Modular Construction of mixed Motives

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1 Motives and their cohomological realizations

In this manuscript I use the concept of motives without defining what I really mean by that. Basically a motive should be a piece in the cohomology of an algebraic variety, but the rules how I get such pieces are not fixed. In any case a motive must have various cohomological realizations, namely the Betti realization, the de-Rham realization and the ℓ-adic realizations for all primes ℓ.

1.1 Pure Motives

I consider smooth projective schemes $X/\mathrm{Spec}(\mathbb{Q})$, we know that we can find a nonempty open subset $V = \mathrm{Spec}(\mathbb{Z}_S) \subset \mathbb{Q}$ such that X extends to a smooth projective scheme $\mathcal{X} \to V$. Let us choose such an extension. We consider the cohomology of this scheme, I denote it by $H^\bullet(X)$ and by this I mean the various realizations:

1.1.1 The cohomological realizations

The Betti-cohomology:

$$H_B^\bullet(X) = H_B^\bullet(X(\mathbb{C}),\mathbb{Z}) = \bigoplus_{m=0}^{2\dim(X)} H^m(X(\mathbb{C}),\mathbb{Z}).$$

This a finitely generated $\mathbb{Z}$ graded $\mathbb{Z}$-module together with the involution F_∞ induced by the complex conjugation on $X(\mathbb{C})$

The de-Rham-cohomology:

The de Rham cohomology is defined as the hypercohomology of the complex of coherent sheaves $0 \to \mathcal{O}_X \to \Omega_X^1 \to \dots \Omega_X^d \to 0$. This cohomology is the cohomology of a double complex

$$\begin{array}{cccccccc}
0 & \to & \mathcal{O}_X & \to & \Omega_X^1 & \to \dots & \Omega_X^d \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & \to & \mathcal{O}_X^{0,1} & \to & \Omega_X^{1,1} & \to \dots & \Omega_X^{d,1} \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & \to & \mathcal{O}_X^{0,2} & \to & \Omega_X^{1,2} & \to \dots & \Omega_X^{d,2} \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow
\end{array}$$

here the vertical complexes are resolutions of the coherent sheaves in the top line by coherent acyclic sheaves. Then

$$H_{dRh}^\bullet(X) = H^\bullet(\Omega^{\bullet,\bullet}(X))$$

These cohomology groups are finite dimensional $\mathbb{Q}$ vector space together with the descending filtration. In degree $\bullet = n$ it is of the form

$$H_{dRh}^n(X) = F^0 H_{dRh}^n(X) \supset F^1 H_{dRh}^n(X) \supset \dots \supset F^n H_{dRh}^n(X) = H^0(X, \Omega_X^n).$$

The next step in the filtration $F^{n+1} H_{dRh}^n(X) = 0$.

We have the comparison isomorphism

$$I_{B-dRh} : H_B^\bullet(X) \otimes \mathbb{C} \xrightarrow{\sim} H_{dRh}^\bullet(X) \otimes \mathbb{C}$$

(**Hodge decomposition + motives of pure weight+Hodge numbers , effective motives?**)

The ℓ-adic cohomology:

For any prime ℓ we have the etale cohomology groups

$$H_{et,\ell}^\bullet(X) = H^\bullet(X \times \bar{\mathbb{Q}}, \mathbb{Z}_\ell)$$

they are modules for the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$

For any embedding $\mathbb{Q}_\ell \hookrightarrow \mathbb{C}$ we have the comparison isomorphism

$$I_\ell : H_{et,\ell}^\bullet(X) \otimes \mathbb{C} \xrightarrow{\sim} H_B^\bullet(X) \otimes \mathbb{C}$$

which is compatible with F_∞

Furthermore the complex conjugation acts on $H^\bullet_B(X(\mathbb{C}),\mathbb{C})$ and $H^\bullet(\Omega^\bullet(X))\otimes \mathbb{C}$ via the complex conjugations c_B and c_{DR}. The comparison isomorphisms satisfy in addition

$$\begin{gathered} I\circ c_B\otimes F_\infty = c_{DR}\circ I \\ I_\ell\circ c = F_\infty\circ I_\ell \end{gathered}$$

If I consider the cohomology in a fixed degree n then I want to call the object $H^n(X)$ a pure motive of weight n. This weight is visible as the length of the filtration of the de-Rham cohomology.

It this weight is also visible in the etale cohomology: For $p\notin S$ and $p\neq \ell$ the modules $H^n(X\times\bar{\mathbb{Q}},\mathbb{Q}_\ell)$ are unramified at p. The characteristic polynomial

$$\det(\mathrm{Id}\; T-\Phi_p^{-1}|H^\bullet(X\times\bar{\mathbb{Q}},\mathbb{Q}_\ell))\in\mathbb{Z}[T]$$

is independent of ℓ and its roots (the eigenvalues of Frobenius Φ_p^{-1}) are of absolute value $p^{n/2}$.

1.2 Some simple pure motives

Now $\mathbb{Z}(-n)=H^{2n}(\mathbb{P}^n,\mathbb{Z})$ is the following object

$$\mathbb{Z}(-n)=\begin{cases} H^{2n}_B(\mathbb{P}^n)=\mathbb{Z}\cdot 1_B\;, F_\infty(1_B)=(-1)^n 1_B \\ H^{2n}_{DR}(\mathbb{P}^n)=\mathbb{Q}\cdot 1_{DR}+\ \text{Filtration}, F^n\mathbb{Q}(-n)=\mathbb{Q}(-n), F^{n+1}\mathbb{Q}(-n)=0 \\ I: H^{2n}_B\otimes_{\mathbb{Z}}\mathbb{C}\xrightarrow{\sim} H^{2n}_{DR}\otimes_{\mathbb{Q}}\mathbb{C} \\ I: 1_B\longrightarrow (\frac{1}{2\pi i})^n\ 1_{DR} \\ I\circ F_\infty\circ c_B = c_{DR}\circ I \\ H^{2n}_{\acute{e}t}(\mathbb{P}^n)=\mathbb{Z}_\ell(-n)\quad \text{Galoismodul} \\ I_\ell: H^{2n}_B(\mathbb{P}^n)\otimes\mathbb{Z}_\ell\xrightarrow{\sim} H^{2n}_{\text{ét}}(\mathbb{P}^n,\mathbb{Z}_\ell) \\ \text{compatible with the action of } F_\infty. \end{cases}$$

It will important that the comparison isomorphism gives us a canonical generator in $\mathbb{Z}_\ell(-n-1)$. This generator can also be seen in the following way. For all m we have the privileged ℓ^m-s root of unity

$$\zeta_{\ell^m}=e^{\frac{2\pi i}{\ell^m}}$$

and the canonical generator in $\mathbb{Z}_\ell(-n)$ is given by $\zeta_{\ell^m}^{\otimes(-n)}$. These motives $\mathbb{Z}(-n)$ for $n\in\mathbb{Z}$ are called *pure Tate motives.*

If we have a finite extension $K/\mathbb{Q}$, then we can consider $\mathbb{P}^n/K$ and the Weil restriction $R_{K/\mathbb{Q}}\mathbb{P}^n$. Then we can consider the motive $H^n(R_{K/\mathbb{Q}}\mathbb{P}^n)=\mathbb{Z}(-n)^{K/\mathbb{Q}}$. Its Betti-cohomology and *é*tale cohomology

$$\mathbb{Z}(-n)_B^{K/\mathbb{Q}}=\sum_{\sigma:K\to\mathbb{C}}\mathbb{Z}1_B,\ \ \mathbb{Z}_\ell(-n)^{K/\mathbb{Q}}=\mathrm{Ind}_{\mathrm{Gal}(\bar{\mathbb{Q}}/K)}^{\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})}\mathbb{Z}_\ell(-n).$$

element $1_B^{K/\mathbb{Q}}=(\dots,1_B,\dots)_{\sigma:K\to\mathbb{C}}$

These motives are Tate motives which are twisted by an Artin motive.

We also want to consider correspondences on $T \subset X \times_{\mathbb{Q}} X$, they induces endomorphisms on the cohomology $H^\bullet(X)$, of course by this we mean that they induce endomorphisms in any of the cohomological realizations. We consider the ring generated by these endomorphisms and we try to find correspondences which are projectors in all cohomological realizations. If we have such an endomorphism q then we also want that $(H^\bullet(X), q)$ is also a pure motive. In any case it has all the cohomological realizations and this is my basic criterion for something being a motive.

If we are lucky then we can find such projectors, which induce the identity on the cohomology $H^n(X)$ in degree n and which are zero in the other degrees. Then we may speak of $H^n(X)$ as a pure motive of weight n. It is also possible that we can find only "projectors with denominators", i.e. endomorphism p which satisfy $p^2 = mp$ with some non zero integer m. In such a case we get a motive with coefficients. (See 1.8.)

1.3 Mixed motives

Now I want to remove a closed subscheme $Y \subset X$. Let $U = X \setminus Y$, this is now quasi projective. I want to consider the cohomology $H^\bullet(U)$ of U and I want explain that we may consider this (under certain conditions) as a mixed motive.

We denote the inclusions $j : U \hookrightarrow X, i : Y \hookrightarrow X$.

Let me assume for simplicity that $Y \subset X$ is smooth, then $Y = \mathrm{Spec}(\emptyset_X/\mathcal{J})$ and we consider the completion

$$\mathcal{N}Y = \mathrm{Spec}(\lim_{\leftarrow}(\mathcal{O}_X/\mathcal{J}^n))$$

which I consider as being a tubular neighborhood of Y. Locally on Y this is of the form $\mathrm{Spec}(\mathcal{O}_Y(W')[[f_1, \dots, f_r]])$, where the f_i are generators of the ideal $\mathcal{J}$ which form a system of local parameters.

If I remove the zero section Y from this scheme I get

$$\overset{\bullet}{\mathcal{N}}\, Y = \mathcal{N}Y \setminus Y$$

I want that the cohomology $H^\bullet(\overset{\bullet}{\mathcal{N}}\, Y)$ is a mixed motive and I will explain why this is not an entirely absurd idea.

Let us start from the case that Y is just a finite number of $\mathbb{Q}$-rational points. In this case and our completion is simply a disjoint union of $B_d = \mathrm{Spec}\mathbb{Q}[[x_1, \dots, x_d]]$ where $d = \dim(X)$. If we stick to one of these points P then we have to understand the cohomology of $B_d \setminus P = \overset{\bullet}{B}_d$.

It is clear that from the point of view of Betti-cohomology this is just a sphere of dimension $2d - 1$ and we say

$$H^p_B(\overset{\bullet}{B}_d) = \begin{cases} \mathbb{Z} \text{ if } p = 0, 2d-1 \\ 0 \;\text{ else} \end{cases}$$

The involution F_∞ acts by the identity in degree zero and by $(-1)^d$ in degree $2d - 1$.

If we want to understand the de-Rham and the etale realization I begin with the case $d = 1$. In this case we consider $\dot{B}_1$ as "homotopy equivalent" to the multiplicative group scheme $\mathbb{G}_m$. If we cover the projective line $\mathbb{P}^1$ by two affine planes U_0, U_1 then $\mathbb{G}_m = U_0 \cap U_1$ and we consider the resulting Mayer-Vietoris sequence in cohomology, it provides and isomorphism

$$H^1(\mathbb{G}_m) \xrightarrow{\sim} H^2(\mathbb{P}^1)$$

Now we remember how we compute the cohomology of a sphere by using the Mayer-Vietoris sequence. In B_d we can define the subschemes $B_d[x_i \neq 0]$ and we can cover B_d by these subschemes. Writing down certain Mayer-Vietoris sequences provides some convincing evidence that

$$H^{2d-1}(B_d) = \mathbb{Z}(-d)$$

Now we consider the general case, our subscheme Y is still smooth. We can view $\dot{\mathcal{N}}\, Y$ as a fibre bundle over Y where the fibres are $\dot{B}_d$ where d is the codimension of Y in X. If we consider the sheaf $\mathbb{Z}$ on $\dot{\mathcal{N}}\, Y$ and the inclusion $\dot{\mathcal{N}}\, Y \hookrightarrow \mathcal{N}Y$ then the direct image functor is not exact we have

$$R^q j_*(\mathbb{Z}) = 0 \text{ if } q \neq 2d-1, 0$$

and in degree zero

$$j_*(\mathbb{Z}) \text{ is the constant sheaf } \mathbb{Z}$$

and

$$R^{2d-1} j_*(\mathbb{Z})$$

is a local system of sheaves with stalk at a point isomorphic to $\mathbb{Z}(-d)$. This is just the local system of the cohomology groups of the fibres $\dot{B}_d$. I claim that this local system is trivial because if we consider the Betti-cohomology, then we have an orientation on the normal bundle and the stalk $R^{2d-1} j_*(\mathbb{Z})_x = \mathbb{Z}$. In the other realizations we get trivializations from the comparison isomorphisms. We get a spectral sequence for the cohomology with E_2-term

$$H^p(Y, R^q j_*(\mathbb{Z})) \Rightarrow H^n(\dot{\mathcal{N}}\, Y, \mathbb{Z})$$

and since there are only two columns we get the Gysin sequence

$$\to H^n(Y, R^0 j_*(\mathbb{Z})) \to H^n(\dot{\mathcal{N}}\, Y, \mathbb{Z}) \to H^{n-2d+1}(Y, R^{2d-1} j_*(\mathbb{Z})) \to H^{n+1}(Y, R^0 j_*(\mathbb{Z}))$$

Now we have to assume that the kernel and the cokernel of

$$H^{n-2d+1}(Y, R^{2d-1} j_*(\mathbb{Z})) \to H^{n+1}(Y, R^0 j_*(\mathbb{Z}))$$

are pure motives. Since the local system $R^{2d-1} j_*(\mathbb{Z})$ is trivial this map

$$H^{n-2d+1}(Y, R^{2d-1} j_*(\mathbb{Z})) = H^{n-2d+1}(Y, \mathbb{Z}(-2d)) \to H^{n+1}(Y, R^0 j_*(\mathbb{Z}))$$

is given by the multiplication by d-th the Chern class of the normal bundle and we see that the map is induced by an algebraic cycle and this makes it clear that we can consider $H^n(\overset{\bullet}{\mathcal{N}}\, Y, \mathbb{Z})$ as a mixed motive. Of course the kernel and the cokernel are just the terms $E_3^{n,0}, E_3^{n-2d+1,2d-1}$.

We want to say a few words about the de-Rham realization. At first we consider the case that Y is of codimension one. We return to the global situation and consider $Y \hookrightarrow X$. In a suitable neighborhood of a point $y_0 \in Y$ the subscheme Y is given by an equation $x_1 = 0$, let $x_1, x_2, \dots, x_{d_0}$ be a set of local coordinates in this neighborhood. Then we define two modified de-Rham complexes:

The first one is

$$j_{*,\log}(\Omega^\bullet)_{y_0} = 0 \to \mathcal{O}_{X,y_0,\log} \to \Omega^1_{y_0,\log} \to \dots \Omega^\nu_{y_0,\log} \to$$

where $\Omega^\nu_{y_0,\log}$ is $\emptyset_{X,y_0}-$ module generated by the forms

$$\frac{dx_1}{x_1} \wedge dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_{\nu-1}}$$

it is slightly larger than $\Omega^\bullet_X$ in all degrees > 0.

The second one is

$$j_{!,\mathrm{zero}}(\Omega^\bullet)_{y_0} = 0 \to x_1\mathcal{O}_{X,y_0} \to \mathcal{O}_{X,y_0}dx_1 \oplus x_1\mathcal{O}_{X,y_0}dx_2 \oplus \dots \to \dots$$

where the differentials in degree ν are generated by the differentials $dx_1 \wedge dx_{i_2} \wedge \dots dx_{i_\nu}$ and $x_1 dx_{i_1} \wedge dx_{i_2} \wedge \dots dx_{i_\nu}$ if all the i_ν are different from the index 1. Now we can define

$$H^\bullet_{DR}(U, \mathbb{Z}) = H^\bullet_{DR}(X, i_*(\mathbb{Z})) = H^\bullet(X, j_{*,\log}(\Omega^\bullet))$$

and

$$H^\bullet_{DR,c}(U, \mathbb{Z}) = H^\bullet_{DR}(X, i_!(\mathbb{Z})) = H^\bullet(X, j_{*,\mathrm{zero}}(\Omega^\bullet))$$

and

$$H^\bullet_{DR}(\overset{\bullet}{\mathcal{N}}\, Y, j_*(\mathbb{Z})) = H^\bullet(X, j_{*,\log}(\Omega^\bullet)/j_{*,\mathrm{zero}}(\Omega^\bullet)).$$

We define the Hodge filtration in the standard way, then we can verify that

$$\begin{array}{c}\mathrm{Im}(H^n_{DR}(U, \mathbb{Z}) \to H^n_{DR}(\overset{\bullet}{\mathcal{N}}\, Y, j_*(\mathbb{Z}))) \cap F^m(H^n_{DR}(\overset{\bullet}{\mathcal{N}}\, Y, j_*(\mathbb{Z}))) = \\ \mathrm{Im}(F^m(H^n_{DR}(U, \mathbb{Z})) \to F^m(H^n_{DR}(\overset{\bullet}{\mathcal{N}}\, Y, j_*(\mathbb{Z})))\end{array} \qquad (Hodge)$$

If Z is not of codimension one, then we blow up X along Y, we get a diagram

$$\begin{array}{ccccc} \hat{Y} & \hookrightarrow & \hat{X} & \leftarrow & U \\ \downarrow & & \downarrow & & \| \\ Y & \hookrightarrow & X & \leftarrow & U \end{array},$$

where now $\hat{Y}$ is of codimension one. The reasoning in SGA4$\frac{1}{2}$ IV.5 shows that we have $H^n(\overset{\bullet}{\mathcal{N}}\, Y, \mathbb{Z}) = H^n(\overset{\bullet}{\mathcal{N}}\, \hat{Y}, \mathbb{Z})$ in the Betti and the ℓ adic realizations, hence we define

$$H^n_{DR}(\overset{\bullet}{\mathcal{N}}\, Y, \mathbb{Z}) = H^n_{DR}(\overset{\bullet}{\mathcal{N}}\, \hat{Y}, \mathbb{Z})$$

and we have constructed all the realizations of our mixed motive $H^n(\overset{\bullet}{\mathcal{N}}\, Y, \mathbb{Z})$. It has a weight filtration coming from our spectral sequence, this weight filtration is visible on all realizations and compatible with the comparison isomorphisms. The weights are n and $n+1$.We get a long exact sequence

$$\to H^n_c(U, \mathbb{Z}) \to H^n(U, \mathbb{Z}) \to H^n(\overset{\bullet}{\mathcal{N}}\, Y, \mathbb{Z}) \to$$

Now we encounter a problem which we have seen in milder form before. We certainly should try to show that the image of

$$H^n(U, \mathbb{Z}) \to H^n(\overset{\bullet}{\mathcal{N}}\, Y, \mathbb{Z})$$

is the cohomology of a mixed motive and we also should show a similar assertion for the kernel the map

$$H^{n-1}(\overset{\bullet}{\mathcal{N}}\, Y, \mathbb{Z}) \to H^n_c(U, \mathbb{Z}).$$

As far as I understand this is one of the major obstacles if we want to construct an abelian category of mixed motives. If we can show that this is so under certain assumptions or in a given concrete situation, then we might be justified to say that

$$\text{image } (H^n_c(U, \mathbb{Z}) \to H^n(U, \mathbb{Z})) = H_!(U, \mathbb{Z})$$

is a pure motive and it sits in an exact sequence

$$0 \to H^n_!(U, \mathbb{Z}) \to H^n(U, \mathbb{Z}) \to \ker(H^n(\overset{\bullet}{\mathcal{N}}\, Y, \mathbb{Z}) \xrightarrow{\delta} H^{n+1}_c(U, \mathbb{Z})) \to 0.$$

The motive $H^n_!(U, \mathbb{Z})$ is pure of weight n the kernel $\ker(\delta) = \ker(H^n(\overset{\bullet}{\mathcal{N}}\, Y, \mathbb{Z}) \xrightarrow{\delta} H^{n+1}_c(U, \mathbb{Z}))$ is mixed of weights $n, n+1$. Hence $H^n(U, \mathbb{Z})$ has a weight filtration with weights $n, n+1$.

If d_0 is the dimension of U, then the dimension of Y is $d_0 - d$. If we assume that $n > 2(d_0 - d)$ then the weight n part in $H^n(\overset{\bullet}{\mathcal{N}}\, Y, \mathbb{Z})$ becomes zero and we have $H^n_!(U, \mathbb{Z}) = H^n(X, \mathbb{Z})$ independently of Y. Furthermore $\ker(H^n(\overset{\bullet}{\mathcal{N}}\, Y, \mathbb{Z}) \to H^{n+1}_c(U, \mathbb{Z}))$ is pure of weight $n+1$ and we get

$$[H^n(U, \mathbb{Z})] \in \mathrm{Ext}^1_{\mathcal{MM}}(\ker(\delta), H^n(X, \mathbb{Z})).$$

Now we assume that the codimension of Y is one and we look at the cohomology in degree $n = 2d_0 - 1$ In this case $H^{n+1}_c(U, \mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}(-d_0)$ and $\ker(\delta) \xrightarrow{\sim} \mathbb{Z}(-d_0)^{r-1}$ where r is the number of connected components of Y. Therefore we end up with an element

$$[H^{2d+1}(U, \mathbb{Z})] \in \mathrm{Ext}^1_{\mathcal{MM}}(\mathbb{Z}(-d_0)^{r-1}, H^{2d_0-1}(X, \mathbb{Z})).$$

.

For any element $D \in \ker(\delta), D \neq 0$ we can consider the line $\mathbb{Z}D \in \ker(\delta)$, and the inverse image of this line provides a subextension

$$[H^n(U, \mathbb{Z})][D] \in \mathrm{Ext}^1_{\mathcal{MM}}(\mathbb{Z}(-d_0), H^n(X, \mathbb{Z})).$$

This construction is due to S. Bloch. If $X/\mathbb{Q}$ is a smooth, projective curve, then the only choice we have is $d = 1$ and Y is simply a set of closed points $\{P_1, P_2, \ldots, P_r\}$. Let $\mathbb{Q}(P_i)$ be the residue-field then we put $n_i = \deg(P_i) = [\mathbb{Q}(P_i) : \mathbb{Q}]$. If all the P_i are rational then $H^0(Y, R^1 j_*(\mathbb{Z})) = \mathbb{Z}(-1)^r$. In the general case we have to twist these Tate motives by a finite dimensional representation of the Galois group.

Now $Y(\mathbb{C})$ is a set of $n = \sum n_i$ points and

$$H^1_B(\overset{\bullet}{\mathcal{N}}\, Y) = H^1(\overset{\bullet}{\mathcal{N}}\, Y(\mathbb{C}), \mathbb{Z}) = \mathbb{Z}^n = \bigoplus_I (\bigoplus_{\sigma:\mathbb{Q}(P_i)\to\mathbb{C}} \mathbb{Z})$$

and an element $D \in H^1(\overset{\bullet}{\mathcal{N}}\, Y(\mathbb{C}), \mathbb{Z})$ is simply a divisor, this divisor is rational over $\mathbb{Q}$ if its coefficients at the points lying over a given closed point P_i are constant and hence all equal to an integer d_i. Hence a divisor $D = \sum_i n_i P_i$ can be viewed as an element in $H^1_B(\overset{\bullet}{\mathcal{N}}\, Y)$ and this element is in the kernel of δ if and only if the degree $\deg(D) = \sum n_i d_i = 0$. Therefore we may remove the points P_i from X, we get an open subscheme $U = X \setminus \{P_1, \ldots, P_r\}$ and

$$[H^n(U, \mathbb{Z})][D] \in \mathrm{Ext}^1_{\mathcal{MM}}(\mathbb{Z}(-1), H^1(X, \mathbb{Z})).$$

Now we can send the divisor D to its class $[D]$ in the Picard group $\mathrm{Pic}(X)(\mathbb{Q})$ and we get a diagram

$$\begin{array}{ccc} \ker(\delta)_{\mathbb{Q}} = \{D = \sum n_i P_i, \deg(D) = 0\} & \rightarrow & \mathrm{Ext}^1_{\mathcal{MM}}(\mathbb{Z}(-1), H^1(X, \mathbb{Z})) \\ & \searrow & \downarrow \\ & & \mathrm{Pic}(X)(\mathbb{Q}) \end{array}$$

S. Bloch formulated the idea, that for a good theory of an abelian category mixed motives the horizontal arrow in the top line should become surjective provided the set of points removed is large enough. The vertical arrow is well defined and should define an isomorphism.

We will also allow subvarieties $Y \subset X$ which are singular, we should have some control over the singularities. For instance the case that Y is a divisor with normal crossings should be accepted.

We modify our construction slightly. We define the derived sheaf $j_*(\mathbb{Z})$. We choose an injective resolution of our sheaf $\mathbb{Z}$ on U

$$\begin{array}{ccccccccc} 0 & \rightarrow & \mathbb{Z} & \rightarrow & 0 & \rightarrow & 0 & \rightarrow \\ & & \downarrow & & & & & \\ 0 & \rightarrow & J^0 & \rightarrow & J^1 & \rightarrow & J^2 & \rightarrow \end{array}$$

the complex of sheaves

$$j_*(\mathbb{Z}) := 0 \rightarrow j_*(J^0) \rightarrow j_*(J^1) \rightarrow j_*(J^2) \rightarrow$$

we restrict this sheaf to Y and hope that we can show that

$$H^\bullet(Y, j_*(\mathbb{Z}))$$

is a mixed motive. I think that this has been proved by Deligne in his papers Weil II and Hodge I-III.. Furthermore we hope that can can identify the kernel and the image of

$$\delta : H^\bullet(Y, j_*(\mathbb{Z})) \to H_c^{\bullet+1}(U, \mathbb{Z})$$

as mixed motives, i.e. we can find certain projectors obtained from correspondences which cut out these kernels and cokernels. I do not think that there is a general theorem which asserts this, so it has to be decided in the given concrete case. If we can do this we again get exact sequences

$$0 \to H_!^n(U, \mathbb{Z}) \to H^n(U, \mathbb{Z}) \to \ker(\delta : H^n(Y, j_*(\mathbb{Z})) \to H_c^{n+1}(U, \mathbb{Z})) \to 0.$$

Now the mixed motives will have longer weight filtrations, because $H^\bullet(Y, j_*(\mathbb{Z}))$ has a weight filtration with many different weights $\geq n$.

We get a second mixed motive, if we consider the cohomology with compact supports, namely

$$0 \to \operatorname{koker}(H^{n-1}(X, \mathbb{Z}) \to H^{n-1}(Y, j_*\mathbb{Z})) \to H_c^n(U, \mathbb{Z}) \to H_!^n(U, \mathbb{Z}) \to 0.$$

At this point it is not clear what it means that we have exact sequences of mixed motives. But in any case we can look at the different realizations of these motives and then we get exact sequences in the category $\mathcal{M}_{\mathcal{B}-\lceil\mathcal{R}\langle}$ and $\mathcal{M}_{\mathrm{Gal}}$ and this are abelian categories.

We briefly discuss an example. We may for instance remove three lines in general position from $X = \mathbb{P}^2$, i.e. we consider

$$U = \mathbb{P}^2 \setminus (l_0 \cup l_1 \cup l_2) = \mathbb{P}^2 \setminus \Delta \xrightarrow{\sim} \mathbb{G}_m \times \mathbb{G}_m$$

Then $U = \mathbb{G}_m \times \mathbb{G}_m$, and the Künneth-formula yields

$$H_c^\bullet(U, \mathbb{Z}) = H_c^2(U, \mathbb{Z}) \oplus H_c^3(U, \mathbb{Z}) \oplus H_c^4(U, \mathbb{Z}) = \mathbb{Z}(0) \oplus \mathbb{Z}(-1)^2 \oplus \mathbb{Z}(-2).$$

For the cohomology without supports we get

$$H^\bullet(U, \mathbb{Z}) = H^0(U, \mathbb{Z}) \oplus H^1(U, \mathbb{Z}) \oplus H^2(U, \mathbb{Z}) = \mathbb{Z}(0) \oplus \mathbb{Z}(-1)^2 \oplus \mathbb{Z}(-2).$$

Hence the map $H_c^\bullet(U, \mathbb{Z}) \to H^\bullet(U, \mathbb{Z})$ is zero and this yields short exact sequences

$$0 \to H^\bullet(U, \mathbb{Z}) \to H^\bullet(\Delta, i^*(j_*(\mathbb{Z})) \to H_c^{\bullet+1}(U, \mathbb{Z}) \to 0$$

The computation of the cohomology sheaves $R^\bullet(i^*(j_*(\mathbb{Z}))$ becomes a little bit more complicated, but we can easily compute the E^2 terms $H^p(\Delta, R^q(i^*(j_*(\mathbb{Z}))$ and get

$$H^n(\Delta, i^*(j_*(\mathbb{Z})) = \begin{cases} \mathbb{Z}(0) & \text{for n=0} \\ \mathbb{Z}(-1)^2 \oplus \mathbb{Z}(0) & \text{for n=1} \\ \mathbb{Z}(-2) \oplus \mathbb{Z}(-1)^2 & \text{for n=2} \\ \mathbb{Z}(-2) & \text{for n=3} \end{cases},$$

Construction principles

Now we give a vague outline how we may extend our construction principles to construct certain objects, which may be called mixed motives. These principles will be applied in concrete situations.

We may consider subvarieties Y which are singular, an interesting case is when Y is a divisor with normal crossings. We may also replace the system of coefficients $\mathbb{Z}$ by something more complicated namely a motivic sheaf $\mathcal{F}$ on U. These motivic sheaves are obtained as follows. We may for instance have a smooth, projective morphism $\pi : Z \to U$. Then the cohomology $R^\bullet\pi_*(\mathbb{Z})$ provides such a motivic sheaf. It may happen that certain correspondences of this morphism define idempotents on $R^\bullet\pi_*(\mathbb{Z})$. In this case the cohomology of $R^\bullet\pi_*(\mathbb{Z})$ decomposes into a direct sum, the summands again define motivic sheaves. Finally we may extend a motivic sheaf $\mathcal{F}$ from U to X. This may be done by requiring support conditions for the extensions. If for instance Y is the disjoint union of to subschemes $Y = Y_1 \cup Y_2$, then we may extend to the points in Y_1 by taking the direct image without support conditions and to Y_2 by taking compact supports. (See the construction of Anderson motives in [Ha-Eis] and also later in this paper.) Then we get certain sheaves $\mathcal{F}^\#$ on X and we consider their cohomology $H^\bullet(X, \mathcal{F}^\#)$. These objects will be "mixed motives." We have to take care that these mixed motives still have cohomological realizations, they must have Betti-de-Rham realizations which are mixed Hodge-structures and the ℓ-adic realisations must be modules for the Galois group. Of course we must be aware that we encounter incredibly complicated objects. These mixed motives have very long weight filtration with with many different weights.

But if we are lucky then we can find correspondences, i.e. finite to finite correspondences $T \subset X \times X$ which respect the subset U and then also the subscheme Y. We have the two projections $p_1, p_2 : T \to X$. If we now have a motivic sheaf $\mathcal{F}^\#$ on X and the resulting mixed motive $H^\bullet(X, \mathcal{F}^\#)$ (or a piece cut out by an idempotent) then we get a morphism $[T] : H^\bullet(X, \mathcal{F}^\#) \to H^\bullet(T, p_1^*(\mathcal{F}^\#))$ Now we have $H^\bullet(T, p_1^*(\mathcal{F}^\#)) = H^\bullet(X, R^\bullet p_{2*}(p_1^*(\mathcal{F}^\#))$ and now we hope or assume that we have a natural morphism $\phi : R^\bullet p_{2*}(p_1^*(\mathcal{F}^\#)) \to \mathcal{F}^\#$. Then it is clear that the pair (T, ϕ) induces an endomorphism

$$[T, \phi] : H^\bullet(X, \mathcal{F}^\#) \to H^\bullet(X, \mathcal{F}^\#).$$

These endomorphisms induce endomorphisms in the realizations we can consider the ring of endomorphism generated by these correspondences. An element q in this ring is called an idempotent if it induces an idempotent in any of the realizations.

Then it is tempting to decompose

$$H^\bullet(X, \mathcal{F}^\#) = H^\bullet(X, \mathcal{F}^\#)[q = 1] \oplus H^\bullet(X, \mathcal{F}^\#)[q = 0],$$

we do not know what the individual summands are. But we consider them as mixed motives and we know that they have cohomological realizations.

If we look at examples of these $H^\bullet(X, \mathcal{F}^\#)$ then we see that they become very large, they have weight filtrations with many steps and the dimensions of the cohomology groups become very large.

We hope to find projectors which cut out summands in our mixed motives $H^\bullet(X, \mathcal{F}^\#)$. For instance we can try to construct mixed motives which have only two steps in the weight filtration and where the filtration steps are Tate motives, i.e. we want construct extension of Tate motives of the form

$$\mathcal{X} = \{0 \longrightarrow \mathbb{Z}(0) \longrightarrow X \longrightarrow \mathbb{Z}(-n-1) \longrightarrow 0\},$$

we write $\mathcal{X} \in \mathrm{Ext}^1_{\mathcal{MM}}(\mathbb{Z}(-n-1), \mathbb{Z}(0))$. Such objects have been constructed in [Ha-Eis], these are the Anderson motives. In the second volume of [Ha-Eis] we will extend this construction of Anderson-motives to other groups.

We may also do the following. Let k be an arbitrary field of characteristic zero. As above we remove the triangle Δ from $\mathbb{P}^2$. Now we pick points $Q_i \in l_i$, these point should be different from the intersection points of the lines $P_c = l_a \cap l_b$. We get a second triangle Δ_2 whose sides are the lines passing through the pairs of points Q_i, Q_j. We blow up the three points Q_i, we get a surfaces X. The triangle Δ_1 can be viewed as a subscheme of X, the inverse image of Δ_2 is a hexagon $\tilde{\Delta}_2$ inside of X. Each line of the triangle Δ_1 meets intersects the hexagon in two points.

We put $V = X \setminus \tilde{\Delta}_2 \cup \Delta_1$ and we introduce the notation

$$j_2 : V \hookrightarrow X \setminus \Delta_1 \xrightarrow{j_1} X.$$

On X we define the sheaf $\mathbb{Z}^{\#} = j_{1,*}j_{2,!}(\mathbb{Z})$. Now I hope that the cohomology $H^2(X, \mathbb{Z}^{\#})$ is a very interesting Tate motive which has a three step filtration

$$0 \subset \mathbb{Z}(0) \subset M \subset H^2(X, \mathbb{Z}^{\#}),$$

where $M/\mathbb{Z}(0) \xrightarrow{\sim} \mathbb{Z}(-1), H^2(X, \mathbb{Z}^{\#})/M = \mathbb{Z}(-2)$. (This hope is supported by some tentative computations). Furthermore I hope that

$$0 \to \mathbb{Z}(0) \to M \to \mathbb{Z}(-1) \to 0$$

is a Kummer motive, hence it corresponds to a number $t \in k^\times$. The other quotient $H^2(X, \mathbb{Z}^{\#})/\mathbb{Z}(0)$ is also a Kummer-motive $\otimes\mathbb{Z}(-1)$. This Kummer-motive should be given by the number $1 - t \in k^\times$. The number t should correspond to the position of the third point $Q_2 \in l_2$. We denote this motive by $\mathcal{M}_{x,1-x}$.

Such a motive is of course not in $\mathrm{Ext}^1_{\mathcal{MM}}(\mathbb{Z}(-n-1), \mathbb{Z}(0))$, but we may form "framed" direct sums

$$framed(\bigoplus_{i=1}^{r} \mathcal{M}_{x_i,1-x_i}).$$

If now $\sum x_i \wedge (1-x_i) = 0$ in $\Lambda^2 k^\times$ then we may hope that we can change the basis in $M/\mathbb{Z}(0) = \oplus\mathbb{Z}(-1)e_i$ in such a way that $0 \to \mathbb{Z}(0) \to M_i \to \mathbb{Z}(-1)e_i \to 0$ splits for $i = 1, \ldots, [r/2]$ and $0 \to M_i/\mathbb{Z}(0) \to H^2(X, \mathbb{Z}^{\#})_i \to \mathbb{Z}(-2) \to 0$ splits for $i = [r/2] + 1, \ldots, r$.

This seems to indicate that in some sense (??)

$$framed(\bigoplus_{i=1}^{r} \mathcal{M}_{x_i,1-x_i}) = \{0 \to \mathbb{Z}(0) \to X \to \mathbb{Z}(-2) \to 0\} \oplus \mathbb{Z}(-2)^r.$$

if $\sum x_i \wedge (1-x_i) = 0$. Hence we can say that r-tuples of elements $x_1, x_2, \ldots, x_r \in k^\times \setminus \{1\}$ which satisfy $\sum x_i \wedge (1 - x_i) = 0$ in $\Lambda^2 k^\times$ produce elements in $\mathrm{Ext}^1(\mathbb{Q}(-2), \mathbb{Q}(0)) = K_3(k)$.

1.5 Extensions

Let us assume, that we produced extensions $\mathcal{X}$, which are sequences

$$0 \longrightarrow \mathbb{Z}(0) \longrightarrow X \longrightarrow \mathbb{Z}(-n-1) \longrightarrow 0.$$

We consider their realizations:

1.5.1 The Betti realization The Betti cohomology X_B is a free $\mathbb{Z}$-module which sits in an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow X_B \longrightarrow \mathbb{Z} \longrightarrow 0.$$

We have an involution F_∞ on X_B which acts by 1 on the left copy of $\mathbb{Z}$ and by $(-1)^{n+1}$ on the right. The extremal modules have canonical generators, in other words as modules they are equal to $\mathbb{Z}$. The *de-Rham realization* yields is

an exact sequence of $\mathbb{Q}$ vector spaces

$$0 \longrightarrow \mathbb{Q}(0) \longrightarrow X_{DR} \longrightarrow \mathbb{Q}(-n-1) \longrightarrow 0$$

together with a descending filtration

$$\begin{array}{ccccccccc} F^0 X_{DR} & \supset & F^1 X_{DR} & = \cdots = & F^{n+1} X_{DR} & \supset & F^{n+2} X_{DR} & = & 0 \\ & & & & \downarrow & & & & \\ & & & & F^{n+1}\mathbb{Q}(-n-1) = \mathbb{Q}. & & & & \end{array}$$

where the downwards arrow is an isomorphism.

We have a comparison isomorphism between the two exact sequences

$$I : X_B \otimes \mathbb{C} \longrightarrow X_{DR} \otimes \mathbb{C}.$$

an this comparison isomorphism satisfies

$$I \circ c_B \circ F_\infty = c_{dRh} \circ I$$

where the $c_{??}$ is always the action of the complex conjugation on the coefficients.

We want to consider these objects $(X_B, F_\infty, X_{DR}, F, I)$ as objects of an abelian category $B\!-\!dRh$ it is related to the category of mixed Hodge-structures. Finally we have the *p-adic realizations*. For each prime p we have an action of

the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $X_B \otimes \mathbb{Z}_p$ and we get an exact sequence

$$0 \longrightarrow \mathbb{Z}_p(0) \longrightarrow X_B \otimes \mathbb{Z}_p \longrightarrow \mathbb{Z}_p(-n-1) \longrightarrow 0$$

and this action is unramified outside of $S \cup \{p\}$.

Again we notice that the comparison isomorphism gives us canonical generators in $\mathbb{Z}_p(0)$ and $\mathbb{Z}_p(-n-1)$

1.5.2 The Betti- de-Rham extension class

We can associate an extension class

$$[X]_{B-dRh} \in \mathrm{Ext}^1_{B-dRh}(\mathbb{Z}(-n-1), \mathbb{Z}(0))$$

to our objects X. To do this we have to understand $\mathrm{Ext}^1_{B-dRh}(\mathbb{Z}(-n-1), \mathbb{Z}(0))$. We distinguish two cases, first we assume that n even. We know that $\mathbb{Z}(-n-1)$

has a canonical generator $1_B^{(-n-1)}$. We have a unique lift of this generator to an element $e_B^{(-n-1)} \in X_B \otimes \mathbb{Q}$ which lies in the -1 eigenspace for F_∞. We also find a unique $e_{DR}^{(-n-1)} \in F^{n+1}X_{DR} \otimes \mathbb{C}$ which maps to the image of $I(1_B^{(-n-1)})$. Then $1_B^{(-n-1)} - I^{-1}(e_{DR}^{(-n-1)})$ maps to zero in $\mathbb{C}(-n-1)$. Hence we see that $1_B^{(-n-1)} - I^{-1}(e_{DR}^{(-n-1)}) \in \mathbb{C}(0) = \mathbb{C}$.

Finally we look at the action of c_B on this class. Since F_∞ acts trivially on $\mathbb{Z}(0)$ we get from the compatibility condition

$$\begin{aligned} c_B(1_B^{(-n-1)} - I^{-1}(e_{DR}^{(-n-1)})) &= F_\infty \circ c_B(1_B^{(-n-1)} - I^{-1}(e_{DR}^{(-n-1)})) \\ &= -(1_B^{(-n-1)} - I^{-1}(e_{DR}^{(-n-1)})) \end{aligned}$$

and therefore we conclude that the extension class lies in $i\mathbb{R}$. If we choose $\frac{1}{2\pi i}$ as a basis element for $i\mathbb{R}$ then we get an identification

$$\mathrm{Ext}^1_{B-dRh}(\mathbb{Z}(-n-1), \mathbb{Z}(0)) = \mathbb{R}.$$

Now we consider the case that n is odd.

Again we know that $\mathbb{Z}(-n)$ has a canonical generator $1_B^{(-n)}$. We have a non unique lift of this generator to an element $e_B^{(-n)} \in X_B \otimes \mathbb{Q}$. We find a unique $e_{DR}^{(-n)} \in F^n X_{DR} \otimes \mathbb{C}$ which maps to the image of $I(1_B^{(-n)})$. Then $1_B^{(-n)} - I^{-1}(e_{DR}^{(-n)})$ maps to zero in $\mathbb{C}(-n)$. Hence we see that $1_B^{(-n)} - I^{-1}(e_{DR}^{(-n)}) \in \mathbb{C}(0) \mod \mathbb{Z} = \mathbb{C} \mod \mathbb{Z}$.

We compute the action of c_B on this class and this time we get

$$\begin{aligned} c_B(1_B^{(-n)} - I^{-1}(e_{DR}^{(-n)})) &= F_\infty \circ c_B(1_B^{(-n)} - I^{-1}(e_{DR}^{(-n)})) \\ &= (1_B^{(-n)} - I^{-1}(e_{DR}^{(-n)})) \end{aligned}$$

and hence

$$\mathrm{Ext}^1_{B-dRh}(\mathbb{Z}(-n), \mathbb{Z}(0)) = \mathbb{R}/\mathbb{Z}.$$

Now we encounter the fundamental question: What are the classes which come from a mixed motive over $\mathbb{Q}$, in other words what is the image

$$\mathrm{Ext}^1_{\mathcal{M}}(\mathbb{Z}(-n-1), \mathbb{Z}(0)) \to \mathrm{Ext}^1_{B-dRh}(\mathbb{Z}(-n-1), \mathbb{Z}(0)).$$

Since the group on the left hand side is not really defined we may ask: How many objects of the form

$$\mathcal{X} = \{0 \longrightarrow \mathbb{Z}(0) \longrightarrow X \longrightarrow \mathbb{Z}(-n-1) \longrightarrow 0\}$$

can we find somewhere in the cohomology of an algebraic variety over $\mathbb{Q}$ and what are the possible values for their extension class in the category $B - dRh$.

The general conjectures about the connection between K-theory and the hypothetical category of mixed motives seems to suggest the following question. The case $n > 0$ even:

Is it true that for any such object $\mathcal{X}$ the extension class

$$[\mathcal{X}_{B-dRH}] = \zeta'(-n)a(\mathcal{X}) \qquad (Ext_{B-dRh})$$

with some rational number $a(\mathcal{X})$? What are the possible denominators, are they bounded?

The case n odd:
Is it true that for any such object $\mathcal{X}$ the extension class

$$[\mathcal{X}_{\mathcal{B}-\lceil\mathcal{R}\mathcal{H}}] \in \mathbb{Q}/\mathbb{Z}$$

in other words we get only torsion elements?

I think that we must be aware, that it is by no means clear, that our construction principles do not go beyond the construction of mixed motives constructed by K-theory, or other known approaches to the category of mixed Tate-motives.

Actually I have the feeling that these two conjectures combined with with the conjectures on the p-adic realisation, which is formulated in 1.6. are really conjectures about some finiteness results, which are a little bit scaring.

1.3.1 The Galois-module extension class

Now we consider the attached sequences of Galois modules

$$0 \longrightarrow \mathbb{Z}_p(0) \longrightarrow X \otimes \mathbb{Z}_p \longrightarrow \mathbb{Z}_p(-n-1) \longrightarrow 0$$

We consider exact sequences of Galois-modules $\mathbb{Z}_p \times \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-Moduln

$$0 \to \mathbb{Z}_p(0) \to X \to \mathbb{Z}_p(-n-1) \to 0.$$

We assume that n is even and $p > 2$. Then we know especially $p-1 \nmid n+1$.

Such a module provides an element X

$$[X] \in \mathrm{Ext}^1_{\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})}(\mathbb{Z}_p(-n-1), \mathbb{Z}_p(0)) = H^1(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \mathbb{Z}_p(n+1)) = \lim_{\substack{\leftarrow \\ m}} H^1(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \mathbb{Z}/p^m\mathbb{Z}(n+1)).$$

It may be helpful if we introduce the notation

$$\mathbb{Z}_p/p^m\mathbb{Z}_p(n+1) = \mu_{p^m}^{\otimes(n+1)}$$

To understand this cohomology we pass to the cyclotomic extensions $\mathbb{Q}(\zeta_{p^m})/\mathbb{Q}$ and we denote their Galoisgroups over $\mathbb{Q}$ by $\bar{\Gamma}_m$.We have the canonical isomorphism

$$\alpha : \bar{\Gamma}_m = \mathrm{Gal}(\mathbb{Q}(\zeta_{p^m})/\mathbb{Q}) \xrightarrow{\sim} (\mathbb{Z}/p^m\mathbb{Z})^*.$$

Our assumptions on p, n imply that we can find an $x \in (\mathbb{Z}/p^m\mathbb{Z})^*$ such that $x^{n+1} \not\equiv 1 \mod p$ and this implies that

$$H^1(\bar{\Gamma}_m, \mu_{p^m}^{\otimes(n+1)}) = H^2(\bar{\Gamma}_m, \mu_{p^m}^{\otimes(n+1)}) = 0$$

and the Hochschild-Serre spectral sequence yields an isomorphism

$$H^1(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \mu_{p^m}^{\otimes(n+1)}) \simeq H^1(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\zeta_{p^m})), \mu_{p^m}^{\otimes(n+1)})^{\bar{\Gamma}_m}.$$

Since the $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\zeta_{p^m}))$-module $\mu_{p^m}^{\otimes(n+1)}$ is trivial we have the Kummer isomorphism

$$H^1(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(\zeta_{p^m})), \mu_{p^m}^{\otimes(n+1)})^{\bar{\Gamma}_m} \simeq ((\mathbb{Q}(\zeta_{p^m})^* \otimes \mu_{p^m}^{\otimes n})^{\bar{\Gamma}_m} =$$

$$\big(\mathbb{Q}(\zeta_{p^m})^* \otimes \mathbb{Z}/p^m\mathbb{Z}\big)(-n) = \{x | x \in \mathbb{Q}(\zeta_{p^m})^* \otimes \mathbb{Z}/p^m\mathbb{Z}, x^\sigma = x^{\alpha(\sigma)^{-n}}\}.$$

An element $\xi \in H^1(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \mathbb{Z}_p(n+1))$ is a sequence of elements

$$\xi = (\dots, \xi_m, \dots)$$

which satisfy

$$\xi_m \in (\mathbb{Q}(\zeta_{p^m})^* \otimes \mu_{p^m}^{\otimes n}))^{\Gamma_m} = H^1(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \mu_{p^m}^{\otimes(n+1)})$$

and are mapped to eachother by the transition map: The homomorphism $\mathbb{Z}/p^{m+1}\mathbb{Z} \to \mathbb{Z}/p^m\mathbb{Z}$ yields the projective system and consequently we get a homomorphism

$$(\mathbb{Q}(\zeta_{p^{m+1}})^* \otimes \mu_{p^{m+1}}^{\otimes n})^{\Gamma_{m+1}} \to (\mathbb{Q}(\zeta_{p^m})^* \otimes \mu_{p^m}^{\otimes n})^{\Gamma_m}$$

and we have to identify this homomorphism. An easy computation yields

$$N_{\mathbb{Q}(\zeta_{p^{m+1}})/\mathbb{Q}(\zeta_{p^m})}(\xi_{m+1}) = \xi_m^p,$$

and we conclude that our homomorphism is given by

$$N_{m+1,m}^{1/p} : (\mathbb{Q}(\zeta_{p^{m+1}})^* \otimes \mu_{p^{m+1}}^{\otimes n})^{\Gamma_{m+1}} \to (\mathbb{Q}(\zeta_{p^m})^* \otimes \mu_{p^m}^{\otimes n})^{\Gamma_m}.$$

With respect to these homomorphisms we have

$$H^1(\mathbb{Q}, \mathbb{Z}_p(n+1) = \varprojlim (\mathbb{Q}(\zeta_{p^m})^* \otimes \mu_{p^m}^{\otimes n}))^{\Gamma_m}$$

We consider the restriction

$$H^1(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \mathbb{Z}_p(n+1)) \to H^1(\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p), \mathbb{Z}_p(n+1)).$$

Since $\bar{\Gamma}_m = \mathrm{Gal}(\mathbb{Q}(\zeta_{p^m}/\mathbb{Q}) = \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^m}/\mathbb{Q}_p)$ our considerations above using the Hochschild -Serre sequence also apply to this situation: we may replace $\mathbb{Q}$ by $\mathbb{Q}_p$. Let

$$U_m^{(1)} \subset \mathcal{O}^* = (\mathbb{Z}_p[\zeta_{p^m}])^*$$

be the group of units congruent $1 \mod \mathfrak{p}$,then

$$(U_m^{(1)} \otimes \mu_{p^m}^{\otimes n}))^{\Gamma_m} \subset (\mathbb{Q}_p(\zeta_{p^m})^* \otimes \mathbb{Z}/p^m\mathbb{Z})(-n).$$

The projective limit

$$\varprojlim_n (U_m^{(1)} \otimes \mathbb{Z}/p^m)(-n) = V_p(-n).$$

and I claim that $V(-n)$ is a free $\mathbb{Z}_p$ -module of rank one .

(The Hilbert symbol yields a pairing

$$(U_m^{(1)} \otimes \mathbb{Z}/p^m\mathbb{Z}) \times (U_m^{(1)} \otimes \mathbb{Z}/p^m\mathbb{Z}) \to \frac{1}{p^m}\mathbb{Z}/\mathbb{Z}(1)$$

and a generator in $V_m(-n)$ yields a homomorphism

$$\delta_{n+1} : U_m^{(1)} \otimes \mathbb{Z}/p^m\mathbb{Z} \to \frac{1}{p^m}\mathbb{Z}/\mathbb{Z}$$

which satisfies

$$\delta_{n+1}(u^\sigma) = \alpha(\sigma)^{n+1}\delta_{n+1}(u)$$

and this must be the Coates-Wiles homomorphism. (Washington, Chap. 13).)

Now we assume that n is even. We introduce the subring $\mathcal{O}_m = \mathbb{Z}[\frac{1}{p}, \zeta_{p^m}]$. We define elements $\zeta_{p^m}^{\otimes n} := \zeta_{p^m} \otimes \zeta_{p^m} \cdots \otimes \zeta_{p^m} \in \mu_p^{\otimes n}$ and we construct the Soulé elements in $(\mathcal{O}_m^* \otimes \mu_{p^m}^{\otimes n})^{\Gamma_m}$:

$$c_{n,m}(p) = \prod_{\substack{(a,p)=1 \\ a \mod p^m}} (1 - \zeta_{p^m}^a)^{a^n} \otimes \zeta_{p^m}^{\otimes n}$$

and $N_{m+1,m}^{1/p}(c_{n,m+1}) = c_{n,m}$. We get an element in the projective limit

$$c_n(p) = (\ldots, c_{n,m}(p), \ldots) \in H^1(\mathbb{Q}, \mathbb{Z}_p(n+1)).$$

These elements $c_{n,m}(p)$ and $c_n(p)$ do not depend on the choice of the primitive p^m-th root of unity, they are canonical elements in $H^1(\mathbb{Q}, \mathbb{Z}_p(n+1))$.

If we send the elements $c_p(n)$ into the local Galois cohomology then they become a multiple of a generator e_n

$$c_n(p) = \ell_p(n) \cdot e_n$$

with $\ell_p(n) \in \mathbb{Z}_p$. I think, that the results on p-adic L-functions and Iwasawas results imply (Washington 13.56) that

$$\ell_p(n) = \zeta_p(n+1) \mod \mathbb{Z}_p^*$$

where

$$\zeta_p(n+1) = \lim_{\alpha \to \infty} \zeta(n+1-(p-1)p^\alpha).$$

I also assume at this point that $\zeta_p(n+1) \neq 0$. In any case it is not clear whether $\lim\limits_{\leftarrow}(\emptyset_m^* \otimes \mu_{p^m}^{\otimes n})^{\Gamma_m}$ has a non zero image in $H^1(\mathbb{Q}, \mathbb{Z}_p(n+1))$ without such an assumption.

1.6 The p-adic extension classes

We assume that $n > 0$, even and that we have constructed a mixed motive over $\mathbb{Q}$

$$\mathcal{X} = \{0 \to \mathbb{Z}(0) \to X \to \mathbb{Z}(-n-1) \to 0\}$$

it provides an extension class

$$[\mathcal{X}]_p \in H^1(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \mathbb{Z}_p(n+1))$$

for all primes p. We can say that these Galois-modules form a "compatible system" of representations for the Galois group, because all the Galois-modules come from the same global object. The Soulé elements allow us to formulate an assertion which makes the above statement precise.

We ask:

Let $\mathcal{X}$ be a mixed motive as above. Is it true that for all primes p

$$[\mathcal{X}]_p = a(\mathcal{X})c_p(n) \qquad (Ext_{p-etale})$$

where $a(\mathcal{X})$ is the same number which occurred in our formula for the Hodge-de Rham extension class? Perhaps it is more reasonable to ask the weaker question whether this relation holds for the image of $[\mathcal{X}]_p$ in $H^1(\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p), \mathbb{Z}_p(n+1))$

Do exotic mixed Tate motives exist?

We call a mixed Tate motive $\mathcal{X}$ exotic if one of the above assertions fails. Clearly there are various qualities of being exotic. In the course of these notes we will construct mixed motives for which we know that (Ext_{B-dRh}) is true, but where we do not know how to prove $(Ext_{p-etale})$. In my lecture notes volume "Eisenstein Kohomologie und die Konstruktion gemischter Motive" I gave the construction of the Anderson motives. For these motives I computed the Hodge-de-Rham extension class and showed that in fact they are of the predicted form $a(\mathcal{X})\zeta'(-n)$.

I also hope that I can compute the p-adic extension classes, so both questions can be answered positively for these motives. In their paper "Dirichlet motives via modular curves. " Ann. Sci. cole Norm. Sup. (4) 32 (1999) A. Huber and G. Kings prove that the p-adic classes have the right form. But they use K-theory and I do not understand completely how the object in K-theory can be compared to the object which I construct.

I also will construct mixed Anderson motives $\mathcal{X}(f)$ for the symplectic group GSp_2, they will be labeled by classical elliptic modular forms f. Again I will compute the Hodge-de-Rham extension class, the computation of the p-adic extension class seems to be even much more difficult.

On the other hand in the volume "The 1-2-3 of modular forms" I pointed out that the non existence of exotic mixed Tate-motives gives us a hint to prove the conjectural congruences in my article " A congruence between a Siegel and an elliptic modular form."

Of course the non existence of exotic Tate motives would be an interesting theorem in arithmetic algebraic geometry. But it seems to me also interesting that it has such concrete consequences which can be checked in examples.

Finally I will construct mixed Anderson motives for the symplectic group GSp_4, and here I find some objects whose Betti-de-Rham probably do not satisfy (Ext_{B-dRh}). These are labeled by eigenclasses in the cohomology of $\mathrm{Gl}_4(\mathbb{Z})$ with suitable coefficients.

1.7 Final remarks At this point I am always a little bit confused. Experts in K-theory keep telling me that the answer to both questions is clearly yes, i.e. there are no exotic mixed Tate motives. They say that this follows if we work in the category of mixed Tate-motives over number fields, which has been constructed by Voevodsky. In this category the computation of the extension groups $\mathrm{Ext}_{\mathcal{MM}/\mathbb{Q}}$ is reduced to the computation of K-groups of number fields, which has been done by Borel.

But it is not clear to me whether the mixed Tate motives which I constructed above can be viewed as objects in Voevodsky's category. Especially this applies to the examples for GSp_4.

We add some further speculations: Assume that we have always the above relation between the Hodge-de-Rham extension class to the p-adic extension class. If this would be the case then we would have a tool to attack the question concerning the denominators of $a(\mathcal{X})$. If for instance $\zeta_p(n+1-(p-1)) \not\equiv 0 \mod p$ we have seen that the image of $c_p(n)$ in $H^1(\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p), \mathbb{Z}_p(n+1))$ is a generator and therefore we can not have a p in the denominator of $a(\mathcal{X})$. But if the $\zeta-$value is zero $\mod p$ we get that $c_p(n)$ is locally at p a $p-$ th power. In such a case the Vandiver conjecture would still imply that $c_p(n)$ itself is not a p-th power.

But independently of the validity of the Vandiver conjecture we can consider we can pick an n as above and a prime p. If $\zeta_p(n+1) \neq 0$ then we would know that the p-denominator in $a(\mathcal{X})$ is at most $p^{\delta_p(n)}$.

But of course at this point we cannot say anything for a single value $n > 0$. We have the remarkable result by Christophe Soulé that for $p > v(n)$ the Vandiver conjecture is true for the n-component which means that we know that $c_p(n,1) \neq 0$ and this is equivalent to $c_p(n)$ is not a p-th power. Hence we have to check a finite number of primes. hence we see that we can bound the denominators and we have to check a finite set of primes. But this finite set is so enormously large that we can not check them all.

We can also speculate what happens if n is odd. Then we have seen that the p adic extension classes are all zero. This applies only to the projective limit, the cohomology on the finite level may be non zero. This supports the idea that the Hodge-de Rham classes should be torsion classes in this case.

2 Modular Construction of mixed Motives

We want to produce situations of this kind in the context of Shimura Varieties.

We consider the case $G = \mathrm{Gl}_2/\mathrm{Spec}(\mathbb{Z})$. For a prime p we define the two subgroups $K_1(p) \subset K_0(p) \subset \mathrm{Gl}_2(\mathbb{Z}_p)$

$$K_0(p) = \{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Gl}_2(\mathbb{Z}_p) |\ c \equiv 0 \mod p\}$$

and for $K_1(p)$ we require $a \equiv 1 \mod p$ in addition.

Let $K_{f,0}(p) \subset K_f = G(\hat{\mathbb{Z}})$ be the subgroup where we replace the factor $G(\mathbb{Z}_p)$ by $K_0(p)$ and accordingly we define $K_{f,1}(p)$.

We pick a prime p_0 which will play an auxiliary role. To this prime I attach the modular curve $Y_0(p_0)$ whose complex points are $Y_0(p_0)(\mathbb{C}) = G(\mathbb{Q})\backslash G(\mathbb{R})/K_\infty \times G(\mathbb{A}_f)/K_{f,0}(p)$. Here $K_\infty = \mathrm{SO}(\mathbb{R}) \times \mathbb{R}^\times_{>0}$. This curve is a curve over $\mathrm{Spec}(\mathbb{Z})$ the fiber over p_0 is singular. It has two cusps, we have the compactification

$$X_0(p_0) = Y_0(p_0) \cup \{0, \infty\}$$

where two cusps are actually rational.

For any even integer $n > 0$ I consider the $SL_2(\mathbb{Z})$-modules

$$\mathcal{M}_n = \left\{ \Sigma a_\nu X^\nu Y^{n-\nu} | a_\nu \in \mathbb{Z} \right\}.$$

Actually I can realize $\mathcal{M}_n$ as a $\mathbb{Z}$-module of homogeneous polynomials $P(c,d)$ of degree n in the lower row entries of the matrix

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The group acts by multiplication from the right and I twist by a power of the determinant, i.e.

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} P(c,d) = P(\alpha c + \gamma d, c\beta + \delta d) \det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-n}$$

(see [Ha] ...). (The variable c corresponds to X and d to y). This has the effect that the central character is

$$\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \longrightarrow z^{-n}.$$

They provide sheaves $\tilde{\mathcal{M}}_n$ on the Riemann surface $\Gamma_0(p_0)\backslash H$. This sheaf has a motivic interpretation: We have the universal family $\pi : \mathcal{E} \to Y_0(p_0)$ and its n-fold fibered product

$$\mathcal{E} \times_{Y_0(p_0)} \mathcal{E} \times \cdots \times_{Y_0(p_0)} \mathcal{E} \xrightarrow{\pi_n} Y_0(p_0).$$

This yields a holomorphic map on the complex valued points. The symmetric group S_n acts on this fibered product and hence it acts on the sheaf of cohomology groups $R^n\pi_{n,*}(\mathbb{Z})$. Then it is well known that

$$\tilde{\mathcal{M}}_n = R^n\pi_{n,*}(\mathbb{Z})_{S_n(\varepsilon)}.$$

Now we choose another prime p which should be different from p_0. We can interpret $\tilde{\mathcal{M}}_{n,p} = \tilde{\mathcal{M}}_n \otimes \mathbb{Z}_p$ as a p-adic sheaf over $Y_0(p_0)$. This prime will be fixed for the rest of this article and we will write $\tilde{\mathcal{M}}$ for $\tilde{\mathcal{M}}_{n,p}$. We have the two cusps ∞ and 0 and let $\dot{D}_\infty, \dot{D}_0$ be small punctured discs around the cusps. We get the exact sequence in cohomology

$$\begin{aligned} H^0(Y_0(p_0) \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \tilde{\mathcal{M}}) &\longrightarrow H^0(\dot{D}_\infty \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \tilde{\mathcal{M}}) \oplus H^0(\dot{D}_0 \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \tilde{\mathcal{M}}) \longrightarrow \\ &\xrightarrow{\delta} H^1_c(Y_0(p_0) \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \tilde{\mathcal{M}}) \longrightarrow H^1(Y_0(p_0) \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \tilde{\mathcal{M}}) \xrightarrow{r} \\ &\longrightarrow H^1(\dot{D}_\infty \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \tilde{\mathcal{M}}) \oplus H^1(\dot{D}_0 \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \tilde{\mathcal{M}}) \longrightarrow \end{aligned} \tag{1}$$

We have the action of the Hecke algebra on all these modules and especially we have the operators T_p. Whenever we have a finitely generated $\mathbb{Z}_p$ module M with an action of the Hecke algebra we can decompose it into

$$M = M_{\text{ord}} \oplus M_{\text{nilpt}}.$$

Here M_{nilpt} is the maximal submodule on which T_p acts nilpotently and M_{ord} is a complement on which T_p acts as an isomorphism. We can say

$$M_{\text{ord}} = \bigcap_{m \geq 1} T_p^m(M).$$

Since we assumed that $n > 0$ the morphism δ is injective and r is surjective if we restrict to the ordinary part.

The Hecke operator acts by the eigenvalue $p^{n+1} + 1$ on the ordinary cohomology at infinity.

All these modules are Galois modules and as $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ modules

$$\begin{array}{rcl} H^0(\dot{D}_0, \tilde{\mathcal{M}}) \oplus H^0(\dot{D}_\infty, \tilde{\mathcal{M}}) & = & \mathbb{Z}_p(0) \oplus \mathbb{Z}_p(0) \\ H^1_{\mathrm{ord}}(\dot{D}_0, \mathcal{M}) \oplus H^1_{\mathrm{ord}}(\dot{D}_\infty, \tilde{\mathcal{M}}) & = & \mathbb{Z}_p(-n-1) \oplus \mathbb{Z}_p(-n-1), \end{array}$$

$$\begin{array}{c} 0 \to \mathbb{Z}_p(0)^2 \to H^1_c(Y_0(p_0) \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \tilde{\mathcal{M}}) \to H^1_!(Y_0(p_0) \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \tilde{\mathcal{M}}) \to 0 \\ 0 \to H^1_!(Y_0(p_0) \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \tilde{\mathcal{M}}) \to H^1(Y_0(p_0) \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \tilde{\mathcal{M}}) \to \mathbb{Z}_p(-n-1)^2 \to 0 \end{array}$$

We can also compute these cohomology groups as the cohomology of the sheaves

$$i_!(\tilde{\mathcal{M}}), Ri_*(\tilde{\mathcal{M}})$$

on the complete curve $X_0(p_0)$.

Now we follow the suggestion of G. Anderson and construct a sheaf $\tilde{\mathcal{M}}^\#$ on $X_0(p_0)$ which is a compromise: We have

$$i_0 : Y_0(p_0) \longrightarrow Y_0'(p_0) = Y_0(p_0) \cup \{0\} \overset{i_\infty}{\hookrightarrow} X_0(p_0)$$

and

$$\tilde{\mathcal{M}}^\# = R^\bullet i_{\infty,*} i_{0,!}(\tilde{\mathcal{M}}).$$

The cohomology

$$H^1(X_0(p_0) \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \tilde{\mathcal{M}}^\#) = H^1(Y_0'(p_0) \times_{\mathbb{Q}} \overline{\mathbb{Q}}, i_{0,!}(\tilde{\mathcal{M}}))$$

has a filtration in three steps:

$$\begin{array}{rl} F^0 = & H^1_{\mathrm{ord}}(X_0(p_0) \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \tilde{\mathcal{M}}^\#) \supset F^1 H^1_{\mathrm{ord}}(X_0(p_0) \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \tilde{\mathcal{M}}^\#) \\ & \supset F^2 H^1_{\mathrm{ord}}(X_0(p_0) \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \tilde{\mathcal{M}}) \supset \{0\} = F^3, \end{array}$$

where

$$\begin{array}{rl} F^0/F^1 & \simeq \mathbb{Z}_p(-n-1) \\ F^1/F^2 & \simeq H^1_!(Y_0(p) \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \tilde{\mathcal{M}})_{\mathrm{ord}} \\ F^2 = F^2/F^3 & \simeq \mathbb{Z}_p(0) \end{array}$$

This filtration is of course compatible with the action of the Hecke algebra and the Galois group.

If we divide by the lowest step in the filtration, we get

$$0 \to H^1_!(Y_0(p_0) \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \tilde{\mathcal{M}})_{\mathrm{ord}} \to H^1(Y_0'(p_0) \times_{\mathbb{Q}} \overline{\mathbb{Q}}, i_{0!}(\tilde{\mathcal{M}}))/\mathbb{Z}_p(0) \to \mathbb{Z}_p(-n-1) \to 0.$$

This sequence has a canonical splitting which is compatible with the action of the Hecke algebra if we tensorize by $\mathbb{Q}_p$. Hence if we choose a generator $\omega_n \in H^1_{\mathrm{ord}}(\dot{D}_\infty, \tilde{\mathcal{M}}_n)$, then

$$H^1_{\mathrm{ord}}(Y_0'(p_0) \times_{\mathbb{Q}} \overline{\mathbb{Q}}, i_{0!}(\tilde{\mathcal{M}}))/\mathbb{Z}_p(0) \subset H^1_!(Y_0(p_0) \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \tilde{\mathcal{M}}) \oplus p^{\delta_p(n)} \cdot \mathrm{Eis}(\omega_n) \cdot \mathbb{Z}_p,$$

where the quotient of the two modules is $\mathbb{Z}/p^{\delta_p(n)}\mathbb{Z}$, and where $p^{\delta_p(n)}$ is the denominator of the Eisenstein class, i.e. $p^{\delta_p(n)} \cdot \mathrm{Eis}(\omega_n)$ is a primitive element, and therefore

$$p^{\delta_p(n)}\mathrm{Eis}(\omega_n) \cdot \mathbb{Z}_p \subset H^1(Y_0'(p_0) \times_{\mathbb{Q}} \overline{\mathbb{Q}}, i_{0,!}(\tilde{\mathcal{M}}))/\mathbb{Z}_p(0).$$

Hence we can construct a submodule

$$H^1_{\mathrm{Eis}}(Y_0(p_0) \times_{\mathbb{Q}} \overline{\mathbb{Q}}, i_{0,!}(\tilde{\mathcal{M}}))$$

which sits in an exact sequence

$$0 \to \mathbb{Z}_p(0) \to H^1_{\mathrm{Eis}}(Y_0'(p_0) \times_{\mathbb{Q}} \overline{\mathbb{Q}}, i_{0!}(\tilde{\mathcal{M}})) \to \mathbb{Z}_p(-n-1) \to 0$$

of Galoismodules and on which T_p acts as a scalar with eigenvalue $p^{n+1}+1$.

In some sense we can say that this Galoismodule is the p-adic realization of a mixed motive

$$H^1_{\mathrm{Eis}}(Y_0'(p_0) \times_{\mathbb{Q}} \overline{\mathbb{Q}}, i_{0!}(\mathcal{M})) = H^1_{\mathrm{Eis}}(X_0(p_0) \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathcal{M}^{\#}), \tag{2}$$

we also have a Betti and a Hodge realization of this object.

Now we want to compute the class

$$[H^1_{\mathrm{Eis}}(Y_0(p_0) \times_{\mathbb{Q}} \overline{\mathbb{Q}}, i_{0!}(\tilde{\mathcal{M}}))] \in H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathbb{Z}_p(n+1)).$$

We have the elements

$$c_p(n) \in H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathbb{Z}_p(n+1))$$

constructed by Ch. Soulé.

Now I make the conjecture:

The extension class of the Galoismodule

$$0 \to \mathbb{Z}_p(0) \to H^1_{\mathrm{Eis}}(Y_0'(p_0), i_{0!}(\tilde{\mathcal{M}})) \to \mathbb{Z}_p(-n-1) \to 0$$

is given by

$$c_p(n)^{\frac{p_0^{n+1}-1}{p_0^{n+2}-1}\frac{p^{\delta_p(n)}}{\zeta(-1-n)}}. \tag{3}$$

This formula should of course be compared with the formula for the Hodge de-Rham extension class in my lecture notes volume [Ha-Eis] . (I have to confess that I made the computation much to complicated, because I always had the idea that the integral structure on the de-Rham cohomology should play some role. This had the effect that I sometimes go back and forth.)

But in principle the computation is correct. I have to integrate the differential form $\mathrm{Eis}(\alpha \times \psi_f)$ against the relative cycle $\mathfrak{Z}_n$ and this is the number I want to compute, it gives the extension class.

$$-\frac{p^{\delta_p(n)}}{4}\frac{p_0^{n+1}-1}{p_0^{n+2}-1}\frac{1}{\zeta(-1-n)}\frac{\zeta'(-n)}{2\pi i}. \tag{4}$$

Then the conjecture above simply says that the mixed motive $H^1_{\mathrm{Eis}}(Y_0'(p_0), i_{0!}(\tilde{\mathcal{M}}))$ is not exotic. In a certain sense the complex number $\frac{\zeta'(-n)}{2\pi i}$ is equal to $\log_p(c_p(n))$.

We had to assume $n > 0$ because otherwise there is not enough room for a mixed motive. But if we pass from $Y_0(p_0)$ to $Y_1(p_0)$ then we get a larger set $\Sigma(p_0) = \Sigma_0(p_0) \cup \Sigma_\infty(p_0)$ of cusps and the contribution of the boundary in the exact sequence (1) becomes larger. This is described in detail in [?] 4.1.1 p. 108. We get

$$H^\bullet(\dot{D}_{\Sigma(p_0)}, \tilde{\mathcal{M}} \otimes F) = \bigoplus_\varphi I_{\varphi_f}^{K_1(p_0)},$$

in degree 0 we have to sum over

$$\varphi = (\eta, \alpha^n) \ \text{ und } \ \varphi = (1, \eta\alpha^n)$$

and in degree 1 over the characters

$$\varphi = (\alpha^{n+1}\eta, \alpha^{-1}) \ \text{ and } \ \varphi = (\alpha^{n+1}, \eta\alpha^{-1}),$$

where $\eta : (\mathbb{Z}/p_0\mathbb{Z})^* \to F^*$, and the parity of η is n.

This allows us to define the mixed motives $H^1_{\mathrm{Eis}}(Y_1'(p_0), i_{0!}(\tilde{\mathcal{M}}))[\eta]$ which are called $H^1(\mathcal{M}^\#)[\eta]$ in [?]. If $\eta \neq 1$ then we can put $n = 0$.

Again we get a formula for the Betti-de-Rham extension class [?] and in if $n = 0$ the results in the Dissertation of Ch. Brinkmann [Br] (see also [?]) give a formula for the p-adic extension class. Therefore we see that under these special assumptions the motives are not exotic.

This argument can be used to prove the conjectured formula for the p-adic extension $\mod p$. We replace our curve $Y_0(p_0)$ by the curve $Y_{0,1}(p_0, p)$ this is the which belongs to the level subgroup $K_{f,p_0,p} = \times K_0(p_0) \times K_1(p) \times$. Inside $K_1(p)$ we have the full congruence subgroup $K(p)$, the quotient $\mathrm{Gl}_2(\mathbb{Z}_p)/K(p) = \mathrm{Gl}_2(\mathbb{F}_p)$. We have the Teichmüller character $\omega_p : \mathbb{F}_p^\times \to \mathbb{Z}_p^\times$ it yields a character $\omega_p^n : K_1(p) \to \mathbb{Z}_p^\times$, we construct the induced representation $\mathrm{Ind}_{B(\mathbb{F}_p)}^{\mathrm{Gl}_2(\mathbb{F}_p)} \omega^n = I_{\omega^n}$ This defines a local coefficient system $\tilde{I}_{\omega^n}$ on $Y_0(p_0)$. We construct the sheaf $\tilde{I}^\#_{\omega^n}$ and we get again the mixed Galois-module

$$0 \to \mathbb{Z}_p(0) \to H^1_{\mathrm{Eis}}(X_0(p_0) \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \tilde{I}^\#_{\omega^n}) \to \mathbb{Z}_p(-1) \otimes \omega_p^n \to 0 \tag{5}$$

This gives us an extension class in the category of Galois-modules which can be computed using the results in [Br]. We assume that $0 < n < p$, in [Ha2] we explain that we have an inclusion $\mathcal{M} \otimes \mathbb{F}_p \hookrightarrow I_{\omega^n} \otimes \mathbb{F}_p$ and that this inclusion induces an isomorphism on the ordinary part of cohomology

$$H^1_{\mathrm{ord}}(Y_0 \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathcal{M} \tilde{\otimes} \mathbb{F}_p) \xrightarrow{\sim} H^1_{\mathrm{ord}}(Y_0 \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \tilde{I}_{\omega^n} \otimes \mathbb{F}_p) \tag{6}$$

This extends if we go the modified sheaves $\tilde{I}^\#_{\omega^n}$ and $\mathcal{M} \tilde{\otimes} \mathbb{F}_p{}^\#$ and from here we get the formula for the extension class $\mod p$.

I made a serious attempt to prove the conjecture modulo higher powers p^n but so far I was not really successful.

Comments: We find the exponent $p^{\delta_p(n)}$ because we have to multiply the generator ω_n of the cohomology at infinity by this power of p to make the Eisenstein class integral. On the other hand we know under our assumptions that $\zeta(-1-n)$ is p-integral, hence

$$\zeta(-1-n) = p^{\delta'_p(n)} \cdot \text{unit}.$$

The truth is that we know $\delta'_p(n) = \delta_p(n)$.

But this formula gives us also a hint why we should have the equality of these two numbers. First of all one can see that $0 \leq \delta'_p(n) \leq \delta_p(n)$. But if we had $\delta'_p(n) < \delta_p(n)$, then it would mean that we can extract a p-th root from $c_p(n)$ which is of course not impossible but it will be a rare event.

2.1 Anderson motives for the symplectic group

I may consider the group $G = GSp_g$ and consider the double quotient

$$G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K_f$$

where K_f is a suitable open compact subgroup in the group of finite adeles and X is the hermitian symmetric domain attached to this group. This quotient is the set of complex valued point of a quasiprojective scheme

$$\mathcal{S}^G_{K_f} = \mathcal{S} \longrightarrow \mathrm{Spec}(\mathbb{Z}[\frac{1}{N}])$$

where N is the product of primes occurring in the congruences defining K_f. Hence the topological space will now be denoted by

$$\mathcal{S}^G_{K_f}(\mathbb{C}) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K_f.$$

Remark: In the following exposition we have a slight notational inconsistency. For any reductive group $M/\mathbb{Q}$ we can define the spaces

$$\mathcal{S}^M_{K^M_f} = M(\mathbb{Q})\backslash M(\mathbb{A})/K^M_\infty \times K^M_f \tag{7}$$

If it happens that $G = \mathrm{GSp}_g$ are more generally leads to a Shimura variety, then we gave a different meaning to $\mathcal{S}^G_{K_f}$ in this case it is a scheme and the resulting locally symmetric space is $\mathcal{S}^G_{K_f}(\mathbb{C})$. Hence $\mathcal{S}^G_{K_f}$ has two different meanings. In the following text $\mathcal{S}^G_{K_f}$ will be most of the time the topological space, and only under certain circumstances we remember that the set of complex points of a scheme, which is denoted by the same letter.

If we consider an irreducible rational representation

$$\rho : G/\mathbb{Q} \longrightarrow GL(\mathcal{M}_{\mathbb{Q}})$$

of the algebraic group $G/\mathbb{Q}$. This representation is given by its highest weight $\lambda = \sum n_i\gamma_i + m\mu$, where the γ_i are the fundamental weights and where μ is the weight character. Then this representation provides a sheaf $\tilde{\mathcal{M}}_{\mathbb{Q}}$ of $\mathbb{Q}$-vector spaces on our complex variety $\mathcal{S}^G_{K_f}(\mathbb{C})$. If we are a little bit careful und if we write

$$\mathcal{S}^G_{K_f}(\mathbb{C}) = \bigcup \Gamma_i\backslash X$$

with some congruence subgroups $\Gamma_i \subset G(\mathbb{Q})$ (or maybe even better $\Gamma_i \subset G(\mathbb{Z})$), then we can choose Γ_i-invariant lattices in $\mathcal{M}_{\mathbb{Z}}$ in $\mathcal{M}$ and this provides sheaves $\tilde{\mathcal{M}}_{\mathbb{Z}}$ on $\mathcal{S}^G_{K_f}(\mathbb{C})$.

During the progress of this notes we have to enlarge the ring $\mathbb{Z}$ to a larger ring R at several occasions. This larger ring R will be obtained from $\mathbb{Z}$ by inverting some primes and and then we take the integral closure of this new ring in an algebraic extension $K/\mathbb{Q}$. We tensorize the sheaves $\tilde{\mathcal{M}}_{\mathbb{Z}}$ by R and the resulting sheaves will be denoted by $\tilde{\mathcal{M}}$.

At the beginning of our discussion we do not know how big we have to choose R, whenever I enlarge it I will say which new primes have to be inverted and which further algebraic extensions have to be taken. These primes will be called *small* primes.

These sheaves are obtained from the universal family of principally polarized abelian varieties and products, symmetric parts and so on – over $\mathcal{S}^G_{K_f}$. Let us denote this motivic sheaf also by $\tilde{\mathcal{M}}$ and rebaptize the old $\tilde{\mathcal{M}}$ to $\tilde{\mathcal{M}}_B$, i.e. $\tilde{\mathcal{M}}_B$ is the sheaf of Betti- cohomology groups of this motive. At this step we may have invert some primes, I think tat the primes which are smaller than the coefficients $n_i$of our highest weight are enough.

We may also consider the sheaf of de-Rham cohomology groups

$$\tilde{\mathcal{M}}_{\mathrm{dRh}}/\mathcal{S}^G_{K_f},$$

it comes with a filtration and the Gauss-Manin connection

$$\nabla : \tilde{\mathcal{M}}_{\mathrm{dRh}} \longrightarrow \tilde{\mathcal{M}}_{\mathrm{dRh}} \otimes \Omega^1_S$$

which is flat and satisfies a Griffith transversality condition.

Finally we can consider consider a prime $\mathfrak{l}$, which lies over a prime ℓ and $\tilde{\mathcal{M}}_{\mathfrak{l}} = \tilde{\mathcal{M}}_B \otimes R_{\mathfrak{l}}$, and this is a system of $\mathfrak{l}$-adic sheaves on $\mathcal{S}^G_{K_f}$. If ℓ is not invertible in R then $R_{\mathfrak{l}}$ is a field.

Let $\mathcal{S} \overset{i}{\hookrightarrow} \mathcal{S}^{\wedge\wedge}$ where $\mathcal{S}^{\wedge\wedge}$ is a smooth compactification obtained by the method of toroidal embeddings (Faltings-Chai). We have that

$$\mathcal{S}^{\wedge\wedge}\backslash\mathcal{S} = \bigcup_{[P]} \mathcal{S}^{\wedge\wedge}_{[P]} = \mathcal{S}^{\wedge\wedge}_{\infty}$$

where $[P]$ runs over the conjugacy classes of maximal parabolic subgroups. The Levi-quotients of these maximal parabolic subgroups are essentially products $Gl_{g-a} \times GSp_a \times G_m$ where a runs from 0 to $g-1$. We call the parabolic for $a = 0$ the Siegel parabolic and for $a = g-1$ the Klingen parabolic. The boundary stratum corresponding to the Klingen parabolic subgroup is a union of Shimura varieties attached to GSp_{g-1} together with their universal abelian variety over it. So it is of codimension 1 and a smooth divisor provided K_f is sufficiently small.

The $\mathcal{S}^{\wedge\wedge}_{[P]}$ attached to the Siegel parabolic is a configuration smooth toroidal varieties of dimension $\frac{g(g+1)}{2} - 1$ with transversal intersections. The combinatorics of this configuration is governed by taking certain cone decompositions for the action of congruence subgroups $\Gamma' \subset Gl_g(\mathbb{Z})$ on the positive definite symmetric matrices in $M_g(\mathbb{R})$. I will come back to this point later. For the other strata we get something in between.

We can construct "motivic sheaves" on $\mathcal{S}^{\wedge\wedge}$ by extending $\tilde{\mathcal{M}}$ from $\mathcal{S}$ to $\mathcal{S}_\infty$ where we require support conditions for these extensions. We are mainly interested in the Siegel parabolic and hence we extend somehow to the strata $S^{\wedge\wedge}_Q$ which are different from the Siegel stratum. Then we take an auxiliary prime p_0 and choose a congruence subgroup $K_f(p_0) \subset K_f$. (This is similar to the construction in my book.) We get a decomposition of $\mathcal{S}^{\wedge\wedge}_{[P]}$ into different connected components. And then according to certain rules we extend $\tilde{\mathcal{M}}$ to $\mathcal{S}^{\wedge\wedge}_{[P]}$.

Of course we may still take the full direct image $i_*(\tilde{\mathcal{M}})$ (here we take the derived functor) and we consider the cohomology

$$H^\bullet(S_\infty, i_*(\tilde{\mathcal{M}}))$$

as a mixed motive over $\mathbb{Z}[\frac{1}{p_0 N}]$ with coefficients in R.

If we consider the Betti cohomology of this motive then we can compute it using the Borel-Serre compactification and we apply our considerations from [MixMot 3.1].

We write the compactification

$$\mathcal{S}^G_{K_f}(\mathbb{C}) \longrightarrow \overline{\mathcal{S}^G_{K_f}}$$

and $\overline{\mathcal{S}^G_{K_f}}$ is a manifold with corners. We have

$$\overline{\mathcal{S}^G_{K_f}} \setminus \mathcal{S}^G_{K_f}(\mathbb{C}) = \bigcup_P \partial_P \mathcal{S} = \partial \mathcal{S}$$

where now P runs over all parabolic subgroups containing a fixed Borel subgroup.

We choose a Borel subgroup B and let us choose P to be the representative of P which contains B.

Then we have a finite coset decomposition

$$G(\mathbb{A}_f) = \bigcup_{\xi_f} P(\mathbb{A}_f)\xi_f K_f$$

and we recall from [MixMot 3.1] that we have

$$H^\bullet(\partial_P \mathcal{S}, \tilde{\mathcal{M}}_R) = \bigcup_{\xi_f} H^\bullet(\mathcal{S}^M_{K_f^M(\xi_f)}, \widetilde{H^\bullet(\mathfrak{u}, \mathcal{M})}_R),$$

$$H^\bullet(\mathfrak{u}, \mathcal{M}) = \bigoplus_{w \in W^P} H^{l(w)}(\mathfrak{u}, \mathcal{M})(w \cdot \lambda),$$

where W^P is the set of Kostant representatives of W/W^M and where $w \cdot \lambda = (\lambda+\rho)^w - \rho$ and ρ is the half sum of positive roots.

At this point I am rather imprecise about which primes should be inverted, a safe choice would be to invert all primes which ae less or equal to the numbers n_i which enter in our highest weight. But I am not so sure whether this choice is too cautious, I will discuss this problem later.

Remark. Let us assume for the moment that $g = 2$, Let P be the Siegel parabolic and Q be the Klingen. I have explained that the strata $S_P^{\wedge\wedge}$ and $S_Q^{\wedge\wedge}$ are different in nature. This is also reflected in the Borel-Serre compactification or better in the cohomology of the two strata. Let M (resp. M_1) be the reductive quotient for the Siegel (resp. Klingen) parabolic. In the following discussion I suppress the ξ_f because what I am saying does not depend on this variable. We form the symmetric spaces $\mathcal{S}^M_{K_f^M}$ and $\mathcal{S}^{M_1}_{K_f^{M_1}}$, then they are of the form

$$\begin{array}{ccc} M(\mathbb{Q}) \setminus M(\mathbb{R})/K_\infty^M & \times & M(\mathbb{A}_f)/K_f^M \\ M_1(\mathbb{Q}) \setminus M_1(\mathbb{R})/K_\infty^{M_1} & \times & M_1(\mathbb{A}_f)/K_f^{M_1} \end{array}$$

and the groups $K_\infty^M, K_\infty^{M_1}$ are the images of $P(\mathbb{R}) \cap K_\infty$, $Q(\mathbb{R}) \cap K_\infty$ respectively.

Both groups M, M_1 are naturally product of the form $GL_2 \times G_m = M^{(1)} \times G_m, M_1^{(1)} \times G_m$. Now $K_\infty \cap M^{(1)}(\mathbb{R})$ is not connected but $K_\infty \cap M_1^{(1)}(\mathbb{R})$ is. This has some influence on the structure of the cohomology. We consider the cohomology

$$H^\bullet(M(\mathbb{Q}) \setminus M(\mathbb{R})/K_\infty^M \times M(\mathbb{A}_f)/K_f, H^\bullet(\mathfrak{u}, \tilde{\mathcal{M}}))$$

as a module under the Hecke algebra

$$\mathcal{H}^M = \mathcal{C}_c(K_f^M \setminus M(\mathbb{A}_f)/K_f^M).$$

If we replace K_∞^M by its connected component $\overset{\circ}{K}{}_\infty^M$, then the cohomology becomes a $\mathcal{H}^M \times \pi_0(M(\mathbb{R}))$-module where $\pi_0(M(\mathbb{R}))$ is as usual the group of connected components. If we restrict to the action of the Hecke algebra, then

$$H^\bullet(M(\mathbb{Q}) \setminus M(\mathbb{R})/\overset{\circ}{K}{}_\infty^M \times M(\mathbb{A}_f)/K_f, \widetilde{H^\bullet(\mathfrak{u}, \mathcal{M})})$$

decomposes under $\mathcal{H}^{\mathcal{M}}$ into

$$\bigoplus_{\sigma_f} H^\bullet(M(\mathbb{Q}) \setminus M(\mathbb{R})/\overset{\circ}{K}{}_\infty^M \times M(\mathbb{A}_f)/K_f, \widetilde{H^\bullet(\mathfrak{u}, \mathcal{M})})(\sigma_f) \bigoplus$$
$$\bigoplus_{\tau_f} H^\bullet(M(\mathbb{Q}) \setminus M(\mathbb{R})/\overset{\circ}{K}{}_\infty^M \times M(\mathbb{A}_f)/K_f, \widetilde{H^\bullet(\mathfrak{u}, \mathcal{M})})(\tau_f)$$

where σ_f, τ_f are irreducible R-modules under the Hecke algebra. Here we must enlarge our ring R. We have to be sure that the eigenvalues of the Hecke-operators (of course we only take $\mathbb{Z}$-valued functions in the Hecke-algebra) lie in R and we need that there are no congruence amoung the moular forms. The isotypical components σ_f have multiplicity two.

Then we know:

The σ_f are modules given by Hecke modules on the space of certain cusp forms where the weight and level are determined by $\mathcal{M}$ and K_f. The τ_f correspond to Hecke modules attached to Eisenstein series of the same weight and level.

Now we know that the σ_f components come with multiplicity two and the τ_f come with multiplicity one. These considerations are valid for M and for M_1.

But now we observe that we have replaced K_∞^M by $\overset{\circ}{K}{}_\infty^M$. It is easy to understand the effect of this manipulation. We recall that we have an action of $\pi_0(G(\mathbb{R}))$ and the image $\pi_0(K)$ of K_∞^M in $\pi_0(G(\mathbb{R}))$ is non trivial (The connected component $\overset{\circ}{K}{}_\infty^M$ goes to zero.). This means

$$\begin{aligned} &H^\bullet(M(\mathbb{Q})\setminus (M(\mathbb{R})/K_\infty^M)\times M(\mathbb{A}_f)/K_f^M, \widetilde{H^\bullet(\mathfrak{u},\mathcal{M})}) = \\ &H^\bullet(M(\mathbb{Q})\setminus (M(\mathbb{R})/\overset{\circ}{K}{}_\infty^M)\times M(\mathbb{A}_f)/K_f^M, \widetilde{H^\bullet(\mathfrak{u},\mathcal{M})})^{\pi_0(K)}. \end{aligned} \tag{8}$$

In the case of the Klingen parabolic subgroup $\pi_0(K) = 1$ but in the case of the Siegel parabolic subgroup the group $\pi_0(K)$ has both eigenvalues ± 1 on the isotypic components

$$H^\bullet(\ \ ,H^\bullet(\ \ ))(\sigma_f) = H^\bullet_+(\ \ ,H^\bullet(\ \ ))(\sigma_f)\oplus H^\bullet_-(\ \ ,H^\bullet(\ \ ))(\sigma_f).$$

We return to the case of a general genus g, we will be mostly interested in the Siegel parabolic in the following we reserve the name P for it, let M be its reductive quotient it is a $\mathrm{Gl}_g\times\mathbb{G}_m$. If we consider the cohomology then we have the surjective map from the cohomology with compact support to the inner cohomology.

$$H_c^\bullet(\mathcal{S}^M_{K_f^M}, \widetilde{H^{l(w)}(\mathfrak{u},\mathcal{M})}(w\cdot\lambda)) \longrightarrow H_!^\bullet(\mathcal{S}^M_{K_f^M}, \widetilde{H^{l(w)}(\mathfrak{u},\mathcal{M})}(w\cdot\lambda)).$$

These modules for the Hecke-algebra $\mathcal{H}^M(M(\mathbb{A})//K_f^M)$, according to a theorem of Franke and Schwermer the surjective map has a canonical rational splitting. If some congruence primes for the cohomology are invertible in R and the quotient field of R is large enough then we get an isotypical decompositions over R

$$H_c^\bullet(\mathcal{S}^M_{K_f^M}, \widetilde{H^{l(w)}(\mathfrak{u},\mathcal{M})}(w\cdot\lambda)) \xrightarrow{\sim} H^\bullet_{\mathrm{Eis}}\oplus H_!^\bullet(\mathcal{S}^M_{K_f^M}, \widetilde{H^{l(w)}(\mathfrak{u},\mathcal{M})}(w\cdot\lambda)).$$

$$H_!^\bullet(\mathcal{S}^M_{K_f^M}, \widetilde{H^{l(w)}(\mathfrak{u},\mathcal{M})}(w\cdot\lambda)) = \bigoplus_{\sigma_f} H_!^\bullet(\mathcal{S}^M_{K_f^M}, \widetilde{H^{l(w)}(\mathfrak{u},\mathcal{M})}(w\cdot\lambda))(\sigma_f),$$

where the σ_f are irreducible modules for the Hecke-algebra $\mathcal{H}^M(M(\mathbb{A})//K_f^M)$. We have also the Hecke -algebra $\mathcal{H}^G(G(\mathbb{A})//K_f)$ and I abbreviate the notation by calling them $\mathcal{H}^M,\mathcal{H}^G$.

$$\mathrm{Ind}_{\mathcal{H}^M}^{\mathcal{H}^G} H_!^\bullet(\mathcal{S}^M_{K_f^M}, \widetilde{H^{l(w)}(\mathfrak{u},\mathcal{M})}(w\cdot\lambda))(\sigma_f) \to H^\bullet(\partial\mathcal{S},\tilde{\mathcal{M}}),$$

and again we may have to invert a few more primes.

The modules σ_f have a central character $\omega(\sigma_f)$ which is an algebraic Hecke-character and the type of this character can be read off from the data λ, w.

From this algebraic Hecke character we get another algebraic Hecke character

$$\tilde{\omega}(\sigma_f) : I_{\mathbb{Q},f} \to R^*$$

whose weight is equal something computed from $w \cdot \lambda$ and perhaps we call it simply $\mathbf{w}(w \cdot \lambda)$.

Now we invoke a theorem of R. Pink which tells us that

The isotypical component $H_!^\bullet(\mathcal{S}^M_{K_f^M}, \widetilde{H^{l(w)}(\mathfrak{u}, \mathcal{M})}(w\cdot\lambda))(\sigma_f)$ *is pure Tate motive*

$$H_!^\bullet(\mathcal{S}^M_{K_f^M}, \widetilde{H^{l(w)}(\mathfrak{u}, \mathcal{M})}(w \cdot \lambda))(\sigma_f) \otimes R(\tilde{\omega}(\sigma_f)).$$

We have to inquire whether this inner cohomology can be non zero. A necessary condition is that the representations of M on the cohomology $\widetilde{H^{l(w)}(\mathfrak{u}, \mathcal{M})}(w\cdot\lambda)$ is self dual. This is of course not a problem if $g = 2$ because in this case the semi simple type of M is A_1. We will discuss later what happens if $g \geq 3$

We want to discuss the construction of sheaves with support conditions on $S^{\wedge\wedge}$. I assume that my subgroup is of the form $K_f = \prod_p K_p$ where K_p is open in $G(\mathbb{Z}_p)$ and equal to it for almost all primes p. I choose an auxiliary prime p_0 for which $K_{p_0} = G(\mathbb{Z}_{p_0})$. I consider a group $K_0(p_0) \subset G(\mathbb{Z}_{p_0})$ whose reduction mod p_0 is a Borel subgroup $\bar{B}(\mathbb{F}_{p_0}) \subset G(\mathbb{F}_{p_0})$. We define the new $K_f(p_0)$ to be the subgroup of K_f where I replace the component at p_0 by $K_f(p_0)$. With respect to this choice of the open compact subgroup I define my space $\mathcal{S}^G_{K_f}$.

The boundary of $\mathcal{S}^G_{K_f}$ will now have a certain combinatorial structure obtained from the prime p_0. We have an action of the group $B(\mathbb{F}_{p_0})$ on the sets of different types of parabolic subgroups. We form the simplicial set $\mathcal{T}$ whose vertices are the maximal parabolic subgroups in $G(\mathbb{F}_{p_0})$ modulo this action and the simplices of maximal dimension are the Borel subgroups modulo this action. If we consider the character module $X^*(T)$ of a maximal torus then the maximal simplices are just the chambers and so on.

We from reduction theory we get a projection map

$$\pi : S_\infty = S^{\wedge\wedge} \setminus S \to \mathcal{T}.$$

If we take a closed subset $\Xi \subset \mathcal{T}$ then the inverse image of this closed subset will be an open subset S_∞ and its union with the interior will provide an open subset $S_\Xi \subset S^{\wedge\wedge}$. We have the chain of inclusions

$$i^\Xi : S \hookrightarrow S_\Xi \text{ and } i_\Xi : S_\Xi \hookrightarrow \mathcal{T}$$

We extend our sheaf $\tilde{\mathcal{M}}$ from $\mathcal{S}^G_{K_f}$ to S_Ξ by zero, i.e. we take the sheaf $i^\Xi_!(\tilde{\mathcal{M}})$ and then we take the full direct image $i_{\Xi,*}(i^\Xi_!(\tilde{\mathcal{M}}))$ This gives us a sheaf $\tilde{\mathcal{M}}_\Xi$ on $S^{\wedge\wedge}$ and we can consider its cohomology

$$H^\bullet(S^{\wedge\wedge}, \tilde{\mathcal{M}}_\Xi).$$

Now we will investigate this sheaf and we want to analyze to what extend we can find mixed Tate motives inside this cohomology.

To understand this we look at the middle dimension first. Let $d = \frac{g(g+1)}{2}$ and we consider the maps in the Betti cohomology

$$H^d(\mathcal{S}^G_{K_f}(\mathbb{C}), \tilde{\mathcal{M}}_B) \to H^d(\partial\mathcal{S}, \tilde{\mathcal{M}}_B)$$
$$H^{d-1}(\partial\mathcal{S}, \tilde{\mathcal{M}}_B) \to H^d_c(\mathcal{S}^G_{K_f}(\mathbb{C}), \tilde{\mathcal{M}}_B)$$

We have the Dynkin Diagram as above but now α_g will denote the long root at the right end. To this root corresponds an injective cocharacter $\chi_g : G_m \to T \subset G$ which is defined by $< \chi_g, \alpha_j >= 2\delta_{jg}$ and by the requirement that it factors through the semisimple part $G^{(1)}$ of G. Hence it is clear that $\chi_g(G_m) = A$ is the central torus of M intersected with $G^{(1)}$.

Let $\gamma_0 : A \to G_m$ be the character for which $\gamma_0 \circ \chi_g(x) = x$.

If as usual $\gamma_1, \gamma_2, \ldots, \gamma_g$ are the dominant weights, the γ_g extends to a character on M and for the restriction to A we have $\gamma_g|A = \gamma_0^g$.

We select a σ_f which occurs in $H^{l(w)}\widetilde{(\mathfrak{u}, \mathcal{M})}(w \cdot \lambda)$ and we assume that

$$< \chi_g, w\cdot\lambda > < \; - < \chi_g, \rho_P >$$

so we are on the left hand side of the central point for cohomology. Then the general results on Eisenstein cohomology tell us that the subspace

$$\mathrm{Ind}_{\mathcal{H}^M}^{\mathcal{H}^G} H^\bullet(\mathcal{S}^M_{K^M_f}, H^{l(w)}\widetilde{(\mathfrak{u}, \mathcal{M})}(w\cdot\lambda)(\sigma_f) \subset H^{\bullet + l(w)}(\partial\mathcal{S}, \tilde{\mathcal{M}})_{\mathbb{C}}$$

is in fact in the image of the cohomology provided the associated Eisenstein series does not have a pole.

Actually what we have to do is to extend σ_f to a representation $\sigma = \sigma_\infty \times \sigma_f$ which now occurs in the cuspidal spectrum $\mathcal{A}_{cusp}(M(\mathbb{Q})\backslash M(\mathbb{A}))$. Let H_σ this isotypical submodule so that we have

$$H^\bullet(\mathcal{S}^M_{K^M_f}, H^{l(w)}\widetilde{(\mathfrak{u}, \mathcal{M})}(w\cdot\lambda)(\sigma_f) = H^\bullet(\mathfrak{m}, K^M, H_\sigma \otimes H^{l(w)}(\mathfrak{u}, \tilde{\mathcal{M}}))$$

We twist this representation by a character

$$\mu_s : \underline{m} \mapsto |\gamma_g(\underline{m})|^s$$

and we consider the induced representation

$$I_{\sigma\otimes s} = \{f : G(\mathbb{A}) \to H_\sigma | f(\underline{pg}) = \sigma(\underline{m})\mu_s)(\underline{p})\}$$

where $\underline{m}$ is the image of $\underline{p}$ in $M(\mathbb{A})$. The functions should satisfy some finiteness conditions.

We can form the Eisenstein series

$$\mathrm{Eis} : I_{\sigma\otimes s} \to \mathcal{A}(\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})$$

given by

$$\mathrm{Eis}(f)(\underline{g}) = \sum_{P\mathbb{Q})\backslash G(\mathbb{Q})} f(g)(\underline{e})$$

which is convergent for $\Re(s) >> 0$.

Let us assume that we are in the holomorphic case, i.e. the Eisenstein operator is holomorphic at $s = 0$ Then we know that the Eisenstein series is actually an intertwining operator

$$\mathrm{Eis} : I_\sigma \to \mathcal{A}(\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A}))$$

we get a homomorphism

$$\mathrm{Eis}^\bullet : H^\bullet(\mathfrak{g}, K_\infty, I_{\sigma_\infty} \otimes \mathcal{M}_\mathbb{C}) \otimes \sigma_f \to H^\bullet(S, \tilde{\mathcal{M}}_\mathbb{C})$$

and if we compose this with the restriction to the boundary then I claim that this composition gives us a surjective map

$$r \circ \mathrm{Eis} : H^\bullet(\mathfrak{g}, K_\infty, I_{\sigma_\infty} \otimes \mathcal{M}_\mathbb{C}) \otimes I_{\sigma_f} \to \mathrm{Ind}_{\mathcal{H}^M}^{\mathcal{H}^G} H^\bullet(\mathcal{S}^M_{K_f^M}, H^{l(w)}(\mathfrak{u}, \mathcal{M})(w \cdot \lambda)(\sigma_f)$$

This means that as long as $w \cdot \lambda$ is far enough to the right we know that after tensorization by $\mathbb{C}$ the subspace

$$\oplus_{\sigma_f} \mathrm{Ind}_{\mathcal{H}^M}^{\mathcal{H}^G} H^\bullet(\mathcal{S}^M_{K_f^M}, H^{l(w)}(\mathfrak{u}, \mathcal{M})(w \cdot \lambda)(\sigma_{f\mathbb{C}})$$

is in the image of the restriction map. If now $\Theta_P \in W^P$ is the longest element then we can consider $\Theta_P \cdot \sigma_f = \sigma_f^\vee$ and this module occurs in the cohomology $H^\bullet(\mathcal{S}^M_{K_f^M}, H^{l(w')}(\mathfrak{u}, \mathcal{M})(w' \cdot \lambda))$ where w' is the dual partner to w. Here we have the opposite inequality

$$< \chi_g, w' \cdot \lambda >> \; - < \chi_g, \rho_P >$$

and for these weights we find that the map

$$\mathrm{Ind}_{\mathcal{H}^M}^{\mathcal{H}^G} H^\bullet(\mathcal{S}^M_{K_f^M}, H^{l(w)}(\mathfrak{u}, \mathcal{M})(w \cdot \lambda)(\sigma_f^\vee) \to H_c^{\bullet+1+l(w)}(S^{\wedge\wedge}, \tilde{\mathcal{M}}_\mathbb{C})$$

is injective if we do not have a pole.

If we pick such a σ_f then the module $\mathrm{Ind}_{\mathcal{H}^M}^{\mathcal{H}^G}(\sigma_f)$ provides a module under the Hecke-algebra $\mathcal{H}^G$. This induced module is of course a restricted tensor product over all primes p of local modules. If we consider the local induced modules at p_0 then we can use our support condition Ξ define submodules

$$\mathrm{Ind}_{\mathcal{H}^M}^{\Xi,\mathcal{H}^G} \subset \mathrm{Ind}_{\mathcal{H}^M}^{\mathcal{H}^G}(\sigma_f)$$

and quotients

$$u(\sigma_f) : \mathrm{Ind}_{\mathcal{H}^M}^{\mathcal{H}^G}(\sigma_f^\vee) \to \mathrm{Ind}_{\mathcal{H}^M}^{\Xi',\mathcal{H}^G},$$

where Ξ' is the complementary support condition. These submodules are not modules for the full Hecke algebra, we have to take the identity element at the prime p_0. We define

$$H^\bullet(S^{\wedge\wedge}, \tilde{\mathcal{M}}_\Xi)(\sigma_f)$$

to be the inverse image of $\mathrm{Ind}_{\mathcal{H}^M}^{\Xi,\mathcal{H}^G} H^\bullet(\mathcal{S}^M_{K_f^M}, H^{l(w)}(\mathfrak{u}, \mathcal{M})(w{\cdot}\lambda)(\sigma_f)$ in $H^\bullet(S^{\wedge\wedge}, \tilde{\mathcal{M}}_\Xi)$ divided by the kernel of $u(\sigma_f)$. Then by construction we have a map

$$r(\sigma_f) : H^\bullet(S^{\wedge\wedge}, \tilde{\mathcal{M}}_\Xi)(\sigma_f) \to H^\bullet(\mathcal{S}^M_{K_f^M}, H^{l(w)}(\mathfrak{u}, \mathcal{M})(w \cdot \lambda)(\sigma_f)$$

which is surjective up to torsion. We also get a map

$$\delta(\sigma_f) : \mathrm{Ind}_{\mathcal{H}^M}^{\mathcal{H}^G}(\sigma_f^\vee) \to H^\bullet(S^{\wedge\wedge}, \tilde{\mathcal{M}}_\Xi)(\sigma_f).$$

If we divide the kernel of $r(\sigma_f)$ by the image of $\delta(\sigma_f)$ then we get the inner cohomology.

Now I want to assume for a moment that Ξ is everything and $\Xi' = \emptyset$. I also assume that

$$r(\sigma_f) : H^\bullet(S^{\wedge\wedge}, \tilde{\mathcal{M}})(\sigma_f) \to H^\bullet(\mathcal{S}^M_{K_f^M}, H^{l(w)}(\mathfrak{u}, \mathcal{M})(w\cdot\lambda)(\sigma_f)$$

is in fact surjective on the integral level. Then we have rationally the Manin-Drinfeld principle, this gives us a canonical section and a decomposition up to isogeny

$$H^\bullet(S^{\wedge\wedge}, \tilde{\mathcal{M}})(\sigma_f) \supset H^\bullet_!(S^{\wedge\wedge}, \tilde{\mathcal{M}})(\sigma_f) \oplus H^\bullet_{\mathrm{Eis}}(\mathcal{S}^M_{K_f^M}, H^{l(w)}(\mathfrak{u}, \mathcal{M})(w\cdot\lambda)(\sigma_f).$$

We can not expect that the restriction

$$H^\bullet_{\mathrm{Eis}}(\mathcal{S}^M_{K_f^M}, H^{l(w)}(\mathfrak{u}, \mathcal{M})(w\cdot\lambda)(\sigma_f) \to H^\bullet(\mathcal{S}^M_{K_f^M}, H^{l(w)}(\mathfrak{u}, \mathcal{M})(w\cdot\lambda)(\sigma_f)$$

is surjective. I explained in [Ha-book], chap3, 6.3 that I believe that the order of the cokernel should be related to a special values of the L function attached to σ_f. More precisely the "arithmetic" of the second constant term should tell us something about this kokernel.

2.1.1 The Anderson motive

I want to explain that the discussion of the mixed Anderson motives gives some further evidence that a result of this kind should be true. We take a suitable Ξ. If we divide $H^\bullet(S^{\wedge\wedge}, \tilde{\mathcal{M}})(\sigma_f)$ by the image of $\delta(\sigma_f)$ then we get as a quotient a submodule of the cohomology namely the inverse image of $H^\bullet(\mathcal{S}^M_{K_f^M}, H^{l(w)}(\mathfrak{u}, \mathcal{M})(w\cdot\lambda)(\sigma_f, \Xi)$ in the cohomology. We get an almost decomposition

$$H^\bullet(S^{\wedge\wedge}, \tilde{\mathcal{M}})(\sigma_f)/\ \mathrm{Im}(\delta(\sigma_f^\vee)) \supset H^\bullet_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}) \oplus H^\bullet_{\mathrm{Eis}}(\mathcal{S}^M_{K_f^M}, H^{l(w)}(\mathfrak{u}, \mathcal{M})(w\cdot\lambda)(\sigma_f, \Xi)$$

and this gives us the subobject $H^\bullet_{\mathrm{Eis}}(S^{\wedge\wedge}, \tilde{\mathcal{M}})(\sigma_f)$ which sits in an exact sequence

$$\begin{aligned} 0 \to \mathrm{Ind}_{\mathcal{H}^M}^{\Xi'\mathcal{H}^G} H^\bullet(\mathcal{S}^M_{K_f^M}, H^{l(w')}(\mathfrak{u}, \mathcal{M})(w\cdot\lambda)(\sigma_f^\vee) \quad &\xrightarrow{\delta} H^\bullet_{\mathrm{Eis}}(S^{\wedge\wedge}, \tilde{\mathcal{M}}_\Xi)(\sigma_f) \\ \xrightarrow{r} \mathrm{Ind}_{\mathcal{H}^M}^{\Xi,\mathcal{H}^G} H^\bullet_{\mathrm{Eis}}(\mathcal{S}^M_{K_f^M}, H^{l(w)}(\mathfrak{u}, \mathcal{M})(w\cdot\lambda)(\sigma_f) \to 0 & \end{aligned}$$

and the term in the middle is a mixed Tate motive $\mathcal{X}[\sigma_f]$ Here we have to observe that δ raises the degree by one and r respects the degree. The map

$$H^\bullet_{\mathrm{Eis}}(\mathcal{S}^M_{K_f^M}, H^{l(w)}(\mathfrak{u}, \mathcal{M})(w\cdot\lambda)(\sigma_f) \to H^\bullet(\mathcal{S}^M_{K_f^M}, H^{l(w)}(\mathfrak{u}, \mathcal{M})(w\cdot\lambda)(\sigma_f)$$

has a finite cokernel, this cokernel will be given by a number $\Delta(\sigma_f)$.

So we assume $g=2$ and we also assume that we do not have a pole of the Eisenstein series.

I want to give some indication how the Hodge-de Rham extension classes can be computed. We apply the same argument as in my SLN. Actually I think I made the computations unnecessarily complicated there. To simplify the considerations I also assume that σ_f is defined over $\mathbb{Q}$ otherwise I have to make a lot of noise about fields of definition and conjugation under Galois.

We follow the advice given by our general discussion of the computation of the Hodge de-Rham Ext-group. We can twist by a Tate motive so that the bottom becomes $\mathbb{Z}(0)$ and then the top will be $\mathbb{Z}(-n-1)$ with n. Let us also assume for simplicity that our choice of K_f is so that $\sigma_f^{K_f^M}$ is of rank one so that $H^\bullet(\mathcal{S}^M_{K_f^M}, H^{l(w)}(\mathfrak{u},\mathcal{M})(w\cdot\lambda)(\sigma_f) = \mathbb{Z}(-n-1)$ is of rank one. Now we assume that

$$H^\bullet(S^{\wedge\wedge},\tilde{\mathcal{M}}_\Xi)(\sigma_f) \xrightarrow{r} \mathrm{Ind}_{\mathcal{H}^M}^{\mathcal{H}^G} H^\bullet(\mathcal{S}^M_{K_f^M}, H^{l(w)}(\mathfrak{u},\mathcal{M})(w\cdot\lambda)(\sigma_f,\Xi)$$

is surjective. Then $\frac{1}{\Delta(\sigma_f)}H^\bullet_{\mathrm{Eis}}(S^{\wedge\wedge},\tilde{\mathcal{M}}_\Xi)(\sigma_f) \to H^\bullet(\mathcal{S}^M_{K_f^M}, H^{l(w)}(\mathfrak{u},\mathcal{M})(w\cdot\lambda)(\sigma_f,\Xi)$ will be surjective. We take a suitable differential form

$$\omega_{\mathrm{top}} \in \mathrm{Hom}_{K_\infty}(\Lambda^3(\mathfrak{g}/\mathfrak{k}), I_{\sigma_\infty}\otimes\mathbb{C})$$

such that the Eisenstein intertwining operator maps $\omega_{\mathrm{top}}\otimes I_{\sigma_f}$ to

$$\mathrm{Ind}_{\mathcal{H}^M}^{\mathcal{H}^G} H^\bullet(\mathcal{S}^M_{K_f^M}, H^{l(w)}(\mathfrak{u},\mathcal{M})(w\cdot\lambda)(\sigma_f,\Xi)$$

and such that complex conjugation acts by -1 since n is even. This is the canonical Betti lift which we described earlier. If we multiply be a denominator $d(n)$ then we will land in the integral cohomology of the boundary. Now we can find (at this point some details have to be fixed) a class ω_{hol} such that ω_{hol} and ω_{top} define the same class in $\mathrm{Hom}_{K_\infty}(\Lambda^3(\mathfrak{g}/\mathfrak{k}), I_{\sigma_\infty}\otimes\mathbb{C})$ and such that $\mathrm{Eis}(\omega_{hol})$ lies in the F^2 filtration step of the de Rham filtration. So this is the de-Rham-lift. According to our rules we have to look at the difference

$$(\mathrm{Eis}(\omega_{\mathrm{hol}}) - \mathrm{Eis}(\omega_{\mathrm{top}}))\times\psi_f \in \mathrm{Ind}_{\mathcal{H}^M}^{\mathcal{H}^G} H^\bullet(\mathcal{S}^M_{K_f^M}, H^{l(w)}(\mathfrak{u},\mathcal{M})(w\cdot\lambda)(\sigma_f)$$

Again this can be computed as an integral against a relative cycle.

First of all we notice that we can write the difference $\omega_{hol}-\omega_{\mathrm{top}}$ as a $d\psi_\infty$ where

$$\psi_\infty \in \mathrm{Hom}_{K_\infty}(\Lambda^2(\mathfrak{g}/\mathfrak{k}), I_{\sigma_\infty}\otimes\mathbb{C})$$

This differential form can be interpreted as a form on

$$P(\mathbb{Q})\backslash X\times G(\mathbb{A}_f)/K_f$$

more or less by construction. We have the level function

$$P(\mathbb{Q})\backslash X\times G(\mathbb{A}_f)/K_f \xrightarrow{|\gamma_g|} \mathbb{R}_{>0}$$

and any level surface is homotopy equivalent to $\partial_P\mathcal{S}$. If we restrict this class to such a level hypersurface it becomes closed and $\psi\times\psi_f$ will be a non zero class in

$H^2(\partial_P\mathcal{S},\tilde{\mathcal{M}}) = \mathrm{Ind}_M^G H1(\mathcal{S}^M_{K^M_f}, H^{l(w)}(\mathfrak{u},\mathcal{M})(w\cdot\lambda)_\Xi)$. Now we can find a 2-cycle $\mathfrak{z}$ which represents a non zero class in

$$[\mathfrak{z}] \in H_2(\partial_P\mathcal{S},\tilde{\mathcal{M}}) = H_1(\mathcal{S}^M_{K^M_f}, H_1(\mathfrak{u},\tilde{\mathcal{M}}))$$

and this cycle can be bounded by a chain $\mathfrak{c}$ inside $\mathcal{S}^G_{K_f}(\mathbb{C})$. Then it is the definition that our extension class is given by the integral

$$\int_{\mathfrak{c}} \mathrm{Eis}((\omega_{hol}-\omega_{\mathrm{top}})\times\psi_f) = \int_{\mathfrak{z}} \mathrm{Eis}(\psi_\infty\times\psi_f)$$

and as in [Ha-book] we find that integral can be computed from the second term in the constant term of the Eisenstein class. We copy the result from SecOPs.pdf

$$\mathcal{X}(f)_{H-dRh} = -C(\sigma_{p_0},\lambda)\Big(\frac{1}{\Omega(\sigma_f)^{\epsilon(k,m)}}\frac{\Lambda^{\mathrm{coh}}(f,n_1+n_2+2)}{\Lambda^{\mathrm{coh}}(f,n_1+n_2+3)}\Big)\frac{1}{\zeta(-1-n_1)}\frac{\zeta'(-n_1)}{i\pi} \tag{9}$$

The factor $C(\sigma_{p_0},\lambda)$ is a local contribution which stems from the auxiliary prime p_0. I have not yet done the computation but I think that up to a power of p_0 it is equal to the inverse of the local Euler factor at p_0 in the ratio of L-values. If a_{p_0} is the p_0-th Fourier coefficient, i.e. the eigenvalue of T_{p_0} then $a_{p_0} = \alpha_{p_0}+\beta_{p_0}, \alpha_{p_0}\beta_{p_0} = p^{k-1}$ and we should have

$$\begin{aligned} C(\sigma_{p_0},\lambda) &= \frac{(1-\alpha_{p_0}p_0^{-n_1-n_2-2})(1-\beta_{p_0}p_0^{-n_1-n_2-2})}{(1-\alpha_{p_0}p_0^{-n_1-n_2-3})(1-\beta_{p_0}p_0^{-n_1-n_2-3})}\,\frac{1}{p_0}\frac{1-p_0^{-n_1-1}}{1-p_0^{-n_1-2}} = \\ &\qquad \frac{1-a_{p_0}p_0^{-n_1-n_2-2}+p_0^{-n_1-1}}{1-a_{p_0}p_0^{-n_1-n_2-3}+p_0^{-n_1-3}}\,\frac{1}{p_0}\frac{1-p_0^{-n_1-1}}{1-p_0^{-n_1-2}} \end{aligned} \tag{10}$$

We should interpret the formula (11) as follows: The last factor factor $\frac{\zeta'(-n_1)}{i\pi}$ is an extension class in $\mathrm{Ext}^1_{B-dRh}(\mathbb{Z}(-2-n_1-n_2),\mathbb{Z}(-1-n_2))$ and the rest of this expression is an algebraic number. Since the period is defined up to a unit, it makes sense to speak of the prime decomposition of this number. Under certain conditions we expect congruences modulo primes which occur in the denominator of this number.(See SecOps.pdf)

2.1.2 Non regular coefficients

So far we discussed only the regular case, this means the case where the Eisenstein series is holomorphic at $s=0$. Our our special case this means that $n_1>0$. If we have $n_1=0$ then we have to study the behavior of the function

$$-C(\sigma_{p_0},\lambda)\Big(\frac{1}{\Omega(\sigma_f)^{\epsilon(k,m)}}\frac{\Lambda^{\mathrm{coh}}(f,n_2+2+s)}{\Lambda^{\mathrm{coh}}(f,n_2+3+s)}\Big)\frac{\zeta(1+s)}{\zeta(2+s)} \tag{11}$$

at $s=0$.

Let us recall that f can be viewed as a modular form of weight $k=4+n_1+2n_2=4+2n_2$. Hence we see in the numerator the expression $\Lambda(f,\frac{k}{2}+s)$

If this does not vanish at $s=0$ then the Eisenstein series has a pole at $s=0$. Taking the residue we get some non zero residual classes in $H^2(\mathcal{S}^G_{K_f},\mathcal{M}_\lambda)$, they are square integrable.

At this moment we are more interested in the case where $\Lambda(f,\frac{k}{2})=0$. Then the Eisenstein class will be holomorphic at $s=0$. Let us assume that we are in the unramified case.

It is already discussed in [Ha1] that in this case the induced module $\mathrm{Ind}(\sigma_f)$ has a unitary quotient $J(\sigma_f)$, and this may have the consequence that

$$\mathrm{Hom}_{\mathcal{H}^G}(\mathrm{Ind}(\sigma_f), H^3_!(\mathcal{S}^G_{K_f},\mathcal{M}_\lambda)\neq 0$$

and hence the Manin-Drinfeld principle is not valid under these circumstances. This issue is discussed in [Ha1]. In the appendix (letter to Goresky and MacPherson) we carry out a lacunary computation which shows that

$$\begin{gathered} J(\sigma_f) \textit{ occurs non trivially in } H^3_!(\mathcal{S}^G_{K_f},\mathcal{M}_\lambda) \\ \textit{if and only if the sign in the functional equation is } -1 \end{gathered} \tag{12}$$

We also discuss the relation between this assertion and the Saito Kurokawa lift.

Remark: At this point we should remark that we tacitly assume that we are in the unramified case. This implies that actually $I(\sigma_f)=J(\sigma_f)$ The assertion that that $I(\sigma_f)$ has a unitary quotient, means that after tensorization with $\mathbb{C}$

a) we have an admissible representation $\tilde{I}(\sigma_f)$ of $G(\mathbb{A}_f)$ whose Hecke-module of K_f invariant vectors is $I(\sigma_f)$,

b) The $G(\mathbb{A}_f)-$ module has a non trivial quotient $\tilde{J}(\sigma_f)$ on which we have a positive definite $G(\mathbb{A}_f)$ invariant hermitian scalar product and $I(\sigma_f)$ injects.

In [Ha1] 3.1.4 we also discuss the construction of a mixed motive attached to σ_f. It follows from Piatetkii-Shapiro that $H^3_!(\mathcal{S}^G_{K_f},\mathcal{M}_\lambda\otimes F)$ contains a submodule $\mathcal{SK}(\sigma_f)$ which consists of two copies of $J(\sigma_f)$ and we get an exact sequence

$$0\to\mathcal{SK}(\sigma_f)\to H^3_!(\mathcal{S}^G_{K_f},\mathcal{M}_\lambda\otimes F)(\sigma_f)\to \mathrm{Ind}(H^3(\mathcal{S}^M_{K^M_f},\mathcal{M}_\lambda(w\cdot\lambda))(\sigma_f))\to 0 \tag{13}$$

The motive $\mathcal{SK}(\sigma_f)=\mathbb{M}(\sigma_f,r_1)$ and it provides an extension class

$$\mathcal{Y}(\sigma_f)\in \mathrm{Ext}^1_{\mathcal{MM}}(\mathbb{Z}(-k),\mathbb{M}(\sigma_f,r_1))=\mathrm{Ext}^1_{\mathcal{MM}}(\mathbb{Z}(-k),\mathcal{SK}(\sigma_f)) \tag{14}$$

In [Ha1] we do not discuss the question of computing this extension class, in a sense we did not know what that meant. But following T. Scholl we can give some kind of an answer to this question. We choose an auxiliary prime p_0 and modify K_f at p_0 to the Iwahori and the level will be $K_f(p_0)$. We modify our sheaf and construct a mixed motive $H^3(\mathcal{S}^G_{K_f},\mathcal{M}^\#_\lambda)(\sigma_f)$. We have

$$\begin{gathered} H^1(\mathcal{S}^M_{K^M_f},\mathcal{M})(w'\cdot\lambda)(\sigma_f)^\# \subset H^3(\mathcal{S}^G_{K_f},\mathcal{M}^\#_\lambda)(\sigma_f) \\ H^3(\mathcal{S}^G_{K_f},\mathcal{M}^\#_\lambda)(\sigma_f)\xrightarrow{r} H^1(\mathcal{S}^M_{K^M_f},\mathcal{M})(w\cdot\lambda)(\sigma_f)^\# \end{gathered} \tag{15}$$

The submodule in the top row is a Tate motive $\mathbb{Z}(-k+1)^a$ the quotient in the bottom row is $\mathbb{Z}(-k)^b$ where $a=1,2,\ b=2,1$ depending on the support conditions defining $\mathcal{M}^\#$. We can write two exact sequences

$$\begin{gathered} 0\to H^1(\mathcal{S}^M_{K^M_f},\mathcal{M})(w'\cdot\lambda)(\sigma_f)^\#\to\ker(r)\to\mathcal{SK}(\sigma_f)\to 0 \\ 0\to\mathcal{SK}(\sigma_f)\to H^3(\mathcal{S}^G_{K_f},\mathcal{M}^\#_\lambda)(\sigma_f)\xrightarrow{r} H^1(\mathcal{S}^M_{K^M_f},\mathcal{M})(w\cdot\lambda)(\sigma_f)^\#\to 0 \end{gathered} \tag{16}$$

these two sequences are obtained from the diagram (15). They provide extension classes

$$\mathcal{Y}(\sigma_f) \in \mathrm{Ext}^1_{\mathcal{MM}}(\mathbb{Z}(-k), \mathcal{SK}(\sigma_f), \mathcal{Y}'(\sigma'_f) \in \mathrm{Ext}^1_{\mathcal{MM}}(\mathcal{SK}(\sigma_f), \mathbb{Z}(-k+1)) \quad (17)$$

Such a pair is a biextension and to such a biextenion T. Scholl attaches an "intersection number " or "height pairing"

$$i[(\mathcal{Y}(\sigma_f), \mathcal{Y}'(\sigma'_f))]$$

, which is well defined modulo an "element in $\mathrm{Ext}^1(\mathbb{Z}(-1), \mathbb{Z}(0))$." To define this pairing Scholl "intergrates" the pair of extension classes $((\mathcal{Y}(\sigma_f), \mathcal{Y}'(\sigma'_f))$ into a diagram of type (15) let us call it

$$((\widetilde{\mathcal{Y}(\sigma_f), \mathcal{Y}'}(\sigma'_f)) \quad (18)$$

and to such an object Scholl attaches an honest number

$$i[((\widetilde{\mathcal{Y}(\sigma_f), \mathcal{Y}'}(\sigma'_f)]$$

The "integral" (18) is only defined modulo an element in $\mathrm{Ext}^1(\mathbb{Z}(-1), \mathbb{Z}(0))$ and this explains the ambiguity in the definition of $i[(\mathcal{Y}(\sigma_f), \mathcal{Y}'(\sigma'_f))]$. Moreover the existence of this integral is conjectural.

But in our case we have an integral $i[((\widetilde{\mathcal{Y}(\sigma_f), \mathcal{Y}'}(\sigma'_f)]$, this is simply the diagram (15). (It is like finding a primitive to a function f which is defined as the derivative of a function F.)

Now an extension of the computations in [Ha1] Kap. IV 4.3.3 and SecOPs.pdf to this case should yield

$$i[((\widetilde{\mathcal{Y}(\sigma_f), \mathcal{Y}'}(\sigma'_f)] \sim L^{\mathrm{coh},\prime}(\sigma_f, r_1, \frac{k}{2}) \quad (19)$$

where $\sim$ means up to some uninteresting non zero factors (JW). This is in a certain sense a formula of Gross-Zagier type.

2.1.3 $g \geq 3$

We consider the case $g = 3$. We start from a highest weight $\lambda = n_1\gamma_1 + n_2\gamma_2 + n_3\gamma_3$ for simplicity we assume that this yields a representation of the group $\mathrm{GSp}_3/\mathbb{G}_m$, then we have $n_1+n_3 \equiv 0 \mod 2$. The group $M = \mathrm{Gl}_3\cdot\mathbb{G}_m$ the locally symmetric space $\mathcal{S}^M_{K^M_f}$ is of dimension 5, we look for cohomology in degree 2 and 3. We have the two interesting Kostant representatives $w' = s_3s_2, w = s_3s_2s_3s_1$. For these two elements we consider the coefficient systems $\mathcal{M}_\lambda(w'\cdot\lambda), \mathcal{M}_\lambda(w\cdot\lambda)$ on $\mathcal{S}^M_{K^M_f}$. Since we want to have non trivial inner cohomology we need to assume that the coefficient systems are self dual and hence we need $n_1 = 1+2n_3$. Then we get for our coefficient systems

$$w'\cdot\lambda = (2+n_2+2n_3)(\gamma_1^M+\gamma_2^M)+(-1+n_3)\gamma_3,\ w\cdot\lambda = (2+n_2+2n_3)(\gamma_1^M+\gamma_2^M)+(-3-n_3)\gamma_3.$$

and we can look for isotypical summands

$$H^3(\mathcal{S}^M_{K^M_f}, \mathcal{M}_\lambda(w'\cdot\lambda))(\sigma'_f),\ H^2(\mathcal{S}^M_{K^M_f}, \mathcal{M}_\lambda(w\cdot\lambda))(\sigma_f) \quad (20)$$

We know that they provide motives, if we assume that K_f is unramified then they are Tate motives of weight $\mathbf{w}(w'\cdot\lambda)$ respectively. Now a simple computation shows that the difference $\mathbf{w}(w\cdot\lambda)-\mathbf{w}(w'\cdot\lambda)$ is even and therefore the extension classes should be torsion and our source for congruences dries out. But this is only good because at the time we can not expect a rationality result for the ratios

$$\frac{\Lambda^{\mathrm{coh}}(\sigma_f,\nu-1)}{\lambda^{\mathrm{coh}}(\sigma_f,\nu)}$$

because the motive $M(\sigma_f)$ should have a non zero middle Hodge number and this puts a parity condition on the critical values, (or kills them all).

The situation changes if we the take the parabolic subgroup given by

$$\alpha_1 \quad - \quad \times \quad <= \quad \alpha_3,$$

the semi simple part is $M=\mathrm{PSl}_2\times\mathrm{Sp}_1$. The first factor has to be viewed as the linear factor and corresponds to α_1, the other factor is the hermitian factor. Hence we see that our locally symmetric space is essentially a product

$$\mathcal{S}^M_{K^M_f}=\mathcal{S}_1\times\mathcal{S}_2. \tag{21}$$

We pick a Kostant representative $w\in W^P$ and write as usual

$$w(\lambda+\rho)-\rho=d_1\gamma_1^M+d_3\gamma_3^M+a(w,\lambda)\gamma_2 \tag{22}$$

The resulting coefficient system is a tensor product of coefficient systems on the two factors and hence we see that in the isotypical decomposition (after a suitable finite extension $F/\mathbb{Q}$)

$$H_!^{\bullet-l(w)}(\mathcal{S}^M_{K^M_f},\mathcal{M}_\lambda(w\cdot\lambda)\otimes F)=\bigoplus_{\sigma_f}H_!^{\bullet-l(w)}(\mathcal{S}^M_{K^M_f},\mathcal{M}_\lambda(w\cdot\lambda))(\sigma_f). \tag{23}$$

The $\sigma_f=\tau_f\times\sigma'_f$ where τ_f resp. σ'_f are simply modular forms f of weight $k_1=d_1+2$ and g of weight $k_3=d_3+2$. We simply write

$$\sigma_f=\tau_f\times\sigma'_f=(f,g).$$

If we now apply the Eisenstein intertwining operator then we have to look at the second term in the constant term. We find the formula for it in chap3.pdf section 6.3. The Dynkin diagram of the semi-simple part of the dual group $M^\vee=\mathrm{Gl}_2\times\mathrm{PSl}_2$ is

$$\alpha_1^\vee \quad - \quad \times \quad >= \quad \alpha_3^\vee,$$

the first factor corresponds to Gl_2 the second to PSl_2. We have to compute the action of ${}^LM$ on the Lie-algebra $\mathfrak{u}_P^\vee$. The roots in $\Delta^+_{U_P^\vee}$ are those $\beta^\vee=a_1\alpha_1^\vee+a\alpha_1^\vee+a_3\alpha_3^\vee$ for which $a>0$. By inspection we get 6 such roots with $a=1$ and one such root with $a=2$. We can easily check that $r_1^{\mathfrak{u}_P^\vee}=r_1\otimes\mathrm{Ad}$ and $r_2^{\mathfrak{u}_P^\vee}=\det$, where det is of course the determinant on the first factor. The highest weight for the representation Ad is $\chi_1=\alpha_1^\vee+\alpha_2^\vee+2\alpha_3^\vee$ and $\chi_2=\alpha_1^\vee+2\alpha_2^\vee+2\alpha_3^\vee$

We compute the second constant term. We are interested in cases where we can construct Anderson mixed motives and this means that we should deal with a pair of Kostant representatives w', w where $l(w) = 4$ and $l(w') = 3$. We have two such pairs

$$\begin{array}{ll} w_1 = s_2 s_1 s_3 s_2 & w_1' = s_2 s_1 s_3 \\ w_2 = s_2 s_3 s_2 s_1 & w_2' = s_2 s_3 s_2 \end{array}.$$

and then

$$w_1(\lambda+\rho) = (3+n_2+2n_3)\gamma_1^M + (3+n_1+n_2+n_3)\gamma_3^M + 1/2(-1-n_2)\gamma_2$$

$$w_1'(\lambda+\rho) = (3+n_2+2n_3)\gamma_1^M + (3+n_1+n_2+n_3)\gamma_3^M + 1/2(+1+n_2)\gamma_2$$

$$w_2(\lambda+\rho) = (5+n_1+2n_2+2n_3)\gamma_1^M + (1+n_3)\gamma_3^M + 1/2(-1-n_1)\gamma_2$$

$$w_2'(\lambda+\rho) = (5+n_1+2n_2+2n_3)\gamma_1^M + (1+n_3)\gamma_3^M + 1/2(1+n_1)\gamma_2 \tag{24}$$

The coefficients of γ_1^M resp.γ_3^M are the numbers d_1+1 resp. d_3+1 in equation (22). Then we find easily that

$$\begin{cases} 2d_3 - d_1 = 2 + 2n_1 + n_2 (\geq 2) & \text{if } w = w_1 \\ d_1 - 2d_3 = 4 + n_1 + 2n_2 (\geq 4) & \text{if } w = w_2 \end{cases} \tag{25}$$

In other words: We give ourselves d_1, d_3 and we look for $w = w_1$ resp. $w = w_2$ and for a solution of the equations in (24) with λ dominant. Then d_1, d_3 determine the choice of w.

In the case $w = w_1$ we have the further constraint $d_1 - d_3 = n_3 - n_1$ which in the case $d_1 \leq d_3$ implies $n_1 \geq -d_1 + d_3$.

Then it becomes clear that the possible solutions for n_2 resp. n_1 are even and $\frac{n_2}{2}$ resp. $\frac{n_1}{2}$ run through an interval

$$[0, c_\lambda] = \begin{cases} [0, \min(\frac{2d_3-d_1-2}{2}, \frac{d_1-2}{2})] & \text{if } w = w_1 \\ [0, \frac{d_1-2d_3-4}{2}] & \text{if } w = w_2 \end{cases} \tag{26}$$

We want to understand the expression in chap3.pdf (100). We get in the two cases

$$\begin{array}{rl} < \chi_1, \tilde{\mu_1}^{(1)} >= \frac{1}{2}(9+2n_1+3n_2+4n_3) & b(w_1,\lambda) = -\frac{1}{2}(1+n_2) \\ < \chi_2, \tilde{\mu_1}^{(1)} >= 0 & 2b(w_1,\lambda) = -(1+n_2) \\ \\ < \chi_1, \tilde{\mu_2}^{(1)} >= \frac{1}{2}(7+n_1+2n_2+4n_3) & b(w_2,\lambda) = -\frac{1}{2}(1+n_1) \\ < \chi_2, \tilde{\mu_2}^{(1)} >= 0 & 2b(w_1,\lambda) = -(1+n_1) \end{array} \tag{27}$$

For w_1 this yields for the the following expression for the second constant term (chap3.pdf (100)and SecOps.pdf).

$$\frac{\pi}{\Omega(\sigma_f)^\epsilon} \frac{\Lambda^{\mathrm{coh}}(\tau \times \sigma'_f, r_1 \times \mathrm{Ad}, 5+n_1+2n_2+2n_3)}{\Lambda^{\mathrm{coh}}(\tau \times \sigma'_f, r_1 \times \mathrm{Ad}, 6+n_1+2n_2+2n_3)} \frac{\zeta(1+n_2)}{\zeta(2+n_2)} C^*(\sigma_\infty, \lambda) T_\infty^{\mathrm{loc}}(\omega) \otimes T_f^{\mathrm{loc}}(\psi_f)) \tag{28}$$

and for w_2

$$\frac{\pi}{\Omega(\sigma_f)^\epsilon}\frac{\Lambda^{\mathrm{coh}}(\tau\times\sigma'_f, r_1\times \mathrm{Ad}, 4+n_1+n_2+2n_3)}{\Lambda^{\mathrm{coh}}(\tau\times\sigma'_f, r_1\times \mathrm{Ad}, 5+n_1+n_2+2n_3)}\frac{\zeta(1+n_1)}{\zeta(2+n_1)}C^*(\sigma_\infty,\lambda)T^{\mathrm{loc}}_\infty(\omega)\otimes T^{\mathrm{loc}}_f(\psi_f)) \tag{29}$$

We give the "arithmetic interpretation" of these two second terms. For the beginning we forget the factors at the right we come from the infinite place.

Again we expect that these second terms give us the Betti-de-Rham extension class of a mixed Tate motive $\mathcal{X}(\sigma_f)$ and if we look at the formulae (24) then we see that we get

$$\mathcal{X}(\sigma_f)\in \mathrm{Ext}^1_{\mathcal{M}\mathcal{M}}(\mathbb{Z}(-1-n),\mathbb{Z}(0)) \tag{30}$$

where $n=n_1$ or n_2 depending on the case in which we are. Since we need non torsion classes we have to assume that n is even. Then we apply the functional equation for the Riemann ζ- function to the ratio of ζ values and get

$$\frac{\zeta(n+1)}{\zeta(n+2)}=-\frac{n+1}{\pi^2}\frac{\zeta'(-n)}{\zeta(-1-n)}$$

and we get a factorization

$$\Big(\frac{1}{\Omega(\sigma_f)^\epsilon}\frac{\Lambda^{\mathrm{coh}}(\cdots)}{\Lambda^{\mathrm{coh}}(\cdots+1)}\Big)\Big(\frac{-n-1}{\zeta(-1-n)}\Big)\Big(\frac{\zeta'(-n)}{\pi}\Big) \tag{31}$$

We assume that n is even. The last factor on the right is interpreted as extension class in $\mathrm{Ext}^1_{B-deRham}(\mathbb{Z}(-1-n),\mathbb{Z}(0))$, the factor in the middle is a rational number. The first factor needs some more explanation. It depends on a pair (f,g) of cusp forms on $\mathrm{Gl}_2(\mathbb{Z})$ of weight k_1 resp. k_3. These weights are the coefficients of γ_1^M resp. γ_3^M in (24) augmented by 1. We have the symmetric square lift of the automorphic form σ' to an automorphic form Π_f on $\mathrm{Gl}_3/\mathbb{Z}$. Let $H=\mathrm{Gl}_2\times\mathrm{Gl}_3$ then this lift provides an isotypical subspace

$$H^\bullet(\mathcal{S}^H_{K_f^H},)(\tau_f\times\Pi_f)\subset H^\bullet(\mathcal{S}^H_{K_f^H},) \tag{32}$$

and then we have more or less by definition

$$\Lambda^{\mathrm{coh}}(\tau\times\sigma'_f, r_1\times\mathrm{Ad}, s)=\Lambda^{\mathrm{coh}}(\tau\times\Pi_f, r_1\times r_2, s) \tag{33}$$

where r_1, r_2 are the two tautological representations(**In chap3.pdf erklären).**

Now we have the results in [Ha-Rag] and we know that for integers ν in a certain interval $[c(w,\lambda), d(w,\lambda)]$ the ratios

$$\frac{1}{\Omega(\sigma_f)^\epsilon}\frac{\Lambda^{\mathrm{coh}}(\nu)}{\Lambda^{\mathrm{coh}}(\nu+1)} \tag{34}$$

are algebraic numbers in F. Here $\Omega(\sigma_f)$ is a period which is well defined up to a unit in $\mathcal{O}_F^\times$ (See [Ha-Rag]). The above intervall $[c(w,\lambda), d(w,\lambda)]$ can be determined from the data w,λ. It is called the interval of critical arguments.

2.1.4 Delignes conjectures

In [Ha-book] , chap3.pdf, 3.1 and 3.1.3. we discussed the hypothetical construction of motives attached to isotypical subspaces in the cohomology of arithmetic groups. In our situation here this is actually not so difficult, we have

$$\mathbb{M}(\sigma_f, r_1 \times \mathrm{Ad}) = \mathbb{M}(\tau_f, r_1) \times \mathbb{M}(\Pi_f, r_2) = \mathbb{M}(\tau_f, r_1) \times \mathrm{Sym}^2((\mathbb{M}(\sigma'_f, r_2)). \tag{35}$$

where the factors $\mathbb{M}(\tau_f, r_1), \mathbb{M}(\sigma'_f, r_1)$ are the Deligne-Scholl motives attached to the modular forms (f, g). Note that the motive attached to $(\sigma_f, r_1 \times \mathrm{Ad})$ does not change if we twist σ_f by a power of $|\delta_{P,f}|$.

For any pure motive $\mathbb{M}$ of weight $\mathbf{w} = \mathbf{w}(\mathbb{M})$ Deligne defines a set of critical arguments. To define this set we look a the Hodge-decomposition

$$\mathbb{M}_B \otimes \mathbb{C} = \bigoplus_{p,q:p+q=\mathbf{w}} \mathbb{M}_B^{p,q} \tag{36}$$

and we say that $\mathbb{M}$ has Hodge numbers (p, q) if $\mathbb{M}_B^{p,q} \neq 0$. We define this set only under the assumption that our motive does not have a middle Hodge number, i.e. $h^{\frac{\mathbf{w}}{2},\frac{\mathbf{w}}{2}} = 0$. We look for the Hodge number (p_c, q_c) with $p_c > q_c$ and p_c minimal, then the set of critical arguments is the interval $[q_c + 1, p_c]$.

Under these conditions Deligne formulates the following conjecture (here we assume that the motive is a motive with coefficients in $\mathbb{Q}$)

There exist two periods $\Omega_\pm \in \mathbb{C}^\times$ which are defined in terms of the comparison of Betti- and de-Rham cohomology and which are unique up to an element in $\mathbb{Q}^\times$ such that for all integers $\nu \in [q_c + 1, p_c]$

$$\frac{\Lambda(\mathbb{M}, \nu)}{\Omega_{\epsilon(\nu)}} \in \mathbb{Q} \tag{37}$$

In our situation the Hodge numbers are

$$(d_1 + 1, 0), (0, d_1 + 1) \text{ for } \mathbb{M}(\tau_f, r_1)$$

and

$$(2d_3 + 2, 0), (d_3 + 1, d_3 + 1), (0, 2d_3 + 2) \text{ for } \mathrm{Sym}^2((\mathbb{M}(\sigma'_f, r_2)).$$

The Hodge numbers for $\mathbb{M}(\sigma_f, r_1 \times \mathrm{Ad})$ are the sums of these Hodge numbers. The motive is pure of weight $\mathbf{w} = d_1 + 2d_3 + 3$ this number is odd and hence we know that for all Hodge numbers we have $p \neq q$. Therefore we get

$$p_c - \frac{\mathbf{w}+1}{2} = \begin{cases} \begin{cases} \frac{d_1}{2} & \text{if } d_1 \leq d_3 \\ d_3 - \frac{d_1}{2} & \text{if } d_1 > d_3 \end{cases} & \text{if } w = w_1 \\ \frac{d_1}{2} - d_3 - 1 & \text{if } w = w_2 \end{cases}. \tag{38}$$

Miraculously (?) this number is the number $c_\lambda + 1$ in (26). Our second term in the constant term becomes

$$\frac{\pi}{\Omega(\sigma_f)^\epsilon} \frac{\Lambda^{\mathrm{coh}}(\sigma_f, r_1 \times \mathrm{Ad}, \frac{\mathbf{w}+1}{2} + \frac{n}{2})}{\Lambda^{\mathrm{coh}}(\sigma_f, r_1 \times \mathrm{Ad}, \frac{\mathbf{w}+1}{2} + \frac{n}{2} + 1)} \frac{\zeta(1+n)}{\zeta(2+n)} C^*(\sigma_\infty, \lambda) T_\infty^{\mathrm{loc}}(\omega) \otimes T_f^{\mathrm{loc}}(\psi_f)) \tag{39}$$

where $n = n_1$ or $n = n_2$ depending on the case. The argument $\frac{\mathbf{w}+1}{2} + \frac{n}{2} + 1$ runs exactly over the right half of the critical arguments.

If we believe in the existence of the motive $\mathbb{M}(\sigma_f, r_1 \times \mathrm{Ad})$ and the equality of the motivic and the cohomological L-function then the conjecture of Deligne predicts that in formula (39) the ratio

$$\frac{1}{\Omega(\sigma_f)^{\epsilon}} \frac{\Lambda^{\mathrm{coh}}(\sigma_f, r_1 \times \mathrm{Ad}, \frac{\mathbf{w}+1}{2} + \frac{n}{2})}{\Lambda^{\mathrm{coh}}(\sigma_f, r_1 \times \mathrm{Ad}, \frac{\mathbf{w}+1}{2} + \frac{n}{2} + 1)} \tag{40}$$

is an algebraic number in F, provided ν and $\nu + 1$ are critical and we choose as period

$$\Omega(\sigma_f)^{\epsilon} = \frac{\Omega(\mathbb{M}(\sigma_f, r_1 \times \mathrm{Ad}))_{\epsilon(\nu+1)}}{\Omega(\mathbb{M}(\sigma_f, r_1 \times \mathrm{Ad}))_{\epsilon(\nu)}}$$

In this form Delignes conjectures are not available, already the existence of the motive is not clear. But there is still another drawback: The periods $\Omega(\mathbb{M}(\sigma_f, r_1 \times \mathrm{Ad}))_{\epsilon(\nu)}$ are only defined modulo an element in $F^{\times}$. The definition of the periods uses the comparison between the Betti and de-Rham cohomology.

In our paper with Raghuram [Ha-Rag] we prove a rationality result about special values of Rankin-Selberg L-functions which is weaker than Delignes conjecture but also in some sense stronger. Applied to our situation here it says that we can define a period $\Omega(\sigma_f)$ which is well defined up to an element in $\mathcal{O}_F^{\times}$. With this definition of the period the numbers

$$\frac{1}{\Omega(\sigma_f)^{\epsilon}} \frac{\Lambda^{\mathrm{coh}}(\sigma_f, r_1 \times \mathrm{Ad}, \frac{\mathbf{w}+1}{2} + \frac{n}{2})}{\Lambda^{\mathrm{coh}}(\sigma_f, r_1 \times \mathrm{Ad}, \frac{\mathbf{w}+1}{2} + \frac{n}{2} + 1)} \frac{1}{\zeta(-1-n)} C^*(\sigma_\infty, \lambda) \tag{41}$$

are in $F^{\times}$ are their prime decomposition is well defined. In [Ha-Rag] we also show that the factor $C^*(\sigma_\infty, \lambda)$ is a non zero rational number. It is an important question to compute this number exactly. In the case $g = 2$ this number in SecOps.pdf and it turns out to be very simple. A similar question arises in [Ha-Mum] and has been solved by Don Zagier in the appendix to that paper.

We are again at the point where we can ask the question whether primes $\mathfrak{l}$ dividing the denominator of the algebraic number in (41) create denominators of the Eisenstein classes and therefore also congruences between eigenvalues of modular forms on different groups.

We return to the ratios of L-values on p.3. The L-functions which occur in these expressions are actually the "automorphic" or "unitary" L functions. But I think that I have strong reasons that we should express them in terms of the "cohomological" L-function. In the case discussed in "Eis-coh..." the arguments of evaluation are exactly the critical points of the Scholl-motive $M(f)$ attached to the automorphic form and this is equal to the cohomological L-function.

In the special case which we consider we started from two modular forms f, g of weights k_1, k_3 respectively. For both of them we have the Scholl-motive $M(f), M(g)$ and the two dimensional ℓ-adic Galois-representations

$$\rho(\tau) : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gl}(M(f))_{\ell}), \rho(\sigma) : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gl}(M(f))_{\ell}),$$

and we have for the Frobenii:

$$\rho(\tau)(\Phi_p^{-1}) \simeq \begin{pmatrix} \alpha_p & 0 \\ 0 & \beta_p \end{pmatrix}, \ \alpha_p + \beta_p = a_p, \alpha_p\beta_p = p^{k_1-1} = p^{d_1+1}$$

$$\rho(\sigma)(\Phi_p^{-1}) \simeq \begin{pmatrix} \gamma_p & 0 \\ 0 & \delta_p \end{pmatrix}, \ \gamma_p + \delta_p = c_p, \gamma_p\delta_p = p^{k_3-1} = p^{d_3+1}$$

where a_p resp. c_p is the $p-th$ Fourier coefficient of f resp. g.

We take the symmetric square of $\rho(\sigma)$ and get

$$\rho(\mathrm{Sym}^2(\sigma)) : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to Gl_3(\mathbb{Z}_\ell)$$

(here we assume that f, g have coefficients in $\mathbb{Z}$.) Then

$$\rho(\mathrm{Sym}^2(\sigma))(\Phi_p^{-1}) \simeq \begin{pmatrix} \gamma_p^2 & 0 & 0 \\ 0 & p^{d_3+1} & 0 \\ 0 & 0 & \delta_p^2 \end{pmatrix}$$

Then we can write the finite part of the L-function as

$$L^{\mathrm{coh}}(\tau \times \Pi, s) = \prod_p \frac{1}{\det\big(\mathrm{Id} - \begin{pmatrix} \alpha_p & 0 \\ 0 & \beta_p \end{pmatrix}) \otimes \begin{pmatrix} \gamma_p^2 & 0 & 0 \\ 0 & p^{d_3+1} & 0 \\ 0 & 0 & \delta_p^2 \end{pmatrix} p^{-s}\big)}$$

Here it becomes clear that this is the motivic L-function of the motive $M(\tau \times \Pi)$. Here the representation r of the dual group is the tensor product of the two tautological representations.

The local Euler-factor is of degree 6 it can be expressed in terms of the eigenvalues a_p, c_p and is given by

$$\big[\big(1 + \left(-a_p c_p^2 + 2a_p p^{-1+h}\right) p^{-s} + (a_p^2 p^{-2+2h} + c_p^4 p^{-1+k} - 4c_p^2 p^{-2+h+k} + 2p^{-3+2h+k})p^{-2s} + (a_p c_p^2 p^{-3+2h+k} + 2a_p p^{-4+3h+k})p^{-3s} + p^{-6+2(2h+k)}p^{-4s}\big)\big) * (1 - a_p p^{h-1}p^{-s} + p^{k+2h-3}p^{-2s})\big]^{-1}$$

Our motives $M(f), M(g)$ have Hodge types $\{(d_1+1, 0), (0, d_1+1), (d_3+1, 0), (0, d_3+1)\}$ and therefore we get for the Hodge type of $M(\tau \times \Pi)$

$$\{(d_1{+}2d_3{+}3, 0), (d_1{+}d_3{+}2, d_3{+}1), (d_1{+}1, 2d_3{+}2), (2d_3{+}2, d_1{+}1), (d_3{+}1, d_1{+}d_3{+}2), (0, d_1{+}2d_3{+}3)\}$$

it is pure of weight $d_1 + 2d_3 + 3$.

We reorder these Hodge type according to the size of the second component and get

$$\{(w, 0), (w-a, a), (w-b, b), (b, w-b), (a, w-a), (0, w)\},$$

where now $0 \leq a \leq b \leq \frac{w}{2}$.

From the the Hodge type or from representation-theoretic considerations we get a Γ factor at infinity which is (if I am not mistaken)

$$L_\infty(\tau \times \Pi, s) = \frac{\Gamma(s)\Gamma(s-a)\Gamma(s-b)}{(2\pi)^{3s}}$$

Again we put

$$\Lambda^{\mathrm{coh}}(\tau \times \Pi, s) = L_\infty(\tau \times \Pi, s) L^{\mathrm{coh}}(\tau \times \Pi, s).$$

This function satisfies a functional equation:

$$\Lambda^{\mathrm{coh}}(\tau \times \Pi, s) = \Lambda^{\mathrm{coh}}(\tau \times \Pi, w + 1 - s)$$

Once we accept this functional equation then we have fast algorithms to compute the values $\Lambda^{\mathrm{coh}}(\tau \times \Pi, s_0)$ at given argument s_0 up to very high precision.

(For classical modular forms f of weight k we have the following formula

$$\Lambda(f, s) = \sum_{n=1}^{\infty} \big((\frac{1}{2\pi})^s \frac{a_n}{n^s} \Gamma(s, 2\pi n A) + (-1)^{\frac{k}{2}} (\frac{1}{2\pi})^{k-s} \frac{a_n}{n^{k-s}} \Gamma(s, 2\pi n/A) \big)$$

where $\Gamma(s, 2\pi n A)$ is the incomplete Γ function and where A is a strictly positive real number. The right hand side is independent of A (this gives a good test that the functional equation is really correct) and $A = 1$ is the best choice. The sum is rapidly converging, because the incomplete Γ goes rapidly to zero.)

I remember that Don Zagier once mentioned that we always have such a formula to compute values of L-functions, once we can guess the functional equation and this formula can be used to confirm the guess.

This has been done by Tim Dokchitser in his Note "Computing special values of motivic L-functions. Experiment. Math. 13 (2004), no. 2, 137–149. "

Finally we discuss the special values. We have the above list of Hodge types, recall that the Hodge types lists those pairs (p, q) with $p + q = w = d_1 + 2d_3 + 2$ for which $h^{p,q}(M) \neq 0$. The Deligne conjecture predicts that we have to look at pairs (p_c, q_c) for which $p_c + q_c = w, p_c > q_c$ for which $h^{p_c,q_c} \neq 0$ and for which $h^{\nu,w-\nu} = 0$ for all $q_c < \nu < p_c$. This is the critical interval $M_{\mathrm{crit}} = [(p_c, q_c), (q_c, p_c)]$ of our motive. One should look at it as an interval on the line $p + q = w$.

We look at our Hodge types

$$\{(d_1{+}2d_3{+}3, 0), (d_1{+}d_3{+}2, d_3{+}1), (d_1{+}1, 2d_3{+}2), (2d_3{+}2, d_1{+}1), (d_3{+}1, d_1{+}d_3{+}2), (0, d_1{+}2d_3{+}3)\}$$

We have to find the interval we have to distinguish cases. The first case is

a)

$$d_1 < 2d_3 + 1$$

Now we have two possibilities for the critical interval, it is either

a1)

$$[(2d_3 + 2, d_1 + 1), (d_1 + 1, 2d_3 + 2)]$$

a2)

$$[(d_1 + d_3 + 2, d_3 + 1), (d_3 + 1, d_1 + d_3 + 2)]$$

depending on which one is smaller.

The second case is
b)

$$d_1 > 2d_3 + 1$$

In this case the critical interval is clearly

$$[(d_1 + 1, 2d_3 + 2), (2d_3 + 2, d_1 + 1)],$$

In the paper with Raghuram [Ha-Rag] we will prove that we can define a period $\Omega(\tau_f \times \Pi_f)$ which under our assumptions (f, g have coefficients in $\mathbb{Q}$) is unique up to a sign such that

$$\Omega(\tau_f \times \Pi_f)^{\epsilon(a)} \frac{\Lambda^{\mathrm{coh}}(\tau \times \Pi, a)}{\Lambda^{\mathrm{coh}}(\tau \times \Pi, a+1)} \in \mathbb{Q} \text{ provided } p_c \geq a+1\ , a \geq q_c + 1$$

From our data $[p_c, q_c]$ and the value of a we can reconstruct the coefficient system λ.

"Large" primes occuring in the denominator of these rational number should produce congruence between eigenvalues of Hecke operators on Siegel modular forms of genus three and certain expressions in eigenvalues on pairs of modular forms of genus one.

The computation of the period is somewhat delicate. We give a definition in [Ha-Rag] and the period is well defined up to a unit (under our special assumptions up to ± 1) But it is not clear from the abstract definition how - given explicit data, i.e. f, g -we can really compute a number with high precision which gives us the value of the period.

There is a way out. Recall that we compute ratios of special values $a, a+1$ where a runs through an interval $[p_c - 1, q_c + 1]$ of integers, this interval can be quite long. So we simply choose our period such that for $a_0 = p_c - 1$

$$\Omega^*(\tau_f \times \Pi_f)^{\epsilon(a_0)} \frac{\Lambda^{\mathrm{coh}}(\tau \times \Pi, a_0)}{\Lambda^{\mathrm{coh}}(\tau \times \Pi, a_0+1)} = 1.$$

The correct period differs from this one by a rational number, which will have some prime factors $\{p_1, p_2, \ldots, p_r\}$ in it. Now we can start to verify the above rationality assertion for all a and we can compute these ratios as rational numbers.

Recall that we are interested in arguments a for which our ratio of L-values divided by the "correct" period has a "large" prime p in its factorization (in the denominator). Now it would be really bad luck, if this prime p would be (always) member of $\{p_1, p_2, \ldots, p_r\}$.

Hence if we find large primes p in the denominator of the ratios

$$\Omega^*(\tau_f \times \Pi_f)^{\epsilon(a)} \frac{\Lambda^{\mathrm{coh}}(\tau \times \Pi, a)}{\Lambda^{\mathrm{coh}}(\tau \times \Pi, a+1)}$$

for some values of a then we can look for congruences $\mod p$ between different kinds of Siegel modular forms.

2.1.5 The Hecke operators on the boundary cohomology

We go back to the very general case that $G/\mathrm{Spec}(\mathbb{Z})$ is a Chevalley scheme and let $P \subset G$ be a maximal parabolic subgroup, here we assume that it is conjugate to its opposite. We assume that $T/\mathrm{Spec}(\mathbb{Z})$ is a maximal split torus and $T \subset B \subset P$. Let $\pi = \{\alpha_1, \alpha_2, \dots, \alpha_r\}$ be the set of simple positive roots, let $\{\gamma_1, \gamma_2, \dots, \gamma_r\}$ be the set of dominant fundamental weights. We have

$$2\frac{< \gamma_i, \alpha_j >}{< \alpha_i, \alpha_i >} = \delta_{ij},$$

the dominant weights are elements in $X^*(T) \otimes \mathbb{Q}$. We also consider the cocharacters $\{\chi_1, \chi_2, \dots, \chi_r\} \in X_*(T) \otimes \mathbb{Q}$, which form the dual basis to the α_i. If we identify $X_*(T) \otimes \mathbb{Q} = X^*(T) \otimes \mathbb{Q}$ via the canonical quadratic form, then $\chi_i = \frac{2\gamma_i}{<\alpha_i, \alpha_i>}$.

We choose a parabolic subgroup P, let α_{i_0} be the erased simple root. We consider the cuspidal (inner ?) cohomology of the boundary stratum attached to P and consider an isotypical subspace

$$H_!^{\bullet - l(\tilde{w})}(S^M, \mathcal{M}(\tilde{w} \cdot \lambda))(\sigma_f)) \subset H^\bullet(\partial_P(S), \mathcal{M}).$$

Actually we should take an induced module on the left hand side, but let us assume that we only look at unramified cohomology, i.e. $K_f = G(\hat{\mathbb{Z}})$. Then induction simply means that we restrict the action of $\mathcal{H}^M$ to the action of $\mathcal{H}^G$ on $H_!^{\bullet - l(w)}(S^M, \mathcal{M}(w \cdot \lambda))$. We want to derive a formula for a "cohomological" Hecke operator in $\mathcal{H}^G$ as a sum over "cohomological" Hecke operator in $\mathcal{H}^M$.

The algebra of Hecke operators is generated by local algebras $\mathcal{H}_p^G$ and these local algebras commute (under our assumption that everything is unramified, they are even commutative).

We fix a prime p. To get Hecke operators we start from cocharacters $\chi = \sum m_i \chi_i : G_m \to T$, where the $m_i \in \mathbb{Z}$. This provides an element $\chi(p) \in T(\mathbb{Q}_p)$, and hence a double coset $K_p \chi(p) K_p$ whose characteristic function is denoted by T_χ. By convolution this defines an operator (also denoted by T_χ) on the cohomology with 3.1.2 rational coefficients

$$T_\chi : H^\bullet(\mathcal{S}_{K_f}^G, \mathcal{M}_\lambda \otimes \mathbb{Q}) \to H^\bullet(\mathcal{S}_{K_f}^G, \mathcal{M}_\lambda \otimes \mathbb{Q}).$$

We have defined the modified operators, which act on the cohomology with integral coefficients

$$T_\chi^{\mathrm{coh}} \text{ " = " } p^{c(\chi,\lambda)} T_\chi : H^\bullet(\mathcal{S}_{K_f}^G, \mathcal{M}_\lambda) \to H^\bullet(\mathcal{S}_{K_f}^G, \mathcal{M}_\lambda).$$

(See chap.3.pdf 3.1.2)

We have a formula for the action of $\mathcal{T}_\chi$ on the unramified spherical functions. We consider unramified characters $\nu_p : T(\mathbb{Q}_p) \to \mathbb{C}^\times$. Since $T(\mathbb{Q}_p) = X_*(T) \otimes \mathbb{Q}_p^\times$ we have for the module of unramified characters

$$\mathrm{Hom}_{un}(T(\mathbb{Q}_p), \mathbb{C}^\times) = \mathrm{Hom}(X_*(T), \mathbb{C}^\times) = X^*(T) \otimes \mathbb{C}^\times$$

If we pick a $\chi \in X_*(T)$ and a $\nu_p =\in \mathrm{Hom}_{un}(T(\mathbb{Q}_p), \mathbb{C}^\times) \nu_p(\chi(p))$ We have the embedding $X^*(T) \hookrightarrow \mathrm{Hom}_{un}(T(\mathbb{Q}_p), \mathbb{C}^\times)$ which is given by $\gamma \mapsto |\gamma|_p =$

$(x \mapsto |\gamma(x)|_p)$. I want to distinguish carefully between the algebraic character and its absolute value. If we have a $\gamma \in X^*(T)$ and a $\chi \in X_*(T)$ then we put

$$|\gamma|_p(\chi(p)) =< \chi, \gamma >_p = p^{-<\chi,\gamma>}$$

Especially we have the half sum of positive roots $\rho_B^G \in X^*(T) \otimes \mathbb{Q}$ and the resulting character $|\rho_B^G|$.

We define the spherical function ψ_{ν_p} by

$$\psi_{\nu_p}(g) = \nu_p(bk) = \nu_p(b)$$

and this will be an eigenfunction for the convolution with a Hecke operator

$$T_\chi * \psi_{\nu_p} = T_\chi^\vee(\nu_p)\psi_{\nu_p}.$$

This spherical function differs from the spherical function in chap3.pdf 2.3.4 they are related by the formula

$$\psi_{\nu_p}(g) = \phi_{\nu_p - |\rho_B^G|_p}(g)$$

We write a formula for $T_\chi^\vee(\nu_p)$ for the case that $\chi = \chi_i$ is one of our basis cocharacters χ_i. We look at the orbit of χ_i under the Weyl group, let W_i be the stabilizer of χ_i in W, then

$$T_\chi^\vee(\nu_p) = p^{<\chi_i,\rho_B^G>} \sum_{W/W_i} < w\chi_i, \nu_p - |\rho_B^G|_p > +\delta(\chi_i),$$

where $\delta(\chi_i)$ is a positive integer. It is zero if for all positive roots α we have $< \chi_i, \alpha >\in \{0,1\}$, i.e. the coefficient of the root α_i in any positive root is always ≤ 1. (This extra term comes bla bla)

If we now have an isotypical submodule $H_!^\bullet(S^G, \mathcal{M}_\lambda)(\pi_f), \pi_f = \otimes_p \pi_p$, and $\pi_p = \mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} \nu_p$ (algebraic induction) then our above formula says

$$T_{\chi_i}^{\mathrm{coh}}(\pi_f) = p^{<\chi_i,\lambda+\rho_B^G>}(\sum_{W/W_i} < w\chi_i, \nu_p - |\rho_B^G|_p >) + p^{<\chi_i,\lambda+\rho_B^G>}\delta(\chi_i). \quad (42)$$

The exponent $< \chi_i, \lambda + \rho_B^G >= c(\chi, \lambda)$, the δ is equal to zero because of our assumption.

Now we ask for a formula for the Hecke operator on T_χ^{coh} on an isotypical piece $H_!^{\bullet - l(\tilde{w})}(S^M, \mathcal{M}(\tilde{w}\cdot\lambda))(\sigma_f))$ in the cohomology of some boundary stratum. We assume that $\sigma_p = \mathrm{Ind}_{B(\mathbb{Q}_p)}^{M(\mathbb{Q}_p)} \nu_p$. The Weyl group W_M acts on W/W_i from the left, let us choose a set of representatives $\{\dots, v, \dots\}$ for this action. Then the sum becomes

$$p^{<\chi_i,\lambda+\rho_B^G>}\Big(\sum_{v\in W_M\backslash W/W_i} \sum_{wv \in W_M/W_{M,i}} < wv\chi_i, \nu_p - |\rho_B^M|_p >) + p^{<\chi_i,\lambda+\rho_B^G>}\delta(\chi_i)\Big) \quad (43)$$

We want to transform this into a sum over Hecke operators acting on $H_!^{\bullet - l(\tilde{w})}(S^M, \mathcal{M}(\tilde{w}\cdot \lambda))(\sigma_f))$ we write

$$p^{<\chi_i,\lambda+\rho_B^G>}\Big(\sum_{v\in W_M\backslash W/W_i}\sum_{wv\in W_M/W_{M,i}} < wv\chi_i,\nu_p-|\rho_B^M|_p-|\rho_P|_p>\Big)+p^{<\chi_i,\lambda+\rho_B^G>}\delta(\chi_i) \tag{44}$$

The character $\rho_P=f_P\gamma_{i_0}$ is invariant under the action of W_M we can pull this factor in front

$$p^{<\chi_i,\lambda+\rho_B^G>}\Big(\sum_{v\in W_M\backslash W/W_i} p^{f_P<v\chi_i,\gamma_{i_0}>}\sum_{w\in W_M/W_{M,i}} < wv\chi_i,\nu_p-|\rho_B^M|_p>\Big)+p^{<\chi_i,\lambda+\rho_B^G>}\delta(\chi_i) \tag{45}$$

For a given $v\in W_M\backslash W$ the inner sum is the value of a Hecke operator on the cohomology $H_!^{\bullet-l(\tilde{w})}(S^M,\mathcal{M}(\tilde{w}\cdot\lambda))(\sigma_f))$ times a correcting factor. To compute this correcting factor we write

$$\tilde{w}(\lambda+\rho_B^G)=\tilde{\mu}^{(1)}_{\tilde{w},\lambda}+b(\tilde{w},\lambda)\gamma_{i_0} \tag{46}$$

Note that this expression is - as it must be- independent of the representative v. If we want to compute the correcting factor we have to choose the representative $v_k=w_kv, w_k\in W_M$ such that $v_k\chi_i$ is in the positive chamber with respect to the given Borel subgroup in M, i.e.

$$< v_k\chi_i,\alpha_\nu>\geq 0 \text{ for all } \nu\neq i_0 \tag{47}$$

this is certainly true if v_k is a Kostant representative.

Then the correcting factor becomes

$$p^{<v_k\chi_i,\tilde{\mu}^{(1)}_{\tilde{w},\lambda}-b(\tilde{w},\lambda)\gamma_{i_0}>} \tag{48}$$

(note the minus sign!) and hence we get

$$p^{<v_k\chi_i,\tilde{\mu}^{(1)}_{\tilde{w},\lambda}-b(\tilde{w},\lambda)\gamma_{i_0}>}\sum_{w\in W_M/W_{v,i}} < wv_k\chi_i,\nu_p-|\rho_B^M|_p>=T^{M,\mathrm{coh}}_{v_k\chi}(\sigma_f) \tag{49}$$

We get for our eigenvalue (??? wo ist das f_P geblieben???????)

$$\sum_{v_k\in W_M\backslash W/W_i} p^{<\chi_i,\lambda+\rho_B^G>-<v_k\chi_i,\tilde{\mu}^{(1)}_{\tilde{w},\lambda}-b(\tilde{w},\lambda)\gamma_{i_0}>}T_{v_k\chi_i}(\sigma_f)\Big)+p^{<\chi_i,\lambda+\rho_B^G>}\delta(\chi_i) \tag{50}$$

and this is equal to

$$\sum_{v_k\in W_M\backslash W/W_i} p^{<\chi_i,\lambda+\rho_B^G-v_k^{-1}(\tilde{\mu}^{(1)}_{\tilde{w},\lambda}-b(\tilde{w},\lambda)\gamma_{i_0})>}) T_{v_k\chi_i}(\sigma_f)\Big)+p^{<\chi_i,\lambda+\rho_B^G>}\delta(\chi_i) \tag{51}$$

We can still write this differently, we have

$$\tilde{\mu}^{(1)}_{\tilde{w},\lambda} - b(\tilde{w},\lambda)\gamma_{i_0} = \tilde{\mu}^{(1)}_{\tilde{w},\lambda} + b(\tilde{w},\lambda)\gamma_{i_0} - 2b(\tilde{w},\lambda)\gamma_{i_0} = \tilde{w}(\lambda+\rho_B^G) - 2b(\tilde{w},\lambda)\gamma_{i_0} \tag{52}$$

and then (51) becomes

$$\sum_{v_k \in W_M\backslash W/W_i} p^{<\chi_i,\lambda+\rho_B^G - v_k^{-1}(\tilde{w}(\lambda+\rho_B^G)-2b(\tilde{w},\lambda)\gamma_{i_0})>}\big(T_{v_k\chi_i}(\sigma_f)\big) + p^{<\chi_i,\lambda+\rho_B^G>}\delta(\chi_i) \tag{53}$$

The factor in front is equal to one if $w = \tilde{w}$ and otherwise the exponent is a strictly positive number. Hence we get

$$T^{G,\mathrm{coh}}_{\chi_i}(\mathrm{Ind}(\sigma_p)) = T^{M,\mathrm{coh}}_{\tilde{w}\chi_i}(\sigma_p) + \qquad \text{Hecke-ind}$$

$$\sum_{w\in W^P/W_i, w\neq\tilde{w}} p^{<\chi_i,(\lambda+\rho_B^G)-w^{-1}(\tilde{w}(\lambda+\rho_B^G)>-2b(\tilde{w},\lambda)\gamma_{i_0})>} T^{M,\mathrm{coh}}_{w\chi_i}(\sigma_p) + p^{<\chi_i,\lambda>}\delta(\chi_i).$$

Let us call the first summand on the right hand side the "main" term. We observe that for $w \neq \tilde{w}$ the exponent $<\chi_i,(\lambda+\rho_B^G) - w^{-1}\tilde{w}(\lambda+\rho_B^G) >> 0$ and if λ is regular this is also true for $<\chi_i,\lambda>$. This tells us that the eigenvalue $T^{M,\mathrm{coh}}_{\chi_i}(\mathrm{Ind}(\sigma_p))$ is a p-adic unit if and only if $T^{M,\mathrm{coh}}_{\tilde{w}\chi}(\sigma_p)$ is a p-adic unit, provided λ is regular or $\delta(\chi_i) = 0$.

(For the special case $G = GSp_2/\mathrm{Spec}(\mathbb{Z})$ and P the Siegel parabolic this yields the formulae in 3.1.2.1 in "Eisenstein Kohomologie...". The formula for $T_{p,\beta}$ is wrong, I overlooked the term $p^{<\chi_i,\lambda>}\delta(\chi_i)$. This was discovered by Gerard, the congruences for the second Hecke operator became wrong.)

2.1.6 The general philosophy

Now we can formulate how the general form of a Ramunujan-type congruence should look like. We start from an isotypical subspace $H^\bullet(S^M,\mathcal{M}(w\cdot\lambda_R))(\sigma_f)$ where $R = \mathbb{Z}[1/N]$ where N is a suitable integer. Let $I_{\sigma_f} \subset \mathcal{H}^M_R$ be the annihilator of σ_f. Then the quotient $\mathcal{H}^M_R/I_{\sigma_f} = R(\sigma_f)$ is an order in an algebraic number field $\mathbb{Q}(\sigma_f)$. We consider the second constant term of the Eisenstein series evaluated at $s_w = 0$ and assume that it is of the form

$$a(\sigma_f)\mathrm{Mot}(\sigma_f)$$

where $a(\sigma_f) \in \mathbb{Q}(\sigma)$ and where $\mathrm{Mot}(\sigma_f)$ has some kind of an interpretation as an element in some $\mathrm{Ext}^1_{\mathcal{MM}}$. Now we assume that a "large" prime $\mathfrak{l} \subset R(\sigma_f)$ divides the denominator of $a(\sigma_f)$. We assume that σ_ℓ is ordinary at $\mathfrak{l}$, i.e. $T^{M,\mathrm{coh}}_{\chi_i}(\sigma_\ell) \notin \mathfrak{l}$ for all i (some i_0 ?).

Then we can hope for an isotypical component Π_f for the Hecke algebra $\mathcal{H}^G_R$ in the cohomology $H^\bullet(S^G,\mathcal{M}_\lambda)(\Pi_f)$, we consider the order $\mathcal{H}^G_R/I_{\Pi_f} = R(\Pi_f)$, we expect to find a prime $\mathfrak{l}_1 \subset R(\Pi_f)$ and an isomorphism between the completions

$$\Phi : R(\Pi_f)_{\mathfrak{l}_1} \xrightarrow{\sim} R(\sigma_f)_{\mathfrak{l}}$$

such that for all primes p

$$\Phi(T^G_{\chi_i}(\Pi_p)) \equiv T^{G,\mathrm{coh}}_{\chi_i}(\mathrm{Ind}(\sigma_p)) \mod \mathfrak{l}.$$

We consider the case where our modular forms f, g have rational coefficients, i.e. are of weight $12, 16, 18, 20, 22, 26$ this means that the values for d_1, d_3 are $10, 14, 16, 18, 20, 24$. Following a notation in representation theory we put

$$w \cdot \lambda = w(\lambda + \rho) - \rho = d_1(w \cdot \lambda)\gamma^M_{\alpha_1} + d_3(w \cdot \lambda)\gamma^M_{\alpha_3} + 1/2(-6 - n_2)\gamma_{\alpha_2}.$$

Given d_1, d_3 a value a in the upper half of the above range, we solve the equations

$$d_1(w_1 \cdot \lambda) = d_1,\ d_3(w_1 \cdot \lambda) = d_3,\ \frac{n_2}{2} + \frac{d_1}{2} + d_3 + 2 = a \qquad (case1)$$

$$d_1(w_2 \cdot \lambda) = d_1,\ d_3(w_2 \cdot \lambda) = d_3,\ \frac{n_1}{2} + \frac{d_1}{2} + d_3 + 3 = a \qquad (case2)$$

We introduce the number

$$\mathbf{w} = d_1 + 2d_3 + 3$$

and observe that $\frac{d_1}{2} + d_3 + 2 = \frac{\mathbf{w}+1}{2}$ is the reflection point of the functional equation. We rewrite our equations a little bit. In $(case1)$

$$\begin{gathered} k_1 - 4 = d_1 - 2 = n_2 + 2n_3 \\ k_3 - 4 = d_3 - 2 = n_1 + n_2 + n_3 \\ 2a - \mathbf{w} - 1 = n_2 \end{gathered}$$

and in (case2)

$$\begin{gathered} k_1 - 6 = d_1 - 4 = n_1 + 2n_2 + 2n_3 \\ k_3 - 4 = d_3 - 2 = n_3 \\ 2a - \mathbf{w} - 3 = n_1 \end{gathered}$$

As it turns out that for our restricted choice of f, g we never have solutions in $(case2)$.

This gives us a unique highest weight $\lambda = \lambda(d_1, d_3, a)$ and a space of holomorphic modular cusp forms $S_{n_1,n_2,4+n_3}$ in which we should look for a cusp form satisfying congruences.

I want to give the precise form for the expected congruences. We choose the Hecke operator T_{χ_3}, this is the operator whose eigenvalues are the traces of the Frobenius, it has also the property that $< \chi_3, \alpha > \in \{-1, 0, 1\}$ for all roots α, and if we identify $X_*(T)_\mathbb{Q} = X_*(T)_\mathbb{Q}$ then $\chi_3 = \gamma_3$.

The Weyl group W is the semidirect product of S_3 and $(\mathbb{Z}/2\mathbb{Z})^3$ and is of order 48. The stabilizer W_3 of χ_3 is the subgroup S_3, this is the Weyl group of A_2. We have to study the double cosets

$$W_M \backslash W / W_3 = W^P / W_3.$$

The quotient W/W_3 has cardinality 8, on this quotient we have the action of W_M, this is the group generated by the reflections s_1, s_3 and hence is of order 4.

It is clear that we have two orbits of length 2 and one orbit of length 4. Hence the sum in (Hecke-ind) has three terms.

The orbit of length 4 gives us the "main" term in our formula (Hecke-ind) and $T^{M,\mathrm{coh}}_{\tilde{w}\chi_i}(\sigma_p) = a_p(f)a_p(g)$, where of course the two factors are the eigenvalues of f, g respectively.

The two other orbits correspond to the Kostant representatives $e = (\mathrm{Id}, \Theta_P$, they are fixed by s_1, hence the W_M orbits are given by $\{(e, s_3), (\Theta_P, s_3\Theta_P)\}$. This means that for choice of w we have $T^{M,\mathrm{coh}}_{w^{-1}\tilde{w}\chi_i}(\sigma_p) = a_p(g)$, it remains to compute the factor in front. For $w = e$ or $w = \Theta_P$ this factor is

$$p^{<(\mathrm{Id}-\tilde{w}^{-1}w)\chi_3,\lambda+\rho>}$$

Our element $\tilde{w}$ is one of the two Kostant representatives w_1, w_2 on p. 1. Then $\tilde{w}^{-1}\Theta_P$ is equal to the the corresponding elements v_1, v_2. We get

$$< (\mathrm{Id} - w_1^{-1})\chi_3, \lambda + \rho >= n_2 + n_3 + 2 \qquad < (\mathrm{Id} - v_1^{-1})\chi_3, \lambda + \rho >= n_3 + 1$$
$$< (\mathrm{Id} - w_2^{-1})\chi_3, \lambda + \rho >= n_1 + n_2 + n_3 + 3 \quad < (\mathrm{Id} - v_2^{-1})\chi_3, \lambda + \rho >= n_2 + n_3 + 2$$

Hence we expect:

We choose triple d_1, d_3, a and a pair of eigenforms f, g with weight $d_1 + 2 = k_1, d_3 + 2 = k_3$. Let λ solve the appropriate equations $(case1), (case2)$. If a prime $\mathfrak{l}$ divides the **denominator** *of*

$$\Omega(\tau_f \times \Pi_f)^{\epsilon(a)} \frac{\Lambda^{coh}(\tau \times \Pi, a)}{\Lambda^{coh}(\tau \times \Pi, a+1)}$$

then we find an isotopical subspace $H^6_!(S^G, \mathcal{M}_\lambda)(\tilde{\Pi}_f)$ and a congruence

$$T^G_{\chi_3}(\tilde{\Pi}_p) \equiv a_p(g)(p^{n_3+1} + a_p(f) + p^{n_2+n_3+2}) \mod \mathfrak{l}$$

in $(case1)$ and

$$T^G_{\chi_3}(\tilde{\Pi}_p) \equiv a_p(g)(p^{n_2+n_3+2} + a_p(f) + p^{n_1+n_2+n_3+3}) \mod \mathfrak{l}$$

in $(case2)$

We compare to TABLE 1. in [BFG]: We have

$$(k_1, k_3) = (m_2, m_1)$$

and

$$r_1 = n_2 + n_3 + 2, r_2 = n_3 + 1 \text{ in } (case1),$$
$$r_1 = n_1 + n_2 + n_3 + 3, r_2 = n_2 + n_3 + 2 \text{ in } (case2).$$

Recall that we are interested in the special value $a+1$, we can say in (case1)

$$a + 1 = \frac{n_2 + 1}{2} + \frac{\mathbf{w}}{2} + 1 = \frac{r_1 - r_2 + \mathbf{w}}{2} + 1$$

and in $(case2)$

$$a + 1 = \frac{n_2 + 1}{2} + \frac{\mathbf{w}}{2} + 1 = \frac{r_1 - r_2 + \mathbf{w}}{2} + 1$$

Now I checked against TABLE1 in [BFG] and Anton's tables and the data match perfectly. We even see some "small" primes providing congruences. We see a 17^2 occuring in the case f of weight 12 and g of weight 18. We observe that both forms are ordinary at 17.

Remark: In our special case the expression for $T^{G,\mathrm{coh}}_{\chi_i}(\mathrm{Ind}(\sigma_p))$ is a sum of three terms, the term in the middle $a_p(g)a_p(f)$ has weight $\frac{d_1+d_2}{2}+1$ the first term has a lower weight the third term has a higher weight. The difference of the weights of the first and third term is up to a shift our evaluation point a. This means: The closer these two weights get, the closer a comes to the center of the L function.

We go to $g=4$. In this case our group $M=\mathrm{Gl}_4\cdot\mathbb{G}_m$. We choose a highest weight $\lambda=n_1\gamma_1+n_2\gamma_2+n_1\gamma_3+n_4\gamma_4$, the central character is trivial. It seems that the interesting Kostant representatives are

$$w'=s_4s_3s_2s_4 \text{ and } w=s_4s_3s_2s_4s_3s_1 \tag{54}$$

We get

$$\begin{aligned} w'(\lambda+\rho^G_B) &= (2+n_1+n_2)(\gamma_1^M+\gamma_3^M)+(3+n_1+2n_4)\gamma_2^M+\tfrac{1}{2}(1+n_1)\gamma_4,\\ w(\lambda+\rho^G_B) &= (2+n_1+n_2)(\gamma_1^M+\gamma_3^M)+(3+n_1+2n_4)\gamma_2^M+\tfrac{1}{2}(-1-n_1)\gamma_4 \end{aligned} \tag{55}$$

We see that $\mathcal{M}_\lambda(w\cdot\lambda)$ is self dual, this is the reason why we have chosen $n_1=n_3$. As usual we define numbers $d_1=d_3,d_2$ by

$$d_1+1=d_3+1=2+n_1+n_2, d_2+1=3+n_1+2n_4 \tag{56}$$

The dimension of $\mathcal{S}^G_{K_f}$ is 20, we look at our fundamental exact sequence

$$\begin{array}{cccc} H^5(\mathcal{S}^M_{K^M_f},\mathcal{M}_\lambda(w'\cdot\lambda)) & \xrightarrow{\delta} H^{10}_c(\mathcal{S}^G_{K_f},\mathcal{M}_\lambda) & \to H^{10}(\mathcal{S}^G_{K_f},\mathcal{M}_\lambda) & \to H^4(\mathcal{S}^M_{K^M_f},\mathcal{M}_\lambda(w\cdot\lambda)) \\ \uparrow & & & \uparrow \\ H^5_!(\mathcal{S}^M_{K^M_f},\mathcal{M}_\lambda(w'\cdot\lambda))(\sigma'_f) & & & H^4_!(\mathcal{S}^M_{K^M_f},\mathcal{M}_\lambda(w\cdot\lambda))(\sigma_f) \end{array} \tag{57}$$

This is the constellation where we can hope for extensions of mixed Tate motives. The difference of the weights of w' and w is two, which seems to be too big. But the cohomology of $\mathcal{S}^M_{K^M_f}$ is concentrated in degree 4 and 5, so we get boundary cohomology in degree 9 and 10.

We have to compute the second constant term. To do this we have to study the representation of the group ${}^LM$ on the Lie-algebra $\mathfrak{u}^\vee_P$. The Dynkin diagram for the Langlands dual group ${}^LG$ is

$$\alpha_1^\vee \quad - \quad \alpha_2^\vee \quad - \quad \alpha_3^\vee \quad >= \quad \alpha_4^\vee,$$

and we get ${}^LM$ if we erase $\alpha_4^\vee$. The representation of ${}^LM$ on $\mathfrak{u}^\vee_P$ decomposes into two irreducible representations, the first one has highest weight

$$\eta_1=\alpha_1^\vee+\alpha_2^\vee+\alpha_3^\vee+\alpha_4^\vee$$

and is up to a twist the tautological representation. The second one has highest weight

$$\eta_2 = \alpha_1^\vee + 2\alpha_2^\vee + 2\alpha_3^\vee + 2\alpha_4^\vee$$

and is (again up to a twist) the Λ^2 of the tautological representation. It is of dimension 6. We recall formula (100) in chap3.pdf. The number a in this formula takes the values 1,2 and we get

$$\begin{array}{c} \frac{\mathbf{w}_1}{2} =< \eta_1, \tilde{\mu}^{(1)} >= \frac{7}{2} + \frac{3}{2}n_1 + n_2 + n_4 = d_1 + \frac{1}{2}d_2 + \frac{3}{2} \\ \frac{\mathbf{w}_2}{2} =< \eta_2, \tilde{\mu}^{(1)} >= 5 + 2n_1 + n_2 + 2n_4 = d_1 + d_2 + 3 \end{array} \tag{58}$$

This implies that the second constant term is

$$\begin{array}{c} \frac{1}{\Omega(\sigma_f)} \frac{\Lambda^{\mathrm{coh}}(\sigma_f, r_{\eta_1}, 4+2n_1+n_2+n_4)}{\Lambda^{\mathrm{coh}}(\sigma_f, r_{\eta_1}, 5+2n_1+n_2+n_4)} \frac{\Lambda^{\mathrm{coh}}(\sigma_f, r_{\eta_2}, 6+3n_1+n_2+2n_4)}{\Lambda^{\mathrm{coh}}(\sigma_f, r_{\eta_2}, 7+3n_1+n_2+2n_4)} \\ = \\ \frac{1}{\Omega(\sigma_f)} \frac{\Lambda^{\mathrm{coh}}(\sigma_f, r_{\eta_1}, \frac{\mathbf{w}_1}{2}+\frac{1}{2}(1+n_1))}{\Lambda^{\mathrm{coh}}(\sigma_f, r_{\eta_1}, \frac{\mathbf{w}_1}{2}+\frac{1}{2}(1+n_1)+1)} \frac{\Lambda^{\mathrm{coh}}(\sigma_f, r_{\eta_2}, \frac{\mathbf{w}_2}{2}+1+n_1)}{\Lambda^{\mathrm{coh}}(\sigma_f, r_{\eta_2}, \frac{\mathbf{w}_2}{2}+2+n_1)} \end{array} \tag{59}$$

Since assume that we are in the unramified case the two isotypical subspaces σ'_f resp. σ_f in (57) provide Tate motives $\mathbb{Z}(-\frac{\mathbf{w}_1}{2} + \frac{1}{2}(n_1+1))$ resp. $\mathbb{Z}(-\frac{\mathbf{w}_1}{2} - \frac{1}{2}(n_1+1))$. Hence our usual construction of Anderson motives will provide elements

$$\mathcal{X}(\sigma_f) \in \mathrm{Ext}^1_{\mathcal{MM}}(\mathbb{Z}(-1-n_1), \mathbb{Z}) \tag{60}$$

Since we want non torsion classes, we assume n_1 even. This implies that d_2 must be even and if we give ourselves $d_1 \geq 1, d_2 \geq 2$ and even, then we see that for our given d_1, d_2 and given Kostant representative $w = s_4s_3s_2s_4s_3s_1$ we find a dominant $\lambda = n_1\gamma_1 + n_2\gamma_2 + n_1\gamma_3 + n_4\gamma_4$ with

$$w(\lambda+\rho) = (d_1+1)(\gamma_1^M + \gamma_3^M) + (d_2+1)\gamma_2^M - \frac{1}{2}(1+n_1)\gamma_4$$

if and only if $n_1 \in [0, \min(d_1-1, d_2-2)]$ and even.

Again we consult [Ha-Rag] and find that the miracle happens again: The numbers $\frac{\mathbf{w}_1}{2} + \frac{1}{2}(1+n_1)+1$ run through the critical arguments, for $n_1 = 0$ the number $\frac{\mathbf{w}_1}{2} + \frac{1}{2}$ is the smallest critical argument to the right from the central argument for the functional equation $(= \frac{\mathbf{w}_1}{2})$.

Hence we know that the factor in front

$$\frac{1}{\Omega(\sigma_f)} \frac{\Lambda^{\mathrm{coh}}(\sigma_f, r_{\eta_1}, \frac{\mathbf{w}_1}{2} + \frac{1}{2}(1+n_1))}{\Lambda^{\mathrm{coh}}(\sigma_f, r_{\eta_1}, \frac{\mathbf{w}_1}{2} + \frac{1}{2}(1+n_1)+1)} \tag{61}$$

is an algebraic number in F. The period $\Omega(\sigma_f)$ is locally well defined up to a unit and hence we can speak of the prime decomposition of this algebraic number. Hence we may apply the principles outlined in 2.1. and ask whether "large" primes $\mathfrak{l}$ which divide the denominator of the expression in (61) create eigenclasses in $H^{10}(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda)$ whose eigenvalues are congruent to the eigenvalues of σ_f modulo $\mathfrak{l}$.

In our heuristic reasoning we encounter new difficulties, before we discuss these problems I want to give the precise form of these congruences in the sense of (**??**,(Hecke-Ind)). The following considerations hold for arbitrary even value of g.

We choose the cocharacter $\chi_g : t \to T$ which satisfies $< \chi, \alpha_i >= 0$ for $i < g$ and $< \chi_g, \alpha_g >= 1$. (This means the first g entries on the diagonal are equal to t the other entries are equal to 1.) The stabilizer of this character in the Weyl group is $S_4 = W_M$ the Weyl group of M. Then we can represent the cosets W/W_M by the 2^g elements which exchange some of the $e_i \to f_i, f_i \to -e_i$ and leave the others fixed. So we can say that W/W_M is equal to the set of subsets of $\{1, 2, \ldots, g\}$. The Weyl group W_M also acts from the left on this coset space and acts transitively transitively on the set of subsets of a fixed cardinality h. Therefore the number of orbits is $g + 1$.

We go back to the case $g = 4$, our cocharacter is χ_4 and our parabolic subgroup is the Siegel parabolic subgroup, i.e. $i_0 = 4$. Let w_P be the longest Kostant representative. We have the choices $v_k = e, v_k = s_4, v_k = s_4 s_3 s_2 s_4 s_3 s_1, w_P s_4, w_P$, they are Kostant representatives and hence they satisfy (47). We choose $\tilde{w} = s_4 s_3 s_2 s_4 s_3 s_1$.

We investigate the expressions

$$< \chi_4, \lambda + \rho_B^G - v_k^{-1}\Big(\tilde{w}(\lambda + \rho_B^G) - 2b(\tilde{w}, \lambda)\gamma_4\Big) > \tag{62}$$

these will give us the exponents in the powers of p which enter in the sum. To do this we have to write

$$\lambda + \rho_B^G - v_k^{-1}\Big(\tilde{w}(\lambda + \rho_B^G) - 2b(\tilde{w}, \lambda)\gamma_4\Big) = \sum m_j \alpha_j \tag{63}$$

and then

$$< \chi_4, \lambda + \rho_B^G - v_k^{-1}\Big(\tilde{w}(\lambda + \rho_B^G) - 2b(\tilde{w}, \lambda)\gamma_4\Big) >= m_4 \tag{64}$$

Perhaps it it even simpler to rewrite this in the terms of the μ-s. We observe that $< \chi_4, \tilde{\mu}^{(1)}_{e,\lambda} >= 0$ and hence

$$m_4 =< \chi_4, b(e, \lambda)\gamma_4 - v_k^{-1}\Big(\tilde{\mu}^{(1)}_{\tilde{w},\lambda} - b(\tilde{w}, \lambda)\gamma_4\Big) >= 2b(e, \lambda) - < \chi_4, v_k^{-1}\Big(\tilde{\mu}^{(1)}_{\tilde{w},\lambda} - b(\tilde{w}, \lambda)\gamma_4\Big) > \tag{65}$$

If we choose $v_k = e$ or $v_k = w_P$ then $v_k^{-1}\tilde{\mu}^{(1)}_{\tilde{w},\lambda} = \tilde{\mu}^{(1)}_{\tilde{w},\lambda}$ and hence $< \chi_4, v_k^{-1}\tilde{\mu}^{(1)}_{\tilde{w},\lambda} >= 0$, so

$$2b(e, \lambda) - < \chi_4, v_k^{-1}\Big(\tilde{\mu}^{(1)}_{\tilde{w},\lambda} - b(\tilde{w}, \lambda)\gamma_4\Big) >= 2b(e, \lambda) \pm b(\tilde{w}, \lambda) \tag{66}$$

These numbers are easy to compute and equal to

$$5 + 2n_1 + n_2 + 2n_4 \pm (1 + n_1) \tag{67}$$

Now we consider the two choices $v_k = s_4, w_P s_4$. In this case we get for the exponents

$$\frac{3}{2} + \frac{n_1}{2} + n_4 \pm \frac{1}{2}(1 + n_1) \tag{68}$$

or more precisely we get

$$m_4 =< \chi_4, b(e,\lambda)\gamma_4 - v_k^{-1}\Big(\tilde{\mu}^{(1)}_{\tilde{w},\lambda} - b(\tilde{w},\lambda)\gamma_4\Big) >= \begin{cases} 1+n_4 & \text{if } v_k = s_4 \\ 2+n_1+n_4 & \text{if } v_k = w_P s_4 \end{cases} \tag{69}$$

Finally we choose $v_k = s_4s_3s_2s_4s_3s_1 = \tilde{w}$. In this case the two μ contributions cancel and one also checks easily that $< \chi_4, v_k^{-1}\gamma_4 >= 0$. Hence the exponent is zero. Since χ_4 is miniscule we get $\delta(\chi_4) = 0$. We conclude that formula (53) yields

The cocharacters $\chi_4, s_4\chi_4, \tilde{w}\chi_4, w_P s_4, w_p$ define conjugacy classes of cocharacters for the group $M = \mathrm{Gl}_4 \cdot \mathbb{G}_m$. Since we assumed for simplicity that the central character of $\mathcal{M}_\lambda$ is trivial we can divide by the factor $\mathbb{G}_m$ and consider these cocharacters as homomorphisms from $\mathbb{G}_m$ to the standard maximal torus of $M = \mathrm{Gl}_4$. The conjugacy classes of of these five cocharacters are $\chi_4, \chi_3, \chi_2, \chi_1$ in the notation of chap3.pdf 3.1.4. and χ_0 which is the trivial character. We observe that $T^{M,\mathrm{coh}}_{\chi_4} = T^{M,\mathrm{coh}}_{\chi_0} = 1$ and therefore we get

$$\begin{array}{rc} T^{G,\mathrm{coh}}_{\chi_4}(\mathrm{Ind}(\sigma_p)) = & (p^{4+n_1+n_2+2n_4} + p^{6+3n_1+n_2+2n_4}) \\ + & p^{1+n_4}T^{M,\mathrm{coh}}_{\chi_3}(\sigma_p) + p^{2+n_1+n_4}T^{M,\mathrm{coh}}_{\chi_1}(\sigma_p) \\ + & T^{M,\mathrm{coh}}_{\chi_2}(\sigma_p) \end{array} \tag{70}$$

Therefore we can express the eigenvalues of the above Hecke operators at a prime p in terms of the Satake parameter $\omega_p = \{\omega_{1,p}, \omega_{2,p}, \omega_{3,p}, \omega_{4,p}\}$ of π_p. We get

$$T^{M,\mathrm{coh}}_{\chi_\nu}(\sigma_p) = \sum_{I:\#I=\nu} p^{<\chi_\nu, \tilde{\mu}^{(1)}_{\tilde{w},\lambda}>}\omega^*_{I,p} \tag{71}$$

This has now the same shape as the expressions which we have seen before. The numbers $T^{M;\mathrm{coh}}_{\tilde{w}\chi_i}(\sigma_f)$ are algebraic integers. We have exactly one term which does not have a strictly positive power of p in front of it. Therefore we may ask whether for a "large" prime $\mathfrak{l} \subset \mathcal{O}_F$, which divides the denominator of

$$\frac{1}{\Omega(\sigma_f)}\frac{\Lambda^{\mathrm{coh}}(\sigma_f, r_{\eta_1}, 4+2n_1+n_2+n_4)}{\Lambda^{\mathrm{coh}}(\sigma_f, r_{\eta_1}, 5+2n_1+n_2+n_4)}$$

"creates" an isomorphism class class Π_f with $H^{10}(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda)(\Pi_f)) \neq 0$ such that

$$T^{G,\mathrm{coh}}_{\chi_4}(\Pi_p) \equiv T^{G,\mathrm{coh}}(\mathrm{Ind}(\sigma_p)) \mod \mathfrak{l} \text{ for all primes } p \tag{72}$$

This is in perfect analogy to the cases $g = 2, 3$ where the congruences have been verified experimentally.

But we may have a problem. We still have the "motivic" factor

$$\frac{\Lambda^{\mathrm{coh}}(\sigma_f, r_{\eta_2}, 6+3n_1+n_2+2n_4)}{\Lambda^{\mathrm{coh}}(\sigma_f, r_{\eta_2}, 7+3n_1+n_2+2n_4)}. \tag{73}$$

In our previous cases this was a ratio

$$\frac{\zeta(\dots)}{\zeta(\dots+1)} \tag{74}$$

and this had an interpretation as an extension class in the Betti-de-Rham realization.

We now **assume** that the analogous computations to the computation in [Ha1], 4.2. and SecOPs.pdf work and especially that the secondary operator is non zero and is given by a "simple" rational number. Then

$$\mathcal{X}_{B-de-Rham}(\sigma_f) \in \mathrm{Ext}^1_{B-dRh}(\mathbb{Z}(-1-n_1), \mathbb{Z}(0)) = i\mathbb{R} \tag{75}$$

is essentially equal to this motivic factor

$$\frac{\Lambda^{\mathrm{coh}}(\sigma_f, r_{\eta_2}, 6+3n_1+n_2+2n_4)}{\Lambda^{\mathrm{coh}}(\sigma_f, r_{\eta_2}, 7+3n_1+n_2+2n_4)}. \tag{76}$$

This may challenge our belief that there are no exotic Tate motives, because otherwise we must have a relation

$$\frac{\Lambda^{\mathrm{coh}}(\sigma_f, r_{\eta_2}, 6+3n_1+n_2+2n_4)}{\Lambda^{\mathrm{coh}}(\sigma_f, r_{\eta_2}, 7+3n_1+n_2+2n_4)} \sim \zeta'(-n_1) \tag{77}$$

where $\sim$ means equal up to an algebraic number. This is hard to believe!

Of course some computations have to be checked. Especially we have to check whether, in analogy with the case $g = 2$, the secondary operator on the cohomology of the relevant Harish-Chandra modules is non zero and has a "reasonable" value. If this is so, then we can say:

If (77) is not true then we can construct a mixed Tate motive $\mathcal{X}(\sigma_f)$ whose extension class in $\mathrm{Ext}^1_{B-deRh}(\mathbb{Z}(-1-n_1), \mathbb{Z}(0))$ is not in the rational line through $\zeta'(-n_1)/i\pi$

This does not destroy our hope for congruences. We may ask for the image of

$$\mathrm{Ext}^1_{\mathcal{MM}}(\mathbb{Z}(-1-n_1), \mathbb{Z}(0)) \xrightarrow{\mathrm{BdR}} \mathrm{Ext}^1_{B-dRh}(\mathbb{Z}(-1-n_1), \mathbb{Z}(0)).$$

If our construction works then it seems to be plausible that this image may generate even an infinite dimensional $\mathbb{Q}$-vector space. But perhaps there is some reason that its image is not infinitely divisible. Assuming this we can ask the question about congruences formulated above.

In principle we can check these questions experimentally. For the congruences Bergström and friends should extend their computations to $g = 4$. More serious is the question whether (77) is true. If we find an algebraic number β such that

$$\frac{\Lambda^{\mathrm{coh}}(\sigma_f, r_{\eta_2}, 6+3n_1+n_2+2n_4)}{\Lambda^{\mathrm{coh}}(\sigma_f, r_{\eta_2}, 7+3n_1+n_2+2n_4)} = \beta\zeta'(-n_1)$$

up to a very high order of precision (high with respect to the height of β), then this does not prove that (77) is true, but it makes us almost sure. If we do not find such a number then it is very likely that (77) is false.

2.1.7 Non regular coefficients again

So far we always assumed that $n_1 > 0$. We expect that in this case the Eisenstein intertwining operator is holomorphic at $s = 0$. The second term in the constant term of the Eisenstein series is a product of two terms and the second factor is a ratio of Λ^2 -L values

$$\frac{\Lambda^{\mathrm{coh}}(\sigma_f, r_{\eta_2}, 5 + 2n_1 + n_2 + 2n_4 + n_1 + 1 + s)}{\Lambda^{\mathrm{coh}}(\sigma_f, r_{\eta_2}, 5 + 2n_1 + n_2 + 2n_4 + n_1 + 2 + s)} \tag{78}$$

where

$$\Lambda^{\mathrm{coh}}(\sigma_f, r_{\eta_2}, s) = \prod_p \prod_{I:\#I=2} \frac{1}{1 - p^{\frac{\mathbf{w}_2}{2}} \omega^*_{I,p} p^{-s}} = \prod_p \prod_{I:\#I=2} \frac{1}{1 - \tilde{\omega}_{I,p} p^{-s}} \tag{79}$$

and $\frac{\mathbf{w}_2}{2} =< \eta_2, \tilde{\mu}^{(1)}_{\tilde{w},\lambda} = 5 + 2n_1 + n_2 + 2n_4$.

Now the situation becomes very unclear. We have to evaluate at $s = \frac{\mathbf{w}_2}{2} + 1 + n_1$. Our estimates for the $\tilde{\omega}_{I,p}$ imply that $\Lambda^{\mathrm{coh}}(\sigma_f, r_{\eta_2}, s)$ is holomorphic at this argument if $n_1 > 0$. But if $n_1 = 0$ then we may have a first order pole. Actually we do not know whether such a pole is a first order pole, we only know from Langlands see **??** that the expression in (59) has at most a first order pole. But let us assume that we have a first order pole. This pole cancels if

$$\Lambda^{\mathrm{coh}}(\sigma_f, r_{\eta_1}, 4 + n_2 + n_4) = 0 \tag{80}$$

This vanishing may be a rare event, but it can happen that for our given σ_f the L-function in the first factor satisfies the functional equation

$$\Lambda^{\mathrm{coh}}(\sigma_f, r_{\eta_1}, 8 + 2n_2 + 2n_4 - s) = -\Lambda^{\mathrm{coh}}(\sigma_f, r_{\eta_1}, s) \tag{81}$$

and then (80) is forced.

The situation is now analogous to the situation in section 2.1.2 and we may ask whether the minus sign in the functional equation implies that we have a submodule $\mathcal{SK}(\sigma_f) \subset H^{10}_!(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda)$ which is a direct sum of copies of $J(\sigma_f)$ and which provides a motive which is isomorphic to $\mathbb{M}(\sigma_f, r_{\eta_2})$.

More precisely we can define $\mathcal{SK}^\bullet(\sigma_f)$ as the image of the tautological map

$$\mathrm{Hom}_{\mathcal{H}^G_{K_f}}(J(\sigma_f), H^\bullet_!(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda)) \otimes J(\sigma_f) \xrightarrow{\mathrm{taut}} H^\bullet_!(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda) \tag{82}$$

For any prime ideal $\mathfrak{l}$ we have an action of the Galois action on

$$H^\bullet_!(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda \otimes F_\mathfrak{l})$$

which commutes with action of the Hecke algebra. (see the remark at the beginning of section 2.1 this induces an action on $\mathcal{SK}^\bullet(\sigma_f) \otimes F_\mathfrak{l}$ and therefore we get an action of the Galois group on

$$W_\mathfrak{l}(\sigma_f) = \mathrm{Hom}_{\mathcal{H}^G_{K_f}}(J(\sigma_f), H^\bullet_!(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda) \otimes F_\mathfrak{l})$$

such that the tautological map becomes an isomorphism of Galois$\times$ Hecke modules.

We still have the congruence relations and they tell us that for all primes p the eigenvalues of the Frobenius Φ_p^{-1} on $W_\mathfrak{l}(\sigma_f)$ have to be taken from the list

$$\mathcal{L}_p(\sigma_f) = \{p^{a(\nu)} p^{<\chi_\nu, \tilde{\mu}^{(1)}_{\tilde{w},\lambda}>} \omega^*_{I,p}\} \tag{83}$$

of summands occurring in the formulas (70) and (71).

This implies that we can have an non trivial $\mathcal{SK}^q(\sigma_f) \subset H^q_!(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda) \otimes F_\mathfrak{l})$ only if we have some summands in our list $\mathcal{L}_p(\sigma_f)$ which satisfy

$$|p^{a(\nu)} p^{<\chi_\nu, \tilde{\mu}^{(1)}_{\tilde{w},\lambda}>} \omega^*_{I,p}| = p^{\frac{q}{2} + + n_2 + 2n_4} = p^{<\chi_2, \tilde{\mu}^{(1)}_{\tilde{w},\lambda}>} \tag{84}$$

In analogy to what we have seen earlier we should choose $q = 10$. If $\mathcal{SK}^q(\sigma_f) \neq 0$ then we get that the members $p^{<\chi_2, \tilde{\mu}^{(1)}_{\tilde{w},\lambda}>} \omega^*_{I,p} \in \mathcal{L}_p(\sigma_f)$, which are also eigenvalues for Φ_p^{-1} must have absolute value $p^{<\chi_2, \tilde{\mu}^{(1)}_{\tilde{w},\lambda}>}$ and hence we get $|\omega^*_{I,p}| = 1$ for those values of I. If we assume that actually all $\omega^*_{I,p} \in \mathcal{L}_p(\sigma_f)$ with $\#I = 2$ occur as Frobenius eigenvalue, then we get the Ramanujan conjecture. Actually it seems to be plausible that each eigenvalue in the sublist where $\#I = 2$ occurs with multiplicity one. then we get that $\dim(W_\mathfrak{l}(\sigma_f)) = 6$. In this case we have found the motive $\mathbb{M}(\sigma_f, r_2)$ in side the cohomology $H^q_!(\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda) \otimes F)$.

If $\Lambda^{\text{coh}}(\sigma_f, r_1, 4 + n_2 + n_4 + s)$ has a first order zero at $s = 0$ then we can construct Anderson mixed motives (as in section 2.1.2), i.e. extensions

$$\begin{aligned} \mathcal{Y}(\sigma_f) &\in \mathrm{Ext}^1_{\mathcal{MM}}(\mathbb{Z}(-k), \mathcal{SK}(\sigma_f)) \\ \mathcal{Y}'(\sigma'_f) &\in \mathrm{Ext}^1_{\mathcal{MM}}(\mathcal{SK}(\sigma_f), \mathbb{Z}(-k+1)) \end{aligned} \tag{85}$$

where now $k = 5 + n_2 + n_4$, and these two extension come with a canonical "integral" to a biextension $(\widetilde{\mathcal{Y}'(\sigma'_f), \mathcal{Y}}(\sigma_f))$ and a computation like the one in SecOPs.pdf should yield

$$i[\widetilde{\mathcal{Y}'(\sigma'_f), \mathcal{Y}}(\sigma_f)] \sim \frac{\Lambda^{\text{coh},\prime}(\sigma_f, r_1, 4 + n_2 + n_4)}{\Omega(\sigma_f)^\epsilon \Lambda^{\text{coh}}(\sigma_f, r_1, 5 + n_2 + n_4)} \mathrm{Res}_{s=0} \frac{\Lambda^{\text{coh}}(\sigma_f, r_{\eta_2}, 6 + n_2 + 2n_4 + s)}{\Lambda^{\text{coh}}(\sigma_f, r_{\eta_2}, 7 + n_2 + 2n_4 + s)} \tag{86}$$

This formula gives us a strong hint that we always should have $\mathcal{SK}^{10}(\sigma_f) \neq 0$, because otherwise

$$i[\widetilde{\mathcal{Y}'(\sigma'_f), \mathcal{Y}}(\sigma_f)] \in \mathrm{Ext}^1_{B-dRh}(\mathbb{Z}(-k), \mathbb{Z}(-k+1))$$

and this last group is hypothetically $\log(\mathbb{Q}^\times_{>0})$ and this is again hard to believe.

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