrak{p}}(\varepsilon)\tilde{\mu}_{1,\mathfrak{p}}(\varepsilon)d^\times\varepsilon \tag{8.214}
$$

This is of course only useful if $\zeta_{\mathfrak{p}}\tilde{\mu}_{1,\mathfrak{p}}$ is unramified, i.e. $\zeta_{\mathfrak{p}}\tilde{\mu}_{1,\mathfrak{p}}(\varepsilon) = 1$. We assume that this is the case, then our pin point $g_{\mathfrak{p}}$ does not put any constraint on $K^T_{\mathfrak{p}}$. Hence we may assume that $K^T_{\mathfrak{p}} \cap T_1(F^\times_{0,\mathfrak{p}}) = \mathcal{O}^\times_{F_0,\mathfrak{p}}$ and the integral is simply equal to 1.

Again we have to discuss different cases. We have seen that $h^{(0)}_{\pi_{\mathfrak{p}}}(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix})$ is supported on an annulus if $\pi_{\mathfrak{p}}$ is a principal series representation and $\chi_{1,\mathfrak{p}}$ is ramified, or if $\pi_{\mathfrak{p}}$ is a discrete series representation. Hence we can choose m_0 so that in the summation the only surviving term is the term $n = 0$ and hence under these conditions we can achieve

$$
I^{\mathrm{loc}}_{\mathfrak{p}}(h^{(0)}_{\pi_{\mathfrak{p}}})(\underline{g}_{\mathfrak{p}}) = 1.
$$

If the function $h^{(0)}_{\pi_{\mathfrak{p}}}(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix})$ is not supported on an annulus, then our computations above show that we can find a m_0 such that

$$
I^{\mathrm{loc}}_{\mathfrak{p}}(h^{(0)}_{\pi_{\mathfrak{p}}})(\begin{pmatrix} \varpi^{m_0}_{\mathfrak{p}} & 0 \\ 0 & 1 \end{pmatrix}) = 1.
$$

We come to the case where $\zeta_{\mathfrak{p}}\tilde{\mu}_{1,\mathfrak{p}}$ is ramified, we have to choose a different pin point. We take

$$
g_{\mathfrak{p}} = \begin{pmatrix} \varpi^{m_0}_{\mathfrak{p}} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \varpi^{-\nu_0}_{\mathfrak{p}} \\ 0 & 1 \end{pmatrix}
$$

where $\nu_0 > 0$. This imposes some restriction on the choice of our level $K^T_{\mathfrak{p}}$, if we want condition b) (for $K_{\mathfrak{p}} = K_{\mathfrak{p},1}(\mathfrak{p}^{n_0})$) we must have:

$$
\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \in K^T_{\mathfrak{p}} \implies \frac{t_1}{t_2} \equiv 1 \mod \mathfrak{p}^{\nu_0}
$$

We compute the value

$$h^{(0)}_{\pi_\mathfrak{p}}(\begin{pmatrix}\varpi_\mathfrak{p}^{n+m_0}\varepsilon & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & \varpi_\mathfrak{p}^{-\nu_0}\\ 0 & 1\end{pmatrix}) = h^{(0)}_{\pi_\mathfrak{p}}(\begin{pmatrix}1 & \varepsilon\varpi_\mathfrak{p}^{-\nu_0+n+m_0}\\ 0 & 1\end{pmatrix}\begin{pmatrix}\varpi_\mathfrak{p}^{n+m_0}\varepsilon & 0\\ 0 & 1\end{pmatrix})$$

$$h^{(0)}_{\pi_\mathfrak{p}}(\begin{pmatrix}\varpi_\mathfrak{p}^{n+m_0} & 0\\ 0 & 1\end{pmatrix})\zeta(\varepsilon)\psi_\mathfrak{p}(\varepsilon\varpi_\mathfrak{p}^{-\nu_0+n+m_0})$$

and hence our expression in (8.213) becomes

$$[T_1(\mathcal{O}_{F_0,\mathfrak{p}}):K_\mathfrak{p}^T]\sum_{n\in\mathbb{Z}} h^{(0)}_{\pi_\mathfrak{p}}(\begin{pmatrix}\varpi_\mathfrak{p}^{n} & 0\\ 0 & 1\end{pmatrix})\int_{\mathcal{O}^\times_{F_0,\mathfrak{p}}}\psi_\mathfrak{p}(\varpi_\mathfrak{p}^{-\nu_0+n}\varepsilon)\zeta_\mathfrak{p}(\varepsilon)\tilde{\mu}_f(\varepsilon)d^\times\varepsilon$$

The integral is a Gaussian sum, the conductor $\mathfrak{f}(\zeta_\mathfrak{p}\tilde{\mu}_{1,\mathfrak{p}})>0$ and we know that the Gaussian sum is zero unless the additive character $u\mapsto\psi_\mathfrak{p}(\varpi_\mathfrak{p}^{-\nu_0+n+m_0}u)$ restricted to $\mathcal{O}_{F_0,\mathfrak{p}}$ has conductor $\mathfrak{f}(\zeta_\mathfrak{p}\tilde{\mu}_{1,\mathfrak{p}})$. Hence the Gaussian sum is only non zero if

$$-\nu_0+n+\mathfrak{f}(\zeta_\mathfrak{p}\tilde{\mu}_{1,\mathfrak{p}})=-d(\psi_\mathfrak{p}).$$

We have normalised $h^{(0)}_{\pi_\mathfrak{p}}(\begin{pmatrix}\varpi_\mathfrak{p}^{-d(\psi_\mathfrak{p})} & 0\\ 0 & 1\end{pmatrix})=1$ hence we have to choose $\nu_0=\mathfrak{f}(\zeta_\mathfrak{p}\tilde{\mu}_{1,\mathfrak{p}})$ and we find

$$I^{\mathrm{loc}}_\mathfrak{p}(h^{(0)}_{\pi_\mathfrak{p}})(\underline{g}_\mathfrak{p})=G(\zeta_\mathfrak{p}\tilde{\mu}_{1,\mathfrak{p}},\psi_\mathfrak{p}) \tag{8.215}$$

The Gaussian sum only depends on the restriction of $\zeta_\mathfrak{p}\tilde{\mu}_{1,\mathfrak{p}}$ to the units $T(\mathcal{O}_{F_0,\mathfrak{p}})$, The values of $\mathfrak{p}\tilde{\mu}_{1,\mathfrak{p}}$ on the units are $(N(\mathfrak{p})-1)\times N(\mathfrak{p})^{\mathfrak{f}(\zeta_\mathfrak{p}\tilde{\mu}_{1,\mathfrak{p}})}-$ th roots of unity, the values of $\psi_\mathfrak{p}$ are also $N(\mathfrak{p})^{\mathfrak{f}(\zeta_\mathfrak{p}\tilde{\mu}_{1,\mathfrak{p}})}-$ th roots of unity. Hence the number $G(\zeta_\mathfrak{p}\tilde{\mu}_{1,\mathfrak{p}},\psi_\mathfrak{p})$ is an algebraic integer, it lies in the field $\mathbb{Q}[\zeta_{N(\mathfrak{p})-1},\zeta_{N(\mathfrak{p})^{\mathfrak{f}(\zeta_\mathfrak{p}\tilde{\mu}_{1,\mathfrak{p}})}}]$ In factorisation of this integer into prime ideals only primes lying above p occur. Hence we have control over the numbers $I^{\mathrm{loc}}_\mathfrak{p}(h^{(0)}_{\pi_\mathfrak{p}})(\underline{g}_\mathfrak{p})$.

The numbers $\frac{\Lambda(\pi\otimes\tilde{\mu},0)}{\Omega(\underline{\epsilon}\times\pi_f)}$ are of arithmetic interest, for instance the factorisation into primes contains information about the structure of the cohomology (see further down). For instance we can ask whether they are integers themselves, or if not what are the denominators. This amounts to the study of the numbers $\Delta(\pi_f,\mu)$ and $I^{\mathrm{loc}}(h^{(0)}_{\pi_f})(\underline{g}_f)=\prod_\mathfrak{p} I^{\mathrm{loc}}(h^{(0)}_{\pi_\mathfrak{p}})(\underline{g}_\mathfrak{p})$.

The number $\Delta(\pi_f,\mu)$ is of global nature, it should be a denominator of the Eisenstein class. We determined this denominator in a very special case in Chapter 5 Theorem 5.1.2, in this case $\mathcal{G}/\mathbb{Z}=\mathrm{Gl}_2/\operatorname{Spec}(\mathbb{Z})$ and the level was $K_f=\mathcal{G}(\hat{\mathbb{Z}})$. It is certainly not to difficult to extend this result to the case of congruence subgroups.

I believe that it is very interesting problem to study the numbers $\Delta(\pi_f,\mu)$ if $F_0/\mathbb{Q}$ is a non trivial extension of $\mathbb{Q}$, for instance simply a real quadratic extension.

8.3.4 Poincare duality and modular symbols.

We consider the second example where can apply the strategy which we outlined in section 8.2.1. We start from an arbitrary quasi split group $H/\mathbb{Q}$. Let $F/\mathbb{Q}$ a minimal normal extension which splits $H/\mathbb{Q}$. Let $T/\mathbb{Q} \subset B/\mathbb{Q}$ a maximal torus in a Borel sub group $B/\mathbb{Q}$. Let us denote the center of $H/\mathbb{Q}$ by $C_H/\mathbb{Q} \subset T/\mathbb{Q}$, for any $\lambda \in X^*(T \times_{\mathbb{Q}} \bar{\mathbb{Q}})$ the restriction of λ to C_H is denoted by λ_{C_H}.

We take $G/\mathbb{Q} = H \times H/\mathbb{Q}$ for our ambient group and we embed $H/\mathbb{Q} \hookrightarrow G/\mathbb{Q}$ diagonally. We follow the steps in 8.2.1. We choose a level subgroup $K_f^H \subset H(\mathbb{A}_f)$ and put $K_f^H \times K_f^H = K_f^G$. Then we have $\mathcal{S}^H_{K_f^H} \times \mathcal{S}^H_{K_f^H} = \mathcal{S}^G_{K_f}$. We choose the base point $e_0 \in H(\mathbb{R})/K_\infty^H = X$ and $x_0 = (e_0, e_0)$. From this we get the map

$$j(x_0, e_f) : \mathcal{S}^H_{K_f^H} = H(\mathbb{Q})\backslash H(\mathbb{R})/K^H_\infty \times H(\mathbb{A}_f)/K_f^H \to \mathcal{S}^G_{K_f}. \tag{8.216}$$

An irreducible representation of $G/\mathbb{Q}$ is of the form $\mathcal{M}_{\underline{\lambda}} = \mathcal{M}_{\lambda_1} \otimes \mathcal{M}_{\lambda_2}$, where λ_1, λ_2 are dominant weights in $X^*(T \times_{\mathbb{Q}} F)$, we view them as modulss over $\mathcal{O}_F$We choose a one dimensional representation $\mu : H/\mathbb{Z} \to \mathcal{O}_F\mu$. We have to understand the module of H-homomorphisms $\mathrm{Hom}_H(\mathcal{M}_{\underline{\lambda}}, \mathcal{O}\mu)$. We know

Proposition 8.3.3. *The module $Hom_H(\mathcal{M}_{\underline{\lambda}}, \mathcal{O}\mu)$. is free of rank*

$$d_{\underline{\lambda},\mu} = \begin{cases} 1 & \text{if } \lambda_1^{(1)} = -w_0(\lambda_1^{(2)}) \text{ and } \lambda^{(1)}_{1,C_H} + \lambda^{(1)}_{2,C_H} = \mu_{C_H} \\ 0 & \text{else} \end{cases} \tag{8.217}$$

here of course w_0 is the element in the Weyl group which sends all positive roots into negative roots.

Proof. This is obvious from the theory of representations of algebraic groups. □

We assume now that $d_{\underline{\lambda},\mu} = 1$ and choose a generator $r_{\underline{\lambda},\mu} \in \mathrm{Hom}_H(\mathcal{M}_{\underline{\lambda}}, \mathcal{O}\mu)$. We get a homomorphism (See 8.117)

$$j(x_\infty, e_f, r_{\underline{\lambda},\mu})^{d_H} : H_c^{d_H}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\underline{\lambda}}) \to H^{d_H}(\mathcal{S}^H_{K_f^H}, \mathcal{O}\mu).$$

We know that the Manin-Drinfeld principle is valid, this means that we get a canonical splitting for the Hecke modules

$$H_c^{d_H}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\underline{\lambda}} \otimes F) = \mathrm{Im}(H^{d_H-1}(\partial(\mathcal{S}^G_{K_f}), \tilde{\mathcal{M}}_{\underline{\lambda}} \otimes F)) \oplus H_!^{d_H}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\underline{\lambda}} \otimes F)$$

Hence we can restrict our homomorphism

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$$j(x_\infty, e_f, r_{\underline{\lambda},\mu})_!^{d_H} : H_!^{d_H}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\underline{\lambda}} \otimes F) \to H^{d_H}(\mathcal{S}^H_{K_f^H}, F\mu).$$

We want to discuss the integral cohomology. We start from the exact sequence and get a diagram

$$\begin{array}{ccccccc} H^{d_H-1}(\partial(\mathcal{S}^G_{K_f}), \tilde{\mathcal{M}}_{\underline{\lambda}}) & \xrightarrow{\delta} & H_c^{d_H}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\underline{\lambda}}) & \to & H_!^{d_H}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\underline{\lambda}}) & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ H^{d_H-1}(\partial(\mathcal{S}^G_{K_f}), \tilde{\mathcal{M}}_{\underline{\lambda}})^{\mathrm{sat}}_{\mathrm{int}} & \xrightarrow{\delta} & H_c^{d_H}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\underline{\lambda}})_{\mathrm{int}} & \to & H_!^{d_H}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\underline{\lambda}})_{\mathrm{int}} & \to & 0 \end{array} \tag{8.218}$$

and the Manin-Drinfeld principle gives us a splitting up to isogeny

$$H_c^{d_H}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\underline{\lambda}})_{\text{ int}} \supset H^{d_H-1}(\partial(\mathcal{S}^G_{K_f}),\tilde{\mathcal{M}}_{\underline{\lambda}})_{\text{ int}} \oplus H_!^{d_H}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\underline{\lambda}})_{\text{ int}} \quad (8.219)$$

the reader should pay attention to the difference between the subscripts $_{\text{int}}$ and $_{\text{int}}$ we have an inclusion $H_!^{d_H}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\underline{\lambda}})_{\mathbf{int}} \subset H_!^{d_H}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\underline{\lambda}})_{\text{ int}}$ the quotient is a finite module, which may be difficult to understand. There is a non zero number $\Delta_\lambda \in \mathcal{O}$ such that $\Delta_\lambda H_!^{d_H}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\underline{\lambda}})_{\text{ int}} \subset H_!^{d_H}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\underline{\lambda}})_{\mathbf{int}}$. Then it is clear that $j(x_\infty,e_f,r_{\underline{\lambda},\mu})_!^{d_H}$ induces Hecke invariant homomorphisms

$$\begin{aligned} j(x_\infty,e_f,r_{\underline{\lambda},\mu})_!^{d_H} &: H_!^{d_H}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\underline{\lambda}})_{\mathbf{int}} \to H_!^{d_H}(\mathcal{S}^G_{K_f},\mathcal{O}\mu) \\ j(x_\infty,e_f,r_{\underline{\lambda},\mu})_!^{d_H} &: H_!^{d_H}(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\underline{\lambda}})_{\text{ int}} \to \tfrac{1}{\Delta_\lambda} H_!^{d_H}(\mathcal{S}^G_{K_f},\mathcal{O}\mu) \end{aligned} \quad (8.220)$$

Assume that the argument, which I have in my mind, is correct, then we may even apply proposition 8.2.1 and then we can take $\Delta_\lambda = 1$.

We can produce classes in $H_!^{d_H}((\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\underline{\lambda}})$. For any $0 \le r \le d_H$ we have the Künneth homomorphism

$$H_!^r(\mathcal{S}^H_{K_f^H},\tilde{\mathcal{M}}_{\lambda_1}) \times H_!^{d_H-r}(\mathcal{S}^H_{K_f^H},\tilde{\mathcal{M}}_{\lambda_2}) \to H_!^{d_H}((\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\underline{\lambda}})$$

and taking the composition with $j(x_\infty,e_f,r_{\underline{\lambda},\mu})_!^{d_H}$ we get

$$J(r): H_!^r(\mathcal{S}^H_{K_f^H},\tilde{\mathcal{M}}_{\lambda_1}) \times H_!^{d_H-r}(\mathcal{S}^H_{K_f^H},\tilde{\mathcal{M}}_{\lambda_2}) \to \frac{1}{\Delta_\lambda} H_!^{d_H}(\mathcal{S}^H_{K_f^H},\mathcal{O}\mu) \quad (8.221)$$

It is clear from the definitions that this homomorphism is the cup product.

If necessary we enlarge our field such that we get decompositions (up to isogeny) into absolutely irreducible Hecke modules

$$H_!^r(\mathcal{S}^H_{K_f^H},\tilde{\mathcal{M}}_{\lambda_1}) \supset \bigoplus\nolimits_{\pi_{1,f}} H_!^r(\mathcal{S}^H_{K_f^H},\tilde{\mathcal{M}}_{\lambda_1})[\pi_{1,f}]$$

$$H_!^{d_H-r}(\mathcal{S}^H_{K_f^H},\tilde{\mathcal{M}}_{\lambda_2}) \supset \bigoplus\nolimits_{\pi_{2,f}} H_!^{d_H-r}(\mathcal{S}^H_{K_f^H},\tilde{\mathcal{M}}_{\lambda_2})[\pi_{2,f}]$$

$$H_!^{d_H}(\mathcal{S}^H_{K_f^H},\mathcal{O}\mu) \supset \bigoplus\nolimits_{\tilde{\mu}:\text{type}(\tilde{\mu})=\mu} \mathcal{O}\tilde{\mu}.$$

Hence we have to compute the pairing

$$H_!^r(\mathcal{S}^H_{K_f^H},\tilde{\mathcal{M}}_{\lambda_1})[\pi_{1,f}] \times H_!^{d_H-r}(\mathcal{S}^H_{K_f^H},\tilde{\mathcal{M}}_{\lambda_2})[\pi_{2,f}] \to \frac{1}{\Delta_\lambda}\mathcal{O}\tilde{\mu} \quad (8.222)$$

This pairing is zero unless $\pi_{1,f},\pi_{2,f}$ are essentially dual, i.e. dual up to a twist. discussion in **??**)

But we still have to go one step further, we to take into account the action of $\pi_0(G(\mathbb{R}))$ on the different cohomology groups, our pairing becomes

$$H^r(\mathcal{S}^H_{K_f^H},\tilde{\mathcal{M}}_{\lambda_1})[\epsilon_1 \times \pi_{1,f}] \times H^{d_H-r}(\mathcal{S}^H_{K_f^H},\tilde{\mathcal{M}}_{\lambda_2})[\epsilon_2 \times \pi_{2,f}] \to \frac{1}{\Delta_\lambda}\mathcal{O}\tilde{\mu} \quad (8.223)$$

and the consistency rule $\epsilon_1\epsilon_2 = \tilde{\mu}_\infty$ should be satisfied.

Here we described a very general situation, it seems to be a very difficult problem to compute this pairing, at the end of section 8.2.2 we formulated the expectation that the value of this pairing should be expressible in terms of L-functions. attached to $\pi_{1,f}$, I have no idea how to do this in general.

A special example

We stop our general reasoning and consider a very special case, we choose a finite extension $F_0/\mathbb{Q}$ and choose $H/\mathbb{Q} = R_{F_0/\mathbb{Q}}(\mathrm{Gl}_2/F)$ and $G/\mathbb{Q} = R_{F/\mathbb{Q}}((\mathrm{Gl}_2 \times \mathrm{Gl}_2)/F_0)$. In this situation we can work with the Whittaker model. In this case $d_H = 2r_1 + 3r_2$, we pick two isomorphism types $\pi_{1,f}, \pi_{2,f}$ which occur in the cuspidal cohomology $H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda_1}), H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda_2})$.

We want to compute the value of the pairing

$$H^{r_1+r_2}(\mathcal{S}^H_{K^H_f}, \tilde{\mathcal{M}}_{\lambda_1})(\underline{\epsilon}_2 \times \pi_{2,f}) \times H^{r_1+2r_2}(\mathcal{S}^H_{K^H_f}, \tilde{\mathcal{M}}_{\lambda_1})(\underline{\epsilon}_2 \times \pi_{2,f}) \to \frac{1}{\Delta(\pi_{1,f}, \pi_{2,f})}\mathcal{O}\tilde{\mu} \tag{8.224}$$

here ϵ_1, ϵ_2 and η are characters on $(\mathbb{Z}/2\mathbb{Z})^{r_1} = \pi_0(H(\mathbb{R}))$. Of course this pairing is zero unless we have $\pi^\vee_{2,f} = \pi_{1,f}$ and $\underline{\epsilon}_1\underline{\epsilon}_2\eta = 1$.

We start from the Whittaker model $\tilde{\mathcal{D}}_\lambda \otimes \mathcal{W}(\pi_f, \psi_f)$ and we choose generators $\omega^{(\dagger,\nu)}_{\underline{\epsilon}} \in \mathrm{Hom}_{K_\infty}(\Lambda^{r_1+\nu r_2}(\mathfrak{g}/\mathfrak{k}), \tilde{\mathcal{D}}^{\epsilon}_{\lambda}, \psi_\infty) \otimes \mathcal{M}_{\underline{\lambda}})$ and $h^{(i)}_f = \prod_{\mathfrak{p}} h^{(i)}_{\mathfrak{p}} \in \prod_{\mathfrak{p}} \mathcal{W}(\pi_{i,\mathfrak{p}}, \psi_{\mathfrak{p}})$, where we even choose $h^{(i)}_{\mathfrak{p}} = h^{(0)}_{\pi_{i,\mathfrak{p}}}$ our previously chosen generators. Then the cup product of the two integral(!) cohomology classes $\boxed{\text{cupff}}$

$$[\frac{1}{\Omega^{(1)}(\underline{\epsilon} \times \pi_f)}\mathcal{F}^{(1)}_1(\omega^{(\dagger,1)}_{\underline{\epsilon}} \times h^{(1)}_f)] \cup [\frac{1}{\Omega^{(2)}(\underline{\epsilon} \times (\pi_f)}\mathcal{F}^{(2)}_1(\omega^{(\dagger,2)}_{\underline{\epsilon}} \times h^{(2)}_f)] \tag{8.225}$$

is given by the integral (see section 6.3.11)

$$\frac{1}{\Omega^{(1)}(\underline{\epsilon} \times \pi_f)}\Omega^{(2)}_{\underline{\epsilon}\times\pi_f)} \int_{\mathcal{S}^H_{K^H_f}} \mathcal{F}^{(1)}_1(\omega^{(\dagger,1)}_{\underline{\epsilon}} \times h^{(1)}_f) \wedge \mathcal{F}^{(1)}_1(\omega^{(\dagger,2)}_{\underline{\epsilon}} \times h^{(2)}_f)),$$

by construction the expression under the integral is a differential form in top degree.

We choose a specific invariant volume form $d\underline{y} = dy_\infty \times d\underline{y}_f$ on $H(\mathbb{A})$. We normalize $\mathrm{vol}_{d\underline{g}_f}(K^H_f) = 1$, and we write $dy_\infty = dx_\infty \times dk_\infty$, we require $\mathrm{vol}_{dk_\infty}(K_\infty) = 1$ and dx_∞ will be the volume form given by the Riemannian metric. To write down the integral explicitly we choose an orthonormal basis of $\mathfrak{p} \otimes \mathbb{R}$. This basis will consist of basises of the $\mathfrak{p}_v$. For the $\mathfrak{p}_v$ we choose the following basis

a) If v is real our basis will be $X_{v,+} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, X_{v,-} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

b) For v complex we have the basis $X_{v,0}, X_{v,1}, X_{v,-1}$

Hence we get a basis

$$\{\dots, X_{v,+}, X_{v,-}, \dots\}_{S^{\mathrm{real}}_\infty} \cup \{\dots, X_{v,0}, X_{v,1}, X_{v,-1}, \dots\}_{S^{\mathrm{comp}}_\infty} \tag{8.226}$$

To evaluate the integral we have to look at the value

$$\omega^{(\dagger,1)}_{\underline{\epsilon}} \wedge \omega^{(\dagger,2)}_{\underline{\epsilon}'}(\Lambda^{2r_1+3r_2}X_\nu) \tag{8.227}$$

where the X_ν run through the above basis. The result is an element in $\mathcal{W}(\mathcal{D}_\lambda) \otimes \mathcal{M}_\lambda$. To compute this value we have to divide the above basis into a subset $A = \{\dots, X_\nu, \dots\}$ consisting of $r_1 + r_2$ elements and a complement $B =$

$\{\ldots, X_\mu^*, \ldots\}$ consisting of $r_1 + 2r_2$ elements of our basis, we have to multiply $\omega_{\underline{\epsilon}}^{(\dagger,1)}(\Lambda X_\nu)\omega_{\underline{\epsilon}'}^{(\dagger,2)}(\Lambda X_\mu)$. Then we have to sum over all these divisions into two subsets.

But looking at the definition of our $\omega_{\underline{\epsilon}}^{(\dagger,1)}, \omega_{\underline{\epsilon}'}^{(\dagger,2)}$ we see that only one division into two disjoint sets can give a non zero contribution. We describe this division. Recall that $\underline{\epsilon} = \{\ldots, \epsilon_v, \ldots\}_{v \in S_\infty^{\mathrm{real}}}$ is an array of signs, and $\underline{\epsilon}'$ is the opposite array. Then the first r_1 elements in A will be the X_{v,ϵ_v} with $v \in S_\infty^{\mathrm{real}}$ and this will be supplemented by the $X_{v,0}^*$ with $v \in S_\infty^{\mathrm{comp}}$, this is the set A_1. The set B_1 is the complement, but we also give the explicit description. The first r_1 elements will be the $X_{v,\epsilon_v'}$ and the second part consists of the elements $\{\ldots, X_{v,1}, X_{v,-1}, \ldots\}$. Hence we see that

$$\begin{gathered}\omega_{\underline{\epsilon}}^{(\dagger,1)}(\wedge\omega_{\underline{\epsilon}'}^{(\dagger,2)}(\Lambda^{2r_1+3r_2}X_\nu)) = \\ \omega_{\underline{\epsilon}}^{(\dagger,1)}((\cdots \wedge X_\mu \wedge \ldots)_{X_\mu \in A_1}))\omega_{\underline{\epsilon}'}^{(\dagger,1)}((\cdots \wedge X_\mu \wedge \ldots)_{X_\mu \in B_1}))\end{gathered} \tag{8.228}$$

To evaluate the above integral we apply a method which goes back to Asai(?). We write the constant function 1 on $H(\mathbb{Q})\backslash H(\mathbb{A})$ as the residue of an Eisenstein series. More precisely for any complex number s we define a "height" function $H_s(\underline{y}) = H_s(\underline{bk}) = \rho(\underline{b})^{2+s}$ where $\underline{y} \in H(\mathbb{A}), \underline{b} \in B_H(\mathbb{A})$ and $\underline{k} \in K_\infty \times K_f$. This function is invariant under $B_H(\mathbb{Q})$ and we define

$$\mathrm{Eis}(s, \underline{y}) = \mathrm{Res}_{s=0} \sum_{\gamma \in B_H(\mathbb{Q})\backslash H(\mathbb{Q})} H_s(\gamma \underline{y})$$

It is well known that this series converges for $\Re(s) >> 0$, hence it defines an analytic function in $\Re(s) >> 0$ and it has a meromorphic continuation into the entire s- plane. It is known that this function in s has a simple pole at $s = 0$ and

$$\mathrm{Res}_{s=0}\ \mathrm{Eis}(s, \underline{y}) = \mathrm{Res}_{s=0} \frac{\zeta_F(s+1)}{\zeta_F(s+2)}$$

especially we see that this residue considered as function in $\underline{y}$ is a constant c_F. Therefore we compute the integral $\boxed{\mathrm{IE}}$

$$\int_0^\infty (\mathit{content})\ \mathrm{Eis}(s, \underline{y}) d\underline{y} \tag{8.229}$$

and compute its residue at $s = 0$.

Let us denote the elements $(\cdots \wedge X_\mu \wedge \ldots)_{X_\mu \in A_1}$ resp. $(\cdots \wedge X_\mu \wedge \ldots)_{X_\mu \in B_1}$ by $\mathcal{X}_{A_1}$ resp. $\mathcal{X}_{B_1}$, then our integral becomes

$$\int_{H(\mathbb{Q})\backslash H(\mathbb{A})} \mathcal{F}(\omega_{\underline{\epsilon}}^{(\dagger,1)} \times h_f^{(1)})(\mathcal{X}_{A_1})(y_\infty, \underline{y}_f)\mathcal{F}(\omega_{\underline{\epsilon}}^{(\dagger,1)} \times h_f^{(2)})(\mathcal{X}_{B_1})(y_\infty, \underline{y}_f))\ \mathrm{Eis}(s, \underline{y}) d\underline{y} =$$

$$\int_{H(\mathbb{Q})\backslash H(\mathbb{A})} \mathcal{F}(\omega_{\underline{\epsilon}}^{(\dagger,1)} \times h_f^{(1)})(\mathcal{X}_{A_1})(y_\infty, \underline{y}_f)\mathcal{F}(\omega_{\underline{\epsilon}}^{(\dagger,1)} \times h_f^{(2)})(\mathcal{X}_{B_1})(y_\infty, \underline{y}_f))(\textstyle\sum_{\gamma \in B_H(\mathbb{Q})\backslash H(\mathbb{Q})} H_s(\gamma \underline{y}))d\underline{y}$$

$$\int_{B_H(\mathbb{Q})\backslash H(\mathbb{A})} \mathcal{F}(\omega_{\underline{\epsilon}}^{(\dagger,1)} \times h_f^{(1)})(\mathcal{X}_{A_1})(y_\infty, \underline{y}_f)\mathcal{F}(\omega_{\underline{\epsilon}}^{(\dagger,1)} \times h_f^{(2)})(\mathcal{X}_{B_1})(y_\infty, \underline{y}_f))H_s(\underline{y})d\underline{y}$$

We recall the decomposition $H(\mathbb{A}) = B_H(\mathbb{A})K_\infty^H K_f^{H,0}$ then our measure $d\underline{y} = d\underline{b} \times (dk_\infty \times d\underline{k}_f)$. Then $H_s(\underline{b}\underline{k}_f)) = H_s(\underline{b})$, the expression under the integral is invariant under the action of K_∞^H from the right and hence our integral becomes

$$\int_{B_H(\mathbb{Q})\backslash B_H(\mathbb{A})} \int_{K_f^{H,0}} \mathcal{F}(\omega_{\underline{\epsilon}}^{(\dagger,1)} \times h_f^{(1)})(\mathcal{X}_{A_1})(y_\infty, \underline{y}_f) \mathcal{F}(\omega_{\underline{\epsilon}}^{(\dagger,1)} \times h_f^{(2)})(\mathcal{X}_{B_1})(y_\infty, \underline{y}_f)) d\underline{k}_f (H_s(\underline{b})) d\underline{b}$$

This integral converges for $\Re(s) >> 0$ and the value of the residue at $s = 0$ is equal to the value of our integral. Now $(\omega_{\underline{\epsilon}}^{(\dagger,1)}(\mathcal{X}_{A_1}) \times h_f^{(1)}, \omega_{\underline{\epsilon}'}^{(\dagger,2)}(\mathcal{X}_{B_1}) \times h_f^{(2)}$ are Whittaker functions (with values in $\mathcal{M}_\lambda^\flat \otimes \mathbb{C}, \mathcal{M}_\lambda \otimes \mathbb{C}$ respectively) and we have the Fourier-expansions

$$\mathcal{F}(\omega_{\underline{\epsilon}}^{(\dagger,1)}(\mathcal{X}_{A_1}) \times h_f^{(1)}(\underline{ut}) = \sum_{a \in F^\times} \omega_{\underline{\epsilon}}^{(\dagger,1)}(\mathcal{X}_{A_1}) \times h_f^{(1)})(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \underline{u} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \underline{t} & 0 \\ 0 & 1 \end{pmatrix})$$

and

$$\mathcal{F}(\omega_{\underline{\epsilon}'}^{(\dagger,2)}(\mathcal{X}_{B_1}) \times h_f^{(2)})(\underline{ut}) = \sum_{a \in F^\times} \omega_{\underline{\epsilon}'}^{(\dagger,2)}(\mathcal{X}_{B_1}) \times h_f^{(2)})(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \underline{u} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \underline{t} & 0 \\ 0 & 1 \end{pmatrix})$$

Since the functions $\omega_{\underline{\epsilon}}^{(\dagger,1)}(\mathcal{X}_{A_1}) \times h_f^{(1)}, \omega_{\underline{\epsilon}'}^{(\dagger,2)}(\mathcal{X}_{B_1}) \times h_f^{(2)}$ are Whittaker functions i.e. they satisfy

$$\omega_{\underline{\epsilon}}^{(\dagger,1)}(\mathcal{X}_{A_1}) \times h_f^{(1)}(\begin{pmatrix} 1 & \underline{u} \\ 0 & 1 \end{pmatrix} \underline{y}) = \psi(\underline{u}) \omega_{\underline{\epsilon}}^{(\dagger,1)}(\mathcal{X}_{A_1}) \times h_f^{(1)}(\underline{y})$$

$$\omega_{\underline{\epsilon}'}^{(\dagger,2)}(\mathcal{X}_{B_1}) \times h_f^{(2)}(\begin{pmatrix} 1 & \underline{u} \\ 0 & 1 \end{pmatrix} \underline{y}) = \psi(\underline{u}) \omega_{\underline{\epsilon}'}^{(\dagger,2)}(\mathcal{X}_{B_1}) \times h_f^{(2)}(\underline{y})$$

Our volume form is $d\underline{y} = c_F |\underline{t}|^{-1} d\underline{u} \times d^\times \underline{t} d\underline{k}$ where all these measures are product over local measures, we require $\mathrm{vol}_{dk_v}(K_v) = 1$ and $\mathrm{vol}_{\underline{u}} U(\mathbb{Q}) \backslash U(\mathbb{A}) = 1$ the constant c_F is essentially he inverse of the discriminant.

Then

$$\int_{U(\mathbb{A})} \omega_{\underline{\epsilon}}^{(\dagger,1)}(\mathcal{X}_{A_1}) \times h_f^{(1)})(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \underline{u} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \underline{t} & 0 \\ 0 & 1 \end{pmatrix} \underline{k}) \omega_{\underline{\epsilon}'}^{(\dagger,2)}(\mathcal{X}_{B_1}) \times h_f^{(2)})(\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \underline{u} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \underline{t} & 0 \\ 0 & 1 \end{pmatrix}) \underline{k}) d\underline{u}$$

$$= \begin{cases} \omega_{\underline{\epsilon}'}^{(\dagger,2)}(\mathcal{X}_{B_1}) \times h_f^{(1)})(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \underline{t} & 0 \\ 0 & 1 \end{pmatrix}) \omega_{\underline{\epsilon}'}^{(\dagger,2)}(\mathcal{X}_{B_1}) \times h_f^{(2)})(\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \underline{t} & 0 \\ 0 & 1 \end{pmatrix}) \underline{k}) & \text{if } a + b = 0 \\ 0 & \text{else} \end{cases} \tag{8.230}$$

and therefore our integral becomes

$$\int_{T(\mathbb{Q})\backslash T(\mathbb{A})} \int_{K_f^{H,0}} \sum_{a \in F^\times} \omega_{\underline{\epsilon}}^{(\dagger,1)}(\mathcal{X}_{A_1}) \times h_f^{(1)})(\begin{pmatrix} a\underline{t} & 0 \\ 0 & 1 \end{pmatrix} \underline{k}_f) \omega_{\underline{\epsilon}'}^{(\dagger,2)}(\mathcal{X}_{B_1}) \times h_f^{(2)}) \begin{pmatrix} -a\underline{t} & 0 \\ 0 & 1 \end{pmatrix}) \underline{k}_f) |\underline{t}|^s d\underline{k}_f d^\times \underline{t} \tag{8.231}$$

and since $T(\mathbb{Q}) = F^\times$ for the value of the integral

$$\int_{T(\mathbb{A})}\int_{K_f^{H,0}} \omega_{\underline{\epsilon}}^{(\dagger,1)}(\mathcal{X}_{A_1})\times h_f^{(1)})(\begin{pmatrix}\underline{t} & 0\\ 0 & 1\end{pmatrix}\underline{k}_f)\omega_{\underline{\epsilon}'}^{(\dagger,2)}(\mathcal{X}_{B_1})\times h_f^{(2)})\begin{pmatrix}-\underline{t} & 0\\ 0 & 1\end{pmatrix})\underline{k}_f)|\underline{t}|^s d\underline{k}_f d^\times \underline{t} \tag{8.232}$$

In the variable $\underline{k}_f$ our functions are right invariant under K_f^H hence the integral over $\underline{k}_f$ is actually a finite sum. Then for a fixed value of $\underline{k}_f$ our functions are products of local Whittaker functions, i.e.

$$\omega_{\underline{\epsilon}}^{(\dagger,1)}(\mathcal{X}_{A_1})\times h_f^{(1)})(\begin{pmatrix}\underline{t} & 0\\ 0 & 1\end{pmatrix}\underline{k}_f) = \omega_{\underline{\epsilon}}^{(\dagger,1)}(\mathcal{X}_{A_1}(\begin{pmatrix}t_\infty & 0\\ 0 & 1\end{pmatrix})\prod_{\mathfrak{p}} h_{\mathfrak{p}}^{(1)}((\begin{pmatrix}t_{\mathfrak{p}} & 0\\ 0 & 1\end{pmatrix}k_{\mathfrak{p}})$$

$$\omega_{\underline{\epsilon}'}^{(\dagger,2)}(\mathcal{X}_{B_1})\times h_f^{(2)})\begin{pmatrix}-\underline{t} & 0\\ 0 & 1\end{pmatrix})\underline{k}_f) = \omega_{\underline{\epsilon}}^{(\dagger,2)}(\mathcal{X}_{B_1}(\begin{pmatrix}t_\infty & 0\\ 0 & 1\end{pmatrix})\prod_{\mathfrak{p}} h_{\mathfrak{p}}^{(2)}((\begin{pmatrix}t_{\mathfrak{p}} & 0\\ 0 & 1\end{pmatrix}k_{\mathfrak{p}})$$

and hence our integral becomes

$$\mathrm{vol}_{d\underline{k}}(K_f^H)\sum_{\underline{k}_f\in K_f^{H,0}/K_f}\prod_v\int_{T(F_v)} h_{\mathfrak{p}}^{(1)}((\begin{pmatrix}t_v & 0\\ 0 & 1\end{pmatrix}k_v)h_{\mathfrak{p}}^{(2)}((\begin{pmatrix}t_v & 0\\ 0 & 1\end{pmatrix}k_v)|t_v|_v^s d^\times t_v$$

The local Whittaker functions are explicitly given to us. We look at the different places. We begin with a finite place $\mathfrak{p}$ and if $\pi_{\mathfrak{p}}$ i unramified, i.e. $K_{\mathfrak{p}}^H$ is maximal. We have to compute

$$\int_{T(F_{\mathfrak{p}})} h_{\mathfrak{p}}^{(1)}((\begin{pmatrix}t_{\mathfrak{p}} & 0\\ 0 & 1\end{pmatrix})h_{\mathfrak{p}}^{(2)}((\begin{pmatrix}t_{\mathfrak{p}} & 0\\ 0 & 1\end{pmatrix})|t_{\mathfrak{p}}|_{\mathfrak{p}}^s d^\times t_{\mathfrak{p}}$$

We recall the explicit formulas for the values of $h_{\mathfrak{p}}^{(1)}((\begin{pmatrix}t_{\mathfrak{p}} & 0\\ 0 & 1\end{pmatrix})$ and $h_{\mathfrak{p}}^{(2)}((\begin{pmatrix}t_{\mathfrak{p}} & 0\\ 0 & 1\end{pmatrix})$, Let $\omega(\pi_{1,\mathfrak{p}})$ be the Satake parameter of $\pi_{1,\mathfrak{p}}$ then - as usual- we define

$$\alpha_{\mathfrak{p}} = N(\mathfrak{p})^{\frac{1}{2}}\omega(\pi_{1,\mathfrak{p}})(\begin{pmatrix}\varpi_{\mathfrak{p}} & 0\\ 0 & 1\end{pmatrix}), \beta_{\mathfrak{p}} = N(\mathfrak{p})^{\frac{1}{2}}\omega(\pi_{1,\mathfrak{p}})(\begin{pmatrix}1 & 0\\ 0 & \varpi_{\mathfrak{p}}\end{pmatrix})$$

we introduced the Euler factor in (4.118)

$$L(\pi_{1,\mathfrak{p}}, s) = \frac{1}{(1-\alpha_{\mathfrak{p}}p^{-s})(1-\beta_{\mathfrak{p}}p^{-s})}.$$

After expanding we get

$$L(\pi_{1,\mathfrak{p}}, s) = \sum_{n=0}^{\infty}(\sum_{\nu=0}^{n}\alpha_{\mathfrak{p}}^{n-\nu}\beta_{\mathfrak{p}}^{\nu})N(\mathfrak{p})^{-ns} = \sum_{n=0}^{\infty} h_{\mathfrak{p}}^{(1)}(\begin{pmatrix}\varpi_{\mathfrak{p}}^{\mathfrak{f}(\pi_{\mathfrak{p}})} & 0\\ 0 & 1\end{pmatrix})N(\mathfrak{p})^{n(1-s)} \tag{8.233}$$

The for the second factor we have $\pi_{2,\mathfrak{p}} = \pi_{1,\mathfrak{p}}^\vee$ hence the Satake parameter is $\omega(\pi_{2,\mathfrak{p}}) = \omega(\pi_{1,\mathfrak{p}})^{-1}$. If we now define

$$\beta'_{\mathfrak{p}} = N(\mathfrak{p})^{\frac{1}{2}}\omega(\pi_{2,\mathfrak{p}})(\begin{pmatrix}\varpi_{\mathfrak{p}} & 0\\ 0 & 1\end{pmatrix}), \alpha'_{\mathfrak{p}} = N(\mathfrak{p})^{\frac{1}{2}}\omega(\pi_{2,\mathfrak{p}})(\begin{pmatrix}1 & 0\\ 0 & \varpi_{\mathfrak{p}}\end{pmatrix}),$$

then we have $\alpha_{\mathfrak{p}}\beta'_{\mathfrak{p}} = \alpha'_{\mathfrak{p}}\beta_{\mathfrak{p}} = N(\mathfrak{p})$ and hence $\alpha_{\mathfrak{p}}\alpha'_{\mathfrak{p}}\ \beta_{\mathfrak{p}}\beta'_{\mathfrak{p}} = N(\mathfrak{p})^2$.

We get

$$L(\pi_{2,\mathfrak{p}}, s) = \sum_{n=0}^{\infty}(\sum_{\nu=0}^{n} {\alpha'}_{\mathfrak{p}}^{n-\nu}{\beta'}_{\mathfrak{p}}^{\nu})N(\mathfrak{p})^{-s} = \sum_{n=0}^{\infty} h_{\mathfrak{p}}^{(2)}(\begin{pmatrix} \varpi_{\mathfrak{p}}^{\mathfrak{f}(\pi_{\mathfrak{p}})} & 0 \\ 0 & 1 \end{pmatrix})N(\mathfrak{p})^{n(1-s)} \tag{8.234}$$

We express the inner sums in terms of the semi-simple Satake parameters (See (**??** and remark after it), we have

$$\alpha_{\mathfrak{p}}^{n-\nu}\beta_{\mathfrak{p}}^{\nu} = (\frac{\alpha_{\mathfrak{p}}}{\beta_{\mathfrak{p}}})^{\frac{n}{2}-\nu}(\alpha_{\mathfrak{p}}\beta_{\mathfrak{p}})^{\frac{n}{2}} = \omega^{(1)}(\pi_{1,\mathfrak{p}})^{\frac{n}{2}-\nu}(\alpha_{\mathfrak{p}}\beta_{\mathfrak{p}})^{\frac{n}{2}},$$

the same holds for $\pi_{2,\mathfrak{p}}$. and therefore

$$h_{\mathfrak{p}}^{(1)}((\begin{pmatrix} t_{\mathfrak{p}} & 0 \\ 0 & 1 \end{pmatrix})h_{\mathfrak{p}}^{(2)}((\begin{pmatrix} t_{\mathfrak{p}} & 0 \\ 0 & 1 \end{pmatrix}) = (\sum_{\nu=0}^{n} \omega^{(1)}(\pi_{\mathfrak{p}})^{\frac{n}{2}-\nu})^2$$

Now we have the following identity in power series ring $\mathbb{Z}[u, 1/u][[t]]$:

$$\frac{1-t^2}{(1-u^2t)(1-t)^2(1-u^{-2}t)} = \sum_{n=0}^{\infty}(\sum_{\nu=0}^{n} u^{n-2\nu})^2 t^n$$

(According to Jacquet ([**?**], ??) the proof is a refreshing exercise.) We put $t = N(\mathfrak{p})^{-1-s}$ and $u = \omega^{(1)}(\pi_{\mathfrak{p}})$ then this identity gives us

$$\frac{1-N(\mathfrak{p})^{-2-s}}{(1-\omega^{(1)}(\pi_{\mathfrak{p}})N(\mathfrak{p})^{-1-s})(1-N(\mathfrak{p})^{-1-s})^2(1-\omega^{(1)}(\pi_{\mathfrak{p}})^{-1}(N(\mathfrak{p})^{-1-s}))} =$$

$$\textstyle\sum_{n=0}^{\infty} \Phi_{\mathfrak{p}}^{(1)}(\begin{pmatrix} \varpi_{\mathfrak{p}}^{\mathfrak{f}(\pi_{\mathfrak{p}})} & 0 \\ 0 & 1 \end{pmatrix})\Phi_{\mathfrak{p}}^{(2)}(\begin{pmatrix} \varpi_{\mathfrak{p}}^{\mathfrak{f}(\pi_{\mathfrak{p}})} & 0 \\ 0 & 1 \end{pmatrix})N(\mathfrak{p})^{n(-1-s)} \tag{8.235}$$

The factor in the numerator is the inverse of the local factor of the Dedekind $\zeta_{\mathfrak{p}}(\ )$ function at $s+2$, the factor $(1-N(\mathfrak{p})^{-1-s})$ in the denominator gives us the local zeta factor $\zeta_{\mathfrak{p}}(1+s)$. The remaining expression gives us local factor of the adjoint L-function, i.e.

$$\frac{1}{(1-\omega^{(1)}(\pi_{\mathfrak{p}})N(\mathfrak{p})^{-1-s})(1-N(\mathfrak{p})^{-1-s})(1-\omega^{(1)}(\pi_{\mathfrak{p}})^{-1}(N(\mathfrak{p})^{-1-s}))} = L(\pi_{\mathfrak{p}}, \mathrm{Ad}, s+1) \tag{8.236}$$

Therefore we get for an unramified $\pi_{\mathfrak{p}}$

$$\frac{\zeta_{\mathfrak{p}}(1+s)}{\zeta_{\mathfrak{p}}(2+s)}L(\pi_{\mathfrak{p}}, \mathrm{Ad}, s+1) = \int_{T(F_{\mathfrak{p}})} h_{\mathfrak{p}}^{(1)}((\begin{pmatrix} t_{\mathfrak{p}} & 0 \\ 0 & 1 \end{pmatrix})h_{\mathfrak{p}}^{(2)}((\begin{pmatrix} t_{\mathfrak{p}} & 0 \\ 0 & 1 \end{pmatrix})|t_{\mathfrak{p}}|_{\mathfrak{p}}^{s} d^{\times} t_{\mathfrak{p}} \tag{8.237}$$

In this book we try to avoid the discussions of the subtle phenomena at ramified $\pi_{\mathfrak{p}}$, therefore we assume that a similar formula also holds at the finite number of ramified places, we may take this as definition of the local Euler factor at these places.

The integral at the archimedean places

We treat the cases of a real and a complex place separately.

A) The place v is real. We have the two generators $\omega^\dagger_{v,\pm}$ (4.152) and the factor at our place v becomes

$$\int_0^\infty <\omega^\dagger_{v,+}(H)(\begin{pmatrix} t & 0\\ 0 & 1\end{pmatrix}), <\omega^\dagger_{v,-}(V)(\begin{pmatrix} t & 0\\ 0 & 1\end{pmatrix}) > t^s\frac{dt}{t}$$

We recall the definition of the generators and then our integral becomes (up to some small power of 2 (to be fixed later))

$$<(X-Y\otimes i)^n,(X+Y\otimes i)^n> \int_0^\infty t^{n+2}e^{-4\pi t}t^s\frac{dt}{t} =<(X-Y\otimes i)^n,(X+Y\otimes i)^n> \frac{\Gamma(n+2+s)}{(4\pi)^{n+2+s}}$$

The factor in front is

$$<(X-Y\otimes i)^n),(X+Y\otimes i)^n>=\sum_{\nu,\mu} i^{\mu-\nu}\binom{n}{\nu}\binom{n}{\mu} <X^{n-\nu}Y^\nu, X^{n-\mu}Y\mu>$$

and by definition we have (See **??**) $<X^{n-\nu}Y^\nu, X^{n-\mu}Y\mu>=0$ unless we have $\nu+\mu=n$ and then

$$<X^{n-\nu}Y^\nu, X^\nu Y^{n-\mu}>=\binom{n}{\nu}^{-1}.$$

Hence we see that one of the binomial factor cancels and we find $<(X-Y\otimes i)^n),(X+Y\otimes i)^n>=(2i)^n$. So we finally get

$$\int_0^\infty <\omega_{v,+}(H)(\begin{pmatrix} t & 0\\ 0 & 1\end{pmatrix}), <\omega_{v,-}(V)(\begin{pmatrix} t & 0\\ 0 & 1\end{pmatrix}) > t^s\frac{dt}{t} = (2i)^n\frac{\Gamma(n+2+s)}{(4\pi)^{n+2+s}} \tag{8.238}$$

Let us call this last expression $\mathfrak{G}_v(n,s)$

B) The place v is a complex place, this case is more difficult (interesting, amusing). In this case we have to evaluate

$$\int_0^\infty <\omega_v^{\dagger,1}(X_{0,v})(\begin{pmatrix} t & 0\\ 0 & 1\end{pmatrix}), \omega_v^{\dagger,2}(X_{1,v},X_{-1,v})(\begin{pmatrix} t & 0\\ 0 & 1\end{pmatrix}) > t^s\frac{dt}{t} =$$

We have the explicit formula (4.178) for these these factors for $Z=X_{v,0}$ or $Z=(X_{1,v},X_{-1,v})$ we have cangen3

$$\omega^{\dagger,\bullet}(Z)=\sum_{\mu=-n}^{n}\Phi_{\lambda,\mu}\otimes(\sum_{\mu_1+\mu_2=|\mu|} e^\flat_{\mu_1}\otimes\bar{e}^\flat_{\mu_2}) \tag{8.239}$$

Hence we multiply and get a sum

$$\sum_{\mu,\mu'}\Phi_{\lambda,\mu}(\begin{pmatrix} t & 0\\ 0 & 1\end{pmatrix})\Phi_{\lambda,\mu}(\begin{pmatrix} t & 0\\ 0 & 1\end{pmatrix})\otimes(\sum_{\mu_1,\mu_1'} T(\mu_1,\mu_1'))$$

where

$$T(\mu_1,\mu_1') =< \rho_\lambda(\begin{pmatrix} t & 0 \\ 0 & 1\end{pmatrix})(e^\flat_{\mu_1}\otimes \bar{e}^\flat_{\mu_2}), \rho_{\lambda^\vee}(\begin{pmatrix} t & 0 \\ 0 & 1\end{pmatrix})(e^\flat_{\mu_1'}\otimes \bar{e}^\flat_{\mu_2'} > .$$

But since our pairing is invariant under the action of $G(\mathbb{R})$ we can ignore the $\rho_\lambda(\begin{pmatrix} t & 0 \\ 0 & 1\end{pmatrix})$ and we find that the value

$$< e^\flat_{\mu_1}\otimes \bar{e}^\flat_{\mu_2}, e^\flat_{\mu_1'}\otimes \bar{e}^\flat_{\mu_2'} >= \begin{cases} \binom{|\mu|}{\mu_1}\binom{|\mu|}{|\mu|-\mu_1}) & \text{if } \mu_1=-\mu_1', \mu_2=-\mu_2' \\ 0 & \text{else}\end{cases} \tag{8.240}$$

and taking into account the formulas for the pairing we get for the integrand

$$\sum_{\mu=n}^{-n} \Phi_{\lambda,\mu}\Phi_{\lambda,-\mu}(\begin{pmatrix} t & 0 \\ 0 & 1\end{pmatrix})(\sum_{\mu_1}\binom{|\mu|}{\mu_1}\binom{|\mu|}{|\mu|-\mu_1})t^s$$

We have our explicit expressions for the $\Phi_{\lambda,\mu}$ we have to compute the Mellin transform

$$\frac{4\pi^{2n}}{\Gamma(n+1)^2}\int_0^\infty K_\mu(2\pi t)K_{-\mu}(2\pi t)t^{2n+4+s}\frac{dt}{t} =$$

$$\frac{4\pi^{2n}}{\Gamma(n+1)^2}\frac{\Gamma(n+1+\mu+s/2)\Gamma(n+1-\mu+s/2)\Gamma(n+2+s/2)^2}{(2\pi)^{2n+4+s}} = \tag{8.241}$$

$$\frac{\Gamma(n+2+s/2)^2}{4\pi^{4+s}\Gamma(n+1)^2}\Gamma(n+1+\mu+s/2)\Gamma(n+1-\mu+s/2)$$

To get the value of the above integral we have to sum over the μ. Hence finally we get

$$\mathfrak{G}_v(n,s) := \int_0^\infty <\omega_v^{\dagger,1}(X_{0,v})(\begin{pmatrix} t & 0 \\ 0 & 1\end{pmatrix}), \omega_v^{\dagger,2}(X_{1,v},X_{-1,v})(\begin{pmatrix} t & 0 \\ 0 & 1\end{pmatrix})> t^s\frac{dt}{t} =$$

$$\frac{1}{4\pi^{s+4}}\frac{\Gamma(n+2+s/2)^2}{\Gamma(n+1)}(\sum_{\mu=-n}^{\mu=n}\Gamma(n+1+\mu+s/2)\Gamma(n+1-\mu+s/2)(\sum_{\mu_1}\binom{|\mu|}{\mu_1}\binom{|\mu|}{|\mu|-\mu_1}))) \tag{8.242}$$

Eventually we are interested in the value at $s=0$, then we have the following identity, which I checked experimentally

$$\sum_{\mu=-n}^{\mu=n}\Gamma(n+1+\mu)\Gamma(n+1-\mu)(\sum_{\mu_1}\binom{|\mu|}{\mu_1}\binom{|\mu|}{|\mu|-\mu_1})) = \Gamma(2n+2) \tag{8.243}$$

so that $\mathfrak{G}_v(n,0) = \frac{(n+1)^2}{4\pi^4}(2n+1)!$ We put $\mathfrak{G}_\infty(\lambda,s) = \prod_{v\in S_\infty}\mathfrak{G}(n_v,s)$
Then we see that the value our integral in (8.229) eventually is given by

$$c_F\frac{\zeta_F(1+s)}{\zeta_F(s+2)}\mathfrak{G}_\infty(\lambda,s)L(\pi_f,\mathrm{Ad},s+1) \tag{8.244}$$

We have to take the residue at $s=0$, we know that all the factors except $\zeta_F(s+1)$ are holomorphic at $s=0$ and hence we get for the cup product of the two cohomology classes

$$\frac{1}{\Omega^{(1)}_{\underline{\epsilon}}(\pi_f)}\mathcal{F}_1^{(1)}(\omega^{(1)}_{\underline{\epsilon}} \times h_f^{(1)}) \cup \frac{1}{\Omega^{(2)}_{\underline{\epsilon}}(\pi_f)}\mathcal{F}_1^{(2)}(\omega^{(2)}_{\underline{\epsilon}} \times h_f^{(1)}) = \frac{1}{\Omega^{(1)}_{\underline{\epsilon}}(\pi_f)\Omega^{(2)}_{\underline{\epsilon}}(\pi_f)} \frac{\mathrm{Res}_{s=0}\zeta_F(1+s)}{\zeta_F(2)} \mathfrak{G}_\infty(\lambda,0)L(\pi_f,\mathrm{Ad},1) \tag{8.245}$$

We know that this number is in $\frac{1}{\Delta_\lambda}\mathcal{O}_F$. Let $\mathfrak{p}$ be a prime in $\mathcal{O}_F[\frac{1}{\Delta_\lambda}]$.which divides this number, i.e.

$$\mathfrak{p}^{\delta(\pi_f)}||\frac{1}{\Omega^{(1)}_{\underline{\epsilon}}(\pi_f)\Omega^{(2)}_{\underline{\epsilon}}(\pi_f)} \frac{\mathrm{Res}_{s=0}\zeta_F(1+s)}{\zeta_F(2)} \mathfrak{G}_\infty(\lambda,0)L(\pi_f,\mathrm{Ad},1).$$

We have the non degenerate pairing

$$H^{r_1+r_2}(\mathcal{S}^H_{K^H_f}, \tilde{\mathcal{M}}_{\lambda_1} \otimes \mathcal{O}_F[\frac{1}{\Delta_\lambda}])_{\mathrm{int},!} \times H^{r_1+2r_2}(\mathcal{S}^H_{K^H_f}, \tilde{\mathcal{M}}_{\lambda_2} \otimes \mathcal{O}_F[\frac{1}{\Delta_\lambda}])_{\mathrm{int},!} \to \mathcal{O}_F[\frac{1}{\Delta_\lambda}]$$

and the decomposition into saturated Hecke submodules

$$H^{r_1+r_2}(\mathcal{S}^H_{K^H_f}, \tilde{\mathcal{M}}_{\lambda_1} \otimes \mathcal{O}_F[\tfrac{1}{\Delta_\lambda}])_{\mathrm{int},!} \supset H^{r_1+r_2}(\mathcal{S}^H_{K^H_f}, \tilde{\mathcal{M}}_{\lambda_1} \otimes \mathcal{O}_F[\tfrac{1}{\Delta_\lambda}])_{\mathrm{int},!}(\pi_{1,f}) \oplus H^{r_1+r_2}(\mathcal{S}^H_{K^H_f}, \tilde{\mathcal{M}}_{\lambda_1} \otimes \mathcal{O}_F[\tfrac{1}{\Delta_\lambda}])_{\mathrm{int},!}(\pi_{1,f})^\perp \tag{8.246}$$

where $^\perp$ means that we take the saturated direct sum over the $\pi'_f \neq \pi_f$. We introduce the quotient

$$\tilde{H}^{r_1+r_2}(\mathcal{S}^H_{K^H_f}, \tilde{\mathcal{M}}_{\lambda_1} \otimes \mathcal{O}_F[\tfrac{1}{\Delta_\lambda}])_{\mathrm{int},!}(\pi_{1,f}) = H^{r_1+r_2}(\mathcal{S}^H_{K^H_f}, \tilde{\mathcal{M}}_{\lambda_1} \otimes \mathcal{O}_F[\tfrac{1}{\Delta_\lambda}])_{\mathrm{int},!}/H^{r_1+r_2}(\mathcal{S}^H_{K^H_f}, \tilde{\mathcal{M}}_{\lambda_1} \otimes \mathcal{O}_F[\tfrac{1}{\Delta_\lambda}])_{\mathrm{int},!}(\pi_{1,f})^\perp \tag{8.247}$$

and the above pairing induces a non degenerate pairing

$$\tilde{H}^{r_1+r_2}(\mathcal{S}^H_{K^H_f}, \tilde{\mathcal{M}}_{\lambda_1} \otimes \mathcal{O}_F[\frac{1}{\Delta_\lambda}])_{\mathrm{int},!}(\pi_{1,f}) \times H^{r_1+2r_2}(\mathcal{S}^H_{K^H_f}, \tilde{\mathcal{M}}_{\lambda_2} \otimes \mathcal{O}_F[\frac{1}{\Delta_\lambda}])_{\mathrm{int},!}(\pi_{2,f}) \to \mathcal{O}_F[\frac{1}{\Delta_\lambda}] \tag{8.248}$$

We choose a character $\underline{\epsilon}'$ then $H^{r_1+2r_2}(\mathcal{S}^H_{K^H_f}, \tilde{\mathcal{M}}_{\lambda_2} \otimes \mathcal{O}_F[\frac{1}{\Delta_\lambda}])_{\mathrm{int},!}(\underline{\epsilon}' \times \pi_{2,f})$ is a free $\mathcal{O}_F[\frac{1}{\Delta_\lambda}]$ module of rank one, a generator is $y_0 = [\frac{1}{\Omega^{(2)}_{\underline{\epsilon}'}(\pi_f)}\mathcal{F}_1^{(2)}(\omega^{(2)}_{\underline{\epsilon}} \times h_f^{(1)})]$. Let x_0 be the corresponding generator in $H^{r_1+r_2}(\mathcal{S}^H_{K^H_f}, \tilde{\mathcal{M}}_{\lambda_1} \otimes \mathcal{O}_F[\frac{1}{\Delta_\lambda}])_{\mathrm{int},!}(\underline{\epsilon} \times \pi_{1,f})$ We can find an element $\tilde{x}_0 \in \tilde{H}^{r_1+r_2}(\mathcal{S}^H_{K^H_f}, \tilde{\mathcal{M}}_{\lambda_1} \otimes \mathcal{O}_F[\frac{1}{\Delta_\lambda}])_{\mathrm{int},!}(\epsilon \times \pi_{1,f})$ such that $< x_0, y_0 >= 1$. Let $\varpi_\mathfrak{p}$ be an uniformizer for $\mathfrak{p}$ we lift x_0 to an element $\tilde{x}_0^* \in H^{r_1+r_2}(\mathcal{S}^H_{K^H_f}, \tilde{\mathcal{M}}_{\lambda_1} \otimes \mathcal{O}_F[\frac{1}{\Delta_\lambda}])_{\mathrm{int},!}$ and we can write (we localize at $\mathfrak{p}$)

$$\tilde{x}_0^* = \frac{x_0 + z}{\varpi_\mathfrak{p}^m} \text{ with } z \in H^{r_1+r_2}(\mathcal{S}^H_{K^H_f}, \tilde{\mathcal{M}}_{\lambda_1} \otimes \mathcal{O}_F[\frac{1}{\Delta_\lambda}])_{\mathrm{int},!}(\pi_{1,f})^\perp. \tag{8.249}$$

Then

$$1 =< \tilde{x}_0, y_0 >= \frac{< x_0, y_0 >}{\varpi_{\mathfrak{p}}^m} \tag{8.250}$$

and this implies $m = \delta(\pi_f)$.

We can slightly modify this argument. Any element $\tilde{x} \in \tilde{H}^{r_1+r_2}(\mathcal{S}^H_{K^H_f}, \tilde{\mathcal{M}}_{\lambda_1} \otimes \mathcal{O}_F[\frac{1}{\Delta_\lambda}])_{\text{ int},!}$ can be written as above in the form

$$\tilde{x} = \frac{x+y}{\varpi_{\mathfrak{p}}^{\delta(\pi_f)}}$$

and then the map $\tilde{x} \mapsto y \mod \varpi_{\mathfrak{p}}^{\delta(\pi_f)} H^{r_1+r_2}(\mathcal{S}^H_{K^H_f}, \tilde{\mathcal{M}}_{\lambda_1} \otimes \mathcal{O}_F[\frac{1}{\Delta_\lambda}])_{\text{ int},!}(\pi_{1,f})^\perp$ yields an inclusion

$$\begin{aligned} &\tilde{H}^{r_1+r_2}(\mathcal{S}^H_{K^H_f}, \tilde{\mathcal{M}}_{\lambda_1})_{\text{ int},!}(\pi_{1,f}) \otimes \mathcal{O}_F[\tfrac{1}{\Delta_\lambda}]/\mathfrak{p}^{\delta(\pi_{\mathfrak{p}})} \hookrightarrow \\ &H^{r_1+r_2}(\mathcal{S}^H_{K^H_f}, \tilde{\mathcal{M}}_{\lambda_1})_{\text{ int},!}(\pi_{1,f})^\perp \otimes \mathcal{O}_F[\tfrac{1}{\Delta_\lambda}]/\mathfrak{p}^{\delta(\pi_{\mathfrak{p}})} \end{aligned} \tag{8.251}$$

This has consequences for congruences, it is clear how to formulate a theorem corresponding to Theorem (3.3.2).

At this place references to Urban, Dimitroff and Namikawa will be added.

8.3.5 Fixing the period

We have mentioned that the prime factorisation of the numbers $\frac{\Lambda(\pi\otimes\tilde{\mu},0)}{\Omega(\underline{\epsilon}\times\pi_f)}$ are of great arithmetical interest. Conjecturally primes dividing these numbers should also divide the denominators of certain Eisenstein classes and hence they should also provide some congruences between eigenvalues of Hecke operators acting on the cohomology of different groups (see ([43])).

I think that it is of great importance to collect experimental data, which verify (or falsify) these conjectures. We will explain in (Mix-Mot) and in [45] that there is a strategy to prove these conjectures if we accept some plausible and fundamental conjectures about mixed motives. This means that the experimental verification of these conjectures would give some support to these motivic conjectures.

We have rather effective algorithms to compute the values $\Lambda(\pi \otimes \tilde{\mu}, 0)$ with very high precision. But then we also need a numerical value for the period $\Omega(\underline{\epsilon} \times \pi_f)$. Of course this period is well defined (up to a unit) but how can we actually compute it? We also have to take into account that we also must compute all the conjugates $\Omega(^\sigma\underline{\epsilon} \times^\sigma \pi_f)$.

In section 5.1.2 we gave a recipe to compute these periods in the special case that our group is $\mathrm{Gl}_2/\mathbb{Z}$ and we look at unramified cohomology, i.e. $K_f = \mathrm{Gl}_2(\hat{\mathbb{Z}})$. In this case we get the periods from the values $\Lambda(\pi_f, \nu)$ themselves (see (5.38)).

This method to fix the periods also works if we allow ramification. In this case we use the lemma of Shapiro (section 2.1.1 (2.6) to "transfer" the ramification into the coefficient system. More precisely we consider free $\mathbb{Z}$-modules $\mathcal{V}$ with an action of $\Gamma/\Gamma(N)$, i.e. a module of congruence origin (see section

1.2.2). Now we return to the situation in section 5.1.2 and replace in (5.6) the coefficient system by $\mathcal{M}_\lambda^\flat \otimes \mathcal{V}$.

We resume the reasoning from section 5.1.2, Again we start from the orbiconvex covering $\Gamma\backslash\mathbb{H} = U_i \cup U_\rho$ in section 2.1.4. As before we get that the map

$$\mathcal{S} : \mathcal{M}_\lambda^\flat \otimes \mathcal{V} \to H_1(\Gamma\backslash\mathbb{H}, \partial(\Gamma\backslash\mathbb{H}), \mathcal{M}_\lambda^\flat \otimes \mathcal{V}) \,; m \mapsto [C_{0,\infty} \otimes m] \tag{8.252}$$

is surjective, here we should assume that the Γ -module $\mathcal{V}$ is irreducible. Then we can find elements $m_1, m_2, \ldots, m_d \in \mathcal{M} \otimes \mathcal{V}$ which generate the kernel of the boundary map $\partial : H_1(\Gamma\backslash\mathbb{H}, \partial(\Gamma\backslash\mathbb{H}), \mathcal{M}_\lambda^\flat \otimes \mathcal{V}) \to H_0(\partial(\Gamma\backslash\mathbb{H}), \mathcal{M}_\lambda^\flat \otimes \mathcal{V})$. We lift these elements to elements $\tilde{m}_1, \ldots, \tilde{m}_d$ and evaluate these lifts on the appropriate cohomology class, i.e. we consider the numbers (ideals)

$$< \mathcal{F}(\frac{\omega_{\underline{\epsilon}}}{\Omega(\underline{\epsilon} \times \pi_f)} \times h^{(0)}_{\pi_f}), \tilde{m}_i > \tag{8.253}$$

These numbers should be expressed in terms of L-values and our choice of the period is the right one if and only if these numbers form a set of coprime integers.

The necessary computations may be a little bit difficult, therefore we discuss a very special case. We choose a prime p, and consider the congruence subgroup $\Gamma_0(p) \subset \Gamma = \mathrm{Gl}_2(\mathbb{Z})$ and the cohomology.

$$H^1(\Gamma_0(p)\backslash\mathbb{H}, \tilde{\mathcal{M}_\lambda^\flat}) = H^1(\Gamma\backslash\mathbb{H}, \widetilde{\mathcal{M}_\lambda^\flat \otimes \mathrm{Ind}_{\Gamma_0(p))}^{\Gamma}} \mathbb{1}). \tag{8.254}$$

Here $\mathrm{Ind}_{\Gamma_0(p))}^{\Gamma}\mathbb{1} = \mathrm{Ind}_{B(\mathbb{F}_p))}^{\mathrm{Gl}_2(\mathbb{F}_p)}\mathbb{1}$ is the induced representation from the trivial representation $\mathbb{1}$. This is of course simply the $\mathbb{Z}$ module of $\mathbb{Z}-$ valued functions on $B(\mathbb{F}_p)\backslash G(\mathbb{F}_p)$. The representation $\mathrm{Ind}_{B(\mathbb{F}_p))}^{\mathrm{Gl}_2(\mathbb{F}_p)}\mathbb{1} \otimes \mathbb{Q}$ of $\mathrm{Gl}_2(\mathbb{F}_p)$ is not irreducible, it contains the trivial representation $\mathbb{1}$ and the irreducible complement is the Steinberg representation St_p. Then

$$H^1(\Gamma_0(p)\backslash\mathbb{H}, \mathcal{M}_\lambda^\flat) = H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}_\lambda^\flat}) \oplus H^1(\Gamma\backslash\mathbb{H}, \widetilde{\mathcal{M}_\lambda^\flat \otimes \mathrm{St}_p}) \tag{8.255}$$

We have the action of the Hecke algebra $\mathcal{H}^{(p)}$ on these cohomology groups and decompose into eigenspaces

$$H^1(\Gamma\backslash\mathbb{H}, \widetilde{\mathcal{M}_\lambda^\flat \otimes \mathrm{St}_p} \otimes F) = \bigoplus_{\pi_f} H^1(\Gamma\backslash\mathbb{H}, \widetilde{\mathcal{M}_\lambda^\flat \otimes \mathrm{St}_p} \otimes F)(\pi_f) \tag{8.256}$$

and we have defined the periods $\Omega(\epsilon \times \pi_f)$ in (8.192). We resume our considerations from section 5.1, we consider the map

$$\delta_1 : H_1(\Gamma\backslash\mathbb{H}), \partial(\Gamma\backslash\mathbb{H})), \mathcal{M}_\lambda^\flat \otimes \mathrm{St}_p) \to H_0(\partial(\Gamma\backslash\mathbb{H})), \mathcal{M}_\lambda^\flat \otimes \mathrm{St}_p) \tag{8.257}$$

We know (see section 3.3.1 that $H_0(\partial(\Gamma\backslash\mathbb{H})), \mathcal{M}_\lambda^\flat \otimes \mathrm{St}_p) = \mathcal{M}_\lambda^\flat \otimes \mathrm{St}_p/(\mathrm{Id} - T_+)\mathcal{M}_\lambda^\flat \otimes \mathrm{St}_p$. We extend our base ring to $R = \mathbb{Z}[\frac{1}{p}, \zeta_p]$ then we have the direct sum decomposition

$$\mathrm{St}_p \otimes R = \oplus_{a=0}^{p-1} R f_a \text{ where } T_+ f_a = \zeta_p^a f_a. \tag{8.258}$$

and the arguments in section 3.3.1 imply $H_0(\partial(\Gamma\backslash\mathbb{H}),\mathcal{M}^\flat_\lambda\otimes Re_a)=0$ if $a\neq 0$. For $a=0$ we have of course $H_0(\partial(\Gamma\backslash\mathbb{H}),\mathcal{M}^\flat_\lambda\otimes Re_0)=H_0(\partial(\Gamma\backslash\mathbb{H}),\mathcal{M}^\flat_\lambda)\otimes R$ this module was computed earlier.

Now we can say that the images of

$$e_\nu\otimes f_a;\ \nu=0\dots,n, a=1,\dots,p-1$$

and

$$e_\nu\otimes f_0;\ \nu=1\dots n-1$$

generate a submodule $\mathcal{S}(\{\dots,e_\nu\otimes e_a,\dots\})$ of finite index in the kernel of δ_1, and the primes dividing the order of the quotient $\mathcal{S}(\dots,\{e_\nu\otimes e_a,\dots\}/\ker\delta_1)$ are the primes which divide some of the $\delta_{0,\infty}(e_\nu)$.

We give the elements f_a explicitly. We know that the elements $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ and $\begin{pmatrix}0&1\\-1&0\end{pmatrix}\cdot\begin{pmatrix}1&u\\0&1\end{pmatrix}$ for $u=0,\dots,p-1$ are representatives for the quotient $B(\mathbb{F}_p)\backslash G(\mathbb{F}_p)$. Then we can choose

$$f_0(\begin{pmatrix}1&0\\0&1\end{pmatrix})=-p, f_0(\begin{pmatrix}0&1\\-1&0\end{pmatrix}\cdot\begin{pmatrix}1&u\\0&1\end{pmatrix})=1 \text{ for all } u\in\mathbb{F}_p \tag{8.259}$$

and for $a=1,\dots,p-1$

$$f_a(\begin{pmatrix}1&0\\0&1\end{pmatrix})=0 \text{ and } f_a(\begin{pmatrix}0&1\\-1&0\end{pmatrix}\cdot\begin{pmatrix}1&u\\0&1\end{pmatrix})=\zeta_p^{au} \tag{8.260}$$

Next we look at the action of the torus $T(\mathbb{F}_p)$ on St_p. The center acts trivially, hence we can restrict this action to $\{\begin{pmatrix}t&0\\0&1\end{pmatrix}\}=\mathbb{F}_p^\times$. We make a list of characters $\chi:\mathbb{F}_p^\times\to\mathbb{C}^\times$

$$\{\chi_0,\chi_1,\dots,\chi_{p-2},\chi_{p-1}\} \tag{8.261}$$

where the χ_b with $b=1,\dots,p-2$ are just the non trivial characters and $\chi_0=\chi_{p-1}$ is the trivial character.. We extend our base ring further and put $R_1:=\mathbb{Z}[\frac{1}{p(p-1)},\zeta_p,\zeta_{p-1}]$ and we define the functions g_b for $b=0,1,\dots,p-2,p-1$:

$$\text{For } b=1,\dots,p-2 \text{ we put } g_b(\begin{pmatrix}1&0\\0&1\end{pmatrix})=0\ ; g_b(\begin{pmatrix}0&1\\-1&0\end{pmatrix}\cdot\begin{pmatrix}1&u\\0&1\end{pmatrix})=\chi_b(u) \tag{8.262}$$

where of course $\chi_b(0)=0$. Finally we put

$$g_0(\begin{pmatrix}1&0\\0&1\end{pmatrix})=-p; g_0(\begin{pmatrix}0&1\\-1&0\end{pmatrix}\cdot\begin{pmatrix}1&u\\0&1\end{pmatrix})=1 \text{ for all } u\in\mathbb{F}_p$$
$$\text{and } g_{p-1}(\begin{pmatrix}1&0\\0&1\end{pmatrix})=0; g_{p-1}(\begin{pmatrix}0&1\\-1&0\end{pmatrix}\cdot\begin{pmatrix}1&u\\0&1\end{pmatrix})=\begin{cases}1 & \text{for } u\neq 0\\ -p+1 & \text{for } u=0\end{cases} \tag{8.263}$$

we notice that $f_0 = g_0$. Let $\mathrm{St}_p^{(0)} \subset \mathrm{St}_p \otimes R_1$ be the submodule of functions with support on the big cell $\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\}_{u=0,1,\dots,p-1}$. Then an elementary computation shows that the g_b with $b = 1, \dots, p-1$ as well as the f_a with $a = 1, \dots, p-1$ form a basis of $\mathrm{St}_p^{(0)}$. More precisely we find

$$g_b = \frac{1}{p} \sum_{a=1}^{p-1} G(b,a) f_a \text{ for } b = 1, \dots, p-2 \text{ and } g_{p-1} = -\sum_{a=1}^{p-1} f_a \qquad (8.264)$$

where $G(b,a) \in \mathbb{Z}[\zeta_p, \zeta_{p-1}]$ are Gaussian sums. We may as well express the f_a in terms of the g_b we get

$$f_a = \frac{1}{p-1} \sum_{b=1}^{p-1} C'(a,b) g_b \text{ and } f_0 = g_0 \qquad (8.265)$$

where again the $C'(a,b)$ are algebraic integers.

We consider the numbers

$$< \mathcal{F}(\frac{\omega_{\underline{\epsilon}}}{\Omega(\underline{\epsilon} \times \pi_f)} \times h_{\pi_f}^{(0)}), e_\nu^\flat \otimes f_a > ; \nu = 0, \dots, n;\ a = 0, \dots, p-1 \qquad (8.266)$$

Then we know that for $a \neq 0$ these numbers are in $\mathcal{O}_F[\frac{1}{p}, \zeta_p]$. We also know that for $a = 0$ the numbers $\mathfrak{n}(\pi_f, \nu) < \mathcal{F}(\frac{\omega_{\underline{\epsilon}}}{\Omega(\underline{\epsilon} \times \pi_f)} \times h_{\pi_f}^{(0)}), e_\nu^\flat \otimes f_0 > \in \mathcal{O}_F[\frac{1}{p}, \zeta_p]$.

A) Hence we can fix the period by requiring that the ideal generated by these numbers is the ring $\mathcal{O}_F[\frac{1}{p}, \zeta_p]$.

Now we consider numbers

$$< \mathcal{F}(\frac{\omega_{\underline{\epsilon}}}{\Omega(\underline{\epsilon} \times \pi_f)} \times h_{\pi_f}^{(0)}), e_\nu^\flat \otimes g_b > ; \nu = 0, \dots, n;\ b = 1, \dots, p-1 \qquad (8.267)$$

These numbers can be expressed in terms of $L-$values. A close inspection shows us that

$$< \mathcal{F}(\frac{\omega_{\underline{\epsilon}}}{\Omega(\underline{\epsilon} \times \pi_f)} \times h_{\pi_f}^{(0)}), e_\nu^\flat \otimes g_b >= \frac{1}{\Omega(\underline{\epsilon} \times \pi_f)} \Lambda(\pi_f \times \chi_b, \nu). \qquad (8.268)$$

Hence we get

B) Another option to fix the periods is to require that the ideal generated by numbers in (8.267) is the ring $\mathcal{O}_F[\frac{1}{p(p-1)}, \zeta_p, \zeta_{p-1}]$.

We could also fix periods $\Omega^A(\epsilon \times \pi_f), \Omega^B(\epsilon \times \pi_f)$, such that if we but them into (8.266) and (8.267) then the ideal we get is simply $\mathcal{O}_F$. These two periods "differ" from the actual period by an ideal which only contains prime factors above $p, p-1$ and $\mathfrak{n}(\pi_f, \nu)$. The periods $\Omega^B(\epsilon \times \pi_f)$ can be computed from the L- values.

It is clear that we have some control over the primes that have to be inverted. We call them *small with respect to* π_f primes.

I made the conjecture that the large primes ℓ dividing these L-values are also dividing denominators of Eisenstein classes for the cohomology of the symplectic group $\mathrm{Sp}_2/\mathbb{Z}$, what this means has been explained in 5.6. A special case of this conjecture has been formulated in [43]. This conjecture implies that we also get congruences $\mod \ell$ between eigenvalues of Hecke operators on the cohomology of the symplectic group and the cohomology of $\mathrm{Gl}_2/\mathbb{Z}$.

These conjectures on congruences have been verified in many cases for a finite (far from empty) set of Hecke operators. The first such verification was done in [25] in the meanwhile many more cases of these congruences have been checked.

But of course thedenominator conjecture is stronger than the conjecture on congruences. The denominator conjecture can be verified (in principle) in any given case. This has been explained in section 3.2.1 and carried out in section 3.3 in a toy case. But in this toy case we profit from the fact that the dimension is low and that we have such an extremely simple covering by orbiconvex sets. The computation of the cohomology is very easy but the computation of the Hecke operator was not easy at all (at least for the two of us: H. Gangl and me).

But this changes dramatically once we leave the group Gl_2 and if we pass for instance to the symplectic group $\mathrm{Sp}_2/\mathbb{Z}, \mathrm{Sp}_3/\mathbb{Z}$. For many cases the congruence have been checked by Bergström, Faber, v.d. Geer, Dummighan and many others, but as far as I know the denominator has not been verified for more complicated groups. On the other hand once the denominator has been verified the congruences follow for all Hecke eigemvalues not only for a finite number of them.

Here the computational complexity increases dramatically if the parameter - the weight, the prime p or the rank of the group become large, but I hope that somebody can write algorithms which verify the denominator conjecture for a few small values of these parameters. So we may for instance take our prime p again and consider the congruence subgroup $\Gamma_0(p) \subset \mathrm{Sp}_2(\mathbb{Z})$ which is the inverse image of $P(\mathbb{F}_p) \subset \mathrm{Sp}_2(\mathbb{F}_p)$ where P is the standard Klingen-parabolic subgroup. Then it may be possible to check the denominator for $H^3(\Gamma_0(p)\backslash\mathbb{H}_2, \tilde{\mathcal{M}}_\lambda)$ for some small primes and small highest weights λ. The case of trivial coefficients, i.e. $\lambda = 0$ is discussed in [45]).

If we only want to verify the congruences then there is the option that we choose a $\mathbb{Q}$-form $G^*/\mathbb{Q}$ such that $G^*(\mathbb{R})$ is compact. Then our locally symmetric space is a finite set. In our special situation we can use some well known arguments from Galois-cohomology to construct a $G^*/\mathbb{Q}$ such that $G^*/\mathbb{Q}$ splits at all finite places except at the chosen place p. Then we can extend the group $G^*/\mathbb{Q}$ to a smooth group scheme $\mathcal{G}^*/\mathbb{Z}$ which is semi simple over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{p}])$ and $\mathcal{G}^*(\mathbb{Z})$ is the so called paramodular group $\Gamma^*(p)$. In terms of the Bruhat-Tits theory this says that $\mathcal{G}^*(\mathbb{Z}_p)$ is a maximal parahoric subgroup which is not hyperspecial. The we put $K_f^*(p) := \mathcal{G}^*(\hat{\mathbb{Z}})$ and then we know

$$H^\bullet(\mathcal{S}^{\mathcal{G}^*}_{K_f^*(p)}, \tilde{\mathcal{M}}) = H^0(\mathcal{S}^{\mathcal{G}^*}_{K_f^*(p)}, \tilde{\mathcal{M}}) = \bigoplus_{\underline{x}_f \in \mathcal{G}^*(\mathbb{Q})\backslash \mathcal{G}^*(\mathbb{A}_f)/K_f^*(p)} \mathcal{M}^{\Gamma_{\underline{x}}} \tag{8.269}$$

For a few small values of p and small modules $\mathcal{M}$ one should be able to get hold of the set $\mathcal{S}^{\mathcal{G}^*}_{K^*_f(p)} = \mathcal{G}^*(\mathbb{Q})\backslash\mathcal{G}^*(\mathbb{A}_f)/K^*_f(p)$.

Hence we get an explicit description of the cohomology, We can apply the method from section 3.2.1 and we get explicit formulas for Hecke operators

$$T^{\mathrm{coh},\lambda}_{\ell,\chi} : H^0(\mathcal{S}^{\mathcal{G}^*}_{K^*_f(p)}, \tilde{\mathcal{M}}) \to H^0(\mathcal{S}^{\mathcal{G}^*}_{K^*_f(p)}, \tilde{\mathcal{M}})$$

for some small values of $\ell \neq p$.

From this we get a list $\mathcal{E}^*(p)$ of eigenspaces π^*_f for the central sub algebra $\mathcal{H}^{(\{p\})}$ which occur non trivially in $H^0(\mathcal{S}^{\mathcal{G}^*}_{K^*_f(p)}, \tilde{\mathcal{M}} \otimes F)$, and each such eigenspace provides a short list of eigenvalues

$$\mathcal{L}(\pi^*_f) := \{\dots, \pi^*_f(T^{\mathrm{coh},\lambda}_{\ell,\chi}), \dots\}_{\text{ finite list of }(\chi,\ell)} \tag{8.270}$$

The Hecke algebra $\mathcal{H}^{(\{p\})}$ also acts on $H^\bullet(\mathcal{S}^G_{K_f(p)}, \tilde{\mathcal{M}}_\lambda)$, here $K_f(p)$ is the product of $\mathcal{G}(\mathbb{Z}_\ell)$ over all $\ell \neq p$ and K_p is the unique maximal parahoric sub group which is not hyperspecial, in other words $G(\mathbb{Q}) \cap K_p = \Gamma_p$ is the paramodular subgroup. Hence we get anther list of eigenspaces π_f, which occur in $H^\bullet(\mathcal{S}^G_{K_f(p)}, \tilde{\mathcal{M}}_\lambda \otimes F)$, Now the principles of functoriality or Arthurs trace formula or the topological trace formula tell us that the two lists of of eigenvalues have many members in common, i.e. the two lists $\mathcal{E}^*(p)$ and $\mathcal{E}(p)$ have many members in common(I do not know whether this comparison has been carried out in the literature and what the precise statement is.)

From the computational point of view it seems to be a lot easier to produce the lists $\mathcal{E}^*(p)$ and $\mathcal{L}(\pi^*_f)$. Assume we find an eigenclass σ_f in some $H^1(\mathcal{S}^{\mathrm{Sl}_2}_{K^0_f(p)}, \tilde{\mathcal{M}}_\mu)$ and a prime ideal $\mathfrak{l}$ dividing $\frac{\Lambda(\sigma\times\chi,\nu)}{\Omega(\epsilon\times\sigma_f)}$. Then we expect to find a congruence between σ_f and a π_f, but as we know the lists $\mathcal{E}(p).\mathcal{L}(\pi_f)$ are difficult to produce. Therefore we may instead look up the lists $\mathcal{E}^*(p).\mathcal{L}(\pi^*_f)$ and try to find a a list $\mathcal{L}(\pi^*_f)$ which satisfies the congruence.

The conjecture concerning the congruences and the denominators of Eisenstein classes can be formulated in a much more general context (see also the next chapter or [?]). We may for instance start from a group $G/\mathbb{Q} = R_{F_0/\mathbb{Q}}\mathrm{Gl}_2/F_0$, where F_0 is an arbitrary number field.

Already in the case that $F_0/\mathbb{Q}$ is a real quadratic field some questions arise which seem to be worth to be investigated. In this case the space $\mathcal{S}^G_{K_f}$ is a Hilbert modular surface, let $T_0/F_0 \subset \mathrm{Gl}_2/F_0$ be the standard split maximal torus and $T = R_{F/\mathbb{Q}}(T_0) \subset G/\mathbb{Q}$. Let us assume we are in the same simple situation as above, we have chosen a prime ideal $\mathfrak{p}$ We may also choose $\mathfrak{p} = \mathcal{O}_{F_0}$. Now we choose the sub group $K_{0,f}(\mathfrak{p}) = \prod_{\mathfrak{q}\neq\mathfrak{p}} \mathrm{Gl}_2(\mathcal{O}_{F_{0_\mathfrak{q}}}) \times K_{0.f}(\mathfrak{p})$. We now want to mimic the above approach. Let $K^T_f \subset T(\mathbb{A}_f)$ be the maximal compact open subgroup we consider the map

$$j(x_\infty, \underline{\epsilon}_f) : T(\mathbb{Q})\backslash T(\mathbb{R}) \times T(\mathbb{A}_f)/K^T_f \to G(\mathbb{Q})\backslash \mathrm{X} \times G(\mathbb{A}_f)/K_{0,f}(\mathfrak{p}). \tag{8.271}$$

We choose a highest weight $\underline{\lambda}$

8.3.6 The L-functions

Again I have to say a few words concerning L-functions.

To automorphic L-functions at the unramified places we have to introduce the dual group $G^\vee(\mathbb{C})$ (this is $\mathrm{Gl}_2(\mathbb{C})$ in this case) and a finite dimensional representation r of this group. The definition of the dual group is designed in such a way that the Satake parameter ω_p of an unramified representation at p can be interpreted as a semi simple conjugacy class in $G^\vee(\mathbb{C})$ (see [La]). Therefore we can form the expression

$$L(\pi_p, r, s) = \det(\mathrm{Id} - r(\omega_p)p^{-s})^{-1}$$

and then the global L function $L(\pi, r, s)$ is defined as the product over all these unramified L -factors times a product over suitable L-factors at the finite primes. If we do this for our automorphic forms on Gl_2 and if $r = r_1$ is the tautological representation of $Gl_2(\mathbb{C})$ then we get the local L-factors

$$L(\pi_p, r_1, s) = \frac{1}{(1 - \lambda_{p,2}(p)p^{-s})\,(1 - \lambda_{p,1}(p)p^{-s})}$$

and we see that it differs by a shift by 1/2 from our previous definition. Our earlier L -function was the motivic L-function, its definition does not require the additional datum r. Our automorphic form π defines a motive $\mathbb{M}(\pi)$. This motive has the disadvantage that it does not occur in the cohomology of a variety, it occurs only after we apply a Tate twist to it. The central character $\omega(\pi)$ has type $x \mapsto x^n$ and defines a Tate motive. The automorphic form $\pi \otimes \omega(\pi)^{-1} = \pi^\vee$ occurs in the cohomology

$$H^1(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}[-n]) \supset H^1(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}[-n])(\pi \otimes \omega(\pi)^{-1}) = H^1(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}[-n])(\pi^\vee)$$

where $\mathcal{M}_n[-n]$ is obtained by twisting the original module by the $-n$-th power of the determinant. (See [36], III). This motive occurs in the cohomology of a quasiprojective scheme (See also [Scholl]) Now we adopt the point of view that π_f is a pair (Π_f, ι) (See 1.2.6) and then $\mathbb{M}(\pi)$ defines a system of $\mathfrak{l}$-adic representations $\rho(\pi)_\mathfrak{l}$ which are also labelled by the $\iota : \mathbb{Q}(\pi_f) \to \bar{\mathbb{Q}}$. Then it is Delignes theorem that for unramified primes

$$L(\pi_p, r_1, s - \frac{1}{2}) = L_p((\mathbb{M}(\pi^\vee), s) = \det(\mathrm{Id} - \rho(F_p)_p^{-1} | \mathbb{M}(\pi^\vee)_\mathfrak{l}\, p^{-s})$$

for a suitable choice of $\ell \neq p$.

Weights and Hodge numbers

We may of course look at the motives $\mathbb{M}(\pi)$ which are attached to an eigenspace in $H^1_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}[-k])(\pi)$ in other words we twisted the natural module $\mathcal{M}_n$ by the $-k$-th power of the determinant. Again we get an $\mathfrak{l}$-adic representation $\rho_\mathfrak{l}$ and the Weil conjectures imply that the eigenvalues of the inverse Frobenius $\rho_\mathfrak{l}(F_p^{-1})$ all have the same absolute value $p^{\frac{2k-n+1}{2}}$. The number $2k - n + 1$ is usually called the weight $w(\rho_\mathfrak{l})$ of the Galois representation or also the weight $w(\mathbb{M}(\pi))$ of the motive $\mathbb{M}(\pi)$.

The central character $\omega(\pi)$ of π has a type and if we make the natural identification of G_m with the centre then the type of $\omega(\pi)$ is an integer type$(\omega(\pi)) \in \mathbb{Z}$ and the formula for the weight is

$$w(\mathbb{M}(\pi)) = -\mathrm{type}(\omega(\pi)) + 1.$$

This weight plays a role if we want to get a first understanding of the analytic properties of the motivic L-functions. Its abcizza of convergence is the line $\Re(s) = w(\mathbb{M}(\pi)) + 1$.

The special case $k = n$ is special, because in this case our motive occurs in the cohomology of a variety. The eigenvalues of the Frobenius are algebraic integers and the non zero Hodge numbers are $h^{n+1,0}$ and $h^{0,n+1}$. If k is arbitrary then the centre acts on $\mathcal{M}_n[-k]$ by the character $t(k) = n - 2k$ and the non zero Hodge numbers will be $h^{1+\frac{n-t(k)}{2}, -\frac{n+t(k)}{2}}$. We notice that for an isotypic component $H^1_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}[-k])(\pi)$ the number $t(k)$ is the type of the central character $\omega(\pi)$.

8.3.7 The special values of L-functions

We now observe that the local L factors $L(\mathbb{M}(\pi^\vee \otimes (\chi^{(1)})^{-1}), s)$ which we introduced in 2.2.6 are actually the local L-factors of the motivic L-function, i.e.

$$L(\mathbb{M}(\pi^\vee \otimes (\chi^{(1)})^{-1}), s) = L(\mathbb{M}(\pi^\vee \otimes (\chi^{(1)})^{-1}), s)$$

Theorem 8.3.2. *With these notations we can give a formula for the composition*

$$J_{c_{\chi,!}} \circ \Omega_\varepsilon(\pi_f)^{-1} \cdot \mathcal{F}_1^{(1)}(\omega_\varepsilon) = \frac{L(\mathbb{M}(\pi^\vee \otimes (\chi^{(1)})^{-1}), 1)}{\Omega_\varepsilon(\pi_f)} \cdot I^{loc}(\pi_f, \chi_f^{-1})$$

Applications

We evaluate this formula at elements $\psi_f \in \mathcal{W}(\pi_f, \tau)_{\mathrm{O}(\pi_f,\chi)}$ and an element $\underline{g}_f \in G(\mathbb{A}_f)$. We get $\Omega_\varepsilon(\pi_f)^{-1} \cdot \mathcal{F}_1^{(1)}(\omega_\varepsilon)(\psi_f) = \tilde{\psi}_f \in H^1_{!,\varepsilon}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})_{\mathrm{O}(\pi_f,\chi)}$ and

$$J_{c_{\chi,!}}(\psi_f)(\underline{g}_f) = \frac{L(\mathbb{M}(\pi^\vee \otimes (\chi^{(1)})^{-1}), 1)}{\Omega_\varepsilon(\pi_f)} \cdot I^{loc}(\pi_f, \chi_f^{-1})(\psi_f)(\underline{g}_f))$$

We have seen that $J_{c_{\chi,!}}(\psi_f)(\underline{g}_f)d(\underline{g}_f)$ (Lemma 2.2) is an integer and it is obvious that $d(\underline{g}_f) = \prod_p d(g_p)$. If we choose for ψ_f an element which is also a product $\psi_f(\underline{g}_f) = \prod_p \psi_p(g_p)$ then we get

$$J_{c_{\chi,!}}(\psi_f)(\underline{g}_f) \prod_p d(g_p) = \frac{L(\mathbb{M}(\pi^\vee \otimes (\chi^{(1)})^{-1}), 1)}{\Omega_\varepsilon(\pi_f)} \cdot \prod_p I_p^{loc}(\pi_p, \chi_p^{-1})(\psi_p)(g_p)d(g_p))$$

The factors in the products over all primes are equal to one at almost all places. Then we have to optimize the choices of ψ_p and g_p. First of all we can choose these data such that all local factors are different from zero. Then we conclude that we have an invariance under Galois for the L-values

$$(\frac{L(\mathbb{M}(\pi^\vee \otimes (\chi^{(1)})^{-1}), 1)}{\Omega_\varepsilon(\pi_f)})^\sigma = \chi^{(1)}(\underline{t}_\sigma)\frac{L(\mathbb{M}((\pi^\vee \otimes (\chi^{(1)})^{-1})^\sigma, 1)}{\Omega_\varepsilon(\pi_f^\sigma)}$$

We may observe that the characters $\chi^{(1)}$ can be written as product of a Dirichlet character and a power of the Tate character, i.e. $\chi^{(1)} = \phi{\cdot}\alpha^{-\nu}$ where $\nu = 0, \dots n$. Now we can write

$$\mathbb{M}(\pi^\vee \otimes (\chi^{(1)})^{-1}) = \mathbb{M}(\pi^\vee \otimes \phi^{-1}) \otimes \mathbb{Z}(\nu)$$

and

$$L(\mathbb{M}(\pi^\vee \otimes (\chi^{(1)})^{-1}), 1) = L(\mathbb{M}(\pi^\vee \otimes \phi^{-1}), 1+\nu)$$

and the arguments $1+\nu$ are exactly the critical arguments for the motive $\mathbb{M}(\pi^\vee \otimes \phi^{-1})$ in the sense of Deligne.

Of course we are now able to prove also some integrality results, it is clear that the left hand side is integral, more precisely it is an element in $\mathcal{O}(\pi_f, \chi_f)$. Now we have to work with local representations to find out under which conditions we can force the product of local factors to be a unit or at least to bound the primes dividing it. Hence we have a tool to show that

$$\frac{L(\mathbb{M}(\pi^\vee \otimes (\chi^{(1)})^{-1}), 1)}{\Omega_\varepsilon(\pi_f)} \in \mathcal{O}(\pi_f, \chi_f)$$

at least if we invert a few more primes.

Chapter 9

Eisenstein cohomology

Our starting point is a smooth group scheme $\mathcal{G}/\operatorname{Spec}(\mathbb{Z})$ whose generic fiber $G = \mathcal{G} \times_{\mathbb{Z}} \mathbb{Q}$ is reductive and quasisplit. We choose a Borel subgroup $B/\mathbb{Q}$ and a torus $T/\mathbb{Q} \subset B/\mathbb{Q}$. Let $F/\mathbb{Q}$ be a smallest extension such that $T \times_{\mathbb{Q}} F$ split, let Σ be a finite set of primes which contains the primes ramified in $F/\mathbb{Q}$. We assume the group scheme is reductive over the open subset $\mathcal{U} = \operatorname{Spec}(\mathbb{Z}) \setminus \Sigma$ and at the places in Σ it is given by a maximal parahoric group scheme structure. If G is split, then we assume that $\mathcal{G}$ is split. Our open compact level subgroup $K_f = \prod_p K_p \subset \mathcal{G}(\hat{\mathbb{Z}}) = \prod_p \mathcal{G}(\mathbb{Z}_p) \subset G(\mathbb{A}_f)$ and $K_p = \mathcal{G}(\mathbb{Z}_p)$ for $p \in \mathcal{U}$.

Our Cartan involution Θ fixes the maximal torus $T \times \mathbb{R}$, it defines the maximal compact subgroup $K_\infty \subset G^{(1)}(\mathbb{R})$.

Let $\lambda \in X^*(T)$ be a highest weight, let $\mathcal{M}_\lambda$ be a highest weight module attached to this weight. It is a $\mathbb{Z}$-module, the module $\mathcal{M}_\lambda \otimes \mathbb{Q}$ is a highest weight module for the group $G/\mathbb{Q}$.

We want to study the map $r : H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) \to H^\bullet(\partial \mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$. It is a central problem in this book to get insight into the nature of this map. This map is the trigger for many arithmetic applications. This is manifested for the group $\mathrm{Sl}_2/\mathbb{Q}$ in the earlier chapters 4 and 5.

The assumption that $G/\mathbb{Q}$ is quasisplit is not essential, a reader who is somewhat familiar with the work of Borel and Tits [10] will not have difficulties to translate the following considerations to the case that $G/\mathbb{Q}$ is any reductive group.

9.1 The Borel-Serre compactification

We consider our space

$$\mathcal{S}^G_{K_f} = G(\mathbb{Q}) \backslash G(\mathbb{A}) / K_\infty K_f$$

and its Borel-Serre compactification

$$i : \mathcal{S}^G_{K_f} \to \bar{\mathcal{S}}^G_{K_f}.$$

Our highest weight module $\mathcal{M}_\lambda$ provides a sheaf $\tilde{\mathcal{M}}_\lambda$ on these spaces.

We have an isomorphism

$$H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) \xrightarrow{\sim} H^\bullet(\bar{\mathcal{S}}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$$

for any coefficient system $\tilde{\mathcal{M}}_\lambda$ coming from a rational representation $\mathcal{M}$ of $G(\mathbb{Q})$. The boundary $\partial\bar{\mathcal{S}}_K$ is a manifold with corners. It is stratified by submanifolds

$$\partial\bar{\mathcal{S}}_K = \bigcup_P \partial_P \mathcal{S}^G_{K_f},$$

where P runs over the $G(\mathbb{Q})$ conjugacy classes of proper parabolic subgroups defined over $\mathbb{Q}$. We identify the set of conjugacy classes of parabolic subgroups with the set of representatives given by the parabolic subgroups that contain our standard Borel subgroup $B/\mathbb{Q}$. Then we have

$$H^\bullet(\partial_P \mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) = H^\bullet(P(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K_f, \tilde{\mathcal{M}}_\lambda)$$

We have a finite coset decomposition

$$G(\mathbb{A}_f) = \bigcup_{\xi_f} P(\mathbb{A}_f)\xi_f K_f,$$

for any $\underline{\xi_f}$ put $K^P_f(\xi_f) = P(\mathbb{A})_f \cap \xi_f K_f \xi_f^{-1}$. Then we have

$$P(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K_f = \bigcup_{\xi_f} P(\mathbb{Q})\backslash X \times P(\mathbb{A}_f)/K^P_f(\xi_f)\xi_f,$$

If $U_P \subset P$ is the unipotent radical, then

$$M = P/U_P$$

is a reductive group. For any open compact subgroup $K_f \subset G(\mathbb{A}_f)$(resp. for $K_\infty \subset G_\infty$) we define $K^M_f(\xi_f) \subset M(\mathbb{A}_f)$(resp. $K^M_\infty \subset M_\infty$) to be the image of $K^P(\xi_f)$ in $M(\mathbb{A}_f)$ (resp. M_∞). We put

$$\mathcal{S}^M_{K_f(\xi_f)} = M(\mathbb{Q})\backslash M(\mathbb{A})/K^M_\infty K^M_f(\xi_f)$$

and get a fibration

$$\pi_P : P(\mathbb{Q})\backslash X \times P(\mathbb{A}_f)/K^P_f(\xi_f) \to M(\mathbb{Q})\backslash M(\mathbb{A})/M(\mathbb{Q})\backslash K^M_\infty \times K^M_f(\xi_f)$$

where the fibers are of the form $\Gamma_U\backslash U_P(\mathbb{R})$ and where $\Gamma_U \subset U(\mathbb{Z})$ is of finite index and defined by some congruence condition dictated by $K^P_f(\xi_f)$. The Lie-algebra $\mathfrak{u}$ of U_P is a free $\mathbb{Z}$-module and it is clear that we have an integral version of the van -Est theorem which says:

If $R = \mathbb{Z}[\frac{1}{N}]$ where a suitable set of primes has been inverted then

$$H^\bullet(\Gamma_U\backslash U_P(\mathbb{R}), \tilde{\mathcal{M}}_R) \xrightarrow{\sim} H^\bullet(\mathfrak{u}, \tilde{\mathcal{M}}_R).$$

More precisely we know that the local coefficient system $R^\bullet\pi_{P*}(\tilde{\mathcal{M}})$ is obtained from the rational representation of M on $H^\bullet(\mathfrak{u}, \mathcal{M})$.

Hence we get

$$H^\bullet(\partial_P\mathcal{S},\tilde{\mathcal{M}}_R)=\bigcup_{\xi_f}H^\bullet(\mathcal{S}^M_{K_f^M(\xi_f)},\widetilde{H^\bullet(\mathfrak{u},\mathcal{M})}_R),$$

and

$$H^\bullet(\mathfrak{u},\mathcal{M}_R)=\bigoplus_{w\in W^P}H^{l(w)}(\mathfrak{u},\mathcal{M}_R)(w\cdot\lambda),$$

where W^P is the set of Kostant representatives of $W^M\backslash W$ and where $w\cdot\lambda=(\lambda+\rho)^w-\rho$ and ρ is the half sum of positive roots.

The primes which we have to be inverted should be those which are smaller than the coefficients of the dominant weights in the highest weight of $\mathcal{M}$. But at this point we may have to enlarge the set of small primes.

We conclude

The cohomology of the boundary strata $\partial_P\mathcal{S}^G_{K_f}$ with coefficients in $\mathcal{M}$ can be computed in terms of the cohomology of the reductive quotients of parabolic subgroups, where we have to take coefficients in the cohomology of the Lie algebra of the unipotent radical with coefficients in $\mathcal{M}$

9.1.1 The two spectral sequences

The covering of the boundary by the strata $\partial_P\mathcal{S}$ provides a spectral sequence, which converges to the cohomology of the boundary. We can introduce the simplex Δ of types of parabolic subgroups, the vertices correspond to the maximal ones and the full simplex corresponds to the minimal parabolic. To any type of a parabolic P let $d(P)$ its index , we make the convention that $d(P)-1$ is equal to the dimension of the corresponding face in the simplex. Let $M=M_P=P/U_P$ be the reductive quotient (the Levi quotient). If $Z_M/\mathbb{Q}$ is the connected component of the identity of the center of $M/\mathbb{Q}$ then $d(P)$ is also the dimension of the maximal split subtorus of $Z_M/\mathbb{Q}$ minus the dimension of the maximal split subtorus of $Z_G/\mathbb{Q}$. The covering yields a spectral sequence whose $E_1^{\bullet,\bullet}$ term together with the differentials of our spectral sequence is given by

$$0\to E_1^{0,q}=\bigoplus_{P,d(P)=1}H^q(\partial_P\mathcal{S}^G_{K_f},\mathcal{M})\xrightarrow{d_1^{0,q}}\bigoplus_{P,d(P)=p+1}H^q(\partial_P\mathcal{S}^G_{K_f},\mathcal{M})\xrightarrow{d_1^{p,q}}\qquad(9.1)$$

We apply Kostants theorem to get a more explicit description of the boundary operators $d_1^{p,q}$. We get

$$d_1^{p,q}:\bigoplus_{P,d(P)=p+1}\bigoplus_{w\in W^P}H^\bullet(\mathcal{S}^{M_P}_{K_f^{M_P}},\tilde{\mathcal{M}}(w\cdot\lambda))\to\bigoplus_{Q,d(Q)=p+2}\bigoplus_{w'\in W^Q}H^\bullet(\mathcal{S}^{M_Q}_{K_f^{M_Q}},\tilde{\mathcal{M}}(w'\cdot\lambda))\qquad(9.2)$$

and the $d_1^{p,q}=\oplus_{w,w'}d^{p,q}_{w,w'}$, we explain these $d^{p,q}_{w,w'}$: Here $w\in W^P,w'\in W^Q$, we have $Q\subset P$. Then $w\in W^Q$ and $w'=vw$ where $v\in W^{\bar{Q}}$ and $\bar{Q}$ is the image of Q in M_P. Then the M_Q module $\mathcal{M}(w'\cdot\lambda) = v\cdot(w\cdot\lambda)$ and the map $d^{p,q}_{w,w'}$ is simply the restriction map to the boundary cohomology

$$d^{p,q}_{w,w'} : H^\bullet(\mathcal{S}^{M_P}_{K_f^{M_P}}, \tilde{\mathcal{M}}(w\cdot\lambda)) \to H^\bullet(\mathcal{S}^{M_Q}_{K_f^{M_Q}}, \tilde{\mathcal{M}}(v\cdot w\cdot\lambda)) \tag{9.3}$$

There is also a homological spectral sequence which converges to the cohomology of the boundary. It can be written as a spectral sequence for the cohomology with compact supports. Let d be the dimension of $\mathcal{S}$ then we have a complex

$$\to \bigoplus_{P,d(P)=p+1} H_c^{d-1-p-q-1}(\partial_P\mathcal{S}^G_{K_f}, \mathcal{M}) \xrightarrow{\delta_1} \bigoplus_{P,d(P)=p} H_c^{d-1-p-q}(\partial_P\mathcal{S}^G_{K_f}, \mathcal{M}) \to \tag{9.4}$$

and therefore the $E^1_{\bullet,\bullet}$ term is

$$E^1_{p,q} = \bigoplus_{P,d(P)=p} H_c^{d-1-p-q}(\partial_P\mathcal{S}^G_{K_f}, \mathcal{M})$$

the (higher) differential go from (p,q) to $(p-r, q+1-r)$.

We write the spectral sequence for the cohomology of the boundary more explicitly. We put $d(P_I) := \#I$ so that P_I becomes maximal if $d(P_I) = 1$. Then we get for the $E_1^{p,q}$ term

$$E_1^{p,q} = \bigoplus_{I,\#I:p+1} H^q(\partial_{P_I}(S^G, \tilde{\mathcal{M}}_\lambda) = \bigoplus_{I,\#I:p+1}\bigoplus_{w\in W^{P_I}} H^{q-l(w)}(S^{M_I}, \tilde{\mathcal{M}}_{w\cdot\lambda}) \tag{9.5}$$

here $\tilde{\mathcal{M}}_{w\cdot\lambda} = H^{l(w)}(\mathfrak{u}_{P_I}.\tilde{\mathcal{M}}_\lambda)$ is the irreducible M_I module with highest weight $w\cdot\lambda = w(\lambda+\rho)-\rho$. We want to write down the differential $d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$ to do this we restrict it to a direct summand, i.e. we fix I and $w \in W^{P_I}$. Then the maximal parabolic subgroups in M_I correspond to the subsets $I' \subset I$ with $\#I' = \#I-1$. The parabolic subgroup $P_{I'}$ induces a maximal parabolic subgroup $\bar{P}_{I'} \subset M_I$ and we get the restriction

$$r_{I,I',w} : H^{q-l(w)}(S^{M_I}, \tilde{\mathcal{M}}_{w\cdot\lambda}) \to \bigoplus_{I'\subset I}\bigoplus_{w'\in W^{\bar{P}_{I'}}} H^{q-l(w)-l(w')}(S^{\bar{M}_{I'}}, \tilde{\mathcal{M}}_{w'w\cdot\lambda}) \tag{9.6}$$

Now we have ordered the simple roots, hence we also ordered the set I and we can define the sign $(-1)^{p(I,I')}$, where $p(I,I')$ is the position of $I\setminus I'$ in I.

Then we find

$$d_1^{p,q} = \sum_{I:\#I=p+1, I'\subset I, w} (-1)^{p(I,I')} r_{I,I',w} \tag{9.7}$$

Hence we see that understanding the map $r : H^\bullet(\mathcal{S}^G, \tilde{\mathcal{M}}_\lambda) \to H^\bullet(\partial(\mathcal{S}^G), \tilde{\mathcal{M}}_\lambda)$ is reduced to the understanding of this map for the reductive quotients of parabolic subgroups and some Weyl group combinatorics, which may become quite complicated.

Now we define $E_2^{p,\cdot}$ to be the cohomology of the complex $\{d_1^{p,|pkt}, E_1^{p,\bullet}\}$, and then the mechanism of al sequences gives us new complexes

$$\to E_2^{p-2,q-1} \xrightarrow{d_2^{p-2,q+1}} E_2^{p,q} \xrightarrow{d_2^{p,q}} E_2^{p+2,q-1} \to \tag{9.8}$$

and the cohomology of these complexes are the $E_3^{p,q}$. Again we have differentials $d_3^{p,q} : E_3^{p,q}. \to: E_3^{p+3,q-2}$. But since $p \leq r-1$ the differentials $d_\nu^{p,q} = 0$ once $\nu > r$. and hence the $E_\nu^{p,q} = E_\infty^{p,q}$ for $\nu > r$.

We get a filtration

$$\begin{aligned} H^\bullet(\partial(\Gamma\backslash X), \tilde{\mathcal{M}}_\lambda) = \mathcal{F}^0 H^\bullet(\partial(\Gamma\backslash X), \tilde{\mathcal{M}}_\lambda) \supset \mathcal{F}^1 H^\bullet(\partial(\Gamma\backslash X), \tilde{\mathcal{M}}_\lambda) \supset \\ \cdots \supset \mathcal{F}^{r-1} H^\bullet(\partial(\Gamma\backslash X), \tilde{\mathcal{M}}_\lambda) \supset \mathcal{F}^r H^\bullet(\partial(\Gamma\backslash X), \tilde{\mathcal{M}}_\lambda) = \{0\} \end{aligned} \tag{9.9}$$

and the quotients

$$\mathcal{F}^\nu H^\bullet(\partial(\Gamma\backslash X), \tilde{\mathcal{M}}_\lambda)/\mathcal{F}^{\nu+1} H^\bullet(\partial(\Gamma\backslash X), \tilde{\mathcal{M}}_\lambda) = E_\infty^{\nu,\bullet-\nu} \tag{9.10}$$

and hence these quotients are subquotients of $E_2^{\nu,\bullet-\nu}$.

Here we are confronted with several very interesting questions.

A) Does the above spectral sequence degenerate at $E_2^{\bullet,\bullet}$ level, i.e. are all differentials $d_\nu^{\bullet,\bullet} = 0$ for $\nu \geq 2$

If our highest weight $\lambda = \sum_i n_i\gamma_i + \delta$ is regular, this means all $n_i > 0$, then degeneration follows from the work of Schwermer and Li that the spectral sequence degenerates at $E_2^{\bullet,\bullet}$ level. This is so because in this situations the Eisenstein series involved are holomorphic at the evaluation point.

This question is closely related to the next one.

B) We consider the image of the map r (see **??**). How does Im(r) interfere with the filtration 9.9, can it happen that $\mathrm{Im(r)} \cap \mathcal{F}^\nu H^\bullet(\partial(\Gamma\backslash X), \tilde{\mathcal{M}}_\lambda) \neq \{(0)\}$ for some $\nu > 0$? What is the largest such ν?

If there are classes $\xi \in \mathrm{Im(r)} \cap \mathcal{F}^\nu H^\bullet(\partial(\Gamma\backslash X), \tilde{\mathcal{M}}_\lambda)$, $\xi \neq 0$ then these classes are called "ghost classes." It follows again from Schwermer and Li that there are no ghost classes if λ is regular.

9.1.2 Induction

The description of the cohomology of a boundary stratum is a little bit clumsy, since we are working with the coset decomposition. The reason is that we are working on a fixed level, if we consider cohomology with integral coefficients. If we have rational coefficients then we can pass to the limit. Then

$$H^\bullet(\partial_P\mathcal{S}, \tilde{\mathcal{M}}) = \lim_{K_f} H^\bullet(P(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K_f, \tilde{\mathcal{M}}) =$$

$$\mathrm{Ind}_{\pi_0(M(\mathbb{R})\times P(\mathbb{A}_f)}^{\pi_0(G(\mathbb{R})\times G(\mathbb{A}_f)} \lim_{K_f^M} H^\bullet(\mathcal{S}^M_{K_f^M}, \widetilde{H^\bullet(\mathfrak{u},\mathcal{M})}) = \mathrm{Ind}_{\pi_0(M(\mathbb{R}))\times P(\mathbb{A}_f)}^{\tilde{G}(\mathbb{A})} H^\bullet(\mathcal{S}^M, \widetilde{H^\bullet(\mathfrak{u},\mathcal{M})}),$$

where the induction is ordinary group theoretic induction (plain induction). We should keep in our mind that the $\pi_0(M(\mathbb{R})) \times P(\mathbb{A}_f)$ -modules are in fact

$\pi_0(M(\mathbb{R})) \times M(\mathbb{A}_f)$-modules. We need some simplification in the notation and we will write for any such $\pi_0(M(\mathbb{R})) \times M(\mathbb{A}_f)$ module H

$$\mathrm{Ind}_{\pi_0(M(\mathbb{R}))\times P(\mathbb{A}_f)}^{\tilde{G}(\mathbb{A})} H = I_M^G H$$

We will use the same notation for an induction from the torus T to M.

Under certain conditions we also have the notion of induction for Hecke - modules and we can work with integral coefficient systems. This will be discussed at another occasion.

But I want to mention that in the case that K_f is a hyperspecial maximal compact subgroup (in the cases where we are dealing with a split semisimple group scheme over $\mathrm{Spec}(\mathbb{Z})$ we can take $K_f = \prod \mathcal{G}(\mathbb{Z}_p)$ (see 1.1)) then $G(\mathbb{Q}_p) = P(\mathbb{Z}_p)K_p = B(\mathbb{Z}_p)K_p$ the group theoretic induction followed by taking K_f invariants gives back the original module. In this case we do not have to induce!

Of course we have to understand the coefficient systems $H^\bullet(\mathfrak{u}, \mathcal{M})$, for this we need the theorem of Kostant which will be discussed in the next section.

9.1.3 A review of Kostants theorem

At this point we can make the assumption that our group $G/\mathbb{Q}$ is quasisplit, we also assume that $G^{(1)}/\mathbb{Q}$ is simply connected. Then we may assume that $\mathcal{M}_\mathbb{Z}$ is irreducible and of highest weight λ. Let $B/\mathbb{Q}$ be a Borel subgroup, we choose a torus $T/\mathbb{Q} \subset B/\mathbb{Q}$. Let $X^*(T) = \mathrm{Hom}(T \times_\mathbb{Q} \bar{\mathbb{Q}}, \mathbb{G}_m \times_\mathbb{Q} \bar{\mathbb{Q}})$ be the character module, it comes with an action of a finite Galois group $\mathrm{Gal}(F/\mathbb{Q})$, here F is the smallest sub field of $\bar{\mathbb{Q}}$ over which $G/\mathbb{Q}$ splits. Let $T^{(1)}/\mathbb{Q} \subset T/\mathbb{Q}$ the maximal torus in $G^{(1)}/\mathbb{Q}$, then $X^*(T^{(1)})$ contains the set Δ of roots, the subset Δ^+ of positive roots (with respect to B.) The set of simple roots is identified to a finite index set $I = \{1, 2, \dots, r\}$, i.e we write the set of simple roots as $\pi = \{\alpha_1, \dots, \alpha_i, \dots, \alpha_r\} \subset \Delta^+$. We assume that the numeration is somehow adapted to the Dynkin diagram. The finite Galois group $\mathrm{Gal}(F/\mathbb{Q})$ acts on I and π by permutations. Furthermore we have the action of the (absolute) Weylgroup W on $X^*(T^{(1)} \times F)$ and we have a positive definite scalar product $< , >= X^*(T^{(1)} \times F) \times X^*(T^{(1)} \times F) \to \mathbb{Z}$. Attached to the simple roots we have the dominant fundamental weights $\{, \dots, \gamma_i, \dots, \gamma_j, \dots\}$ they are related to the simple roots by the rule

$$2\frac{< \gamma_i, \alpha_j >}{< \alpha_j, \alpha_j >} = \delta_{i,j}.$$

The dominant fundamental weights form a basis of $X^*(T^{(1)} \times F)$.

Our maximal torus $T/\mathbb{Q}$ is up to isogeny the product of $T^{(1)}$ and the central torus $C/\mathbb{Q}$, i.e. $T = T^{(1)} \cdot C$ and the restriction of characters yields an injection

$$j : X^*(T) \to X^*(T^{(1)}) \oplus X^*(C),$$

this becomes an isomorphism if we tensorize by the rationals

$$X^*_\mathbb{Q}(T) = X^*(T) \otimes \mathbb{Q} \xrightarrow{\sim} X^*_\mathbb{Q}(T^{(1)}) \oplus X^*_\mathbb{Q}(C).$$

This isomorphism gives us canonical lifts of elements in $X^*(T^{(1)})$ or $X^*(C)$ to elements in $X^*_{\mathbb{Q}}(T)$ which will be denoted by the same letter. Especially the fundamental weights $\gamma_1 \dots, \gamma_i, \dots$ are elements in $X^*_{\mathbb{Q}}(T)$.

Let $\lambda \in X^*(T \times F)$ be a dominant weight, our decomposition allows us to write it as

$$\lambda = \sum_{i \in I} a_i \gamma_i + \delta = \lambda^{(1)} + \delta$$

we have $a_i \in \mathbb{Z}, a_i \geq 0$ and $\delta \in X^*(C)$. To such a dominant weight λ there is an absolutely irreducible $G \times F$ -module $\mathcal{M}_\lambda$. The abelian part δ is rather irrelevant, we can not choose it to be zero because it has to satisfy a parity condition with respect to $\lambda^{(1)}$.

We consider maximal parabolic subgroups $P/\mathbb{Q} \supset B/\mathbb{Q}$. These parabolic subgroups are given by the choice of a $\mathrm{Gal}(F/\mathbb{Q})$ orbit $\tilde{i} = J \subset I$. Such an orbit yields a character $\gamma_J = \sum_{i \in J} \gamma_i$. The parabolic subgroup $P/\mathbb{Q}$ provided by this datum is determined by its root system $\Delta^P = \{\beta \in \Delta | < \beta, \gamma_J > \geq 0\}$. The choice of the maximal torus $T \subset P$ also provides a Levi subgroup $M \subset P$ but actually it is better to consider M as the quotient P/U_P.

The set of simple roots of $M^{(1)}$ is the subset $\pi_M = \{\dots, \alpha_i, \dots\}_{i \in I_M}$, where of course $I_M = I \setminus J$. We also consider the group $G^{(1)} \cap M = M_1$. It is a reductive group, it has $T^{(1)}$ as its maximal torus. We apply our previous considerations to this group M_1. It has a non trivial central torus $C_1/\mathbb{Q}$. This torus has a simple description, we pick a root $\alpha_i, i \in J$, we know that J is an orbit under $\mathrm{Gal}(F/\mathbb{Q})$. We have the subfield $F_{\alpha_i} \subset F$ such that $\mathrm{Gal}(F/F_{\alpha_i})$ is the stabilizer of α_i. Then it is clear that

$$C_1 \xrightarrow{\sim} R_{F_{\alpha_i}/\mathbb{Q}}(\mathbb{G}_m/F_{\alpha_i}),$$

up to isogeny it is a product of an anisotropic torus $C_1^*/\mathbb{Q}$ and a copy of $\mathbb{G}_m$. The character module $X^*_{\mathbb{Q}}(C_1)$ is a direct sum

$$X^*_{\mathbb{Q}}(C_1) = X^*_{\mathbb{Q}}(C_1^*) \oplus \mathbb{Q}\gamma_J. \tag{9.11}$$

Here $X^*_{\mathbb{Q}}(C_1^*) = \{\gamma \in X^*_{\mathbb{Q}}(C_1) \mid < \gamma, \sum_{i \in J} \alpha_i >= 0\}$. The half sum of positive roots in the unipotent radical is

$$\rho_U = f_P \gamma_J \tag{9.12}$$

where $2f_P > 0$ is an integer.

We also have the semi simple part $T^{(1,M)} \subset M^{(1)}$ and again we get the orthogonal decomposition

$$X^*_{\mathbb{Q}}(T^{(1)}) = X^*_{\mathbb{Q}}(T^{(1,M)}) \oplus X^*_{\mathbb{Q}}(C_1) = \bigoplus_{i \in I_M} \mathbb{Q}\alpha_i \oplus \bigoplus_{i \in J} \mathbb{Q}\gamma_i = \bigoplus_{i \in I_M} \mathbb{Q}\gamma_i^M \oplus \bigoplus_{i \in J} \mathbb{Q}\gamma_i.$$

Here we have to observe that the $\gamma_i^M, i \in I_M$ are the dominant fundamental weights for the group $M^{(1)}$, they are the orthogonal projections of the γ_i to the first summand in the above decomposition. We have a relation

$$\gamma_j = \gamma_j^M + \sum_{i \in \tilde{i}} c(j,i)\gamma_i, \text{ for } j \in I_M$$

and we have $c(j,i) \geq 0$ for all $i \in J$.

Let $W_M \subset W$ be the Weyl group of M. For the quotient $W_M \backslash W$ we have a canonical system of representatives

$$W^P = \{w \in W \mid w^{-1}(\pi_M) \subset \Delta^+\}.$$

To any $w \in W$ we define $w \cdot \lambda = w(\lambda + \rho) - \rho$ where ρ us the half sum of positive roots. If we do this with an element $w \in W^P$ then $\mu = w \cdot \lambda$ is a highest weight for $M^{(1)}$ and therefore $w \cdot \lambda$ provides a highest weight module $\mathcal{M}_\lambda(w \cdot \lambda)$ The cohomology $H^\bullet(\mathfrak{u}_P, \mathcal{M}_\lambda)$ is a (graded) M-module we can decompose into irreducibles. Then Kostant's theorem

$$H^\bullet(\mathfrak{u}_P, \mathcal{M}_\lambda) = \bigoplus_{w \in W^P} H^{\ell(w)}(\mathfrak{u}_P, \mathcal{M}_\lambda)(w \cdot \lambda),$$

where the isomorphism type of $H^{\ell(w)}(\mathfrak{u}_P, \mathcal{M}_\lambda)(w \cdot \lambda)$ is $\mathcal{M}_\lambda(w \cdot \lambda)$ but we have always to remember that it sits in degree

$$l(w) = \#\{\alpha \in \Delta^+ | w^{-1}\alpha \in \Delta^-\}. \tag{9.13}$$

Each isomorphism class occurs only once.

We write

$$\begin{array}{lccc} w \cdot \lambda = & \underbrace{\mu^{(1,M)} + \delta_1} & & +\delta \\ & \in X^*_{\mathbb{Q}}(T^{(1,M)}) \oplus X^*_{\mathbb{Q}}(C_1) & & \oplus X^*(C) \end{array} \tag{9.14}$$

We decompose δ_1 and define the numbers $a(w, \lambda)$ (see (9.11))

$$\delta_1 = \delta_1' + a(w,\lambda)\gamma_J.$$

Then we get

$$w(\lambda + \rho) - \rho = \mu^{(1,M)} + a(w,\lambda)\gamma_J \tag{9.15}$$

We also consider the extended Weyl group $\tilde{W}$, this is the group of automorphisms of the root system. Let $w_0 \in W$ be the element sending all positive roots into negative ones. We have an automorphism $\Theta_- \in \tilde{W}$ inducing $t \mapsto t^{-1}$ on the torus. Let $\Theta = w_0 \circ \Theta_-$. This element induces a permutation on the set π of positive roots, which may be the identity and induces -1 on the determinant. Then

$$\Theta\lambda = \sum_{i \in I} a_{\Theta i}\gamma_i - \delta$$

is a dominant weight and the resulting highest weight module is dual module to $\mathcal{M}_\lambda$. Therefore we get a non degenerate pairing

$$H^\bullet(\mathfrak{u}_P, \mathcal{M}_\lambda) \times H^\bullet(\mathfrak{u}_P, \mathcal{M}_{\Theta\lambda}) \to H^{d_{U_P}}(\mathfrak{u}_P, F) = F(-2\rho_U),$$

which respects the decomposition, i.e. we get a bijection $w \mapsto w'$ such that $l(w) + l(w') = d_{U_P}$ and such

$$H^{l(w)}(\mathfrak{u}_P, \mathcal{M}_\lambda)(w \cdot \lambda) \times H^{l(w')}(\mathfrak{u}_P, \mathcal{M}_{\Theta\lambda})(w' \cdot \Theta\lambda) \to H^{d_{U_P}}(\mathfrak{u}_P, F) \qquad (9.16)$$

is non degenerate. We conclude

$$a(w, \lambda) + a(w', \Theta\lambda) = -2f_P. \qquad (9.17)$$

We say that $w \cdot \lambda$ is in the positive chamber if

$$a(w, \lambda) \leq -f_P \qquad (9.18)$$

The element Θ conjugates the parabolic subgroup P into the parabolic subgroup Q, which may be equal to P or not. If $P = Q$ resp. $P \neq Q$ then we say that P is (resp. not) conjugate to its opposite parabolic. If Θ_- is in the Weyl group then all parabolic subgroups are conjugate to their opposite. In this case we have $\Theta = 1$.

Conjugating by the element Θ provides an identification $\theta_{P,Q} : W^P \xrightarrow{\sim} W^{\mathbb{Q}}$. We have two specific Kostant representatives, namely the identity $e \in W^P$ and the element $w_P \in W^P$, this is the element which sends all the roots in U_P to negative roots (the longest element). Its length $l(w_P)$ is equal to the dimension $d_P = \dim(U_P)$.

Any element in $w \in W^P$ can be written as product of reflections

$$w = s_{\alpha_{i_1}} \dots s_{\alpha_{i_\nu}} \qquad (9.19)$$

where $\nu = l(w)$ and the first factor $\alpha_{i_1} \in J$. We always can complement this product to a product giving the longest element

$$s_{\alpha_{i_1}} \dots s_{\alpha_{i_\nu}} s_{\alpha_{i_{\nu+1}}} \dots s_{\alpha_{i_{d_P}}} = w s_{\alpha_{i_{\nu+1}}} \dots s_{\alpha_{i_{d_P}}} = w_P, \qquad (9.20)$$

The inverse of the element $s_{\alpha_{i_{\nu+1}}} \dots s_{\alpha_{i_{d_P}}}$ is

$$w' = s_{\alpha_{i_{d_P}}} \dots s_{\alpha_{i_{\nu+1}}} \in W^Q$$

This defines a second bijection $i_{P,Q} : W^P \xrightarrow{\sim} W^Q$ which is defined by the relation

$$w = w_P \cdot i_{P,Q}(w) = w_P \cdot w', \;\; l(w) + l(w') = d_P \qquad (9.21)$$

The composition $\theta_{P,Q}^{-1} \circ I_{P,Q} : W^P \to W^P$ is the bijection provided by duality.

The element w_P conjugates the Levi subgroup M of P into the Levi subgroup of $Q = w_P P w_P^{-1}$. The element $\tilde{w}_P = \Theta w_P$ conjugates the parabolic subgroup P into its opposite (which is conjugate to Q) and induces an automorphism on the subgroup M which is a common Levi-subgroup of P and its opposite.

If we choose $w = e$ then

$$\sum_{i \in I} a_i \gamma_i + \delta = \sum_{i \in I_M} a_i \gamma_i^M + \sum_{j \in J} (\sum_{i \in I_M} a_i c(i,j) + n_j) \gamma_j + \delta.$$

Since J is the orbit of an element $i \in I$ we see that $< \gamma_j, \alpha_j >$ is independent of j and hence we get easily

$$\sum_{j\in J}(\sum_{i\in I_M} a_i c(i,j) + n_j)\gamma_j = \frac{1}{\#J}(\sum_{j\in J}(\sum_{i\in I_M} a_i c(i,j) + n_j))\gamma_J + \delta'$$

and hence

$$a(e,\lambda) = \frac{1}{\#J}(\sum_{j\in J}(\sum_{i\in I_M} a_i c(i,j) + a_j))$$

If we choose Θ_P then as an M-module $\mathcal{M}_{\Theta_P\cdot\lambda}$ is dual to $\mathcal{M}_{\Theta\lambda}(-2f_J\gamma_J)$. We write $\Theta\lambda + \rho = \sum_{i\in I} a_{\Theta i}\gamma_i - \delta$ and then

$$w_P(\sum_{i\in I} a_i\gamma_i + \delta) = \sum_{i\in I_M} n_{\Theta i}\gamma_i^M - \sum_{j\in J}(\sum_{\Theta i\in I_M} a_{\Theta i}c(\Theta i,\Theta j) + a_{\Theta j})\gamma_j - 2f_J\gamma_J - \delta.$$

and especially we find

$$a(w_P,\lambda) = -(\frac{1}{\#J}(\sum_{j\in J}(\sum_{i\in I_M} a_{\Theta i}c(\Theta i,\Theta j) + a_{\Theta j})) + 2f_J)\gamma_J$$

In general we have the inequalities

$$a(\Theta_P,\lambda) \le a(w,\lambda) \le a(e,\lambda).$$

We can write our relation (9.15) slightly differently. We can move the half sum of positive roots to the right and split into $\rho = \rho^M + f_P\gamma_J$. We put $\tilde{\mu}^{(1)} = \mu^{(1,M)} + \rho^M$ and then we write

$$w(\lambda+\rho) = \tilde{\mu}^{(1)} + (a(w,\lambda) + f_P)\gamma_J = \tilde{\mu}^{(1)} + b(w,\lambda)\gamma_J \tag{9.22}$$

and of course now we have

$$b(w,\lambda) + b(w',\Theta\lambda) = 0. \tag{9.23}$$

9.1.4 The inverse problem

Later we will encounter the following problem. Our data are as above and we start from a highest weight for M, we write

$$\mu = \mu^{(1)} + \delta_1 + a\gamma_J + \delta = \sum_{i\in I_M} n_{\Theta i}\gamma_i^M + \delta_1 + a\gamma_J + \delta.$$

We ask whether we can find a λ such that we can solve the equation $(Kost)$. More precisely: We give ourselves only the semi simple component $\mu^{(1)}$ of μ and we ask for the solutions

$$w(\lambda+\rho) = \tilde{\mu}^{(1)} + \dots$$

where $w \in W^P$ and λ dominant, i.e. we only care for the semi simple component.

Let us consider the case where $J = \{i_0\}$, i.e. it is just one simple root. Then the term δ_1 disappears and our equation becomes

$$w(\lambda+\rho) = \tilde{\mu}^{(1)} + b\gamma_{i_0} + \delta,$$

of course the δ is irrelevant, but we want to know the range of the values $b = b(\lambda, w)$ when $\tilde{\mu}^{(1)}$ is fixed, but λ, w vary. Of course it may be empty. Let us fix a w and let us assume we have solved $w(\lambda+\rho) = \tilde{\mu}^{(1)} + \dots$. Then it is clear that the other solutions are of the form $\lambda+\rho+\nu$ where $w\nu \in \mathbb{Q}\gamma_{i_0}$. These ν are of the form $\nu = c\nu_0$ with $c \in \mathbb{Z}$. We write $\nu_0 = \sum_{i \in I} b_i \gamma_i$ and it is easy to see that there must be some $b_i > 0$ and some $b_j < 0$. This implies that $\lambda + c\nu_0$ is dominant if and only if $c \in [M, N]$, an interval with integers as boundary point. This of course implies that -still for a given w - the values $b = b(\lambda, w)$ also have to lie in a fixed finite interval

$$b = b(w, \lambda) \in [b_{\min}(w, \tilde{\mu}^{(1)}), b_{\max}(w, \tilde{\mu}^{(1)}] = I(w, \tilde{\mu}^{(1)}). \tag{9.24}$$

This will be of importance because these intervals will be related to intervals of critical values of L-functions.

9.2 The goal of Eisenstein cohomology

The goal of the Eisenstein cohomology is to provide an understanding of the restriction map r in theorem (6.2.1). More precisely we assume that we understand (can describe) the cohomology $H^\bullet(\partial\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$, then we want to understand the image $H^\bullet_{\mathrm{Eis}}(\partial\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$ in terms of this description.

It is clear from the previous considerations that understanding of the boundary cohomology $H^\bullet(\partial\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$ requires an understanding the cohomology of $H^\bullet(\mathcal{S}^M_{K^M_f}, \widetilde{H^\bullet(\mathfrak{u}, \mathcal{M}_\lambda)})$, where M runs over the reductive quotients of the different (conjugacy classes) of parabolic subgroups. We have to compute the differentials in the spectral sequence. These differentials will depend on the Eisenstein cohomology of the $H^\bullet(\mathcal{S}^M_{K^M_f}, \widetilde{H^\bullet(\mathfrak{u}, \mathcal{M}_\lambda)})$, in other words we meet the same issue for smaller reductive groups.

The situation simplifies if the highest weight λ is regular. Then we have a good understanding of the Eisenstein cohomology and the spectral sequence degenerates at $E_2^{\bullet,\bullet}$ level (see [73]). I am convinced that the spectral sequence does not degenerate at $E_2^{\bullet,\bullet}$ in general, and this raises the question how high the level of non degeneration may become.

We have to take the action of the Hecke-algebra $\mathcal{H}^G_{K_f}$ on the spectral sequence

$$\{E_r^{\bullet,\bullet},\ d_r^{\bullet,\bullet}\}_{r=1,2,\dots} \tag{9.25}$$

into account, more precisely we have to consider the entire spectral sequence as (multigraded) module for the Hecke algebra. We look for irreducible $\mathcal{H}^M_{K^M_f}$ modules σ_f which occur in some $H^\bullet(\mathcal{S}^M_{K^M_f}, H^\bullet(\mathfrak{u}, \widetilde{\mathcal{M}(w\cdot\lambda)} \otimes F))(\sigma_f) \neq 0$. In general they may pop up in different summands of some $E_1^{p,q}$, these summands are indexed by P, w. Let us denote this direct sum by $E_1^{p,q}(\sigma_f)$.

We want to compute the $E_2^{p,q}(\sigma_f)$ term, this means we have to compute $d_1^{\bullet,\bullet}(\sigma_f) : E_1^{p,q}(\sigma_f) \to E_1^{p+1,q}(\sigma_f)$. We go to the transcendental level, i.e. we tensor everything by $\mathbb{C}$. Then the methods of Eisenstein cohomology will give us some matrix expressions for $d_1^{\bullet,\bullet}(\sigma_f)$ where the matrix coefficients are expressible in terms of certain special values of $L-$ functions $L(\sigma_f, r_\chi, \nu)$ at certain specific (critical) arguments ν. This tells us that we can express the kernel of $d_1^{\bullet,\bullet}(\sigma_f)$ and $E_2^{p,q}(\sigma_f)$ in terms of special values of L-functions. Now the next step will be to compute $d_2^{\bullet,\bullet}(\sigma_f)$.. We know that it is zero if $\lambda$is regular or if the rank of $G/\mathbb{Q}$ is two but to to the best of my knowledge this has not been done in any other case.

Hence we see that the structure of the cohomology of the boundary, the structure of the Eisenstein cohomology and the differentials in the spectral sequence will depend on the behaviour of certain L-functions $L(\sigma_f, r_\chi, z)$ at certain integer arguments $\nu \in \mathbb{Z}$. Sometimes we need to know whether or not $L(\sigma_f, r_\chi, z)$ has a pole at $z = \nu$. We know that in some cases the structure of the Eisenstein cohomology depends on the vanishing of $L(\sigma_f, r_\chi, z)$ at a certain specific argument ν. We do not know whether there are cases where the structure of the boundary cohomology or the vanishing of certain higher differentials even depends on the order of vanishing of the L-function. Actually this would be very exciting.

On the other hand we know that the cohomology is a $\mathbb{Q}$-vector space and $\{E_r^{\bullet,\bullet}(\sigma_f),\ d_r^{\bullet,\bullet}\}_{r=1,2,\dots}$ is a Hecke modules spectral sequence over F. Our transcendental description of $\{E_r^{\bullet,\bullet}(\sigma_f)\otimes\mathbb{C},\ d_r^{\bullet,\bullet}\}$ involves the special values $L(\sigma_f, r_\chi, \nu)$ and hence we must get some algebraicity relations for special values of L function, i.e a certain expression in $L(\sigma_f, r_\chi, \nu)$ with varying ν must be an element of F.

The Eisenstein cohomology itself, i.e. the image of r is defined over $\mathbb{Q}$, but it has a transcendental description involving special values of L functions. This should be another source for rationality theorems on special values. We discuss a typical example of such an algebraicity relation in the next subsection and in [48]. See also section 9.4.5

Finally we want to add a very speculative remark. In principle we can compute the cohomology as Hecke module explicitly in a given case. Hence we know that certain "anomalies" in the structure of the cohomology must be induced by zeros (or higher order zeros) of L-functions. Therefore we might be able to certify a zero (or higher order zero) just by looking deeply into the Hecke-module structure. This is a dream of the author, but the computational difficulties might be insurmountable.

9.2.1 The lowest step

In the following section we apply the above strategy to a very specific part of the cohomology of the boundary. We discuss the special case of rank one Eisenstein cohomology. We call this method the "Cohomological-Langlands-Shahidi" method, in brief the C-L-S-method.

This strategy has been carried out successfully in the case that $G/\mathbb{Q} = \mathrm{Gl}_N/\mathbb{Q}$, the maximal parabolic subgroup P has reductive quotient quotient $\mathrm{Gl}_n \times \mathrm{Gl}_{n'}$ with nn' even in [48].In the following we try to present a kind of axiomatised approach to the proceeding in [48]. All the assumption which we make in the following are verified in [48]. The combinatorial Lemma was proved by U.Weselmann.

We recall the filtration (8.27) of the cohomology, we define the corresponding filtration on the cohomology of the boundary. We study the Eisenstein cohomology on the lowest step of this filtration.

We start from a maximal parabolic subgroup $P/\mathbb{Q} \supset B/\mathbb{Q}$ let $M/\mathbb{Q}$ be its reductive quotient. We define

$$H^\bullet_!(\partial_P \mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) = \bigoplus_{w \in W^P} H^{\bullet - l(w)}_!(\mathcal{S}^M_{K^M_f}, H^{l(w)}(\mathfrak{u}_P, \tilde{\mathcal{M}})(w \cdot \lambda)) \subset H^\bullet(\partial_P \mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) \tag{9.26}$$

We will abbreviate $H^{l(w)}(\mathfrak{u}_P, \tilde{\mathcal{M}})(w \cdot \lambda) = \tilde{\mathcal{M}}(w \cdot \lambda)$ where always keep in mind that the element $w \in W^P$ knows what the actual parabolic subgroup is and that $\tilde{\mathcal{M}}(w \cdot \lambda)$ sits in degree $l(w)$.

By definition the inner cohomology is the image of the cohomology with compact supports. This implies that the submodule

$$\bigoplus_{P: d(P)=1} H^q_!(\partial_P \mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) \subset \bigoplus_{P: d(P)=1} H^q(\partial_P \mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) = E_1^{0,q}$$

is annihilated by all differentials $d_\nu^{0,q}$ and hence we get an inclusion

$$i: \bigoplus_{P: d(P)=1} \bigoplus_{w \in W^P} I_P^G H^{\bullet - l(w)}_!(\mathcal{S}^M_{K^M_f}, \mathcal{M}(w \cdot \lambda)) \to H^\bullet(\partial \mathcal{S}^G_{K_f}, \mathcal{M}_\lambda). \tag{9.27}$$

The image of this inclusion will be called $H^\bullet_!(\partial \mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$. The Hecke algebra acts on these modules. Let us assume that $H^\bullet_!(\partial \mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{Q}) \subset H^\bullet(\partial \mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{Q})$ is complete. This is not an unrealistic assumption, it can be verified in many concrete situations. Then we get a decomposition

$$H^\bullet_!(\partial \mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{Q}) \oplus H^\bullet_{\text{non!}}(\partial \mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{Q}) = H^\bullet(\partial \mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{Q}). \tag{9.28}$$

Now we can ask:

What is the intersection of ${H^\bullet}_{Eis}(\partial \mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{Q})$ *with the first summand, or what amounts to the same, what is* $H^\bullet_{!,\ Eis}(\partial \mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{Q})$.

This is in a sense a first step in our efforts to understand the Eisenstein part of the cohomology.

The Cartan involution Θ sends $P/\mathbb{Q}$ into $\Theta(P) \supset B_-/\mathbb{Q}$ and $\Theta(P)$ is conjugate to a parabolic $[\Theta]P = Q \supset B/\mathbb{Q}$, hence Θ induces an involution $[\Theta]$ on the set of parabolic subgroups containing B (= set of $G(\mathbb{Q})$ conjugacy classes of parabolic subgroups). Two parabolic subgroups $P, Q \supset B$ are called associate if $[\Theta]P = Q$.

We can decompose the cohomology $H^\bullet_!(\partial \mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{Q})$ into summands attached to the classes of associated parabolic subgroups

$$H^\bullet_!(\partial \mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{Q}) =$$

$$\bigoplus_{P: P=[\Theta]P} H^\bullet_!(\partial_P \mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) \oplus \bigoplus_{[P,Q]} H^\bullet_!(\partial_P \mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{Q}) \oplus H^\bullet_!(\partial_Q \mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{Q}) \tag{9.29}$$

where in the second sum $Q = [\Theta]P$. Each summand is a sum over the elements of W^P and then we can decompose under the action of the Hecke algebra. We choose a sufficiently large extension $F/\mathbb{Q}$ and in the case $P = [\Theta]P$ we get

$$H_!^\bullet(\partial_P \mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes F) = \bigoplus_{w \in W^P} \bigoplus_{\sigma_f} H_!^{\bullet - l(w)}(\mathcal{S}^M_{K_f^M}, \tilde{\mathcal{M}}_\lambda(w \cdot \lambda) \otimes F)(\sigma_f) \quad (9.30)$$

In the case $P \neq [\Theta]P = Q$ we group the contributions from the two parabolic subgroups together. To any $w \in W^P$ we have the element $i_{P,Q}(w) = w' \in W^Q$ see 9.21). We also group the terms corresponding to w and w' together. To any σ_f which occurs in $H_!^{\bullet - l(w)}(\mathcal{S}^M_{K_f^M}, H^{l(w)}(\mathfrak{u}_P, \tilde{\mathcal{M}})(w \cdot \lambda) \otimes F)$ we find a $\sigma'_f = \sigma_f^{w_P} |\gamma_{\Theta j}|_f^{2f_Q}$, which occurs in the second summand.

The decomposition into isotypical pieces becomes

$$\bigoplus_{\sigma_f} \big(H_!^{\bullet - l(w)}(\mathcal{S}^M_{K_f^M}, \tilde{\mathcal{M}}(w \cdot \lambda) \otimes F)(\sigma_f) \oplus H_!^{\bullet - l(w')}(\mathcal{S}^{M'}_{K_f^{M'}}, \tilde{\mathcal{M}}(w' \cdot \lambda) \otimes F)(\sigma'_f) \big) \quad (9.31)$$

We can define the second step in the filtration (6.39) as the inverse image of $H_!^\bullet(\partial \mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$ under the restriction r.

Delorme's method

We briefly review some results of Delorme on the $(\mathfrak{g}, K_\infty)$ cohomology of induced representations. We introduce some notation. As usual $P/U = M/\mathbb{Q}$ will be the reductive quotient, Let $\mathfrak{P}, \mathfrak{m}, \mathfrak{u}$ be the corresponding Lie algebras. We also assume that the Cartan involution is defined over $\mathbb{Q}$ and hence K_∞ is the group of $\mathbb{R}$-valued points of a group defined over $\mathbb{Q}$, let $\mathfrak{k}$ be its Lie algebra. The intersection $P \cap \Theta(P)$ is a Levi-subgroup of $P/\mathbb{Q}$, we also denote by $M/\mathbb{Q}$. The intersection $M \cap K_\infty = K^M_\infty$ is a maximal compact subgroup of $M(\mathbb{R})$.

Let $M^{(1)} \subset M$ be the semi simple derived subgroup and let C_M be the connected component of identity of the centre. The corresponding Lie algebras are $\mathfrak{m}^{(0)}$ and $\mathfrak{c}$. The intersection $K_M^{(1)} \cap K_\infty$ is maximal compact in $M^{(1)}(\mathbb{R})$, its Lie-algebra is $\mathfrak{k}_M^{(0)}$.

We have $\mathfrak{g} = \mathfrak{P} + \mathfrak{k} = (\mathfrak{m} \oplus \mathfrak{u}) + \mathfrak{k}$ and $\mathfrak{k}^M = \mathfrak{P} \cap \mathfrak{k} = \mathfrak{m} \cap \mathfrak{k}$. Hence $\mathfrak{g}/\mathfrak{k} = \mathfrak{m}/\mathfrak{k}^M \oplus \mathfrak{u}$. We construct an isomorphism

$$\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}),\ \mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})} H_{\sigma_\infty} \otimes \mathcal{M}_\lambda) \xrightarrow{r_{G,P}} \mathrm{Hom}_{K^M_\infty}(\Lambda^\bullet(\mathfrak{m}/\mathfrak{k}) \otimes \Lambda^\bullet(\mathfrak{u}), H_{\sigma_\infty} \otimes \mathcal{M}_\lambda). \quad (9.32)$$

An element $\omega \in \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}),\ \mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})} H_{\sigma_\infty} \otimes \mathcal{M}_\lambda)$ has a value

$$\omega(X_1, X_2, \ldots, X_n) \in \mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})} H_{\sigma_\infty} \otimes \mathcal{M}_\lambda)$$

on a n -tuple $X_1, X_2, \ldots, X_n \in \mathfrak{g}$. This value is a function $G(\mathbb{R}) \to H_{\sigma_\infty} \otimes \mathcal{M}_\lambda)$ which by definition of Ind satisfies $\omega(X_1, X_2, \ldots, X_n)(pg) = \sigma_\infty(p) \otimes \mathrm{Id}(\omega(X_1, X_2, \ldots, X_n)(g))$ for all $p \in P(\mathbb{R}), g \in G(\mathbb{R})$. Since any $g \in G(\mathbb{R})$ can be written as $g = pk$, with $p \in P(\mathbb{R}), k \in K_\infty$ we see that $\omega(X_1, X_2, \dots, X_n)$ is determined by its values on K_∞. But since ω is K_∞-invariant we have $\omega(\mathrm{Ad}(k)X_1, \dots, \mathrm{Ad}(k)X_n))(k) = \omega(X_1, \dots, X_n)(e_G)$, Therefore we see that ω is completely determined by its values at all n-tuples $X_1, \dots, X_n \in \mathfrak{g}$ evaluated at the identity e_G. If we now take $X_1 = t_1, X_2 = t_2, \dots, X_p = t_p$ with $t_i \in \mathfrak{m}$ and $X_{p+1} = u_1, \dots, X_{p+q} = u_q$ with $u_i \in \mathfrak{u}$, then obviously the map

$$(t_1, t_2, \dots, t_p, u_1, \dots, u_q) \mapsto \omega(t_1, t_2, \dots, t_p,) \mapsto \omega(t_1, t_2, \dots, t_p, u_1, \dots, u_q)(e_G\}$$

is an element in $\mathrm{Hom}_{K_\infty^M}(\Lambda^\bullet(\mathfrak{m}/\mathfrak{k}) \otimes \Lambda^\bullet(\mathfrak{u}), H_{\sigma_\infty} \otimes \mathcal{M}_\lambda)$. It is not too difficult to check that this gives the isomorphism $r_{G,P}$. The complex on the right is a double complex, the differential d is the sum $d = d_M + d_U$.

If we fix $t_1, t_2, \dots, t_p$ then the evalution on this p-tuple gives us a map

$$ev(\underline{t}) : \mathrm{Hom}_{K_\infty^M}(\Lambda^\bullet(\mathfrak{m}/\mathfrak{k}) \otimes \Lambda^\bullet(\mathfrak{u}), H_{\sigma_\infty} \otimes \mathcal{M}_\lambda). \to \mathrm{Hom}(\Lambda^\bullet(\mathfrak{u}).H_{\sigma_\infty} \otimes \mathcal{M}_\lambda)$$

Now $\mathrm{Hom}(\Lambda^\bullet(\mathfrak{u}).H_{\sigma_\infty} \otimes \mathcal{M}_\lambda) = H_{\sigma_\infty} \otimes \mathrm{Hom}(\Lambda^\bullet(\mathfrak{u}).\mathcal{M}_\lambda)$ and we have seen that the complex $\mathrm{Hom}(\Lambda^\bullet(\mathfrak{u}).\mathcal{M}_\lambda)$ has a subcomplex of invariants under U

$$\mathbb{H}^\bullet(\mathfrak{u}, \mathcal{M}_\lambda) = \mathrm{Hom}(\Lambda^\bullet(\mathfrak{u}), \mathcal{M}_\lambda)^U,$$

on this sub complex all the differentials d_U are zero and the inclusion induces an isomorphism $\mathbb{H}^\bullet(\mathfrak{u}, \mathcal{M}_\lambda) \xrightarrow{\sim} H^\bullet(\mathfrak{u}, \mathcal{M}_\lambda)$. We recall the theorem of Kostant which says

$$\mathbb{H}^\bullet(\mathfrak{u}, \mathcal{M}_\lambda) = \bigoplus_{w \in W^P} H^{l(w)}(\mathfrak{u}, \mathcal{M}_\lambda)(w \cdot \lambda), \tag{9.33}$$

where $H^{l(w)}(\mathfrak{u}, \mathcal{M}_\lambda)(w \cdot \lambda)$ is the irreducible highest weight module for M with highest weight $w \cdot \lambda = w(\lambda + \rho) - \rho$. We will denote it by $\mathcal{M}(w \cdot \lambda)$ We always remember that it sits in degree $l(w)$ and we always keep in mind that w is not only an element in W, but that there is a specific P in the background and $w \in W^P$.

Hence we get the theorem of Delorme

Theorem 9.2.1. *We have an injective homomorphism of complexes*

$$Hom_{K_\infty^M}(\Lambda^\bullet(\mathfrak{m}/\mathfrak{k}), H_{\sigma_\infty} \otimes \mathbb{H}^\bullet(\mathfrak{u}, \mathcal{M}_\lambda)) \to Hom_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \; Ind_{P(\mathbb{R})}^{G(\mathbb{R})} H_{\sigma_\infty} \otimes \mathcal{M}_\lambda)$$

which induces an isomorphism in cohomology. Hence we get an isomorphism of cohomology groups

$$\bigoplus_{w \in WP} H^{\bullet - l(w)}(\mathfrak{m}, K_\infty^M, H_{\sigma_\infty} \otimes \mathcal{M}_\lambda(w \cdot \lambda)) \xrightarrow{\sim} H^\bullet(\mathfrak{g}, K_\infty, \; Ind_{P(|R)}^{G(\mathbb{R})} H_{\sigma_\infty} \otimes \mathcal{M}_\lambda)$$

The restriction map r on the level of the de-Rham complexes

We recall the de-Rham isomorphism in section 8.1.3 and (8.5). Of course we have a corresponding formula for the cohomology of a boundary stratum

$$H^\bullet(\partial_P \mathcal{S}_{K_f}^G, \tilde{\mathcal{M}}_\lambda) \xrightarrow{\sim} H^\bullet(\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathcal{C}_\infty(U(\mathbb{A})P(\mathbb{Q}) \backslash G(\mathbb{A})/K_f) \otimes \mathcal{M}_\lambda) \tag{9.34}$$

and the restriction map $r_P : H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) \to H^\bullet(\partial_P\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$ is induced by the constant Fourier coefficient

$$\mathcal{F}^P(f)(\underline{g}) = \int_{U(\mathbb{A})} f(\underline{u}\underline{g})d\underline{u} \tag{9.35}$$

We mentioned that we may have a difference between the cuspidal and the inner cohomology. We consider an irreducible (non zero) module $H_{\pi_\infty} \otimes H_{\pi_f} \subset L^2_{\text{disc}}(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f)$ which has non trivial cohomology $H^\bullet(\mathfrak{g}, K_\infty, H_{\pi_\infty} \otimes \mathcal{M}_\lambda) \neq 0$. Let q_0 be the lowest degree such that $H^{q_0}(\mathfrak{g}, K_\infty, H_{\pi_\infty} \otimes \mathcal{M}_\lambda) \neq 0$. Then we have the following

Proposition 9.2.1. *If $H_{\pi_\infty} \otimes H_{\pi_f} \not\subset L^2_{\text{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f)$, then the composition*

$$H^{q_0}(\mathfrak{g}, K_\infty, H_{\pi_\infty} \otimes \mathcal{M}_\lambda) \otimes H_{\pi_f} \to H^{q_0}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{C}) \xrightarrow{r} H^{q_0}(\partial\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{C}) \tag{9.36}$$

is non zero.

Proof. We have to show that for all proper parabolic subgroups the map $\mathcal{F}^P : H_{\pi_\infty} \otimes H_{\pi_f} \to \mathcal{C}_\infty(U(\mathbb{A})P(\mathbb{Q})\backslash G(\mathbb{A})/K_f$ is zero. If this is not so then there is a parabolic subgroup such that the Fourier coefficient even lands in the space $\mathcal{C}_{\infty,\text{cusp}}(U(\mathbb{A})P(\mathbb{Q})\backslash G(\mathbb{A})/K_f)$. We "decompose" this space, hence we get an embedding

$$\mathcal{F}^P : H_{\pi_\infty} \otimes H_{\pi_f} \hookrightarrow \bigoplus_{\sigma_\infty} \text{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} V_{\sigma_\infty} \otimes V_{\sigma_f}. \tag{9.37}$$

Since we assumed that $H_{\pi_\infty} \otimes H_{\pi_f}$ is irreducible we know that the projection to a suitable summand $\text{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} V_{\sigma_\infty} \otimes V_{\sigma_f}$ $\text{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} V_{\sigma_\infty} \otimes V_{\sigma_f}$ induces already an injection. On the level of complexes we get

$$\begin{gathered}\text{Hom}_{K_\infty}(\ \text{Hom}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), H_{\pi_\infty} \otimes \mathcal{M}_\lambda) \otimes H_{\pi_f} \hookrightarrow \\ \text{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}),\ \text{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})} V_{\sigma_\infty} \otimes \mathcal{M}_\lambda) \otimes\ \text{Ind}_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)} V_{\sigma_f}\end{gathered} \tag{9.38}$$

We apply Delorme's method to factor at infinity on the right hand side. We have the harmonic sub complex

$$\begin{gathered}\bigoplus_{w\in W^P}\ \text{Hom}_{K^M_\infty}(\Lambda^\bullet(\mathfrak{m}^{(1)}/\mathfrak{k}_M), V_{\sigma_\infty} \otimes \mathcal{M}(w\cdot\lambda)) \otimes \Lambda^\bullet(\mathfrak{c}_M) \subset \\ \text{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k})), (\ \text{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}(V_{\sigma_\infty} \otimes \mathcal{M}_\lambda)\end{gathered} \tag{9.39}$$

We also know the that the restriction of σ to the central torus C_M of M is of type $w \cdot \lambda | C_M$. We see that the embedding (9.38) actually factors

$$\begin{gathered}\mathcal{F}^P_h :\ \text{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), H_{\pi_\infty} \otimes \mathcal{M}_\lambda) \otimes H_{\pi_f} \hookrightarrow \\ \bigoplus_{\sigma_\infty} \bigoplus_{w\in W^P}\ \text{Hom}_{K^M_\infty}(\Lambda^\bullet(\mathfrak{m}^{(1)}/\mathfrak{k}_M), V_{\sigma_\infty} \otimes \mathcal{M}(w\cdot\lambda)) \otimes \Lambda^\bullet(\mathfrak{c}_M)\end{gathered} \tag{9.40}$$

But since H_{π_∞} is in L^2 and we have the information about the asymptotic behaviour of the V_{σ_∞} we can conclude that the image of $\mathcal{F}^P_h$ actually lies in the summand with $w = e_P$ the identitiy element. So the constant Fourier coefficient gives us a map

$$\begin{aligned} \mathcal{F}_h^P:\ &\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), H_{\pi_\infty}\otimes\mathcal{M}_\lambda)\otimes H_{\pi_f}\hookrightarrow \\ \bigoplus\nolimits_{\sigma_\infty}\ &\mathrm{Hom}_{K_\infty^M}(\Lambda^\bullet(\mathfrak{m}^{(1)}/\mathfrak{k}_M), V_{\sigma_\infty}\otimes\mathcal{M}(e_P\cdot\lambda))\otimes\Lambda^\bullet(\mathfrak{c}_M)\end{aligned} \tag{9.41}$$

Let q_1 be the smallest degree in which the complex on the right is not zero, then there is a σ_∞ such that $\mathrm{Hom}_{K_\infty^M}(\Lambda^{q_1}(\mathfrak{m}^{(1)}/\mathfrak{k}_M), V_{\sigma_\infty}\otimes\mathcal{M}(e_P\cdot\lambda)) = H^{q_1}(\mathfrak{m}, K_\infty^M, V_{\sigma_\infty}\otimes\mathcal{M}(e_P\cdot\lambda)) \neq 0$ we even know that it is of dimension one. Of course the complex on the left is zero in degree $< q_1$. We restrict the action of $G(\mathbb{R})$ on H_{π_∞} to $M(\mathbb{R})$ then we have a surjective $M(\mathbb{R})$ homomorphism $H_{\pi_\infty}\to V_{\sigma_\infty}$. We further restrict the $M(\mathbb{R})$ to K_∞^M then we find a section $s: V_{\sigma_\infty}\to H_{\pi_\infty}$ This allows us to lift the non zero class in $\mathrm{Hom}_{K_\infty^M}(\Lambda^{q_1}(\mathfrak{m}^{(1)}/\mathfrak{k}_M), V_{\sigma_\infty}\otimes\mathcal{M}(e_P\cdot\lambda))$ to a class in $\mathrm{Hom}_{K_\infty}(\Lambda^{q_1}(\mathfrak{g}/\mathfrak{k}), H_{\pi_\infty}\otimes\mathcal{M}_\lambda)$. This shows $q_1 = q_0$ and the proposition.

□

We return to the issue raised in the section "cuspidal vs. inner." We said that an isomorphism type π_f of a Hecke module which occurs in the inner cohomology is strongly inner, if

$$H^\bullet_{\mathrm{cusp}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda\otimes_F\mathbb{C})(\pi_f) = H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)(\pi_f)\otimes_F\mathbb{C}$$

This is an equality between a Hecke-module defined by transcendental notions and a Hecke module which has a combinatorial definition. But in view of our proposition we also see that π_f is strongly inner if and only if

$$H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda\otimes F)(\pi_f) = H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda\otimes F)(\pi_f) \tag{9.42}$$

and this is now a combinatorial characterisation for the isomorphism type π_f to be strongly inner.

We should be a little bit careful with the notion of cuspidal cohomology. Of course if we given an isomorphism type of Hecke-modules π_f which is strongly inner, then this means that all the isotypic cohomology $H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda\otimes F)(\pi_f)$ is cuspidal.

But if π_f is given and occurs in the inner cohomology then we can find a friend π_∞ and an embedding $\Phi: H_{\pi_\infty}\otimes H_{\pi_f}\hookrightarrow L^2_{\mathrm{disc}}(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f)$. This produces cohomology classes and we call these classes cuspidal if Φ factors through the space of cusp forms. But this notion depends on the choice of σ and Φ. Hence such a space of cuspidal classes is only defined over $\mathbb{C}$.

Of course we have seen in the proof of proposition 9.2.1 that an irreducible Hecke module π_∞ which occurs in the inner cohomology has to be somewhat special if it is not strongly inner. The proof shows that we must have

$$H_{\pi_f}\subset\ \mathrm{Ind}_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)}V_{\sigma_f} \tag{9.43}$$

where σ_f is strongly inner in $H^\bullet(\mathcal{S}^M_{K_f^M}, \mathcal{M}_\lambda(e\cdot\lambda))$ and M the Levi-quotient of a proper parabolic subgroup. This is now a very strong restriction on π_f. If for instance λ is regular this can never happen ([73]).

On the other hand this happens if λ is not regular. In [36] Kapitel III we discuss examples where this happens. There we also discuss the possibility to construct mixed motives from these inner but not strongly inner π_f, and for us this possibility was the main motivation to study these phenomena.

For some more recent developments in this direction we refer to Mix-Mot.pdf on my home page.

9.2.2 Induction and the local intertwining operator at finite places

Our modules σ_f are modules for the Hecke algebras $\mathcal{H}^M_{K^M_f} = \otimes_p \mathcal{H}^M_{K^M_p}$. Therefore we can write them as tensor product $\sigma_f = \otimes_p \sigma_p$. We consider a prime p where σ_f is unramified then we get can give a standard model for this isomorphism class. The module H_{σ_p} is the rank one $\mathcal{O}_F$ -module $\mathcal{O}_F$, i.e. it comes with a distinguished generator 1. The Hecke algebra acts by a homomorphism (See 6.3.2)

$$h(\sigma_p) : \mathcal{H}^{(M,w\cdot\lambda)}_{K^M_p,\mathbb{Z}} \to \mathcal{O}_F \tag{9.44}$$

and gives us the Hecke-module structure on H_{σ_p}. We can induce H_{σ_p} to a $\mathcal{H}^G_{K^G_p}$ module. This is actually the same $\mathcal{O}_F$ module but now with an action of the algebra $\mathcal{H}^{(G,\lambda)}_{K^G_p,\mathbb{Z}}$. We simply observe that we have an inclusion $\mathcal{H}^{(G,\lambda)}_{K^G_p,\mathbb{Z}} \hookrightarrow \mathcal{H}^{(M,w'\cdot\lambda)}_{K^M_p,\mathbb{Z}}$ and induction simply means restriction.

It follows easily from the description of the spherical (unramified) Hecke modules via their Satake-parameters that the induced modules H_{σ_p} and $H_{\sigma'_p}$ are isomorphic as $\mathcal{H}^{(G,\lambda)}_{K^G_p,\mathbb{Z}}$ -modules and hence we get that after induction the two summands in (9.31) become isomorphic. We choose a local intertwining operator

$$T_p^{\text{loc}} : H_{\sigma_p} \to H_{\sigma'_p} \tag{9.45}$$

simply the identity. We do not discuss local intertwining operator at ramified places. But it can be shown that there are non zero local operators T_p^{loc} : $H_{\sigma_p} \to H_{\sigma'_p}$ for all p and we define $T_f^{\text{loc}} = \prod_p T_p^{\text{loc}}$.

9.3 The Eisenstein intertwining operator

Our notations are as above, let H_{σ_f} be an absolutely irreducible module for $\mathcal{H}^M_{K^M_f}$, we assume that it occurs in some $H^{\bullet-l(w)}_!(\mathcal{S}^M_{K^M_f}, \tilde{\mathcal{M}}(w\cdot\lambda)\otimes F)(\sigma_f))$. Here $F \subset \mathbb{C}$ is a finite normal extension of $\mathbb{Q}$. We realise σ_f as a restricted tensor product $H_{\sigma_f} = \bigotimes'_p H_{\sigma_p}$, we assume that for the primes $p \notin \Sigma$ H_{σ_p} is of dimension one. 6.60). Now it follows from Theorem 8.1.1 that there is an irreducible $M(\mathbb{R})$ module H_{σ_∞} such that $H^\bullet(\mathfrak{m}, K^M_\infty, H^{(\infty)}_{\sigma_\infty} \otimes \tilde{\mathcal{M}}(w\cdot\lambda)) \neq 0$ and that we have an embedding

$$\Phi : H_{\sigma_\infty} \otimes H_{\sigma_f} \otimes_F \mathbb{C} \hookrightarrow L^2_{\text{disc}}(M(\mathbb{Q})\backslash M(\mathbb{A})). \tag{9.46}$$

In the following we forget the Φ and consider $H_{\sigma_\infty} \otimes H_{\sigma_f}$ as a subspace of $L^2_{\text{disc}}(M(\mathbb{Q})\backslash M(\mathbb{A}))$. We assume that $w \cdot \lambda$ is in the positive chamber.

We consider the induced module, recall that this is the space of functions

$$\{f : G(\mathbb{A}) \to H_\sigma | f(\underline{p}\underline{g}) = \bar{\underline{p}} f(\underline{g})\} \tag{Ind}$$

where $\bar{\underline{p}}$ is the image of $\underline{p}$ in $M(\mathbb{A})$. We can define the subspace $H_\sigma^{(\infty)}$ consisting of those f which satisfy some suitable smoothness conditions and then we can define a submodule $\operatorname{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} H_\sigma^{(\infty)}$ where the $f(g) \in H_\sigma^{(\infty)}$ and the f themselves also satisfy some smoothness conditions.

We embed this space into the space $\mathcal{A}(P(\mathbb{Q})\backslash G(\mathbb{A}))$ by sending

$$f \mapsto \{g \mapsto f(g)(e_M)\},$$

here $\mathcal{A}$ denotes some space of automorphic forms. This an embedding of $G(\mathbb{A})$-modules or an embedding of Hecke modules if we fix a level.

We have the character $\gamma_P : M \to G_m$, for any complex number z we define the homomorphism $|\gamma_P|^z : M(\mathbb{A}) \to \mathbb{R}^\times$ which is given by $|\gamma_P|^z : \underline{m} \mapsto |\gamma_P(\underline{m})|^z$. As usual we denote it by $\mathbb{C}(|\gamma_P|^z)$ the one dimensional $\mathbb{C}$ vector space on which $M(\mathbb{A})$ acts by the character $|\gamma_P|^z$. Then we may twist the representation H_σ by this character and put $H_\sigma \otimes |\gamma_P|^z = H \otimes \mathbb{C}(|\gamma_P|^z)$. An element $\underline{g} \in G(\mathbb{A})$ can be written as $\underline{g} = \underline{p}\underline{k}, \underline{p} \in P(\mathbb{A}), \underline{k} \in K_f^0$ where $K_f^0 \supset K_f$ is a suitable maximal compact subgroup and now we define $h(\underline{g}) = |\gamma_P|(\underline{p})|$. Eisenstein summation yields embeddings

$$\text{Eis} : \operatorname{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} H_\sigma^{(\infty)} \otimes |\gamma_P|^z \to \mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A})), \tag{9.47}$$

where

$$\text{Eis}(f)(\underline{g}) = \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} f(\gamma \underline{g})(e_M) h(\gamma \underline{g})^z,$$

it is well known that this is locally uniformly convergent provided $\Re(z) >> 0$, (see [50]).

Now we assume that H_σ is even in the cuspidal spectrum. We get important information concerning these Eisenstein series, if we compute their constant Fourier coefficient with respect to parabolic subgroups: For any parabolic subgroup $P_1/\mathbb{Q} \subset G/\mathbb{Q}$ with unipotent radical $U_1 \subset P_1$ we define (See [50], 4)

$$\mathcal{F}^{P_1}(\text{ Eis}(f))(\underline{g}) = \int_{U_1(\mathbb{Q})\backslash U_1(\mathbb{A})} \text{Eis}(f)(\underline{u}_1 \underline{g}) d\underline{u}_1.$$

This only depends on the $G(\mathbb{Q})$-conjugacy class of $P_1/\mathbb{Q}$. It is also in [50] , 4 that this constant term is zero unless P_1 is maximal and the conjugacy class of P_1 is equal to the conjugacy class of $P/\mathbb{Q}$ or the conjugacy class of $Q/\mathbb{Q}$. (which may or may not be equal to the conjugacy class of $P/\mathbb{Q}$.) Here we need that H_σ is in the cuspidal spectrum.

These constant Fourier coefficients have been computed by Langlands, we briefly recall the main steps in this calculation. We refer to [50] Chapter II, §4.

We consider the action of U_B on the quotient $P\backslash G$, we know that we have a Bruhat cell decomposition

$$P(\mathbb{Q})\backslash G(\mathbb{Q}) = \bigcup_{w\in W^P} P(\mathbb{Q})\backslash P(\mathbb{Q})wU^{(w)}(\mathbb{Q})$$

where $U^{(w)} = \prod_{\alpha\in\Delta^+, w\alpha\notin\Delta^P} U_\alpha$. Then the Eisenstein summation becomes

$$\mathrm{Eis}(f)(\underline{g}) = \sum_{w\in W^P} \sum_{u\in U^{(w)}(\mathbb{Q})} f(wu\underline{g})(e_M)h(wu\underline{g})^z, \tag{9.48}$$

and we get for the constant Fourier coefficient

$$\sum_{w\in W^P} \int_{U_1(\mathbb{Q})\backslash U_1(\mathbb{A})} \sum_{u\in U^{(w)}(\mathbb{Q})} f(wu\underline{u}_1\underline{g})(e_M)h(wu\underline{u}_1\underline{g})^z d\underline{u}_1 \tag{9.49}$$

Since we assumed that H_σ is in the space of cusp forms on M it follows

$$\int_{U_1(\mathbb{Q})\backslash U_1(\mathbb{A})} f(wu\underline{u}_1\underline{g})(e_M)h(wu\underline{u}_1\underline{g})^z d\underline{u}_1 = 0 \tag{9.50}$$

unless we have $wU_1w^{-1} \cap P \subset U_P$. (See [50], loc. cit. Lemma 33)

Hence we get a non zero term in the expression (9.49) if

a) The parabolic subgroup $P/\mathbb{Q}$ is conjugate to its opposite parabolic $P_-/\mathbb{Q}$ and $P_1 = P$.

b) $P \neq [\Theta]P$ and $P_1 = P$ or $P_1 = [\Theta]P$.

In case a) we have to take $P_1 = P$ and we have two non zero summands in (9.49), namely we can take $w = e_W$ or $w = w^P$ where $w^P P(w^P)^{-1} = P_-$ the opposite parabolic subgroup. The element w^P induces an automorphism of $M/\mathbb{Q}$. We get a twisted representation $w^P(\sigma)$ of $M(\mathbb{A})$, we have $H_{w^P(\sigma)} = \{\underline{m} \mapsto f(w^P(\underline{m})|f \in H_\sigma\}$. We get two terms in (9.49)

$$\mathcal{F}^P(\ \mathrm{Eis}(f))(\underline{g}) = f(\underline{g})h(\underline{g})^z + \int_{U(\mathbb{A})} f(w^P\underline{u}\underline{g})h(w^P\underline{u}\underline{g})^z d\underline{u} \tag{9.51}$$

and this means that we get an intertwining operator

$$\mathcal{F}^P \circ \ \mathrm{Eis}: \ \ \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} H_\sigma \otimes |\gamma_P|^z \to$$

$$\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} H_\sigma \otimes |\gamma_P|^z \oplus \ \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} H_{w^P(\sigma)} \otimes |\gamma_Q|^{2f_P - z} \subset \mathcal{A}(U_P(\mathbb{A})P(\mathbb{Q})\backslash G(\mathbb{A})). \tag{9.52}$$

In case b) we have to compute the constant Fourier coefficients along P and along $Q = P_1 = [\Theta]P$. In this each of the two integrals only one of the terms in the summation is non zero: For $\mathcal{F}^P$ this is the term $w = e_W$ and for $\mathcal{F}^Q$ it is again the element w^P, but now the conjugacy class of P_- is Q. We get

$$\mathcal{F}^P(\ \mathrm{Eis}(f))(\underline{g}) = f(\underline{g})h(\underline{g})^z \text{ and } \mathcal{F}^Q(\ \mathrm{Eis}(f))(\underline{g}) = \int_{U_Q(\mathbb{A})} f(w^P\underline{u}_1\underline{g})h(w^P\underline{u}\underline{g})^z d\underline{u} \tag{9.53}$$

and hence we get an intertwining operator

$$(\mathcal{F}^P \oplus \mathcal{F}^Q)\circ \text{ Eis}: \ \ \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} H_\sigma \otimes |\gamma_P|^z \rightarrow$$

$$\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} H_\sigma \otimes |\gamma_P|^z \oplus \ \mathrm{Ind}_{Q(\mathbb{A})}^{G(\mathbb{A})} H_{w^P(\sigma)} \otimes |\gamma_Q|^{2f_P - z} \tag{9.54}$$

$$\subset \mathcal{A}(U_P(\mathbb{A})P(\mathbb{Q})\backslash G(\mathbb{A})) \oplus \mathcal{A}(U_Q(\mathbb{A})Q(\mathbb{Q})\backslash G(\mathbb{A})).$$

Regardless of $P = Q$ or $P \neq Q$ we have to consider the intertwining operator (the second term in the constant term)

$$\mathcal{F}^Q \circ \text{ Eis}: \ \ \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} H_\sigma \otimes |\gamma_P|^z \rightarrow \ \mathrm{Ind}_{Q(\mathbb{A})}^{G(\mathbb{A})} H_{w^P(\sigma)} \otimes |\gamma_Q|^{2f_P - z}.$$

We observe that both sides are restricted tensor products taken over all places and therefore we try to define local intertwining operators

$$\begin{aligned} T_p^{\mathrm{loc}}(z): \ &\mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} H_{\sigma_p} \otimes |\gamma_P|_p^z \rightarrow \ \mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} H_{w^P(\sigma_p)} \otimes |\gamma_Q|_p^{2f_P - z} \\ T_\infty^{\mathrm{loc}}(z): \ &\mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R}} H_{\sigma_\infty} \otimes |\gamma_P|_\infty^z \rightarrow \ \mathrm{Ind}_{Q(\mathbb{R})}^{G(\mathbb{R})} H_{w^P(\sigma_\infty)} \otimes |\gamma_Q|_\infty^{2f_P - z} \end{aligned} \tag{9.55}$$

which are holomorphic in a neighbourhood of $z = 0$ and isomorphism for z in a non empty open set. We know that these intertwining operators exist, we have to choose them.

At the unramified finite places the H_{σ_p} is one dimensional and comes with a generator 1 the local operator is constant, i.e. does not depend on z and is equal to T_p^{loc} in section (9.2.2). and $T^{\mathrm{loc}}(0) = \otimes_p T_p^{\mathrm{loct}}$.

At the ramified places $p \in \Sigma$ things are not so easy. We may choose

$$T_p^{\ \mathrm{loc}}(z) = \int_{U_Q(\mathbb{Q}_p)} f(w^P u_p g_p) h(w^P u_p g_p)^z du_p,$$

It is not so difficult to see that $T_p^{\ \mathrm{loc}}(z)$ is holomorphic at $z = 0$ again we use that $w \cdot \lambda$ is in the positive chamber. In view the applications to arithmetic we have to discuss rationality questions, especially we should prove that $T_p^{\ \mathrm{loc}}(0)$ respects the F-vector space structure, i.e. we should prove that

$$T_p^{\ \mathrm{loc}}(0): \ \mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} H_{\sigma_p} \rightarrow \ \mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} H_{w^P(\sigma_p)} \tag{9.56}$$

This issue is discussed in a special case in [48] 7.3.2.1.

We also assume that we have chosen nice realisations $H_{\sigma_\infty}, H_{\sigma'_\infty}$, and an intertwining operator

$$T_\infty^{\ \mathrm{loc}}(z): \ \mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})} H_{\sigma_\infty} \otimes |\gamma_P|_\infty^z \rightarrow \ \mathrm{Ind}_{Q(\mathbb{R})}^{G(\mathbb{R})} H_{\sigma'_\infty} \otimes |\gamma_Q|_\infty^{2f_P - z} \tag{9.57}$$

which is normalised by the requirement that it induces the "identity" on a certain fixed K_∞^M type. It should also satisfy certain rationality conditions. Again this is discussed in a special case [48] Chap. 8 and 9 (Weselmann).

Now we can define $T^{\ \mathrm{loc}}(z) = T_\infty^{\ \mathrm{loc}} \times \prod_p T_p^{\ \mathrm{loc}}(z)$, this is a legal expression since for all $p \notin \Sigma$ it sends the generator to the generator.

We get the classical formula of Langlands for the constant term: For $f \in \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} H_\sigma \otimes |\gamma_P|^z$ we get

$$\mathcal{F}^P \circ \ \mathrm{Eis}(f) = f + C(\sigma, z) T^{\mathrm{loc}}(z)(f), \tag{9.58}$$

where $C(\sigma, \lambda, z)$ is a product of local factors $C_p(\sigma_p, , z)$ over all primes and a factor $C_\infty(\sigma_\infty, z)$. For any $\mathbf{v} = p$ or $\mathbf{v} = \infty$ the function $C_{\mathbf{v}}(\sigma_{\mathbf{v}}, z)$ is a function in z which is holomorphic for $\Re(z) \geq 0$ (here we need that $w \cdot \lambda$ is in the positive chamber.) By definition we have $C_p(\sigma_p, z) = 1$ at the ramified primes. For the unramified primes we get $C_p(\sigma_p, z) = \int_{U_Q(\mathbb{Q}_p)} f_0(w^P u_p g_p) h(w^P u_p g_p)^z du_p$ where f_0 is the spherical function, i.e. the generator of H_{σ_p}.

The computation of this last integral carried out in H. Kims paper in [62], chap. 6. Kim expresses the factor in terms of automorphic $L-$ functions attached to σ_f. To formulate the result of this computation we have to recall the notion of the dual group (7.1). Inside the dual group ${}^L G$ we have the dual group ${}^L M$ which acts by conjugation on the Lie algebra $\mathfrak{u}_P^\vee$. The set of roots $\Delta^+_{U_P^\vee}$ is a set of cocharacters of $T/\mathbb{Q}$, a coroot $\alpha^\vee \in \Delta^+_{U_P}$ defines a one-dimensional root subgroup $\mathfrak{u}^\vee_{P,\alpha^\vee}$. The ${}^L M$ -module $\mathfrak{u}_P^\vee$ decomposes into submodules. We recall that the maximal parabolic subgroup $P/\mathbb{Q}$ was obtained from the choice of a Galois-orbit $\tilde{i} \subset I$ (9.1.3) and any

$$\alpha^\vee \in \Delta^+_{U_P^\vee},\ \alpha^\vee = a(\alpha^\vee, \tilde{i}) \alpha^\vee{}_{\tilde{i}} + \sum_{j \notin \tilde{i}} m_{\tilde{i},j} \alpha^\vee{}_j. \tag{9.59}$$

Here the coefficients are integers ≥ 0 and $a(\alpha^\vee, \tilde{i}) > 0$. For a given integer $a > 0$ we define

$$\mathfrak{u}_P^\vee[a] = \bigoplus_{\alpha^\vee : a(\alpha^\vee, \tilde{i}) = a} \mathfrak{u}^\vee_{P,\alpha^\vee} \tag{9.60}$$

it is rather obvious that $\mathfrak{u}_P^\vee[a]$ is an invariant submodule under the action of $M^\vee$ and actually it is even irreducible. Let us denote the representation of $M/\mathbb{Q}$ on $\mathfrak{u}_P^\vee[a]$ by $r_a^{\mathfrak{u}_P^\vee}$. In the following η_a will be the highest weight of $r_a^{\mathfrak{u}_P^\vee}$.

With these notations we get the following formula for the local factor at an unramified place p (See [62])

$$C_p(\sigma, z) = \prod_{a=1}^{r} \frac{L(\sigma_p, r_a^{\mathfrak{u}_P^\vee}, az + 1)}{L(\sigma_p, r_a^{\mathfrak{u}_P^\vee}, az + 2)} T_p^{\mathrm{loc}}(z)(f). \tag{9.61}$$

If we put $L^{(\Sigma)}(\sigma_f, r_a^{\mathfrak{u}_P^\vee}, z) = \prod_{p \notin \Sigma} L^{(\Sigma)}(\sigma_p, r_a^{\mathfrak{u}_P^\vee}, z)$, then

$$C(\sigma, z) = C(\sigma_\infty, z) \prod_p C_p(\sigma_p, z) = C(\sigma_\infty, z) \prod_{a=1}^{r} \frac{L^{(\Sigma)}(\sigma_f, r_a^{\mathfrak{u}_P^\vee}, az)}{L^{(\Sigma)}(\sigma_f, r_a^{\mathfrak{u}_P^\vee}, az + 1)}$$

The local factor at infinity depends on the choice of T_∞^{loc}, in 1.2.4 we gave some rules how to fix it, if it is not zero on cohomology.

In general people hope that there is a consistent definition of local Euler-factors $L(\sigma_p, r_a^{\mathfrak{u}_P^\vee}, z)$ also for ramified primes and also Euler-factors $L(\sigma_\infty, r_a^{\mathfrak{u}_P^\vee}, z)$ at infinity. Then we define the completed L-function

$$\Lambda(\sigma, r_a^{\mathfrak{u}_P^\vee}, z) = L(\sigma_\infty, r_a^{\mathfrak{u}_P^\vee}, z) \prod L(\sigma_p, r_a^{\mathfrak{u}_P^\vee}, z).$$

Now it makes sense to alter the definition of $T_{\mathbf{v}}^{\mathrm{loc}}(z)$, for $\mathbf{v} = p \in \Sigma$ or $\mathbf{v} = \infty$ the new local operators will be

$$T_{\mathbf{v}}^{\mathrm{loc}}(z) := \frac{L_{\mathbf{v}}(\sigma_{\mathbf{v}}, r_a^{\mathfrak{u}_P^\vee}, az+1)}{L_{\mathbf{v}}(\sigma_{\mathbf{v}}, r_a^{\mathfrak{u}_P^\vee}, az))} \times (\text{ old } T_{\mathbf{v}}^{\mathrm{loc}}(z))$$

and $T^{\mathrm{loc}}(z) = \prod_{\mathbf{v}} T_{\mathbf{v}}^{\mathrm{loc}}(z)$. If now $\mathcal{F} = \mathcal{F}^P$ or $\mathcal{F} = \mathcal{F}^P \oplus \mathcal{F}^Q$ then we get

$$\mathcal{F} \circ \ \mathrm{Eis}(f) = f + \prod_a \frac{\Lambda(\sigma, r_a^{\mathfrak{u}_P^\vee}, az))}{\Lambda(\sigma, r_a^{\mathfrak{u}_P^\vee}, az+1)} T^{\mathrm{loc}}(z)(f), \tag{9.62}$$

It is a theorem of Langlands that the Eisenstein intertwining operator is holomorphic at $z = 0$ if the second term is holomorphic at $z = 0$. Since $w \cdot \lambda$ is in the positive chamber there is no problem with $T^{\mathrm{loc}}(z)(f)$, it is holomorphic at $z = 0$. Hence we see that the Eisenstein intertwining operator is holomorphic at $z = 0$.

Let us assume that $w \cdot \lambda$ or equivalently σ_f are in the positive chamber. In case a) we have holomorphicity at $z = 0$ if the weight λ is regular (See [Schw]) and in case b) the Eisenstein series is always holomorphic at $z = 0$. In this section that we assume that the Eisenstein series is holomorphic at $z = 0$ and hence we can evaluate at $z = 0$ in (9.58) . Then we get an intertwining operator

$$\mathrm{Eis} \circ \Phi : \ \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} H_\sigma \to \mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A})). \tag{9.63}$$

We get a homomorphism on the de-Rham complexes

$$\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \ \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} H_\sigma \otimes_F \mathbb{C} \otimes \mathcal{M}_\lambda) \xrightarrow{\mathrm{Eis}^\bullet} \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A})) \otimes \tilde{\mathcal{M}}_\lambda) \tag{9.64}$$

We introduce the abbreviation $H_{\iota\circ\sigma_f} = H_{\sigma_f} \otimes_{F,\iota} \mathbb{C}$ and decompose $H_\sigma = H_{\sigma_\infty} \otimes H_{\sigma_f}$. We compose (9.64) with the constant term and get

$$\begin{aligned} \mathcal{F} \circ \ \mathrm{Eis}^\bullet : \ & \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \ \mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})} H_{\sigma_\infty} \otimes \mathcal{M}_\lambda) \otimes H_{\iota\circ\sigma_f} \to \\ & \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \ \mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})} H_{\sigma_\infty} \otimes \mathcal{M}_\lambda) \otimes H_{\iota\circ\sigma_f} \\ & \oplus \\ & \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \ \mathrm{Ind}_{Q(\mathbb{R})}^{G(\mathbb{R})} H_{\sigma'_\infty} \otimes \mathcal{M}_\lambda) \otimes H_{\iota\circ\sigma'_f} \end{aligned} \tag{9.65}$$

where $P = Q$ in case a).

We study the contribution from the place ∞

$$T_\infty^{\mathrm{loc},*} : \ \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \ \mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})} H_{\sigma_\infty} \otimes \mathcal{M}_\lambda) \to \ \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \ \mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})} H_{\sigma'_\infty} \otimes \mathcal{M}_\lambda) \tag{9.66}$$

and the resulting operator on the cohomology

$$T_\infty^{\mathrm{loc},\bullet}: H^\bullet(\mathfrak{g}, K_\infty,\ \mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})} H_{\sigma_\infty} \otimes \mathcal{M}_\lambda) \to H^\bullet(\mathfrak{g}, K_\infty,\ \mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})} H_{\sigma'_\infty} \otimes \mathcal{M}_\lambda) \tag{9.67}$$

Our computations above imply that the intertwining operator in (9.67) respects the direct sum decomposition and hence $T_\infty^{\mathrm{loc},\bullet} = \bigoplus_{w\in W^P} T^{\mathrm{loc},\bullet}(w)$ where

$$\begin{aligned} T_\infty^{\mathrm{loc},\bullet}(w) : H^{\bullet - l(w)}(\mathfrak{m}, K_\infty^M, H_{\sigma_\infty} \otimes H^{l(w)}(\mathfrak{u}, \mathcal{M}_\lambda)(w\cdot\lambda)) \to \\ H^{\bullet - l(w')}(\mathfrak{m}, K_\infty^M, H_{\sigma'_\infty} \otimes H^{l(w')}(\mathfrak{u}, \mathcal{M}_\lambda)(w'\cdot\lambda)) \end{aligned} \tag{9.68}$$

The computation of $T_\infty^{\mathrm{loc},\bullet}(w)$ is a problem in representation theory of semisimple groups or the theory of Harish-Chandra modules. A first interesting and amusing case is treated in [44] (Appendix by D. Zagier). This has been generalised in an appendix by U. Weselmann in [48].

Hier muss irgendwo die relative Periode auftauchen

Theorem 9.3.1. *If $w\cdot\lambda$ is in the positive chamber, and if the Eisenstein series is holomorphic at $z = 0$ then the Eisenstein intertwining operator gives us a homomorphism of Hecke-modules*

$$Eis_{\mathbb{C}} \otimes \Phi^q : H^{\bullet - l(w)}(\mathfrak{m}, K_\infty^M, H_{\sigma_\infty} \otimes \mathcal{M}_\lambda(w\cdot\lambda)) \otimes\ Ind_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)} H_{\sigma_f} \otimes_F \mathbb{C} \to H^q(\mathcal{S}_{K_f}^G, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{C})(\sigma_f)$$

The composition with the restriction r to the cohomology of the boundary gives us

$$r\circ\ Eis_{\mathbb{C}} \circ \Phi^q :\ Ind_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)} H_{\sigma_f} \to$$

$$Ind_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)} H_{\mathrm{cusp}}^{q-l(w)})(\mathcal{S}_{K_f^M}^M, \mathcal{M}(w\cdot\lambda)\otimes\mathbb{C})(\sigma_f) \oplus\ Ind_{Q(\mathbb{A}_f)}^{G(\mathbb{A}_f)} H_{\mathrm{cusp}}^{q-l(w'))})(\mathcal{S}_{K_f^M}^M, \mathcal{M}(w'\cdot\lambda)\otimes\mathbb{C})(\sigma'_f)$$

$$r\circ\ Eis_{\mathbb{C}} \circ \Phi^q : \omega\otimes\psi_f \mapsto \omega\otimes\psi_f + \prod_a \frac{\Lambda(\sigma, r_a^{\mathfrak{u}_P^\vee}, 0)}{\Lambda(\sigma, r_a^{\mathfrak{u}_P^\vee}, 1)} T_\infty^{\mathrm{loc},\bullet}(w)(\omega) \otimes T_f^{\mathrm{loc}}(\psi_f) \tag{9.69}$$

We recall the definition $W_{\mathrm{cusp}}(\sigma_\infty \times \sigma_f) :=\ \mathrm{Hom}_{K_f}(H_{\sigma_\infty} \otimes H_{\pi_f}, L^2_{\mathrm{cusp}}(M(\mathbb{Q})\backslash M(\mathbb{A})/K_f^M)$ summing up the Hodge decomposition summands we get an intertwining Eisenstein operator

$$\begin{aligned} \mathrm{Eis}_C : \bigoplus_{\sigma_\infty} W(\sigma_\infty\times\sigma_f) \otimes H^{q-l(w)}(\mathfrak{m}, K_\infty^M, H_{\sigma_\infty} \otimes \mathcal{M}_\lambda(w\cdot\lambda)) \otimes\ \mathrm{Ind}_{P(\mathbb{A}_f)}^{G(A_f)} H_{\sigma_f} \to \\ H^q(\mathcal{S}_{K_f}^G, \tilde{\mathcal{M}}_\lambda) \end{aligned} \tag{9.70}$$

A priori the cuspidal cohomology is defined on the transcendental level, i.e

$$H_{\mathrm{cusp}}^{q-l(w)}(\mathcal{S}_{K_f^M}^M, \mathcal{M}(w\cdot\lambda)\otimes_F\mathbb{C}) \subset H_!^{q-l(w)}(\mathcal{S}_{K_f^M}^M, \mathcal{M}(w\cdot\lambda)) \otimes_F \mathbb{C}.$$

In the following we make the assumption that the cuspidal cohomology $H^{q-l(w)}_{\text{cusp}})(\mathcal{S}^M_{K^M_f},\mathcal{M}(w\cdot\lambda)\otimes\mathbb{C})$ as $\mathcal{H}^M_{K^M_f}-$ module is complete in the inner cohomology. Then we that (See proposition 3.2.3)

$$H^{q-l(w)}_{\text{cusp}}(\mathcal{S}^M_{K^M_f},\mathcal{M}_\lambda(w\cdot\lambda))\otimes\mathbb{C})(\sigma_f)=H^{q-l(w)}_{\text{cusp}}(\mathcal{S}^M_{K^M_f},\mathcal{M}_\lambda(w\cdot\lambda))(\sigma_f)\otimes_F\mathbb{C}).$$

We abbreviate and put

$$\begin{aligned} I^G_PH^q(\sigma_f) &:= \ \text{Ind}^{G(|A)}_{P(\mathbb{A})}H^{q-l(w)}_{\text{cusp}}(\mathcal{S}^M_{K^M_f},\mathcal{M}_\lambda(w\cdot\lambda))(\sigma_f)\\ I^G_QH^q(\sigma'_f) &:= \ \text{Ind}^{G(\mathbb{A}_f)}_{Q(\mathbb{A}_f)}H^{q-l(w')}_{\text{cusp}}(\mathcal{S}^M_{K^M_f},\mathcal{M}(w'\cdot\lambda))(\sigma'_f)\end{aligned}\tag{9.71}$$

then the $\mathcal{H}^G_{K_f}$ Hecke-module

$$I^G_PH^q(\sigma_f)\oplus I^G_QH^q:(\sigma'_f)\subset H_!(\partial\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda)\tag{9.72}$$

is a complete submodule.

If we assume that the same assumptions hold for the dual module $\mathcal{M}_{\lambda^\vee}$ then a simple argument using Poincare duality implies (see (6.3.8)

$$\begin{aligned}&\text{Im}(r\circ\ \text{Eis}_{\mathbb{C}})(\ \text{Ind}^{G(\mathbb{A}_f)}_{P(\mathbb{A}_f)}H^{q-l(w)}_{\text{cusp}})(\mathcal{S}^M_{K^M_f},\mathcal{M}(w\cdot\lambda)\otimes\mathbb{C})(\sigma_f))\\ &=\text{Im}(r(H^q(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda)\otimes\mathbb{C})\cap(I^G_PH^q(\sigma_f)\oplus I^G_QH^q:(\sigma'_f))\end{aligned}\tag{9.73}$$

and this implies that $r(H^q(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda))\cap(I^G_PH^q(\sigma_f)\oplus I^G_QH^q(\sigma'_f))$ is the F-vector space

$$r(H^q(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda))\cap(I^G_PH^q(\sigma_f)\oplus I^G_QH^q(\sigma'_f))=\{\psi_f+T(\sigma_f)(\psi_f)\}\tag{9.74}$$

where $\psi_f\in H^{q-l(w)}_{\text{cusp}})(\mathcal{S}^M_{K^M_f},\mathcal{M}(w\cdot\lambda))(\sigma_f)$ and where $T(\sigma_f)$ is a homomorphism between the two Hecke modules $I^G_PH^q(\sigma_f)$ and $I^G_QH^q(\sigma'_f)$.

We compute the operator $T(\sigma_f)$. This may become a delicate issue. It seems that this operaror may be simply zero if $l(w)>l(w')$, and therefore this operator seems to be totally uninteresting if this is the case. But this is certainly not the case and why this is so will be explained in the following section..

But for the moment we assume $l(w)=l(w')$. Of course we apply the theorem **??** above, then it is essentially a linear algebra problem. We make several assumptions, which can be verified in many cases. We fix σ_f.

a) For σ_∞ we have $\dim W_{\text{cusp}}(\sigma_\infty\times\sigma_f)=0,1$

b) If $\dim W_{\text{cusp}}(\sigma_\infty\times\sigma_f)=1$ then $H^{q-l(w)}(\mathfrak{m},K^M_\infty,H_{\sigma_\infty}\otimes\mathcal{M}_\lambda(w\cdot\lambda))$ is also one dimensional.

This second condition needs to be commented, First of all it means that $q-l(w)$ is the lowest degree in which the cohomology is non zero. Secondly it may happen that $K^M_\infty=P(\mathbb{R})\cap K_\infty$ is not connected. If $K^{M,1}_\infty$ is its connected component of the identity then it may happen that $H^{q-l(w)}(\mathfrak{m}, K_\infty^{M,1}, H_{\sigma_\infty} \otimes \mathcal{M}_\lambda(w\cdot\lambda))$ becomes two dimensional. This is related to the fact that the restriction of H_{σ_∞} to $M^{(1)}(\mathbb{R})$ becomes reducible. (See the discussion of the discrete series in ???). Then the two-group $K_\infty^M/K_\infty^{M,1}$ acts on this space and our cohomology groups are the one dimensional + subspaces.

If we now assume that a) and b) are satisfied for σ_∞ and also for σ'_∞ then we choose inclusions $\Phi_{\sigma_\infty} : H_{\sigma_\infty} \otimes H_{\sigma_f} \hookrightarrow L^2_{\text{cusp}}(M(\mathbb{Q})\backslash M(\mathbb{A})/K_f^M)$ and we also choose generators $\omega_{\sigma_\infty} \in H^{q-l(w)}(\mathfrak{m}, K_\infty^M, H_{\sigma_\infty} \otimes \mathcal{M}_\lambda(w\cdot\lambda))$. We do the same for the $H_{\sigma'_\infty}$.

We briefly return to our isotypical subspace $H^{q-l(w)}_{\text{cusp}}(\mathcal{S}^M_{K_f^M}, \mathcal{M}_\lambda(w\cdot\lambda)))(\sigma_f)$ We remember that originally our coefficient system was obtained from a representation of $\mathcal{G}/\mathcal{O}_{F_0}$, the $\tilde{\mathcal{M}}_\lambda$ are $\mathcal{O}_{F_0}$ modules. Then we know that there is a finite set S of primes such that

$$H^{q-l(w)}_{\text{cusp}}(\mathcal{S}^M_{K_f^M}, \mathcal{M}_\lambda(w\cdot\lambda)) \otimes \mathcal{O}_{F,S})(\sigma_f) \xrightarrow{\sim} \bigoplus_1^{d(\sigma_f)} H_{\sigma_f,\mathcal{O}_{F,S}} \tag{9.75}$$

Here $H_{\sigma_f,\mathcal{O}_{F,S}}$ is e free $\mathcal{O}_{F,S}$ modules with an action of the Hecke algebra $\mathcal{H}^M_{K_f^M}$. Of course $H_{\sigma_\infty} = H_{\sigma_f,\mathcal{O}_{F,S}} \otimes F$, if S is large enough we have

$$H_{\sigma_f,\mathcal{O}_{F,S}} = \mathcal{H}^M_{K_f} e_0 \tag{9.76}$$

Therefore we can rewrite (9.75)

$$H^{q-l(w)}_{\text{cusp}}(\mathcal{S}^M_{K_f^M}, \mathcal{M}_\lambda(w\cdot\lambda)) \otimes \mathcal{O}_{F,S})(\sigma_f) \xrightarrow{\sim} \bigoplus_1^{d(\sigma_f)} \mathcal{H}^M_{K_f} e_\nu \tag{9.77}$$

It follows from our assumptions that $d(\sigma_f)$ is equal to the number of σ_∞ with $W_{\text{cusp}}(\sigma_\infty \times \sigma_f) = 1$ and

$$H^{q-l(w)}_{\text{cusp}}(\mathcal{S}^M_{K_f^M}, \mathcal{M}_\lambda(w\cdot\lambda)) \otimes \mathbb{C})(\sigma_f) = \bigoplus_{\sigma_\infty} \mathcal{H}^M_{K_f} \Phi^q_{\sigma_\infty}(\omega_{\sigma_\infty} \otimes e_0) \tag{9.78}$$

Hence we can write

$$\begin{aligned} e_\nu &= \textstyle\sum_{\sigma_\infty} h_{\nu,\sigma_\infty} \mathbf{\Omega}_{\nu,\sigma_\infty} \Phi^q_{\sigma_\infty}(\omega_{\sigma_\infty} \otimes e_0) \\ \Phi^q_{\sigma_\infty}(\omega_{\sigma_\infty} \otimes e_0) &= \textstyle\sum_\mu g_{\sigma_\infty,\mu} \mathbf{A}_{\sigma_\infty,\mu} e_\mu \end{aligned} \tag{9.79}$$

where $h_{\nu,\sigma_\infty}, g_{\sigma_\infty,\mu} \in \mathcal{H}^M_{K_f}$ and $\mathbf{\Omega}_{\nu,\sigma_\infty}, \mathbf{A}_{\sigma_\infty,\mu} \in \mathbb{C}$. (Here the $\mathbf{A}$ is thecapital greek letter Alpha)

We can to the same for $H^{q-l(w')}_{\text{cusp}}(\mathcal{S}^M_{K_f^M}, \mathcal{M}(w'\cdot\lambda))(\sigma'_f)$ and we get the same relations where we have to put a $'$ at the right spots.

Hence we can write an element $\underline{\psi}_f \in H^{q-l(w)}_{\text{cusp}}(\mathcal{S}^M_{K_f^M}, \mathcal{M}_\lambda(w\cdot\lambda)) \otimes \mathbb{C})(\sigma_f)$ in two ways as an array

$$\underline{\psi}_f = \{\dots, \psi_f^{(\nu)}, \dots\}_{\nu=1,\dots,d(\sigma_f)}\ ,\ \text{or}\ \underline{\psi}_f = \{\dots, \psi_f^{(\sigma_\infty)}, \dots\}_{\sigma_\infty}$$

where $\psi^{(\nu)} \in \operatorname{Ind}_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)} \mathcal{H}_{K_f}^M e_\nu$; $\psi^{(\sigma_\infty)} \in \operatorname{Ind}_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)} \mathcal{H}_{K_f}^M (\omega_{\sigma_\infty} \otimes e_0)$.

We are ready to compute $T(\sigma_f)$. We have chosen $\omega_{\sigma_\infty}, \omega_{\sigma'_\infty}$ and we define numbers $c_\infty(\sigma_\infty, \sigma'_\infty)$ by

$$T_\infty^{\mathrm{loc,q}}(\omega_{\sigma_\infty}) = c_\infty(\sigma_\infty, \sigma'_\infty)\omega_{\sigma'_\infty}. \tag{9.80}$$

Then theorem **??** yields

$$T(\sigma_f)(\Phi^q_{\sigma_\infty}(\omega_{\sigma_\infty} \otimes \psi_f^{(\nu)})) = \prod_a \frac{\Lambda(\sigma, r_a^{\mathfrak{u}_P^\vee} 0)}{\Lambda(\sigma, r_a^{\mathfrak{u}_P^\vee}, 1)} c_\infty(\sigma_\infty, \sigma'_\infty)\Phi^q_{\sigma'_\infty}((\omega_{\sigma'_\infty} \otimes T_f^{\mathrm{loc}}(\psi_f^{(\nu)}))) \tag{9.81}$$

Then (9.79) yields

$$T(\sigma_f)(\underline{\psi}_f) = T(\sigma_f)\{\dots, \psi_f^{(\nu)}, \dots\}_{\nu=1,\dots,d(\sigma_f)} =$$

$$\prod_a \frac{\Lambda(\sigma, r_a^{\mathfrak{u}_P^\vee}, 0)}{\Lambda(\sigma, r_a^{\mathfrak{u}_P^\vee}, 1)} \{\dots, \underset{\substack{\uparrow \\ \nu'\text{- th spot}}}{\sum_{\sigma_\infty} h_{\nu,\sigma_\infty} g_{\sigma'_\infty,\nu'} c_\infty(\sigma_\infty, \sigma'_\infty) \mathbf{\Omega}_{\nu,\sigma_\infty} \mathbf{A}'_{\sigma'_\infty,\nu'}}, \dots\} \tag{9.82}$$

We define the relative period-matrix

$$\mathcal{P}(\sigma_f) = \big(\sum_{\sigma_\infty} h_{\nu,\sigma_\infty} g_{\sigma'_\infty,\nu'} c_\infty(\sigma_\infty, \sigma'_\infty) \mathbf{\Omega}_{\nu,\sigma_\infty} \mathbf{A}'_{\sigma'_\infty,\nu'}\big)_{\nu,\nu'}, \tag{9.83}$$

this is a $(d(\sigma_f), d(\sigma_f))$- matrix with coefficients in $\mathcal{H}_{K_f}^M \otimes \mathbb{C}$. It depends on the choice of the two basis's $\dots e_\nu, \dots$ and e'_μ. hence it is unique up to multiplication from the left and the right by elements in $\mathrm{Gl}_{d(\sigma_f)}(\mathcal{H}_{K_f}^M \otimes \mathcal{O}_{F,S})$.

Our first arithmetic application of Eisenstein cohomology is the following rationality result

Theorem 9.3.2. *The matrix*

$$\mathcal{P}(\sigma_f) \prod_a \frac{\Lambda(\sigma, r_a^{\mathfrak{u}_P^\vee}, 0)}{\Lambda(\sigma, r_a^{\mathfrak{u}_P^\vee}, 1)} \mathcal{P}(\sigma_f) \in \mathcal{H}_{K_f}^M \otimes F$$

We write the values of the complete L functions above as values of the complete cohomological L function. We have written

$$w(\lambda + \rho) = \mu^{(1)} + \rho_M + \delta_1 + \delta = \tilde{\mu}^{(1)} + b(w, \lambda)\gamma_J + \delta_1^* + \delta$$

where $\mu^{(1)}$ is a highest weight for the semi-simple group $M^{(1)}/\mathbb{Q}$ and $b(w, \lambda)\gamma_J + \delta_1 + \delta$ is the abelian part. We denote by χ_a the highest weight of the representation , $r_a^{\mathfrak{u}_P^\vee}$ then we get from the definition

$$\Lambda(\sigma, r_a^{\mathfrak{u}_P^\vee}, s) = \Lambda^{\mathrm{coh}}(\sigma, r_a^{\mathfrak{u}_P^\vee}, s + < \chi_a, \mu^{(1)} + \rho_M - \delta > - b(w, \lambda) < \chi_a, \gamma_J >)$$

here we take into account that $<\chi_a,\delta_1^*>=0$. The ratio of $L-$ values in the theorem above becomes

$$\frac{\Lambda(\sigma,r_a^{\mathfrak{u}_P^\vee},<\chi_a,\rho_M>))}{\Lambda(\sigma,r_a^{\mathfrak{u}_P^\vee},<\chi_a,\rho_M>)+1)}=\frac{\Lambda^{\text{coh}}(\sigma,r_a^{\mathfrak{u}_P^\vee},<\chi_a,\mu^{(1)}-\delta>-b(w,\lambda)<\chi_a,\gamma_J>)}{\Lambda^{\text{coh}}(\sigma,r_a^{\mathfrak{u}_P^\vee},<\chi_a,\mu^{(1)}-\delta>-b(w,\lambda)<\chi_a,\gamma_J>+1)}$$

There is a cancellation of the term $<\chi_a,\rho_M>)$.

Now we remember that the cohomological L function is invariant under twisting (see (7.26) by a Tate character, hence we have

$$\Lambda^{\text{coh}}(\sigma,r_a^{\mathfrak{u}_P^\vee},s)=\Lambda^{\text{coh}}(\sigma\otimes|\gamma_J^M|,r_a^{\mathfrak{u}_P^\vee},s).$$

We see that $H_{\sigma_\infty}\otimes H_{\sigma_f}|\gamma_J^M|$ occurs in $L^2_{\text{cusp}}(M(\mathbb{Q})\backslash M(\mathbb{A})/K_f^M)$ more precisely we can identify

$$W(\sigma,L^2_{\text{cusp}}(M(\mathbb{Q})\backslash M(\mathbb{A})/K_f^M))=W(\sigma|\gamma_J|,L^2_{\text{cusp}}(M(\mathbb{Q})\backslash M(\mathbb{A})/K_f^M)).$$

Hence we get a family of non trivial Hecke modules

$$H^{q-l(w)}(\mathcal{S}^M_{K_f^M},\mathcal{M}_{\mu^{(1)}+\delta_1^*+b\gamma_J^M+\delta})(\sigma_f|\gamma_J|_f^b).$$

and on the other side we get the modules

$$H^{q-l(w')}(\mathcal{S}^M_{K_f^M},\mathcal{M}_{\mu^{(1)}+\delta_1^{*,\prime}-b\gamma_J^M+\delta})(\sigma'_f|\gamma_J|_f^{-b}).$$

By the same procedure as above we can define a relative period $\mathcal{P}(\sigma_f|\gamma_j|_f^b)$, We have to understand how this period varies if b varies. Therefore we consider a covering of of $\mathcal{S}^M_{K_f^M}$ by a narrow space, To be more precise we consider the isogeny $M^{(1)}\times C_1\times C/\mathbb{Q}\to M/\mathbb{Q}$ and we explained in (???) that we get partial coverings

$$\mathcal{S}^{M^{(1)}}_{K_f^{M^{(1)}}}\times\mathcal{S}^{\mathbb{G}_m}_{K_f^{\mathbb{G}_m}}\times\mathcal{S}^{C_1^*}_{K_f^{C_1^*}}\times\mathcal{S}^C_{K_f^C}\to\mathcal{S}^M_{K_f^M}\tag{9.84}$$

where on the left hand side the group $K_\infty^{(1)}\subset M^{(1)}(\mathbb{R})$ is the connected component of the identity of a maximal compact subgroup and on the torus component we may take the connected component of the group of real points of the maximal split subtorus.

We get injective homomorphism between Hecke modules

$$\begin{gathered}H^{q-l(w)}(\mathcal{S}^M_{K_f^M},\mathcal{M}_{\mu^{(1)}+\delta_1^*+b\gamma_J^M+\delta})(\sigma_f|\gamma_J|_f^b)\to\\ \to H^{q-l(w)}(\mathcal{S}^{M^{(1)}}_{K_f^{M^{(1)}}},\mathcal{M}_{\mu^{(1)}})\otimes H^0(\mathcal{S}^{\mathbb{G}_m}_{K_f^{\mathbb{G}_m}},F(b\gamma_J))\otimes...(\sigma_f^{(1)}\otimes|\gamma_J|_f^b\otimes\dots)\end{gathered}\tag{9.85}$$

We can define the period matrices $(\mathbf{\Omega}^{(1)}),(\mathbf{A}^{(1)})$ also for the cohomology of the narrow covering, but these period matrices only depend on the first factor in the Künneth decomposition above. Hence they do not depend on b. Our original period matrices in (9.79) are certain submatrices of these, which submatrix depends on the parity of b. Then a careful inspections shows that

$$\mathcal{P}(\sigma_f|\gamma_J|_f|) = \mathcal{P}(\sigma_f)^{-1} \tag{9.86}$$

If we start from our original σ_f for any value $b \in \mathbb{Z}$ we can consider the expression

$$\prod_a \frac{\Lambda^{\mathrm{coh}}(\sigma, r_a^{\mathfrak{u}_P^\vee}, <\chi_a, \mu^{(1)} - \delta> -b <\chi_a, \gamma_J>)}{\Lambda^{\mathrm{coh}}(\sigma, r_a^{\mathfrak{u}_P^\vee}, <\chi_a, \mu^{(1)} - \delta> -b <\chi_a, \gamma_J> +1)} \mathcal{P}(\sigma_f)^{\pm 1} \tag{9.87}$$

where $\pm 1 = (-1)^{b-b(w,\lambda)}$. Can we prove the assertion of Theorem 9.3.2 for this given value of b?

This is certainly the case if there is a dominant weight λ_1 and a Kostant representative $w_1 \in W^P$ with $l(w_1) = l(w_1')$ such that

$$w_1(\lambda_1 + \rho) = \tilde{\mu}^{(1)} + \delta_1^* + \delta + b\gamma_J$$

We discussed this issue briefly in section 9.1.4 and saw that for a given w_1 the number b has to lie in an interval, which may be empty. But we may of course have several options to choose w_1.

The final answer should be: There is a finite interval o f integers $[a(\mu^{(1)}, b(\mu^{(1)})]$ which are the so called critical arguments for $\tilde{\mu}^{(1)} + \delta_1^*$ (Will be explained in earlier section on the cohomological L-function.) Then we can solve the equation

$$w_1(\lambda_1 + \rho) = \tilde{\mu}^{(1)} + \delta_1^* + \delta + b\gamma_J; w_1 \in W^P,\ l(w_1) = l(w_1'),\ \lambda_1 \text{ dominant} \tag{9.88}$$

if and only if

$$\begin{aligned} \alpha[\mu^{(1)}] \leq\ &<\chi_a, \mu^{(1)} - \delta> -b <\chi_a, \gamma_J> \\ &<\chi_a, \mu^{(1)} - \delta> -b <\chi_a, \gamma_J> +1 \leq b[\mu^{(1)}] \end{aligned} \tag{9.89}$$

This is the Combinatorial Lemma.

We summarise : The C-L-S-method proves - provided certain assumptions are verified - that

$$\prod_a \frac{\Lambda^{\mathrm{coh}}(\sigma, r_a^{\mathfrak{u}_P^\vee}, <\chi_a, \mu^{(1)} - \delta> -b <\chi_a, \gamma_J>)}{\Lambda^{\mathrm{coh}}(\sigma, r_a^{\mathfrak{u}_P^\vee}, <\chi_a, \mu^{(1)} - \delta> -b <\chi_a, \gamma_J> +1)} \mathcal{P}(\sigma_f)^{\pm 1} \in \mathcal{H}_{K_f}^M \otimes F \tag{9.90}$$

provided the arguments $<\chi_a, \mu^{(1)} + \rho_M - \delta> -b <\chi_a, \gamma_J>$, and $<\chi_a, \mu^{(1)} + \rho_M - \delta> -b <\chi_a, \gamma_J> +1$ are critical.

9.3.1 Denominators of Eisenstein classes and Congruences

We drop the assumption $l(w) = l(w')$ and replace it by $l(w) \geq l(w')$ and we also assume that $w \cdot \lambda$ is in the positive chamber. Furthermore we assume that $\mathrm{Ind}_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)} H_{\mathrm{cusp}}^{q-l(w)}(\mathcal{S}_{K_f^M}^M, \mathcal{M}(w \cdot \lambda) \otimes \mathbb{C})(\sigma_f)$ is complete in $H^q(\mathcal{S}_{K_f}^G, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{C})$, i.e. the Manin-Drinfeld principle is valid. This is certainly the case if λ is (sufficiently) regular,

Then we can conclude that $\mathrm{Eis}_{\mathbb{C}} = \mathrm{Eis}_F \otimes_F \mathbb{C}$ where

$$\mathrm{Eis}_F: \ \mathrm{Ind}_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)} H^{q-l(w)}_{\mathrm{cusp}}(\mathcal{S}^M_{K_f^M}, \mathcal{M}(w\cdot\lambda)\otimes F)(\sigma_f) \to H^q(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes F) \tag{9.91}$$

Now we know that for regular representations $\mathcal{M}_\lambda$ the cohomology $H^\nu(\mathfrak{m}, K^M_\infty, H_{\sigma_\infty} \otimes \mathcal{M}(w\cdot\lambda))$ is non zero only for ν in a very narrow interval around the middle degree (see [108], Thm. 5.5). If the difference $l(w) - l(w')$ is greater than the length of this interval, then the following condition is fulfilled

In any degree $T^{\mathrm{loc},\bullet}_\infty(w)$ induces zero on the cohomology. (Tzero)

If we assume (Tzero) and Manin-Drinfeld then we get $r \circ \mathrm{Eis}_F(\psi_f) = \psi_f$, i.e. the restriction of the Eisenstein class to the boundary gives us back the original class. It does snot spread out (see section 7.1.5.).

Then it seems that the second term in the constant term has no influence on the structure of the cohomology. It is a central message of this book that this is not the case. Under certain circumstances the second term is of fundamental arithmetic interest, it contains relevant arithmetic information.

We pass to the integral cohomology, then we may restrict Eis_F to the integral cohomology $\mathrm{Ind}_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)} H^{q-l(w)}_{\mathrm{cusp,\ int}}(\mathcal{S}^M_{K_f^M}, \mathcal{M}(w\cdot\lambda))_{\mathcal{O}_{F,S}}(\sigma_f)$, here S is a controllable finite set of primes, which have to be inverted. Again we come across with the question: Determine the denominator ideal

$$\Delta(\sigma_f) = \{a | a \in \mathcal{O}_{F,S} |$$
$$a\ \mathrm{Eis}_F(\ \mathrm{Ind}_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)} H^{q-l(w)}_{\mathrm{cusp}}(\mathcal{S}^M_{K_f^M}, \mathcal{M}(w\cdot\lambda))_{\mathcal{O}_{F,S}}(\sigma_f)) \subset H^q(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathcal{O}_{F,S})_{\mathrm{int}}. \tag{9.92}$$

We have some heuristic (or speculative) arguments which suggest a relation between this denominator and the arithmetic of the second constant term. This conjectural relationship is also supported by an impressive amount of experimental data.

To get a little bit closer to the formulation of the conjecture we look at the constant term, but not on the level of cohomology , we look at it on the level of complexes. We assume that $l(w) = l(w') + 1$. We decompose the Lie-algebra $\mathfrak{m} = \mathfrak{m}^{(1)} \oplus \mathbb{Q}\, d\gamma_P \oplus \mathfrak{c}_1^* \oplus \mathfrak{c}$ and then the Delorme complex and the local intertwining operator becomes

$$\begin{array}{c} \to \quad \mathrm{Hom}_{K^M_\infty}(\Lambda^\bullet(\mathfrak{m}^{(1)}/\mathfrak{k}^M), H_{\sigma^{(1)}_\infty} \otimes \mathcal{M}(w\cdot\lambda)), \otimes\Lambda^\bullet(\mathbb{Q}\, d\gamma_p) \otimes \Lambda^\bullet(\mathfrak{c}_1^*) \otimes \Lambda^\bullet(\mathfrak{c}) \quad \to \\ \downarrow \\ \to \quad \mathrm{Hom}_{K^M_\infty}(\Lambda^\bullet(\mathfrak{m}^{(1)}/\mathfrak{k}^M), H_{\sigma'_\infty{}^{(1)}} \otimes \mathcal{M}(w'\cdot\lambda)), \otimes\Lambda^\bullet(\mathbb{Q}\, d\gamma_Q) \otimes \Lambda^\bullet(\mathfrak{c}_1^*) \otimes \Lambda^\bullet(\mathfrak{c}) \end{array} \tag{9.93}$$

Let q_M be the lowest degree such that $\mathrm{Hom}_{K_\infty^M}(\Lambda^{q_M}(\mathfrak{m}^{(1)}/\mathfrak{k}^M), H_{\sigma_\infty^{(1)}} \otimes \mathcal{M}(w\cdot\lambda)) \neq 0$, put $q = q_M + l(w)$. Then we have to compute the restriction of $T_\infty^{\mathrm{loc}}(w)$ to the essential piece

$$\begin{array}{c}\mathrm{Hom}_{K_\infty^M}(\Lambda^{q-l(w)}(\mathfrak{m}^{(1)}/\mathfrak{k}^M), H_{\sigma_\infty^{(1)}} \otimes \mathcal{M}(w\cdot\lambda)) \otimes \Lambda^0(\mathbb{Q}\, d\gamma_p) \otimes \Lambda^0(\mathfrak{c}_1^*) \otimes \Lambda^0(\mathfrak{c}) \\ \downarrow T_\infty^{\mathrm{loc}}(w) \\ \mathrm{Hom}_{K_\infty^M}(\Lambda^{\bullet}(\mathfrak{m}^{(1)}/\mathfrak{k}^M), H_{\sigma'_\infty{}^{(1)}} \otimes \mathcal{M}(w'\cdot\lambda)) \otimes \Lambda^1(\mathbb{Q}\, d\gamma_p) \otimes \Lambda^0(\mathfrak{c}_0^*) \otimes \Lambda^0(\mathfrak{c})\end{array} \tag{9.94}$$

We perform the same computation as above, but now we define the number $c_\infty(\sigma_\infty, \sigma'_\infty)$ by

$$T_\infty^{\mathrm{loc},q}(\omega_{\sigma_\infty}) = c_\infty(\sigma_\infty, \sigma'_\infty)\omega_{\sigma'_\infty} \wedge d\gamma_Q, \tag{9.95}$$

We define the relative period matrix $\mathcal{P}(\sigma_f)$ by the same formula, with respect to the bases $\dots e_\nu \dots$ and $\dots e'_{\nu'}$ this gives us a linear map

$$H^{q-l(w)}(\mathcal{S}^M_{K_f^M}, \mathcal{M}(w\cdot\lambda)\otimes F)(\sigma_f) \otimes \mathbb{C} \xrightarrow{\mathcal{P}(\sigma_f)} H^{q-1-l(w')}(\mathcal{S}^M_{K_f^M}, \mathcal{M}(w'\cdot\lambda)\otimes F)(\sigma'_f) \wedge \mathbb{C}d\gamma_Q \tag{9.96}$$

This linear maps induces a Hecke module linear map

$$\begin{array}{c}\tilde{T}^{\mathrm{loc}}(\sigma_f)\ \mathrm{Ind}_{P(\mathbb{A}_f)}^{G(\mathbb{A}_f)} : H^{q-l(w)}(\mathcal{S}^M_{K_f^M}, \mathcal{M}(w\cdot\lambda)\otimes F)(\sigma_f) \to \\ \mathrm{Ind}_{Q(\mathbb{A}_f)}^{G(\mathbb{A}_f)} H^{q-1-l(w')}(\mathcal{S}^M_{K_f^M}, \mathcal{M}(w'\cdot\lambda)\otimes F)(\sigma'_f) \wedge \mathbb{C}d\gamma_Q\end{array} \tag{9.97}$$

Eventually we get for the constant Fourier coefficients of our Eisenstein series

$$\mathcal{F} \circ\ \mathrm{Eis}_{\mathbb{C}}(\psi_f) = \psi_f +$$

$$\prod_a \frac{\Lambda^{\mathrm{coh}}(\sigma, r_a^{\mathfrak{u}_P^\vee}, <\chi_a, \mu^{(1)} - \delta> - b <\chi_a, \gamma_J>)}{\Lambda^{\mathrm{coh}}(\sigma, r_a^{\mathfrak{u}_P^\vee}, <\chi_a, \mu^{(1)} - \delta> - b <\chi_a, \gamma_J> + 1)} \tilde{T}^{\mathrm{loc}}(\sigma_f)(\psi_f) \wedge \mathbb{C}d\gamma_Q \tag{9.98}$$

Here ψ_f is a "harmonic differential" form which represent the cohomology class also called ψ_f. Then $\tilde{T}^{\mathrm{loc}}(\sigma_f)(\psi_f) \wedge \mathbb{C}d\gamma_Q$ is also a harmonic form, but the cohomology class is zero. Hence we do not get any consequences (for instance rationality) for the factor in front.

But there are instances where this second term contains interesting information. We abbreviate and write

$$C(\sigma, b) := \prod_a \frac{\Lambda^{\mathrm{coh}}(\sigma, r_a^{\mathfrak{u}_P^\vee}, <\chi_a, \mu^{(1)} - \delta> - b <\chi_a, \gamma_J>)}{\Lambda^{\mathrm{coh}}(\sigma, r_a^{\mathfrak{u}_P^\vee}, <\chi_a, \mu^{(1)} - \delta> - b <\chi_a, \gamma_J> + 1)} \tag{9.99}$$

Now we write $C(\sigma, b)$ as a product

$$C(\sigma, b) = C^{\mathrm{crit}}(\sigma, b) C^{\mathrm{mot}}(\sigma, b) \tag{9.100}$$

where the first factor collects the critical values and the second factor is gives us something like a motivic extension class. Then we find instances where we can show

$$C^{\mathrm{crit}}(\sigma, b)\mathcal{P}(\sigma_f) \in M_{d(\sigma_f),d(\sigma_f)}(F) \tag{9.101}$$

Since the relative period matrix is well defined modulo action by $\mathrm{Gl}_{d(\sigma_f)}(\mathcal{O}_{F,S})$ from the left and from the right, we can speak of the denominators of this matrix, i.e.

$$\Delta_L(\sigma_f) = \{a | a \in \mathcal{O}_{F,S} | aC^{\mathrm{crit}}(\sigma, b) \in M_{d(\sigma_f),d(\sigma_f)}(\mathcal{O}_{F,S}). \tag{9.102}$$

Now we have some speculative arguments suggesting, that under certain conditions we should have

$$\Delta(\sigma_f) \subset \Delta_L(\sigma_f) \tag{9.103}$$

This is of course a very vague statement, one might even argue that it is empty, if we do not specify the set of exceptional primes S. But in principle it gives us a slightly more precise idea how this relationship between the "arithmetic" of the second constant term and the denominator of the Eisenstein class looks like.

We discuss and verify these conjectures in the Chapters 3-5 in the very special case $G/\mathbb{Z} = \mathrm{Gl}_2/\mathbb{Z}$, the parabolic subgroup $P/\mathbb{Z}$ is of course the standard Borel and we consider unramified cohomology, i.e. $K_f = \mathrm{Gl}_2(\hat{Z})$. In this special situation the group $M/\mathbb{Z} = T/\mathbb{Z}$ thestandard maximal torus. Our coefficient system is the $\mathcal{M}_\lambda$ from section 4.1.1, we assume that n is even and $d = 0$. Then σ is simply the $n+2$-th power of the Tate character, the only possible value of b is $b = n$ and then-after using the functional equation- we get

$$C(\sigma, n) = c(n)\frac{\zeta'(-n)}{\zeta(-1-n)}$$

where $c(n)$ is a rational number with prime factors $\leq n$. (See MixMot). The numerator is the motivic factor, the denominator is the critical factor, we know that $\zeta(-1-n) \in \mathbb{Q}$. In this cases we prove that the exact denominator is the numerator of $\zeta(-1-n)$. (Theorem 5.1.2).)

Unfortunately the proof given there is a "wrong" proof, because it uses fortunate special circumstances (Whittaker model, modular symbols).

In Chapter 3 we describe a computer program (written with the help of H. Gangl) which verifies the conjecture -in the above special case - experimentally (and in principle) in any specific case.

In the last section of this book we discuss several instances of this conjecture, we also discuss the further experimental evidence in these cases.

9.4 The special case Gl_n

In the previous section we considered only the bottom step in the boundary cohomology, and in this bottom step we only considered cuspidal contributions. For the special case $G = \mathrm{Gl}_n/\mathbb{Z}$ we extend the construction of Eisenstein classes which we obtain from parabolic of lower rank. This means that we make an attempt to understand the questions raised above, but we admit that we are very far from a satisfactory answer. We more or less make a start and end up in a combinatorial coppice where we may get completely lost. On the other hand we expect consequences on special values of L-functions like the ones in [48] but it is not clear how much get that goes beyond the results in [48]. In any case I am very optimistic that these considerations will give the corresponding result to the main theorem in [48]in the case nn' odd. In this case there is a non zero Hodge middle Hodge type and this implies that the critical arguments have to satisfy a parity condition. (See [**?**]).

9.4.1 The tempered representations with cohomology at infinity

We consider the group $\mathrm{Gl}_n/\mathbb{R}$, we choose a essentially selfdual highest weight $\lambda = \sum_1^{n-1} a_i\gamma_i + d\delta($ i.e. $a_i = a_{n-i})$. The a_i are integers ≥ 0 and d is a half integer which satisfies the parity condition

$$d \in \mathbb{Z} \text{ if } n \text{ is odd }, a_{\frac{n}{2}} \equiv 2d \mod \mathbb{Z} \text{ if } n \text{ is even}$$

We want to recall the construction of a specific $(\mathfrak{g}, K_\infty)$ -module $\mathbb{D}_\lambda$ with

$$H^\bullet(\mathfrak{g}, K_\infty, \mathbb{D}_\lambda \otimes \mathcal{M}_\lambda) \neq 0$$

and we will also determine the structure of this cohomology. This module is the only tempered Harish-Chandra module which has non trivial cohomology with coefficients in $\mathcal{M}_\lambda$. The center $\mathbb{G}_m$ of Gl_n acts on the module $\mathcal{M}_\lambda$ by the character $\omega_\lambda : x \mapsto x^{nd}$. Since we want no zero cohomology the center $S(\mathbb{R})$ of $\mathrm{Gl}_n(\mathbb{R})$ acts by the central character $(\omega_\lambda)_\mathbb{R}^{-1}$ on $\mathbb{D}_\lambda$. The module $\mathbb{D}_\lambda$ will be essentially unitary with respect to that character.

We construct our representation $\mathbb{D}_\lambda$ by inducing from discrete series representations. We consider the parabolic subgroup ${}^\circ P$ whose simple root system is described by the diagram

$$\circ - \times - \circ - \times - \cdots - \circ(-\times) \tag{9.104}$$

i.e. the set of simple roots $I_{{}^\circ M}$ of the semi simple part of the Levi quotient ${}^\circ M$ is consists of those which have an odd index. Let m be the largest odd integer less or equal to $n-1$ then α_m is the last root in the system of simple roots in $I_{{}^\circ M}$. Of course $m = n-1$ if n is even and $m = n-2$ else.

The reductive quotient is equal to $\mathrm{Gl}_2 \times \mathrm{Gl}_2 \times \ldots \mathrm{Gl}_2(\times \mathbb{G}_m)$, where the last factor occurs if n is odd. This product decomposition of ${}^\circ M$ induces a product decomposition of the standard maximal torus $T = \prod_{i:i\mathrm{odd}} T_i(\times \mathbb{G}_m)$ and for the character module we get

$$X^*(T) = \bigoplus_{i:i\mathrm{odd}} X^*(T_i)(\oplus X^*(\mathbb{G}_m)) \tag{9.105}$$

The semi simple reductive quotient ${}^\circ M^{(1)}(\mathbb{R})$ is $A_1 \times A_1 \times \cdots \times A_1$, the number of factors is

$$^\circ r = (m+1)/2 = \begin{cases} \frac{n}{2} & \text{if n is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

We also introduce the number

$$\epsilon(n) = \begin{cases} 0 & \text{if } n \text{ even} \\ 1 & \text{if } n \text{ odd} \end{cases} \tag{9.106}$$

We have a very specific Kostant representative $w_{\text{un}} \in W^{{}^\circ P}$. The inverse of this permutation it is given by

$$w_{\text{un}}^{-1} = \{1 \mapsto 1, 2 \mapsto n, 3 \mapsto 2, 4 \mapsto n-1 \ldots\}.$$

The length of this element is equal to $1/2$ the number of roots in the unipotent radical of ${}^\circ P$, i.e.

$$l(w_{\text{un}}) = \begin{cases} \frac{1}{4}n(n-2) & \text{if } n \text{ is even} \\ \\ \frac{1}{4}(n-1)^2 & \text{if } n \text{ is odd} \end{cases} \tag{9.107}$$

We compute

$$w_{\text{un}}(\lambda+\rho) - \rho = \sum_{i:i \text{ odd}} b_i \gamma_i^{{}^\circ M^{(1)}} + d\delta = \sum_{i:i \text{ odd}} b_i \frac{\alpha_i}{2} + d\delta = \tilde{\mu}^{(1)} + d\delta. \tag{9.108}$$

(The subscript $_{\text{un}}$ refers to unitary, it refers also to the length $l(w_{\text{in}}$ being half the dimension of the unipotent radical. Here we have to observe that $w \cdot \lambda$ is an element in $X^*(T)$ but the individual summands may only lie in $X^*(T) \otimes \mathbb{Q} = X^*_{\mathbb{Q}}(T)$. Any element $\gamma \in X^*(T)$ also defines a quasicharacter $\gamma_{\mathbb{R}} : T(\mathbb{R}) \to \mathbb{R}^\times$ (by definition). But an element $\gamma \in X^*_{\mathbb{Q}}(T)$ only defines a quasicharacter $|\gamma|_{\mathbb{R}} : T(\mathbb{R}) \to \mathbb{R}^\times_{>0}$ which is defined by $|\gamma|_{\mathbb{R}}(x) = |m\gamma(x)|^{1/m}$.)

To compute the coefficients b_j we use the pairing (See7.1) and observe that $< \chi_i, \gamma_j >= \delta_{i,j}$. Then

$$b_j =< \chi_j, w_{\text{un}}(\lambda+\rho) - \rho >=< w_{\text{un}}^{-1}\chi_j, \lambda+\rho > - < \chi_j, \rho > . \tag{9.109}$$

Now the choice of w_{un} becomes clear. It is designed in such a way that

$$w_{\text{un}}^{-1}\chi_1(t) = \begin{pmatrix} t & 0 & 0 & & \ldots & 0 \\ 0 & \ddots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & & \ddots & 0 \\ 0 & 0 & 0 & \ldots & 0 & t^{-1} \end{pmatrix}, w_{\text{un}}^{-1}\chi_3(t) = \begin{pmatrix} 1 & 0 & 0 & & \ldots & 0 \\ 0 & t & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & & t^{-1} & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}$$

and for the general odd index j we have $w_{\mathrm{un}}^{-1}\chi_j(t) = h_{(j+1)/2}$ where for all $1 \le \nu \le n/2$ we denote by $h_\nu(t)$ the diagonal matrix which has a 1 at all entries different from $\nu, n+1-\nu$ and which has entry t at ν and t^{-1} at $n+1-\nu$. Then $h_\nu = \{t \mapsto h_\nu(t)\}$ is a cocharacter. It is clear that

$$\gamma_i(h_\nu(t)) = \begin{cases} t & \text{if } \nu \le i \le n-\nu \\ 1 & \text{else} \end{cases}$$

This yields for $j = 1, \dots,^\circ r$

$$b_{2j-1} = \sum_\nu (a_\nu + 1) < h_j, \gamma_\nu > - < \chi_j, \rho >= (\sum_{j \le \nu \le n-j} (a_\nu + 1)) - 1.$$

We should keep in mind that we assume $a_\nu = a_{n-\nu}$. Then we can rewrite the expressions for the b_ν :

$$b_{2j-1} = \begin{cases} 2a_j + 2a_{j+1} + \cdots + 2a_{\frac{n}{2}-1} + a_{\frac{n}{2}} + n - 2j & \text{if } n \text{ is even} \\ 2a_j + 2a_{j+1} + \cdots + 2a_{\frac{n-1}{2}} + n - 2j & \text{if } n \text{ is odd} \end{cases} \tag{9.110}$$

The b_{2j+1} will be called the *cuspidal parameters* of λ and we summarise

The b_{2j-1} have the same parity, this parity is odd if n is odd. If n is even then b_{2j-1} has parity of $a_{\frac{n}{2}}$. We have $b_1 > b_3 > \cdots > b_m > 0$. They only depend on the semi simple part $\lambda^{(1)}$.

By Kostants theorem

$$w_{\mathrm{un}} \cdot \lambda = w_{\mathrm{un}}(\lambda + \rho) - \rho$$

is the highest weight of an irreducible representation of $^\circ M$. This irreducible representation occurs with multiplicity one in $H^{l(w_{\mathrm{un}})}(\mathfrak{u}_{^\circ P}, \mathcal{M}_\lambda)$.

The highest weight of this representation is

$$w_{\mathrm{un}} \cdot \lambda = w_{\mathrm{un}}(\lambda + \rho) - \rho = \sum_{i:i \text{ odd}} b_i \gamma_i^{^\circ M^{(1)}} + d\delta - (2\gamma_2 + 2\gamma_4 + \cdots + 2\gamma_{m-1} + \frac{3}{2}\gamma_{m+1}) \tag{9.111}$$

Digression: *Discrete series representations of* $\mathrm{Gl}_2(\mathbb{R})$*, some conventions*

We consider the group $\mathrm{Gl}_2/\operatorname{Spec}(\mathbb{Z})$, the standard torus T and the standard Borel subgroup B. We have $X^*(T) = \{\gamma = a\gamma_1 + d\delta | a \in \mathbb{Z}, d \in \frac{1}{2}\mathbb{Z}; a + 2d \equiv 0 \mod 2\}$ where

$$\gamma(\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}) = t_1^{\frac{a}{2}+d} t_2^{-\frac{a}{2}+d} = (\frac{t_1}{t_2})^{\frac{a}{2}} (t_1 t_2)^d$$

(Note that the exponents in the expression in the middle term are integers)

A dominant weight $\lambda = a\gamma_1 + d\delta$ is a character where $a \ge 0$. These dominant weights parameterize the finite dimensional representations of $\mathrm{Gl}_2/\mathbb{Q}$. The dual representation is given by $\lambda^\vee = a\gamma_1 - d\delta$. But these highest weights also parameterize the discrete series representations of $\mathrm{Gl}_2(\mathbb{R})$, (or better the discrete series

Harish-Chandra modules). The highest weight λ defines a line bundle $\mathcal{L}_{-a\gamma+d\delta}$ on $B\backslash G$ and

$$\mathcal{M}_\lambda = H^0(B\backslash G, \mathcal{L}_{-a\gamma+d\delta})$$

Then we get an embedding and a resulting exact sequence

$$0 \to \mathcal{M}_\lambda \to I_B^G((-a\gamma_1 + d\delta)_{\mathbb{R}}) \to \mathcal{D}_{\lambda^\vee} \to 0$$

and $\mathcal{D}_{\lambda^\vee}$ is the discrete series representation attached to $\lambda^\vee$. (Note the subscript $_{\mathbb{R}}$ can not be pulled inside the bracket!).

A basic argument in representation theory yields a pairing

$$I_B^G((-a\gamma_1 - d\delta)_{\mathbb{R}}) \times I_B^G(((a+2)\gamma_1 + d\delta)_{\mathbb{R}}) \to \mathbb{R}$$

(here observe that $2\gamma_1 = 2\rho \in X^*(T)$).

From this we get another exact sequence which gives the more familiar definition of the discrete series representation

$$0 \to \mathcal{D}_\lambda \to I_B^G(((a+2)\gamma_1 + d\delta)_{\mathbb{R}}) \to \mathcal{M}_\lambda \to 0. \tag{9.112}$$

The module $\mathbb{D}_\lambda$ is also a module for the group $K_\infty = \mathrm{SO}(2)$ and it is well known that it decomposes into K_∞ types

$$D_\lambda = \cdots \oplus \mathbb{C}\psi_\nu \ldots \mathbb{C}\psi_{-a-4} \oplus \mathbb{C}\psi_{-a-2} \oplus \mathbb{C}\psi_{+a+2} \oplus \mathbb{C}\psi_{a+4} \cdots \tag{9.113}$$

(End of digression)

We return to our formula (9.111). The group

$$^\circ M = \prod_{i:i\mathrm{odd}} M_i \times (\mathbb{G}_m)$$

where $M_i = \mathrm{Gl}_2$. If T_i is the maximal torus in the i-th factor, then the highest weight is $\gamma_i^{^\circ M^{(1)}}$ and let δ_i be the determinant on that factor. The indices i run over the odd numbers $1, 3, \ldots, m$. If n is odd then let $\delta_n : T \to \mathbb{G}_m$ be the character given by the last entry. Then we have for the determinant

$$\delta = \delta_1 + \delta_3 + \cdots + \delta_m + \begin{cases} 0 \\ \delta_n \end{cases} \tag{9.114}$$

We want to write the character $2\gamma_2 + 2\gamma_4 + \cdots + 2\gamma_{m-1} + \frac{3}{2}\gamma_{m+1}$ in terms of the δ_i. We recall that

$$\begin{array}{c} \gamma_2 = \delta_1 - \frac{2}{n}\delta \\ \gamma_4 = \delta_1 + \delta_3 - \frac{4}{n}\delta \\ \vdots \\ \gamma_{m-1} = \delta_1 + \delta_3 \cdots + \delta_{m-2} - \frac{m-1}{n}\delta \\ \text{and if } n \text{ is odd} \\ \gamma_{m+1} = \delta_1 + \delta_3 \cdots + \delta_m - \frac{m+1}{n}\delta \end{array} \tag{9.115}$$

Then the summation over the δ-terms on the right hand side yields

$$-\frac{1}{n}(4+8+\cdots+2(m-1)-\begin{cases}0\\ \frac{3}{2}(m+1)\end{cases})=-[\frac{n-1}{2}] \qquad (9.116)$$

and if we take our formula (9.114) into account and also count the number of times a δ_i occurs in the summation we get

$$2\gamma_2+2\gamma_4+\cdots+2\gamma_{m-1}+\frac{3}{2}\gamma_{m+1}=\begin{cases}(\frac{n}{2}-1)\delta_1+(\frac{n}{2}-3)\delta_3+\cdots+(-\frac{n}{2}+1)\delta_{m-2} & n\equiv 0 \mod 2\\ \frac{n-2}{2}\delta_1+\cdots+\frac{-n+4}{2}\delta_m-\frac{n-1}{2}\delta_n & \text{else}\end{cases} \qquad (9.117)$$

Let us denote the coefficient of δ_i in the expressions on the right hand side by $c(i,n.)$ We recall that we still have the summand $d\delta$ in our formula (**??**. Then

$$\underline{\mu}=w_{\mathrm{un}}\cdot\lambda=\sum_{i:i \text{ odd}} b_i\gamma_i^{\circ M^{(1)}}+(c(i,n)+d)\delta_i+\begin{cases}d\delta\\ (-\frac{n-1}{2}+d)\delta_n\end{cases} \qquad (9.118)$$

We claim that the individual summands are in the character modules $X^*(T_i)$ (resp. $X^*(\mathbb{G}_m)$). This means that

$$b_i\gamma_i^{\circ M^{(1)}}+(c(i,n)+d)\delta_i\in X^*(T_i),\ -\frac{n-1}{2}+d\in\mathbb{Z}. \qquad (9.119)$$

We have to verify the parity conditions. If n is odd the parity condition for λ says that $d\in\mathbb{Z}$. On the other hand we know that in this case the b_i are odd and since the $c(i,n)$ are also odd the parity condition is satisfied for the individual summands.

If n is even then the parity condition for for λ says that $\frac{n}{2}a_{\frac{n}{2}}\equiv nd \mod n$. We know that the b_i all have the same parity: $b_i\equiv a_{\frac{n}{2}} \mod 2$. Hence need that $a_{\frac{n}{2}}\equiv 2d \mod 2$, but this is the parity condition for λ.

For any of the characters μ_i we have the induced representations $I_{B_i}^{\circ M_i}(\mu_i+2\rho_i)$ the discrete series representation $\mathcal{D}_{\mu_i}$ and the exact sequence

$$0\to\mathcal{D}_{\mu_i}\to I_{B_i}^{\circ M_i}(\mu_i+2\rho_i)\to\mathcal{M}_{\mu_i}\to 0. \qquad (9.120)$$

The tensor product

$$\mathcal{D}_\mu=\bigotimes_{i:i\text{odd}}\mathcal{D}_{\mu_i}\otimes\mathbb{C}(-\frac{n-1}{2}+d) \qquad (9.121)$$

is a module for ${}^\circ M$.

Here we have to work with $K_\infty^{\circ M}=K_\infty\cap^\circ M$. This compact group is not necessarily connected, its connected component of the identity is

$$K_\infty^{\circ M}\cap^\circ M^{(1)}(\mathbb{R})=\mathrm{SO}(2)\times\mathrm{SO}(2)\times\cdots\times\mathrm{SO}(2)=K_\infty^{\circ M,(1)}.$$

An easy computation shows

$$K_\infty^{\circ M}=\begin{cases}S(\mathrm{O}(2)\times\mathrm{O}(2)\times\cdots\times\mathrm{O}(2)) & \text{if } n \text{ is even}\\ \mathrm{O}(2)\times\mathrm{O}(2)\times\cdots\times\mathrm{O}(2) & \text{if } n \text{ is odd}\end{cases}, \qquad (9.122)$$

since $K_\infty \subset \mathrm{Sl}_n(\mathbb{R})$ we have the determinant condition in the even case, in the odd case we have the $\{\pm 1\}$ in the last factor and this relaxes the determinant condition.

Under the action of $K_\infty^{{}^\circ M,(1)}$ we get a decomposition

$$\mathcal{D}_{\underline{\mu}} = \bigoplus_{\underline{\varepsilon}} \bigotimes_{i=1}^{{}^\circ r} \big(\bigoplus_{\nu_i=0}^{\infty} \mathbb{C}\psi_{\varepsilon_i(b_i+2+2\nu_i)}\big) \tag{9.123}$$

occur with multiplicity one. Here $\underline{\varepsilon} = (\dots, \varepsilon_i, \dots)$ is an array of signs ± 1.

The induced representation (algebraic induction)

$$\mathrm{Ind}_{{}^\circ P(\mathbb{R})}^{G(\mathbb{R})} \mathcal{D}_{\underline{\mu}} = \mathbb{D}_\lambda \tag{9.124}$$

is an irreducible essentially unitary $(\mathfrak{g}, K_\infty)$ -module, this is the module we wanted to construct. (To be more precise: We first construct the induced representation of $G(\mathbb{R})$ where $G(\mathbb{R})$ is acting on vectors space V_∞ consisting of a suitable class of functions from $G(\mathbb{R})$ with values in $\mathcal{D}_{\underline{\mu}}$ and then we take the K_∞ finite vectors in V_∞.) The restriction of this module to $K_\infty^{(1)}$ s given by

$$\mathrm{Ind}_{K_\infty^{{}^\circ M^{(1)}}}^{K_\infty^{(1)}} \mathcal{D}_{\underline{\mu}} = \bigoplus_{\underline{\varepsilon}} \bigotimes_{i=1}^{{}^\circ r} \big(\bigoplus_{\nu_i=0}^{\infty} \mathrm{Ind}_{K_\infty^{{}^\circ M^{(1)}}}^{K_\infty^{(1)}} \mathbb{C}\psi_{\varepsilon_i(b_i+2+2\nu_i)}\big) \tag{9.125}$$

(The last induced module is defined in terms of the theory of algebraic groups. We consider $K_\infty^{(1)}$ as the group of real points of an algebraic group, namely the connected group of the identity of the fixed points under the Cartan involution Θ. Then $K_\infty^{{}^\circ M^{(1)}}$ is the group of real points of a maximal torus. Then

$$\begin{gathered}\mathrm{Ind}_{K_\infty^{{}^\circ M^{(1)}}}^{K_\infty^{(1)}} \mathbb{C}\psi_{\varepsilon_i(b_i+2+2\nu_i)} = \\ \{f | f \text{ regular function } f(tk) = \textstyle\prod_j e_i(t)^{\varepsilon_i(b_i+2+2\nu_i)} f(k), \text{ for all } t \in K_\infty^{{}^\circ M^{(1)}}, k \in K_\infty\}\end{gathered} \tag{9.126}$$

)

We compute the cohomology of this module

$$\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathbb{D}_\lambda \otimes \mathcal{M}_\lambda) = H^\bullet(\mathfrak{g}, K_\infty, \mathbb{D}_\lambda \otimes \mathcal{M}_\lambda),$$

i.e. the differentials in the complex on the left hand side are all zero. (Reference to 4.1.4)

We apply Delorme to compute this cohomology. We can decompose ${}^\circ\mathfrak{m} = {}^\circ\mathfrak{m}^{(1)} \oplus \mathfrak{a}$ then ${}^\circ\mathfrak{k} \subset {}^\circ\mathfrak{m}^{(1)}$ and

$$\begin{aligned}\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathbb{D}_\lambda \otimes \mathcal{M}_\lambda) = {} & \mathrm{Hom}_{K_\infty^{{}^\circ M}}(\Lambda^\bullet({}^\circ\mathfrak{m}/{}^\circ\mathfrak{k}), \mathcal{D}_{\tilde{\mu}} \otimes \mathcal{M}_{w_{\mathrm{un}}\cdot\lambda}) = \\ & \mathrm{Hom}_{K_\infty^{{}^\circ M}}(\Lambda^\bullet({}^\circ\mathfrak{m}^{(1)}/{}^\circ\mathfrak{k}), \mathcal{D}_{\tilde{\mu}} \otimes \mathcal{M}_{w_{\mathrm{un}}\cdot\lambda}) \otimes \Lambda^\bullet(\mathfrak{a}).\end{aligned} \tag{9.127}$$

If we replace $K_\infty^{{}^\circ M}$ on the right hand side by its connected component of the identity then we have an obvious decomposition

$$\mathrm{Hom}_{K_\infty^{{}^\circ M,(1)}}(\Lambda^\bullet({}^\circ\mathfrak{m}^{(1)}/{}^\circ\mathfrak{k}), \mathcal{D}_\mu \otimes \mathcal{M}_{w_{\mathrm{un}}\cdot\lambda}) = \bigotimes_{i:i \text{ odd}} \mathrm{Hom}_{K_\infty^{i,{}^\circ M,(1)}}(\Lambda^\bullet({}^\circ\mathfrak{m}^{(i,1)}/{}^\circ\mathfrak{k}^i), \mathcal{D}_{b_i} \otimes \mathcal{M}_{b_i}) \tag{9.128}$$

the factors on the right hand side are of rank two: We have $K_\infty^{i,{}^\circ M,(1)} = \mathrm{SO}(2)$ and under the adjoint action of $K_\infty^{i,{}^\circ M,(1)}$ the module $\mathfrak{m}^{(i,1)}/{}^\circ\mathfrak{k}^i \otimes \mathbb{C}$ decomposes

$$\mathfrak{m}^{(i,1)}/{}^\circ\mathfrak{k}^i \otimes \mathbb{C} = \mathbb{C}P_{i,+}^\vee \oplus \mathbb{C}P_{i,-}^\vee$$

(See [Sltwo.pdf]) Then the two summands are generated by the tensors

$$\omega_{i,+} = P_{i,+}^\vee \otimes \psi_{b_i+2} \otimes m_{-b_i}, \bar{\omega}_{i,-} = P_{i,-}^\vee \otimes \psi_{-b_-2} \otimes m_{b_i} \qquad (9.129)$$

where $m_{\pm(b_i)}$ is a highest (resp.) lowest weight vector for $K_\infty^{i,{}^\circ M}$ acting on $\mathcal{M}_{w_{\mathrm{un}}\cdot\lambda}$. On the tensor product on the right we have an action of the maximal compact subgroup $\mathrm{O}(2)\times\mathrm{O}(2)\times\cdots\times\mathrm{O}(2)$ and under this action it decomposes into eigenspaces of dimension one. These eigenspaces are given by the product of sign characters $\underline{\epsilon} = (\epsilon_1, \epsilon_2, \dots)$.

Then it becomes clear that $\mathrm{Hom}_{K_\infty^{\circ M}}(\Lambda^\bullet({}^\circ\mathfrak{m}^{(1)}/{}^\circ\mathfrak{k}), \mathcal{D}_{\underline{\mu}} \otimes \mathcal{M}_{w_{\mathrm{un}}\cdot\lambda})$ is of rank one if n is odd and for n even it decomposes into two eigenspaces for the action of the group $\mathrm{O}(2)\times\mathrm{O}(2)\times\cdots\times\mathrm{O}(2)/S(\mathrm{O}(2)\times\mathrm{O}(2)\times\cdots\times\mathrm{O}(2)) = \{\pm 1\}$

$$\mathrm{Hom}_{K_\infty^{\circ M}}(\Lambda^\bullet({}^\circ\mathfrak{m}^{(1)}/{}^\circ\mathfrak{k}), \mathcal{D}_{\underline{\mu}} \otimes \mathcal{M}_{w_{\mathrm{un}}\cdot\lambda}) =$$

$$\mathrm{Hom}_{K_\infty^{\circ M}}(\Lambda^\bullet({}^\circ\mathfrak{m}^{(1)}/{}^\circ\mathfrak{k}), \mathcal{D}_{\underline{\mu}} \otimes \mathcal{M}_{w_{\mathrm{un}}\cdot\lambda}))_+ \oplus \mathrm{Hom}_{K_\infty^{\circ M}}(\Lambda^\bullet({}^\circ\mathfrak{m}^{(1)}/{}^\circ\mathfrak{k}), \mathcal{D}_{\underline{\mu}} \otimes \mathcal{M}_{w_{\mathrm{un}}\cdot\lambda}))_-$$

We have to recall that $\mathcal{M}_{\lambda^{\mathrm{un}}_{{}^\circ M}} = H^{l(w_{\mathrm{un}})}(\mathfrak{u}_{{}^\circ P}, \mathcal{M}_\lambda)$ is a cohomology group in degree $l(w_{\mathrm{un}})$. The classes in the factors of the last tensor product lie in degree 1, hence the multiply up to classes in degree ${}^\circ r$. This means that

$$H^q(\mathfrak{g}, K_\infty, \mathbb{D}_\lambda \otimes \mathcal{M}_\lambda) \neq 0 \text{ exactly for } q \in [l(w_{\mathrm{un}}) +^\circ r, l(w_{\mathrm{un}}) + n] \qquad (9.130)$$

in the minimal degree $l(w) +^\circ r$ it is of rank 2 or 1 depending on the parity of n.

9.4.2 The lowest K_∞ type in $\mathbb{D}_\lambda$

The maximal compact subgroup K_∞ is the fixed group of the standard Cartan-involution $\Theta : g \mapsto {}^t g^{-1}$. The subgroup ${}^\circ M$ is fixed under Θ and the subgroup $\mathrm{SO}(2)\times\mathrm{SO}(2)\times\cdots\times\mathrm{SO}(2) = K_\infty^{{}^\circ M,(1)} = T_1^c$ is a maximal torus in K_∞. It is the stabilizer of a direct sum decompositions of $\mathbb{R}^n$ into two dimensional oriented euclidian planes V_i plus a line $\mathbb{R}z$ if n is odd, we write

$$\mathbb{R}^n = \bigoplus V_i \oplus (\mathbb{R}z) \qquad (9.131)$$

The Cartan involution is the identity on our torus $T_1^c/\mathbb{R}$. This torus can be supplemented to a $\Theta-$ stable maximal torus by multiplying it by the torus $T_{1,\mathrm{split}}$ which is the product of the diagonal tori acting on the V_i in (9.131) times another copy of $\mathbb{G}_m$ acting on $\mathbb{R}z$ (if necessary). So we get a maximal torus $T_1 = T_1^c \cdot T_{1,\mathrm{split}}$. Obviously T_1 is the centralizer of T_1^c and the centralizer of $T_{1,\mathrm{split}}$ is the group ${}^\circ M$.

If we base change to $\mathbb{C}$ then T_1^c splits. We identify

$$\mathrm{SO}(2) \xrightarrow{\sim} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \tag{9.132}$$

and then the character group $X^*(T_1^c \times \mathbb{C}) = \oplus \mathbb{Z} e_\nu$ where on the ν-th component $e_\nu : \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mapsto a + bi = a + b\sqrt{-1}$. Then this choice provides a Borel subgroup $B_c \supset T_1^c \times \mathbb{C}$, for which the simple roots $\alpha_1^c, \alpha_2^c, \ldots, \alpha_{{}^\circ r}^c$ are

$$\begin{cases} e_1 - e_2, e_2 - e_3, \ldots, e_{{}^\circ r-1} - e_{{}^\circ r}, e_{{}^\circ r-1} + e_{{}^\circ r} & \text{for } n \text{ even} \\ e_1 - e_2, e_2 - e_3, \ldots, e_{{}^\circ r} & \text{if } n \text{ is odd} \end{cases}$$

(See [Bou]). For n even we get the fundamental dominant weights

$$\gamma_\nu^c = \begin{cases} e_1 + e_2 + \cdots + e_\nu, & \text{if } \nu <^\circ r - 1 \\ \frac{1}{2}(e_1 + e_2 + \cdots + e_{{}^\circ r-1} - e_{{}^\circ r}) & \text{if } \nu =^\circ r - 1 \\ \frac{1}{2}(e_1 + e_2 + \cdots + e_{{}^\circ r-1} + e_{{}^\circ r}) & \text{if } \nu =^\circ r \end{cases} \tag{9.133}$$

and for n odd we get

$$\gamma_\nu^c = \begin{cases} e_1 + e_2 + \cdots + e_\nu, & \text{if } \nu <^\circ r \\ \frac{1}{2}(e_1 + e_2 + \cdots + e_{{}^\circ r}) & \text{last weight} \end{cases} \tag{9.134}$$

An easy calculation shows

$$\sum_{i=1}^{{}^\circ r} g_i e_i = \begin{cases} (g_1 - g_2)\gamma_1^c + (g_2 - g_3)\gamma_2^c + \cdots + (g_{{}^\circ r-1} - g_{{}^\circ r})\gamma_{{}^\circ r-1}^c + (g_{{}^\circ r-1} + g_{{}^\circ r})\gamma_{{}^\circ r}^c & n \text{ even} \\ (g_1 - g_2)\gamma_1^c + (g_2 - g_3)\gamma_2^c + \cdots + (g_{{}^\circ r-1} - g_{{}^\circ r})\gamma_{{}^\circ r-1} + 2g_{{}^\circ r}\gamma_{{}^\circ r}^c & n \text{ odd} \end{cases} \tag{9.135}$$

The character $\sum_{i=1}^{{}^\circ r} g_i e_i$ is dominant (with respect to B_c) if

$$\begin{cases} g_1 \geq g_2 \geq \ldots g_{{}^\circ r-1} \geq \pm g_{{}^\circ r} & \text{if } n \text{ is even} \\ g_1 \geq g_2 \geq \cdots \geq g_{{}^\circ r-1} \geq g_{{}^\circ r} \geq 0 \end{cases} \tag{9.136}$$

Under the action of $K_\infty^{(1)}$ the $(\mathfrak{g}, K_\infty^{(1)})$- module $\mathbb{D}_\lambda$ decomposes into a direct sum

$$\mathbb{D}_\lambda = \bigoplus_{\mu^c} \mathbb{D}_\lambda(\Theta_{\mu^c}) \tag{9.137}$$

where $\mu^c \in X^*(T^c \times \mathbb{C})$ is a highest weight, Θ_{μ^c} is the resulting irreducible K_∞-module and $\mathbb{D}_\lambda(\Theta_{\mu^c})$ is the isotypical component.

We introduce the highest weight (see (9.110))

$$\mu_0^c(\lambda) = (b_1 + 2)e_1 + (b_3 + 2)e_2 + \cdots + (b_{2^\circ r-1} + 2)e_{{}^\circ r} \tag{9.138}$$

and in terms of our dominant weight λ this is

$$\mu_0^c(\lambda) = \begin{cases} 2(a_1 + 1)\gamma_1^c + \cdots + 2(a_{{}^\circ r-1} + 1)\gamma_{{}^\circ r-1}^c + 2(a_{{}^\circ r-1} + a_{{}^\circ r} + 3)\gamma_{{}^\circ r}^c & \text{if } n \text{ is even} \\ 2(a_1 + 1)\gamma_1^c + \cdots + 2(a_{{}^\circ r} + 3)\gamma_{{}^\circ r}^c & \text{if } n \text{ is odd} \end{cases} \tag{9.139}$$

For $\lambda = 0$ we get an expression (not depending on the parity of n)

$$\mu_0^c(0) = 2\gamma_1^c + \cdots + 2\gamma_{\circ r-1}^c + 6\gamma_{\circ r}^c \tag{9.140}$$

In the case that n is even the group K_∞ contains the element θ which maps $e_i \to e_i$ for $i \leq^\circ r-1$ and $e_{\circ r} \to -e_{\circ r}$ or what amounts to the same exchanges $\gamma_{\circ r-1}^c$ and $\gamma_{\circ r}^c$ and fixes the other fundamental dominant weights. Then

$$\bar{\mu}_0^c(\lambda) := \vartheta(\mu_0^c(\lambda)) = 2\gamma_1^c + \cdots + 6\gamma_{\circ r-1}^c + 2\gamma_{\circ r}^c + \vartheta(\lambda^c) \tag{9.141}$$

Proposition 9.4.1. *If n is odd then the $K_\infty^{(1)}$- type $\Theta_{\mu_0^c(\lambda)}$ occurs in $\mathbb{D}_\lambda$ with multiplicity one. All other occurring $K_\infty^{(1)}$ types are "larger", i.e. their highest weight satisfies $\mu^c = \mu_0^c(\lambda) + \sum n_i\alpha_i^c$ with $n_i \geq 0$. We have*

$$H^\bullet(\mathfrak{g}, K_\infty, \mathbb{D}_\lambda \otimes \mathcal{M}_\lambda) = \ Hom_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \Theta_{\mu_0^c(\lambda)} \otimes \mathcal{M}_\lambda)$$

If n is even then the $(\mathfrak{g}, K_\infty^{(1)})$ module $\mathbb{D}_\lambda$ decomposes into two irreducible sub modules

$$\mathbb{D}_\lambda = \mathbb{D}_\lambda^+ \oplus \mathbb{D}_\lambda^-.$$

The $K_\infty^{(1)}$ types $\Theta_{\mu_0^c(\lambda)}$ resp. $\Theta_{\bar{\mu}_0^c(\lambda)}$ occur with multiplicity one (resp. zero) in $\mathbb{D}_\lambda^+$(resp. $\mathbb{D}_\lambda^-$). They are the lowest $K_\infty^{(1)}$ types respectively. We have

$$H^\bullet(\mathfrak{g}, K_\infty^{(1)}, \mathbb{D}_\lambda \otimes \mathcal{M}_\lambda) = H^\bullet(\mathfrak{g}, K_\infty^{(1)}, \mathbb{D}_\lambda^+ \otimes \mathcal{M}_\lambda) \oplus H^\bullet(\mathfrak{g}, K_\infty^{(1)}, \mathbb{D}_\lambda^- \otimes \mathcal{M}_\lambda) =$$

$$Hom_{K_\infty^{(1)}}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \Theta_{\mu_0^c(\lambda)} \otimes \mathcal{M}_\lambda) \oplus \ Hom_{K_\infty^{(1)}}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \Theta_{\bar{\mu}_0^c(\lambda)} \otimes \mathcal{M}_\lambda)$$

Proof. For two fundamental weights we write $\mu^c \geq \mu_1^c$ if μ^c is "larger" than μ_1^c in the above sense. We start from (9.125) and consider a single summand $\mathrm{Ind}_{K_\infty^\circ M^{(1)}}^{K_\infty^{(1)}} \mathbb{C}\psi_{\varepsilon_i(b_i+2+2\nu_i)}$. This induced module decomposes into isotypical modules

$$\mathrm{Ind}_{K_\infty^\circ M^{(1)}}^{K_\infty^{(1)}} \mathbb{C}\psi_{\varepsilon_i(b_i+2+2\nu_i)} = \bigoplus_{\mu^c} \mathrm{Ind}_{K_\infty^\circ M^{(1)}}^{K_\infty^{(1)}} \mathbb{C}\psi_{\varepsilon_i(b_i+2+2\nu_i)}(\Theta_{\mu^c}) \tag{9.142}$$

where μ^c runs over the set of dominant weights, where Θ_{μ^c} is the irreducible module of highest weight μ^c and where $\mathrm{Ind}_{K_\infty^\circ M^{(1)}}^{K_\infty^{(1)}} \mathbb{C}\psi_{\varepsilon_i(b_i+2+2\nu_i)}(\Theta_{\mu^c})$ is the isotypical component. If we pick any dominant weight μ^c then Frobenius reciprocity yields that

$$\begin{aligned} &\Theta_{\mu^c} \text{ occurs in } \mathrm{Ind}_{K_\infty^\circ M^{(1)}}^{K_\infty^{(1)}} \mathbb{C}\psi_{\varepsilon_i(b_i+2+2\nu_i)} \text{ with multiplicity } k \iff \\ &\quad t \mapsto \prod_j e_i(t)^{\varepsilon_i(b_i+2+2\nu_i)} \text{ occurs in } \Theta_{\mu^c} \text{ with multiplicity } k \end{aligned} \tag{9.143}$$

and if $k > 0$ this implies $\mu^c \geq t \mapsto \prod_j e_i(t)^{\varepsilon_i(b_i+2+2\nu_i)}(t)$. It it easy to see that we get minimal $K_\infty^{(1)}$ types only if all $\nu_i = 0$. But

$$t \mapsto \prod_j e_i(t)^{\varepsilon_i(b_i+2)} \text{ is dominant } \iff \begin{cases} \varepsilon = (1,1,\ldots,1,\pm 1) \text{ if } n \text{ even} \\ \varepsilon = (1,1,\ldots,1,1) \text{ if } n \text{ odd} \end{cases} \tag{9.144}$$

and in the n even case these two characters are exactly $\mu_0^c(\lambda)$ and $\bar{\mu}_0^c(\lambda)$ and in the n odd case this character is $\mu_0^c(\lambda)$.

□

9.4.3 The unitary modules with cohomology, cohomological induction.

We start from an essentially self dual highest weight λ and the attached highest weight module $\mathcal{M}_\lambda$. In their paper [108] Vogan and Zuckerman construct a finite family of $(\mathfrak{g}, K_\infty)$ modules denoted by $A_\mathfrak{q}(\lambda)$ which have non trivial cohomology with coefficients in $\mathcal{M}_\lambda$, i.e.

$$H^\bullet(\mathfrak{g}, K_\infty, A_\mathfrak{q}(\lambda) \otimes \mathcal{M}_\lambda) \neq 0$$

They also show that all unitary irreducible $(\mathfrak{g}, K_\infty)$ -modules with non trivial cohomology in with coefficients in $\mathcal{M}_\lambda$. are of this form. We briefly recall their construction and translate it into our language and our way of thinking about these issues.

We introduce the torus $\mathbb{S}^1/\mathbb{R}$ whose group of real points is the unit circle in $\mathbb{C}^\times$ and we chose once for all the isomorphism

$$i_0 : \mathbb{S}^1 \times_\mathbb{R} \mathbb{C} \xrightarrow{\sim} \mathbb{G}_m/\mathbb{C}, \tag{9.145}$$

which sends $z \in \mathbb{S}^1(\mathbb{R})$ to $z \in \mathbb{C}^\times$. We identify $\mathbb{S}^1 = \mathrm{SO}(2)$ by sending $z = x+iy$ to $\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$. We consider the free $\mathbb{Z}$ module

$$\mathrm{Hom}_\mathbb{R}(\mathbb{S}^1, T_1^c) = \ \mathrm{Hom}_\mathbb{R}(\mathbb{S}^1, T_1) = X_*(T_1^c \times_\mathbb{R} \mathbb{C})$$

where of course the last identification depends on the choice of i_0. We have the standard pairing $< \ , \ > : X_*(T_1 \times_\mathbb{R} \mathbb{C}) \times X^*(T_1 \times_\mathbb{R} \mathbb{C}) \to \mathbb{Z}$.

The first ingredient in the construction of an $A_\mathfrak{q}(\lambda)$ is the choice of a cocharacter $\chi : \mathbb{S}^1 \to T_c$ (defined over $\mathbb{R}$). From this cocharacter we get the centralizer Z_χ , this is a reductive subgroup whose set of roots is

$$\Delta_\chi = \{\alpha \in \Delta \subset X^*(T_1 \times_\mathbb{R} \mathbb{C}) | \ < \chi, \alpha >= 0\}.$$

We can also define

$$\Delta_\chi^+ = \{\alpha | < \chi, \alpha > > \ 0\},$$

this set depends on the choice of i_0 (see (3.29)). This provides a parabolic subgroup $P_\chi \subset G \times_\mathbb{R} \mathbb{C}$ whose system of roots is $\Delta_\chi \cup \Delta_\chi^+$. Clearly $\Theta(P_\chi) = P_\chi$ hence P_χ is the Θ-stable parabolic subgroup attached to the datum χ. This parabolic subgroup is only defined over $\mathbb{C}$, if we intersect it with its conjugate $\bar{P}_\chi$ then we get the centralizer Z_χ of χ. We relate this to the notations in [108]: the $\mathfrak{q}$ in $A_\mathfrak{q}(\lambda)$ is the Lie-algebra of P_χ, the group Z_χ is the L. Let $\mathfrak{u}_\chi$ be the Lie algebra of U_χ. The datum χ determines the $\mathfrak{q}$ in $A_\mathfrak{q}(\lambda)$. We could introduce the notation $A_\mathfrak{q}(\lambda) = A_\chi(\lambda)$. Since T_1 is the centralizer of T_c we can find a generic cocharacter χ_{gen} such that $P_{\chi_{\mathrm{gen}}} = B_c$ our chosen Borel subgroup in ${}^\circ M$.

To a highest weight λ which is trivial on the semi-simple part $Z_\chi^{(1)}$ Vogan-Zuckerman attach an irreducible unitary $(\mathfrak{g}, K_\infty)$ module $A_\mathfrak{q}(\lambda)$ such that

$$H^\bullet(\mathfrak{g}, K_\infty, A_\mathfrak{q}(\lambda) \otimes \mathcal{M}_\lambda) \neq 0.$$

Vogan and Zuckerman show (based on results of many others) that all the unitary irreducible $(\mathfrak{g}, K_\infty)$ modules with non trivial cohomology in $\mathcal{M}_\lambda$ are isomorphic to an $A_\mathfrak{q}(\lambda)$.

Furthermore they give a description of the K_∞ types occurring in $A_\mathfrak{q}(\lambda)$ especially they show that $A_\mathfrak{q}(\lambda)$ contains a lowest K_∞ type. This lowest K_∞-type is given by a dominant weight which obtained by the following rule:

We consider the action of the group K_∞ on the unipotent radical U_χ and on the Lie algebra $\mathfrak{u}_\chi$ and the restriction of this action to T_1^c. The torus T_1 also acts on $\mathfrak{u}_\chi$ and under this action we get a decomposition into one dimensional eigenspaces

$$\mathfrak{u}_\chi = \bigoplus_{\alpha \in \Delta_\chi^+} \mathfrak{u}_\alpha$$

let us choose generators X_α in these eigenspaces. We observe that the roots $\alpha, \Theta\alpha \in \Delta^+$ induce the same root α_c on T_1^c. The vector $V_{\alpha_c} = X_\alpha - \Theta X_\alpha \in \mathfrak{u}_\chi$ is a non zero eigenvector for T_1^c and

$$\mathfrak{u}_\chi \cap (\mathfrak{p} \otimes \mathbb{C}) = \bigoplus_{(\alpha, \Theta\alpha) \in \Delta_\chi^+} \mathbb{C} V_{\alpha_c}$$

the sum runs over the unordered pairs. Then

$$\mu_c(\chi, \lambda) = \sum_{(\alpha,\Theta\alpha)\in\Delta_\chi^+} \alpha_c + \lambda_c \tag{9.146}$$

is a highest weight of a representation $\Theta_{\mu_c(\chi,\lambda)}$ of $K_\infty^{(1)}$ and this is the lowest $K_\infty^{(1)}$ type in $A_\mathfrak{q}(\lambda)$. We get

$$H^\bullet(\mathfrak{g}, K_\infty^{(1)}, A_\mathfrak{q}(\lambda) \otimes \mathcal{M}_\lambda) = \mathrm{Hom}_{K_\infty^{(1)}}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), A_\mathfrak{q}(\lambda) \otimes \mathcal{M}_\lambda) = \mathrm{Hom}_{K_\infty^{(1)}}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \Theta_{\mu_c(\chi,\lambda)} \otimes \mathcal{M}_\lambda) \tag{9.147}$$

The module is determined by these properties:

1) It has non trivial cohomology with coefficients in $\mathcal{M}_\lambda$

2) It has $\mu_c(\chi, \lambda)$ as highest weight of a minimal K_∞ type. (See Thm. 5. 3 in [108].)

We return to our group $\mathrm{Gl}_n/\mathbb{Z}$, , then a cocharacter $\chi : \mathbb{S}^1 \to T_1^c$ is of the form

$$\chi : z \mapsto \begin{pmatrix} z^{n_1} & 0 & 0 & \dots & \\ 0 & z^{n_2} & \dots & \dots & \\ 0 & 0 & \ddots & 0 & \dots \\ 0 & & \dots & z^{n_r} & \\ 0 & & & & (1) \end{pmatrix} \tag{9.148}$$

where the entry (1) occurs if n is odd. Such a character is regular if and only if the integers n_i are pairwise different.

It is known that $A_\chi(\lambda)$ is tempered if and only if χ is regular or what amounts to the same when the corresponding Θ-stable parabolic subgroup is a Borel subgroup. The set of regular χ has several connected components, these components are open convex cones, whose faces are given by hyperplanes $n_i = n_j$ for some $ii \neq j$. The Θ-stable parabolic subgroup B_χ only depends on the connected component which contains χ.

Furthermore we have an action of Weyl group W_{K_∞} on these connected components. This Weyl group is the semi direct product of the symmetric group S_r and by by involutions $n_\nu \mapsto (-1)^{\epsilon_\nu} n_\nu$ with the constraint $\sum_\nu \epsilon_\nu \equiv 0 \mod 2$ in case n even.

We can summarise

Given our dominant autodual λ there are (resp.there is) exactly two tempered isomorphism types $A_{\chi^\pm}(\lambda)$ if n is even (resp. one tempered $A_\chi(\lambda)$ if n is odd.) If we look at their lowest K_∞ type we see that these modules are isomorphic to the modules $\mathbb{D}^\pm_\lambda$ for n even (resp. $\mathbb{D}_\lambda$)for n odd.

Cuspidal cohomology for Gl_n

Grobners objection

A theorem of Wallach asserts that in a cuspidal representation $H_{\pi_\infty} \otimes H_{\pi_\infty} \subset L^2_{\text{cusp}}(\mathrm{Gl}_n(\mathbb{Q})\backslash \mathrm{Gl}_n(\mathbb{A}))$ with $H^\bullet(\mathfrak{g}, K_\infty, H_{\pi_\infty} \otimes \mathcal{M}_\lambda) \neq 0$ the component H_{π_∞} must be tempered (See [**?**],), hence it must be an $A_\chi(\lambda)$ with χ regular.

On the other hand Theorem 5.2 in [48] says that for any other $\Phi : H_{\pi'_\infty} \otimes H_{\pi_f} \hookrightarrow L^2_{\text{disc}}(\mathrm{Gl}_n(\mathbb{Q})\backslash \mathrm{Gl}_n(\mathbb{A})$ the homomorphism Φ factors through the space of cusp forms, in other words any Φ provides cuspidal classes. Hence this means that a Hecke-module π_f which occurs in the inner cohomology $H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes F)$ and which provides cuspidal classes is necessarily strongly inner.

It this point a little warning is appropriate. In [48] in the proof of Theorem 4.5 the authors refer to a preliminary version of this book (and also a paper of Grobner). Here - in this book - the Theorem 4.5 is replaced by the much more general and much more simpler to prove proposition ??? so the reference becomes obsolete.

9.4.4 Eisenstein classes

In the previous section we considered only the bottom step in the boundary cohomology, and in this bottom step we only considered cuspidal contributions. We now extend the construction of Eisenstein classes to obtained from parabolic of lower rank.

Our group is $\mathrm{Gl}_n/\mathbb{Q}$ and we choose a parabolic subgroup $P/\mathbb{Q}$ containing the standard Borel subgroup and with reductive quotient $M = \mathrm{Gl}_{n_1} \times \mathrm{Gl}_{n_2} \times \cdots \times \mathrm{Gl}_{n_r}$.

We want to construct Eisenstein cohomology classes in $H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,\mathbb{C}})$ starting from inner classes in

$$H^\bullet_{!!}(\partial_P \mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,\mathbb{C}}) = \bigoplus_{w \in W^P} H^{\bullet - l(w)}_{!!}(\mathcal{S}^M_{K^M_f}, H^{l(w)}(\mathfrak{u}_P, \mathcal{M}_\lambda).(w \cdot \lambda))$$

We pick an element $w \in W^P$, and write

$$w(\lambda + \rho) = \tilde{\mu}^{(1)} - b_1(w,\lambda)\gamma_{n_1} - b_2(w,\lambda)\gamma_{n_1+n_2} + \cdots - b_r(w,\lambda)\gamma_{n_1+\cdots+n_{r-1}} + d\delta, \tag{9.149}$$

here the $\gamma_{n_1+\dots n_\nu} \in \mathrm{Hom}(P,\mathbb{G}_m)\otimes\mathbb{Q}$ are the dominant fundamental weights(see 1.76) und $\tilde{\mu}^{(1)}$ is the semi simple part (with respect to M), i.e.

$$\tilde{\mu}^{(1)} = ((a_1+1)\gamma_1^M+\cdots+(a_{n_1-1}+1)\gamma_{n_1-1}^M)+((a_{n_1+1}+1)\gamma_{n_1+1}^M+\dots(a_{n_1+n_2+1}+1)\gamma_{n_1+n_2-1}^M)+\dots$$
$$= \tilde{\mu}_1^{(1)}+\cdots+\tilde{\mu}_r^{(1)}. \tag{9.150}$$

We assume that $b_i^M(w,\lambda)\geq 0$ i.e. $w(\lambda+\rho)$ is in the negative chamber and we also assume that the $\tilde{\mu}_i^{(1)}$ are self dual, this is a condition on λ, w. We pass to a suitable finite normal extension $F/\mathbb{Q}$ and decompose the strongly inner cohomology

$$H^\bullet_{!!}(\partial_P\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\lambda,F}) = \bigoplus_{w\in W^P}\bigoplus_{\underline{\sigma}_f}\mathrm{Ind}_P^G H^{\bullet-l(w)}_{!!}(\mathcal{S}^M_{K_f^M},\tilde{\mathcal{M}}_{w\cdot\lambda})(\underline{\sigma}_f) \tag{9.151}$$

where the σ_f are absolutely irreducible. The Künneth-theorem implies that $\underline{\sigma}_f=\sigma_{1,f}\otimes\sigma_{2,f}\otimes\cdots\otimes\sigma_{r,f}$. At an unramified place p this module has a Satake parameter

$$\omega_p(\sigma_f)=\{\omega_{1,p},\dots,\omega_{n_1,p},\omega_{n_1+1,p},\dots,\omega_{n_1+n_2,p},\dots\}$$

where the first n_1 entries are the Satake parameters of $\sigma_{1,f}$ and so on.

We choose an $\iota: F\to\mathbb{C}$. We take an irreducible submodule

$$H_{\underline{\sigma}_f}\subset\ \mathrm{Ind}_P^G H^{\bullet-l(w)}_{!!}(\mathcal{S}^M_{K_f^M},\tilde{\mathcal{M}}_{w\cdot\lambda})(\underline{\sigma}_f),$$

then there is an irreducible $(\mathfrak{m},K_\infty^M)$-module $H_{\underline{\sigma}_\infty}$ and an embedding

$$\Phi: H_{\underline{\sigma}_\infty}\otimes H_{\underline{\sigma}_f}\otimes_{F,\iota}\mathbb{C}=H_{\underline{\sigma}}\hookrightarrow L^2_{\mathrm{cusp}}(M(\mathbb{Q})\backslash M(\mathbb{A})) \tag{9.152}$$

such that $H_{\underline{\sigma}_f}\otimes_{F,\iota}\mathbb{C}\subset H^\bullet(\mathfrak{m},K_\infty^M,H_{\sigma_\infty})\otimes H_{\sigma_f}$. (see Theorem 8.1.1).

For $\underline{z}=(z_1,z_2,\dots,z_{r-1}), z_i\in\mathbb{C}$ we define the character

$$|\gamma_P|^{\underline{z}}=|\gamma_{n_1}|^{z_1}|\gamma_{n_1+n_2}|^{z_2}\dots|\gamma_{n_1+n_2+\cdots+n_{r-1}}|^{z_{r-1}}: M(\mathbb{A})\to\mathbb{C}^\times$$

By the usual summation process we get an Eisenstein intertwining operator

$$\mathrm{Eis}(\underline{\sigma},\underline{z}): I_P^G H_{\underline{\sigma}}\otimes|\gamma_P|^{\underline{z}}\to\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A})) \tag{9.153}$$

the series is locally uniformly converging in a region where all $\Re(z_i)>>0$ and hence the Eisenstein intertwining operator is holomorphic in this region. We know that it admits a meromorphic extension into the entire $\mathbb{C}^{r-1}$.

We want to evaluate at $\underline{z}=0$, this is possible if $\mathrm{Eis}(\underline{\sigma},\underline{z})$ is holomorphic at $\underline{z}=0$. To find out what happens at $\underline{z}=$ we have to consider the constant term (constant Fourier coefficient) of $\mathrm{Eis}(\underline{\sigma},\underline{z})$ along parabolic subgroups P_1. (See [50]) These constant Fourier coefficients a given by integrals

$$\mathcal{F}^{P_1}: f(\underline{g})\mapsto\int_{U_{P_1}(\mathbb{Q})\backslash U_{P_1}(\mathbb{A})} f(\underline{u}\underline{g})d\underline{u}. \tag{9.154}$$

It suffices to compute these constant terms only for parabolic subgroups containing our given maximal torus. It is shown in [50] that the constant term evaluated at $\mathrm{Eis}(\underline{\sigma},\underline{z})(f)$ is zero unless P and P_1 are associate, this means that the Levi subgroups M and M_1 are isomorphic. (For this we need the cuspidality condition (See [50],)But then we can find an element in the Weyl group which conjugates M into M_1 and hence we may assume that P and P_1 both contain our given Levi subgroup M. Of course now P_1 may not contain the standard Borel subgroup.

We may also assume that $n_1 = n_2 = \cdots = n_{j_1} < n_{j_1+1} = \cdots = n_{j_1+j_2} < \cdots < n_{j_1+\ldots j_{s-1}+1} = \cdots = n_{j_1+\cdots+j_s} = n_r$. Then it is easy to see that the number of conjugacy classes of parabolic subgroups which contain M is equal to $r!/j_1!j_2!...j_s!$.

We compute $\mathcal{F}^{P_1}\circ\ \mathrm{Eis}(\underline{\sigma},\underline{z})(f)$ following [50], . By definition (adelic variables in $U(\mathbb{A}), P(\mathbb{A}),$ ...are underlined)

$$\mathcal{F}^{P_1}\circ\ \mathrm{Eis}(\underline{\sigma},\underline{z})(f)(\underline{g}) = \int_{U_{P_1}(\mathbb{Q})\backslash U_{P_1}(\mathbb{A})}\sum_{a\in P(\mathbb{Q})\backslash G(Q)} f_{\underline{z}}(a\underline{u}\underline{g})d\underline{u} \tag{9.155}$$

Let W_M be the Weyl group of M, the Bruhat decomposition yields $G(\mathbb{Q}) = \bigcup_{w\in W} P(\mathbb{Q})\backslash wP_1(\mathbb{Q})$, put $P_1^{(w)}(\mathbb{Q}) = w^{-1}P(\mathbb{Q})w\cap P_1(\mathbb{Q})$ then our expression becomes (we pull the summation over W to the front)

$$\mathcal{F}^{P_1}\circ\ \mathrm{Eis}(\underline{\sigma},\underline{z})(f)(\underline{g}) = \sum_{W_{M_1}\backslash W/W_M}\int_{U_{P_1}(\mathbb{Q})\backslash U_{P_1}(\mathbb{A})}\sum_{b\in P_1^{(w)}(\mathbb{Q})\backslash P_1(\mathbb{Q})} f_{\underline{z}}(wb\underline{u}\underline{g})d\underline{u} \tag{9.156}$$

where W_M is the Weyl group of M. If now for a given w the intersection of algebraic groups $w^{-1}U_1w\cap M = V$ has dimension > 0, then this intersection is the unipotent radical of a proper parabolic subgroup of M. Since σ is cuspidal the integral over $V(\mathbb{Q})\backslash V(\mathbb{A})$ is zero, therefore this w contributes by zero. Hence we can restrict our summation over those $w\in W$ which satisfy $wMw^{-1} = M_1$. let us call this set W^{M,M_1}. But then

$$P_1^{(w)}(\mathbb{Q})\backslash P_1(\mathbb{Q}) = w^{-1}U_P(\mathbb{Q})w\cap U_{P_1}(\mathbb{Q})\backslash U_{P_1}(\mathbb{Q})$$

and the above expression becomes

$$\mathcal{F}^{P_1}\circ\ \mathrm{Eis}(\underline{\sigma},\underline{z})(f)(\underline{g}) =$$

$$\textstyle\sum_{W_M\backslash W^{M,M_1}/W_M}\int_{U_{P_1}(\mathbb{Q})\backslash U_{P_1}(\mathbb{A})}\sum_{v\in U_{P_1}^{(w)}(\mathbb{Q})\backslash U_{P_1}(\mathbb{Q})} f_{\underline{z}}(wv\underline{u}\underline{g})d\underline{u} = \tag{9.157}$$

$$\textstyle\sum_{W_{M_1}\backslash W^{M,M_1}/W^M} / \int_{(w^{-1}U_Pw\cap U_{P_1}\backslash U_{P_1})(\mathbb{A})} f_{\underline{z}}(w\underline{u}\underline{g})d\underline{u}$$

Our parabolic subgroup P contains the standard Borel subgroup, let U_P^- be the unipotent radical of the opposite group. In the argument of $f_{\underline{z}}$ we conjugate by w, then $U_P\cap wU_{P_1}w^{-1}\ \backslash wU_{P_1}w^{-1} = wU_{P_1}w^{-1}\cap U_P^- = U_{P,P_1}^{-,w}$.

$$\mathcal{F}^{P_1}\circ\ \mathrm{Eis}(\underline{\sigma},\underline{z})(f)(\underline{g}) = \sum_{W_{M_1}\backslash W^{M,M_1}/W_M}\int_{U_{P,P_1}^{-,w}(\mathbb{A})} f_{\underline{z}}(\underline{u}w\underline{g})d\underline{u} \tag{9.158}$$

We pick a $w \in W^{M,M_1}$ the group M acts by the adjoint action on $w^{-1}U^{-,w}_{P,P_1}w$ and hence by a character $\delta^{(w)}_{P.P_1}$ on the highest exterior power of the Lie-algebra of this group. Then this operator sends

$$\mathcal{F}^{P_1,w} \circ \ \mathrm{Eis}(\underline{\sigma},\underline{z}) : I_P^G H_{\underline{\sigma}} \otimes |\gamma_P|^{\underline{z}} \to I_{P_1}^G H_{\underline{\sigma}^{w^{-1}}} \otimes (|\gamma_P|^{\underline{z}})^{w^{-1}} |\delta^{(w)}_{P.P_1}| \tag{9.159}$$

The integral is a product of local integrals over all places, we may assume that $f_{\underline{z}} = f_{\infty,\underline{z}} \prod_{p:prime} f_{p,\underline{z}}$. and then

$$\int_{U^{-,w}_{P,P_1}(\mathbb{A})} f_{\underline{z}}(\underline{u}w\underline{g})d\underline{u} = \int_{U^{-,w}_{P,P_1}(\mathbb{R})} f_{\infty,\underline{z}}(u_\infty w g_\infty) \prod_p \int_{U^{-,w}_{P,P_1}(\mathbb{Q}_p)} f_{p,\underline{z}}(u_p w g_p) \tag{9.160}$$

and for $bv = \infty$ or $\mathbf{v} = p$ here the local integrals yield intertwining operators

$$T_v^{P,P_1,w}(\sigma_{\mathbf{v}},\underline{z}) : I_P^G H_{\underline{\sigma}_{\mathbf{v}}} \otimes |\gamma_P|^{\underline{z}}_{\mathbf{v}} \to I_{P_1}^G H_{\underline{\sigma}^{w^{-1}}_{\mathbf{v}}} \otimes |\gamma_P|^{w^{-1}\underline{z}}_{\mathbf{v}} \otimes |\delta^{(w)}_{P.P_1}|_{\mathbf{v}} \tag{9.161}$$

Proposition 9.4.2. *There are local intertwining operators*

$$T_{\mathbf{v}}^{P,P_1,w,\mathrm{loc}}(\sigma_{\mathbf{v}},\underline{z}) : I_P^G H_{\underline{\sigma}_{\mathbf{v}}} \otimes |\gamma_P|^{\underline{z}}_{v} \to I_{P_1}^G H_{\underline{\sigma}^{w^{-1}}_{\mathbf{v}}} \otimes |\gamma_P|^{w^{-1}\underline{z}}_{\mathbf{v}} \otimes |\delta^{(w)}_{P.P_1}|_{\mathbf{v}} \tag{9.162}$$

which have the following properties

a) They are holomorphic and nowhere zero in $\Re z_i \geq 0$ (we are still assuming that $\tilde{\mu}$ is in the negative chamber.)

b) They have a certain rationality property ([48], for the case of finite places 7.3.2.1, for the infinite places 8.4.5 and Chapter 9 (Weselmann.)

c) At the unramified primes $\mathbf{v} = p$ they map the spherical vector to the spherical vector.

Finally we have

$$\mathcal{F}^{P_1,w} \circ \ Eis(\underline{\sigma},\underline{z}) = C(w,P,P_1,\underline{\sigma},\underline{z})\, T_\infty^{P,P_1,w,\mathrm{loc}}(\sigma_\infty,\underline{z}) \otimes \bigotimes_{p:\mathrm{primes}}{}' \ T_p^{P,P_1,w,\mathrm{loc}}(\sigma_p,\underline{z}) \tag{9.163}$$

where $C(w,P,P_1,\underline{\sigma},\underline{z})$ is a meromorphic function in the variable $\underline{z}$. Therefore these functions $C(w,P,P_1,\underline{\sigma},\underline{z})$ decide whether $Eis(\underline{\sigma},\underline{z})$ is holomorphic at $\underline{z} = 0$, the poles of $Eis(\underline{\sigma},\underline{z})$ at $\underline{z}$ are the poles of the $C(w,P,P_1,\underline{\sigma},\underline{z})$.

We compute these factors $C(w,P,P_1,\underline{\sigma},\underline{z})$. By definition the group $U^{-,w}_{P,P_1}$ is a subgroup of U_P^- and as such it it easy to describe. Recall that our our group M is $\mathrm{Gl}_{n_1} \times \cdots \times \mathrm{Gl}_{n_r}$ and this corresponds to a decomposition of $\mathbb{Q}^n = X_1 \oplus X_2 \oplus \cdots \oplus X_r$ into subspaces and for any two indices $1 \leq i < j \leq r$ we define $G_{i,j}$ to be the subgroup $\mathrm{Gl}(X_i \oplus X_j)$ acting trivially on all other summands. For all pairs i,j we define the cocharacters $\chi_{i,j} : \mathbb{G}_m \to T$ where $\chi_{i,j}(t)$ is the diagonal matrix having t as entry at place i, and t^{-1} at place j and 1 everywhere else. We define $\mathbf{w}_{i,\mathrm{j}} :=< \chi_{i,j}, \tilde{\mu}^{(1)} >$.

The intersection $G_{i,j} \cap U^{-,w}_{P,P_1}$ is either trivial or it is the full left lower block unipotent group $U^-_{i,i+1}$.

This tells us that the above integral can be written as iterated integral over subgroups of the form $U_{\nu,\mu}(\mathbb{A})$. To be more precise: If $U^{-,w}_{P,P_1} \neq 1$ then we find an index i such that $U_{i,i+1}$ is not trivial. In a first step we compute the local integral $\int_{U_{i,i+1}(\mathbb{Q}_p)} f^{(0)}_{p,\underline{z}}(u_p w g_p) du_p$ at finite places where our representation $\underline{\sigma}_p$ unramified. We are basically in the situation, that our parabolic subgroup is maximal. The group $P' = P \cap G_{i,i+1}$ contains the standard Borel subgroup, $P'_1 = P_1 \cap G_{ii,i+1}$ is the opposite and $w = e$. Then

$$C_p(e, P', P'_1, \underline{\sigma}, \underline{z}) = \frac{L^{\mathrm{coh}}(\sigma_{i,p} \times \sigma^{\vee}_{i+1,p}, \frac{\mathbf{w}_{i,i+1}}{2} + b_i(w,\lambda) + < \chi_{i,i+1}, \underline{z} > -1)}{L^{\mathrm{coh}}(\sigma_{i,p} \times \sigma^{\vee}_{i+1,p}, \frac{\mathbf{w}_{i,i+1}}{2} + b_i(w,\lambda) + < \chi_{i,i+1}, \underline{z} >)} \tag{9.164}$$

A standard argument (See Langlands, Kim, Shahidi) tells us that we can reduce the computation of the iterated integral to situations like the one above and then we get at unramified places

$$C_p(w, P, P_1, \underline{\sigma}, \underline{z}) = \prod_{i,j} \frac{L^{\mathrm{coh}}(\sigma_{i,p} \times \sigma^{\vee}_{j,p}, \frac{\mathbf{w}_{i,j}}{2} + b_{i,j}(w,\lambda) + < \chi_{i,j}, \underline{z} > -1)}{L^{\mathrm{coh}}(\sigma_{i,p} \times \sigma^{\vee}_{j,p}, \frac{\mathbf{w}_{i,j}}{2} + b_{i,j}(w,\lambda) + < \chi_{i,j}, \underline{z} >} \tag{9.165}$$

Here the indices i, j run over those indices for which $U_{i,j} \subset U^{-,w}_{P,P_1}$, and $b_{i,j}(w,\lambda) =< \chi_{i,j}, \underline{\mu}^{\mathrm{ab}} >$. At ramified primes p we also have a definition of the local Euler-factors $L^{\mathrm{coh}}(\sigma_{i,p} \times \sigma^{\vee}_{j,p}, z)$ (Shahidi's book) and hence we can define $C_p(w, P, P_1, \underline{\sigma}, \underline{z})$ by the same expression.

At the infinite place the Rankin-Selberg local Euler factors are also defined. We introduce the modified Γ- function $\Gamma_{\mathbb{C}}(z) = \frac{2}{(2\pi)^z}\Gamma(z)$. Then

$$L^{\mathrm{coh}}(\sigma_{i,\infty} \times \sigma^{\vee}_{j,\infty}, z) = \prod_{\nu} \Gamma_{\mathbb{C}}(z + p_\nu)$$

where the p_ν are (half)-integers which are computed from the coefficients of $w \cdot \lambda$ by mysterious rules. ([48].)

Now we define $C_v(w, P, P_1, \underline{\sigma}, \underline{z})$ for all places v by the above expression, where we express the cohomological L factor by the automorphic Rankin-Selberg L factor with the shift in the variable s. We go back to equation (9.163) and define

$$C(w, P, P_1, \underline{\sigma}, \underline{z}) = \prod_{\mathbf{v}} C_{\mathbf{v}}(w, P, P_1, \underline{\sigma}, \underline{z}). \tag{9.166}$$

We from the above proposition (9.4.2) that the factors $C(w, P, P_1, \underline{\sigma}, \underline{z})$ determine the analytic behavior of $Eis(\underline{\sigma}, \underline{z})$ at $\underline{z} = 0$. We have to exploit the analytic properties of the Rankin-Selberg L-functions. Here we have to use Shahidi's theorem which yields -(always remember that $\tilde{\mu}$ is in the negative chamber-)

$$\begin{gathered} \Lambda^{\mathrm{coh}}(\sigma \times \sigma^{\vee}, \tfrac{\mathbf{w}_{i,j}}{2} + b_{i,j}(w,\lambda) + < \chi_{i,j}, \underline{z} > -1) = \\ \prod_{\mathbf{v}} L^{\mathrm{coh}}(\sigma_{i,\mathbf{v}} \times \sigma^{\vee}_{j,\mathbf{v}}, \tfrac{\mathbf{w}_{i,j}}{2} + b_{i,j}(w,\lambda) + < \chi_{i,j}, \underline{z} > -1) \end{gathered} \tag{9.167}$$

is holomorphic at $\underline{z} = 0$ unless we are in the following special case:

a) In the product in formula (9.165) we have $j = i+1$ and where $n_i = n_{i+1}, \mu_i^{(1)} = \mu_{i+1}^{(1)}$ and $b_i(w,\lambda) = 1$.

b) The pair $\sigma_i \times \sigma_{i+1}$ is a segment, this means that $\sigma_i \otimes \det_i = \sigma_{i+1}$

If these two conditions are fulfilled then $C(w, P, P_1, \underline{\sigma}, \underline{z})$ has first order pole along $z_i = 0$.

The denominator is always holomorphic and never zero at $\underline{z} = 0$. (This is a deep theorem: it is the prime number theorem for Rankin-Selberg L-functions.)

9.4.5 Rationality of L-values and some questions

We see that we get an abundant supply of cohomology classes: Starting from any parabolic P and an isotypical subspace $\mathrm{Ind}_P^G H^{\bullet - l(w)}_{\mathrm{cusp}}(\mathcal{S}^M_{K_f^M}, \tilde{\mathcal{M}}_{w\cdot\lambda})(\underline{\sigma}_f)$ we get the Eisenstein intertwining operator (See equation (9.153)). We analyse what happens at $\underline{z} = 0$. If it is holomorphic we get a Hecke invariant homomorphism

$$\mathrm{Eis}^\bullet(0) : H^\bullet(\mathfrak{g}, K_\infty,\ \mathrm{Ind}_P^G \sigma_\infty \otimes \tilde{\mathcal{M}}) \otimes\ \mathrm{Ind}_P^G H_{\underline{\sigma}_f} \to H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\mathbb{C}}). \quad (9.168)$$

We can restrict these cohomology classes to the boundary and even to boundary strata $\partial_Q(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})$ where Q runs over the parabolic subgroups associate to P, or more generally those parabolic subgroups which contain an associate to P. This means that the class "spreads out" over different boundary strata These restrictions to these other strata are given by certain linear maps which are product of "local intertwining operators" times certain special values of L functions.

In certain cases this "spreading out" is highly non trivial. We have to clarify some local issues. First of all we have to find out whether the local intertwining operators are non zero and have certain rationality properties. Especially we have to show that these local operators at the infinite places induce non zero maps between the cohomology groups of certain induced Harish-Chandra modules. And we have to show that these maps on the level of cohomology have rationality properties. ([48],7.3,)

If these local issues are settled then we can argue: The image of the cohomology $H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}})$ in the cohomology of the boundary is defined over $\mathbb{Q}$ (or some number field depending on our data). Since the $L-$ values enter in the description of this image we get rationality statements for special values of L-functions.

This has been exploited in some cases ([35], [37], [44]]) and a very general result in this direction is in [48](see previous section).

But in case we have a pole we may also produce cohomology classes by taking residues, again starting from one boundary stratum. The restriction of these classes to the boundary will spread out over other strata in the boundary and we may play the same game. In this case the non vanishing issue of intertwining operators on cohomological level comes up again and will be discussed in the following section. (see Thm. **??**)

We also will encounter situations where a pole along a plane $z_i = 0$ (or may be even several such planes) "fights" with a zero along some other planes containing zero. Then this influences the structure of the cohomology. But how? This question has been discussed in [37]. Is the order of vanishing along this zero visible in the structure of the cohomology? Or is it visible in the structure of the cohomology of the boundary, or in the spectral sequence?

9.5 Residual classes

We have seen that our Eisenstein classes may be singular at $\underline{z} = 0$. In this section we look at the extremal case that $\mathrm{Eis}(\sigma, \underline{z})$ has simple poles along the lines $z_i =< \chi_{n_i,n_i+1}, \underline{z} >= 0$, In this case we call these Eisenstein classes residual. It follows from the work of Moeglin-Waldspurger[76] that this can only happen under some very special conditions.

We start from a factorization $n = uv$ we look the parabolic subgroup $P_{u,v}$ which contains the standard Borel subgroup and has reductive quotient $\mathrm{Gl}_u \times \mathrm{Gl}_u \times \cdots \times \mathrm{Gl}_u$. The standard maximal torus is a product $T = \prod_{i=1}^{i=v} T_i$ and accordingly we have $X^*(T) = \bigoplus_{i=1}^{i=v} X^*(T_i)$. We have an obvious identification $T_i = \mathbb{G}_m^u$.

We choose a highest weight $\lambda = \sum a_i\gamma_i + d\delta$, we assume that it is self dual, i.e. $a_i = a_{n-i}$. We have a restriction on the character $\mu = w\cdot\lambda = w(\lambda+\rho_N)-\rho_N$, we must have

$$\begin{aligned} w(\lambda + \rho_N) - \rho_N = b_1\gamma_1^M + b_2\gamma_2^M + \cdots + b_{u-1}\gamma_{u-1}^M - d_0\mathrm{det}^{(1)} \\ +b_1\gamma_{1+u}^M + b_2\gamma_{2+u}^M + \cdots + b_{u-1}\gamma_{2u-1}^M - (d_0+1)\mathrm{det}^{(2)} + \dots \\ \dots \\ b_1\gamma_{(v-1)u+1}^M + b_2\gamma_2^M + \cdots + b_{u-1}\gamma_{vu-1}^M - (d_0+v-1)\mathrm{det}^{(v)} \end{aligned} \tag{9.169}$$

where $\mathrm{det}^{(\nu)}$ is the determinant on the ν-th block. In other words our highest weight is a sum $\mu = \sum \mu_i$ where

$$\mu_i = \mu^{(1)} - (d_0 + i - 1)\mathrm{det}^{(i)} \tag{9.170}$$

where the semi simple component $\mu^{(1)} = b_1\gamma_1^M + b_2\gamma_2^M + \cdots + b_{u-1}\gamma_{u-1}^M = b_1\gamma_{1+u}^M + b_2\gamma_{2+u}^M + \cdots + b_{u-1}\gamma_{2u-1}^M \dots$ is "always the same". We notice that of course we have the self duality condition $b_i = b_{u-i}$. Furthermore we have $\sum d_i = -d$.

We define

$$\mathbb{D}_\mu = \bigotimes_{i=1}^{i=v} \mathbb{D}_{\mu_i} \tag{9.171}$$

and start from our isotypical $H^\bullet_{\mathrm{cusp}}(\mathcal{S}^M_{K_f^M}, \mathcal{M}_{w\cdot\lambda})(\sigma_f)$. The Künneth formula yields that we can write $\sigma_f = \sigma_{1,f} \times \sigma_{2,f} \times \cdots \times \sigma_{v,f}$ where all the $\sigma_{i,f}$ occur in the cuspidal cohomology of Gl_u, hence they may be compared. The relation (9.170) allows us to require that $\sigma_{i+1,f} = \sigma_{i,f} \otimes |\delta|$. If this is satisfied we say that σ_f is a segment. We assume $v \neq 1$ and hence $P \neq G$.

We know that under the assumption that σ_f is a segment (and only under this assumption) the factor $C(\sigma, w_P, \underline{z})$ has a simple poles along the lines $z_i = 0$, and this is the only term in (??) having these poles. The operator $T^{\mathrm{loc}}(\sigma, \underline{s})$ is a product of local operators at all places

$$T^{\mathrm{loc}}(\sigma, \underline{z}) = T^{\mathrm{loc}}_{\infty}(\sigma_\infty, \underline{s}) \times \prod_p T^{\mathrm{loc}}_p(\sigma_p, \underline{z}),$$

and the local factors are holomorphic as long as $\Re(z_i) \geq 0$. We take the residue at $\underline{z} = 0$ i.e. we evaluate

$$(\prod z_i)\mathcal{F}^P \circ \ \mathrm{Eis}(\sigma \otimes \underline{s})|_{\underline{z}=0} = (\prod z_i) C(\sigma, w_P, \underline{z})|_{\underline{z}=0} T^{\mathrm{loc}}(\sigma, w_P, \underline{0})(f) \quad (9.172)$$

This tells us that the residue of the Eisenstein class gives us an intertwining operator

$$\mathrm{Res}_{\underline{z}=0} \ \mathrm{Eis}(\sigma \otimes \underline{z}) : {}^{\mathrm{a}}\mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})} \mathbb{D}_\mu \otimes V_{\sigma_f} \to L^2_{\mathrm{disc}}(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f, \omega^{-1}_{\mathcal{M}_\lambda}|_{S(\mathbb{R})^0}) \quad (9.173)$$

The image $J_{\sigma_\infty} \otimes J_{\sigma_f}$ is an irreducible module (this is a Langlands quotient) and via the constant Fourier coefficient it injects into ${}^{\mathrm{a}}\mathrm{Ind}_{P(\mathbb{A}))}^{G(\mathbb{A})} \mathbb{D}_{\mu'} \otimes V_{\sigma_f}$. At the infinite place we get a diagram

$$\begin{array}{ccc} \mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})} \mathbb{D}_\mu & \stackrel{T^{(\mathrm{loc})}(D_\mu)}{\longrightarrow} & J_{\sigma_\infty} \\ & & \downarrow \\ & & \mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})} \mathbb{D}_{\mu'} \end{array} \quad (9.174)$$

It is a - not completely trivial - exercise to write down the solutions for the system of equations (9.169). This means starting from our highest weight μ we have to find w, λ. The answer is that λ must be of the special form

$$\lambda = a_1\gamma_v + a_2\gamma_{2v} + \cdots + a_{u-1}\gamma_{(u-1)v} + d\delta \quad (9.175)$$

which in addition is essentially self dual, i.e. $a_i = a_{v-i}$ the number d is uninteresting and only serves to satisfy the parity condition.

We choose a specific Kostant representative $w'_{u,v} \in W^P$, it is the permutation in the letters $1, 2, \ldots, n$ given by the following rule: write $\nu = i + (j-1)v$ with $1 \leq i \leq v$ then $w'_{u,v}(\nu) = j + (i-1)v$. Then we compute $w'_{u,v}(\lambda + \rho_N) - \rho_N \in X^*(T \times E)$ and we get

$$\begin{gathered} (w'_{u,v}(\lambda+\rho_N) - \rho_N) = \\ (a_1+v-1)\gamma_1^M + (a_2+v-1)\gamma_2^M + (a_{u-1}+v-1)\gamma_{u-1}^M \\ (a_1+v-1)\gamma_{1+u}^M + (a_2+v-1)\gamma_{2+u}^M + (a_{u-1}+v-1)\gamma_{u-1+u}^M \\ \vdots \\ (a_1+v-1)\gamma_{1+(v-1)u}^M + (a_2+v-1)\gamma_{2+(v-1)u}^M + \cdots + (a_{u-1}+v-1)\gamma_{u-1+(v-1)u}^M)+ \\ -(u-1)(\gamma_u + \gamma_{2u} + \cdots + \gamma_{(v-1)u}) + d\delta \end{gathered} \quad (9.176)$$

The length of this Kostant representative is

$$l(w'_{u,v}) = n(u-1)(v-1)/4.$$

Let w_P be the longest Kostant representative which sends all the roots in U_P to negative roots. Then we define the (reflected) Kostant representative $w_{u,v} = w_P w'_{u,v}$. We get

$$\begin{aligned} w_{u,v}(\lambda+\rho)-\rho = \mu = (a_1+v-1)(\gamma_1^M+\gamma_{1+u}^M+\cdots+\gamma_{1+(v-1)u}^M)+ \\ (a_2+v-1)(\gamma_2^M+\gamma_{2+u}^M+\cdots+\gamma_{2+(v-1)u}^M)+ \\ \vdots \\ (a_{u-1}+v-1)(\gamma_{u-1}^M+\gamma_{u-1+u}^M+\cdots+\gamma_{u-1+(v-1)u}^M)+ \\ -(u+1)(\gamma_u+\gamma_{2u}+\cdots+\gamma_{(v-1)u})+d\delta. \end{aligned} \quad (9.177)$$

Hence we see that we the semi simple component stays the same and the abelian parts differ by $2(\gamma_u+\gamma_{2u}+\cdots+\gamma_{(v-1)u}))$ We see that we can solve (9.169) provided $b_i \geq v-1$.

The identification $J_{\sigma_\infty} \xrightarrow{\sim} A_{\mathfrak{q}}(\lambda))$

Of course we expect

$$H^\bullet(\mathfrak{g}, K_\infty, J_{\sigma_\infty} \otimes \mathcal{M}_\lambda) \neq 0. \quad (9.178)$$

In the paper [108] the authors give a list of irreducible $(\mathfrak{g}, K_\infty)$ modules $A_{\mathfrak{q}}(\lambda)$ which have non trivial cohomology $H^\bullet(\mathfrak{g}, K_\infty, A_{\mathfrak{q}}(\lambda) \otimes \mathcal{M}_\lambda) \neq 0$. This list contains all unitary modules having this property. On the other hand we know that any such unitary $A_{\mathfrak{q}}(\lambda)$ can be written as a Langlands quotient. In the paper of Vogan and Zuckerman it is explained how we can get a given unitary $A_{\mathfrak{q}}(\lambda)$ as Langlands quotient, basically this means we construct a diagram of the form (9.174) but where now we have $A_{\mathfrak{q}}(\lambda)$ in the upper right corner instead of J_{σ_∞}. In the following section we describe a specific $A_{\mathfrak{q}}(\lambda)$ and write it as a Langlands quotient (i.e. we find its Langlands parameters) this means we determine the upper left and lower right entries and then check that these entries are the ones in diagram (9.174). From this we will derive the following

The map

$$H^\bullet(\mathfrak{g}, K_\infty, J_{\sigma_\infty} \otimes \mathcal{M}_\lambda) \otimes J_{\sigma_f} \to H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) \quad (9.179)$$

is non zero in degree $l(w'_{u,v}) = n(u-1)(v-1)/4$.

See Theorem (**??**)

Attaching motives to σ_f???

The condition (*NUQuot*)) will be true if λ is sufficiently regular but for non regular weights it fails. If the weight is not regular then we may have a pole of the Eisenstein series at $z=0$. This possibility has to be discussed, it can only happen if we have *(UQuot)*. But even if we have *(UQuot)* we may not have a pole.

Let us assume that we have *(UQuot)* and the Eisenstein operator is holomorphic at $z=0$. Then we may have several copies of $J(\sigma_f)$ in $H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{C})$. This defines again an isotypical submodule $H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes F)(\bar{\sigma_f})$. We get an exact sequence

$$0 \to H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes F)(\bar{\sigma_f}) \to \mathcal{X}(\sigma_f) \to J(\sigma_f) \to 0 \tag{9.180}$$

This is a sequence of Hecke-modules over F, the section (9.69) provides a section over $\mathbb{C}$.

If our locally symmetric space $\mathcal{S}^G_{K_f}$ the set of complex points of a Shimura variety then we can interpret this sequence as a mixed motive. This motive has an extension class in the category of mixed Hodge-structures

$$[\mathcal{X}(\sigma_f)]_{B-dRh} \in \mathrm{Ext}^1_{B-dRh}(J(\sigma_f), H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes F)(\bar{\sigma_f})) \tag{9.181}$$

and in some cases we can compute this class (we have to look at a suitable bi-extension) and express it in terms of the second term in the constant term (See [MixMot-2013.pdf].)

We have seen that in many situations the space $\mathcal{S}^M_{K^M_f}$ is not the set of complex points of a Shimura variety and therefore we do not know how to attach a motive or an ℓ adic Galois representation to it. (Sometimes we know how to attach a motive to it but it is simply a Tate motive). But if it happens that the module $J(\sigma_f)$ produces a non trivial submodule $H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes F)(\bar{\sigma_f}))$ then the situation changes and we can attach a Galois-module $H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes F_\lambda)(\bar{\sigma_f}))$ to it which contains a lot of information about σ_f. Again we refer to ([MixMot-2013.pdf].) We have seen in [36] (3.1.4.) that this can happen.

The motivic interpretation of Shahidis theorem

We go back to a general submodule $\sigma_f = \sigma_f^{(1)} \times \sigma_f^{(2)} = \sigma_f \in \mathrm{Coh}(H^\bullet_{\mathrm{cusp}}(\mathcal{S}^M_{K^M_f}, \tilde{\mathcal{M}}_{w\cdot\lambda})$, we drop the assumptions above. We assume that we can attach motives $\mathbb{M}(\sigma_f^{(1)}, r_1), \mathbb{M}(\sigma_f^{(2)}, r_1)$ where r_1 is the tautological representation. (Actually we do not need the motives it suffices to have the compatible systems of $\mathfrak{l}$-adic representations) Then we can attach the Rankin-Selberg motive to this pair

$$\mathbb{M}_{\mathrm{RS}}(\sigma_f, \mathrm{Ad}) = \mathbb{M}(\sigma_f^{(1)}, r_1) \times \mathbb{M}(\sigma_f^{(2)}, r_1)^\vee = \mathrm{Hom}(\mathbb{M}(\sigma_f^{(2)}, r_1), \mathbb{M}(\sigma_f^{(1)}, r_1)) \otimes \mathbb{Z}(-\mathbf{w}(\mu^{(2)}, r_2)) \tag{9.182}$$

Under the assumption of the theorem the we have $\mathbb{M}(\sigma_f^{(1)}, r_1) \xrightarrow{\sim} \mathbb{M}(\sigma_f^{(2)}, r_1)$ and we see that the Galois module $\mathrm{Hom}(\mathbb{M}(\sigma_f^{(2)}, r_1), \mathbb{M}(\sigma_f^{(1)}, r_1))$ contains a copy of $\mathbb{Z}_\mathfrak{l}(0)$ and therefore we get an exact sequence of Galois modules

$$0 \to \mathbb{Z}(-\mathbf{w}(\mu^{(2)}, r_2)) \to \mathbb{M}_{\mathrm{RS}}(\sigma_f, \mathrm{Ad})_{\text{ét},\mathrm{Ad}} \to \mathbb{M}^{(0)}_{\mathrm{RS}}(\sigma_f, \mathrm{Ad})_{\text{ét},\mathrm{Ad}} \to 0$$

Hence the motivic L function is a product

$$L(\mathbb{M}_{\mathrm{RS}}(\sigma_f, \mathrm{Ad})_{\text{ét},\mathrm{Ad}}, s) = L(\mathbb{Z}(-\mathbf{w}(\mu^{(2)}), s) L(\mathbb{M}^{(0)}_{\mathrm{RS}}(\sigma_f, \mathrm{Ad})_{\text{ét},\mathrm{Ad}}, s)$$

If we substitute for s the expression

$$\frac{\mathbf{w}(r_1, \mu_1^{(1)}) + \mathbf{w}(r_2, \mu_2^{(1)})}{2} - b(w, \lambda) + s = \mathbf{w}(r_2, \mu_2^{(1)}) - b(w, \lambda) + s$$

then we find

$$L(\mathbb{M}_{\mathrm{RS}}(\sigma_f,\mathrm{Ad})_{\text{ét},\mathrm{Ad}},s)=\zeta(-b(w,\lambda)+s)L(\mathbb{M}^{(0)}_{\mathrm{RS}}(\sigma_f,\mathrm{Ad})_{\text{ét},\mathrm{Ad}},s)$$

Then the motivic interpretation of Shahidis theorem is, that $L(\mathbb{M}^{(0)}_{\mathrm{RS}}(\sigma_f,\mathrm{Ad})_{\text{ét},\mathrm{Ad}},\mathbf{w}(r_2,\mu_2^{(1)})-b(w,\lambda)+s)$ is holomorphic at $s=0$ and non zero (this is in a sense the prime number theorem for this L function) and therefore - if we have $b(w,\lambda)=-1$- the pole comes from the first order pole of the Riemann -ζ function. If now $\sigma_f^{(1)}\times\sigma_f^{(2)}=\sigma_f$ occurs in the cuspidal cohomology then we have an inclusion

$$\mathbb{D}_\mu\times H_{\sigma_f}\hookrightarrow\mathcal{A}(M(\mathbb{Q})\backslash M(\mathbb{A})/K_f^M)$$

We form the Eisenstein intertwining operator and compose it with constant Fourier coefficient, then we get

$$\mathcal{F}^P\circ\ \mathrm{Eis}(s):f\mapsto f+C(\sigma,s)T^{\mathrm{loc}}(s)(f) \tag{9.183}$$

The operator $T^{\mathrm{loc}}(s)=T^{\mathrm{loc}}_\infty(s)\otimes\bigotimes T^{\mathrm{loc}}_p(s)$ is holomorphic at $s=0$. Under our assumptions the function $C(\sigma,s)$ has a first order pole at $s=0$ and we get a residual intertwining operator

$$\mathrm{Res}_{s=0}:\ \mathrm{Ind}_P^G\mathbb{D}_\mu\times H_{\sigma_f}\otimes(0)\to\mathcal{A}^{(2)}(G(\mathbb{Q})\backslash G(\mathbb{A})/K_f) \tag{9.184}$$

Rationality results

Ist das nicht schon diskutiert?? Finally we want to discuss the case that $P\neq\Theta(P)=Q$. If this happens then $\mathcal{S}^G_{K_f}$ is never a Shimura variety. We have isotypical pieces (see (9.31))

$$H_!^{\bullet-l(w)}(\mathcal{S}^M_{K_f^M},\tilde{\mathcal{M}}(w\cdot\lambda)\otimes F)(\sigma_f)\oplus H_!^{\bullet-l(w')}(\mathcal{S}^{M'}_{K_f^{M'}},\tilde{\mathcal{M}}(w'\cdot\lambda)\otimes F)(\sigma'_f) \tag{9.185}$$

and we know that component of the Eisenstein cohomology consists of the classes

$$\{\psi_f\oplus\mathcal{L}(\sigma_f){T_f}^{\mathrm{loc}}(\psi_f)\} \tag{9.186}$$

where $\mathcal{L}(\sigma_f)$ is an element of F and for all $\iota:F\to\mathbb{C}$ we have

$$\iota(\mathcal{L}(\sigma_f))=\frac{1}{\Omega(\iota\circ\sigma_f)}C(\sigma_\infty,\lambda)C(\iota\circ\sigma_f,\lambda) \tag{9.187}$$

If the factor at infinity $C(\sigma_\infty,\lambda)\neq 0$ then we get from this rationality results for the ratios of L-values. (See [44],[48]) These rationality results will be important when we discuss the arithmetic nature of the numbers in**??**

Combining the results of Borel–Garland [8] and Mœglin–Waldspurger [76] we get that the homomorphism

$$\bigoplus_{u|n}\ \bigoplus_{\sigma_f:\text{segment}} H^\bullet(\mathfrak{g},K_\infty;A_{\mathfrak{q}}(\lambda)\otimes\mathcal{M}_\lambda)\otimes J_{\sigma_f}\to H^\bullet_{(2)}(\mathcal{S}^G_{K_f},\mathcal{M}_\lambda) \tag{9.188}$$

is surjective. This gives us the decomposition into isotypical spaces of $H^\bullet_{(2)}(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda)$. We separate the cuspidal part ($v = 1$) from the residual part and get

$$H^\bullet_{(2)}(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda) = \bigoplus_{\pi_f:\text{cuspidal}} H^\bullet_{\text{cusp}}(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda)(\pi_f) \oplus \bigoplus_{\substack{u|n \\ u<n}} \bigoplus_{\sigma_f:\text{segment}} \overline{H^\bullet(\mathfrak{g}, K_\infty; A_\mathfrak{q}(\lambda) \otimes \mathcal{M}_\lambda)} \otimes J_{\sigma_f},$$

where the bar on top means we have gone to its image via the map in (9.188). It follows from the theorem of Jacquet–Shalika [61] that there are no intertwining operators between the summands.

In the extremal case $u = n, v = 1$ the parabolic subgroup P is all of G and $A_\mathfrak{q}(\lambda) = \mathbb{D}_\lambda$. In this case and only this case the representation $A_\mathfrak{q}(\lambda)$ is tempered, and the lowest degree of nonvanishing cohomology is the number b^F_n. An easy computation shows that in the case $v > 1$ the number $q < b^F_n$. Then our theorem above implies that in degree q

$$H^q(\mathfrak{g}, K_\infty; A_\mathfrak{q}(\lambda) \otimes \mathcal{M}_\lambda) \otimes J_{\sigma_f} \to H^q(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda)$$

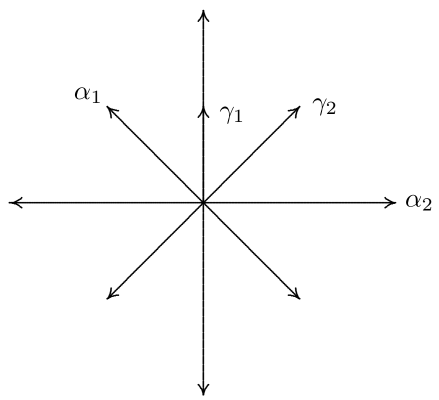
is injective. This has also been proved by Grobner [30]. The above result, which we announced earlier (**??**), can be sharpened as in the following theorem. During the induction argument we use Thm. **??** for the reductive quotients M of the parabolic subgroups.

9.6 Some examples where we expect denominators

We will discuss some specific examples where we can make the ideas alluded to in section (9.3.1) more explicit. In many of these examples the congruences can be verified. The ambient reductive group will be $\mathrm{Sp}_2/\mathbb{Z}$ or $\mathrm{Sp}_3/\mathbb{Z}$.

9.6.1 Some notations and structural data

The roots and weights-diagram for $\mathrm{Sp}_2/\mathbb{Z}$ looks like this.



The maximal torus is

$$T/\mathbb{Z} = t = \{ \begin{pmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 & t_1^{-1} \end{pmatrix} \}$$

Let $B\mathbb{Z} \supset T/\mathbb{Z}$ be the Borel subgroup hose positive simple roots are

$$\alpha_1(t) = t_1/t_2, \alpha_2(t) = t_2^2.$$

The fundamental dominant weights are

$$\gamma_1(t) = t_1, \gamma_2(t) = t_1 t_2.$$

We are mainly interested in the Siegel parabolic subgroup $P \supset B$, its reductive quotient $M = P/U$ has the roots $\alpha_2, -\alpha_2$. The fundamental weight for M is

$$2\gamma_1^M = t_1/t_2$$

We choose a highest weight $\lambda = n_1\gamma_1 + n_2\gamma_2$, we assume $n_1 \equiv 0 \mod 2$, let $\mathcal{M}_\lambda$ be a resulting module for $G/\operatorname{Spec}(\mathbb{Z})$. We get the following list of Kostant representatives for the Siegel parabolic subgroup P and they provide the following list of weights.

$$\begin{aligned} 1 \cdot \lambda &= \lambda = \tfrac{1}{2}(2n_2 + n_1)\gamma_2 + n_1\gamma_1^M \\ s_2 \cdot \lambda &= \tfrac{1}{2}(-2 + n_1)\gamma_2 + (2n_2 + n_1 + 2)\gamma_1^M \\ s_2 s_1 \cdot \lambda &= \tfrac{1}{2}(-4 - n_1)\gamma_2 + (2n_2 + n_1 + 2)\gamma_1^M \\ s_2 s_1 s_2 \cdot \lambda &= \tfrac{1}{2}(-6 - 2n_2 - n_1)\gamma_2 + n_1\gamma_1^M, \end{aligned}$$

We choose for $K_\infty \subset \mathrm{Sp}_2(\mathbb{R})$ the standard maximal compact subgroup $U(2)$, it is the centraliser of the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

which defines a complex structure. We consider the cohomology $H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$ to simplify the exposition we assume that $K_f = \mathrm{Sp}_2(\hat{\mathbb{Z}})$.

9.6.2 The cuspidal cohomology of the Siegel-stratum

We consider the fundamental exact sequence. Inside the cohomology $H^\bullet(\partial(\mathcal{S}^G_{K_f}), \tilde{\mathcal{M}}_\lambda)$ we have the strongly inner part $H^\bullet_{!!}\partial_P(\mathcal{S}^G_{K_f}), \tilde{\mathcal{M}}_\lambda)$ where we inverted a controlled finite set of primes. If we invert a certain controlled finite set of primes S then

$$H^\bullet_{!!}(\partial_P(\mathcal{S}^G_{K_f}), \tilde{\mathcal{M}}_\lambda \otimes \mathbb{Z}_S) = \bigoplus_{w \in W^P} H^{\bullet - l(w)}_{!!}(\mathcal{S}^M_{K_f^M}, H^{l(w)}(\mathfrak{u}_P, \mathcal{M}_\lambda \otimes \mathbb{Z}_S)(w \cdot \lambda) \tag{9.189}$$

With respect to the Hecke-module structure this module will be a complete direct summand in the cohomology of the boundary, provided S is chosen properly, We have to avoid the inner congruences and the denominators of the Eisenstein classes. (See Chap. V)

We look at the special Kostant representant $w = s_2 s_1$ in this case we know how to describe the corresponding summand in terms of automorphic forms on Gl_2. We introduce the usual abbreviation $H^{l(w)}(\mathfrak{u}_P, \mathcal{M}_\lambda \otimes \mathbb{Z}_S) = \mathcal{M}_\lambda(w \cdot \lambda) \otimes \mathbb{Z}_S$. Our coefficient modules are the modules attached to the highest weight

$$\mu = w \cdot \lambda = (2 + 2n_2 + n_1)\gamma_1^{M_1} + \frac{1}{2}(-4 - n_1)\gamma_2 \tag{9.190}$$

Let us put $k = 4 + 2n_2 + n_1$ and $m = \frac{1}{2}n_1$. It will be of great importance that we can vary λ and still get the same value of k, i.e. the same semi simple component and m varies from 0 to $k - 2$ (See 9.24).

The relative period

Let us look at the space $\mathcal{S}^M_{K^M_f}$. The group $M/\operatorname{Spec}(\mathbb{Z})$ is isomorphic to Gl_2, it is the Levi-quotient of the Siegel parabolic. The group K^M_∞ is the image of $P(\mathbb{R}) \cap K_\infty$ under the projection $P(\mathbb{R}) \to M(\mathbb{R})$. This is the group $\mathbb{O}(2)$ its connected component of the identity is $K^M_\infty(1) = \mathrm{SO}(2)$ as a subgroup of index 2. Hence we get a covering of degree 2

$$\mathcal{S}^{\tilde{M}}_{K^M_f} = M(\mathbb{Q})\backslash M(\mathbb{R})/K^M_\infty(1) \times M(\mathbb{A}_f)/K^M_f \to \mathcal{S}^M_{K^M_f} \tag{9.191}$$

and an inclusion

$$i : H^1(\mathcal{S}^M_{K^M_f}, \mathcal{M}_\lambda(w \cdot \lambda)) \hookrightarrow H^1(\mathcal{S}^{\tilde{M}}_{K^M_f}, \mathcal{M}_\lambda(w \cdot \lambda)). \tag{9.192}$$

On the cohomology on the right we have the action of $\mathbb{O}(2)/\mathrm{SO}(2) = \mathbb{Z}/2\mathbb{Z}$ and the cohomology decomposes into a $+$ and a $-$ eigenspace. The inclusion i provides an isomorphism of the left hand side and the $+$ eigenspace.

This inclusion is of course compatible with the action of the Hecke algebra. If we pass to a suitable extension $F/\mathbb{Q}$ we get the decompositions into isotypic subspaces if we tensor our coefficient system by F. An isomorphism type σ_f occurs with multiplicity one on the left hand side and with multiplicity two on the right hand side. Over the ring $\mathcal{O}_{F,S}$ the modules $H^1_\pm(\mathcal{S}^M_{K^M_f}, \mathcal{M}_\lambda(w \cdot \lambda)_{\mathcal{O}_{F,S}})(\sigma_f)$ are of rank one, hence we can find locally in the base $\operatorname{Spec}(\mathcal{O}_F)$ an isomorphism

$$T^{\mathrm{arith}}(\sigma_f) : H^1_+(\mathcal{S}^{\tilde{M}}_{K^M_f}, \mathcal{M}_\lambda(w \cdot \lambda)_{\mathcal{O}_{F,S}}))(\sigma_f) \xrightarrow{\sim} H^1_-(\mathcal{S}^{\tilde{M}}_{K^M_f}, \mathcal{M}_\lambda(w \cdot \lambda)_{\mathcal{O}_{F,S}}))(\sigma_f) \tag{9.193}$$

this isomorphism is unique up to an element in $\mathcal{O}^\times_{F,S}$.

We have to understand how this period varies if we twist by a power of the determinant, i.e. by a multiple of γ_2. We recall the isomorphism (see(7.26))

$$c_2 : H^1(\mathcal{S}^{\tilde{M}}_{K^M_f}, \tilde{\mathcal{M}}_\mu) \xrightarrow{\cup e_{\gamma_2}} H^1(\mathcal{S}^{\tilde{M}}_{K^M_f}, \tilde{\mathcal{M}}_{\mu+\gamma_2})$$

this isomorphism is compatible with the action of the Hecke-algebra but it interchanges the + and the − eigenspace. Hence we can arrange our arithmetic intertwining operator such that it satisfies

$$T^{\text{arith}}(\sigma_f \otimes \gamma_{2,f})^{-1} = c_2 T^{\text{arith}}(\sigma_f) c_2^{-1} \tag{9.194}$$

Now we consider the transcendental description of the cohomology groups

$$H^1(\mathcal{S}^{\tilde{M}}_{K_f^M}, \mathcal{M}_\lambda(w\cdot\lambda)_{\mathbb{C}}) = \bigoplus_{\sigma_f} H^1_+(\mathcal{S}^{\tilde{M}}_{K_f^M}, \mathcal{M}_\lambda(w\cdot\lambda)_{\mathbb{C}})(\sigma_f) \oplus H^1_-(\mathcal{S}^{\tilde{M}}_{K_f^M}, \mathcal{M}_\lambda(w\cdot\lambda)_{\mathbb{C}})(\sigma_f) \tag{9.195}$$

We consider the character $\mu_{\mathbb{R}} : B(\mathbb{R}) \to \mathbb{C}^\times$ which yields the Harish-Chandra module $\mathfrak{I}_B^G \mu_{\mathbb{R}}$ which contains contains the sum of the two discrete representations $\mathcal{D}_{\mu_{\mathbb{R}}} \supset \mathcal{D}^+_{\mu_{\mathbb{R}}} \oplus \mathcal{D}^-_{\mu_{\mathbb{R}}}$. We have the decomposition

$$\mathcal{D}_{\mu_{\mathbb{R}}} = \bigoplus_{\nu\equiv 0(2), |\nu|\geq k} F\phi_{\mu,\nu}, \mathcal{D}^+_{\mu_{\mathbb{R}}} = \bigoplus_{\nu\equiv 0(2), \nu\geq k} F\phi_{\mu,\nu}$$

where

$$\phi_{\mu,\nu}(g) = \phi_{\mu,\nu}(b\begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi \end{pmatrix}) = \mu_{\mathbb{R}}(b) e^{2\pi i\nu\phi}.$$

Of course $K_\infty^{M,0} = T_1(\mathbb{R}) = \{e(\phi) = \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi \end{pmatrix}\}$ and we can write $e(\phi)^\nu = e^{2\pi\nu i\phi}$. We have the well known formula for the $(\mathfrak{m}, K_\infty^{M,0}-)$ cohomology

$$H^1((\mathfrak{m}, K_\infty^{M,0}), \mathcal{D}_{\mu_{\mathbb{R}}} \otimes \mathcal{M}_\lambda(w\cdot\lambda)) = \operatorname{Hom}_{K_\infty^{M,0}}(\Lambda^1(\mathfrak{m}/\mathfrak{k}^M), \mathcal{D}_{\mu_{\mathbb{R}}} \otimes \mathcal{M}_\lambda(w\cdot\lambda)) =$$

$$\mathbb{C}P_+^\vee \otimes \phi_{\chi,-k} \otimes v_{k-2} + \mathbb{C}P_-^\vee \otimes \phi_{\chi,k} \otimes v_{-k+2} = \mathbb{C}\omega_{k,m} + \mathbb{C}\bar{\omega}_{k,m}. \tag{9.196}$$

Here $v_{k-2} = (X+iY)^{k-2}$, resp. $v_{2-k} = (X-iY)^{k-2}$ are two carefully chosen highest (resp. lowest) weight vectors with respect to the action of $K_\infty^{M,0}$. The elements $P_\pm$ are the usual elements in $\mathfrak{m}/\mathfrak{k}$. We choose a model space H_{σ_f} for σ_f i.e. a free rank one $\mathcal{O}_F$ -module on which the Hecke algebra acts by the homomorphism $\sigma_f : \mathcal{H}^M_{K_f^M} \to \mathcal{O}_F$. We also choose and embedding $\iota : F \hookrightarrow \mathbb{C}$ and an $(\mathfrak{m}, K_\infty^{M,0}) \times K_\infty^M \times \mathcal{H}^M_{K_f^M}$- invariant embedding

$$\Phi : \mathcal{D}_{\mu_{\mathbb{R}}} \otimes H_{\sigma_f} \otimes_{F,\iota} \mathbb{C} \to L^2_0(M(\mathbb{Q})\backslash M(\mathbb{A})) \tag{9.197}$$

this is unique up to a scalar in $\mathbb{C}^\times$ because the representation is irreducible and occurs with multiplicity one in the right hand side. This yields an isomorphism

$$\Phi^1_\iota : H^1((\mathfrak{m}, K_\infty^{M,0}), \mathcal{D}_{\mu_{\mathbb{R}}} \otimes \mathcal{M}_\lambda(w\cdot\lambda)) \otimes H_{\sigma_f} \otimes_{F,\iota} \mathbb{C} \xrightarrow{\sim} H^1(\mathcal{S}^{\tilde{M}}_{K_f^M}, \mathcal{M}_\lambda(w\cdot\lambda)_{\mathbb{C}})(\iota\circ\sigma_f)$$

We observe that the element $\epsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in K_\infty^M$ has the following effect

$$\operatorname{Ad}(\epsilon)(P_+) = P_- \text{ , } \epsilon(\phi_{\chi,k}) = \phi_{\chi,-k} \text{ and } \epsilon(v_{k-2}) = (-1)^m v_{2-k} \tag{9.198}$$

Hence we see that

$$\omega_{k,m}^{(+)} = \omega_{k,m} + (-1)^m \bar{\omega}_{k,m} \text{ resp. } \omega_{k,m}^{(-)} = \omega_{k,m} - (-1)^m \bar{\omega}_{k,m} \tag{9.199}$$

are generators of the $+$ and the $-$ eigenspace in $H^1(\mathfrak{m}, K_\infty^{M,0}, \mathcal{D}_{\mu_\mathbb{R}} \otimes \mathcal{M}_\lambda(w\cdot\lambda))$. Therefore our map Φ and the choice of these generators provide isomorphisms

$$\Phi_\iota^{(+)} : H_{\sigma_f} \otimes_{F,\iota} \mathbb{C} \xrightarrow{\sim} H^1_+(\mathcal{S}^{\tilde{M}}_{K_f^M}, \mathcal{M}_\lambda(w\cdot\lambda)_\mathbb{C})(\iota\circ\sigma_f), \tag{9.200}$$

$$\Phi_\iota^{(-)} : H_{\sigma_f} \otimes_{F,\iota} \mathbb{C} \xrightarrow{\sim} H^1_-(\mathcal{S}^{\tilde{M}}_{K_f^M}, \mathcal{M}_\lambda(w\cdot\lambda)_\mathbb{C})(\iota\circ\sigma_f) \tag{9.201}$$

The choice of P_+, P_- and $\phi_{\chi,-\nu}$ is canonical, hence we see that the identifications depend only on Φ_ι , which is unique up to a scalar. This means that the composition $T^{\text{trans}}(\iota\circ\sigma_f) = \Phi_\iota^{(-)} \circ (\Phi_\iota^{(+)})^{-1}$ yields a second (canonical) identification between the $\pm$ eigenspaces in the cohomology:

$$T^{\text{trans}}(\iota\circ\sigma_f) : H^1_+(\mathcal{S}^{\tilde{M}}_{K_f^M}, \mathcal{M}_\lambda(w\cdot\lambda)_\mathbb{C})(\iota\circ\sigma_f) \xrightarrow{\sim} H^1_-\mathcal{S}^{\tilde{M}}_{K_f^M}, \mathcal{M}_\lambda(w\cdot\lambda)_\mathbb{C})(\iota\circ\sigma_f)$$

Our arithmetic intertwining operator (See (9.193) yields an array - labeled by the embeddings $\iota : F \to \mathbb{C}$- of intertwining operators

$$T^{\text{arith}}(\sigma_f) \otimes_{F,\iota} \mathbb{C} : H^1_+(\mathcal{S}^{\tilde{M}}_{K_f^M}, \mathcal{M}_\lambda(w\cdot\lambda)_F))(\sigma_f) \otimes_{F,\iota} \mathbb{C} \xrightarrow{\sim} H^1_-(\mathcal{S}^{\tilde{M}}_{K_f^M}, \mathcal{M}_\lambda(w\cdot\lambda)_F))(\sigma_f) \otimes_{F,\iota} \mathbb{C} \tag{9.202}$$

Hence get an array of periods which compare these two arrays of intertwining operators

$$\Omega(\sigma_f,\iota) T^{\text{trans}}(\iota\circ\sigma_f) = T^{\text{arith}}(\sigma_f) \otimes_{F,\iota} \mathbb{C} \tag{9.203}$$

Our formula (9.194) tells us that we can arrange the intertwining operators such that

$$\Omega(\sigma_f \otimes \gamma_{2,f}, \iota) = \Omega(\sigma_f,\iota)^{-1} \tag{9.204}$$

These periods are uniquely defined up to a unit in $\mathcal{O}_{F,S}^\times$. We also see that the period only depends on the parity of $n_1/2 = m$, hence define

$$\Omega(\sigma^{(1)}, \epsilon(m)) := \Omega(\sigma_f,\iota) \text{ where } \epsilon(m) = (-1)^m \tag{9.205}$$

The Eisenstein intertwining

We pick a σ_f which occurs in $H^1_!(\mathcal{S}^{\tilde{M}}_{K_f^M}, \mathcal{M}_\lambda(w\cdot\lambda)_F))$, we choose a $\iota : F \hookrightarrow \mathbb{C}$ and we choose an embedding

$$\Phi_\iota : \mathcal{D}_{\mu_\mathbb{R}} \otimes H_{\sigma_f} \otimes_{F,\iota} \mathbb{C} \hookrightarrow L^2_{\text{cusp}}(M(\mathbb{Q})\backslash M(\mathbb{A})). \tag{9.206}$$

We assume $n_1 > 0$ then the Eisenstein series converges for $z = 0$ and we get the Eisenstein intertwining operator

$$\text{Eis}(0) \circ \Phi_\iota : \text{ Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}(\mathcal{D}_{\mu_\mathbb{R}}) \otimes H_{\sigma_f} \otimes_{F,\iota} \mathbb{C} \to \mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A})) \tag{9.207}$$

(Here we use that $K_f = \mathrm{Sp}_2(\hat{\mathbb{Z}})$.) This induces

$$\begin{aligned}\mathrm{Eis}^\bullet(0)\circ\Phi_\iota:\ &\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), I_P^G(\mathcal{D}_{\mu_\mathbb{R}})\otimes\mathcal{M}_\lambda)\otimes H_{\sigma_f}\otimes_{F,\iota}\mathbb{C}\to\\ &\to\ \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}),\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))\otimes\mathcal{M}_\lambda)\end{aligned}\tag{9.208}$$

and this induces a homomorphism in cohomology

$$H^3(\mathfrak{g},K_\infty, I_P^G(\mathcal{D}_{\mu_\mathbb{R}})\otimes\mathcal{M}_\lambda\otimes H_{\sigma_f}\otimes_{F,\iota}\mathbb{C})\to H^3(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_{\lambda,\mathbb{C}}).\tag{9.209}$$

We want to compose it with the restriction to the cohomology of the boundary. We have to compose it with theconstant Fourier coefficient $\mathcal{F}^P:\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))\to\mathcal{A}(P(\mathbb{Q})U(\mathbb{A})\backslash G(\mathbb{A}))$. We know that $\mathcal{F}^P$ maps into the subspace

$$I_P^G\mathcal{D}_{\mu_\mathbb{R}}\otimes H_{\sigma_f}\otimes_{F,\iota}\mathbb{C}\xrightarrow{\mathcal{F}^P} I_P^G\mathcal{D}_{\mu_\mathbb{R}}\otimes H_{\sigma_f}\otimes_{F,\iota}\mathbb{C}\bigoplus I_P^G\mathcal{D}_{\mu'_\mathbb{R}}\otimes H_{\sigma_f^{w_P}|\gamma_{P,f}|^{2f_P}}\otimes_{F,\iota}\mathbb{C}\tag{9.210}$$

where $\mu'=w_Pw\cdot\lambda=s_2\cdot\lambda=(2+2n_2+n_1)\gamma_1^{M_1}+\frac{1}{2}(-2+n_1)\gamma_2$. More precisely we know that for $h\in I_P^G\mathcal{D}_{\mu_\mathbb{R}}\otimes H_{\sigma_f}\otimes_{F,\iota}\mathbb{C}$

constcoeff1

$$\mathcal{F}^P(h)=h+\frac{\Lambda(\sigma,r_1,-1)}{\Lambda(\sigma,r_1,0)}\frac{\Lambda(\sigma,\Lambda^2(r_1),-1)}{\Lambda(\sigma,\Lambda^2(r_1),0)}\times T^{\,\mathrm{loc}}(0)(h)\tag{9.211}$$

where $T^{\,\mathrm{loc}}(0)=T_\infty^{\mathrm{loc}}\otimes\otimes_pT_p^{\mathrm{loc}}$. The local intertwining operators at the finite primes are normalised, they map the standard spherical function into the standard spherical function. The operator T_∞^{loc} is actually equal to the operator T^{trans} above. This means that we can replace T_∞^{loc} by $\frac{1}{\Omega(\sigma^{(1)},m)}T^{\mathrm{arith}}$. We want to express this in terms of values of the cohomological L-functions, which means that we have to shift the argument $c(\chi,\mu)$ (see **??** where χ is the highest weight of r_1 or $\Lambda^2(r_1)$ respectively, i.e. $\chi_1:=t\mapsto\begin{pmatrix}1&0\\0&1\end{pmatrix}$ or $\chi_2:=t\mapsto\begin{pmatrix}t&0\\0&t\end{pmatrix}$. and this gives us $c(\chi,\mu)=<\chi_1,\mu^{(1)}>-<\chi_1,\delta>=3+n_2+n_1$ and $c(\chi_2,\mu)=2+n_1$. Therefore we get in terms of the cohomological $L-$function constcoeff2

$$\mathcal{F}^P(h)=h+\frac{1}{\Omega(\sigma^{(1)},m)\}}\frac{\Lambda^{\mathrm{coh}}(\sigma,r_1,n_1+n_2+2)}{\Lambda^{\mathrm{coh}}(\sigma,r_1,n_1+n_2+3)}\frac{\zeta(n_1+1)}{\zeta(n_1+2)}\times T^{\mathrm{arith}}(0)(h)\tag{9.212}$$

To our irreducible Hecke module σ_f corresponds a modular cusp $\mathbf{f}$ form of weight k (see **??**). We know that the (completed) cohomological L -function is equal to the classical (completed) L function defined by Hecke. Then $\mathbf{f}$ has a Fourier expansion $\mathbf{f}(q):=q+\sum_{n=2}^\infty a_nq^n$ where the coefficients $a_n\in\mathcal{O}_F$, and

$$\Lambda(\mathbf{f},s)=\frac{\Gamma(s)}{(2\pi)^s}\prod_p\frac{1}{1-a_pp^{-s}+p^{k-1}p^{-2s}}$$

The theorems of Manin and Shimura imply that there are two real numbers (the periods) $\Omega_\pm(\mathbf{f})$ such that for the *critical arguments* $\nu=1,2,\dots k-1$ the value

$$\frac{1}{\Omega_{\epsilon(\nu)}}\Lambda(\mathbf{f},\nu)\in F.$$

and our relative period $\Omega(\sigma^{(1)}, m)$ is equal to the ratio of these two periods up to an element in $F^\times$. But remember the relative period is defined up to a unit in $\mathcal{O}_F^\times$.

If now $\nu = n_1 + n_2 + 2$ then we see that ν then we see that ν assumes the values $\frac{k}{2}, \frac{k}{2} - 1, \ldots, k-2$, here we allow $n_1 = 0$, hence we see that the values $\nu, \nu+1$ are exactly half of the critical arguments of the Hecke L-function. **Discuss briefly the case $n_1 = 0$?**

The intertwining operator $T_\infty^{\mathrm{loc}} : I_P^G \mathcal{D}_{\mu_\mathbb{R}} \to I_P^G \mathcal{D}_{\mu'_\mathbb{R}}$ has a kernel $\mathbb{D}_{\mu_\mathbb{R}}$, this is the sum of two discrete series representations. We know that

$$\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathbb{D}_{\mu_\mathbb{R}} \otimes \mathcal{M}_\lambda) = \mathrm{Hom}_{K_\infty}(\Lambda^3(\mathfrak{g}/\mathfrak{k}), \mathbb{D}_{\mu_\mathbb{R}} \otimes \mathcal{M}_\lambda) = \quad (9.213)$$

$$H^3(\mathfrak{g}, K_\infty, \mathbb{D}_{\mu_\mathbb{R}} \otimes \mathcal{M}_\lambda) = \mathbb{C}\Omega_{2,1} \oplus \mathbb{C}\Omega_{1,2} \quad (9.214)$$

where $\Omega_{2,1}$(resp. $\Omega_{1,2}$) are generators which map to $\omega_{k,m}$(resp. $\bar{\omega}_{k,m}$). Therefore $\Omega_{21} + (-1)^m \Omega_{12}$ maps to the generator $[\omega^+_{k,m}] \in H^3(\mathfrak{g}, K_\infty^M, I_P^G \mathcal{D}_{\mu_\mathbb{R}} \otimes \mathcal{M}_\lambda) \otimes_{F,\iota} \mathbb{C})$ whereas $\Omega_{21} - (-1)^m \Omega_{12}$ maps to the zero class in $H^3(\mathfrak{g}, K_\infty^M, I_P^G \mathcal{D}_{\mu_\mathbb{R}} \otimes \mathcal{M}_\lambda) \otimes_{F,\iota} \mathbb{C}$. This shows that in (9.210) the composition $\mathcal{F}^P \circ \mathrm{Eis}(0)$ is the projection to the first summand because T_∞^{loc} kills the factor at infinity. The Eisenstein intertwining operator provides a section from a certain piece of the boundary cohomology back to $H^3(\mathcal{S}_{K_f}^G, \tilde{\mathcal{M}}_\lambda) \otimes_{F,\iota} \mathbb{C}$. We are in the case (*Tzero*).. Apparently the second term in (9.210) does not play any role.

Since $n_1 > 1$ we can apply the Manin-Drinfeld principle and we can conclude that this section is defined over F, if we define

$$H^3(\mathcal{S}_{K_f}^G, \tilde{\mathcal{M}}_\lambda \otimes F)(\sigma_f) = r^{-1}(H^1_!(\mathcal{S}^{\tilde{M}}_{K_f^M}, \mathcal{M}_\lambda(w \cdot \lambda)_F))(\sigma_f)) \quad (9.215)$$

(Induction does not play a role since the level is one) then we get the decomposition

$$H^3_!(\mathcal{S}_{K_f}^G, \tilde{\mathcal{M}}_\lambda \otimes F) \oplus H^3_{\mathrm{Eis}}(\mathcal{S}_{K_f}^G, \tilde{\mathcal{M}}_\lambda \otimes F)(\sigma_f) = H^3_!(\mathcal{S}_{K_f}^G, \tilde{\mathcal{M}}_\lambda \otimes F)(\sigma_f) \quad (9.216)$$

The denominator of the Eisenstein class

We restrict this decomposition

$$H^3_{\mathrm{int}}(\mathcal{S}_{K_f}^G, \tilde{\mathcal{M}}_\lambda \otimes \mathcal{O}_{F,S})(\sigma_f) \supset H^3_{!,\,\mathrm{int}}(\mathcal{S}_{K_f}^G, \tilde{\mathcal{M}}_\lambda \otimes \mathcal{O}_{F,S})(\sigma_f) \oplus H^3_{\mathrm{int,\ Eis}}(\mathcal{S}_{K_f}^G, \tilde{\mathcal{M}}_\lambda \otimes \mathcal{O}_{F,S})(\sigma_f) \quad (9.217)$$

The image of $H^3_{\mathrm{int,\ Eis}}(\mathcal{S}_{K_f}^G, \tilde{\mathcal{M}}_\lambda \otimes \mathcal{O}_{F,S})(\sigma_f)$ under r is a submodule of finite index in $H^1_{!,\,\mathrm{int}}(\mathcal{S}^{\tilde{M}}_{K_f^M}, \mathcal{M}_\lambda(w \cdot \lambda)_{\mathcal{O}_{F,S}}))(\sigma_f))$ and the quotient is

$$H^3_{\mathrm{int}}(\mathcal{S}_{K_f}^G, \tilde{\mathcal{M}}_\lambda \otimes \mathcal{O}_{F,S})(\sigma_f)/(H^3_{!,\,\mathrm{int}}(\mathcal{S}_{K_f}^G, \tilde{\mathcal{M}}_\lambda \otimes \mathcal{O}_{F,S})(\sigma_f) \oplus H^3_{\mathrm{int,\ Eis}}(\mathcal{S}_{K_f}^G, \tilde{\mathcal{M}}_\lambda \otimes \mathcal{O}_{F,S})(\sigma_f))$$

$$H^1_{!\ \mathrm{int}}(\mathcal{S}^{\tilde{M}}_{K_f^M}, \mathcal{M}_\lambda(w \cdot \lambda)))(\sigma_f))/\mathrm{image}(r). \quad (9.218)$$

The quotient on the right hand side is $\mathcal{O}_{F,S}/\Delta_S(\sigma_f)$ where $\Delta_S(\sigma_f)$ is the denominator ideal. Tensoring the exact sequence

$$0 \to H^3_{!,\,\mathrm{int}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathcal{O}_{F,S})(\sigma_f) \oplus H^3_{\mathrm{int,\,Eis}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathcal{O}_{F,S})(\sigma_f) \to H^1_{\mathrm{int}}(\mathcal{S}^{\tilde{M}}_{K^M_f}, \mathcal{M}_\lambda(w\cdot\lambda) \otimes \mathcal{O}_{F,S}))(\sigma_f)) \to \mathcal{O}_{F,S}/\Delta_S(\sigma_f) \to 0 \tag{9.219}$$

by $\mathcal{O}_F/\Delta(\sigma_f)$ yields an inclusion

$$\mathrm{Tor}^1_{\mathcal{O}_F}(\mathcal{O}_{F,S}/\Delta(\sigma_f), \mathcal{O}_F/\Delta(\sigma_f) = \mathcal{O}_{F,S}/\Delta(\sigma_f)) \hookrightarrow H^3_{!,\,\mathrm{int}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathcal{O}_{F,S})(\sigma_f) \otimes \mathcal{O}_{F,S}/\Delta(\sigma_f). \tag{9.220}$$

We are now in the same situation as in Chapter 3-5, we would like to prove a theorem analogous to Theorem 5.1.2. In principle we could write a computer program which computes $\Delta_S(\sigma_f)$ in any case, as we did this for $\mathrm{Sl}_2(\mathbb{Z})$ in Chapter 3 with the help of H. Gangl. But to the best of my knowledge such a program is not yet written. I encouraged several postdocs and PhD-students to write such a program, but in the meanwhile I realise that I underestimated the difficulties.

In the section below we formulate a rather precise conjecture for the denominator.

The secondary class

For the following we refer to SecOps.pdf. We can write

$$\Omega_{2,1} - (-1)^m \Omega_{1,2} = d\psi \text{ where } \psi \in \mathrm{Hom}_{K_\infty}(\Lambda^2(\mathfrak{g}/\mathfrak{k}), I_P^G \mathcal{D}_{\mu_\mathbb{R}} \otimes \mathcal{M}_\lambda)$$

and

$$\psi' = T_\infty^{\mathrm{loc},2}(\psi) \in \mathrm{Hom}_{K_\infty}(\Lambda^2(\mathfrak{g}/\mathfrak{k}), I_P^G \mathcal{D}_{\mu'_\mathbb{R}} \otimes \mathcal{M}_\lambda)$$

is a closed form, hence it provides a cohomology class. This class is a multiple of $\omega^+_{k,m'}$ we write

$$[\psi'] = c(k,m)[\omega^+_{k,m'}]$$

where $c(k,m)$ is a non zero rational number. (This is the number c on p. 12 iin [43]). The computation of this number is not trivial. In SecOps.pdf I make an attempt to compute this number, assuming that this computation is correct we get

$$c(k,m)\frac{\zeta(n_1+1)}{\zeta(n_1+2)} = \frac{1}{\zeta(-1-n_1)}\frac{\zeta'(-n_1)}{\pi} \times \text{ small power of 2} \tag{9.221}$$

Now we can formulate a conjecture (*Denom*):

In the expression

$$\Big(\frac{1}{\Omega(\sigma_f,\iota))^{\epsilon(k,m)}}\frac{\Lambda^{coh}(\iota\circ f, n_1+n_2+2)}{\Lambda^{coh}(\iota\circ f, n_1+n_2+3)}\frac{1}{\zeta(-1-n_1)}\Big)\frac{\zeta'(-n_1)}{\pi}\Big)$$

the factor inside the large brackets is in F and behaves invariantly under the action of the Galois group. The denominator of this number divides $\Delta_S(\sigma_f)$

This is of course not very precise, we must have some information about the set S. In [43] we consider this conjecture for a very special case and there we say that S should be a set of small primes, and we decided that 41 is not small. We also tacitly assumed that the number c does not have a 41 in its numerator and we also assumed that 41 does not occur in the torsion of $H^4_c(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda)$.

In [43] I give a heuristic argument why I believe that the conjecture should be true. This argument is assuming some plausible - but apparently very deep -conjectures about mixed Tate-motives. These mixed Tate-motives here are not the ones in the current literature, they are Grothendieck mixed Tate motives. (See [Mix-Mot,]) [45] . This argument also gives us some hints which should be the primes in our set S.

We give a very vague outline of this argument. In [Mix-Mot] we will construct a so called Anderson-Tate mixed motive $\mathcal{H}(\Delta_S(\sigma_f)\sigma_f)$ with coefficients in F. This motive has a three step filtration

$$\begin{array}{c}\{0\} \subset \mathcal{O}_{F,S}(-n_2-1) \subset \mathcal{H}_!(\Delta_S(\sigma_f)\sigma_f) \subset \mathcal{H}(\Delta_S(\sigma_f)\sigma_f) \\ \text{and} \\ \mathcal{H}(\Delta_S(\sigma_f)\sigma_f)/\mathcal{H}_!(\Delta_S(\sigma_f)\sigma_f) \xrightarrow{\sim} \mathcal{O}_{F,S}(-n_1-n_2-2).\end{array} \tag{9.222}$$

Furthermore we know that the sequence

$$0 \to \mathcal{H}_!(\Delta_S(\sigma_f)\sigma_f)/\mathcal{O}_{F,S}(-n_2-1) \to \mathcal{H}(\Delta_S(\sigma_f)\sigma_f)/\mathcal{O}_{F,S}(-n_2-1) \to \mathcal{O}_{F,S}(-n_1-n_2-2) \to 0$$

splits (this is Manin-Drinfeld and the definition of $\Delta_S(\sigma_f)$.). This splitting is canonical and hence we can construct a mixed motive $\mathcal{X}(\Delta_S(\sigma_f)\sigma_f)$ which now sits in an exact sequence

$$0 \to \mathcal{O}_{F,S}(-n_2-1) \to \mathcal{X}(\Delta_S(\sigma_f)\sigma_f) \to \mathcal{O}_{F,S}(-n_2-n_1-2) \to 0. \tag{9.223}$$

Actually we do not need to know what it means that $\mathcal{X}(\Delta_S(\sigma_f)\sigma_f)$ is a mixed Anderson-Tate motive, the only thing we need to know that has a Betti-de-Rham realisation

$$0 \to \mathcal{O}_{F,S}(-n_2-1)_{\mathrm{B-dRh}} \to \mathcal{X}(\Delta_S(\sigma_f)\sigma_f)_{\mathrm{B-dRh}} \to \mathcal{O}_{F,S}(-n_2-n_1-2)_{\mathrm{B-dRh}} \to 0. \tag{9.224}$$

and for each prime $\mathfrak{l}$ in $\mathcal{O}_F$ we get ct sequence of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ -modules

$$0 \to \mathcal{O}_{F_\mathfrak{l},S}(-n_2-1) \to \mathcal{X}(\Delta_S(\sigma_f)\sigma_f) \otimes \mathcal{O}_{F_\mathfrak{l},S} \to \mathcal{O}_{F_\mathfrak{l},S}(-n_2-n_1-2) \to 0 \tag{9.225}$$

In [Mix-Mot] we will explain that we can attach extension classes to these mixed motives, we have the Betti-de-Rham extension class

$$[\mathcal{X}(\Delta_S(\sigma_f)\sigma_f)_{\mathrm{B-dRh}}] \in$$

$$\mathrm{Ext}^1_{\mathrm{B-dRh}}(\mathcal{O}_{F,S}(-n_2-n_1-2)_{\mathrm{B-dRh}}, \mathcal{O}_{F,S}(-n_2-1)_{\mathrm{B-dRh}}) = \mathbb{R}\frac{\zeta'(-n_1)}{\pi} \tag{9.226}$$

and for each ℓ and $\mathfrak{l}|\ell$ we have the Galois -module extension class

$$\mathcal{X}(\Delta_S(\sigma_f)\sigma_f)_{et-\mathfrak{l})}] \in \mathrm{Ext}^1_{et-\mathfrak{l}}(\mathcal{O}_{F,S}(-n_2-n_1-2)_{et-\mathfrak{l})}, \mathcal{O}_{F,S}(-n_2-1)_{et-\mathfrak{l})}) =$$

$$H^1(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \mathcal{O}_{F_\mathfrak{l},S}(n_1+1)) = H^1(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \mathbb{Z}_\ell(n_1+1)) \otimes \mathcal{O}_{F_\mathfrak{l},S}. \tag{9.227}$$

In [36] I explain how we can compute the Betti-de-Rham extension class of this mixed motive in a simpler case (Gl_2 instead of GSp_2) but mutatis mutandis this method can be transferred to this situation here and we get (see [Mix-Mot])

$$[\mathcal{X}(\Delta_S(\sigma_f)\sigma_f)_{\mathrm{B-dRh}}] =$$

$$\frac{c(k,m)\Delta(\sigma_f)}{\Omega(\sigma_f,\iota))^{\epsilon(k,m)}}\frac{\Lambda^{\mathrm{coh}}(\sigma_f,n_1+n_2+2)}{\Lambda^{\mathrm{coh}}(\sigma_f),n_1+n_2+3)}\frac{\zeta(n_1+1)}{\zeta(n_1+2)}) = \tag{9.228}$$

$$(\frac{\Delta(\sigma_f)}{\Omega(\sigma_f,\iota))^{\epsilon(k,m)}}\frac{\Lambda^{\mathrm{coh}}(\sigma_f,n_1+n_2+2)}{\Lambda^{\mathrm{coh}}(\sigma_f,n_1+n_2+3)}\frac{1}{\zeta(-1-n_1)})\frac{i\zeta'(-n_1)}{\pi})$$

Now our conjecture follows once we can prove

$$(\frac{\Delta(\sigma_f)}{\Omega(\sigma_f,\iota))^{\epsilon(k,m)}}\frac{\Lambda^{\mathrm{coh}}(\sigma_f,n_1+n_2+2)}{\Lambda^{\mathrm{coh}}(\sigma_f,n_1+n_2+3)}\frac{1}{\zeta(-1-n_1)}) \in \mathcal{O}_{F,S} \tag{9.229}$$

We can make an attempt to prove this last assertion, for this we look at the Galois-module extension class. Here we encounter the main stumbling block: We have the collection of extension classes and we expect that they are not independent, they " know of each other". More precisely we hope that the Betti-de-Rham extension class determines the $\ell - et$ Galois-module extension classes.

In [Mix-Mot] we introduce the Soule- element $c_{n_1}(\ell) \in H^1(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}),\mathbb{Z}_\ell(n_1+1))$, we consider its restriction to $H^1(\mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell),\mathbb{Z}_\ell(n_1+1)) \xrightarrow{\sim} \mathbb{Z}_\ell$. Now we dare to make the conjecture that this restriction is given by

$$\mathcal{X}(\Delta_S(\sigma_f)\sigma_f)_{et-\mathfrak{l})}] = c_{n_1}(\ell)\otimes(\frac{\Delta(\sigma_f)}{\Omega(\sigma_f,\iota))^{\epsilon(k,m)}}\frac{\Lambda^{\mathrm{coh}}(\sigma_f,n_1+n_2+2)}{\Lambda^{\mathrm{coh}}(\sigma_f,n_1+n_2+3)}\frac{1}{\zeta(-1-n_1)} \tag{9.230}$$

We make a small detour and discuss some heuristic arguments which provide some support for this last conjecture. In [?] and [Mix-Mot] we discuss the analogous situation for the group Gl_2, we construct certain Anderson-Tate mixed motives

$$[H^1_{\mathrm{Eis}}(X_0(p_0),\tilde{\mathcal{M}}^{\#}_{n,\mathbb{Z}})] \in \mathrm{Ext}^1_{\mathcal{MM}}(\mathbb{Z}(-n-1),\mathbb{Z}(0)),$$

We can compute their Betti-de-Rham extension class (see [36],[Mix-Mot].)

$$[H^1_{\mathrm{Eis}}(X_0(p_0),\tilde{\mathcal{M}}^{\#}_{n,\mathbb{Z}})]_{B-dRh} = \frac{p_0^{n+1}-1}{p_0^{n+2}-1}\frac{1}{\zeta(-1-n)}(\frac{-2i}{\pi}\zeta'(-n)) \tag{9.231}$$

here p_0 is an auxiliary prime, which is suppressed in our considerations for Sp_2. We notice that it is of the form rational number times $(\frac{-2i}{\pi})$, it is essentially the same shape as in (9.228). The difference is that here the Hecke eigenspace σ_f is replaced by a Hecke character and the ratio of L-values attached to σ_f is missing

In [Mix-Mot] we make an attempt to compute the $\ell - et$ again we formulate the conjecture:

$$[H^1_{\text{Eis}}(X_0(p_0), \tilde{\mathcal{M}}_n^{\#} \otimes \mathbb{Z}_\ell)]_{\ell-et} = \frac{p_0^{n+1}-1}{p_0^{n+2}-1}\frac{1}{\zeta(-1-n)}c_\ell(n) \tag{9.232}$$

so basically the factor $(\frac{-2i}{\pi}\zeta'(-n_1))$ is replaced by $c_\ell(n_1)$.

In [Mix-Mot] we prove the above conjecture for $n = 0$ (Anderson-Kummer Motives) and make an attempt to prove the conjecture by ℓ-adically approximating $[H^1_{\text{Eis}}(X_0(p_0), \tilde{\mathcal{M}}_n^{\#} \otimes \mathbb{Z}_\ell)]_{\ell-et}$ by Anderson-Kummer motives. So far this attempt fails but at least we get the above conjecture $\mod \ell$. This is the end of our detour.

Assuming the conjecture (9.228) our conjecture (*Denom*)) follows provided we know in addition $c_\ell(n)$ is a generator. Hence we put this into the assumptions in our conjecture.

Here we enter the realm of cyclotomic fields and we refer to [111]. We encounter another Wieferich dilemma. We have a very simple criterion to make sure that $c_\ell(n)$ is a generator. We consider the image $c_{1,\ell}(n)$ of $c_\ell(n)$ in $H^1(\text{ Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell), \mathbb{F}_\ell(n_1+1))$. It is clear that $c_\ell(n)$ is a generator if $c_{1,\ell}(n) \neq 0$. We refer to the computation in [Mix-Mot], then we can formulate an elementary criterion for $c_{1,\ell}(n) \neq 0$.

We consider the truncated polynomial ring $\mathbb{F}_\ell[x]/(x^{\ell-1})$. We define the following element

$$B_{\ell,n}(x) = \prod_{a=1}^{\ell-1}(\sum_{\mu=0}^{\ell-2}(\sum_{\nu=1}^{a-\mu}\binom{a-\mu}{\nu})x^\mu)^{a^{n_1}})$$

I leave it as an exercise for the reader to show that

$$B_{\ell,n_1}(x) = b_0 + b_k x^k + b_{k+1}x^{k+1} + b_{\ell-2}x^{\ell-2} \text{ where } k = \ell-2-n$$

Then $b_k \in \mathbb{F}_\ell$ is an elementary expression and we can expect that the numbers b_k are randomly distributed $\mod \ell$.

$$c_{1,\ell}(n) = 0 \iff b_k = 0 \iff B_{\ell,n_1}(x) = b_0$$

Therefore we can expect that $c_{1,\ell}(n) = 0$ is a rare event, but we do not know whether it is almost always the case.

Perhaps I should have mentioned that $c_{1,\ell}(n) = 0$ is equivalent to $\ell|\zeta(-1-n_1)$. Hence we see that $c_{1,\ell}(n) \neq 0$ if ℓ is a regular prime. The Wieferich dilemma is that we cannot prove that the set of regular primes is infinite.

We do not care so much about this issue. In this volume the experimental aspect of the subject plays a significant role. With the help of the computer we verified the analogous conjecture (*Denom*) in the case $G = \mathrm{Sl}_2/\mathbb{Z}$ and coefficients $\mathcal{M}_n$ in many cases and we see this verification as a model for other cases. We think that it is of interest to accumulate experimental evidence for our conjecture. If we succeed doing these computations we will (almost) never encounter a case where $c_{1,\ell}(n) = 0$. In the case here it is of course also of interest that the conjecture follows from a conjecture about mixed Anderson-Tate motives (for this see also [45]).

We discuss the set of exceptional primes S of course we should keep it as small as possible. If we want to check the conjecture (*Denom*) numerically our data ℓ, λ should not be too large, even small primes like $\ell = 2, 3, 5$, could occasionally show up in the race, i. e. they are not necessarily in S.

In our construction of $\mathcal{H}(\Delta_S(\sigma_f)\sigma_f)$ we need at some point that primes ℓ which occur in the torsion of $H^4_c(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda)$ should be in S. Hence if we want to be on the safe side we put those primes into S.

Of course we know that $\mathcal{M}_\lambda$ is not determined by its highest weight. Actually it easy to see that there is a smallest $\mathcal{M}^\flat_\lambda$ which is generated by the highest weight vector. We have the invariant pairing $<;>: \mathcal{M}_\lambda \otimes \mathbb{Q} \times \mathcal{M}^\flat_{-w_0\lambda} \otimes \mathbb{Q} \to \mathbb{Q}$ and hence we see that we get a biggest module $\mathcal{M}_\lambda$ if we define

$$\mathcal{M}_\lambda{}^\sharp := \{m \in \mathcal{M}_\lambda \otimes \mathbb{Q} | < m, \mathcal{M}^\flat_{-w_0\lambda} > \in \mathbb{Z}\}$$

This suggests that S should contain those primes which divide the order of $\mathcal{M}_\lambda{}^\sharp / \mathcal{M}_\lambda{}^\flat$, but this is definitely not a good idea. We pick a prime ℓ, let $\mathbb{Z}_{(\ell)} \in \mathbb{Q}$ be thelocal ring at (ℓ), we choose a Hecke operator $T^{\text{coh},\lambda}_{\ell,\chi}$ where we assume $< \chi, \alpha_i >> 0$ for all simple roots. In our paper [**?**] we define-following Hida - the ℓ ordinary part $H^\bullet_{\text{ord}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{Z}_{(\ell)})) \subset H^\bullet_{\text{ord}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{Z}_{(\ell)}))$, this is the largest submodule such that

$$T^{\text{coh},\lambda}_{\ell,\chi} : H^\bullet_{\text{ord}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{Z}_{(\ell)})) \xrightarrow{\sim} H^\bullet_{\text{ord}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{Z}_{(\ell)})) \tag{9.233}$$

is an isomorphism. This module does not depend on χ. Then it follows easily from the definition (see 6.51) or (**??**) that $T^{\text{coh},\lambda}_{\ell,\chi}$ acts nilpotently on $H^\bullet(\mathcal{S}^G_{K_f}, \widetilde{\mathcal{M}^\#_\lambda / \mathcal{M}^\flat_\lambda})$. This implies of course immediately that the map

$$H^\bullet_{\text{ord}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}^\flat_\lambda \otimes \mathbb{Z}_{(\ell)})) \xrightarrow{\sim} H^\bullet_{\text{ord}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}^\#_\lambda \otimes \mathbb{Z}_{(\ell)})) \tag{9.234}$$

is an isomorphism and this implies that the ordinary part of the cohomology does not depend on the choice of the integral structure $\mathcal{M}_\lambda$. This suggests that we should include those primes ℓ into our set S for which σ_f is not ordinary. This is also consistent with the data in [43].

For me the most difficult part in the calculation is the treatment of the intertwining operator at ∞, this is carried out in SecOps.pdf. At the end of SecOps.pdf. I discuss the arithmetic applications and the conjectural relationship between the primes dividing the denominator of the expression in the large brackets and the denominators of the Eisenstein classes in (9.6.2).

9.6.3 Higher rank examples

In section 9.3.1 we a somewhat vague description of the relationship between prime ideals which divide certain values of L-functions and denominators of Eisenstein classes. Here we want to make this more precise in another case namely for the group $\mathcal{G}/\mathbb{Z} = \mathrm{GSp}_3/\mathbb{Z}$. We make some assumptions for which we have experimental data.

We consider the group $G/\operatorname{Spec}(\mathbb{Z}) = GSp_3/\operatorname{Spec}(\mathbb{Z})$ with Dynkin-Diagram

$$\alpha_1 \quad - \quad \alpha_2 \quad <= \quad \alpha_3$$

and choose a highest weight $\lambda = n_1\gamma_1 + n_2\gamma_2 + n_3\gamma_3$, here we assume $n_1 \equiv n_3 \mod 2$ so that we may consider it as highest weight for $G/\operatorname{Spec}(\mathbb{Z})$ which is trivial on the centre. The highest weight module $\mathcal{M}_\lambda$ provides a sheaf $\tilde{\mathcal{M}}_\lambda$ on $S^G_{K_f} = Sp_3(\mathbb{Z})\backslash\mathbb{H}_3$.

We consider the cohomology of the boundary, we select the boundary stratum corresponding to the parabolic subgroup P given by $\alpha_1 \quad - \quad \times \quad <= \quad \alpha_3$, the semi simple part is $M = \mathrm{Gl}_2 \times \mathrm{Sl}_2$. The first factor has to be viewed as the linear factor and corresponds to α_1, the other factor is the hermitian factor.

We look at the Kostant representatives w, w' with $l(w) = 4, l(w') = 3$, we have two such pairs

$$\begin{array}{cc} w_1 = s_2s_1s_3s_2 & w_2 = s_2s_3s_2s_1 \\ v_1 = s_2s_1s_3 & v_2 = s_2s_3s_2 \end{array}. \tag{9.235}$$

If Θ_P is the longest Kostant representatives then $w_1 = \Theta_P v_1, w_2 = \Theta_P v_2$ and we get

$$\begin{aligned} w_1(\lambda+\rho)-\rho &= (2+n_2+2n_3)\gamma^M_{\alpha_1} + (2+n_1+n_2+n_3)\gamma^M_{\alpha_3} + 1/2(-6-n_2)\gamma_{\alpha_2} \\ v_1(\lambda+\rho)-\rho &= (2+n_2+2n_3)\gamma^M_{\alpha_1} + (2+n_1+n_2+n_3)\gamma^M_{\alpha_3} + 1/2(-4+n_2)\gamma_{\alpha_2} \\ w_2(\lambda+\rho)-\rho &= (4+n_1+2n_2+2n_3)\gamma^M_{\alpha_1} + n_3\gamma^M_{\alpha_3} + 1/2(-6-n_1)\gamma_{\alpha_2} \\ v_2(\lambda+\rho)-\rho &= (4+n_1+2n_2+2n_3)\gamma^M_{\alpha_1} + n_3\gamma^M_{\alpha_3} + 1/2(-4+n_1)\gamma_{\alpha_2} \\ \Theta_P(\lambda+\rho)-\rho &= n_1\gamma^M_{\alpha_1} + n_3\gamma^M_{\alpha_3} + \tfrac{1}{2}(-5-\tfrac{n_1}{2}-n_2-n_3)\gamma_{\alpha_2} \end{aligned} \tag{9.236}$$

We denote the two coefficients at $\gamma^M_{\alpha_1}$(resp.)$\gamma^M_{\alpha_3}$ by d_1(resp.)d_3. Then the cohomology $H^2(S^M_{K^M_f}, \mathcal{M}(w\cdot\lambda))$ is given by the Künneth-formula, the factors are given by holomorphic modular forms of weight $d_1+2 = k_1, d_3+2 = k_3$. Since we want the boundary cohomology to be non zero and since work on level 1 these weights k_1, k_2 must be even. This implies that we should require that n_2 is even and $n_1 \equiv n_3 \mod 2$ in the first case (i.e. $w = w_1$) and n_1, n_3 even in the second case ($w = w_2$).

Given d_1, d_3 we find

$$\begin{aligned} \lambda &= (d_3 - \tfrac{d_1+2+n_2}{2})\gamma_{\alpha_1} + n_2\gamma_{\alpha_2} + \tfrac{d_1-2-n_2}{2}\gamma_{\alpha_3} \text{ in case 1} \\ \lambda &= (d_1 - 2d_3 - 4 - 2n_2)\gamma_{\alpha_1} + n_2\gamma_{\alpha_2} + d_3\gamma_{\alpha_3} \text{ in case 2} \end{aligned} \tag{9.237}$$

In the first case the coefficient n_2 has to lie in a string of even integers

$$n_2 \in \{\min(d_1-2, 2d_3-d_1-2), \ldots, 2, 0\},\ n_2 \equiv 0 \mod 2 \tag{9.238}$$

and this means for the coefficient in front of γ_{α_2}

$$b(w,\lambda) = 1/2(-6-n_2) \in \{-3,\dots,-2-\frac{\min(d_1,2d_3-d_1)}{2}\} \tag{9.239}$$

This string of integers is not empty if and only if the minimum is > 0. If we want to solve equation (9.236) in the first case (of course with the above constraints) we need

$$d_1 \geq 2 \text{ and } 2d_3 \geq d_1+2 \tag{9.240}$$

In the second case we easily see that

$$n_1 \in \{d_1-4-2d_3,\dots,2,0\} \tag{9.241}$$

and if we want to find solutions we need the inequality

$$d_1 \geq 2d_3+4 \tag{9.242}$$

Then we see that the factor in front of γ_{α_2} runs through the interval

$$b(w,\lambda) \in \{-3,\dots,-1-\frac{d_1}{2}+d_3\} \tag{9.243}$$

Now we consider the Eisenstein cohomology in degree 6

$$\begin{array}{ccc} H^6(S^G_{K_f},\mathcal{M}_\lambda) & \xrightarrow{r} & H^6(\partial(S^G_{K_f}),\tilde{\mathcal{M}}_\lambda) \\ & & \uparrow i_P \\ & & H^2_!(S^M_{K^M_f},\tilde{\mathcal{M}}(w\cdot\lambda)) \end{array} \tag{9.244}$$

where w is one of the two elements of length 4 and where i_P is an inclusion. An eigenspace

$$H^2_!(S^M_{K^M_f},\tilde{\mathcal{M}}(w\cdot\lambda))(\tau\times\sigma) \subset (H^2_!(S^M_{K^M_f},\tilde{\mathcal{M}}(w\cdot\lambda)) \tag{9.245}$$

is essentially given by a pair $\mathbf{f}$ resp.$\mathbf{g}$ of holomorphic cusp forms of weight k_1 resp. k_3. For simplicity we assume that these forms are unramified and have rational Fourier coefficients.

We consider the Eisenstein cohomology given by this pair of forms, i.e we study the map

$$\mathrm{Eis}(0): H^2_!(S^M_{K^M_f},\tilde{\mathcal{M}}(w\cdot\lambda)\otimes\mathbb{Q})(\tau\times\sigma) \to H^6(S^G_{K_f},\mathcal{M}_\lambda\otimes\mathbb{Q}) \tag{9.246}$$

we assume that $n_1 > 0$, in this case the Eisenstein series is holomorphic at $z=0$ and the Manin-Drinfeld principle is valid.

Again we want to understand the denominator of Eis(0). We study the factor in front of the second term in the constant term. (See (9.62), We apply [**?**], we look at the dual group $G^\vee$ and the action of $M^\vee = \mathrm{Gl}_2 \times \mathrm{PSl}_2$ on $\mathfrak{u}_P^\vee$, the Lie algebra unipotent radical of $P^\vee$. This Lie algebra is of dimension 7 and a closer kook shows that as a $M^\vee$ module we have

$$\mathfrak{u}_P^\vee = r_1 \otimes \mathrm{ad} \oplus \det \otimes \mathbf{1} = \mathfrak{u}_P^\vee[1] + \mathfrak{u}_P^\vee[2[\tag{9.247}$$

here $r_1 = r_{\chi_1}$ where $\chi_1 : t \mapsto \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{Gl}_2$ and $\mathrm{ad} = r_{\chi_3}$ where $\chi_3 : t \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in \mathrm{Sl}_2$. . And finally $\det = r_{\chi_0}$ where $\chi_0 : t \to \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} \subset \mathrm{Gl}_2 \subset M$. Then we get for the constant term

$$\mathcal{F}(h) = h + \frac{\Lambda(\tau \times \sigma, r_{\chi_1} \otimes r_{\chi_3}, -1)\zeta(n_i+1)}{\Lambda(\tau \times \sigma, r_{\chi_1} \otimes r_{\chi_3}, 0)\zeta(n_i+2)} T^{\mathrm{loc}}(0)(h) \tag{9.248}$$

here the index $i = 1, 2$ depending on the case. Again we assume $n_i > 0$.

We rewrite this in terms of the cohomological L-function. We know from the work of Gelbart-Jacquet that we can lift σ to an automorphic form $\Pi = \mathrm{Sym}^2(\sigma)$ on $H = \mathrm{Gl}_3/\mathbb{Q}$. This form is again cuspidal unless σ is a CM-form (this will never happen under the present circumstances). Translated into cohomology this means the following: Let $\gamma_\alpha, \gamma_\beta$ be the two fundamental dominant weights and $\mu = d_3(\gamma_\alpha + \gamma_\beta)$, let $\mathcal{M}_\mu$ be the resulting highest weight module on H. Then we find a non trivial

$$H^2_!(S^{\mathrm{Gl}_2 \times H}_{K_f^{\mathrm{Gl}_2 \times H}}, \widetilde{\mathcal{M}_{d_1\gamma_1} \otimes \mathcal{M}_\mu})(\tau \times \Pi_f) \subset H^2_!(S^{\mathrm{Gl}_2 \times H}_{K_f^{\mathrm{Gl}_2 \times H}}, \widetilde{\mathcal{M}_{d_1\gamma_1} \otimes \mathcal{M}_\mu}), \tag{9.249}$$

and for all primes p the Satake parameter ω_p^H of Π_p equals the image of the Satake parameter ω_p of σ_p. Then we get an equality of L-functions

$$\Lambda(\tau \times \sigma_f, r_{\chi_1} \otimes r_{\chi_3}, z) = \Lambda(\tau \times \Pi, r_{\chi_1} \otimes r_{\chi_3'}, z),$$

here $\chi_3' : t \mapsto \begin{pmatrix} t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then we write this in terms of the cohomological L-function, we have to make a shift in the variable z by

$$< \chi_1 + \chi_3', d_1\gamma_1 + \mu - b(w,\lambda)\gamma_{\alpha_2} >= \frac{d_1}{2} + d_3 + \frac{1}{2}(6 + n_i) \tag{9.250}$$

hence

$$\Lambda(\tau \times \Pi, r_{\chi_1} \otimes r_{\chi_3'}, z) = \Lambda^{\mathrm{coh}}(\tau \times \Pi, r_{\chi_1} \otimes r_{\chi_3'}, z + \frac{d_1}{2} + d_3 + \frac{1}{2}(6 + n_i) \tag{9.251}$$

The same kind of reasoning as above gives us

$$\mathcal{F}(h) = h + \frac{1}{\Omega(\tau \times \Pi)^{\pm 1}} \frac{\Lambda^{\mathrm{coh}}(\tau \times \Pi, r_{\chi_1} \otimes r_{\chi_3}, \frac{d_1}{2} + d_3 + \frac{1}{2}(6+n_i) - 1)}{\Lambda^{\mathrm{coh}}(\tau \times \Pi, r_{\chi_1} \otimes r_{\chi_3}, \frac{d_1}{2} + d_3 + \frac{1}{2}(6+n_i))} \frac{\zeta(n_i+1)}{\zeta(n_i+2)} T^{\mathrm{arith}}(0)(h) \tag{9.252}$$

here $\Omega(\tau \times \Pi)$ is the relative period defined in [48].

Before we continue I want to say a few words concerning the (cohomological) Rankin-Selberg L-function $\Lambda^{\mathrm{coh}}(\tau\times\Pi, r_{\chi_1}\otimes r_{\chi_3}, z)$. In our special case which we consider we started from two modular forms $\mathbf{f},\mathbf{g}$ of weights k_1,k_3 respectively. For both of them we have the Scholl-motive $M(\mathbf{f}), M(\mathbf{g})$ and the two dimensional ℓ-adic Galois-representations

$$\rho(\tau):\ \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})\to Gl(M(\mathbf{f})_\ell), \rho(\sigma):\ \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})\to Gl(M(\mathbf{g})_\ell),$$

and we have for the Frobenii:

$$\rho(\tau)(\Phi_p^{-1})\simeq\begin{pmatrix}\alpha_p & 0\\ 0 & \beta_p\end{pmatrix},\ \alpha_p+\beta_p=a_p, \alpha_p\beta_p=p^{k_1-1}=p^{d_1+1}$$

$$\rho(\sigma)(\Phi_p^{-1})\simeq\begin{pmatrix}\gamma_p & 0\\ 0 & \delta_p\end{pmatrix},\ \gamma_p+\delta_p=c_p, \gamma_p\delta_p=p^{k_3-1}=p^{d_3+1}$$

where a_p resp. c_p is the $p-th$ Fourier coefficient of f resp. g.

We take the symmetric square of $\rho(\sigma)$ and get

$$\rho(\mathrm{Sym}^2(\sigma)):\ \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})\to Gl_3(\mathbb{Z}_\ell)$$

(here we assume that f,g have coefficients in $\mathbb{Z}$.) Then

$$\rho(\mathrm{Sym}^2(\sigma))(\Phi_p^{-1})\simeq\begin{pmatrix}\gamma_p^2 & 0 & 0\\ 0 & p^{d_3+1} & 0\\ 0 & 0 & \delta_p^2\end{pmatrix}$$

Then we can write the finite part of the cohomological L-function as

$$L^{\mathrm{coh}}(\tau\times\Pi,s)=\prod_p\frac{1}{\det(\mathrm{Id}-\begin{pmatrix}\alpha_p & 0\\ 0 & \beta_p\end{pmatrix}\otimes\begin{pmatrix}\gamma_p^2 & 0 & 0\\ 0 & p^{d_3+1} & 0\\ 0 & 0 & \delta_p^2\end{pmatrix}p^{-s})}\tag{9.253}$$

-

Our motives $M(\mathbf{f}),M(\mathbf{g})$ have Hodge types $\{(d_1+1,0),(0,d_1+1),(d_3+1,0),(0,d_3+1)\}$ and therefore we get for the Hodge type of $M(\tau\times\Pi)$

$$\{(d_1{+}2d_3{+}3,0),(d_1{+}d_3{+}2,d_3{+}1),(d_1{+}1,2d_3{+}2),(2d_3{+}2,d_1{+}1),(d_3{+}1,d_1{+}d_3{+}2),(0,d_1{+}2d_3{+}3)\}$$

it is pure of weight $\mathbf{w}=d_1+2d_3+3$. We reorder these Hodge type according to the size of the second component and get

$$\{(\mathbf{w},0),(\mathbf{w}-a,a),(\mathbf{w}-b,b),(b,\mathbf{w}-b),(a,\mathbf{w}-a),(0,\mathbf{w})\},$$

where now $0<a\le b<\frac{\mathbf{w}}{2}$. From the Hodge type or from representation-theoretic considerations we get a Γ factor at infinity which is (if I am not mistaken)

$$L_\infty(\tau\times\Pi,s)=\frac{\Gamma(s)\Gamma(s-a)\Gamma(s-b)}{(2\pi)^{3s}}$$

Again we put

$$\Lambda^{\mathrm{coh}}(\tau\times\Pi,s)=L_\infty(\tau\times\Pi,s)L^{\mathrm{coh}}(\tau\times\Pi,s).$$

This function satisfies a functional equation:

$$\Lambda^{\mathrm{coh}}(\tau\times\Pi,s)=\Lambda^{\mathrm{coh}}(\tau\times\Pi,\mathbf{w}+1-s). \tag{9.254}$$

In [23] Tim Dokchitser outlines an effective algorithm which computes the value $\Lambda^{\mathrm{coh}}(\tau\times\Pi,z_0)$ at a given argument z_0 with arbitrary high precision. This algorithm uses the functional equation.

We notice that the central point $\frac{\mathbf{w}+1}{2}$ is an integer. The conjecture of Deligne on special values predicts that there are two periods $\Omega(\tau\times\Pi)_\pm$ such that the values at critical arguments

$$\frac{1}{\Omega(\tau\times\Pi)_{\epsilon(\nu)}}\Lambda^{\mathrm{coh}}(\tau\times\Pi,\frac{\mathbf{w}+1}{2}+\nu)\in F \tag{9.255}$$

arguments is the set of integers where the critical arguments are the integers $\mu=\frac{\mathbf{w}+1}{2}+\nu$ which satisfy $b<\mu\le\mathbf{w}-b$. Let us call $(b,\mathbf{w}-b]$ the critical interval. In [48] it is proved that there is a period $\Omega(\tau\times\Pi)$ such that the numbers

$$\frac{1}{\Omega(\tau\times\Pi)^{\pm1}}\frac{\Lambda^{\mathrm{coh}}(\tau\times\Pi,r_{\chi_1}\otimes r_{\chi_3},\mu-1)}{\Lambda^{\mathrm{coh}}(\tau\times\Pi,r_{\chi_1}\otimes r_{\chi_3},\mu)}\in F \tag{9.256}$$

provided μ and $\mu-1$ lie in the critical interval. This period is well defined up to a unit in $\mathcal{O}_F^\times$. We apply this to the ratios of L-values in (9.252)

$$\frac{1}{\Omega(\tau\times\Pi)^{\pm1}}\frac{\Lambda^{\mathrm{coh}}(\tau\times\Pi,r_{\chi_1}\otimes r_{\chi_3},\frac{\mathbf{w}+1}{2}+\frac{n_i}{2})}{\Lambda^{\mathrm{coh}}(\tau\times\Pi,r_{\chi_1}\otimes r_{\chi_3},\frac{\mathbf{w}+1}{2}+\frac{n_i}{2}+1)} \tag{9.257}$$

and here the n_i are just the numbers in (9.238),(9.241). This means $n_i=n_2$ in the first case and $n_i=n_1$ in the second case. But then it is clear that the arguments $\frac{\mathbf{w}+1}{2}+\frac{n_i}{2},\frac{\mathbf{w}+1}{2}+\frac{n_i}{2}+1$ are critical if and only if the numbers n_i are the numbers in the two lists (9.238),(9.241). Therefore we know that the numbers in (9.257) are in F and since the period is unique up to a unit, it makes sense to speak of their factorisation into prime ideals.

Again we can formulate a conjecture. For simplicity we assume that $\mathbf{f},\mathbf{g}$ have rational Fourier coefficients (then we only have finitely many cases)

If ℓ is prime and $\mathbf{f},\mathbf{g}$ are ordinary at ℓ and if

$$\ell^m|\mathrm{Denominator}(\frac{1}{\Omega(\tau\times\Pi)^{\pm1}}\frac{\Lambda^{coh}(\tau\times\Pi,r_{\chi_1}\otimes r_{\chi_3},\frac{\mathbf{w}+1}{2}+\frac{n_i}{2}))}{\Lambda^{coh}(\tau\times\Pi,r_{\chi_1}\otimes r_{\chi_3},\frac{\mathbf{w}+1}{2}+\frac{n_i}{2}+1)}) \tag{9.258}$$

then ℓ^m divides the denominator $\Delta(\tau\times\sigma)$ of the Eisenstein class. If we are a little bit more courageous we may even conjecture the this is the exact power of ℓ which divides the denominator.

Of course this implies again that we must have congruences.

If we have a non zero isotypical subspace $H^2_!(S^M_{K^M_f}, \tilde{\mathcal{M}}(w\cdot\lambda)\otimes\mathbb{Q})(\tau_f\times\sigma_f) \subset H^6(\partial(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda\otimes\mathbb{Q})$ *and* ℓ *divides the denominator in (9.258) then there should be an non zero isotypical subspace* $H^6(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda\otimes F)(\tilde{\Pi}_f)$ *and a prime ideal* $\mathfrak{l}\subset\mathcal{O}_F)$ *such that for all primes* p *and all Hecke operators* $T^{\mathrm{coh}}_{\chi,p}$

$$\tilde{\Pi}_f(T^{\mathrm{coh}}_{\chi,p}) \equiv (\tau_f\times\sigma_f)(T^{\mathrm{coh}}_{\chi,p}) \mod \mathfrak{l} \tag{9.259}$$

We make this a little bit more concrete, as cocharacter we choose χ_3 which is defined by $<\chi_3,\alpha_1>=1$ and $<\chi_3,\alpha_2>=<\chi_3,\alpha_3>=0$. Then we have an explicit formula for $(\tau_f\times\sigma_f)(T^{\mathrm{coh}}_{\chi,p})$ and this gives us

$$\begin{aligned} &T^G_{\chi_3,p}(\tilde{\Pi}_p) \equiv a_p(\mathbf{g})(p^{n_3+1}+a_p(\mathbf{f})+p^{n_2+n_3+2}) \mod \mathfrak{l} \text{ in } (case1) \text{ and}\\ &T^G_{\chi_3,p}(\tilde{\Pi}_p) \equiv a_p(\mathbf{g})(p^{n_2+n_3+2}+a_p(\mathbf{f})+p^{n_1+n_2+n_3+3}) \mod \mathfrak{l} \text{ in } (case2) \end{aligned} \tag{9.260}$$

We have some experimental data supporting this conjecture in some cases for some small primes p. We know 6 unramified modular forms which have rational coefficients, they are of weight $12, 16, 18, 20, 22, 26$ this means that the values for d_1, d_3 are $10, 14, 16, 18, 20, 24$. For any pair (d_1, d_3) we have exactly one pair $(\mathbf{f}, \mathbf{g})$ of modular forms. since we want to avoid the pole of the ζ function we also require $n_2>0$. In the second case we do not find a solution λ with $n_1>0$.

In this special situation the space $H^2(\mathcal{S}^M_{K^M_f}, \mathcal{M}_\lambda(w_1\cdot\lambda))$ is isotypical, hence the Hecke algebra acts by scalars, the formula for the eigenvalue of $T^{\mathrm{coh}}_{\chi_3,p}(\tau_f\times\sigma_f)$. We are always in the case 1 and in (9.237) we give the formula for the highest weight λ for which we get the diagram (9.244). In [3] the authors find many cases of a λ for which $H^6_!(S^G_{K_f}, \mathcal{M}_\lambda)$ is isotypical, i.e. the Hecke operators act as scalars, and they produce short lists of eigenvalues. They find many cases in which they find congruences of the type (**??**). (Table 3 in [3]). Here I want to draw the readers attention to the case $d_1=10, d_3=16$, in this case $\mathbf{f}=\Delta$. Here the authors of [3] find a congruence of the above type modulo 17^2.

Anton Mellit applied the algorithm of Dokchitzer to compute the critical values in 9.255) for all pairs $(\mathbf{f},\mathbf{g})$ with $d_1, d_3\le 18$ and $d_1\ne d_3$. For all primes ℓ for which we have a congruence in Table 3 in [3] and for which Mellit computes the critical values, we find find a n_2 such that

$$\ell|\text{Denominator}(\frac{1}{\Omega(\tau\times\Pi)^{\pm1}}\frac{\Lambda^{\mathrm{coh}}(\tau\times\Pi, r_{\chi_1}\otimes r_{\chi_3}, \frac{\mathbf{w}+1}{2}+\frac{n_2}{2}))}{\Lambda^{\mathrm{coh}}(\tau\times\Pi, r_{\chi_1}\otimes r_{\chi_3}, \frac{\mathbf{w}+1}{2}+\frac{n_2}{2}+1)}) \tag{9.261}$$

and again for $\ell=17$ and $d_1=10, d_3=16$ we even find a divisibility by 17^2.

Of course there are still some issues which have to be discussed. In [3] the authors say that the prime ℓ should be "large" and they are not very precise what this should mean. In any case we should fix in advance a finite set $S=S(\mathbf{f},\mathbf{g})$ of exceptional primes ℓ for which the congruences might not be true. This set may depend on the choice of the periods. The period $\Omega(\tau_f\times\Pi_f)$ is a well defined positive real number (under the present assumptions). In our concrete situation it follows from the results in [48] (5.2, 8.4.7) that the relative period

$$\Omega(\tau_f \times \Pi_f)^{\pm 1} = \frac{\Omega(\tau_f)_+}{\Omega(\tau_f)_-} \tag{9.262}$$

where the two periods $\Omega(\tau_f)_\pm$, these periods have been fixed in section (8.3.5). Mellit choses slightly different periods but his periods differ from our period by a product of powers of $2, 3, 5, 7, 11, 13$ and for the d_1, d_3 he considers these primes are not ordinary, i.e. they lie in $S(\mathbf{f}, \mathbf{g})$. But the 17 is in fact ordinary $d_1 = 10, d_3 = 16$.

Mellit's tables predict many more congruences than those which have been found in [3]. The reason for this is that one encounters serious difficulties if $H^6_!(S^G_{K_f}, \mathcal{M}_\lambda)$ is not isotypical, which probably means that the dimension of this inner cohomology is greater than 8 (this is the length of the Hodge filtration and also equal to $\#W_G/W_K$.) If this is the case we need a finite extension to decompose $H^6_!(S^G_{K_f}, \mathcal{M}_\lambda \otimes F)$ into isotypical pieces

$$H^6_!(S^G_{K_f}, \mathcal{M}_\lambda \otimes F) = \bigoplus_{\tilde{\Pi}_f} H^6_!(S^G_{K_f}, \mathcal{M}_\lambda \otimes F)(\tilde{\Pi}_f) \tag{9.263}$$

The method of counting $\mathbb{F}_p$ valued points which is applied in [3] may give a way to compute the trace $\mathrm{tr}(T^G_{\chi_3} | H^6_!(S^G_{K_f}, \mathcal{M}_\lambda \otimes F))$ and this of course equal to the sum of the traces on the right hand side and here summands are equal to $T^G_{\chi_3,p}(\tilde{\Pi}_p) \dim(H^6_!(S^G_{K_f}, \mathcal{M}_\lambda \otimes F)(\tilde{\Pi}_f))$. But from here it is difficult to get the values $T^G_{\chi_3,p}(\tilde{\Pi}_p)$. In principle one could try to count $\mathbb{F}_{p^r}$ valued points for some $r = 1, 2, 3 \ldots$ but this is definitely not easy.

In Mellit's tables we find for the case $d_1 = 16, d_3 = 20$ a divisibility

$$333769 \mid \text{Denominator}\Big(\frac{1}{\Omega(\tau \times \Pi)^{\pm 1}} \frac{\Lambda^{\mathrm{coh}}(\tau \times \Pi, r_{\chi_1} \otimes r_{\chi_3}, 30)}{\Lambda^{\mathrm{coh}}(\tau \times \Pi, r_{\chi_1} \otimes r_{\chi_3}, 31)}\Big) \tag{9.264}$$

and it would be nice if the expected congruence could be verified for some small values of p. (there is still the very unlikely possibility that $H^7_c(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$ has 333769 torsion or $\mathbf{f}$ or $\mathbf{g}$ is not ordinary for 333769.)

Of course it would be still nicer if we could verify the denominator conjecture in this case.

Bibliography

[1] BATEMAN MANUSCRIPT PROJECT

[2] J. Bellaïche , R. Pollack, *Congruences with Eisenstein series and ?-invariants* Compos. Math. 155

[3] J. Bergström, C. Faber, , G. van der Geer *Siegel modular forms of degree three and the cohomology of local systems.*
Selecta Math. (N.S.) 20 (2014), no. 1, 83-124.

[4] Bergström, Jonas; Dummigan, Neil; Mégarbané, Thomas; Ibukiyama, Tomoyoshi; Katsurada, Hidenori *Eisenstein congruences for SO(4,3), SO(4,4), spinor, and triple product L-values.*
Exp. Math. 27 (2018), no. 2, 230-250.

[5] J.Bewersdorff *Eine Lefschetzsche Fixpunktformel für Hecke-Operatoren.* Dissertation, University of Bonn, Bonn, 1985. Bonner Mathematische Schriften 164. Universität Bonn, Mathematisches Institut, Bonn, 1985

[6] A. Borel, *Regularization theorems in Lie algebra cohomology. Applications.* Duke Math. J. 50 (1983), no. 3, 605–623.

[7] A. Borel and W. Casselman, *L2-cohomology of locally symmetric manifolds of finite volume.* Duke Math. J. 50 (1983), no. 3, 625-647.

[8] A. Borel and H. Garland, *Laplacian and the discrete spectrum of an arithmetic group.* Amer. J. Math. 105 (1983), no. 2, 309–335.

[9] A.Ḃorel and Harish-Chandra *Arithmetic subgroups of algebraic groups.* Ann. of Math. (2) 75 (1962), 485?535

[10] A. Borel and J. Tits *Groupes réductifs.* Inst. Hautes Études Sci. Publ. Math. No. 27 (1965), 55-150

[11] A. Borel and N. Wallach, *Continuous cohomology, discrete subgroups, and representations of reductive groups.* Second edition. Mathematical Surveys and Monographs, 67. American Mathematical Society, Providence, RI, 2000.

[12] A. Borel and J.-P. Serre, *Corners and arithmetic groups,* Commen. Math. Helvetici 48 (1973), 436-491.

[13] N. Bourbaki, *Groupes et algebres de Lie* Chapitres 4,5, et 6 Elements de Mathematiques

[14] Bruhat, F. Tits, J. *Groupes algébriques sur un corps local. Chapitre III.* J. Fac. Sci. Univ. Tokyo Sect. IA Math. 34 (1987), no. 3, 671?698.

[15] J.P. Buhler, D. Harvey *Irregular primes to 163 million* Math. Comp. 80 (2011), 2435-2444

[16] L. Carlitz, *A generalization of Maillet's determinant and a bound for the first factor of the class number.* Proc. Amer. Math. Soc. 12 (1961), 256?261.

[17] W. Casselman *On Some Results of Atkin and Lehner* Math. Ann. 201,301–3t4 (1973)

[18] G. Chenevier, J. Lannes *Automorphic Forms and Even Unimodular Lattices* Ergebnisse der Mathematik und ihrer Grenzgebiete Springer 2019

[19] L. Clozel, *Motifs et formes automorphes: applications du principe de fonctorialité.* Automorphic forms, Shimura varieties, and L-functions, Vol. I (Ann Arbor, MI, 1988), 77–159, Perspect. Math., 10, Academic Press, Boston, MA, 1990.

[20] P. Deligne, *Formes modulaires et représentationsℓ- adiques.*] Séminaire Bourbaki. Vol. 1968/69: Exposés 347-363, Exp. No. 355, 139-172, Lecture Notes in Math., 175, Springer, Berlin, 1971

[21] P. Deligne, *Valeurs de fonctions L et périodes d'intégrales (French),* With an appendix by N. Koblitz and A. Ogus. Proc. Sympos. Pure Math., XXXIII, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Oregon, 1977), Part 2, pp. 313–346, Amer. Math. Soc., Providence, R.I., 1979.

[22] T. Deng *Torsion in Cohomology of $Sl_2(\mathbb{Z})$ and Congruence of Modular Forms* arXiv:1905.05260

[23] Dokchitser, T. *Computing special values of motivic L-functions.* Experiment. Math. 13 (2004), no. 2, 137–149

[24] Elstrodt, J. , Grunewald, F., Mennicke, J. *Groups Acting on Hyperbolic Space* Springer Monographs in Mathematics

[25] C. Faber ; G. van der Geer, *Sur la cohomologie des systèmes locaux sur les espaces de modules des courbes de genre 2 et des surfaces abéliennes.I,* C. R. Math. Acad. Sci. Paris 338 (2004), no. 5, 381–384. II, no. 6, 467–470

[26] J. Franke *Harmonic analysis in weighted L2-spaces.* Ann. Sci. École Norm. Sup. (4) 31 (1998), no. 2, 181?279.

[27] Gelbart,S.-Jacquet,H, *A relation between automorphic representations of* GL(2) *and* GL(3). Ann. Sci. cole Norm. Sup. (4) 11 (1978), no. 4, 471-542

[28] R. Godement, *Lecture Notes on Jacquet-Langlands* IAS Princeton

[29] M. Goresky, ; R. MacPherson, *The topological trace formula.* J. Reine Angew. Math. 560 (2003), 77-150

[30] H. Grobner, *A cohomological injectivity result for the residual automorphic spectrum of GL(n).* Pacific J. Math., 268 (2014), no. 1, 33–46.

[31] B.H. Gross *On the Satake Isomorphism*

[32] K.Ḣaberland, *Perioden von Modulformen einer Variabler and Gruppencohomologie. I, II, III]* Math. Nachr. 112 (1983), 245-282, 283-295, 297-315.

[33] G. Harder, *A Gauss-Bonnet formula for discrete arithmetically defined groups* Annales scientifiques de l'É.N.S. 4e série, tome 4, no 3 (1971), p. 409-455

[34] Harder, G., *General aspects in the theory of modular symbols.* Seminarberichte des Séminaire DPP, Birkhäuser, 73 - 88, 1983

[35] G. Harder, *Eisenstein cohomology of arithmetic groups. The case* GL_2. Invent. Math. 89, no. 1, 37–118 (1987).

[36] G.Ḣarder *Eisensteinkohomologie und die Konstruktion gemischter Motive,* Springer Lecture Notes 1562

[37] G. Harder, *Some results on the Eisenstein cohomology of arithmetic subgroups of* GL_n, in Cohomology of arithmetic groups and automorphic forms (ed. J.-P. Labesse and J. Schwermer), Springer Lecture Notes in Mathematics, vol. 1447, pp. 85–153 (1990).

[38] G. Harder- N.Schappacher *Special values of Hecke L-functions and abelian integrals.* Workshop Bonn 1984 (Bonn, 1984), 17-49, Lecture Notes in Math., 1111, Springer, Berlin, 1985.

[39] G.Ḣarder *Eisenstein cohomology of arithmetic groups and its applications to number theory.* Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990), 779?790, Math. Soc. Japan, Tokyo, 1991.

[40] G. Harder, *Lectures on Algebraic Geometry I* Aspects of Mathematics, E 35 viewegteubner (second edition)

[41] G. Harder, *Lectures on Algebraic Geometry II* Aspects of Mathematics, E 35 viewegteubner

[42] G. Harder, *Interpolating coefficient systems and p-ordinary cohomology of arithmetic groups.* Groups Geom. Dyn., 5 (2011), no. 2, 393–444.

[43] G. Harder *A congruence between a Siegel and an Elliptic Modular Form* The 1-2-3 of Modular Forms, Springer Unitext

[44] G. Harder *Arithmetic Aspects of Rank One Eisenstein Cohomology* Proceedings of the International Colloquium on Cycles, Motives and Shimura Varieties, Mumbai 2008

[45] G. Ḣarder *Can we compute the denominator of Eisenstein classes?* Tiflis 2021

[46] G. Harder *The topological trace formula* `http://www.math.uni-bonn.de/people/harder/Manuscripts/buch/`

[47] G. Harder, R. Pink, *Modular konstruierte unverzweigte abelsche p-Erweiterungen von* $\mathbb{Q}(\zeta_p)$ *und die Struktur ihrer Galoisgruppen.* Math. Nachr. 159 (1992), 83?99

[48] G. Harder and A. Raghuram, *Eisenstein cohomology and ratios of critical values of Rankin–Selberg L-functions.* Annals of mathematics Studies, Princeton University Press.

[49] G. Harder *Cohomology of arithmetic groups,* 51-82, Springer Proc. Math. Stat., 245, Springer, Cham, 2018.

[50] Harish-Chandra, *Automorphic Forms on Semissimple Lie Groups* Springer Lecture Notes in Mathematics 62, 1968

[51] E. Hecke . Über Modulfunktionen und die Dirichletschen Reihen mit Eulerscher Produktentwicklung. I. (German) Math. Ann. 114 (1937), no. 1, 1?28

[52] S. Helgason, *Differential geometry and symmetric spaces.* Pure and Applied Mathematics, Vol. XII. Academic Press, New York-London 1962

[53] Hida, Haruzo *Congruence of cusp forms and special values of their zeta functions.* Invent. Math. 63 (1981), no. 2, 225-26 1.

[54] Iwasawa, Kenkichi *Lectures on p-adic L-functions.* Annals of Mathematics Studies, No. 74. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1972. vii+106 pp.

[55] H. Jacquet, *Principal L-functions of the linear group.* Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, pp. 63-86, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979.

[56] H. Jacquet, *On the residual spectrum of GL(n).* Lie group representations, II (College Park, Md., 1982/1983), 185-208, Lecture Notes in Math., 1041, Springer, Berlin, 1984.

[57] Jacquet- Langlands, *Automorphic forms on* Gl_2, Springer Lecture Notes 114

[58] H. Jacquet and J.A. Shalika, *On Euler products and the classification of automorphic representations. I.* Amer. J. Math. 103 (1981), no. 3, 499–558.

[59] H. Kasten and C.-G. Ṡchmidt *The critical values of Rankin-Selberg convolutions.* Int. J. Number Theory 9 (2013), no. 1, 205?256

[60] D. Kazhdan; B. Mazur and C.-G. Schmidt *Relative modular symbols and Rankin-Selberg convolutions.* J. Reine Angew. Math. 519 (2000), 97?141, p. 97-141

[61] H. Jacquet and J.A.,Shalika, *On Euler products and the classification of automorphic forms. II.* Amer. J. Math. 103 (1981), no. 4, 777–815.

[62] H. Kim, *Automorphic L-functions.* Lectures on automorphic L-functions, 97–201, Fields Inst. Monogr., 20, Amer. Math. Soc., Providence, RI, 2004.

[63] B. Kostant, *Lie algebra cohomology and the generalized Borel-Weil theorem* Annals of Math., Vol 74, No. 2 (1961) 329–387.

[64] A. Knapp, *Local Langlands correspondence: the Archimedean case,* Motives (Seattle, WA, 1991), 393–410, Proc. Sympos. Pure Math., 55, Part 2, Amer. Math. Soc., Providence, RI, 1994.

[65] M.K̇neser, *Strong approximation in Algebraic Groups and Discontinuous Subgroups* (Proc. Sympos. Pure Math., Boulder, Colo., 1965) pp. 187-196 Amer. Math. Soc., Providence, R.I.

[66] T. Kubota, ; H-W. Leopoldt, *Eine p-adische Theorie der Zetawerte. I. Einführung der p-adischen Dirichletschen L-Funktionen.* J. Reine Angew. Math. 214(215) (1964), 328?339.

[67] J.-P. Labesse, R. Langlands, *L-indistinguishability for SL(2)*, Canadian J. Math. 31 (1979), pp. 726?785

[68] R. Langlands, *Euler products,* Yale University Press (1971).

[69] R. Langlands, *On the functional equations satisfied by Eisenstein series.* Lecture Notes in Mathematics, Vol. 544., Springer-Verlag, Berlin-New York, 1976.

[70] Langlands, R. P. *Representations of abelian algebraic groups.* Olga Taussky-Todd: in memoriam. Pacific J. Math. 1997, Special Issue, 231-250

[71] R. Langlands *Automorphic representations, Shimura varieties, and motives. Ein Märchen.* Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, pp. 205-246, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979

[72] R. Lee, ; R. H Szczarba,. *On the homology and cohomology of congruence subgroups* . Invent. Math. 33 (1976), no. 1, 15 – 53.

[73] J.-S. Li, J.Schwermer, *On the Eisenstein cohomology of arithmetic groups.* Duke Math. J. 123 (2004), no. 1, 141-169.

[74] Mazur, B. (; Wiles, A. *Class fields of abelian extensions of Q.* Invent. Math. 76 (1984), no. 2, 179?330

[75] L. J. Mordell, *On Mr. Ramanujan's empirical expansions of modular functions,* Proc. Cambridge Philos. Soc. 19 (1917) 117-124.

[76] C. Mœglin and J.-L. Waldspurger, *Le spectre résiduel de GL(n).* Ann. Sci. École Norm. Sup. (4) 22 (1989), no. 4, 605–674.

[77] G. Nebe, B. Venkov, *On Siegel modular forms of weight 12,* J. reine angew. Math. 351 (2001), pp. 49-60.

[78] J. Neukirch *Algebraic Number Theory* Grundlehren der mathematischen Wissenschaften, Springer 1999

[79] H.-V. Niemeier, *Definite quadratische Formen der Dimension 24 und Diskriminante 1.* J. Number Theory 5 (1973), 142-178

[80] T. Ono *Arithmetic of algebraic tori.* Ann. of Math. (2) 74 (1961), 101-139.

[81] R. Pink, *On l-adic sheaves on Shimura varieties and their higher direct images in the Baily-Borel compactification.* Math. Ann. 292 (1992), no. 2, 197–240.

[82] V. P. Platonov, (1969), *The problem of strong approximation and the Kneser-Tits hypothesis for algebraic groups* Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya, 33: 1211-1219

[83] V. Platonov-A. Rapinchuk Algebraic Groups and Number Theory ACADEMIC PRESS, INC.

[84] Raghunathan, M. S. *A note on quotients of real algebraic groups by arithmetic subgroups.* Invent. Math. 4 (1967/68), 318?335

[85] S. Ramanujan, *On certain arithmetical functions,* Trans. Cambridge Philos. Soc. 22 (1916) 159?184. [Col- lected Papers of Srinivasa Ramanujan, AMS Chelsea Publishing, American Mathematical Society, Providence, RI, 2000, pp. 136-162.]

[86] G, De-Rham, *Differentiable manifolds. Forms, currents, harmonic forms* Springer Grundlehren, 266, 1984

[87] K.- A. Ribet, *A modular construction of unramified p-extensions of* $\mathbb{Q}(\mu_p)$. Inv. Math. 34 (3): 151-162

[88] L. Saper *$\mathcal{L}$-modules and the conjecture of Rapoport and Goresky-MacPherson.* Automorphic forms. I. Astérisque No. 298 (2005)

[89] Satake, I.*Spherical functions and Ramanujan conjecture.* Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math.,Boulder, Colo., 1965) pp. 258-264 Amer. Math. Soc., Providence, R.I.

[90] I. Satake, *The Gauss-Bonnet theorem for V-manifolds.* J. Math. Soc. Japan 9 (1957), 464-492.

[91] Scholl, *Motives for modular forms.* Invent. Math. 100 (1990), no. 2, 419–430.

[92] J. Schwermer, *Kohomologie arithmetisch definierter Gruppen und Eisensteinreihen.* Lecture Notes in Mathematics, 988. Springer-Verlag, Berlin, 1983. iv+170 pp.

[93] J. Schwermer, *Cohomology of arithmetic groups, automorphic forms and L-functions.* Cohomology of arithmetic groups and automorphic forms (Luminy-Marseille, 1989), 1–29, Lecture Notes in Math., 1447, Springer, Berlin, 1990.

[94] J. Schwermer, *On Euler products and residual Eisenstein cohomology classes for Siegel modular varieties.* Forum Math. 7 (1995), no. 1, 1–28.

[95] J.-P. Serre , *A Course in Arithmetic*,
Graduate Texts in Mathematics, 7, Springer-Verlag

[96] J.-P. Serre , *Cohomologie Galoisienne* ,
Lecture Notes in Mathematics 5, Springer-Verlag

[97] J.-P. Serre, *Facteurs locaux des fonctions zeta des variétés algébriques (définitions et conjectures)*, in Séminaire Delange-Pisot-Poitou, 1969/70, Collected Papers II, vol. 19, p. 581–592.

[98] F. Shahidi, *Whittaker models for real groups.* Duke Math. J. 47 (1980), no. 1, 99–125.

[99] F. Shahidi, *Local coefficients as Artin factors for real groups.* Duke Math. J. 52 (1985), no. 4, 973–1007.

[100] F. Shahidi, *Eisenstein series and automorphic L-functions.* American Mathematical Society Colloquium Publications, 58. American Mathematical Society, Providence, RI, (2010).

[101] G. Shimura, *The special values of the zeta functions associated with Hilbert modular forms.* Duke Math. J. 45 (1978), no. 3, 637–679.

[102] C.L. Siegel *Gesammelte Werke*

[103] T.A. Springer *Linear Algebraic Groups* Modern Birkhäuser Classics

[104] R. Steinberg *Lectures on Chevalley Groups* Yale University 1967

[105] H.P.F. Swinnerton-Dyer *On ℓ-adic representations and congruences for coefficients of modular forms* In Modular Functions of One Variable III, SLN 350

[106] Tate, J. T. *Fourier analysis in number fields, and Hecke's zeta-functions.* Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965), 305-347, Thompson, Washington, D.C., 1967.

[107] Vatsal, V. *Congruences for the special values of modular L-functions.* C. R. Math. Acad. Sci. Soc. R. Can. 20 (1998), no. 3, 77?82.

[108] D. Vogan and G. Zuckerman, *Unitary representations with nonzero cohomology.* Comp. Math. 53 (1984), no. 1, 51–90.

[109] Wagstaff, Samuel S., Jr. *Zeros of p-adic L-functions.* Math. Comp. 29 (1975), no. 132, 1138?1143.

[110] N. Wallach *On the constant term of a square integrable automorphic form.* In: Arsene, G. et al. (eds.) Operator Algebras and Group Representations. II. (Monogr. Stud. Maths., vol. 18, pp. 227- 237) London: Pitman 1984

[111] L. Ẇashington *Introduction into cyclotomic fields* Springer

[112] R. Weissauer, *Endoscopy for GSp(4) and the Cohomology of Siegel Modular Threefolds* Springer Lecture Notes 1968 (2009)

[113] U. Weselmann, *Siegel Modular Varieties and the Eisenstein Cohomology of* PGL_{2g+1}, Manuscripta Math. 145 (2014), no. 1-2, 175–220.

[114] A. Wiles *Modular elliptic curves and Fermat's last theorem.* Ann. of Math. (2) 141 (1995), no. 3, 443?551.

[115] E. Witt *Über eine Identität zwischen Modulformen zweiten Grades* Gesammelte Abhandlungen, Springer Verlag

References

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[MixMot-2015.pdf] Modular Construction of Mixed Motives

[SecOps.pdf] Secondary Operations on the Cohomology of Harish-Chandra Modules

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