CHAPTER II

Cohomology of arithmetic groups

Contents

Abstract

We explain some basic facts in the cohomology theory of arithmetic groups. For this need some concepts and results from homological algebra and from the cohomology theory of sheaves, for this I refer to the first four chapters of the [book].This part of the book can be considered as chapter I of this volume.

We also need some concepts and results from the theory of linear algebraic groups. I will explain these facts in terms of various examples and I hope that this discussion of examples will generate enough familiarity with these ideas. For the details I refer to the literature for instance the book of A. Borel or J. Humphreys.

1 Affine algebraic groups over $\mathbb{Q}$.

A linear algebraic group $G/\mathbb{Q}$ is a subgroup $G \subset GL_n$, which is defined as the set of common zeroes of a set of polynomials in the matrix coefficients where in addition these polynomials have coefficients in $\mathbb{Q}$. Of course we cannot take just any set of polynomials the set has to be somewhat special before its common zeroes form a group. The following examples will clarify what I mean:

1.) The group GL_n is an algebraic group itself, the set of equations is empty. It has the subgroup $\mathrm{Sl}_n \subset GL_n$, which is defined by the polynomial equation

$$\mathrm{Sl}_n = \{x \in GL_n \mid \det(x) = 1\}$$

2.) Another example is given by the orthogonal group of a quadratic form

$$f(x_1, \dots, x_n) = \sum_{i=}^{n} a_i x_i^2$$

where $a_i \in \mathbb{Q}$ and all $a_i \neq 0$ (this is actually not necessary for the following). We look at all matrices

$$\alpha = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

which leave this form invariant, i.e.

$$f(\alpha \underline{x}) = f(\underline{x})$$

for all vectors $\underline{x} = (x_1, \dots, x_n)$. This defines a set of polynomial equations for the coefficient a_{ij} of α.

3.) Instead of taking a quadratic form — which is the same as taking a symmetric bilinear form — we could take an alternating bilinear form

$$\begin{aligned} \langle \underline{x}, \underline{y} \rangle =& \langle x_1, \dots, x_{2n}, y_1, \dots y_{2n} \rangle = \\ & \sum_{i=1}^{n} (x_1 y_{i+n} - x_{i+n} y_i \rangle = f \langle \underline{x}, \underline{y} \rangle . \end{aligned}$$

This form defines the symplectic group:

$$Sp_n = \left\{ \alpha \in GL_{2n} \mid \langle \alpha \underline{x}, \alpha \underline{y} \rangle = \langle \underline{x}, \underline{y} \rangle \right\} .$$

1.1. Important remark: The reader may have observed that I did not specify a field (or a ring) from which I take the entries of the matrices. This is done intentionally, because we may take the entries from any ring R containing the rational numbers $\mathbb{Q}$. In other words: for any algebraic group $G/\mathbb{Q} \subset GL_n$ and any ring R containing $\mathbb{Q}$ we may define

$$G(R) \subset GL_n(R)$$

as the group of those matrices whose coefficients satisfy the required polynomial equations.

Adopting this point of view we can say that

A linear algebraic group $G/\mathbb{Q}$ defines a functor from the category of $\mathbb{Q}$-algebras (i.e. commutative rings containing $\mathbb{Q}$) into the category of groups.

4.) Another example is obtained by the so-called restriction of scalars. Let us assume we have a finite extension $K/\mathbb{Q}$, we consider the vector space $V = K^n$. This vector space may also be considered as a $\mathbb{Q}$-vector space V_0 of dimension $n[K:\mathbb{Q}] = N$. We consider the group

$$GL_N/\mathbb{Q}.$$

Our original structure as a K-vector space may be considered as a map

$$\Theta : K \longrightarrow \mathrm{End}_{\mathbb{Q}}(V_0),$$

and the group $GL_n(K)$ is then the subgroup of elements in $GL_N(\mathbb{Q})$ which commute with all the elements of $\Theta(x), x \in K$. Hence we define the subgroup

$$G/\mathbb{Q} = R_{K/\mathbb{Q}}(GL_n) = \{\alpha \in GL_N \mid \alpha \text{ commutes with } \Theta(K)\}$$

and $G(\mathbb{Q}) = GL_n(K)$. For any $\mathbb{Q}$-algebra R we get

$$G(R) = GL_n(K \otimes_{\mathbb{Q}} R).$$

This can also be applied to an algebraic subgroup $H/K \hookrightarrow GL_n/K$, i.e. a subgroup that is defined by polynomial equations with coefficients in K.

Our definition of a linear algebraic group is a little bit provisorial. If we consider for instance the two linear algebraic groups

$$\begin{aligned} G_1/\mathbb{Q} &= \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right\} \subset \mathrm{Gl}_2 \\ G_2/\mathbb{Q} &= \left\{ \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \subset GL_3 \end{aligned}$$

then we would like to say, that these two groups are isomorphic. They are two different "realizations" of the additive group $G_a/\mathbb{Q}$. We see that the same linear algebraic group may be realized in different ways as a subgroup of different GL_N's.

Of course there is a concept of linear algebraic group which does not rely on embeddings. The understanding of this concept requires a little bit of affine algebraic geometry. The drawback of our definiton here is that we cannot define morphism between linear algebraic group. Especially we do not know when they are isomorphic. I assert the reader that the general theory implies that a morphism between two algebraic groups is the same thing as a morphism between the two functors form $\mathbb{Q}$-algebras to groups. In some sense it is enough to give this functor. For instance, we have the multiplicative group $\mathbb{G}_m/\mathbb{Q}$ given by the functor

$$R \longrightarrow R^\times$$

and the additive group $G_a/\mathbb{Q}$ given by $R \to R^+$.

We can realize (represent is the right term) the the group $\mathbb{G}_m/\mathbb{Q}$ as

$$\mathbb{G}_m/\mathbb{Q} = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right\} \subset \mathrm{Gl}_2$$

1.1 Affine groups schemes

We say just a few words concerning the systematic development of the theory of linear algebraic groups.

For any field k an affine k-algebra is a finitely generated algebra A/k, i.e. it is a commutative ring with identity, containing k, the identity of k is equal to the identity of A, which is finitely generated over k as an algebra. In other words

$$A = k[x_1, x_2, \ldots, x_n] = k[X_1, X_2, \ldots, X_n]/I,$$

where they X_i are independent and where I is a finitely generated ideal of polynomials in $k[X_1, \ldots, X_n]$.

Such an affine k-algebra defines a functor from the category of k algebras to the category of sets

$$B \mapsto \operatorname{Hom}_k(A, B).$$

A structure of a group scheme on A/k consists of the following data:

a) A k homomorphism $m : A \to A \otimes_k A$ (the comultiplication)

b) A k-valued point $e : A \to k$ (the identity element)

c) An inverse $inv : A \to A$,

which satisfy certain requirements:

We have $\operatorname{Hom}_k(A \otimes_k A, B) = \operatorname{Hom}_k(A, B) \times \operatorname{Hom}_k(A, B)$ and hence m defines a map ${}^t m : \operatorname{Hom}_k(A, B) \times \operatorname{Hom}_k(A, B) \to \operatorname{Hom}_k(A, B)$.

The requirement is that for all B this composition map ${}^t m$ defines a group structure on $\operatorname{Hom}_k(A, B)$. The k valued point e is the identity and inv yields the inverse.

I leave it to the audience to figure out what this means for m, e, inv. An affine k together with such a collection m, e, inv is called an affine group scheme.

Now it is clear what a homomorphism between affine group schemes is.

It is a not entirely obvious theorem that for any affine group scheme $G/k = (A/k, m, e, inv)$ we can find a faithful representation $i : G/k \hookrightarrow \mathrm{Gl}(V)$.

We may also consider linear algebraic group over other fields K. This means that we only require the coefficients of the defining polynomials to be in this other field. We write G/K for a group defined over K. Then we have the permission to consider the groups $G(R)$ for any ring containing K.

If we have a field $L \supset K$ and a linear group G/K then the group $G/L = G \times_K L$ is the group over L where we forget that the coefficients of the equations are contained in K. The group $G \times_K L$ is the *base extension* from G/K to L

1.1.1 Tori, their character module,...

A special class of algebraic groups is given by the *tori.* An algebraic group T/K over a field K is called a *split torus* if it is isomorphic to a product of $\mathbb{G}_m$-s. It is called a torus if it becomes a split torus after a suitable finite extension of the ground field, i.e we have $T \times_K L \xrightarrow{\sim} \mathbb{G}_m^r/L$.

If we take an arbitrary finite field extension $L/\mathbb{Q}$ we may consider the functor

$$R \to (L \otimes_{\mathbb{Q}} R)^{\times}.$$

It is not hard to see that this functor can be represented by an algebraic group over $\mathbb{Q}$, which is denoted by $R_{L/\mathbb{Q}}(\mathbb{G}_m/L)$ and called the Weil restriction of $\mathbb{G}_m/L$. We propose the notation

$$R_{L/\mathbb{Q}}(\mathbb{G}_m/L) = \mathbb{G}_m^{L/Q} \tag{1}$$

The reader should try to prove that for a finite extension $\tilde{L}/L$ which is normal over $\mathbb{Q}$ we have

$$\mathbb{G}_m^{L/Q} \times_{\mathbb{Q}} \tilde{L} \xrightarrow{\sim} (\mathbb{G}_m/\tilde{L})^{[L:\mathbb{Q}]}$$

and this shows that $\mathbb{G}_m^{L/Q}$ is a torus .

A torus T/K is called *anisotropic* if is does not contain a non trivial split torus. Any torus C/K contains a maximal split torus S/K and a maximal torus S/K. The multiplication induces a map

$$m : S \times C_1 \to C$$

this is a surjective (in the sense of algebraic groups) homomorphism whose kernel is a finite algebraic group. We call such map an *isogeny* and write that $C = S \cdot C_1$.

We give an example. Our torus $R_{L/\mathbb{Q}}(\mathbb{G}_m/L)$ contains $\mathbb{G}_m/\mathbb{Q}$ as a subtorus: For any ring R containing $\mathbb{Q}$ we have $R^{\times} = \mathbb{G}_m(R) \in (\mathbb{R} \otimes L)^{\times}$. On the other and we have the norm map $N_{L/\mathbb{Q}} : (\mathbb{R} \otimes L)^{\times} \to R^{\times}$ and the kernel defines a subgroup

$$R^{(1)}_{L/\mathbb{Q}}(\mathbb{G}_m/L) \subset R_{L/\mathbb{Q}}(\mathbb{G}_m/L)$$

and it is clear that

$$m : \mathbb{G}_m \times R^{(1)}_{L/\mathbb{Q}}(\mathbb{G}_m/L) \to R_{L/\mathbb{Q}}(\mathbb{G}_m/L)$$

has a finite kernel which is the finite algebraic group of $[L : \mathbb{Q}]$-th roots of unity.

For any torus $T = \mathbb{G}_m^r$ we define the character module as the group of homomorphisms

$$X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m).. \tag{2}$$

If the torus is split, i.e. $T = \mathbb{G}_m^r$ then $X^*(T) = \mathbb{Z}^r$ and the identification is given by $(n_1, n_2, \ldots, n_r) \mapsto \{(x_1, x_2, \ldots, x_r) \mapsto x_1^{n_1} x_2^{n_2} \ldots x_r^{n_r}\}$.

It is a theorem that for any torus we can find a finite, separable, normal extension L/K such that $T \times_K L$ splits. Then it is easy to see that we have an action of the Galois group $\mathrm{Gal}(L/K)$ on $X^*(T \times_K L) = \mathbb{Z}^r$. If we have two tori $T_1/K, T_2/K$ which split over L

$$\mathrm{Hom}_K(T_1, T_2) \xrightarrow{\sim} \mathrm{Hom}_{\,\mathrm{Gal}(L/K)}(X^*(T_2 \times_K L), X^*(T_1 \times_K L)) \tag{3}$$

To any $\mathrm{Gal}(L/K)-$ action on $\mathbb{Z}^n$ we can find a torus T/K which splits over L and which realizes this action.

A homomorphism $\phi : T_1/K \to T_2$ is called an isogeny if $\dim(T_1) = \dim(T_2)$ and if ${}^t\phi : X^*(T_2) \to X^*(T_1)$ is injective.

1.1.2 Semi-simple groups, reductive groups,.

An important class of linear algebraic groups is formed by the *semisimple* and the *reductive* groups. I do not want to give the precise definition here. Roughly, a linear group is reductive if it does not contain a non trivial normal subgroup which is isomorphic to a product of groups of type G_a. A group is called semisimple, if it is reductive and does not contain a non trivial torus in its centre.

For example the groups Sl_n, Sp_n are semi simple. The groups $\mathrm{SO}(f)$ are semi-simple provided $n \geq 3$. The groups Gl_n and especially the multiplicative group $\mathrm{Gl}_1/\mathbb{Q} = \mathbb{G}_m/\mathbb{Q}$ are reductive.

Any reductive group $G/\mathbb{Q}$ (or over any field of characteristic zero) has a central torus $C/\mathbb{Q}$ and this central torus contains a maximal split torus S. The derived $G^{(1)}/\mathbb{Q}$ is semi simple and we get an isogeny

$$G^{(1)} \times C_1 \times S \to G$$

or briefly $G = G^{(1)} \cdot C_1 \cdot S$.

If for instance $G = \mathbb{R}_{L/\mathbb{Q}}(\mathrm{Gl}_n/L)$ then $G^{(1)} = \mathbb{R}_{L/\mathbb{Q}}(\mathrm{Sl}_n/L)$ and $C = R_{L/\mathbb{Q}}(\mathbb{G}_m/L)$ and this yields the product decomposition up to isogeny

$$G = G^{(1)} \cdot R^{(1)}_{L/\mathbb{Q}}(\mathbb{G}_m/L) \cdot \mathbb{G}_m.$$

1.2 k-forms of algebraic groups

Exercise: 1) *Consider the following two quadratic forms over $\mathbb{Q}$:*

$$f(x,y,z) = x^2 + y^2 - z^2\ ,\ f_1(x,y,z) = x^2 + y^2 - 3z^2.$$

Prove that the first form is isotropic. This means there exists a vector $(a,b,c) \in \mathbb{Q}^3 \setminus \{0\}$ with

$$f(a,b,c) = 0.$$

Show that the second form is anisotropic, i.e. it has no such vector.

2) Prove that the two linear algebraic group $G/\mathbb{Q} = SO(f)/\mathbb{Q}$ and $G_1/\mathbb{Q} = SO(f_1)/\mathbb{Q}$ cannot be isomorphic. (Hint: This is not so easy since we did not define when two groups are isomorphic.)

Here is some advice: In general we call an element $e \neq u \in G(\mathbb{Q})$ unipotent if it is unipotent in $GL_n(\mathbb{Q})$ where we consider $G/\mathbb{Q} \hookrightarrow GL_n/\mathbb{Q}$. It turns out that this notion of unipotence does not depend on the embedding.

Now it is possible to show that our first group $G(\mathbb{Q}) = SO(f)(\mathbb{Q})$ has unipotent elements, and $G_1(\mathbb{Q})$ does not. Hence these two groups cannot be isomorphic.

3) Prove that the two algebraic groups $G \times_{\mathbb{Q}} \mathbb{R}$ and $G_1 \times_{\mathbb{Q}} \mathbb{R}$ are isomorphic, and therefore the two groups $G(\mathbb{R})$ and $G_1(\mathbb{R})$ are isomorphic.

In this example we see, that we may have two groups $G/k, G_1/k$ which are not isomorphic but which become isomorphic over some extension L/k. Then we say that the groups are *k-forms* of each other. To determine the different forms of a given group G/k is sometimes difficult one has to use the concepts of Galois cohomology.

For a separable normal extension L/k we have the almost tautological description

$G(k) = \{g \in G(L) | \sigma(g) = g$ for all elements in the Galois group $\mathrm{Gal}(L/k)\}$.

Now let we can consider the functor $\mathrm{Aut}(G)$: It attaches to any field extension L/k the group of automorphisms $\mathrm{Aut}(G)(L)$ of the algebraic group $G \times_k L$. We denote this action by $g \mapsto \sigma(g) = g^\sigma$. Note that this notation gives us the rule $g^{(\sigma\tau)} = (g^\tau)^\sigma$. A 1-cocycle of $\mathrm{Gal}(L/k)$ with values in $\mathrm{Aut}(G)$ is a map $c : \sigma \mapsto c_\sigma \in \mathrm{Aut}(G)(L)$ which satisfies the cocycle rule

$$c_{\sigma\tau} = c_\sigma c_\tau^\sigma$$

Now we define a new action of $\mathrm{Gal}(L/k)$ on $G(L)$: An element σ acts by

$$g \mapsto c_\sigma g^\sigma g_\sigma^{-1}$$

We define a new algebraic group G_1/k: For any extension E/k we have an action of $\mathrm{Gal}(L/k)$ on $E \otimes_k L$ and we put

$$G_1(E) = \{g \in G(E \otimes_k L) | g = c_\sigma g^\sigma g_\sigma^{-1}\}$$

For the trivial cocycle $\sigma \mapsto 1$ this gives us back the original group.

It is plausible and in fact not very difficult to show that $E \to G_1(E)$ is in fact represented by an algebraic group. This group is clearly a k-form of G/k.

We can define an equivalence relation on the set of cocycles, we say that

$$\{\sigma \mapsto c_\sigma\} \simeq \{\sigma \mapsto c'_\sigma\}$$

if and only if we can find a $a \in G(L)$ such that

$$c'_\sigma = a^{-1} c_\sigma a^\sigma \text{ for all } \sigma \in \mathrm{Gal}(L/k)$$

We define $H^1(L/k, \mathrm{Aut}(G))$ as the set of 1-cocycles modulo this equivalence relation. If we have a larger normal separable extension $L' \supset L \supset k$ then we get an inclusion $H^1(L/k, \mathrm{Aut}(G)) \hookrightarrow H^1(L'/k, \mathrm{Aut}(G))$. If $\bar{k}_s$ is a separable closure of k we can form the limit over all finite extensions $k \subset L \subset \bar{k}_s$ and put

$$H^1(\bar{k}_s/k, \mathrm{Aut}(G)) = \lim_{\rightarrow} H^1(L/k,\ \mathrm{Aut}(G))$$

This set is isomorphic to the set of isomorphism classes of k-forms of G/k.

We may apply the same concepts in a slightly different situation. A $k-$ algebra D over the field k is called a central simple algebra, if it has a unit element $\neq 0$, if it is finite dimensional over k, if its centre is k (embedded via the unit element) and if it has no non trivial two sided ideals. It is a classical theorem, that such an algebra over a separably closed field is isomorphic to a full matrix algebra $M_n(k)$. Hence we can say over an arbitrary field k, that the central simple algebra of dimension n^2 are the k-forms of $M_n(k)$.

For any algebraic group G/k we may consider the adjoint group $\mathrm{Ad}(G)$, this is the quotient of G/k by its center. It can be shown, that this is again an algebraic group over k. It is clear that we have an embedding

$$\mathrm{Ad}(G) \to \ \mathrm{Aut}(G)$$

which for any $g \in \mathrm{Ad}(G)(L)$ is given by

$$g \mapsto \{x \mapsto g^{-1} x g\}.$$

A form G_1/k of a group G/k is called an *inner k-form*, if it is in the image of
$$H^1(\bar{k}_s/k, \mathrm{Ad}(G)) \to H^1(\bar{k}_s/k, \mathrm{Aut}(G)).$$

We call a semi simple group G/k *anisotropic* if it does not contain a non trivial split torus (See exercise 1.2.1.) In our example below the group of elements of norm 1 is semi simple and anisotropic if and only if $D(a,b)$ is a field.

I want to give an example, we consider the algebraic group $\mathrm{Gl}_2/\mathbb{Q}$ we consider two integers $a, b \neq 0$, for simplicity we assume that b is not a square. Then we have the quadratic extension $L = \mathbb{Q}(\sqrt{b})$. The element $\begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}$ defines the inner automorphism

$$\mathrm{Ad}(\begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}) : g \mapsto \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix} g \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}^{-1}$$

of the group Gl_2, let σ be its non trivial automorphism. Then $\sigma \mapsto \mathrm{Ad}(\begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix})$ and $\mathrm{Id}_{\mathrm{Gal}(L/k)} \mapsto \mathrm{Id}_{\mathrm{Aut}(\mathrm{Gl}_2)(L)}$ is a 1-cocycle and we get a $\mathbb{Q}$ form of our group.

Hence we get a $\mathbb{Q}$ form $G_1 = G(a,b)/\mathbb{Q}$ of our group Gl_2. It is an inner form.

Now we can see easily that group of rational points of our above group $G(a,b)(\mathbb{Q})$ is the multiplicative group of a central simple algebra $D(a,b)/\mathbb{Q}$. To get this algebra we consider the algebra $M_2(L)$ of (2,2)-matrices over L. We define

$$D(a,b) = \{x \in M_2(L) | x = \mathrm{Ad}(\begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}) x^\sigma \mathrm{Ad}(\begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix})^{-1}\}.$$

We have an embedding of the field L into this algebra, which is given by

$$u \mapsto \begin{pmatrix} u & 0 \\ 0 & u^\sigma \end{pmatrix}$$

Let u_b the image of $\sqrt{b}$ under this map. We also have the element $u_a = \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}$ in this algebra.

Now I leave it as an exercise to the reader that as a $\mathbb{Q}$ vctor space

$$D(a,b) = \mathbb{Q} \oplus \mathbb{Q}u_b \oplus \mathbb{Q}u_a \oplus \mathbb{Q}u_a u_b$$

We have the relation $u_a^2 = a, u_b^2 = b, u_a u_b = -u_b u_a$.

Of course we should ask ourselves: When is $D(a,b)$ split, this means isomorphic to $M_2(\mathbb{Q})$. To answer this question we consider the norm homomorphism, which is defined by

$$x+yu_b+zu_a+wa_a u_b \mapsto (x+yu_b+zu_a+wa_a u_b)(x-yu_b-zu_a-wa_a u_b) = x^2-y^2b-z^2a+w^2ab.$$

It is easy to see that $D(a,b)$ splits if and only if we can find a non zero element whose norm is zero.

If we do this with $\mathbb{R}$ as base field and if we take $a = -1, b = -1$ then we get the Hamiltonian quaternions, which is non split.

We may also look at the p-adic completions $\mathbb{Q}_p$ of our field. Then it is not difficult to see that $D(a,b)$ splits over $\mathbb{Q}_p$ if $p \neq 2$ and $p \nmid ab$. Hence it is clear that there is only a finite number of primes p for which $D(a,b)$ does not split.

If we consider $\mathbb{R}$ as completion at the infinite place, and the $\mathbb{Q}_p$ as the completions at the finite places, then we have

The algebra $D(a,b)$ splits if and only if it splits at all places. The number of places where it does not split is always even.

The first assertion is the so called Hasse-Minkowski principle, the second assertion is essentially equivalent to the quadratic reciprocity law.

1.3 The Lie-algebra

We need some basic facts about the Lie-algebras of algebraic groups.

For any algebraic group G/k we can consider its group of points with values in $k[\epsilon] = k[X]/(X^2)$. We have the homomorphism $k[\epsilon] \to k$ sending ϵ to zero and hence we get an exact sequence

$$0 \to \mathfrak{g} \to G(k[\epsilon]) \to G(k) \to 1.$$

The kernel $\mathfrak{g}$ is a k-vector space, if the characteristic of k is zero, then its dimension is equal to the dimension of G/k. It is denoted by $\mathfrak{g} = \mathrm{Lie}(G)$.

Let us consider the example of the group $G = SO(f)$, where $f : V \times \to k$ is a non degenerate symmetric bilinear form. In this case an element in $G(k[\epsilon])$ is of the form $\mathrm{Id} + \epsilon A, A \in \ \mathrm{End}(V)$ for which

$$f((\mathrm{Id} + \epsilon A)v, (\mathrm{Id} + \epsilon A)w) = f(v, w)$$

for all $v, w \in V$. Taking into account that $\epsilon^2 = 0$ we get

$$\epsilon(f(Av, w) + f(v, Aw)) = 0,$$

i.e. A is skew with respect to the form, and $\mathfrak{g}$ is the k-vector space of skew endomorphisms. If we give V a basis and if $f = \sum x_i^2$ with respect to this basis then this means the the matrix of A is skew symmetric.

If we consider $G = \mathrm{Gl}_n/k$ then $\mathfrak{g} = M_n(k)$, the Lie-bracket is given by

$$(A, B) \mapsto AB - BA \tag{4}$$

We have some kind of a standard basis for our Lie algebra

$$\mathfrak{g} = \bigoplus_{i=1}^{n} kH_i \oplus \bigoplus_{i,j,i\neq j} kE_{i,j} \tag{5}$$

where H_i (resp.$E_{i,j}$) are the matrices

$$H_i = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 \\ & & & & & \\ 0 & 0 & \ddots & & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ & & & & & \\ 0 & 0 & 0 & \dots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ resp. } E_{i,j} = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 \\ & & & & & \\ 0 & 0 & \ddots & & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & \\ 0 & 0 & 0 & \dots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and the only non zero entries (=1) is at (i,i) on the diagonal (resp. and (i,j) off the diagonal.)

For the group Sl_n/k the Lie-algebra is $\mathfrak{g}^{(0)} = \{A \in M_n(k) | \operatorname{tr}(A) = 0\}$ and again we have a standard basis

$$\mathfrak{g}^{(0)} = \bigoplus_{i=1}^{n-1} k(H_i - H_{i+1}) \oplus \bigoplus_{i,j,i\neq j} kE_{i,j} \tag{6}$$

A *representation* of a group scheme G/k is a k-homomorphism

$$\rho : G \to \mathrm{Gl}(V)$$

where V/k is a $k-$ vector space. Then it is clear from our considerations above that we have a "derivative" of the representation

$$d\rho : \mathfrak{g} = \mathrm{Lie}(G/k) \to \mathrm{Lie}(\mathrm{Gl}(V)) = \mathrm{End}(V)$$

this is k-linear.

Every group scheme G/k has a very special representation, this is the the *Adjoint representation.* We observe that the group acts on itself by conjugation, this is the morphism

$$Inn : G \times_k G \to G$$

which on T valued points is given by

$$Inn(g_1, g_2) \mapsto g_1 g_2 (g_1)^{-1}.$$

This action clearly induces a representation

$$\mathrm{Ad} : G/k \to \mathrm{Gl}(\mathfrak{g})$$

and this is the adjoint representation. This adjoint representation has a derivative and this is a homomorphism of k vector spaces

$$D_{\mathrm{Ad}} = \mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}).$$

We introduce the notation: For $T_1, T_2 \in \mathfrak{g}$ we put

$$[T_1, T_2] := \mathrm{ad}(T_1)(T_2).$$

Now we can state the famous and fundamental result

Theorem 1.1. *The map $(T_1, T_2) \mapsto [T_1, T_2]$ is bilinear and antisymmetric. It induces the structure of a Lie-algebra on $\mathfrak{g}$, i.e. we have the Jacobi identity*

$$[T_1, [T_2, T_3]] + [T_2, [T_3, T_1]] + [T_3, [T_1, T_2]] = 0.$$

We do not prove this here. In the case $G/k = \mathrm{Gl}(V)$ and $T_1, T_2 \in \mathrm{Lie}(\mathrm{Gl}(V) = \mathrm{End}(V)$ we have $[T_1, T_2] = T_1T_2 - T_2T_1$ and in this case the Jacobi Identity is a well known identity.

On any Lie algebra we have a symmetric bilinear form (the Killing form)

$$B : \mathfrak{g} \times \mathfrak{g} \to k \tag{7}$$

which is defined by the rule

$$B(T_1, T_2) = \mathrm{trace}(\mathrm{ad}(T_1) \circ \mathrm{ad}(T_2))$$

A simple computation shows that for the examples $\mathfrak{g} = \mathrm{Lie}(\mathrm{Gl}_n)$ and $\mathfrak{g}^{(0)} = \mathrm{Lie}(\mathrm{Sl}_n)$ we have

$$B(T_1, T_2) = 2n\ \mathrm{tr}(T_1 T_2) - 2\ \mathrm{tr}(T_1)\ \mathrm{tr}(T_2) \tag{8}$$

we observe that in case that one of the T_i is central, i.e.$= u\mathrm{Id}$ we have $B(T_1, T_2) = 0$. In the case of $\mathfrak{g}^{(0)}$ the second term is zero.

It is well known that a linear algebraic group is semi-simple if and only if the Killing form B on its Lie algebra is non degenerate.

1.4 Structure of semisimple groups over $\mathbb{R}$ and the symmetric spaces:

We need some information concerning the structure of the group $G_\infty = G(\mathbb{R})$ for semisimple groups over $G/\mathbb{R}$. We will provide this information simply by discussing a series of examples.

Of course the group $G(\mathbb{R})$ is a topological group, actually it is even a Lie group. This means it has a natural structure of a $\mathcal{C}_\infty$ -manifold with respect to this structure. Instead of $G(\mathbb{R})$ we will very often write G_∞. Let G^0_∞ be the connected component of the identity in G_∞. It is an open subgroup of finite index. We will discuss the

Theorem of E. Cartan: *The group G^0_∞ always contains a maximal compact subgroup $K \subset G^0_\infty$ and all maximal compact subgroups are conjugate under G^0_∞. The quotient space $X = G^0_\infty/K$ is again a $\mathcal{C}_\infty$-manifold. It is diffeomorphic to an $\mathbb{R}^n$ and carries a Riemannian metric which is invariant under the operation of G^0_∞ from the left. It has negative sectional curvature. The maximal compact subgroup $K \subset G^0_\infty$ is connected and equal to its own normalizer. Therefore the space X can be viewed as the space maximal compact subgroups in G^0_∞.*

This theorem is fundamental. To illustrate this theorem we consider a series of examples:

1.4.1 The groups $\mathbf{Sl}_d(\mathbb{R})$ and $\mathbf{Gl}_n(\mathbb{R})$:

The group $\mathrm{Sl}_d(\mathbb{R})$ is connected. If $K \subset \mathrm{Sl}_d(\mathbb{R})$ is a closed compact subgroup, then we can find a positive definite quadratic form

$$f : \mathbb{R}^n \to \mathbb{R},$$

such that $K \subset SO(f, \mathbb{R})$. since the group $SO(f, \mathbb{R})$ itself is compact, we have equality. Two such forms f_1, f_2 define the same maximal compact subgroup if thre is a $\lambda > 0$ in $\mathbb{R}$ such that $\lambda f_1 = f_2$.

This is rather clear, if we believe the first assertion about the existence of f. The existence of f is also easy to see if one believes in the theory of integration on K. This theory provides a positive invariant integral

$$\begin{array}{ccc} \mathcal{C}_c(K) & \longrightarrow & \mathbb{R} \\ \varphi & \longrightarrow & \displaystyle\int_K \varphi(k)dk \end{array}$$

with $\int \varphi > 0$ if $\varphi \geq 0$ and not identically zero (positivity), $\int \varphi(kk_0)dk = \int \varphi(k_0k)dk = \int \varphi(k)dk$ (invariance).

To get our form f we start from any positive definite form f_0 on $\mathbb{R}^n$ and put

$$f(\underline{x}) = \int_K f_0(k\underline{x})dk.$$

A positive definite quadratic form on $\mathbb{R}^n$ is the same as a symmetric positive definite bilinear form. Hence the space of positive definite forms is the same as the space of positive definite symmetric matrices

$$\tilde{X} = \left\{A = (a_{ij}) \mid A =^t A, A > 0\right\}.$$

Hence we can say that the space of maximal compact subgroups in $\mathrm{Sl}_d(\mathbb{R})$ is given by

$$X = \tilde{X}/\mathbb{R}^*_{>0}.$$

It is easy to see that a maximal compact subgroup $K \subset \mathrm{Sl}_d(\mathbb{R})$ is equal to its own normalizer (why?). If we view X as the space of positive definite symmetric matrices with determinant equal to one, then the action of $\mathrm{Sl}_d(\mathbb{R})$ on $X = \mathrm{Sl}_d(\mathbb{R})/K$ is given by

$$(g, A) \longrightarrow g\ A\ {}^t g,$$

and if we view it as the space of maximal compact subgroups, then the action is conjugation.

There is still another interpretation of the points $x \in X$. In our above interpretation a point was a symmetric, positive definite bilinear form $<\ ,\ >_x$ on $\mathbb{R}^n$ up to a homothety. This bilinear form defines a transpose $g \mapsto^{t_x} g$ and hence an involution

$$\Theta_x : g \mapsto \left({}^{t_x} g\right)^{-1} \tag{9}$$

Then the corresponding maximal compact subgroup is

$$K_x = \{g \in \mathrm{Sl}_n(\mathbb{R}) | \Theta_x(g) = g\} \tag{10}$$

This involution Θ_x is a Cartan involution, it also induces an involution also called Θ_x on the Lie-algebra and it has the property that (See 7)

$$(u, v) \mapsto B(u, \Theta_x(v)) = B_{\Theta_x}(u, v) \tag{11}$$

is negative definite. This bilinear form is K_x invariant. All these Cartan involutions are conjugate.

If we work with $\mathrm{Gl}_n(\mathbb{R})$ instead then we have some freedom to define the symmetric space. In this case we have the non trivial center $\mathbb{R}^\times$ and it is sometimes useful to define

$$X = \mathrm{Gl}_n(\mathbb{R})/\mathrm{SO}(\mathbb{R}) \cdot \mathbb{R}^\times_{>0} \tag{12}$$

then our symmetric space has two components, a point is pair (Θ_x, ϵ) where ϵ is an orientation. If we do not divide by $\mathbb{R}^\times_{>0}$ then we multiply the Riemannian manifold X by a flat subspace and we get the above space $\tilde{X}$.

A Cartan involution on $\mathrm{Gl}_n(\mathbb{R})$ is an involution which induces a Cartan involution on $\mathrm{Sl}_n(\mathbb{R})$ and which is trivial on the center.

Proposition 1.1. *The Cartan involutions on $Gl_n(\mathbb{R})$ are in one to one correspondence to the euclidian metrics on $\mathbb{R}^n$ up to conformal equivalence.*

Finally we recall the Iwasawa decomposition. Inside $\mathrm{Gl}_n(\mathbb{R})$ we have the standard Borel- subgroup $B(\mathbb{R})$ of upper triangular matrices and it is well known that

$$\mathrm{Gl}_n(\mathbb{R}) = B(\mathbb{R}) \cdot \mathrm{SO}(\mathbb{R}) \cdot \mathbb{R}^{\times}_{>0} \tag{13}$$

and hence we see that $B(\mathbb{R})$ acts transitively on X.

1.4.2 The Arakelow- Chevalley scheme $(\mathbf{Gl}_n/\mathbb{Z}, \Theta_0)$

We consider the case $G = \mathrm{Gl}_n$ and the special Cartan involution $\Theta_0(g) = ({}^t g)^{-1}$ and look at it from a slightly different point of view.

We start from the free lattice $L = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \cdots \oplus \mathbb{Z}e_n$ and we think of $\mathrm{Gl}_n/\mathbb{Z}$ as the scheme of automorphism of this lattice. If we choose an euclidian metric $<\ ,\ >$ on $L \otimes \mathbb{R}$ then we call the pair $(L, <\ ,\ >)$ an Arakelow vector bundle. up to homothety, we get a Cartan involution Θ on $\mathrm{Gl}_n(\mathbb{R})$. We choose the standard euclidian metric with respect to the given basis, i.e. $< e_i, e_j >= \delta_{i,j}$. The the resulting Cartan involution is the standard one: $\Theta_0 : g \mapsto ({}^t g)^{-1}$. This pair $(\mathrm{Gl}_n/\mathbb{Z}, \Theta_0)$ is called an Arakelow- Chevalley scheme. (In a certain sense the integral structure of $\mathrm{Gl}_n/\mathbb{Z}$ and the choice of the Cartan involution are "optimally adapted")

In this case we find for our basis elements in (5)

$$B_{\Theta_0}(H_i, H_j) = -2n\delta_{i,j} + 2; B_{\Theta_0}(E_{i,j}, E_{k,l}) = -2n\delta_{i,k}\delta_{j,l} \tag{14}$$

hence the $E_{i,j}$ are part of an orthonormal basis.

We propose to call a pair $(L, <\ ,\ >_x)$ an Arakelow vector bundle over $\mathrm{Spec}(\mathbb{Z}) \cup \{\infty\}$ and $(\mathrm{Gl}_n, \Theta_x)$ an Arakelow group scheme. The Arakelow vector bundles modulo conformal equivalence are in one-to one correspondence with the Arakelow group schemes of type Gl_n.

1.4.3 The group $\mathbf{Sl}_d(\mathbb{C})$

We now consider the group $G/\mathbb{R}$ whose group of real points is $G(\mathbb{R}) = \mathrm{Sl}_d(\mathbb{C})$ (see 1.1 example 4)).

A completely analogous argument as before shows that the maximal compact subgroups are in one to one correspondence to the positive definite hermitian forms on $\mathbb{C}^n$ (up to multiplication by a scalar). Hence we can identify the space of maximal compact subgroup $K \subset G(\mathbb{R})$ to the space of positive definite hermitian matrices

$$X = \left\{ A \mid A = {}^t\,\overline{A}\ ,\ A > 0\ ,\ \det A = 1 \right\}.$$

The action of $\mathrm{Sl}_d(\mathbb{C})$ by conjugation on the maximal compact subgroups becomes

$$A \longrightarrow g\ A\ {}^t\overline{g}$$

on the space of matrices.

1.4.4 The orthogonal group:

The next example I want to discuss is the example of an orthogonal group of a quadratic form

$$f(x_1,\dots,x_n) = x_1^2 + \dots + x_m^2 - x_{m+1}^2 - \dots - x_n^2.$$

Since at this point we consider only groups over the real numbers, we may assume that our form is of this type.

In this case one has the usual notation

$$SO(f,\mathbb{R}) = SO(m,n-m).$$

Of course we can use the same argument as before and see that for any maximal compact subgroup $K \subset SO(f,\mathbb{R})$ we may find a positive definite form ψ

$$\psi : \mathbb{R}^n \longrightarrow \mathbb{R}$$

such that $K = SO(f,\mathbb{R}) \cap SO(\psi,\mathbb{R})$. But now we cannot take all forms ψ, i.e. only special forms ψ provide maximal compact subgroup.

We leave it to the reader to verify that any compact subgroup K fixes an orthogonal decomposition $\mathbb{R}^n = V_+ \oplus V_-$ where our original form f is positive definite on V_+ and negative definite on V_-. Then we can take a ψ which is equal to f on V_+ and equal to $-f$ on V_-.

Exercise 3 *a) Let $V/\mathbb{R}$ be a finite dimensional vector space and let f be a symmetric non degenerate form on V. Let $K \subset SO(f)$ be a compact subgroup. If f is not definite then the action of K on V is not irreducible.*

b) We can find a K invariant decomposition $V = V_- \oplus V_+$ such that f is negative definite on V_- and positive definite on V_+

In this case the structure of the quotient space $G(\mathbb{R})/K$ is not so easy to understand. We consider the special case of the form

$$x_1^2 + \dots + x_n^2 - x_{n+1}^2 = f(x_1,\dots,x_{n+1}).$$

We consider in $\mathbb{R}^{n+1}$ the open subset

$$X_- = \{v = (x_1 \dots x_{n+1}) \mid f(v) < 0\}.$$

It is clear that this set has two connected components, one of them is

$$X_-^+ = \{v \in X_- \mid x_{n+1} > 0\}$$

Since it is known that $SO(n,1)$ acts transitively on the vectors of a given length, we find that $SO(n,1)$ cannot be connected. Let $G_\infty^0 \subset SO(n,1)$ be the subgroup leaving X_-^+ invariant.

Now it is not to difficult to show that for any maximal compact subgroup $K \subset G_\infty^0$ we can find a ray $\mathbb{R}_{>0}^* \cdot v \subset X_-^{(+)}$ which is fixed by K.

(Start from $v_0 \in X_-^{(+)}$ and show that $R_{>0}^* K v_0$ is a closed convex cone in $X_-^{(+)}$. It is K invariant and has a ray which has a "centre of gravity" and this is fixed under K.)

For a vector $v = (x_1,\dots,x_{n+1}) \in X_-^{(+)}$ we may normalize the coordinate x_{n+1} to be equal to one; then the rays $\mathbb{R}_{>0}^+ v$ are in one to one correspondence with the points of the ball

$$\overset{\circ}{D}_n = \left\{(x_1,\dots,x_n) \mid x_1^2 + \dots + x_n^2 < 1\right\} \subset X_-^{(+)}.$$

This tells us that we can identify the set of maximal compact subgroups $K \subset G_\infty^0$ with the points of this ball. The first conclusion is that $G_\infty^0/K \simeq$

D^n is topologically a cell (diffeomorphic to $\mathbb{R}^n$). Secondly we see that for a $v \in X_-^+$ we have an orthogonal decompositon with respect to f

$$\mathbb{R}^{n+1} = \langle v \rangle + \langle v \rangle^\perp,$$

and the corresponding maximal compact subgroup is the orthogonal group on $\langle v \rangle^\perp$.

1.4.5 Special low dimensional cases

1) We consider the group $\mathrm{Sl}_2(\mathbb{R})$. It acts on the upper half plane

$$H = \{z \mid z \in \mathbb{C}, \Im(z) > 0\}$$

by

$$(g, z) \longrightarrow \frac{az+b}{cz+b}, \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}_2(\mathbb{R}).$$

It is clear that the stabilizer of the point $i \in H$ is the standard maximal compact subgroup

$$K = SO(2) = \left\{ \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \right\}.$$

Hence we have $H = \mathrm{Sl}_2(\mathbb{R})/K$. But this quotient has been realized as the space of symmetric positive definite 2×2-matrices with determinant equal to one

$$x = \left\{ \begin{pmatrix} y_1 & x_1 \\ x_1 & y_2 \end{pmatrix} \mid y_1 y_2 - x_1^2 = 1, y_1 > 0 \right\}.$$

It is clear how to find an isomorphism between these two explicit realizations. The map

$$\begin{pmatrix} y_1 & x_1 \\ x_1 & y_2 \end{pmatrix} \longrightarrow \frac{i + x_1}{y_2},$$

is compatible with the action of $\mathrm{Sl}_2(\mathbb{R})$ on both sides and sends the identity $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ to the point i.

If we start from a point $z \in H$ the corresponding metric is as follows: We identify the lattices $\langle 1, z \rangle = \{a + bz \mid a, b \in \mathbb{Z}\} = \Omega$ to the lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$ by sending $1 \to \binom{1}{0}$ and $z \to \binom{0}{1}$. The standard euclidian metric on $\mathbb{C} = \mathbb{R}^2$ induces a metric on $\Omega \subset \mathbb{C}$, and this metric is transported to $\mathbb{R}^2$ by the identification $\Omega \otimes \mathbb{R} \to \mathbb{R}^2$.

2) The two groups $\mathrm{Sl}_2(\mathbb{R})$ and $P\mathrm{Sl}_2(\mathbb{R}) = \mathrm{Sl}_2(\mathbb{R})/\{\pm\mathrm{Id}\}$ give rise to the same symmetric space. The group $P\mathrm{Sl}_2(\mathbb{R})$ acts on the space $M_2(\mathbb{R})$ of 2×2-matrices by conjugation (the group $\mathrm{Gl}_2(\mathbb{R})$ acts by conjugation and the centre acts trivially) and leaves invariant the space

$$\{A \in M_2(\mathbb{R}) \mid \mathrm{trace}(A) = 0\} = M_2^0(\mathbb{R}).$$

On this three-dimensional space we have a symmetric quadratic form

$$\begin{aligned} B &: M_2^0(\mathbb{R}) \longrightarrow \mathbb{R} \\ B &: A \longrightarrow \frac{1}{2} \text{ trace } (A^2) \end{aligned}$$

and with respect to the basis $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $e_+ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $e_- = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, this form is $x_1^2 + x_2^2 - x_3^2$.

Hence we see that $SO(M_2^0(\mathbb{R}), B) = SO(2,1)$, and hence we have an isomorphism between $P\mathrm{Sl}_2(\mathbb{R})$ and the connected component of the identity $G_\infty^0 \subset SO(2,1)$. Hence we see that our symmetric space $H = \mathrm{Sl}_2(\mathbb{R})/K = P\mathrm{Sl}_2(\mathbb{R})/\overline{K}$ can also be realized (see ....) as disc

$$D = \{(x_1, x_2) \mid x_1^2 + x_2^2 < 1\}$$

where we normalized $x_3 = 1$ on $X_-^{(+)}$ as in ....

1.4.6 The group $\mathrm{Sl}_2(\mathbb{C})$.

Recall that in this case the symmetric space is given by the positive definite hermitian matrices

$$A = \left\{ \begin{pmatrix} y_1 & z \\ \overline{z} & y_2 \end{pmatrix} \mid \det(A) = 1, y_1 > 0 \right\}.$$

In this case we have also a realization of the symmetric space as an upper half space. We send

$$\begin{pmatrix} y_1 & w \\ \overline{w} & y_2 \end{pmatrix} \longmapsto \left(\frac{w}{y_2}, \frac{1}{y_2} \right) = (z, \zeta) \in \mathbb{C} \times \mathbb{R}_{>0}$$

The inverse of this isomorphism is given by

$$(z, \zeta) \mapsto \begin{pmatrix} \zeta + z\overline{z}/\zeta & z/\zeta \\ \overline{z}/\zeta & 1/\zeta \end{pmatrix}$$

As explained earlier, the action of $\mathrm{Gl}_2(\mathbb{C})$ on the maximal compact subgroup given by conjugation yields the action

$$G(\mathbb{R}) \times X \longrightarrow X,$$

$$(g, A) \longrightarrow gA^t\overline{g},$$

on the hermitian matrices. Translating this into the realization as an upper half space yield the slightly scaring formula

$$G \times (\mathbb{C} \times \mathbb{R}_{>0}) \longrightarrow \mathbb{C} \times \mathbb{R}_{>0},$$

$$(g, (z, \zeta)) \longrightarrow \left(\frac{(az+b)\,\overline{(cz+d)} + a\overline{c}\,\zeta^2}{(az+d)\,\overline{(cz+d)} + c\overline{c}\,\zeta^2}, \frac{\zeta}{(az+d)\,\overline{(cz+d)} + c\overline{c}\,\zeta^2} \right)$$

1.3.4. The Riemannian metric: It was already mentioned in the statement of the theorem of Cartan that we always have a G_∞^0 invariant Riemannian metric on X. It is not to difficult to construct such a metric which in many cases is rather canonical.

In the general case we observe that the maximal compact subgroup is the stabilizer of the point $x_0 = e \cdot K \in G_\infty^0/K = X$. Hence it acts on the tangent space of x_0, and we can construct a k-invariant positive definite quadratic form on this tangent sapce. Then we use the action of G_∞^0 on X to transport this metric to an arbitrary point in X: If $x \in X$ we find a g so that $x = gx_0$, it defines an isomorphism between the tangent space at x_0 and the tangent space at x. Hence we get a form on the tangent space at x, which will not depend on the choice of $g \in G^0_\infty$.

In our examples this metric is always unique up to scalars.

a) In the case of the group $\mathrm{Sl}_d(\mathbb{R})$ we may take as a base point $x_0 \in X$ the identity $\mathrm{Id} \in \mathrm{Sl}_d(\mathbb{R})$. The corresponding maximal compact subgroup is the orthogonal group $SO(n)$. The tangent space at Id is given by the space

$$\mathrm{Sym}^0_n(\mathbb{R}) = T^X_{\mathrm{Id}}$$

of symmetric matrices with trace zero. On this space we have the form

$$Z \longrightarrow \mathrm{trace}(Z^2),$$

which is positive definite (a symmetric matrix has real eigenvalues). It is easy to see that the orthogonal group acts on this tangent space by conjugation, hence the form is invariant.

b) A similar argument applies to the group $G_\infty = \mathrm{Sl}_d(\mathbb{C})$. Again the identity Id is a nice positive definite hermitian matrix. The tangent space consists of the hermitian matrices

$$T^X_{\mathrm{Id}} = \left\{ A \mid A =^t \overline{A} \text{ and } \ \mathrm{tr}(A) = 0 \right\},$$

and the invariant form is given by

$$A \longrightarrow \ \mathrm{tr}(A\overline{A}).$$

c) In the case of the group $G^0_\infty \subset SO(f)$ where f is the quadratic form

$$f(x_1, \ldots, x_{n+1}) = x_1^2 + \ldots + x_n^2 - x_{n+1}^2.$$

We realized the symmetric space as the open ball

$$\overset{\circ}{D}_n = \{(x_1, \ldots, x_n) \mid x_1^2 + \ldots + x_n^2 < 1\}.$$

The orthogonal group $SO(n,1)$ is the stabilizer of $0 \in \overset{\circ}{D}_n$, and hence it is clear that the Riemannian metric has to be of the form

$$h(x_1^2 + \ldots + x_n^2)(dx_1^2 + \ldots dx_n^2)$$

(in the usual notation). A closer look shows that the metrics has to be

$$\frac{dx_1^2 + \ldots + dx_n^2}{\sqrt{1 - x_1^2 - \ldots - x_n^2}}.$$

In our two low dimensional spacial examples the metric is easy to determine. For the action of the group $\mathrm{Sl}_2(\mathbb{R})$ on the upper half plane H we observe that for any point $z_0 = x + iy \in H$ the tangent vectors $\frac{\partial}{\partial x}|_{z_0}$, $\frac{\partial}{\partial y}|_{z_0}$ form a basis of the tangent spaces at z_0.

If we take $z_0 = i$ then the stabilizer is the group $SO(2)$ and for

$$e(\varphi) = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}.$$

We have

$$e(\varphi) \cdot \left(\frac{\partial}{\partial x} \ |_i \right) = \cos 2\varphi \cdot \frac{\partial}{\partial x} \ |_i + \sin 2\varphi \frac{\partial}{\partial y} \ |_i$$
$$e(\varphi) \left(\frac{\partial}{\partial y} \ |_i \right) = \sin 2\varphi \cdot \frac{\partial}{\partial x} \ |_i + \cos 2\varphi \frac{\partial}{\partial y} \ |_i.$$

Hence we find that $\frac{\partial}{\partial x} \mid_i$ and $\frac{\partial}{\partial y} \mid_i$ have to be orthogonal and of the same length.

Now the matrix

$$\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$$

sends i into the point $z = x + iy$. It sends $\frac{\partial}{\partial x} \mid_i$ and $\frac{\partial}{\partial y} \mid_i$ into $y \cdot \frac{\partial}{\partial x} \mid_z$ and $y \cdot \frac{\partial}{\partial y} \mid_z$, and hence we must have for our invariant metric

$$\langle \frac{\partial}{\partial x} \mid_z, \frac{\partial}{\partial y} \mid_z \rangle = 0 \; ; \; \langle \frac{\partial}{\partial x} \mid_z, \frac{\partial}{\partial x} \mid_z \rangle = \frac{1}{y^2} \; ; \; \langle \frac{\partial}{\partial y} \mid_z, \frac{\partial}{\partial y} \mid_z \rangle = \frac{1}{y^2},$$

and this is in the usual notation the metric

$$ds^2 = \frac{1}{y^2}(dx^2 + dy^2).$$

A completely analogous argument yields for the space $\mathbb{H}_3$ the metric

$$\frac{1}{\zeta^2}\,(d\zeta^2 + dx^2 + dy^2).$$

2 Arithmetic groups

If we have a linear algebraic group $G/\mathbb{Q} \hookrightarrow GL_n$ we may consider the group $\Gamma = G(\mathbb{Q}) \cap GL_n(\mathbb{Z})$. This is the first example of an *arithmetic* group. It has the following fundamental property:

Proposition: *The group Γ is a discrete subgroup of the topological group $G(\mathbb{R})$.*

This is rather easily reduced to the fact that $\mathbb{Z}$ is discrete in $\mathbb{R}$. Actually our construction provides a big family of arithmetic groups. For any integer $m > 0$ we have the homomorphism of reduction $\mod m$, namely

$$GL_n(\mathbb{Z}) \longrightarrow GL_n(\mathbb{Z}/m\mathbb{Z}).$$

The kernel $GL_n(\mathbb{Z})(m)$ of this homomorphism has finite indesx in $GL_n(\mathbb{Z})$ and hence the intersection $\Gamma' = GL_n(\mathbb{Z})(m) \cap \Gamma$ has finite index in Γ.

Definition 2.1.: *A subgroup Γ'' of Γ is called a congruence subgroup, if we can find an integer m such that*

$$GL_n(\mathbb{Z})(m) \cap \Gamma \subset \Gamma'' \subset \Gamma.$$

At this point a remark is in order. I explained already that a linear algebraic group $G/\mathbb{Q}$ may be embedded in different ways into different groups GL_n, i.e.

$$\begin{array}{ccc} & \hookrightarrow & GL_{n_1} \\ G & & \\ & \hookrightarrow & GL_{n_2} \end{array}$$

In this case we may get two different congruence subgroups

$$\Gamma_1 = G(\mathbb{Q}) \cap GL_{n_1}(\mathbb{Z}), \Gamma_2 = G(\mathbb{Q}) \cap GL_{n_2}(\mathbb{Z}).$$

It is not hard to show that in such a case we can find an $m > 0$ such that

$$\begin{aligned} \Gamma_1 &\supset \Gamma_2 \cap GL_{n_2}(\mathbb{Z})(m) \\ \Gamma_2 &\supset \Gamma_1 \cap GL_{n_1}(\mathbb{Z})(m) \quad . \end{aligned}$$

From this we conclude that the notion of congruence subgroup does not depend on the way we realized the group $G/\mathbb{Q}$ as a subgroup in the general linear group.

Now we may also define the notion of an *arithmetic* subgroup. A subgroup $\Gamma' \subset G(\mathbb{Q})$ is called arithmetic if for any congruence subgroup $\Gamma \subset G(\mathbb{Q})$ the group $\Gamma' \cap \Gamma$ is of finite index in Γ' and Γ. (We say that Γ' and Γ are commensurable.) By definition all congruence subgroups are arithmetic subgroups.

The most prominent example of an arithmetic group is the group

$$\Gamma = \mathrm{Sl}_2(\mathbb{Z}).$$

Another example is obtained as follows. We defined for any number field $K/\mathbb{Q}$ the group

$$G/\mathbb{Q} = R_{K/\mathbb{Q}}(\mathrm{Sl}_d)$$

for which $G(\mathbb{Q}) = \mathrm{Sl}_d(K)$. If $\mathcal{O}_K$ is the ring of integers in K, then $\Gamma = \mathrm{Sl}_d(\mathcal{O}_K)$ (and also $\tilde{\Gamma} = GL_n(\mathcal{O}_K)$) is a congruence (and hence arithmetic) subgroup of $G(\mathbb{Q})$.

It is very interesting that the groups $\Gamma = \mathrm{Sl}_2(\mathbb{Z})$ and $\mathrm{Sl}_2(\mathcal{O}_K)$ for imaginary quadratic $K/\mathbb{Q}$ always contain arithmetic subgroups $\Gamma' \subset \Gamma$ which are not congruence subgroups. This means that in general the class of arithmetic subgroups is larger than the class of congruence subgroups. We will prove this assertion in (See ......).

If only the group $G(\mathbb{R})$ is given (as the group of real points of a group $G/\mathbb{Q}$ or perhaps only as a Lie group, then the notion of arithmetic group $\Gamma \subset G(\mathbb{R})$ is not defined. The notion of an arithmetic subgroup $\Gamma \subset G(\mathbb{R})$ requires the choice of a group scheme $G/\mathbb{Q}$ such that the group $G(\mathbb{R})$ is the group of real points of this group over $\mathbb{Q}$. The exercise in 1.1.2. shows that different $\mathbb{Q}$- forms provide different arithmetic groups.

Exercise 2 *If $\gamma \in GL_n(\mathbb{Z})$ is a nontrivial torsion element and if $\gamma \equiv \mathrm{Id} \mod m$ then $m = 1$ or $m = 2$. In the latter case the element γ is of order 2. This implies that for $m \geq 3$ the congruence subgroup $GL_n(\mathbb{Z})(m)$ of $GL_n(\mathbb{Z})$ is torsion free.*

This implies of course that any arithmetic group has a subgroup of finite index, which is torsion free.

2.1 The locally symmetric spaces

We start from a semisimple group $G/\mathbb{Q}$. To this group we attached the the group of real points $G(\mathbb{R}) = G_\infty$. In G_∞ we have the connected component G^0_∞ of the identity and in this group we choose a maximal compact subgroup K. The quotient space $X = G_\infty/K$ is a symmetric space which now may have several connected components. On this space we have the action of an arithmetic group Γ.

We have a fundamental fact:

The action of Γ on X is properly discontinuous, i.e. for any point

$x \in X$ there exists an open neighborhood U_x such that for all $\gamma \in \Gamma$ we have

$$\gamma U_x \cap U_x = \emptyset \quad \text{or} \quad \gamma x = x.$$

Moreover for all $x \in X$ the stabilizer

$$\Gamma_x = \{\gamma \mid \gamma x = x\}$$

is finite.

This is easy to see: If we consider the projection $p : G(\mathbb{R}) \to G(\mathbb{R})/K = X$, then the inverse image $p^{-1}(U_x)$ of a relatively compact neighborhood U_x of $x = g_0K$ is of the form $V_{g_0} \cdot K$, where V_{g_0} is a relatively compact neighborhood of g_0. Hence we look for the solutions of the equation

$$\gamma vk = v'k', \gamma \in \Gamma, v, v' \in V_{g_0}, k, k' \in K.$$

Since Γ is discrete in $G(\mathbb{R})$ there are only finitely many possibilities for γ and they can be ruled out by shrinking U_x with the exception of those γ for which $\gamma x = x$.

If Γ has no torsion then the projection

$$\pi : X \longrightarrow \Gamma\backslash X$$

is locally a $\mathcal{C}_\infty$-diffeomorphism. To any point $x \in \Gamma\backslash X$ and any point $\tilde{x} \in \pi^{-1}(x)$ we find a neighborhood $U_{\tilde{x}}$ such that

$$\pi : U_{\tilde{x}} \xrightarrow{\sim} U_x.$$

Hence the space $\Gamma\backslash X$ inherits the Riemannian metric and the quotient space is a locally symmetric space.

If our group Γ has torsion, then a point $\tilde{x} \in X$ may have a nontrivial stabilizer $\Gamma_{\tilde{x}}$. Then it is not difficult to prove that $\tilde{x}$ has a neighborhood $U_{\tilde{x}}$ which is invariant under $\Gamma_{\tilde{x}}$ and that for all $\tilde{y} \in U_{\tilde{x}}$ the stabilizer $\Gamma_{\tilde{y}} \subset \Gamma_{\tilde{x}}$. This gives us a diagram

$$\begin{array}{ccc} U_{\tilde{x}} & \longrightarrow & \Gamma_{\tilde{x}}\backslash U_{\tilde{x}} = U_x \\ \downarrow & & \downarrow \\ X & \xrightarrow{\pi} & \Gamma\backslash X \end{array}$$

i.e. the point $x \in \Gamma\backslash X$ has a neighborhood which is the quotient of a neighborhood $U_{\tilde{x}}$ by a finite group.

In this case the quotient space $\Gamma\backslash X$ may have singularities. Such spaces are called orbifolds. They have a natural stratification. Any point x defines a Γ conjugacy class $[\Gamma_{\tilde{x}}]$ of finite subgroups $\Gamma_{\tilde{x}} \subset \Gamma$. On the other hand a conjugacy class $[c]$ of finite subgroups $H \subset \Gamma$ defines the (non empty) subset (stratum) $\Gamma\backslash X([c])$ of those points $x \in \Gamma\backslash X$ for which $\Gamma_{\tilde{x}} \in [c]$.

These strata are easy to describe. We observe that for any finite $H \subset \Gamma$ the fixed point set X^H intersected with a connected component of X is contractible. Let $x_0 \in X^H$ be a point with $\Gamma_{x_0} = H$. Then any other point $x \in X^H$ is of the form $x = gx_0$ with $g \in G(\mathbb{R})$. This implies that $g \in N(H)(\mathbb{R})$, where $N(H)$ is the normaliser of H, it is an algebraic subgroup. Then $N(H)(\mathbb{R}) \cap K = K^H$ is compact subgroup, put $\Gamma^H = \Gamma \cap N(H)(\mathbb{R})$, and we get an embedding

$$\Gamma^H\backslash X^H \hookrightarrow \Gamma\backslash X.$$

This space contains the open subset $(\Gamma^H\backslash X^H)^{(0)}$ of those x where $H \in [\Gamma_{\tilde{x}}]$ and this is in fact the stratum attached to the conjugacy class of H.

We have an ordering on the set of conjugacy classes, we have $[c_1] \leq [c_2]$ if for any $H_1 \in [c_1]$ there exists a subgroup $H_2 \in [c_2]$ such that $H_1 \subset H_2$. These strata are not closed, the closure $\overline{\Gamma\backslash X([c])}$ is the union of lower dimensional strata.

If we start investigating the stratification above we immediately hit upon number theoretic problems.Let us pick a prime p and we consider the group $\Gamma = \mathrm{Sl}_{p-1}[\mathbb{Z}]$ and the ring of p-th roots of unity $\mathbb{Z}[\zeta_p]$ as a $\mathbb{Z}$-module is free of rank $p-1$ and hence we get an element

$$\zeta_p \in \mathrm{Sl}(\mathbb{Z}[\zeta_p]) = \mathrm{Sl}_{p-1}(\mathbb{Z})$$

and hence a cyclic subgroup of order p. But clearly we have many conjugacy classes of elements of order p in Γ because any ideal $\mathfrak{a}$ is a free $\mathbb{Z}$-module. If we want to understand the conjugacy classes of elements of order p or the conjugacy classes of cyclic subgroups of order p in $\mathrm{Sl}_{p-1}(\mathbb{Z})$ we need to understand the ideal class group.

In the next section we will discuss two simple cases.

These quotient spaces $\Gamma\backslash X$ attract the attention of various different kinds of mathematicians. They provide interesting examples of Riemannian manifolds and they are intensively studied from that point of view. On the other hand number theoretic data enter into their construction. Hence any insight into the structure of these spaces contains number theoretic information.

It is not difficult to see that any arithmetic group Γ contains a normal congruence subgroup Γ' which does not have torsion. This can be deduced easily from the exercise .... at the end of this section. Hence we see that $\Gamma'\backslash X$ is a Riemannian manifold which is a finite cover of $\Gamma\backslash X$ with covering group Γ/Γ'.

The following general theorem is due to Borel and Harish-Chandra:

The quotient $\Gamma\backslash X$ *always has finite volume with respect to the Riemannian metric. The quotient space* $\Gamma\backslash X$ *is compact if and only if the group* $G/\mathbb{Q}$ *is anisotropic.*

We will give some further explanation below.

2.1.1 Low dimensional examples

We consider the action of the group $\Gamma = \mathrm{Sl}_2(\mathbb{Z}) \subset \mathrm{Sl}_2(\mathbb{R})$ on the upper half plane

$$X = \mathbb{H} = \{z \mid \Im(z) = y > 0\} = \mathrm{Sl}_2(\mathbb{R})/SO(2).$$

As we explained in .... we may consider the point $z = x + iy$ as a positive definite euclidian metric on $\mathbb{R}^2$ up to a positive scalar. We saw already that this metric can be interpreted as the metric on $\mathbb{C}$ induced on the lattice $\Omega = \langle 1, z\rangle$. The action of $\mathrm{Sl}_2(\mathbb{Z})$ on the upper half plane corresponds to changing the basis $1, z$ of Ω into another basis and then normalizing the first vector of the new basis to length equal one.

This means that under the action of $\mathrm{Sl}_2(\mathbb{Z})$ we may achieve that the first vector 1 in the lattice is of shortest length. In other words $\Omega = \langle 1, z\rangle$ where now $|z| \geq 1$.

Since we can change the basis by $1 \to 1$ and $z \to z+n$. We still have $|z+n| \geq 1$. Hence see that this condition implies that we can move z by these translation into the strip $-1/2 \leq \Re(z) \leq 1/2$ and since 1 is still the shortest vector we end up in the classical fundamental domain:

$$\mathcal{F} = \{z | -1/2 \leq \Re(z) \leq 1/2, |z| \geq 1\}$$

Two points $z_1, z_2 \in \mathcal{F}$ are inequivalent under the action of $\mathrm{Sl}_2(\mathbb{Z})$ unless they differ by a translation. i.e.

$$z_1 = -\frac{1}{2} + it \ , \ z_2 = z_1 + 1 = \frac{1}{2} + it,$$

or we have $|z_1| = 1$ and $z_2 = -\frac{1}{z_1}$. Hence the quotient $\mathrm{Sl}_2(\mathbb{Z})\backslash\mathbb{H}$ is given by the following picture

It turns out that this quotient is actually a Riemann surface, i.e. the finite stabilizers at i and ρ do not produce singularities. As a Riemann surface the quotient is the complex plane or better the projective line $\mathbb{P}^1(\mathbb{C})$ minus the point at infinity.

It is clear that the points i and $\rho = +\frac{1}{2} + \frac{1}{2}\sqrt{-3}$ in the upper half plane are the only points with non-trivial stabilizer up to conjugation by an element $\gamma \in \mathrm{Sl}_2(\mathbb{Z})$. Actually the stabilizers are given by

$$\Gamma_i = \left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\} \quad , \quad \Gamma_\rho = \left\{\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}\right\}.$$

We denote the matrices

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ ; \ R = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

The second example is given by the group $\Gamma = \mathrm{Sl}_2(\mathbb{Z}[i]) \subset \mathrm{Sl}_2(\mathbb{C}) = G_\infty$ (see ....). Here we should remember that the choice of $G_\infty = \mathrm{Sl}_2(\mathbb{C})$ allows a whole series of arithmetic groups. For any imaginary quadratic extension $K = \mathbb{Q}(\sqrt{-d})$ with $\mathcal{O}_K$ as its ring of integers we may embed K into $\mathbb{C}$ and get

$$\mathrm{Sl}_2(\mathcal{O}_K) = \Gamma \subset G_\infty.$$

If the number d becomes larger then the structure of the group Γ becomes more and more complicated. We discuss only the simplest case.

We will construct a fundamental domain for the action of Γ on the three-dimensional hyperbolic space $\mathbb{H}_3 = \mathbb{C} \times \mathbb{R}_{>0}$.

We identify $\mathbb{H}_3$ with the space of positive definite hermitian matrices

$$X = \{A \in M_2(\mathbb{C}) \mid A =^t \overline{A}, A > 0, \det(A) = 1\}.$$

We consider the lattice

$$\Omega = \mathbb{Z}[i] \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathbb{Z}[i] \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

in $\mathbb{C}^2$ and view A as a hermitian metric on $\mathbb{C}^2$ where $\mathbb{C}/\Omega$ has volume 1. Let $e_1' = \binom{\alpha}{\beta}$ be a vector of shortest length. We can find a second vector $e_2' = \binom{\gamma}{\delta}$ so that $\det\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = 1$. This argument is only valid because

$\mathbb{Z}[i]$ is a principal ideal domain. We consider the vectors $e_2' + \nu e_1'$ where $\nu \in \mathbb{Z}[i]$. We have

$$\langle e_2'+\nu e_1', e_2'+\nu e_1'\rangle_A = \langle e_2'+\nu e : 1'\rangle_A + \nu\langle e_1', e_2'\rangle_A + \overline{\nu}\langle e_2', e_1'\rangle_a + \nu\overline{\nu}\langle e_1', e_1'\rangle_A.$$

Since we have the the euclidean algorithm in $\mathbb{Z}[i]$ we can choose ν such that

$$-\frac{1}{2}\langle e_1', e_1'\rangle \leq \mathrm{Re}\langle e_1', e_2'\rangle_A, \Im\langle e_1', e_2'\rangle_A \leq \frac{1}{2}\langle e_1', e_1'\rangle_A.$$

If we translate this to the action of $\mathrm{Sl}_2(\mathbb{Z}[i])$ on $\mathbb{H}_3$ then we find that every point $x = (z;\zeta) \in \mathbb{H}_3$ is equivalent to a point in the domain

$$\tilde{F} = \{(z,\zeta) \mid -\frac{1}{2} \leq \mathrm{Re}(z), \Im(z) \leq \frac{1}{2}; z\overline{z} + \zeta^2 \geq 1\}.$$

Since we have still the action of the matrix $\begin{pmatrix} i & 0 \\ 0 & -1 \end{pmatrix}$ we even find a smaller fundamental domain

$$F = \{(z,\zeta) \mid -\frac{1}{2} \leq \mathrm{Re}(z), \Im(z) \leq \frac{1}{2}; z\overline{z} + \zeta^2 \geq 1 \text{ and } \mathrm{Re}(z) + \Im(z) \geq 0\}.$$

I want to discuss also the extension of our considerations to the case of the reductive group $\mathrm{Gl}_2(\mathbb{C})$. In such a case we have to enlarge the maximal compact subgroup. In this case the group $\tilde{K} = \mathrm{Sl}_1(2) \cdot \mathbb{C}^* = K \cdot \mathbb{C}^*$ is a good choice where $\mathbb{C}^*$ is the centre of $\mathrm{Gl}_2(\mathbb{C})$. Then we get

$$\mathbb{H}_3 = \mathrm{Sl}_2(\mathbb{C})/K = \mathrm{Gl}_2(\mathbb{C})/\tilde{K}$$

i.e. we have still the same symmetric space. But the group $\tilde{\Gamma} = \mathrm{Gl}_2(\mathbb{Z}[i])$ is still larger. We have an exact sequence

$$1 \to \Gamma \to \tilde{\Gamma} \to \{i^\nu\} \to 1.$$

The centre $Z_{\tilde{\Gamma}}$ of $\tilde{\Gamma}$ is given by the matrices $\left\{\begin{pmatrix} i^v & 0 \\ 0 & i^v \end{pmatrix}\right\}$. The centre Z_Γ has index 2 in $Z_{\tilde{\Gamma}}$. Since the centre acts trivially on the symmetric space, hence the above fundamental domain will be "cut into two halfes" by the action of $\tilde{\Gamma}$. the matrices $\begin{pmatrix} i^v & 0 \\ 0 & 1 \end{pmatrix}$ induce rotation of $\nu \cdot 90^\circ$ around the axis $z = 0$ and therefore it becomes clear that the region

$$F_0 = \{(z,\zeta) \mid 0 \leq \Im(z), \mathrm{Re}(z) \leq \frac{1}{2}, z\overline{z} + \zeta^2 \geq 1\}$$

is a fundamental domain for $\tilde{\Gamma}$.

The translations $z \to z + 1$ and $z \to z + i$ identify the opposite faces of F. This induces an identification on F_0, namely

$$\left(\frac{1}{2} + iy, \zeta\right) \longrightarrow \left(-\frac{1}{2} + iy, \zeta\right) \longrightarrow \left(y + \frac{i}{2}, \zeta\right).$$

On the bottom of the domain F_0, namely

$$F_0(1) = \{(z,\zeta) \in F_0 \mid z\overline{z} + \zeta^2 = 1\}$$

we have the further identification

$$(z,\zeta) \longrightarrow (i\overline{z}, \zeta).$$

Hence we see that the quotient space $\tilde{\Gamma}\backslash\mathbb{H}_3$ is given by the following figure.

I want to discuss the fixed points and the stabilizers of the fixed points of $\tilde{\Gamma}$. Before I can do that, I need some simple facts concerning the structure of Gl_2.

The group $\mathrm{Gl}_2(K)$ acts upon the projective line $\mathbb{P}^1(K) = (K^2 \backslash \{0\})/K^*$. We write

$$\mathbb{P}^1(K) = (K) \cup \{\infty\}\ ;\ K(xe_1 + e_2) = x, Ke_1 = \infty.$$

It is quite clear that the action of $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{Gl}_2(K)$ is given by

$$gx = \frac{\alpha x + \beta}{\gamma x + \delta}.$$

The action of $\mathrm{Gl}_2(K)$ on $\mathbb{P}^1(K)$ is transitive. For a point $x \in \mathbb{P}^1(K)$ the stabilizer B_x is clearly a linear subgroup of Gl_2/K. If $x = \infty$, then this stabilizer is the subgroup

$$B_\infty = \left\{ \begin{pmatrix} a & u \\ 0 & b \end{pmatrix} \right\},$$

and for $x = 0$ we get

$$B_0 = \left\{ \begin{pmatrix} a & 0 \\ u & b \end{pmatrix} \right\}.$$

It is clear that these subgroups B_x are conjugate under the action of $\mathrm{Gl}_2(K)$. They are in fact maximal solbable subgroups of Gl_2.

If we have two different points $x_1, x_2 \in \mathbb{P}^1(K)$, then this corresponds to a choice of a basis where the basis vectors are only determined up to scalars. Then the intersection of the two groups $B_{x_1} \cap B_{x_2}$ is a so-called maximal torus. If we choose $x_1 = Ke_1$, $x_2 = Ke_2$, then

$$B_{x_1} \cap B_{x_2} = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right\}.$$

Any other maximal torus of the form B_{x_1}, B_2 is conjugate to T_0 under $\mathrm{Gl}_2(K)$.

Now we assume $K = \mathbb{C}$. We compactify the three dimensional hyperbolic space by adding $\mathbb{P}^1(\mathbb{C})$ at infinity, i.e.

$$\mathbb{H}_3 \hookrightarrow \overline{\mathbb{H}}_3 = \mathbb{H}_3 \cup \mathbb{P}^1(\mathbb{C}) = \mathbb{C} \times \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

(The reader should verify that there is a natural topology on $\overline{\mathbb{H}}_3$ for which the space is compact and for which $\mathrm{Gl}_2(\mathbb{C})$ acts continuously.)

Now let us assume that $a \in \mathrm{Gl}_2(\mathbb{C})$ is an element which has a fixed point on $\mathbb{H}_3$ and which is not central. Since it lies in a maximal compact subgroup times $\mathbb{C}^x$ we see that this element a can be diagonalized

$$a \longrightarrow g_0\ a\ g_0^{-1} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} = a'$$

with $\alpha \neq \beta$ and $|\alpha/\beta| = 1$.

Then it is clear that the fixed point set for a' is the line

$$\mathrm{Fix}\ (a') = \{(0, \zeta) \mid \zeta \in \mathbb{R}_{>0}\},$$

i.e. we do not get an isolated fixed point but a full fixed line.

The element a' has the two fixed points $\infty, 0$ in $\mathbb{P}^1(\mathbb{C})$, and hence ist defines the torus $T_0(\mathbb{C})$. Then it is clear that

$$\mathrm{Fix}(a') = \{(0, \zeta) \mid \zeta > 0\} = T_0(\mathbb{C}) \cdot (0, 1)$$

i.e. the fixed point set is an orbit under the action of $T_0(\mathbb{C})$.

2.1.2 Fixed point sets and stabilizers for $\mathbf{Gl}_2(\mathbb{Z}[i]) = \tilde{\Gamma}$

If we want to describe the stabilizers up to conjugation, we can focus our attention on F_0.

If we have an element $\gamma \in \tilde{\Gamma}$, γ not central and if we assume that γ has fixed points on $\mathbb{H}_3$, then we know that γ defines a torus $T_\gamma =$ centralizer$_{\mathrm{Gl}_2}(\gamma) =$ stabilizer of $x_\gamma, x_{\gamma'} \in \mathbb{P}^1(\mathbb{C})$. This torus is defined over $\mathbb{Q}(i)$, but it is not necessarily diagonalizable over $\mathbb{Q}(i)$, it may be that the coordinates of $x_\gamma, x_{\gamma'}$ lie in a quadratic extension of $F/\mathbb{Q}(i)$. This is the quadratic extension defined by the eigenvalues of γ.

We look at the edges of the fundamental domain F_0. We saw that they consist of connected pieces of the straight lines

$$G_1 = \{(z,\zeta) \mid z = 0\}, G_2 = \{(z,\zeta) \mid z = \frac{1}{2}\}, G_3 = \{(z,\zeta) \mid z = \frac{1+i}{2}\},$$

and the circles (these circles are euclidean circles and geodesics for the hyperbolic metric)

$$D_1 = \{(z,\zeta) \mid z\overline{z}+\zeta^2 = 1, \Im(z) = \mathrm{Re}(z)\}, D_2 = \{(z,\zeta) \mid z\overline{z}+\zeta^2 = 1, \Im(z) = 0\},$$

$$D_3 = \{(z,\zeta) \mid z\overline{z}+\zeta^2 = 1, \mathrm{Re}(z) = \frac{1}{2}\}.$$

The pour of points $(\infty, (z_0, 0)) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ has as its stabilizer

$$T_{z_0}(\mathbb{C}) = \begin{pmatrix} 1 & z_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & -z_0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & z_0(\beta-\alpha) \\ 0 & \beta \end{pmatrix},$$

the straight line $\{(z_0,\zeta) \mid \zeta > 0\}$ is an orbit u nder $T_{z_0}(\mathbb{C})$ and it consists of fixed points for

$$T_{z_0}(\mathbb{C})(1) = \left\{ \begin{pmatrix} \alpha & z_0(\beta-\alpha) \\ 0 & \beta \end{pmatrix} \middle| \ \alpha/\beta \in S^1 \right\}.$$

We can easily compute the pointwise stabilizer of G_1, G_2, G_3 in $\tilde{\Gamma}$. They are

$$\begin{aligned}
\tilde{\Gamma}_{G_1} &= \left\{ \begin{pmatrix} i^\nu & 0 \\ 0 & i^\mu \end{pmatrix} \right\} = \left\{ \begin{pmatrix} i^\nu & 0 \\ 0 & i \end{pmatrix} \right\} \cdot z_{\tilde{\Gamma}} \\
\Gamma_{\tilde{G}_2} &= \left\{ \begin{pmatrix} i^\nu & \frac{1-i^\nu}{2} \\ 0 & 1 \end{pmatrix} \middle| \frac{1-i^\nu}{2} \in \mathbb{Z}[i] \right\} \cdot Z_{\tilde{\Gamma}} = \left\{ \begin{pmatrix} \pm 1 & \frac{1\pm 1}{2} \\ 0 & 1 \end{pmatrix} \right\} \cdot Z_{\tilde{\Gamma}} \\
\Gamma_{\tilde{G}_3} &= \left\{ \begin{pmatrix} i^\nu & \frac{(1-i^\nu)(1+i)}{2} \\ 0 & 1 \end{pmatrix} \right\} \cdot Z_{\tilde{\Gamma}},
\end{aligned}$$

where in the last case we have to take into account that $\frac{(1-i^\nu)(1+i)}{2} \in \mathbb{Z}[i]$ for all ν.

Hence modulo the centre $Z_{\tilde{\Gamma}}$ these stabilizers are cyclic groups of order $4, 2, 4$.

The arcs D_i are also pointwise fixed under the action of certain cyclic groups, namely

$$\begin{aligned}
D_1 &= \mathrm{Fix}\left(\begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix} \right) \\
D_2 &= \mathrm{Fix}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \\
D_3 &= \mathrm{Fix}\left(\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \right),
\end{aligned}$$

and we check easily that these arcs are geodesics joining the following points in the boundary

$$\begin{aligned}
&D_1 \text{ runs from } \sqrt{i} \text{ to } -\sqrt{i}\\
&D_2 \text{ runs from } i \text{ to } -i\\
&D_3 \text{ runs from } e = e^{\frac{1\pi i}{6}} = e^{\frac{\pi i}{3}} \text{ to } \overline{\rho}.
\end{aligned}$$

The corresponding tori are

$$\begin{aligned}
T_1 =& \mathrm{Stab}(-1,1) = \left\{\begin{pmatrix}\alpha & i\beta\\ \beta & \alpha\end{pmatrix}\right\}\\
T_2 =& \mathrm{Stab}(-\sqrt{i},\sqrt{i}) = \left\{\begin{pmatrix}\alpha & \beta\\ \beta & \alpha\end{pmatrix}\right\}\\
T_3 =& \mathrm{Stab}(\rho,\overline{\rho}) = \left\{\begin{pmatrix}\delta-\beta & \beta\\ -\beta & \delta\end{pmatrix}\right\}.
\end{aligned}$$

The torus T_2 splits over $\mathbb{Q}(i)$, the other two tori split over an quadratic extension of $\mathbb{Q}(i)$.

Now it is not difficult anymore to describe the finite stabilizers and the corresponding fixed point sets. If $x \in \mathbb{H}_3$ for which the stabilizer is bigger than $Z_{\tilde{\Gamma}}$, then we can conjugate x into F_0. It is very easy to see that x cannot lie in the interior of F_0 because then we would get an identification of two points nearby x and hence still in F_0 under $\tilde{\Gamma}$.

If x is on one of the lines D_1, D_2, D_3 or on one of the arcs G_1, G_2, G_3 but not on the intersection of two of them, then the stabilizer Γ_x is equal to $Z_{\tilde{\Gamma}}$ times the cyclic group we attached to the line or the arc earlier. Finally we are left with the three special points

$$\begin{aligned}
x_{12} =& D_1 \cap D_2 \cap G_1 = \{(0,1)\}\\
x_{13} =& D_1 \cap D_3 \cap G_3 = \left\{\left(\frac{1+i}{2}, \frac{\sqrt{2}}{2}\right)\right\}\\
x_{23} =& D_2 \cap D_3 \cap G_2 = \left\{\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)\right\}.
\end{aligned}$$

In this case it is clear that the stabilizers are given by

$$\begin{aligned}
\tilde{\Gamma}_{x_{12}} =& \langle\begin{pmatrix}0 & i\\ 1 & 0\end{pmatrix}, \begin{pmatrix}i & 0\\ 0 & 1\end{pmatrix}\rangle = D_4\\
\tilde{\Gamma}_{x_{13}} =& \langle\begin{pmatrix}0 & i\\ 1 & 0\end{pmatrix}, \begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}i & 1\\ 0 & 1\end{pmatrix}\rangle = S_4\\
\tilde{\Gamma}_{x_{23}} =& \langle\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}\rangle = S_3.
\end{aligned}$$

2.2 Compactification of $\Gamma\backslash X$

Our two special low dimensional examples show clearly that the quotient spaces $\Gamma\backslash X$ are not compact in general. There exist various constructions to compactify them.

If, for instance, $\Gamma \subset \mathrm{Sl}_2(\mathbb{Z})$ is a subgroup of finite index, then the quotient $\Gamma\backslash\mathbb{H}$ is a Riemann surface. It can be embedded into a compact Riemann surface by adding a finite number of points. this is a special case of a more general theorem of Satake and Baily-Borel: If the symmetric space X is actually hermitian symmetric (this means it has a complex structure) then we have the structure of a quasi-projective variety on $\Gamma\backslash X$. This is the so-called Baily-Borel compactification. It exists only under special circumstances.

I will discuss the process of compactification in some more detail for our special low dimensional examples.

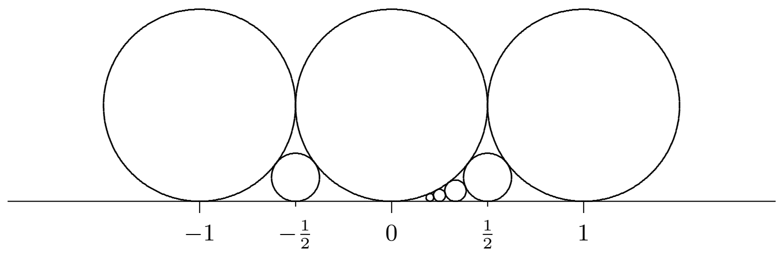
2.2.1 Compactification of $\mathrm{Sl}_2(\mathbb{Z})\backslash\mathbb{H}$ by adding points

Let $\Gamma \subset \mathrm{Sl}_2(\mathbb{Z})$ be any subgroup of finite index. The group Γ acts on the rational projective line $\mathbb{P}^1(\mathbb{Q})$. We add it to the upper half plane and form

$$\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q}),$$

and we extend the action of Γ to this space. Since the full group $\mathrm{Sl}_2(\mathbb{Z})$ acts transitively on $\mathbb{P}^1(\mathbb{Q})$ we find that Γ has only finitely many orbits on $\mathbb{P}^1(\mathbb{Q})$.

Now we introduce a topology on $\overline{\mathbb{H}}$. We defined a system of neighborhoods of points $\frac{p}{q} = r \in \mathbb{P}^1(\mathbb{Q})$. We define the Farey circles $S\left(c, \frac{p}{q}\right)$ which touch the real axis in the point $r = p/q$ $(p,q) = 1$ and have the radius $\frac{c}{2q^2}$. For $c = 1$ we get the picture



Let us denote by $D\left(c, \frac{p}{q}\right) = \cup_{c':0<c'\leq c} S\left(c', \frac{p}{q}\right)$ the Farey disks. For $c \to 0$ these Farey disks $D\left(c, \frac{p}{q}\right)$ define a system of neighborhoods of the point $r = p/q$. The Farey disks at $\infty \in \mathbb{P}^1(\mathbb{Q})$ are given by the regions

$$D(T, \infty) = \{z \mid \Im(z) \geq T\}.$$

It is easy to check that an element $\gamma \in \mathrm{Sl}_2(\mathbb{Z})$ which sends $\infty \in \mathbb{P}^1(\mathbb{Q})$ into the point $r = \frac{p}{q}$ sends $D(T,\infty)$ to $D\left(\frac{1}{T}, \frac{p}{q}\right)$. These Farey disks $D(c, r)$ do not meet provided we take $c < 1$. The considerations in 1.6.1 imply that the complement of the union of Farey disks is relatively compact modulo Γ, and since Γ has finitely many orbits on $\mathbb{P}^1(\mathbb{Q})$, we see easily that

$$Y_\Gamma = \Gamma\backslash\overline{\mathbb{H}}$$

is compact (which means of course also Hausdorff).

It is essential that the set of Farey circles $D(c, r)$ and $D\left(\frac{1}{c}, \infty\right)$ is invariant under the action of Γ on the one hand and decomposes into several connected components (which are labeled by the point $r \in \mathbb{P}^1(\mathbb{Q})$) on the other hand. Hence

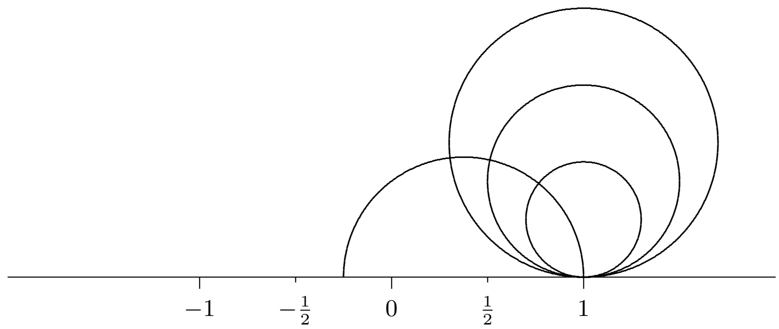
$$\Gamma\backslash \bigcup_r D(c,r) = \bigcup \Gamma_{r_i}\backslash D(c,r_i)$$

where r_i is a set of representatives for the action of Γ on $\mathbb{P}^1(\mathbb{Q})$ and where Γ_{r_i} is the stabilizer of r_i in Γ.

It is now clear that $\Gamma_{r_i}\backslash D(c,r_i)$ is holomorphically equivalent to a punctured disc and hence the above compactification is obtained by filling the point into this punctured disc and this makes it clear that Y_Γ is a Riemann surface.

2.2.2 The Borel-Serre compactification of $\mathrm{Sl}_2(\mathbb{Z})\backslash\mathbb{H}$

There is another construction of a compactification. We look at the disks $D(c,r)$ and divide them by the action of Γ_r. For any point $y \in S(c',r) - \{r\}$ there exists a unique geodesic joining r and y, passing orthogonally through $S(c',r)$ and hitting the projective line in another point y_∞ ($= -1/4$ in the picture below)



If $r = \infty$, then this system of geodesics is given by the vertical lines $\{y \cdot I + x \mid x \in \mathbb{R}\}$.. This allows us to write the set

$$D(c,r) - \{r\} = X_{\infty,r} \times [c,0)$$

where $X_{\infty,r} = \mathbb{P}^1(\mathbb{R}) - \{r\}$. The stabilizer Γ_r acts $D(c,r)$ and on the right hand side of the identification it acts on the first factor, the quotient $\Gamma_r\backslash X_{\infty,r}$ is a circle. Hence we can compactify the quotient

$$\Gamma_r\backslash D(c,r) - \{r\} \hookrightarrow \Gamma_r\backslash X_{\infty,r} \times [c,0].$$

This gives us a second way to compactify $\Gamma\backslash\mathbb{H}$, we apply this process to a finite set of representatives of $\mathbb{P}^1(\mathbb{Q}) \mod \Gamma$.

There is a slightly different way of looking at this. We may form the union

$$\mathbb{H} \cup \bigcup_r X_{\infty,r} = \tilde{\mathbb{H}}$$

and topologize it in such a way that

$$D(c,r) = X_{\infty,r} \times [c,0) \subset X_{\infty,r} \times [c,0]$$

is a local homeomorphism. Then we see that the compactification above is just the quotient

$$\Gamma\backslash\tilde{\mathbb{H}}.$$

This compactification is called the Borel-Serre compactification. Its relation to the Baily-Borel is such that the latter is obtained by the former by collapsing the circles at infinity to a point.

It is quite clear that a similar construction applies to the action of a group $\Gamma \subset \mathrm{Sl}_2(\mathbb{Z}[i])$ on the three-dimensional hyperbolic space. The Farey circles will be substituted by spheres $S(c,\alpha)$ which touch the complex plane $\{(z,0) \mid z \in \mathbb{C}\} \subset \overline{\mathbb{H}}_3$ in the point $(\alpha, 0), \alpha \in \mathbb{P}^1(\mathbb{Q}(i))$ and for $\alpha = \infty$ the Farey sphere is the horizontal plane $S(\infty, \zeta_0) = \{(z, \zeta_0) \mid z \in \mathbb{C}\}$. An element $\gamma \in \Gamma$ which maps $(0, \infty)$ to α maps $S(\infty, \zeta_0)$ to $S(c, \alpha)$, where $c = 1/\zeta_0$. For a given α we may identify the different spheres if we vary c and for any point $\alpha \in \mathbb{P}^1(\mathbb{Q}(i))$ we define $X_{\infty,\alpha} = \mathbb{P}^1(\mathbb{C}) \setminus \{\alpha\}$. Again we can identify

$$D(c,\alpha) \setminus \{\alpha\} = X_{\infty,\alpha} \times (0, c] \subset \overline{D(c,\alpha) \setminus \{\alpha\}} = X_{\infty,\alpha} \times [0, c]$$

The stabilizer Γ_α acts on $D(c,\alpha) \setminus \{\alpha\}$ and again this yields an action on the first factor. If we choose $\alpha = \infty$ then

$$\Gamma_\infty = \{\begin{pmatrix} \zeta & a \\ 0 & \zeta^{-1} \end{pmatrix} | \zeta \text{ root of unity}, a \in M_\infty\}$$

where M_∞ is a free rank 2 module in $\mathbb{Z}[i]$. If ζ does not assume the value i then $\Gamma_\infty\backslash X_{\infty,\infty}$ is a two-dimensional torus, a product of two circles. If ζ assumes the value i then $\Gamma_\infty\backslash X_{\infty,\infty}$ is a two dimensional sphere. If course we get the same result for an arbitrary α.

Then we get an action of the group Γ on $\tilde{\mathbb{H}}_3 = \mathbb{H}_3 \cup \bigcup_{\alpha \in \mathbb{P}^1(K)} \overline{D(c,\alpha) \setminus \{\alpha\}}$ and the quotient is compact.

The the set of orbits of Γ on $\mathbb{P}^1(\mathbb{Q}(i))$ is finite, these orbits are called the cusps.

2.2.3 The Borel-Serre compactification, reduction theory of arithmetic groups

The Borel-Serre compactification works in complete generality for any semi-simple or reductive group $G/\mathbb{Q}$. To explain it, we need the notion of a parabolic subgroup of $G/\mathbb{Q}$.

A subgroup $P/\mathbb{Q} \hookrightarrow G/\mathbb{Q}$ is parabolic if the quotient variety in the sense of algebraic geometry is a projective variety. We mentioned already earlier that for the group $\mathrm{Gl}_2/\mathbb{Q}$ we have an action of Gl_2 on the projective line $\mathbb{P}^1$ and the stabilizers B_x of the points $x \in \mathbb{P}^1(\mathbb{Q})$ are the so-called Borel subgroups of $\mathrm{Gl}_2/\mathbb{Q}$. They are maximal solvable subgroups and

$$\mathrm{Gl}_2/B_x = \mathbb{P}^1,$$

hence they are also parabolic.

More generally we get parabolic subgroups of $\mathrm{Gl}_n/\mathbb{Q}$, if we choose a flag on the vector space $V = \mathbb{Q}^n = \mathbb{Q}e_1 \oplus \cdots \oplus \mathbb{Q}e_n$. This is an increasing sequence of subspaces

$$\mathcal{F} : (0) = \{(0)\} = V_0 \subset V_1 \subset V_2 \ldots V_k = V.$$

The stabilizer P of such a flag is always a parabolic subgroup; the quotient space

$$G/P = \text{Variety of all flags of the given type},$$

where the type of the flag is the sequence of the dimensions $n_i = \dim V_i$.

These flag varieties (the Grassmannians) are smooth projective schemes over $\mathrm{Spec}(\mathbb{Z})$ and this implies that any flag $\mathcal{F}$ is induced by a flag

$$\mathcal{F}_{\mathbb{Z}} : (0) = \{(0)\} = L_0 \subset L_1 \subset L_2 \subset \ldots \subset L_k = L = \mathbb{Z}^n \tag{15}$$

where $L_i = V_i \cap L$, and of course $L_i \otimes \mathbb{Q} = V_i$, this is the elementary fact which we will use later.

If our group $G/\mathbb{Q}$ is the orthogonal group of a quadratic form

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} a_i x_i^2$$

with $a_i \in K^*$. Then we have to replace the flags by sequences of subspaces

$$\mathcal{F} : 0 \subset W_1 \subset W_2 \ldots \qquad\qquad \subset W_2^{\perp} \subset W_1^{\perp} \subset V,$$

where the W_i are isotropic spaces for the form f, i.e. $f \mid W_i \equiv 0$, and where the $W_i^{\perp}$ are the orthogonal complements of the subspaces. Again the stabilizers of these flags are the parabolic subgroups defined over $\mathbb{Q}$.

Especially, if the form f is anisotropic over $\mathbb{Q}$, i.e. there is no non-zero vector $\underline{x} \in K^n$ with $f(\underline{x}) = 0$, then the group $G/\mathbb{Q}$ does not have any parabolic subgroup over $\mathbb{Q}$. This equivalent to the fact that $G(\mathbb{Q})$ does not have unipotent elements.

These parabolic subgroups always have a unipotent radical U_P which is always the subgroup which acts trivially on the successive quotients of the flag. The unipotent radical is a normal subgroup, the quotient $P/U_P = M$ is a reductive group again, it is called the Levi-quotient of P.

We stick to the group $\mathrm{Gl}_n/\mathbb{Q}$. It contains the standard maximal torus whose R valued points are

$$T_0(R) = \{ \begin{pmatrix} t_1 & 0 & \ldots & 0 \\ 0 & t_2 & \ldots & 0 \\ & & \ddots & \\ 0 & 0 & & 0 \\ 0 & 0 & 0 & t_n \end{pmatrix} \quad \mid \quad t_i \in R^{\times}, \prod t_i = 1\} \tag{16}$$

It is a subgroup of the Borel subgroup (maximal solvable subgroup or minimal parabolic subgroup)

$$B_0(R) = \{ \begin{pmatrix} t_1 & u_{1,2} & \ldots & u_{1,n} \\ 0 & t_2 & \ldots & u_{2,n} \\ & & \ddots & \\ 0 & 0 & & u_{n-1,n} \\ 0 & 0 & 0 & t_n \end{pmatrix} \quad \mid \quad t_i \in R^{\times}, \prod t_i = 1\} \tag{17}$$

and its unipotent radical U_0 consists of those $b \in B_0$ where all the $t_i = 1$. This unipotent radical contains the one dimensional root subgroups

$$U_{i,j} = \{ \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & \ddots & x & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \} \tag{18}$$

where $i < j$, these one dimensional subgroups are isomorphic to the one dimensional additive group $\mathbb{G}_a$. They are normalized by the torus, for an element $t \in T(R)$ and $x_{i,j} \in U_{i,j}(R) = R$ we have

$$tx_{i,j}t^{-1} = t_i/t_j x_{i,j}. \tag{19}$$

For $i = 1, \ldots, n, j = 1, \ldots, n, i \neq j$ (resp. $i < J$) characters $\alpha_{i,j}(t) = t_i/t_j$ are called the roots (resp. positive roots) of T_0 in Gl_n. We denote these systems of roots by Δ^{Gl_n} (resp)$\Delta_+^{\mathrm{Gl}_n}$. The one dimensional subgroups $U_{i,j}, i \neq j$ are called the root subgroups.

Inside the set of positive roots we have the set of simple roots

$$\pi = \pi^{\mathrm{Gl}_n} = \{\alpha_{1,2}, \ldots, \alpha_{i,i+1}, \ldots, \alpha_{n-1,n}\} \tag{20}$$

The Borel subgroup B_0 is the stabilizer of the "complete" flag

$$\{0\} \subset \mathbb{Q}e_1 \subset \mathbb{Q}e_1 \oplus \mathbb{Q}e_2 \subset \cdots \subset \mathbb{Q}e_1 \oplus \mathbb{Q}e_2 \oplus \cdots \oplus \mathbb{Q}e_n, \tag{21}$$

the parabolic subgroups $P_0 \supset B_0$ are the stabilizers of "partial" flags

$$\{0\} \subset \mathbb{Q}e_1 \oplus \cdots \oplus \mathbb{Q}e_{n_1} \subset \mathbb{Q}e_1 \oplus \cdots \oplus \mathbb{Q}e_{n_1} \oplus \mathbb{Q}e_{n_1+1} \oplus \cdots \oplus \mathbb{Q}e_{n_1+n_2} \subset \cdots \subset \mathbb{Q}^n. \tag{22}$$

The parabolic subgroup P_0 also acts on the direct sum of the successive quotients

$$\mathbb{Q}e_1 \oplus \cdots \oplus \mathbb{Q}e_{n_1} \oplus \mathbb{Q}e_{n_1+1} \oplus \cdots \oplus \mathbb{Q}e_{n_1+n_2} \oplus \ldots \tag{23}$$

and this yields a homomorphism

$$r_{P_0} : P_0 \to M_0 = \mathrm{Gl}_{n_1} \times \mathrm{Gl}_{n_2} \times \ldots \tag{24}$$

hence M_0 is the Levi quotient of P_0. By definition the unipotent radical U_{P_0} of P_0 is the kernel of r_0.

A parabolic subgroups $P_0 \supset B_0$ defines a subset

$$\Delta^{P_0} = \{\alpha_{i,j} \in \Delta^{\mathrm{Gl}_n} \mid U_{i,j} \subset P_0\}$$

and the set decomposes int two sets

$$\Delta^{U_{P_0}} = \{\alpha_{i,j} \mid U_{i,j} \subset \Delta^{U_{P_0}}\}, \Delta^{M_0} = \{\alpha_{i,j} \mid U_{i,j}, U_{j,i} \subset \Delta^{P_0}\} \tag{25}$$

Intersecting this decomposition with the set π^{Gl_n} yields a disjoint decomposition

$$\pi^{\mathrm{Gl}_n} = \pi^{M_0} \cup \pi^U \tag{26}$$

where $\pi^U = \{\alpha_{n_1.n_1+1}, \alpha_{n_1+n_2,n_1+n_2+1}, \ldots, \}$. In turn any such decomposition of π^{Gl_n} yields a well defined parabolic $P_0 \supset B_0$.

If we choose another maximal split torus T_1 and a Borel subgroup $B_1 \supset T_1$ then this amounts to the choice of a second ordered basis $v_1, v_2, \ldots, v_n$ the v_i are given up to a non zero scalar factor. We can find a $g \in \mathrm{Gl}_n(\mathbb{Q})$ which maps $e_1, e_2, \ldots, e_n$ to $v_1, v_2, \ldots, v_n$, and hence we can conjugate the pair (B_0, T_0) to (B_1, T_1) and hence the parabolic subgroups containing B_0 into the parabolic subgroups containing B_1. The conjugating element g also identifies

$$i_{T_0,B_0,T_1,B_1} : X^*(T_0) \xrightarrow{\sim} X^*(T_1)$$

and this identification does not depend on the choice of the conjugating element g. This allows us to identify the two set of positive simple roots

$\pi^{\mathrm{Gl}_n} \subset X^*(T_0)$ and $\pi \subset X^*(T)$. Eventually we can speak of the set π of simple roots of Gl_n. Hence we have the fundamental fact

The $Gl_n(\mathbb{Q})$ conjugacy classes of parabolic subgroups $P/\mathbb{Q}$ are in one to one correspondence with the subsets $\pi' \subset \pi$ where π' is the set of those simple roots $\alpha_{i,i+1}$ for which $U_{i,i+1} \subset U_P$, the unipotent radical of P. Then number of elements in π is called the rank of P, the set π is called the type of P.

We formulated this result for $\mathrm{Gl}_n/\mathbb{Q}$ but we can replace $\mathbb{Q}$ by any field k and Gl_n by any reductive group G/k. We have to define the system of relative simple positive roots π^G for any G/k (See [B-T]).

The group G/k itself is also a parabolic subgroup it corresponds to $\pi' = \emptyset$. We decide that we do not like it and we consider only proper parabolic subgroups $P \neq G$, i.e. $\pi' \neq \emptyset$. We can define the Grassmann variety $\mathrm{Gr}^{[\pi']}$ of parabolic subgroups of type π. This is a smooth projective variety and $\mathrm{Gr}^{[\pi']}(\mathbb{Q})$ is the set of parabolic subgroups of type π.

There is always a unique minimal conjugacy class it corresponds to $\pi = \pi^G$. (In our examples this minimal class is given by the maximal flags, i.e. those flags where the dimension of the subspaces increases by one at each step (until we reach a maximal isotropic space in the case of an orthogonal group)). The (proper) maximal parabolic subgroups are those for which $\pi = \{\alpha_{i,i+1}\}$, i.e. π consist of one element.

We go back to the special case $\mathrm{Gl}_n/\mathbb{Q}$, the following results are true in general but their formulation is just a little bit more involved.

For a maximal parabolic subgroup we consider the module $\mathrm{Hom}(P, \mathbb{G}_m) \otimes \mathbb{Q} \subset X^*(T) \otimes \mathbb{Q}$. Of course it always contains the determinant. For a maximal parabolic subgroup $P/\mathbb{Q}$ of type $\{\alpha_{i,i+1}\}$ we have

$$\mathrm{Hom}(P, \mathbb{G}_m) \otimes \mathbb{Q} = \mathbb{Q}\gamma_i \oplus \mathbb{Q}\det$$

where γ_i is

$$\gamma_i(t) = (\prod_{\nu=1}^{\nu=i} t_\nu) \det(t)^{-i/n}. \tag{27}$$

These γ_i are the dominant fundamental weights.

If our maximal parabolic subgroup is $P/\mathbb{Q}$ is defined as the stabilizer of a flag $0 \subset W \subset V = \mathbb{Q}^n$, then the unipotent radical is $U = \mathrm{Hom}(V/W, W)$. An element $y \in P(\mathbb{Q})$ induces linear maps $y_W, y_{V/W}$ and hence $\mathrm{Ad}(y)$ on $U = \mathrm{Hom}(V/W, W)$. We get a character $\gamma_P(y) = \det(\mathrm{Ad}(y)) \in \mathrm{Hom}(P, \mathbb{G}_m)$ which is called the sum of the positive roots. An easy computation shows that

$$\gamma_i^n = \gamma_P \tag{28}$$

We add points at infinity to our symmetric space: We consider the disjoint union $\cup_{\pi \neq \pi_G} \mathrm{Gr}^{[\pi']}(\mathbb{Q})$ and form the space

$$\overline{X} = X \cup \bigcup_{\pi' \neq \emptyset} \mathrm{Gr}^{[\pi']}(\mathbb{Q}).$$

This is the analogue of or $\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$ in our first example, it is now more complicated because we have several Grassmannians, and we also have maps

$$r_{\pi_1,\pi_2} \mathrm{Gr}^{[\pi_1]}(\mathbb{Q}) \to \mathrm{Gr}^{[\pi_2]}(\mathbb{Q}) \text{ if } \pi_2 \subset \pi_1.$$

Our first aim is to put a topology on this space such that $\Gamma\backslash\overline{X}$ becomes a compact Hausdorff space.

In our first example we interpreted the Farey circle $D\left(c, \frac{p}{q}\right)$ with $0 < c < 1$ as an open subset of points in $\mathbb{H}$, which are close to the point $\frac{p}{q}$, but far away from any other point in $\mathbb{P}^1(\mathbb{Q})$.

The point of reduction theory is that for any parabolic $P \in \mathrm{Gr}^{[\pi']}(\mathbb{Q})$ (here we also allow $P = G$) we define open sets

$$X^P(\underline{c}_{\pi'}, r(\underline{c}_{\pi'})) \subset X \tag{29}$$

which depend on certain parameters $\underline{c}_P, r(\underline{c}_P))$ The points in $X^P(\underline{c}_{\pi'}, r(\underline{c}_{\pi'}))$ should be viewed as the points, which are "very close" to the parabolic subgroup P (controlled by $\underline{c}_{\pi'}$ but "keep a certain distance" (controlled by $r(\underline{c}_{\pi'})$) to the parabolic subgroups $Q \not\supset P$. They are the analogues of the Farey circles. We will see:

a)This system of open sets is invariant under the action $\mathrm{Gl}_n(\mathbb{Z})$

b) For $P = G$ the set $X^G(\emptyset, r_0)$ is relatively compact modulo the action of $\mathrm{Gl}_n(\mathbb{Z})$.

c) Any subgroup $\Gamma \subset \mathrm{Gl}_n(\mathbb{Z})$ has only finitely many orbits on any $\mathrm{Gr}^{[\pi']}(\mathbb{Q})$

d) For a suitable choice of the the parameters $\underline{c}_{\pi'}$, and $r(\underline{c}'_\pi)$ we have :

$$X = \bigcup_P X^P(\underline{c}_{\pi'}, r_{\underline{c}'_\pi}) = X^G(\emptyset, r_0) \cup \bigcup_{P:P\text{proper}} X^P(\underline{c}_{\pi'}, r(\underline{c}_{\pi'}))$$

and if P and P_1 are conjugate and $P \neq P_1$ then $X^P(\underline{c}'_\pi, r(\underline{c}_{\pi'})) \cap X^{P_1}(\underline{c}_{\pi'}, r(\underline{c}_{\pi'})) = \emptyset$.

Let us assume that we have constructed such a system of open sets, then c) and d) impliy that for a given type π' we have

$$\Gamma\backslash \bigcup_{P:\mathrm{type}(\pi')=\pi} X^P(\underline{c}'_\pi, r(\underline{c}_{\pi'})) = \bigcup \Gamma_{P_i}\backslash X^{P_i}(\underline{c}'_\pi, r(\underline{c}_{\pi'}))$$

where $\{\dots, P_i, \dots\} = \Sigma(\pi, \Gamma)$ is a set of representatives of $\mathrm{Gr}^{[\pi']}(\mathbb{Q})$ modulo the action of Γ and $\Gamma_{P_i} = \Gamma \cap P_i(\mathbb{Q})$.

This tells us that we have a covering

$$\Gamma\backslash X = \Gamma\backslash X^G(\emptyset, r_0) \cup \bigcup_{\pi'\neq\emptyset} \bigcup_{P\in\Sigma(\pi',\Gamma)} \Gamma_P\backslash X^P(\underline{c}_{\pi'}, r(\underline{c}_{\pi'})) \tag{30}$$

The essential points of the philosophy of reduction theory are that $\Gamma\backslash X^G(\emptyset, r_0)$ is relatively compact and that we have an explicit description of the sets $\Gamma_P\backslash X^P(\underline{c}'_\pi, r(\underline{c}_{\pi'}))$ as fiber bundles with nil manifolds as fiber over the locally symmetric spaces $\Gamma_M\backslash X^M$.

We give the definition of the sets $X^P(\underline{c}'_\pi, r(\underline{c}'_\pi)$. We stick to the case that $G = \mathrm{Gl}_n/\mathbb{Q}$ and $\Gamma \subset \Gamma_0 = \mathrm{Gl}_n(\mathbb{Z})$. is a (congruence) subgroup of finite index. We defined the positive definite bilinear form (See 11)

$$\tilde{B}_{\Theta_x} = -\frac{1}{2n} B_{\Theta_x} : \mathfrak{g}_\mathbb{R} \times \mathfrak{g}_\mathbb{R} \to \mathbb{R}$$

and we have the identification $\mathfrak{g}_{\mathbb{R}} \xrightarrow{\sim} T_e^{\mathbb{G}(\mathbb{R})}$, and hence we get a euclidian metric on the tangent space $T_e^{\mathbb{G}(\mathbb{R})}$ at the identity e. This extends to a left invariant Riemannian metric on $G(\mathbb{R})$, we denote it by $d_{\Theta_x} s^2$. Hence we get a volume form $d^{\Theta_x}_{\mathrm{vol_H}}$ on any closed subgroup $H(\mathbb{R}) \subset G(\mathbb{R})$.

For any point $x \in X$ and any parabolic subgroup $P/\mathbb{Q}$ with unipotent radical $U/\mathbb{Q}$) we define

$$p_P(x,P) = \mathrm{vol_U}^{\Theta_x}(\Gamma_0 \cap U(\mathbb{R}))\backslash U(\mathbb{R})) \tag{31}$$

For the Arakelow-Chevalley scheme $(\mathrm{Gl}_n/\mathbb{Z}, \Theta_0)$ See(1.4.2) we have that $\tilde{B}_{\Theta_0}(E_{i,j}) = 1$. We have by construction

$$U_{i,j}(\mathbb{Z})\backslash U_{i,j}(\mathbb{R}) = \mathbb{R}/\mathbb{Z} \tag{32}$$

and under this identification $E_{i,j}$ maps to $\frac{\partial}{\partial x}$. Hence we get

$$d^{\Theta_0}_{\mathrm{vol_{U_{i,j}}}}(U_{i,j}(\mathbb{Z})\backslash U_{i,j}(\mathbb{R})) = 1$$

and from this we get immediately

Proposition 2.1. *For any parabolic subgroup P_0 containing the torus T_0 we have*

$$p_P(\Theta_0, P) = 1.$$

Let $(L, <\ ,\ >_x)$ be an Arakelow vector bundle and $(\mathrm{Gl}_n, \Theta_x)$ the corresponding Arakelow group scheme (of type Gl_n) let

$$\mathcal{F}_{\mathbb{Z}} : (0) = \{(0)\} = L_0 \subset L_1 \subset L_2 \subset \ldots \subset L_k = L = \mathbb{Z}^n$$

be a flag and $P/\mathbb{Z}$ the corresponding parabolic subgroup. Then we have the homomorphism

$$r_P : P/\ \mathrm{Spec}(\mathbb{Z}) \to M/\mathbb{Z} = \prod_{i=1}^{i=k} \mathrm{Gl}(L_i/L_{i-1}) \tag{33}$$

with kernel $U_P/\mathbb{Z}$. The metric $<\ ,\ >_x$ on $L \otimes \mathbb{R}$ yields an orthogonal decomposition

$$L \otimes \mathbb{R} = \bigoplus_{i=1}^{i=k} L_i/L_{i-1} \otimes \mathbb{R}$$

and hence an Arakelow bundle structure $(L_i/L_{i-1}, (\Theta_x)_i)$ for all i, and therefore an Arakelow group scheme structure on $M/\mathbb{Z}$.

Hence we get

Proposition 2.2. *If (Gl_n, Θ) is an Arakelow group scheme then Θ induces an Arakelow group scheme structure Θ^M on any reductive quotient $M = P/U$.*

Definition : A pair $(\mathrm{Gl}_n/\mathbb{Z}, \Theta)$ is called stable (resp. semi stable) if for any proper parabolic subgroup $P/\mathbb{Q} \subset \mathrm{Gl}_n/\mathbb{Q}$ we have

$$p_P(\Theta, P) > 1 \tag{34}$$

In our example in (2.2.1) the stable points are those outside the union of the closed Farey circles.

To get a better understanding of these numbers we have to do some computations with roots and weights. Let us start from an Arakelow vector bundle $(L=\mathbb{Z}^d,<\ ,\ >)$ and let us assume that L is equipped with a complete flag

$$\mathcal{F}_0=\{0)\}=L_0\subset L_1\subset\cdots\subset L_{d-1}\subset L_d \tag{35}$$

which defines a Borel subgroup $B/\mathbb{Z}$. The quotients $(L_i/L_{i-1},<\ ,\ >_i)$ are Arakelow line bundles over $\mathbb{Z}$ or in a less sophisticated language they are free modules of rank one and the generating vector $\bar{e}_i$ has a length $\sqrt{<\bar{e}_i,\bar{e}_i>_i}$. This length is of course also the volume of $(L_i/L_{i-1}\otimes\mathbb{R})/(L_i/L_{i-1})$.

The unipotent radical $U/\mathbb{Z}\subset B/\mathbb{Z}$ has a filtration $\{(0)\}\subset V_1\subset\ldots,V_{n(n-1)/2-1}\subset V_{n(n-1)/2}=U$ by normal subgroups, the successive quotients are isomorphic to $\mathbb{G}_a$ and the torus $T=B/U$ acts by a positive root $\alpha_{i,j}$ and this is a one to one correspondence between the subquotients and the positive roots. Then it is clear: If ν corresponds to (i,j) then

$$(V_\nu/V_{\nu+1},\Theta_\nu)=(L_i/L_{i-1},<\ ,\ >_i)\otimes(L_j/L_{j-1},<\ ,\ >_j)^{-1}. \tag{36}$$

Moreover the quotients $(V_\nu/V_{\nu+1},\Theta_\nu)$ depend only on the conformal class of $<\ ,\ >$ and hence only on the resulting Cartan involution Θ.

The unipotent subgroup $U/\mathbb{Z}$ contains the one parameter subgroup $U_{i,j}/Z$ and this one parameter subgroup maps isomorphically to $(V_\nu/V_{\nu+1})$. Hence our construction defines the Arakelow line bundle $(U_{i,j},\Theta_{i,j})$.

If we now define $n_{\alpha_{i,j}}(x,B)=\ \mathrm{vol}_{\Theta_{i,j}}(U_{i,j}(\mathbb{R})/U_{i,j}(\mathbb{Z}))$ then it is clear that

$$p_B(x,B)=\prod_{i<j}n_{\alpha_{i,j}}(x,B) \tag{37}$$

If $P\supset B$ then its unipotent radical $U_P\subset U$ and we defined the set Δ^{U_P} as the set of positive roots for which $U_{i,j}\subset U_P$. Then we have

$$p_P(x,P)=\prod_{(i,j)\in\Delta^{U_P}}n_{\alpha_{i,j}}(x,B) \tag{38}$$

We follow a convention and put $2\rho_P=\sum_{(i,j)\in\Delta^{U_P}}\alpha_{i,j}$ so that ρ_P is the half sum of positive roots in in the unipotent radical. This character is equal to γ_P in formula (28) and hence we know for any maximal parabolic subgroup P_{i_0}

$$2\rho_{P_{i_0}}=\sum_{i\le i_0,j\ge i_0+1}\alpha_{i,j}=n\gamma_{i_0} \tag{39}$$

Since the numbers $n_{\alpha_{i,j}}(x,B)$ are positive real numbers we can define for any $\gamma=\sum x_i\alpha_{i,i+1}\in X^*(T)\otimes\mathbb{R}$

$$n_\gamma(x,B)=\prod_{i=1}^{n-1}n_{\alpha_{i,j}}(x,B)^{x_i}. \tag{40}$$

then we get

$$p_P(x,P)=\prod_{\nu=1}^{s}p_{P_{i_\nu}}(x,P_{i_\nu})^{r'_\nu} \tag{41}$$

where the $r'_\nu > 0$. This implies that $p_P(x, P) > 1$ if all $p_{P_{i_\nu}}(x, P_{i_\nu}) > 1$. If our parabolic subgroup P is not maximal, and $P = P_{i_1} \cap P_{i_2} \cdots \cap P_{i_s}$ then a simple computation shows that

$$\gamma_P = r_1\gamma_{i_1} + r_2\gamma_{i_2} + \cdots + r_s\gamma_{i_s}. \tag{42}$$

where the r_i are strictly positive rational numbers. Then we get

$$p_P(x, P) = n_{\gamma_P}(x, B_1)$$

where $B_1 \subset P$ is any Borel subgroup in P. The formula (42) implies

The Arakelow scheme $(\mathrm{Gl}_n/\mathbb{Z}, \Theta)$ *is stable if for all maximal parabolic subgroups* $p_{P_i}(\Theta, P_i) = n_{\gamma_i}(\Theta, P_i)^n > 1$.

We need a few more formulas relating roots and weights. For any parabolic subgroup we have the division of the set of simple roots into two parts

$$\pi = \pi^M \cup \pi^{U_P}.$$

This induces a splitting of the character module

$$X^*(T) \otimes \mathbb{Q} = \bigoplus_{\alpha_{i,j} \in \pi^M} \mathbb{Q}\alpha_{i,i+1} \oplus \bigoplus_{\alpha_{i,j} \in \pi^{U_P}} \mathbb{Q}\gamma_i \tag{43}$$

where γ_i is the dominant fundamental weight attached to $\alpha_{i,i+1}$ (See (27)).

If now $\alpha_{i,i+1} \in \pi^{U_P}$ then we can project $\alpha_{i,i+1}$ to the second component, this projection

$$\alpha^P_{i,i+1} = \alpha_{i,i+1} + \sum_{\alpha_{\nu,\nu+1} \in \pi^M} c_{i,\nu}\alpha_{\nu,\nu+1} \tag{44}$$

Here an elementary - but not completely trivial - computation shows that

$$c_{i,\nu} \geq 0 \tag{45}$$

We now state the two fundamental theorems of reduction theory

Theorem 2.1. *For any Arakelow group scheme* (Gl_n, Θ) *we can find a Borel subgroup* $B \subset Gl_n$ *for which*

$$n_{\alpha_{i,i+1}}(\Theta, B) > \frac{\sqrt{3}}{2} \text{ for all } i = 1, \ldots, n-1$$

Theorem 2.2. *For any Arakelow group scheme* (Gl_n, Θ) *we can find a a unique parabolic subgroup such that for all* $\alpha_{i,i+1} \in \pi^{U_P}$ *we have*

$$n_{\alpha^P_{i,i+1}}(\Theta, P) < 1$$

and such that the reductive quotient (M, Θ^M) *is semi stable.*

The first theorem is due to Minkowski, the second theorem is proved in [Stu], [Gray].

This parabolic subgroup is called the canonical destabilizing group. If (G, x) is semi stable then $P = G$. We denote it by $P(x)$. This gives us a dissection of X into the subsets

$$X = \bigcup_{P\text{: parabolic subgroups of } G/\mathbb{Q}} X^{[P]} = \{x \in X \mid P(x) = P\} \tag{46}$$

Clearly $\gamma X^{[P]} = X^{\gamma P \gamma^{-1}]}$ If we divide by the group Γ the we get

$$\Gamma\backslash X = \bigcup_{P \in Par} \Gamma_P \backslash X^{[P]} \tag{47}$$

where $Par\ (\Gamma)$ is a set of representatives of Γ conjugacy classes of parabolic subgroups of $\mathrm{Gl}_n/\mathbb{Q}$. This is a decomposition of $\Gamma\backslash X$ into a disjoint union of subsets. The subset $\Gamma\backslash X^{[\mathrm{Gl}_n]}$ is compact, it is the set of semi stable pairs (x, Gl_n), the subsets $\Gamma_P\backslash X^{[P]}$ for $P \neq G$ are in a certain sense "open in some directions" and "closed in some other direction". Therefore this decomposition is not so useful for the study of cohomology groups. Do remedy this we introduce larger subsets. For a real number $r_0, 0 < r_0 < 1$ (but close to 1) we define

$$X^{\mathrm{Gl}_n}(r) = \{x \in X |\ n_{\alpha^{P(x)}}(x, \mathrm{Gl}_n, P(x)) > r, \text{ for all } \alpha \in \pi^{U_{P(x)}}).$$

It contains the set of semi-stable (x, Gl_n). Together with the first theorem this has a consequence

Proposition 2.3. *The quotient* $\Omega^{Gl_n}(r) = \Gamma\backslash X^{Gl_n}(r)$ *is relatively compact open subset of* $\Gamma\backslash X$, *It contains the set of semi-stable* (x, Gl_n).

If we choose $r < 1$ very close to one then the the elements in $\Omega^{\mathrm{Gl}_n}(r)$ are only a "little bit unstable".

We start from a parabolic subgroup P and choose vectors $\underline{c}_P = (\dots, c_\alpha, \dots)_{\alpha \in \pi^{U_P}}$ where all $0 < c_\alpha < 1$. Furthermore we choose a number $r(\underline{c}_P) < 1$ and define

$$X^P(\underline{c}_P, r(\underline{c}_P)) = \{x |\ n_{\alpha^P}(x, P) < c_\alpha \text{ for all } \alpha \in \pi^{U_P}; x^M \in X^M(r(\underline{c}_P))\} \tag{48}$$

Proposition 2.4. *For a given choice of* $\underline{c}_P$ *we can find a number* $r(\underline{c}_P) < 1$ *such that that for any* $x \in X^P(\underline{c}_P, r(\underline{c}_P))$ *the destabilizing parabolic subgroup* $P(x) \subset P$.

To see this we have to look at the canonical subgroup $\bar{Q} \subset (x_M, M)$. Its inverse image $Q \subset P$ is a parabolic subgroup of Gl_n. The reductive quotient $(x_{\bar{M}}, \bar{M})$ is stable. We want to show that Q is canonical parabolic of (x, Gl_n), i.e. we have to show that $n_{\alpha^Q}(x, \mathrm{Gl}_n, Q) > 0$ for all $\alpha \in \pi^{U_Q} = \pi^{U_P} \cup \pi^{M, U_{\bar{Q}}}$.

For $\in \pi^{M, U_{\bar{Q}}}$ this is true by definition. For $\alpha \in \pi^{U_P}$ we have

$$\alpha^P = \alpha + \sum_{\beta \in \pi^M} a_{\alpha,\beta}\beta \text{ and } \alpha^Q = \alpha + \sum_{\beta \in \pi^{\bar{M}}} a'_{\alpha,\beta}\beta,$$

where $a_{\alpha,\beta} \geq 0$. The roots $\beta \in \pi^{M, U_{\bar{Q}}}$ can be expressed in terms of the $\beta^{\bar{Q}} = \beta^Q$:

$$\beta^Q = \beta + \sum_{\beta' \in \pi^{\bar{M}}} a^*_{\beta,\beta'}\beta' \tag{49}$$

and hence

$$\alpha^Q = \alpha^P - \sum_{\beta \in \pi^{M, U_{\bar{Q}}}} a_{\alpha,\beta}\beta^Q + \sum_{\beta' \in \pi^{\bar{M}}} c_{\alpha\beta'}\beta'. \tag{50}$$

The last sum is zero because $\alpha^Q, \alpha^P, \beta^Q$ are orthogonal to the module $\oplus_{\beta'}\mathbb{Z}\beta'$. We choose a reduced Borel subgroup $\bar{\mathcal{B}} \subset \mathcal{M}$, let $\mathcal{B}$ be its inverse image in Q. We get the relation

$$n_{\alpha^Q}(x, \mathrm{Gl}_n, Q) = n_{\alpha^P}(x, \mathrm{Gl}_n, P) \cdot \prod_{\beta \in \pi^{M, U_{\bar{Q}}}} n_{\beta^Q}(x, \mathrm{Gl}_n, Q)^{-a_{\alpha,\beta}} \tag{51}$$

Now $n_{\alpha^P}(x, \mathrm{Gl}_n, P) < c_\alpha$ and $n_{\beta^Q}(x, \mathrm{Gl}_n, Q) > r(\underline{c}_P)$. If we choose $r(\underline{c}_P)$ close enough to one then it follows that $n_{\alpha^Q}(x, \mathrm{Gl}_n, Q) < 1$ and hence our claim.

We can choose a family of parameters

$$(\dots, \underline{c}_P, \dots)_{P\text{: parabolic over } \mathbb{Q}}, r(\underline{c}_P)$$

which only depend on the type of P and such that we get a covering

$$X = \bigcup_P X^P(\underline{c}_P, r(\underline{c}_P)))$$

and hence

$$\Gamma\backslash X = \bigcup_P \Gamma_P\backslash X^P(\underline{c}_P, r(\underline{c}_P)).$$

Here we have to start from $P = \mathrm{Gl}_n$, in this case $\pi^{U_P} = \emptyset$ and we choose a small positive number $r_0 < 1$ and put $L(\emptyset) = r_0$. Now the rest is clear. Therefore we now constructed the covering which satisfies the necessary requirements. Since the $\underline{c}_P, r(\underline{c}_P)$ only depend on the type of P, we change the notation: $\underline{c}_P \to \underline{c}'_\pi, r(\underline{c}_P) \to r(\underline{c}'_\pi)$.

We have a very explicit description of these sets $\Gamma_P\backslash X^P(\underline{c}'_\pi, r(\underline{c}'_\pi))$. We consider the evaluation map

$$\begin{gathered} n^{\pi'} : \Gamma_P\backslash X^P(\underline{c}'_\pi, r(\underline{c}'_\pi)) \to \prod_{\alpha\in\pi'}(0, c_\alpha) \\ x \mapsto (\dots, n_{\alpha^P}(x, P), \dots) \end{gathered} \tag{52}$$

Of course we also have the homomorphism

$$|\alpha^{\pi'}| : P(\mathbb{R}) \to \{\dots, |\alpha^P|, \dots\}_{\alpha\in\pi'} \tag{53}$$

and the multiplication by an element $y \in P(\mathbb{R})$ induces an isomorphisms of the fibers

$$(n_X^{[\pi']})^{-1}(c_1) \xrightarrow{\sim} (n_X^{[\pi']})^{-1}(c_2) \text{ if } |\alpha^{\pi'}|(y) \cdot c_1 = c_2$$

where the multiplication is taken componentwise. This identification depends on the choice of y.

To get a canonical identification we use the geodesic action which is introduced in the paper by Borel and Serre. We define an action of $A = (\prod_{\alpha\in\pi\pi'} \mathbb{R}^\times_{>0})$ on X. This action depends on P and we denote it by

$$(a, x) \mapsto a \bullet x$$

A point $x \in X$ defines a Cartan involution Θ_x and then the parabolic subgroup P^{Θ_x} of $G \times \mathbb{R}$ is opposite to $P \times \mathbb{R}$ and $P \times \mathbb{R} \cap P^{\Theta_x} = M_x$ is a Levi factor, the projection $P \to M$ induces an isomorphism

$$\phi_x : M \times \mathbb{R} \xrightarrow{\sim} M_x.$$

The character $\alpha^{\pi'}$ induces an isomorphism

$$s_x : A \xrightarrow{\sim} S_x(\mathbb{R})^{(0)}$$

where Hence we $S_x(\mathbb{R})^{(0)}$ is the connected component of the identity of the center $M_x(\mathbb{R}) \cap \mathrm{Sl}_n(\mathbb{R})$ and we put

$$a \bullet x = s_x(a)x$$

We have to verify that this is indeed an action. This is clear because for the Cartan-involution $\Theta_{a\bullet x}$ we obviously have

$$P^{\Theta_x} = P^{\Theta_{a\bullet x}}.$$

It is also clear that this action commutes with the action of $P(\mathbb{R})$ on X because

$$ys_x(a)x = s_{yx}(a)yx \text{ for all } y \in P(\mathbb{R}), x \in X.$$

It follows from the construction that the semigroup $A_- = \{\dots, a_\nu, \dots\}$- where $0 < a_\nu \leq 1$ - acts via the geodesic action on $X^P(c_\pi, r(\underline{c}_{\pi'}))$ and that for $a \in A_-$ we get an isomorphism

$$(n^{[\pi']})^{-1}(c_1) \xrightarrow{\sim} (n^{[\pi']})^{-1}(ac_1).$$

This yields a decomposition as product

$$X^P(c'_\pi, r(\underline{c}_{\pi'})) = (n^{[\pi']})^{-1}(c_0) \times \prod_{\alpha\in\pi'} (0, c_\alpha]$$

where c_0 is an arbitrary point in the product.

Since we know that $|\alpha^{\pi'}|$ is trivial on Γ_P and since the action of P commutes with the geodesic action we conclude

$$\Gamma_P\backslash X^P(c'_\pi, r(\underline{c}_{\pi'})) = \Gamma_P\backslash(n^{\pi'})^{-1}(c_0) \times \prod_{\alpha\in\pi'} (0, c_\alpha] \tag{54}$$

Let $P^{(1)}(\mathbb{R}) = \ker(\alpha^{\pi'})$ then the fiber $(n^{\pi'})^{-1}(c_0)$ is a homogenous space under $P^{(1)}(\mathbb{R})$ We have the projection map $p_{P,M} : X \to X^M$ where X^M is the space of Cartan involutions on the reductive quotient M. Hence we get a map

$$p^*_{P,M} = p_{P,M} \times n^{[\pi']} : X \to X^M \times \prod_{\alpha\in\pi'} (0, c_\alpha] \tag{55}$$

On the product $X^M \times \prod_{\alpha\in\pi'}(0, c_\alpha]$ the geodesic action only acts on the second factor the map $p^*_{P,M}$ commutes with the geodesic action.

The group $U_P(\mathbb{R})$ acts simply transitively on the fibers of this projection, and hence

$$q_{P,M} : \Gamma_P\backslash X^P(c'_\pi, r(\underline{c}_{\pi'})) \to \Gamma_M\backslash X^M(r(\underline{c}_P)) \times \prod_{\alpha\in\pi'} (0, c_\alpha] \tag{56}$$

is a fiber bundle with fiber isomorphic $\Gamma_U\backslash U(\mathbb{R})$. If we pick a point $\tilde{x} \in \Gamma_M\backslash X^M(r(\underline{c}_P)) \times \prod_{\alpha\in\pi'}(0, c_\alpha]$ then the identification of $q^{-1}_{P,M}(\tilde{x})$ with $\Gamma_U\backslash U(\mathbb{R})$ depends on the choice of a point $x \in X^P(c'_\pi, r(\underline{c}_{\pi'}))$ which maps to $\tilde{x}$.

(**The next requires a little revision**) This can now be compactified, we embed it into

$$\overline{\Gamma_P\backslash X^P(c'_\pi, r(\underline{c}_P)} = \Gamma_P\backslash (n^{[\pi']})^{-1}(c_0) \times \prod_{\nu\in\pi_G\backslash\pi} [0, c'_\pi].$$

We define

$$\partial\overline{r(\underline{c}_P)} = \overline{\Gamma_P\backslash X^P(c'_\pi, \Omega_\pi)} \setminus \Gamma_P\backslash X^P(c_\pi, \Omega_\pi)$$

this is equal to

$$\partial\overline{\Gamma_P\backslash X^P(c'_\pi, r(\underline{c}_P)} = \Gamma_P\backslash (n^{[\pi]})^{-1}(c_0) \times \partial(\prod_{\nu\in\pi_G\backslash\pi} [0, c_\pi])$$

where of course $\partial(\prod_{\nu\in\pi_G\backslash\pi}[0, c_\pi]) \subset \prod_{\nu\in\pi_G\backslash\pi}[0, c_\pi]$ is the subset where at least one of the coordinates is equal to zero.

We form the disjoint union of of these boundaries over the π and set of representatives of Γ conjugacy classes, this is a compact space. Now there is still a minor technical point. If we have two parabolic subgroups $Q \subset P$ then the intersection $X^P(\underline{c}_P, r(\underline{c}_P) \cap X^Q(\underline{c}_Q, r(\underline{c}_Q)) \neq \emptyset$. If we now have points

$$x \in \partial\overline{\Gamma_P\backslash X^P(c_\pi, r(\underline{c}_P)}, y \in \partial\overline{\Gamma_Q\backslash X^Q(c_{\pi'}, r(\underline{c}_{P'})}$$

then we identify these two points if we have a sequence of points $\{x_n\}_{n\in\mathbb{N}}$ which lies in the intersection $X^P(c_\pi, r(\underline{c}_P)) \cap X^Q(c_{\pi'}, r(\underline{c}_{P'}))$ and which converges to x in $\overline{\Gamma_P\backslash X^P(c_\pi, r(\underline{c}_P)}$ and to y in $\overline{\Gamma_Q\backslash X^Q(c_{\pi'}, r(\underline{c}_{P'})}$. A careful inspection shows that this provides an equivalence relation $\sim$, and we define

$$\partial(\Gamma\backslash X) = \bigcup_{\pi', P\in \mathrm{Par}(\Gamma)} \partial\overline{\Gamma_P\backslash X^P(c_\pi, r(\underline{c}_P)} / \sim$$

and the Borel-Serre compactification will be the manifold with corners

$$\overline{\Gamma\backslash X} = \Gamma\backslash(X \cup \bigcup_{P: P \mathrm{proper}} \overline{X^P(\underline{c}_{\pi'}, r(\underline{c}_P)}). \tag{57}$$

We define a "tubular" neighborhood of the boundary we put

$$\mathcal{N}(\Gamma\backslash X) = \Gamma\backslash \bigcup_{P: P \mathrm{proper}} \overline{X^P(\underline{c}_{\pi'}, r(\underline{c}_P)}) \tag{58}$$

and then we define the "punctured tubular" neighborhood as

$$\dot{\mathcal{N}}(\Gamma\backslash X) = \Gamma\backslash \bigcup_{P: P \mathrm{proper}} X^P(\underline{c}_{\pi'}, r(\underline{c}_P)) = \Gamma\backslash X \cap \mathcal{N}(\Gamma\backslash X) \tag{59}$$

Eventually we want to use the above covering as a tool to understand cohomology (See) But then it is also necessary to understand the intersections

$$X^{P_1}(c_{\pi_1}, r(\underline{c}_{\pi_1})) \cap \cdots \cap X^{P_\nu}(c_{\pi_\nu}, r(\underline{c}_{\pi_\nu})) \tag{60}$$

Our proposition 2.4 implies that for any point x in the intersection the destabilizing parabolic subgroup $P(x) \subset P_1 \cap \cdots \cap P_\nu$. Hence we see that the above intersection can only be non empty if $Q = P_1 \cap \cdots \cap P_\nu$ is a parabolic subgroup.

Now we look at the product $\prod_{\alpha\in\pi}\mathbb{R}^\times_{>0}$ here it seems to be helpful to identify it - using the logarithm - with $\mathbb{R}^d$:

$$\log : \prod_{\alpha\in\pi}\mathbb{R}^\times_{>0} \xrightarrow{\sim} \mathbb{R}^d \tag{61}$$

If G is one of our reductive groups Gl_n, M let X be the symmetric space of Cartan involutions- If we have a point $x \in X$ and P a parabolic subgroup such that $P(x) \subset P$ then the number $n_{\alpha P}(x, P)$ is defined and < 1. If $P(x) \notin P$ then we put $n_{\alpha P}(x, P) = 1$, so that $n_{\alpha P}(x, P)$ is always defined.

Hence we defined a function

$$N^Q(\ ,Q) : X \to \mathbb{R}^d; x \mapsto \{\dots, -\log(n_{\alpha Q}(x, Q)), \dots\}_{\alpha\in\pi} = \{\dots, N_{\alpha Q}(x, Q), \dots\}_{\alpha\in\pi}. \tag{62}$$

a close look shows that the image is a convex set $C(\underline{\tilde{c}}) \subset \mathbb{R}^d$ because it is an intersection of half spaces defined by hyperplanes. In the target space we can project to the unipotent roots, i.e. we look at the projection

$$r_Q : \underline{x} = \{\dots, x_\alpha, \dots\}_{\alpha\in\pi} \mapsto \{\dots, x_\alpha, \dots\}_{\alpha\in\pi^{U_Q}}.$$

Then we can consider the composition $r_Q \circ N^Q(\ ,Q)$ and the image under this composition is a cone $C_{U_Q}(\underline{\tilde{c}})$ in $\mathbb{R}^{d_1}_{>0}$.Then

$$X^{P_1}(c_{\pi_1}, r(\underline{c}_{\pi_1})) \cap \dots \cap X^{P_\nu}(c_{\pi_\nu}, r(\underline{c}_{\pi_\nu})) = X^Q(C(\underline{\tilde{c}})) \to C_{U_Q}(\underline{\tilde{c}}) \tag{63}$$

is a fiber bundle over the base $C_{U_Q}(\underline{\tilde{c}})$.

3 Cohomology of arithmetic groups as cohomology of sheaves on $\Gamma\backslash X$

We are now in the position to unify — for the special case of arithmetic groups — the two cohomology theories from our chapter II and chapter IV of the [book]. (Lectures on Algebraic Geometry I)

We start from a semi simple group $G/\mathbb{Q}$ and we choose an arithmetic subgroup $\Gamma \subset G(\mathbb{Q})$. Let $X = G(\mathbb{R})/K$ as before.

Let $\mathcal{M}$ is a Γ-module then we can attach a sheaf $\tilde{\mathcal{M}}$ onM$\Gamma\backslash X$ to it. To do this we have to define the group of sections for any open subset $U \subset X$. We start from the projection

$$\pi : X \longrightarrow \Gamma\backslash X$$

and define

$$\tilde{\mathcal{M}}(U) = \{f : \pi^{-1}(U) \to \mathcal{M} \mid f \text{ is locally constant } f(\gamma u) = \gamma f(u)\}.$$

This is clearly a sheaf. For any point $x \in \Gamma\backslash X$ we can find a neighborhood V_x with the following property: If $\tilde{x} \in \pi^{-1}(x)$, then $\tilde{x}$ has a contractible $\Gamma_{\tilde{x}}$-invariant neighborhood $U_{\tilde{x}}$ and $U_x = \Gamma_{\tilde{x}}\backslash U_{\tilde{x}}$. Then it is clear that

$$\tilde{\mathcal{M}}(V_x) = \mathcal{M}^{\Gamma_{\tilde{x}}}.$$

Since x has a cofinal system of neighborhoods of this kind, we see that we get an isomorphism

$$j_{\tilde{x}} : \tilde{\mathcal{M}}(V_x) = \tilde{\mathcal{M}}_x \xrightarrow{\sim} \mathcal{M}^{\Gamma_{\tilde{x}}}.$$

The last isomorphism depends on the choice of $\tilde{x}$. If we are in the special case that Γ has no fixed points then we can cover $\Gamma\backslash X$ by open sets U so that $\tilde{\mathcal{M}}/U$ is isomorphic to a constant sheaf $\underline{\mathcal{M}}_U$. These sheaves are called local systems.

We will denote the functor, which sends $\mathcal{M}$ to $\tilde{\mathcal{M}}$ by

$$\mathrm{sh}_\Gamma : \mathbf{Mod}_\Gamma \to \mathcal{S}_{\Gamma\backslash X},$$

occasionally we will write $\mathrm{sh}_\Gamma(\mathcal{M})$ instead of $\tilde{\mathcal{M}}$, especially in situations where we work with several discrete subgroups.

The motivations for these constructions are

1) The spaces $\Gamma\backslash X$ are interesting examples of so-called locally symmetric spaces (provided Γ has no torsion). Hence they are of interest for differential geometers and analysts.

2) If we have some understanding of the geometry of the quotient space $\Gamma\backslash X$ we gain some insight into the structure of Γ. This will become clear when we discuss the examples in ...x.y.z.

3) The cohomology groups $H^\bullet(\Gamma, \mathcal{M})$ are closely related and in many cases even isomorphic to the sheaf cohomology groups $H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}})$. Again the geometry provides tools to compute these cohomology groups in some cases (see x.y.z.).

4) If the Γ-module $\mathcal{M}$ is a $\mathbb{C}$-vector space and obtained from a rational representation of $G/\mathbb{Q}$, then we can apply analytic tools to get insight (de Rham cohomology, Hodge theory).

3.1 The relation between $H^\bullet(\Gamma, \mathcal{M})$ and $H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}})$.

In general the spaces X will have several connected components. In this section we assume that X is connected and Γ fixes it.

Then it is clear that

$$H^0(\Gamma\backslash X, \tilde{\mathcal{M}}) = \mathcal{M}^\Gamma.$$

Hence we can write our functor $\mathcal{M} \to \mathcal{M}^\Gamma$ from the category of Γ-modules to $\mathbf{Ab}$ as a composite of

$$\mathrm{sh}_\Gamma : \mathcal{M} \longrightarrow \tilde{\mathcal{M}} \text{ and } H^0 : \tilde{\mathcal{M}} \to H^0(\Gamma\backslash X, \tilde{\mathcal{M}}).$$

We want to apply the method of spectral sequences. In a first step we want to convince ourselves that sh_Γ sends injective Γ-modules to acyclic sheaves.

In [book], 2.2.4. we constructed for any Γ module $\mathcal{M}$ the induced Γ -module $\mathrm{Ind}_{\{1\}}^\Gamma \mathcal{M}$. This is the module of functions $f : \Gamma \to \mathcal{M}$ and $\gamma_1 \in \Gamma$ acts on this module by $(\gamma_1 f)(\gamma) = f(\gamma\gamma_1)$. We want to prove that for any such induced module the sheaf $\mathrm{sh}_\Gamma(\ \mathrm{Ind}_{\{1\}}^\Gamma \mathcal{M})$. is acyclic.

We have a little

Lemma: *Let us consider the projection $\pi : X \to \Gamma\backslash X$ and the constant sheaf $\underline{\mathcal{M}}_X$ on X. Then we have a canonical isomorphism of sheaves*

$$\pi_*(\underline{\mathcal{M}}_X) \xrightarrow{\sim} \widetilde{\mathrm{Ind}_{\{1\}}^\Gamma} \mathcal{M}.$$

Proof: This is rather obvious. Let us consider a small neighborhood U_x of a point x, such that $\pi^{-1}(U_x)$ is the disjoint union of small contractible neighborhoods $U_{\tilde{x}}$ for $\tilde{x} \in \pi^{-1}(x)$. Then for all points $\tilde{x}$ we have $\underline{\mathcal{M}}_X(U_{\tilde{x}}) = \mathcal{M}$ and

$$\pi_*(\underline{\mathcal{M}}_X)(U_x) = \prod_{\tilde{x} \in \pi^{-1}(x)} \mathcal{M}.$$

On the other hand

$$\widetilde{\mathrm{Ind}_{\{1\}}^{\Gamma}}\mathcal{M}(U_x) = \left\{ h : \pi^{-1}(U_x) \to \ \mathrm{Ind}_{\{1\}}^{\Gamma}\mathcal{M} \mid h \text{ is locally constant } h(\gamma u) = \gamma h(u) \right\}$$

For $u \in \pi^{-1}(U_x)$ the element $h(u)$ itself is a map

$$f(u) : \Gamma \longrightarrow \mathcal{M},$$

and $(\gamma h(u))(\gamma_1) = h(u)(\gamma_1 \gamma)$ (here $\gamma_1 \in \Gamma$ is the variable.)

Hence we know the function $u \to f(u)$ from $\pi^{-1}(U_x)$ to $\mathrm{Ind}_{\{1\}}^{\Gamma}\mathcal{M}$ if we know its value $f(u)(1)$ and this value can be prescribed on the connected components of $\pi^{-1}(U_x)$. On these connected components it is constant, we may take its value at $\tilde{x}$ and hence

$$f \longrightarrow (\ldots, f(\tilde{x})(1), \ldots)_{\tilde{x} \in \pi^{-1}(x)}$$

yields the desired isomorphism.

Now we get the acyclicity. We apply example d) in [book], 4.6.3 (section on application of spectral sequences) to this situation. The fibre of π is a discrete space and hence

$$\pi_*(\underline{\mathcal{M}}_X) = \widetilde{\mathrm{Ind}_{\{1\}}^{\Gamma}}\mathcal{M}$$

and $R^q(\pi_*)(\underline{\mathcal{M}}_X) = 0$ for $q > 0$. Therefore the spectral sequence yields

$$H^q(X, \underline{\mathcal{M}}_X) = H^q(\Gamma\backslash X, \pi_*(\underline{\mathcal{M}}_X)) = H^q\left(\Gamma\backslash X,\ \widetilde{\mathrm{Ind}_{\{1\}}^{\Gamma}}\mathcal{M}\right),$$

and since X is a cell, we see that this is zero for $q \geq 1$.

We apply this to the case that $m = \mathcal{I}$ is an injective Γ-module. Clearly we can always embed $\mathcal{I} \longrightarrow \ \mathrm{Ind}_{\{1\}}^{\Gamma}\mathcal{I}$. But this is now a direct summand; hence it follows from the acyclicity of $\widetilde{\mathrm{Ind}_{\{1\}}^{\Gamma}}\mathcal{I}$ that also $\tilde{\mathcal{I}}$ must be acyclic.

Hence we get a spectral sequence with E_2 term

$$H^p(\Gamma\backslash X, R^q(\mathrm{sh}_\Gamma)(\mathcal{M})) \Rightarrow H^n(\Gamma, \mathcal{M}).$$

The edge homomorphism yields a homomorphism

$$H^n(\Gamma\backslash X, \mathrm{sh}_\Gamma(\mathcal{M})) \to H^n(\Gamma, \mathcal{M})$$

which in general is neither injective nor surjective.

Of course it is clear that the stalk $R^q(\mathrm{sh}_\Gamma)(\mathcal{M})_x = H^q(\Gamma_{\tilde{x}}, \mathcal{M})$. If we make the assumption that the action of Γ is faithful, this means that any element γ different from the identity acts a non trivially on X, then $R^q(\mathrm{sh}_\Gamma)(\mathcal{M})$ is supported on a lower dimensional closed subset.

If we have a commutative ring R in which the orders of all the finite stabilizers $\Gamma_{\tilde{x}}$ are invertible and if we only consider $R-\Gamma$ modules $\mathcal{M}$, then of course $R^q(\mathrm{sh}_\Gamma)(\mathcal{M}) = 0$ for $q > 0$ and then the edge homomorphism becomes an isomorphism.

3.1.1 Functorial properties of cohomology

We investigate the functorial properties of the cohomology with respect to the change of Γ. If $\Gamma' \subset \Gamma$ is a subgroup of finite index, then we have, of course, the functor

$$\mathbf{Mod}_\Gamma \longrightarrow \mathbf{Mod}_{\Gamma'},$$

which is obtained by restricting the Γ-module structure to Γ'. Since for any Γ-module $\mathcal{M}$ we have $\mathcal{M}^\Gamma \longrightarrow \mathcal{M}^{\Gamma'}$, we obtain a homomorphism

$$\text{res } : H^i(\Gamma, \mathcal{M}) \longrightarrow H^i(\Gamma', \mathcal{M}).$$

We give an interpretation of this homomorphism in terms of sheaf cohomology. We have the diagram

$$\begin{array}{ccc} & X & \\ \pi_{\Gamma'} \swarrow & & \searrow \pi_\Gamma \\ \pi_1 = \pi_{\Gamma,\Gamma'} : \Gamma'\backslash X & \longrightarrow & \Gamma\backslash X \end{array}$$

and a Γ-module $\mathcal{M}$ produces sheaves $\mathrm{sh}_\Gamma(\mathcal{M}) = \tilde{\mathcal{M}}$ and $\mathrm{sh}_{\Gamma'}(\mathcal{M}) \tilde{=} \mathcal{M}'$ on $\Gamma'\backslash X$ and $\Gamma\backslash X$ respectively. It is clear that we have a homomorphism

$$\pi_1^*(\tilde{\mathcal{M}}) \longrightarrow \tilde{\mathcal{M}}'.$$

To get this homomorphism we observe that for $y_1 \in \Gamma'\backslash X$ we have $\pi_1^*(\tilde{\mathcal{M}})_{y_1} = \tilde{\mathcal{M}}_{\pi_1(y_1)}$, and this is

$$\{f : \pi^{-1}(\pi_1(y)) \to \mathcal{M} \mid f(\gamma\tilde{y}) = \gamma f(\tilde{y}) \text{ for all } \gamma \in \Gamma, \tilde{y} \in \pi^{-1}(\pi(y_1))\}$$

and

$$\tilde{\mathcal{M}}'_{y_1} = \{fg : (\pi')^{-1}(y_1) \to \mathcal{M} \mid f(\gamma'\tilde{y}) = \gamma' f(\tilde{y}) \text{ for all } \gamma \in \Gamma', \tilde{y} \in (\pi')^{-1}(y_1)\},$$

and if we pick a point $\tilde{y} \in (\pi')^{-1}(y_1) \subset \pi^{-1}(\pi_1(y_1))$ then

$$\pi_1^*(\mathcal{M})_{y_1} \simeq \mathcal{M}^{\Gamma_{\tilde{y}_1}} \subset \tilde{\mathcal{M}}'_{y_1} = \mathcal{M}^{\Gamma'_{\tilde{y}_1}}.$$

Hence we get (or define) our restriction homomorphism as (see I, ...)

$$H^i(\Gamma\backslash X, \mathrm{sh}_\Gamma(\mathcal{M})) \longrightarrow H^i(\Gamma'\backslash X, \pi_1^*(\mathrm{sh}_\Gamma(\mathcal{M})) \longrightarrow H^i(\Gamma'\backslash X, \mathrm{sh}_{\Gamma'}(\mathcal{M})).$$

There is also a map in the opposite direction.
Since the fibres of π_1 are discrete we have

$$H^i(\Gamma'\backslash X, \tilde{\mathcal{M}}) \xrightarrow{\sim} H^i(\Gamma\backslash X, \pi_{1,*}(\tilde{\mathcal{M}})).$$

But the same reasoning as in the previous section yields an isomorphism

$$\pi_{1,*}(\tilde{\mathcal{M}}) \xrightarrow{\sim} \widetilde{\mathrm{Ind}_{\Gamma'}^\Gamma \mathcal{M}}.$$

Hence we get an isomorphism

$$H^i(\Gamma'\backslash X, \tilde{\mathcal{M}}) \xrightarrow{\sim} H^i(\Gamma\backslash X, \widetilde{\mathrm{Ind}_{\Gamma'}^\Gamma \mathcal{M}})$$

which is well known as Shapiro's lemma. But we have a Γ-module homomorphism

$$e : \mathrm{Ind}_{\Gamma'}^\Gamma \mathcal{M} \longrightarrow \mathcal{M}$$

which sends an $f : \Gamma \to \mathcal{M}$, in $f \in \operatorname{Ind}_{\Gamma'}^{\Gamma}\mathcal{M}$ to the sum

$$\operatorname{tr}(f) = \sum \gamma_i^{-1} f(\gamma_i)$$

where the γ_i are representatives for the classes of $\Gamma'\backslash\Gamma$. This homomorphism induces a map in the cohomology. We get a compositon

$$\pi_{1,\bullet} : H^i(\Gamma'\backslash X, \tilde{\mathcal{M}}) \longrightarrow H^i(\Gamma\backslash X, \tilde{\mathcal{M}}).$$

It is not difficult to check that

$$\pi_{1,\bullet} \circ \pi_1^\bullet = [\Gamma : \Gamma'].$$

3.2 How to compute the cohomology groups $H^i(\Gamma\backslash X, \tilde{\mathcal{M}})$?

3.2.1 The Čech complex of an orbiconvex Covering

We return to the beginning of this note. We want to find a finite set of points $\tilde{x}_1, \dots, \tilde{x}_i, \dots, \tilde{x}_r$ and open sets $\tilde{U}_{\tilde{x}_i}, \tilde{x}_i \in \tilde{U}_{\tilde{x}_i}$ such that the following conditions are true

a) For $\gamma \in \Gamma$ we have $\gamma\tilde{U}_{\tilde{x}_i} \cap \tilde{U}_{\tilde{x}_i} = \emptyset$ unless we have $\gamma\tilde{x}_i = \tilde{x}_i$, i.e. $\gamma \in \Gamma_{\tilde{x}_i}$

b) The map $\bigcup \tilde{U}_{\tilde{x}_i} \to \Gamma\backslash X$ is surjective

c) For all i we have a $\Gamma_{\tilde{x}_i}$ equivariant homotopy contracting $\tilde{U}_{\tilde{x}_i}$ to $\tilde{x}_i$.

d) For any non empty finite intersection $\dots\cap\tilde{U}_{\tilde{x}_i}\cap\dots\cap\tilde{U}_{\tilde{x}_j}\cap\dots$ we can find a point $\tilde{x}_{\underline{i}}$ in this intersection which is fixed by $\dots\cap\Gamma_{\tilde{x}_i}\cap\dots\Gamma_{\tilde{x}_j} = \Gamma_{\tilde{x}_{\underline{i}}}$ and such that we have a $\Gamma_{\tilde{x}_{\underline{i}}}$ equivariant homotopy contracting $\tilde{U}_{\tilde{x}_i}$ to $\tilde{x}_i$.

We know that the Čzech complex

$$C^\bullet(\mathfrak{U}, \tilde{\mathcal{M}}) \quad := 0 \to \bigoplus_{i\in I} \tilde{\mathcal{M}}(U_{x_i}) \xrightarrow{d_0} \bigoplus_{i<j} \tilde{\mathcal{M}}(U_{x_i} \cap U_{x_j}) \to \tag{64}$$

computes the cohomology provided we know that the intersections $U_{\underline{i}} = U_{x_{i_1}} \cap U_{x_{i_2}} \cap \dots \cap U_{x_{i_q}}$ are acyclic, i.e. $H^m(U_{\underline{i}}, \tilde{\mathcal{M}}) = 0$ for $m > 0$.

For the implementation on a computer we need to resolve the definition of the spaces of sections and the definition of the boundary maps. (By this I mean that we have to write explicitly

$$\tilde{\mathcal{M}}(U_{\underline{i}}) = \bigoplus_{\eta} \mathcal{M}_\eta$$

where η runs through an index set and $\mathcal{M}_\eta$ are explicit subspaces of $\mathcal{M}$ and then we have to write down certain explicit linear maps $\mathcal{M}_\eta \to \mathcal{M}_{\eta'}$.)

To be more precise: We have to write $U_{\underline{i}} = \cup U_\eta$ as the union of its connected components, we have to choose a connected component $\tilde{U}_\eta$ in $\pi^{-1}(U_\eta)$ for each value of η, and then the evaluation of a section $m \in \tilde{\mathcal{M}}(U_{\underline{i}})$ on these $\tilde{U}_\eta$ yields an isomorphism

$$\oplus ev_{\tilde{U}_\eta} : \tilde{\mathcal{M}}(U_{\underline{i}}) \xrightarrow{\sim} \bigoplus_{\eta} \mathcal{M}^{\Gamma_\eta}.$$

If we replace $\tilde{U}_\eta$ by $\gamma\tilde{U}_\eta$ then we get for $m \in \tilde{\mathcal{M}}(\pi(\tilde{U}_\eta))$ the equality

$$\gamma ev_{\tilde{U}_\eta}(m) = ev_{\gamma\tilde{U}_\eta} \tag{65}$$

Especially the choice of the $\tilde{x}_i$ yields an identification

$$ev_{U_{x_i}} : \tilde{\mathcal{M}}(U_{x_i}) \xrightarrow{\sim} \mathcal{M}^{\Gamma_{\tilde{x}_i}} \tag{66}$$

this gives us the first term in the complex.

The computation of the second term is a little bit more delicate, the discussion in Chap.II is not correct. The point is that the intersections $U_{x_i} \cap U_{x_j}$ may not be connected. To get these connected components we have to find the elements $\gamma \in \Gamma$ for which

$$\tilde{U}_{\tilde{x_i}} \cap \gamma(\tilde{U}_{\tilde{x_j}}) \neq \emptyset \tag{67}$$

It is clear that this gives us a finite set $G_{i,j}$ of elements $\gamma \in \Gamma/\Gamma_{x_j}$. We have a little lemma

Lemma 3.1. *The images $\pi(\tilde{U}_{\tilde{x_i}} \cap \gamma(\tilde{U}_{\tilde{x_j}}))$ are the connected components of $U_{x_i} \cap U_{x_j}$, two elements γ, γ_1 give the same connected component if and only if $\gamma_1 \in \Gamma_{x_i}\gamma\Gamma_{x_j}$.*

Let $F_{i,j} \subset G_{i,j}$ be a set of representatives for the action of Γ_{x_1} on $G_{i,j}$ this set can be identified to the set of connected components. Of course the set $\tilde{U}_{\tilde{x_i}} \cap \gamma(\tilde{U}_{\tilde{x_j}})$ may have a non trivial stabilizer $\Gamma_{i,j,\gamma}$ and then we get an identification

$$\oplus_{\gamma \in F_{i,j}} ev_{\tilde{U}_{x_i} \cap \gamma \tilde{U}_{x_j}} : \tilde{\mathcal{M}}(U_{x_i} \cap U_{x_j}) \xrightarrow{\sim} \bigoplus_{\gamma \in F_{i,j}} \mathcal{M}^{\Gamma_{i,j,\gamma}} \tag{68}$$

This is now an explicit (i.e. digestible for a computer) description of the second term in our complex above. We still need to give the explicit formula for d_0 in the complex

$$0 \to \bigoplus_{i \in I} \mathcal{M}^{\Gamma_{\tilde{x}_i}} \xrightarrow{d_0} \bigoplus_{i<j} \bigoplus_{\gamma \in F_{i,j}} \mathcal{M}^{\Gamma_{i,j,\gamma}} \tag{69}$$

Looking at the definition it is clear that this map is given by

$$(\dots, m_i, \dots, m_j, \dots) \mapsto (\dots, m_i - \gamma m_j, \dots) \tag{70}$$

Here we have to observe that $\gamma \in \Gamma/\Gamma_{x_j}$ but this does not matter since $m_j \in \mathcal{M}^{\Gamma_{\tilde{x}_j}}$. So we have an explicit description of the beginning of the Cech complex.

A little reasoning shows of course that a different choice $F'_{i,j}$ of the representatives provides an isomorphic complex.

Now it is clear, how to proceed. At first we have to understand the combinatorics of the covering $\mathfrak{U} = \{U_{x_i}\}_{i \in I}$.

We consider sets

$$G_{\underline{i}} = \{\underline{\gamma} = (e, \gamma_1, \dots, \gamma_q) | \gamma_i \in \Gamma/\Gamma_{x_i}; \tilde{U}_{\tilde{x}_0} \cap \dots \cap \gamma_i \tilde{U}_{\tilde{x_i}} \cap \gamma_q \tilde{U}_{\tilde{x}_q} \neq \emptyset\}$$

on these sets we have an action of Γ_{x_0} by multiplication from the left. Again let $F_{\underline{i}}$ be a system of representatives modulo the action of Γ_{x_0}.

We abbreviate

$$\tilde{U}_{\underline{i},\underline{\gamma}} = \tilde{U}_{\tilde{x}_0} \cap \dots \cap \gamma_i \tilde{U}_{\tilde{x_i}} \cap \gamma_q \tilde{U}_{\tilde{x}_q},$$

let $\Gamma_{\underline{i},\underline{\gamma}}$ be the stabilizer of $\tilde{U}_{\underline{i},\underline{\gamma}}$.

The images $\pi(\tilde{U}_{\underline{i},\underline{\gamma}})$ under the projection map π are the connected components $\pi(\tilde{U}_{\underline{i},\underline{\gamma}}) = U_{\underline{i},\underline{\gamma}} \subset U_{\underline{i}} = U_{x_{i_0}} \cap \dots \cap U_{x_{i_\nu}} \cap \dots U_{x_{i_q}}$. On the other hand each set $\tilde{U}_{\underline{i},\underline{\gamma}}$ is a connected component in $\pi^{-1}(U_{\underline{i},\underline{\gamma}})$. We get an isomorphism

$$\bigoplus_{\underline{\gamma}\in F_{\underline{i}}} ev_{\tilde{U}_{\underline{i},\gamma}} : \tilde{\mathcal{M}}(U_{\underline{i}}) = \tilde{\mathcal{M}}(U_{x_{i_0}} \cap \dots \cap U_{x_{i_\nu}} \cap \dots U_{x_{i_q}}) \xrightarrow{\sim} \bigoplus_{\underline{\gamma}\in F_{\underline{i}}} \mathcal{M}^{\Gamma_{\underline{i},\underline{\gamma}}}. \tag{71}$$

We need to give explicit formulas for the boundary maps

$$\bigoplus_{\underline{i}\in I^q} \tilde{\mathcal{M}}(U_{\underline{i}}) \xrightarrow{d_q} \bigoplus_{\underline{i}\in I^{q+1}} \tilde{\mathcal{M}}(U_{\underline{i}}).$$

Abstractly this boundary operator is defined as follows: We look at pairs $\underline{i} \in I^{q+1}, \underline{i}^{(\nu)} \in I^q$ where $\underline{i}^{(\nu)}$ is obtained from $\underline{i}$ by deleting the ν-th entry. Then we have $U_{\underline{i}} \subset U_{\underline{i}^{(\nu)}}$ and from this we get the resulting restriction homomorphism $R_{\underline{i}^{(\nu)},\underline{i}} : \tilde{\mathcal{M}}(U_{\underline{i}^{(\nu)}}) \to \tilde{\mathcal{M}}(U_{\underline{i}})$. Then

$$d_q = \sum_{\underline{i}} \sum_{\nu=0}^{q} (-1)^\nu R_{\underline{i}^{(\nu)},\underline{i}}$$

and hence we have to give an explicit description of $R_{\underline{i}^{(\nu)},\underline{i}}$ with respect to the isomorphism in the diagram (71).

We pick two connected components $\pi(\tilde{U}_{\underline{i},\underline{\gamma}}) \subset U_{\underline{i}}$ and $\pi(\tilde{U}_{\underline{i}^{(\nu)},\underline{\gamma}'} \subset U_{\underline{i}^{(\nu)}})$, then we know that

$$\tilde{U}_{\underline{i},\underline{\gamma}} \subset \tilde{U}_{\underline{i}^{(\nu)},\underline{\gamma}'} \iff \exists\, \eta_{\gamma,\gamma'} \in \Gamma \text{ such that } \eta_{\gamma,\gamma'}\gamma'_\mu = \gamma_\mu \text{ for all } \mu \neq \nu$$

and then the restriction of $R_{\underline{i}^{(\nu)},\underline{i}}$ to these two components is given by

$$\begin{array}{ccc} \tilde{\mathcal{M}}(\pi(\tilde{U}_{\underline{i}^{(\nu)},\underline{\gamma}'})) & \xrightarrow{ev_{\tilde{U}_{\underline{i}^{(\nu)},\underline{\gamma}'}}} & \mathcal{M}^{\Gamma_{\underline{i}^{(\nu)},\underline{\gamma}'}} \\ \downarrow R_{\underline{i}^{(\nu)},\underline{i}} & & \downarrow\ \eta_{\gamma,\gamma'} \\ \tilde{\mathcal{M}}(\pi(\tilde{U}_{\underline{i},\underline{\gamma}})) & \xrightarrow{ev_{\tilde{U}_{\underline{i},\underline{\gamma}}}} & \mathcal{M}^{\Gamma_{\underline{i},\underline{\gamma}}} \end{array} \tag{72}$$

Here the two horizontal maps are isomorphisms, we observe that $\eta_{\gamma,\gamma'}$ is unique up to an element in $\Gamma_{\underline{i}^{(\nu)},\underline{\gamma}'}$ and hence the vertical arrow $\eta_{\gamma,\gamma'}$ is well defined.

Now we can write down the complex explicitly.

We will show that it follows from reduction theory that

Theorem 3.1. *We can construct a finite covering $\Gamma\backslash X = \cup_{i\in E} U_{x_i} = \mathfrak{U}$ by orbiconvex sets.*

This of course implies the following theorem of Raghunathan

Theorem 3.2. *If R is any commutative ring with identity and if $\mathcal{M}$ is a finitely generated $R-\Gamma-$ module then the total cohomology*

$$\bigoplus_{q\in\mathbb{N}} H^q(\Gamma\backslash X, sh_\Gamma(\mathcal{M}))$$

is a finitely generated R-module

2.1.3 Special examples in low dimensions.

We consider the group $\Gamma = \mathrm{Sl}_2(\mathbb{Z})/\{\pm\mathrm{Id}\}$ and its action on the upper half planes H. We want to investigate the cohomology groups $H^i(\Gamma\backslash H, \tilde{\mathcal{M}})$ for any module Γ-module $\mathcal{M}$. The special points i and ρ in $\Gamma\backslash\mathbb{H}$ are the only points which are fixed points. We construct two nice orbiconvex neighborhoods of these two points, which will cover $\Gamma\backslash\mathbb{H}$. We drop the notation with the tilde and consider i, ρ as points in the upper half plane and as points on $\Gamma\backslash\mathbb{H}$. The stabilizers Γ_i, resp. Γ_ρ are cyclic and generated by the two elements

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, R = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \qquad (RS)$$

respectively.

Now we consider i. In the fundamental domain we consider a strip $V_i = \{z \mid -1/2 + \epsilon \leq \Re(z) \leq 1/2 - \epsilon\}$. To this strip we apply the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, we take the union $V_i \cup \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} V_i$ and we get our orbiconvex neighborhood U_i of i. Let us look at ρ. In the fundament domain $\mathcal{F}$ we consider the subset $V_\rho^- = \{z \in \mathcal{F} | \epsilon \leq \Re(z) \leq 1/2\}$ We should also consider we consider the corresponding subset V_ρ^+ containing ρ^2, (Here we have an ambiguity, we have two points in the fundamental domain lying over the fixed point ρ.) we translate this set by the translation by one, then we get the the set $V_\rho = V_\rho^- \cup (V_\rho^+ + 1)$. To this set we apply the elements the stabiliser and the union of the images under the action of the stabiliser of ρ we get a nice orbiconvex neighborhood U_ρ. If we take our $\epsilon > 0$ small enough then clearly

$$\Gamma\backslash\mathbb{H} = U_i \cup U_\rho \qquad (Cov)$$

and we get a resolution of a sheaf $\mathrm{sh}_\Gamma(\mathcal{M}) = \tilde{\mathcal{M}}$

$$0 \to \tilde{\mathcal{M}} \to \tilde{\mathcal{M}}_i \times \tilde{\mathcal{M}}_\rho \to \tilde{\mathcal{M}}_{i,\rho} \to 0$$

and hence the cohomology groups are given by the cohomology of the complex

$$0 \to \mathcal{M}^{\Gamma_i} \oplus \mathcal{M}^{\Gamma_\rho} \to \mathcal{M} \to 0.$$

Then $H^0(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) = \mathcal{M}^\Gamma = \mathcal{M}^{\Gamma_i} \cap \mathcal{M}^{\Gamma_\rho}$. Since this is true for any Γ module we easily conclude that Γ is generated by Γ_i, Γ_ρ.

We get

$$H^1(\mathrm{Sl}_2(\mathbb{Z})\backslash\mathbb{H}, \tilde{\mathcal{M}}_\mathbb{Z}) = \mathcal{M}/(\mathcal{M}_\mathbb{Z}^{(<S>} \oplus \mathcal{M}_\mathbb{Z}^{<R>}),$$
$$\forall p \;\; T_p : H^1(\mathrm{Sl}_2(\mathbb{Z})\backslash\mathbb{H}, \tilde{\mathcal{M}}_\mathbb{Z}) \to H^1(\mathrm{Sl}_2(\mathbb{Z})\backslash\mathbb{H}, \tilde{\mathcal{M}}_\mathbb{Z}).$$

and the cohomology vanishes in higher degrees.

Exercise 1: *Let* $\Gamma' \subset \Gamma = \mathrm{Sl}_2(\mathbb{Z})/\pm\mathrm{Id}$ *be a subgroup of finite index. Prove*

ii) We have (Shapiros lemma)

$$H^1(\Gamma'\backslash\mathbb{H}, \mathbb{Z}) = H^1\Gamma\backslash\mathbb{H}, \widetilde{\mathrm{Ind}_{\Gamma'}^{\Gamma}\mathbb{Z}}).$$

These cohomology groups are free of rank

$$[\Gamma : \Gamma'] - n_i - n_\rho + 1$$

where n_i (resp. n_ρ) is the number of orbits of Γ_i (resp. Γ_ρ) on $\Gamma'\backslash\Gamma$. If Γ' is torsion free then

$$\operatorname{rank}(H^1\Gamma\backslash\mathbb{H},\ \widetilde{\operatorname{Ind}_{\Gamma'}^{\Gamma}\mathbb{Z}}) = \frac{1}{6}[\Gamma:\Gamma'] + 1$$

The Euler-characteristic of $\Gamma'\backslash\mathbb{H}$ is $\frac{1}{6}[\Gamma:\Gamma']$.

Exercise 2: *Let $\mathcal{M}_n$ be the module of homogenous polynomials in the two variables X, Y and coefficients in $\mathbb{Z}$. We have an action of $\Gamma = \mathrm{Sl}_2(\mathbb{Z})$ on this module by*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} P(X,Y) = P(aX + cY, bX + dY).$$

these modules define a sheaf $\tilde{\mathcal{M}}_n$ on $\Gamma\backslash\mathbb{H}$, and we want to investigate their cohomology groups.

Prove:
i) If n is odd, then $\mathcal{M}_n = 0$.
Hence we assume $n \geq 2$ and n even from now on.
ii) $H^0(\Gamma\backslash\mathbb{H}, \mathcal{M}_n) = 0$.
iii) If we tensorize by $\mathbb{Q}$, then $H^1(\Gamma\backslash\mathbb{H}, \mathcal{M}_n \otimes \mathbb{Q})$ is a vector space of rank $n - 1 - 2\left[\frac{n}{4}\right] - 2\left[\frac{n}{6}\right]$.

Hint: *Diagonalize the action of Γ_i and Γ_ρ on $\mathcal{M}_n \otimes \overline{\mathbb{Q}}$ separately and look at the eigenspaces. To say it differently: Over $\bar{\mathbb{Q}}$ we can conjugate the matrices $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ into the diagonal maximal torus $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$, and then look at the decomposition of $\mathcal{M}_n$ into weight spaces.*

iv) Investigate the torsion in $H^1(\Gamma\backslash\mathbb{H}, \mathcal{M}_n)$. (Start from the sequence $0 \to \mathcal{M}_n \to \mathcal{M}_n \to \mathcal{M}_n/\ell\mathcal{M}_n \to 0$.)

v) Now we consider $\Gamma = \mathrm{Sl}_2(\mathbb{Z})$. The two matrices $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $R = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ are generators of the stabilisers of i and ρ respectively.

We take for our module M the cyclic group $\mathbb{Z}/12\mathbb{Z}$,consider the spectral sequence

$$H^p(\Gamma\backslash\mathbb{H}, R^q(\mathrm{sh}_\Gamma)(\mathbb{Z}/12\mathbb{Z}).$$

Show that $H^0(\Gamma\backslash\mathbb{H}, R^1(\mathrm{sh}_\Gamma)(\mathbb{Z}/12\mathbb{Z}) = \mathbb{Z}/12\mathbb{Z}$. Show that the differential

$$H^0(\Gamma\backslash\mathbb{H}, R^1(\mathrm{sh}_\Gamma)(\mathbb{Z}/12\mathbb{Z}) \to H^2(\Gamma\backslash\mathbb{H}, \mathrm{sh}_\Gamma(\mathbb{Z}/12\mathbb{Z})$$

vanishes and conclude

$$H^1(\Gamma, \mathbb{Z}/12\mathbb{Z}) = \mathbb{Z}/12\mathbb{Z}.$$

3.2.2 The group $\Gamma = \mathbf{Sl}_2(\mathbb{Z}[i])$

A similar computation can be made up to compute the cohomology in the case of $\tilde{\Gamma} = \mathrm{Gl}_2(\mathcal{O})$. We have the three special points x_{12}, x_{13} and x_{23} (See(2.1.2), and we choose closed sets A_{ij} containing these points which just leave out a small open strip containing the opposite face. If $\tilde{A}_{ij}$ is a component of the inverse image of A_{ij} in $\mathbb{H}_3$, then

$$A_{ij} = \Gamma_{ij}\backslash\tilde{A}_{ij}.$$

The intersections $A_{ij} \cap A_{i'j'} = A_\nu$ are closed sets. They are of the form

$$A_\nu = \Gamma_\nu \backslash \tilde{A}_\nu$$

where Γ_ν is the stabilizer of the arc joining x_{ij} and $x_{i'j'}$. The restrictions of our sheaves $\tilde{\mathcal{M}}$ to the A_{ij} and A_ν and to $A = A_{12} \cap A_{23} \cap A_{13}$ are acyclic and hence we get a complex

$$0 \longrightarrow \tilde{\mathcal{M}} \longrightarrow \bigoplus_{(i,j)} \tilde{\mathcal{M}}_{A_{ij}} \longrightarrow \bigoplus \tilde{\mathcal{M}}_{A_\nu} \longrightarrow \tilde{\mathcal{M}}_A \longrightarrow 0$$

where the $\tilde{\mathcal{M}}_?$ are the restrictions of $\tilde{\mathcal{M}}$ to ??? and then extended to the space again.

Hence we find that our cohomology groups are equal to the cohomology groups of the complex

$$0 \longrightarrow \bigoplus_{(i,j)} \mathcal{M}^{\Gamma_{ij}} \xrightarrow{d^1} \bigoplus_{\nu} \mathcal{M}^{\Gamma_\nu} \xrightarrow{d^2} \mathcal{M} \longrightarrow 0$$

with boundary maps

$$\begin{aligned} d^1 :& (m_{12}, m_{13}, m_{23}) \longmapsto (m_{12} - m_{13}, m_{23} - m_{12}, m_{13} - m_{23}) \\ d^2 :& (m_1, m_2, m_3) \longmapsto m_1 + m_2 + m_3. \end{aligned}$$

If we take for instance $\tilde{\mathcal{M}} = \mathbb{Z}$ then we get $H^0(\tilde{\Gamma}\backslash\mathbb{H}_3, \mathbb{Z}) = \mathbb{Z}$ and $H^i(\tilde{\Gamma}\backslash\mathbb{H}_3, \mathbb{Z}) = 0$ for $i > 0$ as it should be.

3.2.3 Homology, Cohomology with compact support and Poincaré duality.

Here we have to use the theory of compactifications. For any locally symmetric space we can embed $\Gamma\backslash X$ into its Borel-Serre compactification

$$i : \Gamma\backslash X \longrightarrow \Gamma\backslash\overline{X}_{BS},$$

and this process was explained in detail for our low dimensional examples. If we have a sheaf $\tilde{\mathcal{M}}$ on $\Gamma\backslash X$ we can extend it to the compactification by using the functor i_*. We get a sheaf

$$i_*(\tilde{\mathcal{M}}) \quad \text{on} \quad \Gamma\backslash\overline{X}_{BS}.$$

It will be shown later that for this particular compactification the functor i_* is exact. This is not true for the Baily-Borel compactification.

Our construction $\mathcal{M} \to \tilde{\mathcal{M}}$ can be extended to the action of Γ on $\overline{X}_{BS}$ and

$$i_*(\tilde{\mathcal{M}}) = \text{ result of the construction } \mathcal{M} \to \tilde{\mathcal{M}} \text{ on } \Gamma\backslash\overline{X}_{BS}.$$

Hence we get from our general results in Chapter I, ... that

$$H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}) = H^\bullet(\Gamma\backslash\overline{X}_{BS}, i_*(\tilde{\mathcal{M}})).$$

But we have another construction of extending the sheaf $\tilde{\mathcal{M}}$ from $\Gamma\backslash X$ to $\Gamma\backslash\overline{X}_{BS}$. This is the so called extension by zero. We define the sheaf $i_!(\tilde{\mathcal{M}})$ on $\Gamma\backslash\overline{X}_{BS}$ by giving the stalks. For $x \in \Gamma\backslash\overline{X}_{BS}$ we put

$$i_!(\tilde{\mathcal{M}})_x = \begin{cases} \tilde{\mathcal{M}}_x & \text{if} \quad x \in \Gamma\backslash X \\ 0 & \text{if} \quad x \notin \Gamma\backslash X \end{cases}.$$

It is clear that $i_!$ is an exact functor sending sheaves on $\Gamma\backslash X$ to sheaves on $\Gamma\backslash\overline{X}_{BS}$, and we have for an arbitrary sheaf

$$H^0(\Gamma\backslash\overline{X}_{BS}, i_!(\mathcal{F})) = H^0_c(\Gamma\backslash X, \mathcal{F})$$

where $H^0_c(\Gamma\backslash X, \mathcal{F})$ is the abelian group of those sections $s \in H^0(\Gamma\backslash X, \mathcal{F})$ for which the support

$$\text{supp }(s) = \{x \mid s_x \neq 0\}$$

is compact.

Hence we define the cohomology with compact supports as

$$H^q_c(\Gamma\backslash X, \mathcal{F}) = H^q(\Gamma\backslash\overline{X}_{BS}, i_!\mathcal{F})).$$

If $\tilde{\mathcal{M}}$ is a sheaf on $\Gamma\backslash X$ which is obtained from a Γ-module $\mathcal{M}$, then it is quite clear that

$$H^0_c(\Gamma\backslash X, \tilde{\mathcal{M}}) = 0,$$

provided our quotient $\Gamma\backslash X$ is not compact.

The cohomology with compact supports is actually related to the homology of the group: I want to indicate that we have a natural isomorphism

$$H_i(\Gamma, \mathcal{M}) \simeq H^{d-i}_c(\Gamma\backslash X, \tilde{\mathcal{M}})$$

under the assumption that X is connected and the orders of the stabilizers are invertible in R.

This is the analogous statement to the theorem .... which we discussed when we introduced cohomology.

Our starting point is the fact that the projective Γ-modules have analogous vanishing properties as the induced modules.

Lemma: *Let us assume that Γ acts on the connected symmetric space X. If P if a projective module then*

$$H^i_c(\Gamma\backslash X, \tilde{P}) = \begin{cases} 0 & \textit{if} \quad i \neq \dim X \\ \\ P_\Gamma & \textit{if} \quad i = \dim X. \end{cases}$$

Let us believe this lemma. Then it is quite clear that

$$H_i(\Gamma, \mathcal{M}) \simeq H^{d-i}_c(\Gamma\backslash X, \tilde{P}),$$

because both sides can be computed from a projective resolution.

We have still another description of the homology.

We form the singular chain complex

$$\to C_i(X) \to C_{i-1}(X) \to \ldots \to C_0(X) \to 0.$$

This is a complex of Γ-modules, and we can form the tensor product with $\mathcal{M}$. We get a complex of Γ-modules

$$\longrightarrow C_i(X) \otimes \mathcal{M} \longrightarrow C_{i-1}(X) \otimes \mathcal{M} \longrightarrow \ldots.$$

We define the chain complex

$$C_\bullet(\Gamma\backslash X, \underline{\mathcal{M}}),$$

simply a resulting complex if we take the Γ-coinvariants.

But we may choose for our module $\mathcal{M}$ simply the group ring. Then we have clearly

$$(C_\bullet(X) \otimes \mathbb{Z}[\Gamma])_\Gamma \simeq C_\bullet(X),$$

and hence we have, since X is a cell, that

$$H_i(\Gamma\backslash X, \underline{\mathbb{Z}[\Gamma]}) = 0 \qquad \text{for} \qquad i > 0.$$

On the other hand we have

$$H_0(\Gamma\backslash X, \underline{\mathcal{M}}) = \mathcal{M}_\Gamma.$$

This follows directly from looking at the complex

$$(C_1(X) \otimes \mathcal{M})_\Gamma \longrightarrow (C_0(X) \otimes \mathcal{M})_\Gamma.$$

First of all we observe that 0-cycles

$$x_1 \otimes m - x_0 \otimes m$$

are boundaries since X is pathwise connected. On the other hand we have that

$$x_0 \otimes m - \gamma x_0 \otimes \gamma m \in C_0(X) \otimes \mathcal{M}$$

becomes zero if we go to the coinvatiants and this implies the assertion.

If we have in addition that the orders of the stabilizers are invertible in R than it is clear that a short exact sequence of R-Γ-modules

$$0 \longrightarrow \mathcal{M}' \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}'' \longrightarrow 0$$

leads to an exact sequence of complexes

$$0 \longrightarrow C_\bullet(\Gamma\backslash X, \underline{\mathcal{M}'}) \longrightarrow C_\bullet(\Gamma\backslash X, \underline{\mathcal{M}}) \longrightarrow C_\bullet(\Gamma\backslash X, \underline{\mathcal{M}''}) \longrightarrow 0,$$

and hence to a long exact cohomology sequence

$$H_i(\Gamma\backslash X, \underline{\mathcal{M}'}) \longrightarrow H_i(\Gamma\backslash X, \underline{\mathcal{M}}) \longrightarrow H_i(\Gamma\backslash X, \underline{\mathcal{M}''}) \longrightarrow H_{i-1}(\Gamma\backslash X, \underline{\mathcal{M}'}).$$

Now it is clear that

$$H_i(\Gamma, \mathcal{M}) \simeq H_i(\Gamma\backslash X, \underline{\mathcal{M}}) \simeq H_c^{d-i}(\Gamma\backslash X, \tilde{\mathcal{M}}).$$

3.2.4 The fundamental exact sequence

By construction we have the exact sequence

$$0 \to i_!(\tilde{\mathcal{M}}) \to i_*(\tilde{\mathcal{M}}) \to i_*(\tilde{\mathcal{M}})/i_!(\tilde{\mathcal{M}}) \to 0$$

of sheaves and clearly $i_*(\mathcal{M})/i_!(\mathcal{M})$ is simply the restriction of $i_*(\tilde{\mathcal{M}})$ to the boundary extended by zero to the entire space. This yields the *fundamental exact sequence*

$$\to H^{q-1}(\partial(\Gamma\backslash X), \tilde{\mathcal{M}}) \to H_c^q(\Gamma\backslash X, \tilde{\mathcal{M}}) \to H^q(\Gamma\backslash \bar{X}, \tilde{\mathcal{M}}) \to H^q(\partial(\Gamma\backslash X), \tilde{\mathcal{M}}) \to \dots$$

We define the "inner cohomology" $H_c^q(\Gamma\backslash X, \tilde{\mathcal{M}})$ as the image of $H_c^q(\Gamma\backslash X, \tilde{\mathcal{M}}) \to H^q(\Gamma\backslash X, \tilde{\mathcal{M}})$. (This a little bit misleading because these groups are not honest cohomology groups. An exact sequence of sheaves $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ does not provide an exact sequence for the $H_!$ groups.)

We want to have a slightly different look at this sequence. We recall the covering (See 58,59)

$$\Gamma\backslash X = \Gamma\backslash X(r) \cup \overset{\bullet}{\mathcal{N}}(\Gamma\backslash X) = \Gamma\backslash X(r) \cup \bigcup_{P:P\text{proper}} \Gamma_P\backslash X^P(\underline{c}_{\pi'}, r(\underline{c}_{\pi'})) \tag{73}$$

where the union runs over Γ conjugacy classes of parabolic subgroups over $\mathbb{Q}$ and $\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X)$ is a punctured tubular neighborhood of ∞, i.e. the boundary of the Borel-Serre compactification.

It is well known (See for instance [book] vol I , 4.5) that from a covering $\Gamma\backslash X = \bigcup_i V_i$ we get a Čzech complex and a spectral sequence with $E_1^{p,q}$- term

$$\prod_{\underline{i}=\{i_0,i_1\dots,i_p\}} H^q(V_{\underline{i}}, \tilde{\mathcal{M}}) \tag{74}$$

where $V_{\underline{i}} = V_{i_0} \cap \cdots \cap V_{i_p}$. The boundary in the Čzech complex gives us the differential

$$d_1^{p,q}: \prod_{\underline{i}=\{i_0,i_1\dots,i_p\}} H^q(V_{\underline{i}}, \tilde{\mathcal{M}}) \to \prod_{\underline{j}=\{j_0,j_1\dots,j_{p+1}\}} H^q(V_{\underline{j}}, \tilde{\mathcal{M}}) \tag{75}$$

Here we work with the alternating Čzech complex, we also assume that we have an ordering on the set of simple positive roots. If such a $V_{\underline{i}}$ is non empty then it of the form $\Gamma_Q\backslash X^Q(C(\underline{\tilde{c}}))$.

We return to the diagram (63), on the left hand side we can divide by Γ_Q. We have the map which maps a Cartan involution on X to a Cartan-involution on M. Then we get a diagram

$$\begin{array}{ccc} f^\dagger: X^Q(C(\underline{\tilde{c}})) & \to & X^M(r)\times C_{U_Q}(\underline{\tilde{c}}) \\ \downarrow p_Q & & \downarrow p_M \\ f:\Gamma_Q\backslash X^Q(C(\underline{\tilde{c}})) & \to & \Gamma_M\backslash X^M(r)\times C_{U_Q}(\underline{\tilde{c}})) \end{array} \tag{76}$$

where the bottom line is a fibration. To describe the fiber in a point $\tilde{x}$ we pick a point $x \in (p_m \circ f^\dagger)^{-1}$. Then $U_Q(\mathbb{R})$ acts simply transitively on the fiber $(f^\dagger)^{-1}(f^\dagger(x))$ hence $U_Q(\mathbb{R}) = (f^\dagger)^{-1}(f^\dagger(x))$. Then $p_Q: U_Q(\mathbb{R}) \to \Gamma_{U_Q}\backslash U_Q(\mathbb{R})$ yields the identification $i_x: \Gamma_{U_Q}\backslash U_Q(\mathbb{R}) \xrightarrow{\sim} f^{-1}(\tilde{x})$. If we replace x by $\gamma x = x_1$ with $\gamma \in \Gamma_{U_Q}$ then we get $i_{x_1} = \mathrm{Ad}(\gamma)\circ i_x$ where for $u \in U_{U_Q}$ $\mathrm{Ad}(\gamma)(u) = \gamma u \gamma^{-1}$ where for $u \in U_Q(\mathbb{R})$, under this action of Γ_Q.

We have the spectral sequence

$$H^p(\Gamma_M\backslash X^M(r), R^q f_*(\tilde{\mathcal{M}})) \Rightarrow H^{p+q}(\Gamma_Q\backslash X^Q(C(\underline{c}_{\pi_1},\dots,c_{\pi_\nu})), \tilde{\mathcal{M}})$$

and clearly $R^q f_*(\tilde{\mathcal{M}})$ is a locally constant sheaf. This sheaf is easy to determine. Under the above identification we get an isomorphism

$$i_x^\bullet: H^\bullet(\Gamma_{U_Q}\backslash U_Q(\mathbb{R}), \tilde{\mathcal{M}})) \xrightarrow{\sim} R^\bullet(\tilde{\mathcal{M}})_{\tilde{x}}.$$

The adjoint action $\mathrm{Ad}: \Gamma_Q \to \mathrm{Aut}(\Gamma_{U_Q}\backslash U_Q(\mathbb{R}))$ induces an action of Γ_Q on the cohomology $H^\bullet((\Gamma_{U_Q}\backslash U_Q(\mathbb{R})), \tilde{\mathcal{M}})$. Since the functor cohomology is the derived functor of taking Γ_{U_Q} invariants it follows that the restriction of Ad to Γ_{U_Q} acts trivially on $H^\bullet(\Gamma_{U_Q}\backslash U_Q(\mathbb{R}),\tilde{\mathcal{M}})$. Consequently $H^\bullet((\Gamma_{U_Q}\backslash U_Q(\mathbb{R})),\tilde{\mathcal{M}})$ is a Γ_M- module. We get

$$R^\bullet f_*(\tilde{\mathcal{M}}) \xrightarrow{\sim} H^\bullet(\Gamma_{U_Q}\widetilde{\backslash U_Q}(\mathbb{R}),\tilde{\mathcal{M}})$$

and our spectral sequence becomes

$$H^p(\Gamma_M\backslash X^M(r), H^\bullet((\Gamma_{U_Q}\widetilde{\backslash U_Q}(\mathbb{R})),\tilde{\mathcal{M}})) \Rightarrow H^{p+q}(\Gamma_Q\backslash X^Q(C(\underline{\tilde{c}})),\tilde{\mathcal{M}})$$

We can take the composition $r_Q \circ f$. Then it is obvious that for any point $c_0 \in C_{U_Q}(\underline{\tilde{c}}))$ the restriction map

$$H^\bullet(X^Q(C(\underline{\tilde{c}})),\tilde{\mathcal{M}}) \to H^\bullet(X^Q((r_Q\circ f)^{-1}(c_0),\tilde{\mathcal{M}}) \tag{77}$$

is an isomorphism. On the other hand it is clear that we may vary our parameter $\underline{\tilde{c}}$ we may assume that the $C_{U_Q}(\underline{\tilde{c}})$ go to infinity. Then we may enlarge the parameter r without violating the assumptions in proposition 2.3. Hence we get that the inclusion $\Gamma_Q\backslash X^Q(C(\underline{\tilde{c}})) \subset \Gamma_Q\backslash X^Q$ induces an isomorphism in cohomology

$$H^\bullet(\Gamma_Q\backslash X^Q(C(\underline{\tilde{c}})),\tilde{\mathcal{M}}) \xrightarrow{\sim} H^\bullet(\Gamma_Q\backslash X,\tilde{\mathcal{M}}) \tag{78}$$

We choose a total ordering on the set of Γ conjugacy classes of parabolic subgroups, i.e. we enumerate them by a finite interval of integers $[1,N]$. We also enumerate the set of simple roots $\{\alpha_1,\dots,\alpha_d)$ in our special case $\alpha_i = \alpha_{i,i+1}$. For any conjugacy class $[P]$ we define the type of P to be $t(P) = \pi^{U_P}$ the subset of unipotent simple roots and $d(P) = \#\pi^{U_P}$ the cardinality of this set. If $P_{i_1},\dots,P_{i_r}$ are maximal, $i_1 < i_2\dots < i_r$ and if $P_{i_1}\cap,\dots\cap P_{i_r} = Q$ is a parabolic subgroup then we require that $t(P_{i_1}) < \dots < t(P_{i_r})$.

The indexing set $\mathrm{Par}(\Gamma)$ of our covering is the Γ conjugacy classes of parabolic subgroups over $\mathbb{Q}$. If we have a finite set $[P_{i_0}],[P_{i_1}],\dots,[P_{i_p}]$ of conjugacy classes then we say $[Q]\in[P_{i_0}],[P_{i_1}],\dots,[P_{i_p}]$ if we can find representatives $P'_{i_\nu}\in[P_{i_\nu}]$ and $Q'\in[Q]$ such that $Q' = P'_{i_0}\cap\dots P'_{i_p}$.

Hence we see that the $E_1^{\bullet,q}$ complex in our spectral sequence (75) is given by

$$\prod_i H^q(\Gamma_{Q_i}\backslash X^{Q_i}(C(\underline{\tilde{c}})),\tilde{\mathcal{M}}) \to \prod_{i<j}\prod_{[R]\in[Q_i]\cap[Q_j]} H^q(\Gamma_R\backslash X^R(C(\underline{\tilde{c}})),\tilde{\mathcal{M}}) \to \tag{79}$$

this obtained from our covering (59). Now we replace our covering by a simplicial space, i.e. we consider the diagram of maps between spaces

$$\mathfrak{Par} := \prod_i \Gamma_{Q_i}\backslash X \underset{\xleftarrow{p_2}}{\xleftarrow{p_1}} \prod_{i<j}\prod_{[R]\in[Q_i]\cap Q_j]} \Gamma_R\backslash X \begin{smallmatrix}\longleftarrow\\ \longleftarrow\\ \longleftarrow\end{smallmatrix} \tag{80}$$

this yields a spectral sequence with $E_1^{\bullet,q}$ term

$$\prod_i H^q(\Gamma_{Q_i}\backslash X,\tilde{\mathcal{M}}) \xrightarrow{d^{(0)}} \prod_{i<j}\prod_{[R]\in[P_i]\cap[P_j]} H^q(\Gamma_R\backslash X^R,\tilde{\mathcal{M}}) \xrightarrow{d^{(1)}} \tag{81}$$

Our covering also yields a simplicial space which is a subspace of (80) we get a map from (75) to (81) and this map is an isomorphism of complexes.

We replace $\mathfrak{Par}$ by another simplicial complex

$$\mathfrak{Parmax} := \prod_{[P]:d(P)=1} \Gamma_P\backslash X \overset{p_1}{\underset{p_2}{\leftleftarrows}} \prod_{[Q]:d(Q)=2} \Gamma_Q\backslash X \begin{smallmatrix}\longleftarrow\\ \longleftarrow\\ \longleftarrow\end{smallmatrix} \tag{82}$$

We have an obvious projection $\Pi : \mathfrak{Par} \to \mathfrak{Parmax}$ which induces a homomorphism

$$\begin{array}{ccccc} \prod_i H^q(\Gamma_{Q_i}\backslash X, \tilde{\mathcal{M}}) & \xrightarrow{d^{(0)}} & \prod_{i<j}\prod_{[R]\in[P_i]\cap[P_j]} H^q(\Gamma_R\backslash X^R, \tilde{\mathcal{M}}) & \xrightarrow{d^{(1)}} \\ \uparrow & & \uparrow & \\ \prod_{[P]:d(P)=1} H^q(\Gamma_P\backslash X, \tilde{\mathcal{M}}) & \xrightarrow{d^{(0)}} & \prod_{[R]:d(R)=2} H^q(\Gamma_R\backslash X^R, \tilde{\mathcal{M}}) & \xrightarrow{d^{(1)}} \end{array} \tag{83}$$

and an easy argument in homological algebra shows that this induces an isomorphism in cohomology or in other words an isomorphism of the $E_2^{p,q}$ terms of the two spectral sequences.

We had the covering

$$\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X) = \bigcup_{P:P\text{proper}} \Gamma_P\backslash X^P(\underline{c}_{\pi'}, r(\underline{c}_{\pi'})) \tag{84}$$

which gives us the spectral sequence converging to $H^\bullet(\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X), \tilde{\mathcal{M}})$ with

$$E_1^{p,q} = \bigoplus_{i_0<i_1<\cdots<i_p} \bigoplus_{[Q]\in[P_{i_0}]\cap[P_{i_1}]\cap\cdots\cap[P_{i_p}]} H^q(\Gamma_Q\backslash X^Q(\underline{c}_{\pi'}, r(\underline{c}_{\pi'}))) \tag{85}$$

Our covering of $\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X)$ gives us a simplicial space $\mathfrak{Cov}(\overset{\bullet}{\mathcal{N}})\Gamma\backslash X)$ and we have maps

$$\mathfrak{Cov}(\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X)) \hookrightarrow \mathfrak{Par} \to \mathfrak{Parmax}. \tag{86}$$

We saw that the resulting maps induced an isomorphism in the $E_2^{p,q}$ terms of the spectral sequences. Hence we see that $\mathfrak{Parmax}$ yields a spectral sequence

$$E_1^{p,q} = \bigoplus_{[P]:d(P)=p+1} H^q(\Gamma_P\backslash X, \tilde{\mathcal{M}}) \Rightarrow H^{p+q}(\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X), \tilde{\mathcal{M}})) \tag{87}$$

At this point we want to raise an interesting question

Does this spectral sequence degenerate at $E_2^{p,q}$ level?

The author of this book is hoping that the answer to this question is no! And this is so for interesting reasons! We come back to this question when we discuss the Eisenstein cohomology.

The complement of $\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X)$ is a relatively compact open set $V \subset \Gamma\backslash X$, this map contains the stable points. We define $\tilde{\mathcal{M}}_V^! = i_{V,!}(\tilde{\mathcal{M}})$ then we get an exact sequence

$$0 \to \tilde{\mathcal{M}}_V^! \to \tilde{\mathcal{M}} \to \tilde{\mathcal{M}}/\tilde{\mathcal{M}}_V^! \to 0 \tag{88}$$

and $\tilde{\mathcal{M}}/\tilde{\mathcal{M}}^!_V$ is obviously the extension of the restriction of $\tilde{\mathcal{M}}$ to $\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X)$ and the extended by zero to $\Gamma\backslash X$. We claim (easy proof later) that

$$H^\bullet_c(\Gamma\backslash X, \tilde{\mathcal{M}}) = H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}^!_V) \tag{89}$$

and this gives us again the fundamental exact sequence

$$H^{q-1}(\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X), \tilde{\mathcal{M}}) \to H^q(\Gamma\backslash X, \tilde{\mathcal{M}}^!_V) \to H^q(\Gamma\backslash X, \tilde{\mathcal{M}}) \to H^q(\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X), \tilde{\mathcal{M}}) \to \tag{90}$$

3.2.5 How to compute the cohomology groups $H^q_c(\Gamma\backslash X, \tilde{\mathcal{M}})$

We apply the considerations in 4.8 from the [book]. Again we cover $\Gamma\backslash X$ by orbiconvex open neighborhoods U_{x_i}, and now we define

$$\tilde{\mathcal{M}}^!_{\underline{x}} = (i_{\underline{x}})_! i^*_{\underline{x}}(\tilde{\mathcal{M}}).$$

These sheaves have properties, which are dual to those of the sheaves $\tilde{\mathcal{M}}_{ulx}$. If $\underline{x} = (x_1, \dots, x_s)$ and if we add another point $\underline{x}' = (x_1, \dots, x_s, x_{s+1})$ then we have the restriction $\tilde{\mathcal{M}}_{\underline{x}} \to \tilde{\mathcal{M}}_{\underline{x}'}$, which were used to construct the Čech resolution.

Now let $d = \dim(X)$. For the ! sheaves we get (See [book] , loc. cit.) get a morphism $\tilde{\mathcal{M}}^!_{\underline{x}'} \to \tilde{\mathcal{M}}^!_{\underline{x}}$. For $\underline{x} = (x_1, \dots, x_s)$ we define the degree $d(\underline{x}) = d + 1 - s$. Then we construct the Čech-coresolution (See [book], 4.8.3)

$$\to \prod_{\underline{x}:d(\underline{x})=q} \tilde{\mathcal{M}}^!_{\underline{x}} \to \cdots \to \prod_{(x_i,x_j)} \tilde{\mathcal{M}}^!_{x_i,x_j} \to \prod_{x_i} \tilde{\mathcal{M}}^!_{x_i} \to i_!(\tilde{\mathcal{M}}) \to 0.$$

Now we have a dual statement to the proposition with label **acyc**

Proposition: **(acyc!)** *If $d = \dim(X)$ then*

$$H^q(U_{\tilde{\underline{x}}}, \tilde{\mathcal{M}}^!_{\underline{x}}) = \begin{cases} \mathcal{M}_{\Gamma_{\tilde{y}}} & q = d \\ 0 & q \neq d \end{cases}$$

Hence the above complex of sheaves provides a complex of modules $C^\bullet_!(\mathfrak{U}, \tilde{\mathcal{M}})$:

$$\to \prod_{\underline{x}:d(\underline{x})=q} H^d(U_{\underline{x}}, \tilde{M}^!_{\underline{x}}) \to \cdots \to \prod_{(x_i,x_j)} H^d(U_{x_i,x_j}, \tilde{\mathcal{M}}^!_{x_i,x_j}) \to \prod_{x_i} \tilde{H}^d(U_{x_i}, \tilde{\mathcal{M}}^!_{x_i}) \to 0.$$

Now it is clear that

$$H^q(\Gamma\backslash X, i_!(\tilde{\mathcal{M}})) = H^q_c(\Gamma\backslash X, \tilde{\mathcal{M}}) = H^q(C^\bullet_!(\mathfrak{U}, \tilde{\mathcal{M}})).$$

Now let us assume that $\mathcal{M}$ is a finitely generated module over some commutative noetherian ring R with identity. Then clearly all our cohomology groups will be R-modules.

Our Theorem A above implies

Theorem (Raghunathan) *Under our general assumptions all the cohomology groups $H^q_c(\Gamma\backslash X, \tilde{\mathcal{M}}), H^q(\Gamma\backslash X, \tilde{\mathcal{M}}), H^q_!(\Gamma\backslash X, \tilde{\mathcal{M}}), H^q(\partial(\Gamma\backslash X), \tilde{\mathcal{M}})$ are finitely generated R modules.*

3.2.6 The case $\Gamma = \mathbf{Sl}_2(\mathbb{Z})$

In the following $\mathcal{M}$ can be any Γ-module. We investigate the fundamental exact sequence for this special group.

Of course we start again from our covering $\Gamma\backslash\mathbb{H} = U_i \cup U_\rho$. The cohomology with compact supports is the cohomology of the complex

$$0 \to H^2(U_i \cap U_\rho, \tilde{\mathcal{M}}^!_{i,\rho}) \to H^2(U_i \ , \tilde{\mathcal{M}}^!_i) \oplus H^2(U_\rho, \tilde{\mathcal{M}}^!_\rho) \to 0.$$

Now we have $H^2(U_i \cap U_\rho, \tilde{\mathcal{M}}^!_{i,\rho}) = M, H^2(U_i \ , \tilde{\mathcal{M}}^!_i) = \mathcal{M}_{\Gamma_i} = \mathcal{M}/(\mathrm{Id}-S)M, H^2(U_\rho \ , \tilde{\mathcal{M}}^!_\rho) = M_{\Gamma_\rho} = \mathcal{M}/(\mathrm{Id}-R)\mathcal{M}$ and hence we get the complex

$$0 \to \mathcal{M} \to \mathcal{M}_{\Gamma_i} \oplus \mathcal{M}_{\Gamma_\rho} \to 0$$

and from this we obtain

$$H^1(\Gamma\backslash\mathbb{H}, i_!(\mathcal{M})) = \ \ker(\mathcal{M} \to (M/(\mathrm{Id}-S)\mathcal{M} \oplus \mathcal{M}/(\mathrm{Id}-R)\mathcal{M}))$$

and

$$H^0(\Gamma\backslash\mathbb{H}, i_!(\mathcal{M})) = 0, H^2(\Gamma\backslash\mathbb{H}, i_!(\mathcal{M})) = \mathcal{M}_\Gamma$$

We discuss the fundamental exact sequence in this special case. To do this we have to understand the cohomology of the boundary $H^\bullet(\partial(\Gamma\backslash\mathbb{H}, \tilde{M})$. We discussed the Borel-Serre compactification and saw that in this case we get this compactification if we add a circle at infinity to our picture of the quotient. But we may as well cut the cylinder at any level $c > 1$, i.e. we consider the level line $\mathbb{H}(c) = \{z = x + ic | z \in \mathbb{H}\}$ and divide this level line by the action of the translation group

$$\Gamma_U = \{\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} | n \in \mathbb{Z}\} = \{\begin{pmatrix} \epsilon & n \\ 0 & \epsilon \end{pmatrix} | n \in \mathbb{Z}, \epsilon = \pm 1\}/\{\pm \mathrm{Id}\}.$$

But this quotient is homotopy equivalent to the cylinder

$$\Gamma_U\backslash\mathbb{H} \simeq \Gamma_U\backslash\mathbb{H}(c).$$

We apply our general consideration on cohomology of arithmetic groups to this situation and find

$$H^\bullet(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}) = H^\bullet(\Gamma_U\backslash\mathbb{H}, \mathrm{sh}_{\Gamma_U}(\mathcal{M})) = H^\bullet(\Gamma_U\backslash\mathbb{H}(c), \mathrm{sh}_{\Gamma_U}(\mathcal{M})).$$

This cohomology is easy to compute. The group Γ_U is generated by the element $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. It is rather clear that

$$H^0(\Gamma_U\backslash\mathbb{H}, \mathrm{sh}_{\Gamma_U}(\mathcal{M})) = \mathcal{M}^{\Gamma_U}, H^1(\Gamma_U\backslash\mathbb{H}, \mathrm{sh}_{\Gamma_U}(\mathcal{M})) = \mathcal{M}_{\Gamma_U} = \mathcal{M}/(\mathrm{Id}-T)\mathcal{M}.$$

Then our fundamental exact sequence becomes

$$0 \to \mathcal{M}^\Gamma \to \mathcal{M}^{\Gamma_U} \to \ \ker(\mathcal{M} \to (\mathcal{M}/(\mathrm{Id}-S)\mathcal{M} \oplus \mathcal{M}/(\mathrm{Id}-R)\mathcal{M})) \xrightarrow{j} \mathcal{M}/(\mathcal{M}^{\Gamma_i} \oplus \mathcal{M}^{\Gamma_\rho}) \xrightarrow{r}$$
$$\mathcal{M}/(\mathrm{Id}-T)\mathcal{M} \to \mathcal{M}_\Gamma \to 0$$

Now it may come as a little surprise to the readers, that we can formulate a little exercise which is not entirely trivial

Exercise: *Write down explicitly all the arrows in the above fundamental sequence*

We give the answer without proof. I change notation slightly and work with the matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, R = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

and we have the relation

$$RS = T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Then $\Gamma_i =< S >, \Gamma_\rho =< R >$. The map

$$\mathcal{M}/(\mathcal{M}^{<S>} \oplus \mathcal{M}^{<R>}) \to \mathcal{M}/(\mathrm{Id} - T)\mathcal{M}$$

is given by

$$m \mapsto m - Sm$$

We have to show that this map is well defined: If $m \in \mathcal{M}^{<S>}$ then $m \mapsto 0$. If $m \in \mathcal{M}^{<R>}$ then

$$m - Sm = m - SR^{-1}m = m - Tm$$

and this is zero in $\mathcal{M}/(\mathrm{Id} - T)\mathcal{M}$.

The map

$$\ker(\mathcal{M} \to (\mathcal{M}/(\mathrm{Id} - S)\mathcal{M} \oplus \mathcal{M}/(\mathrm{Id} - R)\mathcal{M})) \to \mathcal{M}/(\mathcal{M}^{<S>} \oplus \mathcal{M}^{<R>})$$

is a little bit delicate. We pick an element m in the kernel, hence we can write it as

$$m = m_1 - Sm_1 = m_2 - R^{-1}m_2$$

and send $m \mapsto m_1 - m_2$ (Here we have to use the orientation). If we modify m_1, m_2 to $m_1' = m_1 + n_1, m_2' = m_2 + n_2$ then $m_1' - m_2'$ gives the same element in $\mathcal{M}/(\mathcal{M}^{<S>} \oplus \mathcal{M}^{<R>})$.

This answer can only be right if $m_1 - m_2$ goes to zero under the map r, i.e. we have to show that

$$m_1 - m_2 - S(m_1 - m_2) \in (\mathrm{Id} - T)\mathcal{M}$$

We compute

$$m_1 - m_2 - S(m_1 - m_2) = m - m_2 + Sm_2 = m - m_2 + R^{-1}m_2 - R^{-1}m_2 + Sm_2 =$$

$$-R^{-1}m_2 + Sm_2 = -T^{-1}Sm_2 + Sm_2 \in (\mathrm{Id} - T)\mathcal{M}$$

Finally we claim that the map $\mathcal{M}^{<T>} \to \ker(\mathcal{M} \to (\mathcal{M}/(\mathrm{Id} - S)\mathcal{M} \oplus \mathcal{M}/(\mathrm{Id} - R)\mathcal{M}))$ is given by $m \mapsto m - Sm = m - R^{-1}T^{-1}m = m - R^{-1}m$.

Final remark: The reader may get the impression that it is easy to compute the cohomology, but the contrary is true. In the case $\Gamma = \mathrm{Sl}_2(\mathbb{Z})/\pm\mathrm{Id}$ we found formulae for the rank of the cohomology groups, this seems to be a satisfactory answer, but it is not. The point is that in the next section we will introduce the Hecke operators, these Hecke operators form an algebra of endomorphisms of the cohomology groups. It is a fundamental question (see further down) to understand the cohomology as a module under the action of this Hecke algebra. It is difficult to write down the effect of a Hecke operator on a module like $\mathcal{M}/(\mathcal{M}^{\Gamma_i}+\mathcal{M}^{\Gamma_\rho})$. We will discuss an explicit example in (4.3.2.)

The situation is even worse if we consider the case $\Gamma = \mathrm{Gl}_2(\mathbb{Z}[i])/\{(i^\nu \mathrm{Id})\}$. First of all we notice that it is not possible to read off the dimensions of the individual groups $H^i(\Gamma\backslash\mathbb{H}_3,\tilde{M})$ from the complex in 3.2.2) . Of course we can compute them in any given case, but our method does not give any kind of theoretical insight.

We will see later that we can prove vanishing theorems $H^i(\tilde{\Gamma}\backslash\mathbb{H}_3,\tilde{M}_{\mathbb{C}})$ for certain coefficient systems $\tilde{M}_{\mathbb{C}}$ by transcendental means. These results can not be obtained by our elementary methods.

4 Hecke Operators

4.1 The construction of Hecke operators

We mentioned already that the cohomology and homology groups of an arithmetic group has an additional structure. We have the action of the so-called Hecke algebra. The following description of the Hecke algebra is somewhat provisorial, we get a richer Hecke algebra, if we work in the adelic context (See Chap III). But the desription here is more intuitive.

We start from the arithmetic group $\Gamma\subset G(\mathbb{Q})$ and an arbitrary Γ-module $\mathcal{M}$. The module $\mathcal{M}$ is also a module over a ring R which in the beginning may be simply $\mathbb{Z}$.

At this point it is better to have a notation for this action

$$\Gamma\times\mathcal{M}\to\mathcal{M},(\gamma,m)\mapsto r(\gamma)(m)$$

where now $r:\Gamma\to\mathrm{Aut}(\mathcal{M})$.

We assume that $\mathcal{M}$ is a module over a ring R in which we can invert the orders of the stabilizers of fixed points of elements $\gamma\in\Gamma$.

If we have a subgroup $\Gamma'\subset\Gamma$ of finite index, then we constructed maps

$$\begin{aligned}\pi^\bullet_{\Gamma',\Gamma}: & H^\bullet(\Gamma\backslash X,\tilde{\mathcal{M}})\longrightarrow H^\bullet(\Gamma'\backslash X,\tilde{\mathcal{M}})\\ \pi_{\Gamma',\Gamma,\bullet}: & H^\bullet(\Gamma\backslash X,\tilde{\mathcal{M}})\longrightarrow H^\bullet(\Gamma'\backslash X,\tilde{\mathcal{M}})\end{aligned}$$

(see 2.1.1).

We pick an element $\alpha\in G(\mathbb{Q})$. The group

$$\Gamma(\alpha^{-1})=\alpha^{-1}\Gamma\alpha\cap\Gamma$$

is a subgroup of finite index in Γ and the conjugation by α induces an isomorphism

$$\mathrm{inn}(\alpha):\Gamma(\alpha^{-1})\longrightarrow\Gamma(\alpha).$$

We get an isomorphism

$$j(\alpha):\Gamma(\alpha^{-1})\backslash X\longrightarrow\Gamma(\alpha)\backslash X$$

which is induced by the map $x\longrightarrow\alpha x$ on the space X. This yields an isomorphism of cohomology groups

$$j(\alpha)^\bullet: H^\bullet(\Gamma(\alpha^{-1})\backslash X,\tilde{\mathcal{M}})\longrightarrow H^\bullet(\Gamma(\alpha)\backslash X,j(\alpha)_*(\tilde{\mathcal{M}})).$$

We compute the sheaf $j(\alpha)_*(\tilde{\mathcal{M}})$. For a point $x \in \Gamma(\alpha)\backslash X$ we have $j(\alpha)_*(\tilde{\mathcal{M}})_x = \tilde{\mathcal{M}}_{x'}$ where $j(\alpha)(x') = X$. We have the projection $\pi_{\Gamma(\alpha^{-1})} : X \to \Gamma(\alpha^{-1})\backslash X$, and the definition yields

$$(\tilde{\mathcal{M}})'_x = \left\{ s : \pi^{-1}_{\Gamma(\alpha^{-1})}(x') \to \mathcal{M} \mid s(\gamma m) = \gamma s(m) \text{ for all } \gamma \in \Gamma(\alpha^{-1} \right\}$$

The map $z \longrightarrow \alpha z$ provides an identification $\pi^{-1}_{\Gamma(\alpha^{-1})}(x') \xrightarrow{\sim} \pi^{-1}_{\Gamma(\alpha)}(x)$ in in terms of this fibre we can describe the stalk at x as

$$j(\alpha)_*(\tilde{\mathcal{M}})_x = \left\{ s : \pi^{-1}_{\Gamma(\alpha)}(x) \to \mathcal{M} \mid s(\gamma v) = \alpha^{-1}\gamma\alpha s(v) \text{ for all } \gamma \in \Gamma(\alpha) \right\}.$$

Hence we see: We may use α to define a new $\Gamma(\alpha)$-module $\mathcal{M}^{(\alpha)}$: The underlying abelian group of $\mathcal{M}^{(\alpha)}$ is $\mathcal{M}$ but the operation of $\Gamma(\alpha)$ is given by

$$(\gamma, m) \longrightarrow (\alpha^{-1}\gamma\alpha)m = \gamma *_\alpha m.$$

Then we have obviously that the sheaf $j(\alpha)_*(\tilde{\mathcal{M}})$ is equal to $\tilde{\mathcal{M}}^{(\alpha)}$. Hence we see that every element

$$u_\alpha \in \operatorname{Hom}_{\Gamma(\alpha)}(\mathcal{M}^{(\alpha)}, \mathcal{M})$$

defines a map $\tilde{u}_\alpha : j(\alpha)_*(\tilde{\mathcal{M}}) \to \tilde{\mathcal{M}}$. Hence we get a diagram

$$\begin{array}{ccccc}
H^\bullet(\Gamma(\alpha^{-1})\backslash X, \tilde{\mathcal{M}}) & \xrightarrow{j(\alpha)^\bullet} & H^\bullet(\Gamma(\alpha)\backslash X, j(\alpha)_*(\tilde{\mathcal{M}})) & \xrightarrow{\tilde{u}_\alpha^\bullet} & H^\bullet(\Gamma(\alpha)\backslash X, \tilde{\mathcal{M}}) \quad (91) \\
\uparrow \pi^\bullet & & & & \downarrow \pi_\bullet \quad (92) \\
H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}) & & \xrightarrow{\qquad T(\alpha, u_\alpha) \qquad} & & H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}) \quad (93)
\end{array}$$

where the operator on the bottom line is the Hecke operator. It depends on two data, namely, the element $\alpha \in G(\mathbb{Q})$ and the choice of $u_\alpha \in \operatorname{Hom}_{\Gamma(\alpha)}(\mathcal{M}^{(\alpha)}, \mathcal{M})$.

It is not difficult to show that the operator $T(\alpha, u_\alpha)$ depends only on the double coset $\Gamma\ \alpha\ \Gamma$, provided we adapt the choice of u_α. To be more precise if

$$\alpha_1 = \gamma_1\alpha\gamma_2 \qquad\qquad \gamma_1, \gamma_2 \in \Gamma,$$

then we have an obious bijection

$$\Phi_{\gamma_1,\gamma_2} : \operatorname{Hom}_{\Gamma(\alpha)}(\mathcal{M}^{(\alpha)}, \mathcal{M}) \longrightarrow \operatorname{Hom}_{\Gamma(\alpha_1)}(\mathcal{M}^{\alpha_1)}, \mathcal{M})$$

which is given by

$$\Phi_{\gamma_1,\gamma_2}(u_\alpha) = u_{\alpha_1} = \gamma_1 u_\alpha \gamma_2.$$

The reader will verify without difficulties that

$$T(\alpha, u_\alpha) = T(\alpha_1, u_{\alpha_1}).$$

(Verify this for H^0 and then use some kind of resolution)

There is a case where we have also a rather obvious choice of u_α. This is the case if $R \subset \mathbb{Q}$ and our Γ-module $\mathcal{M}$ is a R-lattice in the $\mathbb{Q}$-vector space $\mathcal{M}_\mathbb{Q}$, where $\mathcal{M}_\mathbb{Q}$ is a rational $G(\mathbb{Q})$ module, i.e. is obtained from a rational (finite dimensional) representation of our group $G/\mathbb{Q}$.

Then we have the canonical choice of an

$$u_{\alpha,\mathbb{Q}} : \mathcal{M}_\mathbb{Q}^{(\alpha)} \longrightarrow \mathcal{M}_\mathbb{Q},$$

which is given by $m \mapsto \alpha m$. But this morphism will not necessarily map the lattice $\mathcal{M}^{(\alpha)}$ into $\mathcal{M}$. It is also bad if $u_{\alpha,\mathbb{Q}}$ maps $\mathcal{M}^{(\alpha)}$ into $b\mathcal{M}$, where b is an integer > 1. But then we can find a unique rational number $d(\alpha) > 0$ for which

$d(\alpha) \cdot u_{\alpha,\mathbb{Q}} : \mathcal{M}^{(\alpha)} \longrightarrow \mathcal{M}$ and $d(\alpha) \cdot u_{\alpha,\mathbb{Q}}(\mathcal{M}^{(\alpha)}) \not\subset b\mathcal{M}$ for any integer $b > 1$.

Then $u_\alpha = d(\alpha) \cdot u_{\alpha,\mathbb{Q}}$ is called the *normalized* choice. The canonical choice defines endomorphisms on the rational cohomology, i.e. the cohomology with coefficients in $\tilde{\mathcal{M}}_{\mathbb{Q}}$ whereas the normalized Hecke operators induce endomorphism of the integral cohomology.

We see that we can construct many endomorphisms $T(\alpha, u_\alpha) : H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}) \to H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}})$. These endomorphisms will generate an algebra

$$\mathcal{H}_{\Gamma,\tilde{\mathcal{M}}} \subset \operatorname{End}(H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}})).$$

This is the so-called Hecke algebra. We can also define endomorphisms $T(\alpha, u_\alpha)$ on the cohomology with compact supports, on the inner cohomology and the cohomology of the boundary. Since the operators are compatible with all the arrows in the fundamental exact sequence we denote them by the same symbol.

We now assume that $\mathcal{M}$ is a finitely generated R module where R is the ring of integers in an algebraic number field K/Q. Then our cohomology groups $H^q(\Gamma\backslash X, \tilde{\mathcal{M}})$ are finitely generated R-modules with an action of the algebra $\mathcal{H}$ on it. The Hecke algebra also acts on the inner cohomology $H^q_!(\Gamma\backslash X, \tilde{\mathcal{M}})$ If we tensorize our coefficient system with any number field $L \supset K$, then we write $M_L = M \otimes L$.

We state without proof :

Theorem 4.1. *Let $\mathcal{M}$ be a module obtained by a rational representation. For any extension $L/K/\mathbb{Q}$ the $\mathcal{H}_\Gamma \otimes L$ module $H^q_!(\Gamma\backslash X, \tilde{\mathcal{M}}_L)$ is semi simple, i.e. a direct sum of irreducible $\mathcal{H}_\Gamma$ modules.*

The proof of this theorem will be discussed in Chap.3, it requires some input from analysis. We tensorize our coefficient system by $\mathbb{C}$, i.e. we consider $\mathcal{M}_L \otimes_L \mathbb{C} = \mathcal{M}_{\mathbb{C}}$. Let us assume that Γ is torsion free. First of all start from the well known fact, that the cohomology $H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}_{\mathbb{C}})$ can be computed from the de-Rham-complex

$$H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}_{\mathbb{C}}) = H^\bullet(\Omega^\bullet \otimes \tilde{\mathcal{M}}_{\mathbb{C}}(\Gamma\backslash X)).$$

We introduces some specific positive definite hermitian form on $\mathcal{M}_{\mathbb{C}}$ and this allows us to define a hermitian scalar product between two $\tilde{\mathcal{M}}_{\mathbb{C}}$ -valued p-forms

$$< \omega_1, \omega_2 >= \int_{\Gamma\backslash X} \omega_1 \wedge *\omega_2,$$

provided one of the forms is compactly supported.

This will give us a positive definite scalar product on $H^p_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{n,\mathbb{C}})$, In the classical case of Gl_2 this is the Peterson scalar product. Finally we show that $\mathcal{H}_\Gamma$ is self adjoint with respect to this scalar product, and then semi-simplicity follows from the standard argument.

4.1.1 Commuting relations

We want to say some words concerning the structure of the Hecke algebra.

To begin we discuss the action of the Hecke-algebra on $H^0(\Gamma\backslash X, \tilde{\mathcal{M}})$. We have to do this since we defined the cohomology in terms of injective (or acyclic) resolutions and therefore the general results concerning the structure of the Hecke algebra can be reduced to this special case.

If we have a Γ-module $\mathcal{M}$ and if we look at the diagram defining the Hecke operators, then we see that we get in degree 0

$$\begin{array}{ccc} \mathcal{M}^{\Gamma(\alpha^{-1})} & \longrightarrow (\mathcal{M}^{(\alpha)})^{\Gamma(\alpha)} \xrightarrow{u_\alpha} & \mathcal{M}^{\Gamma(\alpha)} \\ \uparrow & & \downarrow \\ \mathcal{M}^{\Gamma} & \xrightarrow{\quad T(\alpha,u_\alpha) \quad} & \mathcal{M}^{\Gamma} \end{array}$$

where the first arrow on the top line is induced by the identity map $\mathcal{M} \to \mathcal{M}^{(\alpha)} = \mathcal{M}$ and the second by a map $u_\alpha \in \operatorname{Hom}_{\mathbf{Ab}}(\mathcal{M}, \mathcal{M})$ which satisfies $u_\alpha((\alpha\gamma\alpha^{-1})m) = \gamma u_\alpha(m)$. Recalling the definition of the vertical arrow on the right, we find

$$T(\alpha, u_\alpha)(v) = \sum_{\gamma\in\Gamma/\Gamma(\alpha)} \gamma \cdot u_\alpha(v).$$

We are interested to get formulae for the product of Hecke oprators, so, for instance, we would like to show that under certian assumptions on α, β and certain adjustment of u_α, u_β and $u_{\alpha\beta}$ we can show

$$T(\alpha, u_\alpha) \cdot T(\beta, u_\beta) = T(\beta, u_\beta) \cdot T(\alpha, u_\alpha) = T(\alpha\beta, u_{\alpha\beta}).$$

It is easy to see what the conditions are if we want such a formula to be true. We look at what happens in H^0 and get

$$T(\alpha, u_\alpha) \cdot T(\beta, u_\beta)(v) = \sum_{\gamma\in\Gamma/\Gamma(\alpha)} \sum_{\eta\in\Gamma/\Gamma(\beta)} \gamma u_\alpha \cdot \eta u_\beta(v).$$

We rewrite the right hand slightly illegally:

$$\sum_{\gamma\in\Gamma/\Gamma(\alpha)} \sum_{\eta\in\Gamma/\Gamma(\beta)} \gamma u_\alpha \eta u_\alpha^{-1} u_\alpha u_\beta(v),$$

where we have to take into account that this does not make sense because the term $\gamma u_\alpha \eta u_\alpha^{-1}$ is not defined. But let us assume that (i) for each η

we can find an η' such that

$$\eta' \circ u_\alpha = u_\alpha \circ \eta,$$

where these η' also form a sytem of representatives for $\Gamma/\Gamma(\beta)$ (ii) The

elements $\gamma\eta'$ and $\eta'\gamma$ form a system of representatives for $\Gamma/\Gamma(\alpha\beta)$ (iii)

$u_\alpha u_\beta(v) = u_\beta u_\alpha(v) = u_{\alpha\beta}(v)$, then we get a legal rewrite

$$T(\alpha, u_\alpha) \cdot T(\beta, u_\beta)(v) = \sum_{\gamma\in\Gamma/\Gamma(\alpha)} \sum_{\eta'\in\Gamma/\Gamma(\beta)} \gamma\eta' u_\alpha u_\beta(v) = \sum_{\xi\in\Gamma/\Gamma(\alpha\beta)} \xi u_{\alpha\beta}(v) =$$

$$T(\alpha\beta, u_{\alpha\beta})(v)$$

We want to explain in a special case that we may have relations like the one above.

Let S be a finite set of primes, let $|S|$ be the product of these primes. Then we define $\Gamma_S = G(\mathbb{Z}[\frac{1}{|S|}])$. We say that $\alpha \in G(\mathbb{Q})$ has support in S if $\alpha \in G(\mathbb{Z}[\frac{1}{|S|}])$.

We take the group $\Gamma = \mathrm{Sl}_d(\mathbb{Z})$, and we take two disjoint sets of primes S_1, S_2. For the group Γ one can prove the so-called strong approximation theorem which asserts that for any natural number m the map

$$\mathrm{Sl}_d(\mathbb{Z}) \longrightarrow \mathrm{Sl}_d(\mathbb{Z}/m\mathbb{Z})$$

is surjective. (This special case is actually not so difficult. The theorem holds for many other arithmetic groups, for instance for simply connected Chevalley schemes over $\mathrm{Spec}(\mathbb{Z})$.)

We consider the case

$$\alpha = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_d \end{pmatrix} \in \Gamma_{S_1}, \beta = \begin{pmatrix} b_1 & & & \\ & b_2 & & \\ & & \ddots & \\ & & & b_d \end{pmatrix} \in \Gamma_{S_2},$$

where $a_d | a_{d-1} \dots | a_1$ and $b_d | b_{d-1} | \dots | b_1$. It is clear that we can find integers n_1 and n_2 which are only divisible by the primes in S_1 and S_2 respectively, so that

$$\Gamma(n_i) \subset \Gamma(\alpha^{-1}), \Gamma(n_2) \subset \Gamma(\beta^{-1}),$$

where the $\Gamma(n_i)$ are the full congruence subgroups $\mod n_1$ and n_2 respectively. Since we have

$$\mathrm{Sl}_d(\mathbb{Z}/n\mathbb{Z}) = \mathrm{Sl}_d(\mathbb{Z}/n_1\mathbb{Z}) \times \mathrm{Sl}_d(\mathbb{Z}/n_2\mathbb{Z})$$

we get

$$\Gamma/\Gamma(\alpha^{-1}\beta^{-1}) \xrightarrow{\sim} \Gamma/\Gamma(\alpha^{-1}) \times \Gamma/\Gamma(\beta^{-1}).$$

On the right hand side we can chose representatives γ for $\Gamma/\Gamma(\alpha^{-1})$ which satisfy $\gamma \equiv \mathrm{Id} \mod n_2$ and η for $\Gamma/\Gamma(\beta^{-1})$ which satisfy $\eta \equiv \mathrm{Id}$ $\mod n_1$. Then the products $\gamma\eta$ will form a system of representatives for $\Gamma/\Gamma(\alpha^{-1}\beta^{-1})$. But then we clearly have $u_\alpha \eta = \eta u_\alpha$ and we see that (i) and (ii) above are true. Then we can put $u_{\alpha\beta} = u_\alpha u_\beta$.

We consider the case that our module $\mathcal{M}$ is a R-lattice in $\mathcal{M}_\mathbb{Q}$, where $\mathcal{M}_\mathbb{Q}$ is a rational $G(\mathbb{Q})$-module. Then we saw that we can write

$$u_\alpha = d(\alpha) \cdot \alpha$$

where $d(\alpha)$ will be a product of powers of the primes p dividing n_1 and an anologous statement can be obtained for β and n_2.

Since we have $\alpha\beta = \beta\alpha$ and since clearly $d(\alpha)d(\beta) = d(\alpha\beta)$ we also get the commutation relation.

Of course we have to be careful here. We only proved it for the rather uninteresting case of $H^0(\Gamma\backslash X, \mathcal{M})$. If we want to prove it for cohomology in higher degrees, we have to choose an acyclic resolution

$$0 \longrightarrow \mathcal{M} \longrightarrow A^0 \longrightarrow A^1 \longrightarrow \dots,$$

We have to extend the maps u_α, u_β to this complex

$$\begin{array}{ccccc} 0 \longrightarrow & \mathcal{M}^{(\alpha)} \longrightarrow & (A^\bullet)^{(\alpha)} \\ & \Big\downarrow u_\alpha & \Big\downarrow u_\alpha \\ 0 \longrightarrow & \mathcal{M} \longrightarrow & A^\bullet, \end{array}$$

and we have to prove that the relation

$$u_\alpha \eta u_\beta = \eta' u_\alpha u_\beta = \eta' u_{\alpha\beta}$$

also holds on the complex. If we can prove this, it becomes clear that the commutation rule also holds in higher degrees.

We choose the special resolution

$$0 \longrightarrow \mathcal{M} \longrightarrow \ \mathrm{Ind}_1^\Gamma \mathcal{M} \longrightarrow .$$

It is clear that if suffices to show: If we selected the u_α, u_β in such a way that we have the condition (i), (ii) and (iii) above satisfied, then we can choose extensions $u_\alpha, u_\beta, u_{\alpha\beta}$ to $\mathrm{Ind}_1^\Gamma \mathcal{M}$ so that (i), (ii) and (iii) are also satisfied. Once we have done this we can proceed by induction.

We have the diagram of $\Gamma(\alpha)$-modules

$$\begin{array}{ccccc} 0 \longrightarrow & \mathcal{M}^{(\alpha)} \longrightarrow & (\ \mathrm{Ind}_1^\Gamma \mathcal{M})^{(\alpha)} \\ & \Big\downarrow u_\alpha & \Big\downarrow ? \\ 0 \longrightarrow & \mathcal{M} \longrightarrow & \mathrm{Ind}_1^\Gamma \mathcal{M}, \end{array}$$

and we are searching for a suitable vertical arrow ?. The horizontal arrows are given by (as before)

$$i : m \longrightarrow f_m : \{\gamma \longrightarrow \gamma m\}.$$

To get a map

$$? \in \ \mathrm{Hom}_{\Gamma(\alpha)}\left(\left(\ \mathrm{Ind}_1^\Gamma m\right)^{(\alpha)},\ \mathrm{Ind}_1^\Gamma \mathcal{M}\right)$$

we apply Frobenius reciprocity: We choose representatives $\gamma_1 \dots \gamma_m$ of $\Gamma/\Gamma(\alpha)$; then our $\Gamma(\alpha)$-module in the second argument is

$$\mathrm{Ind}_1^\Gamma \mathcal{M} \simeq \bigoplus_{\gamma_i} \ \mathrm{Ind}_1^{\Gamma(\alpha)} \mathcal{M}$$

where $f \in \ \mathrm{Ind}_1^\Gamma \mathcal{M}$ is mapped to $(f_1, \dots, f_m) \in \ \mathrm{Ind}_1^{\Gamma(\alpha)}$, and where

$$f_i(\gamma) = f(\gamma_i \gamma).$$

Hence we have

$$\mathrm{Hom}_{\Gamma(\alpha)}\left(\left(\ \mathrm{Ind}_1^\Gamma \mathcal{M}\right)^{(\alpha)},\ \mathrm{Ind}_1^\Gamma \mathcal{M}\right) \simeq \bigoplus_{x_i} \ \mathrm{Hom}_{\{1\}}\left(\ \mathrm{Ind}_1^\Gamma \mathcal{M}, \mathcal{M}\right).$$

an element $\Phi_{\gamma_i} :\ \mathrm{Hom}_{\{1\}}(\ \mathrm{Ind}_1^\Gamma \mathcal{M}, \mathcal{M})$ is a collection of homomorphisms

$$\varphi_{\gamma_i,\gamma} : \mathcal{M} \longrightarrow \mathcal{M},$$

so that almost all of them are zero on $\Phi_{\gamma_i}(f) = \varphi_{\gamma_i,\gamma}(f(\gamma))$. The homomorphism

$$i \circ u_\alpha \in \ \mathrm{Hom}_{\Gamma(\alpha)}(\mathcal{M}^{(\alpha)},\ \mathrm{Ind}_1^\Gamma \mathcal{M}) = \ \mathrm{Hom}_{\{1\}}(\mathcal{M}, \oplus_{\gamma_1}\mathcal{M})$$

is by definition given by the vector of maps

$$m \longrightarrow (\ldots, f_{u_\alpha(m)}(\gamma_i), \ldots) = (\ldots \gamma_i u_\alpha(m), \ldots).$$

Hence we define ??? by the conditions that

$$\varphi_{\gamma_i,\gamma} : m \longrightarrow \begin{cases} \gamma_i u_\alpha(m) & \text{for } \gamma = 1 \\ \\ 0 & \text{for } \gamma \neq 1, \end{cases}$$

and we get the required commutative diagram. This morphism ? is now the extension of $u_\alpha : \mathcal{M}^{(\alpha)} \to \mathcal{M}$ to $(\ \mathrm{Ind}_{\{1\}}^\Gamma \mathcal{M})^{(\alpha)} \longrightarrow \ \mathrm{Ind}_{\{1\}}^\Gamma \mathcal{M}$. It is clear that under the assumption (i), (ii), (iii) for the morphisms $u_\alpha : \mathcal{M}^{(\alpha)} \to \mathcal{M}$ and $u_\beta : \mathcal{M}^{(\beta)} \to \mathcal{M}$ the extensions also satisfy (i), (ii), (iii).

Hence we see that under our special assumptions on α, β we have

$$T(\beta, u_\beta) \cdot T(\alpha, u_\alpha) = T(\beta\alpha, u_{\beta\alpha})$$

on all the cohomology groups $H^\bullet(\mathrm{Sl}_d(\mathbb{Z})\backslash X, \tilde{\mathcal{M}})$.

4.1.2 Relations between Hecke operators

We attach a Hecke operator to any coset $\Gamma\alpha\Gamma$ where $\alpha \in \mathrm{Gl}_2^+(\mathbb{Q})$ (i.e. $\det(\alpha) > 0$, we want α to act on the upper half plane). The center of $\mathrm{Gl}_2(\mathbb{Q})$ is $\mathbb{Q}^\times$. It acts trivially on $\mathcal{M}_n$ this will have te effect that α and $\lambda\alpha$ with $\lambda \in \mathbb{Q}^*$ define the same operator. (Of course here we assume that $m = -n/2$.) Hence we may assume that the matrix entries of α are integers. The theorem of elementary divisors asserts that the double cosets

$$\Gamma \cdot M_n(\mathbb{Z})_{\det \neq 0} \cdot \Gamma \subset \mathrm{Gl}_2^+(\mathbb{Q})$$

are represented by matrices of the form

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

where $b \mid a$. But here we can divide by b, and we are left with the matrix

$$\alpha = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad a \in \mathbb{N}.$$

We can attach a Hecke operator to this matrix provided we choose u_α. We see that α induces on the basis vectors

$$X^\nu Y^{n-\nu} \longrightarrow a^{\nu - n/2} \cdot X^\nu Y^{n-\nu}.$$

Hence we see that we have the following natural choice for u_α

$$u_\alpha : P(X.Y) \longrightarrow a^{n/2}\alpha \cdot P(X, Y).$$

(See the general discussion of the Hecke operators)

Hence we get a family of endomorphisms

$$T\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, u_{\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}} \right) = T(a)$$

of the cohomology $H^i(\Gamma\backslash\mathbb{H},\mathcal{M}_n)$.

We have seen already that we have $T_aT_b = T_{ab}$ if a,b are coprime.

Hence we have to investigate the local algebra $\mathcal{H}_p$ which is generated by the

$$T_{p^r} = T\left(\begin{pmatrix} p^r & 0\\ 0 & 1\end{pmatrix}, u_{\begin{pmatrix} p^r & 0\\ 0 & 1\end{pmatrix}}\right)$$

for the special case of the group $\Gamma = \mathrm{Sl}_2(\mathbb{Z})$ and the coefficient system $\mathcal{M}_n$. To do this we compute the product

$$T_{p^r}\cdot T_p = T\left(\begin{pmatrix} p^r & 0\\ 0 & 1\end{pmatrix}, u_{\alpha_p^r}\right)\cdot T\left(\begin{pmatrix} p & 0\\ 0 & 1\end{pmatrix}, u_{\alpha_p}\right)$$

where the $u'_\alpha r$ are the canonical choices.

Again we investigate first what happens in degree zero, i.e. on $H^0(\Gamma\backslash\mathbb{H},\tilde{I})$ where I is any Γ-module.

Let $\alpha = \begin{pmatrix} p & 0\\ 0 & 1\end{pmatrix}$, then we have

$$T(\alpha^r,u_{\alpha^r})T(\alpha,u_\alpha)\xi = (\sum_{\gamma\in\Gamma/\Gamma(\alpha^r)}\gamma u_{\alpha^r})(\sum_{\eta\in\Gamma/\Gamma(\alpha)}\eta u_\alpha)(\xi)$$

We have the classical system of representatives

$$\Gamma/\Gamma(\alpha^r) = \bigcup_{j \mod p^r}\begin{pmatrix} 1 & j\\ 0 & 1\end{pmatrix}\Gamma(\alpha^r)\quad\bigcup\quad\bigcup_{j' \mod p^{r-1}}\begin{pmatrix} 1 & 0\\ j'p & 1\end{pmatrix}\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}\Gamma(\alpha^r)$$

Then our product of Hecke operators becomes

$$(\sum_{j \mod p^r}\begin{pmatrix} 1 & j\\ 0 & 1\end{pmatrix}+\sum_{j' \mod p^{r-1}}\begin{pmatrix} 1 & 0\\ j'p & 1\end{pmatrix}\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}u_{\alpha^r})(\sum_{j_1 \mod p}\begin{pmatrix} 1 & j_1\\ 0 & 1\end{pmatrix}+\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix})u_\alpha(\xi) =$$

$$(\sum_{j \mod p^r, j_1 \mod p}\begin{pmatrix} 1 & j\\ 0 & 1\end{pmatrix}u_{\alpha^r}\begin{pmatrix} 1 & j_1\\ 0 & 1\end{pmatrix}u_\alpha)(\xi)$$
$$+(\sum_{j \mod p^r}\begin{pmatrix} 1 & j\\ 0 & 1\end{pmatrix}u_{\alpha^r}\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix})u_\alpha(\xi)+$$
$$(\sum_{j' \mod p^{r-1}, j_1 \mod p}\begin{pmatrix} 1 & 0\\ j'p & 1\end{pmatrix}\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}u_{\alpha^r}\begin{pmatrix} 1 & j_1\\ 0 & 1\end{pmatrix}u_\alpha)(\xi)+$$
$$(\sum_{j' \mod p^{r-1}}\begin{pmatrix} 1 & 0\\ j'p & 1\end{pmatrix}\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}u_{\alpha^r}\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}u_\alpha)(\xi)$$

Now we have to assume the validity of certain commutation rules

$$\begin{aligned} u_{\alpha^r}\begin{pmatrix} 1 & j_1\\ 0 & 1\end{pmatrix} &= \begin{pmatrix} 1 & j_1p^r\\ 0 & 1\end{pmatrix}u_{\alpha^r}\\ u_{\alpha^r}\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}u_\alpha\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix} &= p^n u_{\alpha^{r-1}}\end{aligned}\qquad (*)$$

which are obviously valid for the canonical choices in the case $I = \mathcal{M}_k[m]$ (here m is arbitrary). We also have $u_{\alpha^r}u_\alpha = u_{\alpha^{r+1}}$. If we exploit the first commutation relation then we get as the sum of the first summand and the third summand

$$\sum_{j \mod p^r, j_1 \mod p} \begin{pmatrix} 1 & j+p^r j_1 \\ 0 & 1 \end{pmatrix} u_{\alpha^{r+1}} +$$

$$\sum_{j' \mod p^{r-1}, j_1 \mod p} \begin{pmatrix} 1 & 0 \\ (j'+p^{r-1}j_1)p & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} u_{\alpha^{r+1}},$$

and this is $T_{p^{r+1}}$. To compute the contribution of the second and the fourth summand we observe that $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \Gamma$ and hence we have $w\xi = \xi$. Now the second commutation relation yields for the sum of the second term and the fourth term

$$p^n (\sum_{j \mod p^r} \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} u_{\alpha^{r-1}} + \sum_{j' \mod p^{r-1}} \begin{pmatrix} 1 & 0 \\ j'p & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} u_{\alpha^{r-1}})$$

If we take into account that our summation over the $j($ resp. $j')$ is $\mod p^r($ resp. $\mod p^{r-1})$, then we see that this second expression yields $p^{n+1}T_{p^{r-1}}$, provided $r > 1$. If $r = 1$ then the summation over p^{r-1} is the same as the summation over p^{r-2} and then the second term is $(1+1/p)T_{p^0}$

If we put $e(r) = 0$ for $r > 1$ and $e(1) = 1$ then we arrive at the formula

$$T_{p^r} \cdot T_p = T_{p^{r+1}} + (1 + \frac{e(r)}{p})p^{n+1}T_{p^{r-1}}$$

This formula is valid for all values of $r \geq 0$ if we put $T_{p^{-1}} = 0$.

We proved the formulae for the $H^0(\Gamma\backslash\mathbb{H}, \tilde{I})$ for any Γ module I for which we can choose the u_α satisfy the commutation rules (*). These commutation rules are satisfied for the canonical choice in the case of $I = \mathcal{M}_n[m]$. But then it is not so difficult to see that we can embed $\mathcal{M}_n$ into an acyclic Γ-module I_0 such that we can extend the $u_\alpha : \mathcal{M}_n^{(\alpha)} \to \mathcal{M}_n$ to $I_0^{(\alpha)} \to I_0$ such that the commutation rules are still valid. Then we get induced morphisms $u_\alpha : (I_0/\mathcal{M}_n)^{(\alpha)} \to (I_0/\mathcal{M}_n)$ and also on these quotient the commutation rules hold. Then we see from the resulting exact sequence that our formulae for the Hecke operators are also true for the action on $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_n)$.

It may be illustrative to generalize a little bit. We choose an integer $N > 1$ and we take as our arithmetic group the congruence group $\Gamma = \Gamma(N)$. For any prime $p \not| N$ the $T(\alpha, u_\alpha)$ with $\alpha \in \mathrm{Gl}_2^+(\mathbb{Z}[1/p])$ form a commutative subalgebra $\mathcal{H}_p$ which is generated by T_p. For $p|N$ we can also consider the $T(\alpha, u_\alpha)$ with $\alpha \in \mathrm{Gl}_2^+(\mathbb{Z}[1/p])$. They will also generate a local algebra $\mathcal{H}_p$ of endomorphisms in any of our cohomology groups, but this algebra will not necessarily be commutative. But we saw that the $\mathcal{H}_p, \mathcal{H}_{p_1}$ commute with eachother for two different primes p, p_1. All these algebras $\mathcal{H}_p$ have an identity element e_p, we form the algebra

$$\mathcal{H}_\Gamma = \bigotimes_p{}' \mathcal{H}_p$$

where the superscript indicates that a tensor has an e_p for almost all p. This algebra acts on all our cohomology groups. The algebra $\mathcal{H}$ of endomorphism of one of our cohomology group is a homomorphic image of $\mathcal{H}_\Gamma$.

We come back to this after a brief recapitulation of the theory of semi simple modules.

4.2 Some results on semi-simple modules for algebras

We need a few results from the theory of algebras $\mathfrak{A}$ acting on finite dimensional vector spaces over a field L. Let $\bar{L}$ be an algebaic closure of L.

Let be a finite dimensional vector space V over some field L and an L- algebra $\mathfrak{A}$ with identity acting on V by endomorphisms. We say that the action of $\mathfrak{A}$ on V is semisimple, if the action of $\mathfrak{A}\otimes\bar{L}$ on $V\otimes\bar{L}$ is semi simple and this means that any $\mathfrak{A}$ submodule $W\subset V\otimes\bar{L}$ has a complement. Then it is clear that we get a decomposition indexed by a finite set E

$$V\otimes\bar{L}=\bigoplus_{i\in E}W_i$$

where the W_i are irreducible submodules, i.e. they do not contain any non trivial $\mathfrak{A}$ submodule.

This decomposition will not be unique in general. For any two W_i, W_j of these submodules we have (Schur lemma)

$$\mathrm{Hom}_{\mathfrak{A}}(W_i,W_j)=\begin{cases}\bar{L} & \text{if they are isomorphic as } \mathfrak{A} \text{ -modules}\\ 0 & \text{else}\end{cases}$$

We decompose the indexing set $E=E_1\cup E_2\cup..\cup E_k$ according to isomorphism types. For any E_ν we choose an $\mathfrak{A}$ module $W_{[\nu]}$ of this given isomorphism type. Then by definition

$$\mathrm{Hom}_{\mathfrak{A}}(W_{[\nu]},W_j)=\begin{cases}\bar{L} & \text{if } j\in E_\nu\\ 0 & \text{else}\end{cases}.$$

Now we define $H_{[\nu]}=\ \mathrm{Hom}_{\mathfrak{A}}(W_{[\nu]},V\otimes\bar{L})$ we get an inclusion $H_{[\nu]}\otimes W_{[\nu]}$ whose image X_ν will be an $\mathfrak{A}$ submodule, which is a direct sum of copies of $W_{[\nu]}$.

We get a direct sum decomposition

$$V\otimes\bar{L}=\bigoplus_{\nu}\bigoplus_{i\in E_\nu}W_i=\bigoplus_{\nu}X_\nu$$

then this last decomposition is easily seen to be unique, it is called the isotypical decomposition.

If V is a semi simple $\mathfrak{A}$ module then any submodule $W\subset V$ also has a complement (this is not entirely obvious because by definition only $W_{\bar{L}}$ has a complement in $V_{\bar{L}}$. But a small moment of meditation gives us that finding such a complement is the same as solving an inhomogenous system of linear equations over L. If this system has a solution over $\bar{L}$ it also has a solution over L.) and hence we also can decompose the $\mathfrak{A}$ module V into irreducibles. Again we can group the irreducibles according to isomorphism types and we get an isotypical decomposition

$$V=\bigoplus_{i\in E}U_i=\bigoplus_{\nu}\bigoplus_{i\in E_\nu}U_i=\bigoplus_{\nu}Y_\nu.$$

But an irreducible $\mathfrak{A}$ module W may become reducible if we extend the scalars to $\bar{L}$. So it may happen that som of our U_i decompose further. Since it is clear that for any two $\mathfrak{A}$- modules V_1, V_2 we have

$$\mathrm{Hom}_{\mathfrak{A}}(V_1, V_2) \otimes \bar{L} = \mathrm{Hom}_{\mathfrak{A}\otimes\bar{L}}(V_1 \otimes \bar{L}, V_2 \otimes \bar{L})$$

we know that we get the isotypical decomposition of $V \otimes \bar{L}$ by taking the isotypical decomposition of the $Y_\nu \otimes \bar{L}$ and then taking the direct sum over ν.

Example: Let L_1/L be a finite extension of degree > 1, then we put $\mathfrak{A} = L_1$ and $V = L_1$, the action is given by multiplication. Clearly V is irreducible, but $V \otimes \bar{L}$ is not. If L_1/L is separable then the module is semisimple, otherwise it is not.

We say that the $\mathfrak{A}$ - module V is absolutely irreducible, if the $\mathfrak{A} \otimes \bar{L}$-module $V \otimes \bar{L}$ is irreducible. In this case it we have a classical result:

Proposition. *Let V be a semi simple $\mathfrak{A}$ module. Then the following assertions are equivalent*
i) The $\mathfrak{A}$ module V is absolutely irreducible
ii) The image of $\mathfrak{A}$ in the ring of endomorphisms is $End(V)$
iii) The vector space of $\mathfrak{A}$ endomorphisms $End_{\mathfrak{A}}(V) = L$.

This can be an exercise for an algebra class. Where do we need the assumption that V is semi simple?

Proposition: *For any semi -simple $\mathfrak{A}$ module V we can find a finite extension L_1/L such that the irreducible sub modules in the decomposition into irreducibles are absolutely irreducible.*

Let us now assume that we have two algebras $\mathfrak{A}, \mathfrak{B}$ acting on V, let us assume that these two operations commute i.e. for $A \in \mathfrak{A}, B \in \mathfrak{B}, v \in V$ we have $A(Bv) = B(Av)$. This structure is the same as having a $\mathfrak{A} \otimes_L \mathfrak{B}$ structure on V. Let us assume that $\mathfrak{A}$ acts semi simply on V and let us assume that the irreducible $\mathfrak{A}$ submodules of V are absolutely irreducible. Then it is clear that the isotypical summands $Y_\nu = \bigoplus W_i$ are invariant under the action $\mathfrak{B}$. Now we pick an index i_0 then the evaluation maps gives us a homomorpism

$$W_{i_0} \otimes \mathrm{Hom}_{\mathfrak{A}}(W_{i_0}, Y_\nu) \to Y_\nu.$$

Under our assumptions this is an isomorphism. Then we see that we get

$$V = \bigoplus_\nu W_{i_\nu} \otimes \mathrm{Hom}_{\mathfrak{A}}(W_{i_0}, Y_\nu)$$

where i_ν is any element in E_ν and where $\mathfrak{A}$ acts upon the first factor and $\mathfrak{B}$ acts upon the second factor via the action of $\mathfrak{B}$ on Y_ν.

Especially we see:

Proposition *If V is an absolutely irreducible $\mathfrak{A} \otimes_L \mathfrak{B}$ module then $V \xrightarrow{\sim} X \otimes Y$, where X (resp, Y) is an absolutely $\mathfrak{A}$ (resp. $\mathfrak{B}$) module*

We apply these considerations to get

Theorem 3: *For any L we can decompose :*

$$H^1_!(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_{n,L}) = \bigoplus H^1_!(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_{n,L})(\Pi_f)$$

where this is the isotypical decomposition and the Π_f are isomorphism classes of irreducible modules. There is a finite extension $L/\mathbb{Q}$ such that all the isomorphism classes of isotypical modules which occur are actually absolutely irreducible.

If Π_f is an absolutely irreducible $\mathcal{H}_\Gamma$ module then it is the tensor product $\Pi_f = \otimes\pi_p$ where the π_p are absolutely irreducible $\mathcal{H}_p$ modules. For $p \not| N$ the modules π_p are of dimension one (see above theorem) and they are determined by a number $\lambda(\pi_p) \in \mathcal{O}_L$ which is the eigenvalue of T_p on π_p.

This follows easily from our previous considerations. The eigenvalues $\lambda(\pi_p)$ are algebraic integers because T_p induces an endomorphism of $H^1_!(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_{n,\mathcal{O}_L})$ which after tensorization with L becomes the T_p on the rational vector space. The above field extension is called the splitting field of $\mathcal{H}_\Gamma$.

These two theorems 2 and 3 are special cases of more general results. We can start from an arbitrary reductive groups over $\mathbb{Q}$, arbitrary congruence subgroups $\Gamma \subset G(\mathbb{Q})$ and arbitrary coefficient systems $\mathcal{M}$ obtained from a rational representation of $G/\mathbb{Q}$, they are finitely generated modules over $\mathbb{Z}$. Then we can consider certain symmetric spaces $X = G(\mathbb{R})/K_\infty$ and we have the cohomology groups $H^\bullet(\Gamma\backslash X,\tilde{\mathcal{M}})$, they are finitely generated $\mathbb{Z}$ modules. Again we can define an action of the Hecke algebra $\mathcal{H}_\Gamma$ and this Hecke algebra acts semi simply on the inner cohomology $H^\bullet_!(\Gamma\backslash X,\tilde{\mathcal{M}}_\mathbb{Q})$. (theorem 2) Again this Hecke algebra is the tensor product of local Hecke algebras where for almost all primes these local Hecke algebras $\mathcal{H}_p$ are polynomial rings in a certain number of variables. Then the theorem 3 is also valid in this situation. We resume this theme in Chap.III.

4.2.1 Hecke operators for Gl_2:

We consider the classical case. Our group $G/\mathbb{Q}$ is the group $\mathrm{Gl}_2/\mathbb{Q}$ and $K = SO(2) \subset G_\infty$. Then $X = G_\infty/K$ is the union of an upper and a lower half plane. We choose $\tilde\Gamma = \mathrm{Gl}_2(\mathbb{Z})$, then

$$\tilde\Gamma\backslash G_\infty/K = \Gamma\backslash\mathbb{H},$$

where $\Gamma = \mathrm{Sl}_2(\mathbb{Z})$ and H is the upper half plane.

As Γ-modules we consider the $\mathbb{Z}$ –module

$$\mathcal{M}_n = \left\{\sum_{\nu=0}^{n} a_v X^\nu Y^{n-\nu} \mid a_\nu \in \mathbb{Z}\right\}.$$

The group Γ acts by

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix} X^\nu Y^{n-\nu} = (aX+cY)^\nu(bX+dY)^{n-\nu}.$$

We observe that the associated sheaf $\mathcal{M}_n$ becomes trivial if $n \neq 0 \mod 2$ hence we assume that n is even. We define a rational representation of $\mathrm{Gl}_2(\mathbb{Q})$ on $\mathcal{M}_{n,\mathbb{Q}}$, which we choose to be

$$\alpha\cdot P(X,Y) = \begin{pmatrix} a & b \\ c & d\end{pmatrix} P(X,Y) = P(aX+cY,bX+dY)\det\begin{pmatrix} a & b \\ c & d\end{pmatrix}^{-n/2}.$$

Here we may also multiply by another power $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}^m$ of the determinant factor. We call the resulting module $\mathcal{M}_{n,\mathbb{Q}}[m]$, later it will turn out that $m = -n$ is the optimal choice. At this present moment our module is $\mathcal{M}_{n,\mathbb{Q}}[-n/2]$, this choice of the exponent m has the advantage that the center acts trivially.

4.3 The case $\Gamma = \mathbf{Sl}_2(\mathbb{Z})$.

We refer to Chap.II 2.1.3. We have the two open sets $\tilde{U}_{\mathbf{i}}$, resp. $\tilde{U}_\rho \subset \mathbb{H}$, they are fixed under

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } R = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix},$$

respectively. We also will use the elements

$$\begin{array}{l} T_+ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ S_1^+ = T_- S T_-^{-1} = \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix} \in \Gamma_0^+(2) \\ \\ T_- = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},\ S_1^- = T_+ S T_+^{-1} = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} \in \Gamma_0^-(2) \end{array}.$$

The elements S_1^+ and S_1^- are elements of order four, i.e. $(S_1^+)^2 = (S_1^-)^2 = -\mathrm{Id}$, the corresponding fixed points are $\frac{\mathbf{i}+1}{2}$ and $\mathbf{i}+1$ respectively. Hence S_1^- fixes the sets $\alpha\tilde{U}_{\frac{\mathbf{i}+1}{2}}$ and $\tilde{U}_{\mathbf{i}+1}$, this is the only occurrence of a non trivial stabilizer.

4.3.1 Explicit formulas for the Hecke operators, a general strategy.

In the following section we discuss the Hecke operators and for numerical experiments it is useful to have an explicit procedure to compute them in a given case. The main obstruction to get such an explicit procedure is to find an explicit way to compute the arrow $j^\bullet(\alpha)$ in the top line of the diagram (91). (we change notation $j(\alpha)$ to $m(\alpha)$).

Let us assume that we have computed the cohomology groups on both sides by means of orbiconvex coverings $\mathfrak{V} : \cup_{i\in I} V_{y_i} = \Gamma(\alpha^{-1})\backslash X$ and $\mathfrak{U} : \cup_{j\in J} U_{y_j} = \Gamma(\alpha)\backslash X$.

The map $m(\alpha)$ is an isomorphism between spaces and hence $m(\alpha)(\mathfrak{V})$ is an acyclic covering of $\Gamma(\alpha)\backslash X$. This induces an identification

$$C^\bullet(\mathfrak{V}, \tilde{\mathcal{M}}) = C^\bullet(m(\alpha)(\mathfrak{V}), \tilde{\mathcal{M}}^{(\alpha)})$$

and the complex on the right hand side computes $H^\bullet(\Gamma(\alpha)\backslash X, \tilde{\mathcal{M}}^{(\alpha)})$. But this cohomology is also computable from the complex $C^\bullet(\mathfrak{U}, \tilde{\mathcal{M}}^{(\alpha)})$. We take the disjoint union of the two indexing sets $I \cup J$ and look at the covering $m_\alpha(\mathfrak{V}) \cup \mathfrak{U}$. (To be precise: We consider the disjoint union $\tilde{I} = I \cup J$ and define a covering $\mathfrak{W}_{\mathrm{i}}$ indexed by $\tilde{I}$. If $i \in \tilde{I}$ then $W_i = m(\alpha)(V_{y_i})$ and if $i \in J$ then we put $W_i = U_{x_i}$. We get a diagram of Czech complexes

$$\begin{array}{ccccc} \to & \bigoplus_{\underline{i}\in I^q} \tilde{\mathcal{M}}^{(\alpha)}(W_{\underline{i}}) & \to & \bigoplus_{\underline{i}\in I^{q+1}} \tilde{\mathcal{M}}^{(\alpha)}(W_{\underline{i}}) \to \\ & \uparrow & & \uparrow \\ \to & \bigoplus_{\underline{i}\in \tilde{I}^q} \tilde{\mathcal{M}}^{(\alpha)}(W_{\underline{i}}) & \to & \bigoplus_{\underline{i}\in \tilde{I}^{q+1}} \tilde{\mathcal{M}}^{(\alpha)}(W_{\underline{i}}) \to \\ & \downarrow & & \downarrow \\ \to & \bigoplus_{\underline{i}\in J^q} \tilde{\mathcal{M}}^{(\alpha)}(W_{\underline{i}}) & \to & \bigoplus_{\underline{i}\in J^{q+1}} \tilde{\mathcal{M}}^{(\alpha)}(W_{\underline{i}}) \to \end{array} \quad (94)$$

The sets $I^\bullet, J^\bullet$ are subsets of $\tilde{I}^\bullet$ and the up- and down-arrows are the resulting projection maps. We know that these up- and down-arrows induce isomorphisms in cohomology.

Hence we can start from a cohomology class $\xi \in H^q(\Gamma(\alpha)\backslash X, \tilde{\mathcal{M}}^{(\alpha)})$, we represent it by a cocycle

$$c_\xi \in \bigoplus_{\underline{i} \in I^q} \tilde{\mathcal{M}}^{(\alpha)}(W_{\underline{i}}).$$

Then we can find a cocycle $\tilde{c}_\xi \in \bigoplus_{\underline{i} \in \tilde{I}^q} \tilde{\mathcal{M}}^{(\alpha)}(W_{\underline{i}})$ which maps to c_ξ under the uparrow. To get this cocycle we have to do the following: our cocycle c_ξ is an array with components $c_\xi(\underline{i})$ for $\underline{i} \in I^q$. We have $d_q(c_\xi) = 0$. To get $\tilde{c}_\xi$ we have to give the values $\tilde{c}_\xi(\underline{i})$ for all $\underline{i} \in \tilde{I}^q \setminus I^q$. We must have

$$d_q \tilde{c}_\xi = 0.$$

this yields a system of linear equations for the remaining entries. We know that this system of equations has a solution -this is then our $\tilde{c}_\xi$ - and this solution is unique up to a boundary $d_{q-1}(\xi')$. Then we apply the downarrow to $\tilde{c}_\xi$ and get a cocycle $c_\xi^\dagger$, which represents the same class ξ but this class is now represented by a cocycle with respect to the covering $\mathfrak{U}$. We apply the map $\tilde{u}^\alpha : \tilde{\mathcal{M}}^{(\alpha)} \to \tilde{\mathcal{M}}$ to this cocycle and then we get a cocycle which represents the image of our class ξ under T_α.

4.3.2 The special case Sl_2

Let $\pi_1 : \mathbb{H} \to \Gamma\backslash\mathbb{H}$ be the projection. We get a covering $\Gamma\backslash\mathbb{H} = \pi_1(\tilde{U}_{\mathbf{i}}) \cup \pi_1(\tilde{U}_\rho) = U_i \cap U_\rho$. From this covering we get the Czech complex

$$\begin{array}{ccccc} 0 & \to & \tilde{\mathcal{M}}(U_{\mathbf{i}}) \oplus \tilde{\mathcal{M}}(U_\rho) & \to & \tilde{\mathcal{M}}(U_{\mathbf{i}} \cap U_\rho) & \to 0 \\ & & \downarrow ev_{\tilde{U}_{\mathbf{i}}} \oplus ev_{\tilde{U}_\rho} & & \downarrow ev_{\tilde{U}_{\mathbf{i}} \cap \tilde{U}_\rho} & \\ & & \mathcal{M}^{<S>} \oplus \mathcal{M}^{<R>} & \to & \mathcal{M} & \to 0 \end{array} \tag{95}$$

and this gives us our formula for the first cohomology

$$H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) = \mathcal{M}/(\mathcal{M}^{<S>} \oplus \mathcal{M}^{<R>}) \tag{96}$$

We want to discuss the Hecke operator T_2. To do this we pass to the subgroups

$$\begin{aligned} \Gamma_0^+(2) &= \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c \equiv 0 \mod 2\} \\ \Gamma_0^-(2) &= \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid b \equiv 0 \mod 2\} \end{aligned} \tag{97}$$

we form the two quotients and introduce the projection maps $\pi_2^\pm : \mathbb{H} \to \Gamma_0^\pm(2)\backslash\mathbb{H}$. We have an isomorphism between the spaces

$$\Gamma_0^+(2)\backslash\mathbb{H} \xrightarrow{\alpha_2} \Gamma_0^-(2)\backslash\mathbb{H}$$

which is induced from the map $m_2 : z \mapsto \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} z = 2z$. This map induces an isomorphism

$$\alpha_2^\bullet : H^1(\Gamma_0^+(2)\backslash\mathbb{H},\tilde{\mathcal{M}}) \xrightarrow{\sim} H^1(\Gamma_0^-(2)\backslash\mathbb{H},\tilde{\mathcal{M}}^{(\alpha)}). \tag{98}$$

We also have the map between sheaves $u_2 : m \mapsto \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} m$ and the composition with this map induces a homomorphism in cohomology

$$H^1(\Gamma_0^+(2)\backslash\mathbb{H},\tilde{\mathcal{M}}) \xrightarrow{u_2^\bullet\circ\alpha_2^\bullet} H^1(\Gamma_0^-(2)\backslash\mathbb{H},\tilde{\mathcal{M}}). \tag{99}$$

This is the homomorphism we need for the computation of the Hecke operator; it is easy to define but it may be difficult in practice to compute it.

4.3.3 The boundary cohomology

We can also look at the same problem for the cohomology of the boundary, then the situation becomes much simpler. Each of the spaces $\Gamma_0^+(2)\backslash\mathbb{H},\Gamma_0^-(2)\backslash\mathbb{H}$ has two cusps which can be represented by the points $\infty, 0 \in \mathbb{P}^1(\mathbb{Q})$. The stabilizers of these two cusps in $\Gamma_0^+(2)$ resp. $\Gamma_0^-(2)$ are

$$< T_+ > \times\{\pm\mathrm{Id}\} \text{ and } < T_-^2 > \times\{\pm\mathrm{Id}\} \subset \Gamma_0^+(2)$$

resp.

$$< T_+^2 > \times\{\pm\mathrm{Id}\} \text{ and } < T_- > \times\{\pm\mathrm{Id}\} \subset \Gamma_0^-(2)$$

the factor $\{\pm\mathrm{Id}\}$ can be ignored. Then we get

We know that

$$H^1(\partial(\Gamma_0^+(2)bs\mathbb{H}),\tilde{\mathcal{M}}) \xrightarrow{\sim} \mathcal{M}/(\mathrm{Id}-T_+)\mathcal{M}\oplus\mathcal{M}/(\mathrm{Id}-T_-^2)\mathcal{M}$$

$$H^1(\partial(\Gamma_0^-(2)\backslash\mathbb{H}),\tilde{\mathcal{M}}) \xrightarrow{\sim} \mathcal{M}/(\mathrm{Id}-T_+^2)\mathcal{M}\oplus\mathcal{M}/(\mathrm{Id}-T_-)\mathcal{M}.$$

But now it is obvious that α maps the cusp ∞ to ∞ and 0 to 0 and then it is also clear that for the boundary cohomology the map

$$\alpha_2^\bullet : \mathcal{M}/(\mathrm{Id}-T_+)\mathcal{M}\oplus\mathcal{M}/(\mathrm{Id}-T_-^2)\mathcal{M} \to \mathcal{M}/(\mathrm{Id}-T_+^2)\mathcal{M}\oplus\mathcal{M}/(\mathrm{Id}-T_-)\mathcal{M}$$

is simply the map which is induced by $u_2 : \mathcal{M} \to \mathcal{M}$. If we ignore torsion then the individual summands are infinite cyclic.

Our module $\mathcal{M}$ is the module of homogenous polynomials of degree n in 2 variables X, Y with integer coefficients. Then the classes $[Y^n], [X^n]$ of the polynomials Y^n (resp.) X^n are generators of $(\mathcal{M}/(\mathrm{Id}-T_+^\nu)\mathcal{M})/\mathrm{tors}$ resp. $(\mathcal{M}/(\mathrm{Id}-T_+^\nu)\mathcal{M})/\mathrm{tors}$ where $\nu = 1$ resp. 2. Then we get for the homomorphism $\alpha_2^\bullet$

$$\alpha_2^\bullet : [Y^n] \mapsto [Y^n], \alpha_2^\bullet : [X^n] \mapsto 2^n[X^n]. \tag{100}$$

4.3.4 The explicit description of the cohomology

We give the explicit description of the cohomology $H^1(\Gamma_0^+(2)\backslash\mathbb{H},\tilde{\mathcal{M}})$. We introduce the projections

$$\mathbb{H} \xrightarrow{\pi_2^+} \Gamma_0^+(2)\backslash\mathbb{H};\ \mathbb{H} \xrightarrow{\pi_2^-} \Gamma_0^-(2)\backslash\mathbb{H}$$

and get the covering $\mathfrak{U}_2$

$$\Gamma_0^+(2)\backslash\mathbb{H} = \pi_2^+(\tilde{U}_{\mathrm{i}}) \cup \pi_2^+(T_-\tilde{U}_{\mathrm{i}}) \cup \pi_2^+(\tilde{U}_\rho) = \pi_2^+(\tilde{U}_{\mathrm{i}}) \cup \pi_2^+(\tilde{U}_{\frac{\mathrm{i}+1}{2}}) \cup \pi_2^+(\tilde{U}_\rho)$$

where we put $T_-\tilde U_{\mathbf{i}} = \tilde U_{\frac{\mathbf{i}+1}{2}}$. Our set $\{x_\nu\}$ of indexing points is $\mathbf{i}, \frac{\mathbf{i}+1}{2}, \rho$, we put $U^+_{x_i} = \pi_2^+(\tilde U_{x_i})$. Note $T_- \notin \Gamma_0^+(2), T_+ \in \Gamma_0^+(2)$.

Again the cohomology is computed by the complex

$$0 \to \tilde{\mathcal M}(U^+_{\mathbf{i}}) \oplus \tilde{\mathcal M}(T_-\tilde U^+_{\mathbf{i}}) \oplus \tilde{\mathcal M}(U^+_\rho) \to \tilde{\mathcal M}(U^+_{\mathbf{i}} \cap U^+_\rho) \oplus \tilde{\mathcal M}(T_-\tilde U^+_{\mathbf{i}} \cap U^+_\rho) \to 0$$

we have to identify the terms as submodules of some $\bigoplus \mathcal M$ and write down the boundary map explicitly. We have

$$\begin{array}{ccc}
\tilde{\mathcal M}(U^+_{\mathbf{i}}) \oplus \tilde{\mathcal M}(U^+_{\frac{\mathbf{i}+1}{2}}) \oplus \tilde{\mathcal M}(U^+_\rho) & \xrightarrow{d_0} & \tilde{\mathcal M}(U^+_i \cap U^+_\rho) \oplus \tilde{\mathcal M}(U^+_{\frac{\mathbf{i}+1}{2}} \cap U^+_\rho) \\
\downarrow ev_{\tilde U_{\mathbf{i}}} \oplus ev_{T_-\tilde U_{\mathbf{i}}} \oplus ev_{\tilde U_\rho} & & \downarrow ev_{\tilde U_{\mathbf{i}} \cap \tilde U_\rho} \oplus ev_{\tilde U_{\mathbf{i}} \cap T_+^{-1}\tilde U_\rho} \oplus ev_{T_-\tilde U_{\mathbf{i}} \cap \tilde U_\rho} \\
\mathcal M \oplus \mathcal M^{<S_1^+>} \oplus \mathcal M & \xrightarrow{\bar d_0} & \mathcal M \oplus \mathcal M \oplus \mathcal M
\end{array} \tag{101}$$

where the vertical arrows are isomorphisms. The boundary map $\bar d_0$ in the bottom row is given by

$$(m_1, m_2, m_3) \mapsto (m_1 - m_3, m_1 - T_+^{-1}m_3, m_1 - m_2) = (x, y, z)$$

We may look at the (isomorphic) sub complex where $x = z = 0$ and $m_1 = m_2 = m_3$ then we obtain the complex

$$0 \to \mathcal M^{<S_1^+>} \to \mathcal M \to 0;\ m_2 \mapsto m_2 - T_+^{-1}m_2$$

which provides an isomorphism

$$H^1(\Gamma_0^+(2)\backslash\mathbb H, \tilde{\mathcal M}) \xrightarrow{\sim} \mathcal M/(\mathrm{Id} - T_+^{-1})\mathcal M^{<S_1^+>}. \tag{102}$$

A simple computation shows that the cohomology class represented by the class (x, y, z) is equal to the class represented by $(0, y-x+T_+^{-1}z-z, 0)$ we write

$$[(x, y, z)] = [(0, y - x + T_+^{-1}z - z, 0)] \tag{103}$$

4.3.5 The map to the boundary cohomology

We have the restriction map for the cohomology of the boundary

$$\begin{array}{ccc}
H^1(\Gamma_0^+(2)\backslash\mathbb H, \tilde{\mathcal M}) & \xrightarrow{\sim} & \mathcal M/(\mathrm{Id} - T_+^{-1})\mathcal M^{<S_1^+>} \\
\downarrow & & r^+ \oplus r^- \downarrow \\
H^1(\partial(\Gamma_0^+(2)\backslash\mathbb H), \tilde{\mathcal M}) & \xrightarrow{\sim} & \mathcal M/(\mathrm{Id} - T_+)\mathcal M \oplus \mathcal M/(\mathrm{Id} - T_-^2)\mathcal M
\end{array} \tag{104}$$

we give a formula for the second vertical arrow. We represent a class $[m]$ by an element $m \in \mathcal M$ and send m to its class in in each the two summands, respectively. This is well defined, for r^+ it is obvious, while for r^- we observe that if $m = x - T_+^{-1}x$ and $S_1^+x = x$ then $m = x - T_+^{-1}S_1^+x = x - T_-^2x$.

4.3.6 Restriction and Corestriction

Now we have to give explicit formulas for the two maps π^*, π_* in the big diagram on p. 50 in Chap2.pdf. Here we should change notation: The map π in Chap.2 will now be denoted by :

$$\varpi_2^+ : \Gamma_0^+(2)\backslash\mathbb{H} \to \Gamma\backslash\mathbb{H} \tag{105}$$

We have the two complexes which compute the cohomology $H^1(\Gamma_0^+(2)\backslash\mathbb{H}, \tilde{\mathcal{M}})$ and $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})$, and we have defined arrows between them. We realized these two complexes explicitly in (101) resp. (95) and we have

$$\begin{array}{ccc} \tilde{\mathcal{M}}(U_i^+) \oplus \tilde{\mathcal{M}}(U^+_{\frac{i+1}{2}}) \oplus \tilde{\mathcal{M}}(U_\rho^+) & \xrightarrow{d_0} & \tilde{\mathcal{M}}(U_i^+ \cap U_\rho^+) \oplus \tilde{\mathcal{M}}(U^+_{\frac{i+1}{2}} \cap U_\rho^+) \\ (\varpi_2^+)^{(0)} \uparrow \downarrow (\varpi_2^+)_{(0)} & & (\varpi_2^+)^{(1)} \uparrow \downarrow (\varpi_2^+)_{(1)} \\ \tilde{\mathcal{M}}(U_i) \oplus \tilde{\mathcal{M}}(U_\rho) & \xrightarrow{d_0} & \tilde{\mathcal{M}}(U_i \cap U_\rho) \end{array} \tag{106}$$

and in terms of our explicit realization in diagram (101) this gives

$$\begin{array}{ccc} \mathcal{M} \oplus \mathcal{M}^{<S_1>} \oplus \mathcal{M} & \xrightarrow{d_0} & \mathcal{M} \oplus \mathcal{M} \oplus \mathcal{M} \\ (\varpi_2^+)^{(0)} \uparrow \downarrow (\varpi_2^+)_{(0)} & & (\varpi_2^+)^{(1)} \uparrow \downarrow (\varpi_2^+)_{(1)} \\ \mathcal{M}^{<S>} \oplus \mathcal{M}^{<R>} & \xrightarrow{d_0} & \mathcal{M} \end{array} \tag{107}$$

Looking at the definitions we find

$$\begin{gathered} (\varpi_2^+)^{(0)} : (m_1, m_2) \mapsto (m_1, T_- m_1, m_2) \\ (\varpi_2^+)_{(0)} : (m_1, m_2, m_3) \mapsto (m_1 + Sm_1 + T_-^{-1} m_2, (1 + R + R^2) m_3) \end{gathered} \tag{108}$$

and we check easily that the composition $(\varpi_2^+)_{(0)} \circ (\varpi_2^+)^{(0)}$ is the multiplication by 3 as it should be, since this is the index of $\Gamma_0(2)^+$ in Γ.

For the two arrows in degree one we find

$$\begin{gathered} (\varpi_2^+)^{(1)} : m \mapsto (m, Sm, T_- m) \\ (\varpi_2^+)_{(1)} : (m_1, m_2, m_3) \mapsto (m_1 + Sm_2 + T_-^{-1} m_3) \end{gathered} \tag{109}$$

We apply equation (103) and we see that $(\varpi_2^+)^{(1)}(m)$ is represented by

$$[(\varpi_2^+)^{(1)}(m)] = [0, Sm + T_+^{-1} T_- m - m - T_- m, 0] \tag{110}$$

We do the same calculation for $\Gamma_0^-(2)$. As before we start from a covering

$$\Gamma_0^-(2)\backslash\mathbb{H} = \pi_2^-(\tilde{U}_{\mathbf{i}}) \cup \pi_2^-(T_+ \tilde{U}_{\mathbf{i}}) \cup \pi_2^-(\tilde{U}_\rho) = \pi_2^-(\tilde{U}_{\mathbf{i}}) \cup \pi_2^-(\tilde{U}_{i+1}) \cup \pi_2^-(\tilde{U}_\rho)$$

and as before we put $U^-_{y_\nu} = \pi_2^-(\tilde{U}_{y_\nu})$. In this case $\tilde{U}_{i+1} = T_+\tilde{U}_i$ is fixed by $S_1^- = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} \in \Gamma_0^-(2)$ and we get a diagram for the Czech complex

$$\begin{array}{ccc} \tilde{\mathcal{M}}(U^-_{\mathbf{i}}) \oplus \tilde{\mathcal{M}}(U^-_{\mathbf{i}+1}) \oplus \tilde{\mathcal{M}}(U^-_{\rho}) & \xrightarrow{d_0} & \tilde{\mathcal{M}}(U^-_{\mathbf{i}} \cap U^-_{\rho}) \oplus \tilde{\mathcal{M}}(U^-_{\mathbf{i}+1} \cap U^-_{\rho}) \\ ev_{\tilde{U}_{\mathbf{i}}} \oplus ev_{\tilde{U}_{\mathbf{i}+1}} \downarrow \oplus ev_{\tilde{U}_\rho} & & ev_{\tilde{U}_{\mathbf{i}}\cap\tilde{U}_\rho} \oplus ev_{\tilde{U}_{\mathbf{i}}\cap T_-^{-1}\tilde{U}_\rho} \downarrow \oplus ev_{\tilde{U}_{\mathbf{i}+1}\cap\tilde{U}_\rho} \\ \mathcal{M} \oplus \mathcal{M}^{<S_1^->} \oplus \mathcal{M} & \xrightarrow{\bar{d}_0} & \mathcal{M} \oplus \mathcal{M} \oplus \mathcal{M} \end{array} \tag{111}$$

Again we can modify this complex and get

$$H^1(\Gamma_0^-(2)\backslash\mathbb{H}, \tilde{\mathcal{M}}) \xrightarrow{\sim} \mathcal{M}/(\mathrm{Id} - T_-^{-1})\mathcal{M}^{<S_1^->}. \tag{112}$$

We compute the arrows $(\varpi_2^-)^*, (\varpi_2^-)_*$ in degree one

$$\begin{aligned} (\varpi_2^-)^{(1)} &: m \mapsto (m, Sm, T_+m), \\ (\varpi_2^-)_{(1)} &: (m_1, m_2, m_3) \mapsto (m_1 + Sm_2 + T_+^{-1}m_3). \end{aligned} \tag{113}$$

4.3.7 The computation of $\alpha_2^\bullet$.

We recall our isomorphism α between the spaces and the resulting isomorphism (98). The identity map of the module $\mathcal{M}$ and the isomorphism α on the space identifies the two complexes

$$\begin{array}{ccc} \tilde{\mathcal{M}}(U^+_{\mathbf{i}}) \oplus \tilde{\mathcal{M}}(U^+_{\frac{\mathbf{i}+1}{2}}) \oplus \tilde{\mathcal{M}}(U^+_{\rho}) & \xrightarrow{d_0} & \tilde{\mathcal{M}}(U^+_{\mathbf{i}} \cap U^+_{\rho}) \oplus \tilde{\mathcal{M}}(U^+_{\frac{\mathbf{i}+1}{2}} \cap U^+_{\rho}) \\ \tilde{\mathcal{M}}^{(\alpha)}(\alpha(U^+_{\mathbf{i}})) \oplus \tilde{\mathcal{M}}^{(\alpha)}(\alpha(U^+_{\frac{\mathbf{i}+1}{2}})) \oplus \tilde{\mathcal{M}}^{(\alpha)}(\alpha(U^+_{\rho})) & \xrightarrow{d_0} & \tilde{\mathcal{M}}^{(\alpha)}(\alpha(U^+_{\mathbf{i}} \cap U^+_{\rho})) \oplus \tilde{\mathcal{M}}^{(\alpha)}(\alpha(U^+_{\frac{\mathbf{i}+1}{2}} \cap U^+_{\rho})) \end{array} \tag{114}$$

and if we consider their explicit realization then this identification is given by the equality of $\mathbb{Z}$ modules $\mathcal{M} = \mathcal{M}^{(\alpha)}$. This equality of complexes expresses the identification (98). We can compute the cohomology $H^1(\Gamma_0^-(2)\backslash\mathbb{H}, \tilde{\mathcal{M}}^{(\alpha)})$ from any of the two coverings

$$\begin{aligned} &\Gamma_0^-(2)\backslash\mathbb{H} = \alpha(U^+_{\mathbf{i}}) \cup \alpha(U^+_{\frac{\mathbf{i}+1}{2}}) \cup \alpha(U^+_{\rho}) = U_{x_1} \cup U_{x_2} \cup U_{x_3} \\ &\text{and} \\ &\Gamma_0^-(2)\backslash\mathbb{H} = U^-_{\mathbf{i}} \cup U^-_{\mathbf{i}+1} \cup U^-_{\rho} = U_{x_4} \cup U_{x_5} \cup U_{x_6}. \end{aligned} \tag{115}$$

We have to pick a class $\xi \in H^1(\Gamma_0^-(2)\backslash\mathbb{H}, \tilde{\mathcal{M}}^{(\alpha)})$ and represent it by a cocycle

$$c_\xi \in \bigoplus_{1\le i<j\le 3} \tilde{\mathcal{M}}^{(\alpha)}(U_{x_i} \cap U_{x_j})$$

(The cocycle condition is empty since $U_{x_1} \cap U_{x_2} \cap U_{x_3} = \emptyset$.)

Then we have to produce a cocycle

$$c_\xi^\alpha \in \bigoplus_{4\le i<j\le 6} \tilde{\mathcal{M}}^{(\alpha)}(U_{x_i} \cap U_{x_j})$$

which represents the same class.

To get this cocycle we write down the three complexes

$$\begin{array}{ccc}
\bigoplus_{1\le i<j\le 3}\tilde{\mathcal{M}}^{(\alpha)}(U_{x_i}\cap U_{x_j}) & \to & 0\\
\uparrow & & \\
\bigoplus_{1\le i<j\le 6}\tilde{\mathcal{M}}^{(\alpha)}(U_{x_i}\cap U_{x_j}) & \to & \bigoplus_{1\le i<j<k\le 6}\tilde{\mathcal{M}}^{(\alpha)}(U_{x_i}\cap U_{x_j}\cap U_{x_k})\\
\downarrow & & \\
\bigoplus_{4\le i<j\le 6}\tilde{\mathcal{M}}^{(\alpha)}(U_{x_i}\cap U_{x_j}) & \to & 0
\end{array} \tag{116}$$

for our cocycle c_ξ we find a cocycle $c_\xi^\dagger$ in the complex in the middle which maps to c_ξ under the upwards arrow and this cocycle is unique up to a coboundary. Then we project it down by the downwards arrow, i.e. we only take its $4\le i<j\le 6$ components, and this is our cocycle $c_\xi^{(\alpha)}$.

We write down these complexes explicitly. For any pair $\underline{i}=(i,j), i<j$ of indices we have to compute the set $\mathcal{F}_{\underline{i}}$. We drew some pictures and from these pictures we get (modulo errors) the following list (of lists):

$$\begin{array}{cccc}
\mathcal{F}_{1,2}=\emptyset & \mathcal{F}_{1,3}=\{\mathrm{Id},T_+^{-2}\} & \mathcal{F}_{1,4}=\{\mathrm{Id}\} & \mathcal{F}_{1,5}=\{\mathrm{Id},T_+^{-2}\}\\
\mathcal{F}_{1,6}:=\{\mathrm{Id},T_-^{-1}\} & \mathcal{F}_{2,3}=\{\mathrm{Id}\} & \mathcal{F}_{2,4}=\{\mathrm{Id},T_-\} & \mathcal{F}_{2,5}=\{\mathrm{Id}\}\\
\mathcal{F}_{2,6}=\{\mathrm{Id}\} & \mathcal{F}_{3,4}=\{\mathrm{Id},T_+^{2}\} & \mathcal{F}_{3,5}=\{\mathrm{Id}\} & \mathcal{F}_{3,6}=\{\mathrm{Id},S_1^-\}\\
\mathcal{F}_{4,5}=\emptyset & \mathcal{F}_{4,6}=\{\mathrm{Id},T_-^{-1}\} & \mathcal{F}_{5,6}=\{\mathrm{Id}\} &
\end{array} \tag{117}$$

Now we have to follow the rules in the first section and we can write down an explicit version of the diagram (116) . Here we have to be very careful, because the sets $\tilde{U}_{\tilde{x_2}},\tilde{U}_{\tilde{x_5}}$ have the non-trivial stabilizer $<S_1^->$ and we have to keep track of the action of $\Gamma_{\tilde{x}_{2,5}}$: the set $\mathcal{F}_{i,j}\subset\Gamma_{\tilde{x}_i}\backslash\Gamma/\Gamma_{\tilde{x}_j}$. Therefore we have to replace the group elements $\gamma\in\mathcal{F}_{i,j}$ by sets $\Gamma_{\tilde{x}_i}\gamma\Gamma_{\tilde{x}_j}$. In the list above we have taken representatives.

$$\begin{array}{ccc}
\bigoplus_{1\le i<j\le 3}\bigoplus_{\gamma\in\mathcal{F}_{i,j}}(\mathcal{M}^{(\alpha)})^{\Gamma_{i,j,\gamma}} & \to & 0\\
\uparrow & & \\
\bigoplus_{1\le i<j\le 6}\bigoplus_{\gamma\in\mathcal{F}_{i,j}}(\mathcal{M}^{(\alpha)})^{\Gamma_{i,j,\gamma}} & \to & \bigoplus_{1\le i<j<k\le 6}\bigoplus_{\gamma\in\mathcal{F}_{i,j,k}}(\mathcal{M}^{(\alpha)})^{\Gamma_{i,j,k,\gamma}}\\
\downarrow & & \\
\bigoplus_{4\le i<j\le 6}\bigoplus_{\gamma\in\mathcal{F}_{i,j}}(\mathcal{M}^{(\alpha)})^{\Gamma_{i,j,\gamma}} & \to & 0
\end{array} \tag{118}$$

Here we have to interpret this diagram. The module $\mathcal{M}^{(\alpha)}$ is equal to $\mathcal{M}$ as an abstract module, but an element $\gamma\in\Gamma_0^-(2)$ acts by the twisted action (See ChapII, 2.2)

$$m\mapsto\gamma *_\alpha m=\alpha^{-1}\gamma\alpha * m$$

here the $*$ denotes the original action. Hence we have to take the invariants $(\mathcal{M}^{(\alpha)})^{\Gamma_{i,j,\gamma}}$ with respect to this twisted action. In our special situation this has very little effect since almost all the $\Gamma_{i,j,\gamma}$ are trivial, except for the intersection $\alpha(\tilde{U}_{\frac{\mathbf{i}+1}{2}}) \cap \tilde{U}_{\mathbf{i}}$ in which case $\Gamma_{i,j,\gamma} =< S_1^- >$. Hence

$$(\mathcal{M}^{(\alpha)})^{<S_1^->} = \mathcal{M}^{<S_1^+>}.$$

Each of the complexes in (118) compute the cohomology group $H^1(\Gamma_0^-(2)\backslash\mathbb{H}, \tilde{\mathcal{M}})$ and the diagram gives us a formula for the isomorphism in (98). To get $u_\alpha^\bullet$ in (98) we apply the multiplication $m_2{:}m \mapsto \alpha m$ to the complex in the middle and the bottom. Then the cocycle c_ξ^α is now an element in $\bigoplus \tilde{\mathcal{M}}^{(\alpha)}$ and αc_ξ^α represents the cohomology class $u_\alpha^\bullet(\xi) \in H^1(\Gamma_0^-(2)\backslash\mathbb{H}, \tilde{\mathcal{M}})$.

Now it is clear how we can compute the Hecke operator

$$T_2 = T_{\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}} : \mathcal{M}/(\mathcal{M}^{<S>} \oplus \mathcal{M}^{<R>}) \to \mathcal{M}/(\mathcal{M}^{<S>} \oplus \mathcal{M}^{<R>})$$

We pick a representative $m \in \mathcal{M}$ of the cohomology class. We apply $(\varpi_2^+)^{(1)}$ in the diagram (107) to it and this gives the element $(Sm, m, T_- m) = c_\xi$. We apply the above process to compute $c_\xi^{(\alpha)}$. Then $\alpha c_\xi^{(\alpha)} = (m_1, m_2, m_3)$ is an element in $\tilde{\mathcal{M}}(U_{\mathbf{i}}^- \cap U_\rho^-) \oplus \tilde{\mathcal{M}}(U_{\mathbf{i}+1}^- \cap U_\rho^-)$ and this module is identified with $\mathcal{M} \oplus \mathcal{M} \oplus \mathcal{M}$ by the vertical arrow in (111). To this element we apply the trace

$$(\varpi_2^-)_{(1)}(m_1, m_2, m_3) = m_1 + m_2 + T_+^{-1} m_3$$

and the latter element in $\mathcal{M}$ represents the class $T_2([m])$.

We have written a computer program which for a given $\mathcal{M} = \mathcal{M}_n$, i.e. for a given even positive integer n, computes the module $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})$ and the endomorphism T_2 on it.

Looking our data we discovered the following (surprising?) fact: We consider the isomorphism in equation (98). We have the explicit description of the cohomology in (102)

$$H^1(\Gamma_0^+(2)\backslash\mathbb{H}, \tilde{\mathcal{M}}) \xrightarrow{\sim} \mathcal{M}/(\mathrm{Id} - T_+^{-1})\mathcal{M}^{<S_1^+>}$$

and

$$H^1(\Gamma_0^-(2)\backslash\mathbb{H}, \tilde{\mathcal{M}}^{(\alpha)}) \xrightarrow{\sim} \mathcal{M}/(\mathrm{Id} - T_-^{-1})(\mathcal{M}^{(\alpha)})^{<S_1^->}$$

We know that we may represent any cohomology class by a cocycle

$$c_\xi = (0, c_\xi, 0) \in \mathcal{M}^{(\alpha)}(\pi_2^-(\alpha(U_{\mathbf{i}}) \cap \alpha(U_\rho)) \oplus \mathcal{M}^{(\alpha)}(\pi_2^-(\alpha(U_{\mathbf{i}}) \cap \alpha(T_+^{-1} U_\rho)) \oplus \mathcal{M}^{(\alpha)}(\pi_2^-(\alpha(U_{\frac{\mathbf{i}+1}{2}}) \cap \alpha(T_+^{-1} U_\rho))$$

so it is non zero only in the middle component and then it is simply an element in $\mathcal{M}$. If we now look at our data, then it seems to by so that $c_\xi^{(\alpha)}$ is also non zero only in the middle, hence

$$c_\xi^{(\alpha)} \in (0, c'_\xi, 0) \in 0 \oplus \mathcal{M}^{(\alpha)}(\pi_2^-(U_{\mathbf{i}} \cap T_-^{-1} U_\rho)) \oplus 0$$

hence it is also in $\mathcal{M}^{(\alpha)}$ and then our data seem to suggest that

$$c'_\xi = c_\xi$$

Hence we see that the homomorphism in equation (99) is simply given by

$$X^\nu Y^{n-\nu} \mapsto 2^\nu X^\nu Y^{n-\nu}.$$

Is there a kind of homotopy argument (- 2 moves continuously to 1)-, which explains this?

We get an explicit formula for the Hecke operator T_2 : We pick an element $m \in \mathcal{M}$ representing the class $[m]$. We send it by $(\varpi_2^+)^{(1)}$ to $H^1(\Gamma_0^+(2)\backslash\mathbb{H}, \tilde{\mathcal{M}})$, i.e.

$$(\varpi_2^+)^{(1)} : m \mapsto (m, Sm, T_- m) \tag{119}$$

We modify it so that the first and the third entry become zero see(103)

$$[(m, Sm, T_- m)] = [(0, Sm - m + T_+^{-1}T_- m - T_- m, 0)] \tag{120}$$

To the entry in the middle we apply $M_2 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and then apply $(\varpi_2^-)_{(1)}$

and get

$$T_2([m]) = [S \cdot M_2(Sm - m + T_+^{-1}T_- m - T_- m)] \tag{121}$$

4.3.8 The first interesting example

We give an explicit formula for the cohomology in the case of $\mathcal{M} = \mathcal{M}_{10}$. We define the sub-modul

$$\mathcal{M}^{\mathrm{tr}} = \bigoplus_{\nu=0}^{5} \mathbb{Z} Y^{10-\nu} X^\nu$$

and we have the truncation operator

$$\mathrm{trunc} : Y^{10-\nu} X^\nu \mapsto \begin{cases} Y^{10-\nu} X^\nu & \text{if } \nu \leq 5, \\ (-1)^{\nu+1} Y^\nu X^{10-\nu} & \text{else,} \end{cases}$$

which identifies the quotient module $\mathcal{M}/\mathcal{M}^{<S>}$ to $\mathcal{M}^{\mathrm{tr}}$. To get the cohomology we have to divide by the relations coming from $\mathcal{M}^{<R>}$, i.e. we have to divide by the submodule $\mathrm{trunc}(\mathcal{M}^{<R>}.)$ The module of these relations is generated by

$$\begin{gathered} \mathrm{R}_1 = 10Y^9X + 20Y^7X^3 + Y^5X^5 \\ \mathrm{R}_2 = 9Y^8X^2 - 36Y^7X^3 + 14Y^6X^4 - 45Y^5X^5 \\ \mathrm{R}_3 = 8Y^7X^3 + 10Y^5X^5 \end{gathered}$$

and then

$$H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) = \bigoplus_{\nu=0}^{5} \mathbb{Z} Y^{10-\nu} X^\nu / \{\mathrm{R}_1, \mathrm{R}_2, \mathrm{R}_3\} \tag{122}$$

We simplify the notation and put $e_\nu = Y^\nu X^{n-\nu}$. Using R_1 we can eliminate $e_5 = -10e_9 - 20e_7$ and then

$$H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) = \bigoplus_{\nu=10}^{\nu=6} \mathbb{Z} e_\nu / \{-50e_9 + 9e_8 - 96e_7 + 14e_6, -100e_9 - 192e_7\} \tag{123}$$

introduce a new basis $\{f_{10}, f_9, f_8, f_7, f_6, f_5\}$ of the $\mathbb{Z}$ module $\mathcal{M}^{\mathrm{tr}}$:

$$\begin{gathered} f_{10} = e_{10}; f_8 = -2e_8 - 3e_6; f_6 = 9e_8 + 14e_6 \\ f_9 = -12e_9 - 23e_7; f_7 = 25e_9 + 48e_7; f_5 = 10e_9 + 20e_7 + e_5 \end{gathered} \tag{124}$$

and hence in the quotient we get $\bar{f}_5 = 0$ and $2\bar{f}_7 = \bar{f}_6$ and therefore

$$H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) = \mathbb{Z}\bar{f}_{10} \oplus \mathbb{Z}\bar{f}_9 \oplus \mathbb{Z}\bar{f}_8 \oplus \mathbb{Z}/(4)\bar{f}_7 \tag{125}$$

(If we invert the primes < 12 then we we can work with e_{10}, e_9, e_8 and in cohomology $e_6 = -\frac{9}{14}e_8, e_5 = \frac{5}{12}e_9, e_7 = -\frac{25}{48}e_9$.)

If we can apply the above procedure to compute the action of T_2 on cohomology we get the following matrix for T_2 :

$$T_2 = \begin{pmatrix} 2049 & -68040 & 0 & 0 \\ 0 & -24 & 0 & 0 \\ 0 & 0 & -24 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \tag{126}$$

Hence we see that it is non trivial on the torsion subgroup. If we divide by the torsion then the matrix reduces to a (3,3)-matrix and this matrix gives us the endomorphism on the "integral" cohomology which is defined in generality by

$$H^\bullet_{\mathrm{int}}(\Gamma\backslash X, \tilde{\mathcal{M}}) = H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}})/\mathrm{tors} \subset H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}_{\mathbb{Q}}) \tag{127}$$

here we should be careful: the functor $H^\bullet \to H^\bullet_{\mathrm{int}}$ is not exact. In our case we get (perhaps up to a little piece of 2-torsion) exact sequences of Hecke modules

$$\begin{array}{ccccccc} 0 & \to & \mathbb{Z}f_9 \oplus \mathbb{Z}f_8 & \to \mathbb{Z}f_{10} \oplus \mathbb{Z}f_9 \oplus \mathbb{Z}f_8 \xrightarrow{r} & \mathbb{Z}\bar{f}_{10} \to & 0 \\ & & \| & \| & \| & \\ 0 & \to & H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) \to & H^1_{\mathrm{int}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) \xrightarrow{r} & H^1_{\mathrm{int},!}(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}) \to & 0 \end{array} \tag{128}$$

where $T_2(\bar{f}_{10}) = (2^{11}+1)\bar{f}_{10}$. If we tensor by $\mathbb{Q}$ then we can find an element (the Eisenstein class) $f^\dagger_{10} \in H^1_{\mathrm{int}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) \otimes \mathbb{Q}$ which maps to $\bar{f}_{10}$ This element is not necessarily integral, in our case an easy computation shows that $691f^\dagger \in H^1_{\mathrm{int}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})$. This means that 691 is the denominator of $f^\dagger_{10}$, i.e. 691 is the *denominator* of the Eisenstein class $f^\dagger_{10}$.

The exact sequence $\mathcal{X}_{10}$ in (128) is an exact sequence of modules for the Hecke algebra $\mathcal{H} \supset \mathbb{Z}[T_2]$ and hence it yields an element

$$[\mathcal{X}_{10}] \in \mathrm{Ext}^1_{\mathcal{H}}(\mathbb{Z}f_{10}, H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})), \tag{129}$$

and an easy calculation shows that this Ext^1 group is cyclic of order 691 and that it is generated by $\mathcal{X}_{10}$.

We can go one step further and reduce mod 691. Since there is at most 2 torsion we get an exact sequence of Hecke-modules

$$0 \to H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{F}_{691}) \to H^1_{\mathrm{int}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{F}_{691}) \xrightarrow{r} H^1_{\mathrm{int},!}(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}} \otimes \mathbb{F}_{691}) \to 0. \tag{130}$$

The matrix giving the Hecke operator mod 691 becomes

$$T_2 = \begin{pmatrix} 667 & 369 & 0 \\ 0 & 667 & 0 \\ 0 & 0 & 667 \end{pmatrix} \tag{131}$$

This implies that the extension class $[\mathcal{X}_{10} \otimes \mathbb{F}_{691}]$ is a element of order 691. This implies that 691 divides the order of $[\mathcal{X}_{10}]$ and hence divides the order of the denominator of the Eisenstein class.

4.3.9 The general case

Now we describe the general case $\mathcal{M} = \mathcal{M}_n$ where n is an even integer. We define $\mathcal{M}^{\mathrm{tr}}$ as above, if $n/2$ is even, then we leave out the summand $X^{n/2}Y^{n/2}$, then we get

$$\mathcal{M}^{\mathrm{tr}} = \mathcal{M}/\mathcal{M}^{<S>}.$$

This gives us for the cohomology and the restriction to the boundary cohomology

$$\begin{array}{ccc} H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) & \xrightarrow{\sim} & \mathcal{M}^{\mathrm{tr}}/\mathrm{Rel} \\ \downarrow & & \downarrow \\ H^1(\partial(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) & \xrightarrow{\sim} & \mathcal{M}/(\mathrm{Id}-T)\mathcal{M}. \end{array} \tag{132}$$

We have the basis

$$e_n = \mathrm{trunc}(Y^n), e_{n-1} = \mathrm{trunc}(Y^{n-1}X), \ldots, \begin{cases} Y^{n/2}X^{n/2} & n/2 \text{ odd} \\ 0 & \text{else} \end{cases}$$

for $\mathcal{M}^{\mathrm{tr}}$. Let us put $n_2 = n/2$ or $n/2-1$. Then the algorithm *Smith-normalform* provides a second basis $f_n = e_n, f_{n-1}, \ldots, f_{n_2}$ such that the module of relations becomes

$$d_n f_n = 0, d_{n-1}f_{n-1} = 0, \ldots, d_t f_t = 0, \ldots, d_{n_2} f_{n_2} = 0$$

where $d_{n_2}|d_{n_2+1}|\ldots|d_n$. We have $d_n = d_{n-1} = \cdots = d_{n-2s} = 0$ where $2s+1 = \dim H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) \otimes \mathbb{Q}$ and $d_{n-2s-1} \neq 0$.

With respect to this basis the Hecke operator T_2 is of the form

$$T_2(f_i) = \sum_{j=n}^{j=n_2} t_{i,j}^{(2)} f_j \tag{133}$$

where we have (the numeration of the rows and columns is downwards from n to n_2)

$$\begin{array}{c} t_{\nu,n}^{(2)} = 0 \text{ for } \nu < n \text{ and } t_{i,j}^{(2)} \in \ \mathrm{Hom}(\mathbb{Z}/(d_i), \mathbb{Z}(d_j)) \\ \text{and } t_{i,j}^{(2)} = 0 \text{ for } i \geq n-2s, j < n-2s \end{array} \tag{134}$$

If we divide by the torsion then we get for the restriction map to the boundary cohomology

$$H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})_{\mathrm{int}} = \bigoplus_{\nu=n}^{n-2s} \mathbb{Z}f_\nu \xrightarrow{r} H^1(\partial(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})_{\mathrm{int}} = \mathbb{Z}Y^n \tag{135}$$

where $f_n \mapsto Y^n$ and $T_2(Y^n) = (2^{n+1}+1)Y^n$. The Manin-Drinfeld principle implies that we can find a vector

$$\mathrm{Eis}_n = f_n + \sum_{\nu=n-1}^{\nu=n-2s} x_\nu f_\nu, \;\; x_\nu \in \mathbb{Q} \tag{136}$$

which is an eigenvector for T_2 i.e.

$$T_2(\mathrm{Eis}_n) = (2^{n+1}+1)\mathrm{Eis}_n \tag{137}$$

The least common multiple $\Delta(n)$ of the denominators of the x_ν is the denominator of the Eisenstein class, it is the smallest positive integer for which

$$\Delta(n)\mathrm{Eis}_n \in H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})_{\mathrm{int}}. \tag{138}$$

This denominator is of great interest and our computer program allows us to compute it for any given not to large n. We have to compute the x_ν.

We define $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})_{\mathrm{int},!}$ to be the kernel of r, this is equal to $\bigoplus_{\nu=n-1}^{n-2s} \mathbb{Z} f_\nu$ and the Hecke operator defines an endomorphism

$$T_2^{\mathrm{cusp}} : H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})_{\mathrm{int},!} \to H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})_{\mathrm{int},!} \tag{139}$$

which is given by the matrix $(t^{(2)}_{i,j})$ where $n-1 \geq i,j,n-2s$, i.e. we delete the "first" (i.e.the n-th) row and column.

Now we know that $T_2(f_n) = (2^{n+1}+1)f_n + \sum_{\mu=n-1}^{\mu=n-2s} t^{(2)}_{n,\mu} f_\mu$. Then the x_μ are the unique solution of

$$\sum_{\nu=n-1}^{\nu=n-2s} ((2^{n+1}+1)\delta_{\nu,\mu} - t^{(2)}_{\nu,\mu})x_\nu = t^{(2)}_{n,\mu}; \{\mu = n-1, \dots, n-2s\} \tag{140}$$

These denominators are closely related to values of the Riemann ζ function, it seems that

$$\Delta(n) = \mathrm{numerator}(\zeta(-1-n)). \tag{141}$$

This has been verified up to $n \leq 150$ by a computer. We found some handwritten notes (from about 1980) where this is actually proved by using modular symbols, but this proof has to be checked again.

4.3.10 Computing $\mod p$

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Of course the coefficients $t^{(2)}_{\nu,\mu}$ become very large if n becomes larger, hence we can verify (141) only in a very small range of degrees n.

But if we are a little bit more modest we may be able check experimentally whether a given - perhaps large- prime p, which divides a numerator$\zeta(-1-n)$ for a very large n actually divides $\Delta(n)$. Here we need a little bit of luck.

Assume that we have such a pair (p,n). We want to show that the prime p divides the lcm of the denominators of the x_ν in (140) and this means that the equation (140) has no solution in $\mathbb{Z}_{(p)}$, the local ring at p. This is of course clear if the $\mod p$ reduced equation

$$\sum_{\nu=n-1}^{\nu\equiv n-2s} ((2^{n+1}+1)\delta_{\nu,\mu} - t^{(2)}_{\nu,\mu})x_\nu \equiv t^{(2)}_{n,\mu} \mod p \tag{142}$$

has no solution. (Of course the converse is not true, therefore we need just a little bit of luck!). In this computation the numbers become much smaller. In fact this has now been checked for all $n \leq 100$ we can can easily go much further.

4.3.11 Higher powers of p

This reasoning can also be applied if we look at higher powers of p dividing a numerator$\zeta(-1-n)$. Let us assume that $p^{\delta_p(n)}|$numerator$\zeta(-1-n)$. We have to show that $p^{\delta_p(n)}$ divides the lcm of the denominators of the x_ν in equation (140). This follows if we show that the equation

$$\sum_{\nu=n-1}^{\nu\equiv n-2s} ((2^{n+1}+1)\delta_{\nu,\mu} - t^{(2)}_{\nu,\mu})x_\nu \equiv p^{\delta_p(n)-1} t^{(2)}_{n,\mu} \mod p^{\delta_p(n)} \tag{143}$$

has no solution. This in turn means that the class

$$[\mathcal{X}_n \otimes \mathbb{Z}/p^{\delta_p(n)}\mathbb{Z}] \in \mathrm{Ext}^1_{\mathcal{H}}((\mathbb{Z}/p^{\delta_p(n)}\mathbb{Z})(-1-n), H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes (\mathbb{Z}/p^{\delta_p(n)}\mathbb{Z}))$$

has exact order $p^{\delta_p(n)}$.

Interesting cases to check are $p = 37, 59$ then we have

$$\zeta(-31) \equiv 0 \mod 37;\ \zeta(-283) \equiv 0 \mod 37^2;\ \zeta(-37579) \equiv 0 \mod 37^3;\ \zeta(-1072543) \equiv 0 \mod 37^4; \dots$$

$$\zeta(-43) \equiv 0 \mod 59;\ \zeta(-913) \equiv 0 \mod 59^2$$

Here our computations have a surprising outcome. For $\zeta(-283)$ resp. $\zeta(-913)$ it has been checked that the order of the extension class is 37 resp. 59 so it is smaller than expected. This is not in conflict with the assertion that the denominator is of order $37^2, 59^2$. In fact it turns out that the determinant of the matrix on the left hand side in (143) is $(37^3)^2 = 37^6$ where the denominator only predicts 37^4. Is this always so and is this also true for other Hecke operators?

4.3.12 The denominator and the congruences

For the following we assume that (141) is correct. We discuss the denominator of the Eisenstein class in this special case. In [Talk-Lille] this is discussed in a more abstract way, so here we treat basically the simplest example of 4.3 in [Talk-Lille].

We have the fundamental exact sequence

$$0 \to H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) \to H^1_{\mathrm{int}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) \xrightarrow{r} H^1_{\mathrm{int}}(\partial(\Gamma\backslash\mathbb{H}), \tilde{\mathcal{M}}) = \mathbb{Z}e_n \to 0 \tag{144}$$

and we know that $T_2(e_n) = (2^{n+1}+1)e_n$. We get a submodule

$$H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) \oplus \mathbb{Z}\tilde{e}_n \subset H^1_{\mathrm{int}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) \tag{145}$$

where $\tilde{e}_n$ is primitive and $T_2\tilde{e}_n = (2^{n+1}+1)\tilde{e}_n$. We have $r(\tilde{e}_n) = \Delta(n)e_n$ and

$$H^1_{\mathrm{int}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}})/(H^1_{\mathrm{int},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) \oplus \mathbb{Z}\tilde{e}_n) = \mathbb{Z}/\Delta(n)\mathbb{Z} \tag{146}$$

Any $m \in \mathbb{Z}/\Delta(n)\mathbb{Z}$ can be written as

$$m = r(\frac{y' + m\tilde{e}_n}{\Delta(n)}) \tag{147}$$

and this yields an inclusion $\mathbb{Z}/\Delta(n)\mathbb{Z} \hookrightarrow H^1_{\text{int}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) \otimes \mathbb{Z}/\Delta(n)\mathbb{Z}$.

Hence

Theorem 4.2. *The Hecke module $H^1_{\text{int}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) \otimes \mathbb{Z}/\Delta(n)\mathbb{Z}$ contains a cyclic submodule $\mathbb{Z}/\Delta(n)\mathbb{Z}(-1-n)$ on which the Hecke operator T_p acts by the eigenvalue $p^{n+1}+1 \mod \Delta(n)$ for all primes p*

We can find a finite normal field extension $F/\mathbb{Q}$ such that

$$H^1_{\text{int},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}) \otimes F = \bigoplus_{\pi_f} H^1_{\text{int}}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes F)[\pi_f] \tag{148}$$

where π_f is a homomorphism from the Hecke algebra to $\mathcal{O}_f$ and $H^1..[\pi_f]$ is the rank 2 eigenspace for π_f.

The decomposition induces a Jordan-Hölder filtration on the integral cohomology

$$(0) \subset \mathcal{JH}^{(1)}H^1_{\text{int},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{\mathcal{O}_F}) \subset \mathcal{JH}^{(2)}H^1_{\text{int},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{\mathcal{O}_F}) \subset \cdots \subset \mathcal{JH}^{(r)}H^1_{\text{int},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{\mathcal{O}_F}) \tag{149}$$

where the subquotients a locally free $\mathcal{O}_F$ modules of rank 2 and after tensoring with F they become isomorphic to the different eigenspaces.

We choose a prime p which divides $\Delta(n)$, let $p^{\delta_p(n)}||\Delta(n)$. Let $\mathfrak{p}$ be a prime in $\mathcal{O}_F$ which lies above p. If e_p is the ramification index then we have

$$\mathcal{O}_F/\mathfrak{p}^{e_p\delta_p(n)}(-1-n) \subset H^1_{\text{int},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{\mathcal{O}_F}) \otimes \mathcal{O}_F/\mathfrak{p}^{e\delta_p(n)} \tag{150}$$

The above Jordan-Hölder filtration induces a Jordan-Hölder filtration on the homology $\mod \mathfrak{p}^{e_p\delta_p(n)}$ we have

$$\mathcal{O}_F/\mathfrak{p}^{e_p\delta_p(n)}(-1-n) \hookrightarrow \mathcal{JH}^{(1)}H^1_{\text{int},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{\mathcal{O}_F}) \otimes \mathcal{O}_F/\mathfrak{p}^{e_p\delta_p(n)} \hookrightarrow \mathcal{JH}^{(2)}\dots \tag{151}$$

where again the subquotients are free $\mathcal{O}_F/\mathfrak{p}^{e_p\delta_p(n)}$ modules. This implies

Theorem 4.3. *We can find $\pi_{f,1}, \pi_{f,2} \dots, \pi_{f,r}$ in the above decomposition and numbers $f_1 > 0, f_2 > 0, \dots, f_r > 0$ such that $\sum f_i = e_p\delta_p(n)$ and we have the congruence*

$$\pi_{f,i}(T_\ell) \equiv \ell^{n+1} + 1 \mod \mathfrak{p}^{f_i} \tag{152}$$

for all primes ℓ.

We write $n = n_0 + (p-1)\alpha$ where $0 < n_0 < p-1$, we know that $p|Num(\zeta(-1-n_0))$. We apply the above theorem an find $\pi^{(0)}_{1,f}, \dots, \pi^{(0)}_{r_0,f}$ and a prime $\mathfrak{p}_0 \subset \mathcal{O}_{F_0}$ such that

$$\pi^{(0)}_{f,i}(T_\ell) \equiv \ell^{n+1} + 1 \mod \pi^{(0)}_{f,i} \tag{153}$$

for all indices i. Now it seems to be very likely that the Hecke-module $H^1_{\text{int},!}(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{n_0} \otimes \mathbb{Q})$ is irreducible (Maeda's conjecture , this means that $\text{Gal}(F_0/\mathbb{Q})$ acts transitively on the absolutely irreducible constituents π_f.) We may take F_0 minimal, this means that this action on the set of π_f is faithful. Moreover it is also very likely that the primes $\mathfrak{p}_0$ are unramified and split in F_0.

4.4 Harish-Chandra modules with cohomology

In Chap.III , section 4 we will give a general discussion of the tools from representation theory and analysis which help us to understand the cohomology of arithmetic groups. Especially in Chap.III 4.1.4 we will recall the results of Vogan-Zuckerman on the cohomology of Harish-Chandra modules.

Here we specialize these results to the specific cases $G = \mathrm{Gl}_2(\mathbb{R})$ (case A)) and $G = \mathrm{Gl}_2(\mathbb{C})$ (case B)). For the general definition of Harish-Chandra modules and for the definition of $(\mathfrak{g}, K_\infty)$ cohomology we refer to Chap.III, 4.

4.4.1 The finite rank highest weight modules

We consider the case A), in this case our group $G/\mathbb{R}$ is the base extension of the the reductive group scheme $\mathcal{G} = \mathrm{Gl}_2/\ \mathrm{Spec}(\mathbb{Z})$. (See Chap. IV for the notion of reductive group scheme.) In principle this pretentious language at this point means that we can speak of $\mathcal{G}(R)$ for any commutative ring R with identity. Sometimes in the following we will replace $\mathrm{Spec}(\mathbb{Z})$ by $\mathbb{Z}$) We have the maximal torus $\mathcal{T}/\mathbb{Z}$ and the Borel subgroup $\mathcal{B}/\mathbb{Z}$. We consider the character module $X^*(\mathcal{T}) = X^*(T \times \mathbb{C})$. This character module is $\mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ where

$$e_i : \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \mapsto t_i \tag{154}$$

Any character can be written as $\lambda = n\gamma + d\det$ where $\gamma = \frac{e_1 - e_2}{2} (\notin X^*(\mathcal{T})\ !), \det = e_1 + e_2$ and where $n \in \mathbb{Z}, d \in \frac{1}{2}\mathbb{Z}$ and where $n \equiv 2d \mod 2$. To any such character λ we want to attach a highest weight module $\mathcal{M}_\lambda$. We assume that λ is dominant, i.e. $n \geq 0$ and consider the $\mathbb{Z}-$ module of polynomials

$$\mathcal{M}_n = \{P(X,Y) \mid P(X,Y) = \sum_{\nu=0}^{n} a_v X^\nu Y^{n-\nu}, a_\nu \in \mathbb{Z}\}.$$

To a polynomial $P \in \mathcal{M}_n$ we attach the regular function (See Chap. IV)

$$f_P(\begin{pmatrix} x & y \\ u & v \end{pmatrix}) = P(u,v) \det(\begin{pmatrix} x & y \\ u & v \end{pmatrix})^{\frac{n}{2}+d_1} \tag{155}$$

and we obviously have

$$f_P(\begin{pmatrix} t_1 & w \\ 0 & t_2 \end{pmatrix}\begin{pmatrix} x & y \\ u & v \end{pmatrix}) = t_2^n (t_1 t_2)^d f_P(\begin{pmatrix} x & y \\ u & v \end{pmatrix}) = \lambda^-(\begin{pmatrix} t_1 & w \\ 0 & t_2 \end{pmatrix}) f_P(\begin{pmatrix} x & y \\ u & v \end{pmatrix}) \tag{156}$$

where $\lambda^- = -n\gamma + (\frac{n}{2} + d_1)\det = -n\gamma + d\det$ considered as a character on $\mathcal{B}$. This is a module for the group scheme $\mathcal{G}/\mathbb{Z}$ it is called the highest weight module for λ and is denoted by $\mathcal{M}_\lambda$. The action of $\mathcal{G}$ is of course the action by right translations,i.e.

$$\rho_\lambda(\begin{pmatrix} a & b \\ c & d \end{pmatrix})(f)(\begin{pmatrix} x & y \\ u & v \end{pmatrix}) = f(\begin{pmatrix} x & y \\ u & v \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}) \tag{157}$$

Comment: When we say that $\mathcal{M}_\lambda$ is a module for the group scheme $\mathcal{G}/\mathbb{Z}$ we mean nothing more than that for any commutative ring R with identity we have an action of $\mathcal{G}(R)$ on $\mathcal{M}_n \otimes R$ which is given by (155) and depends functorially on R. We can "evaluate" at $R = \mathbb{Z}$ and get the $\Gamma = \mathrm{Gl}_2(\mathbb{Z})$ module $\mathcal{M}_{\lambda,\mathbb{Z}}$. (Actually we should not so much distinguish between the $\mathrm{Gl}_2(\mathbb{Z})$ module $\mathcal{M}_{\lambda,\mathbb{Z}}$ and $\mathcal{M}_\lambda$)

Remark: There is a slightly more sophisticated interpretation of this module. We can form the flag manifold $\mathcal{B}\backslash\mathcal{G} = \mathbb{P}^1/\mathbb{Z}$ and the character λ yields a line bundle $\mathcal{L}_{\lambda^-}$. The group scheme $\mathcal{G}$ is acting on the pair $(\mathcal{B}\backslash\mathcal{G}, \mathcal{L}_{\lambda^-})$ and hence on $H^0(\mathcal{B}\backslash\mathcal{G}, \mathcal{L}_{\lambda^-})$ which is tautologically equal to $\mathcal{M}_\lambda$ (Borel-Weil theorem).

We can do essentially the same in the case B) . In this case we start from an imaginary quadratic extension $F/\mathbb{Q}$ and let $\mathcal{O} = \mathcal{O}_F \subset F$ its ring of integers. We form the group scheme $\mathcal{G}/\mathbb{Z} = R_{\mathcal{O}/\mathbb{Z}}(\mathcal{G}/\mathcal{O})$. Then $\mathcal{G}(\mathcal{O}) = \mathrm{Gl}_2(\mathcal{O}\otimes\mathcal{O}) \subset \mathrm{Gl}_2(\mathcal{O}) \times \mathrm{Gl}_2(\mathcal{O})$. The base change of the maximal torus $T/\mathbb{Q} \subset \mathcal{G} \times_{\mathbb{Z}} \mathbb{Q}$ is the product $T_1 \times T_2/F$ where the two factors are the standard maximal tori in the two factors Gl_2/F.

We get for the character module

$$X^*(T \times F) = X^*(T_1) \oplus X^*(T_2) = \{n_1\gamma + d_1 \det\} \oplus \{n_2\bar{\gamma} + d_2\bar{\det}\} \tag{158}$$

where we have to observe the parity conditions $n_1 \equiv 2d_1 \mod 2, n_2 \equiv 2d_2 \mod 2$ Then the same procedure as in case a) provides a free $\mathcal{O}$- module $\mathcal{M}_\lambda$ with an action of $\mathcal{G}(\mathbb{Z})$ on it. To see this action we embed the group $\mathcal{G}(\mathbb{Z}) = \mathrm{Gl}_2(\mathcal{O})$ into $\mathrm{Gl}_2(\mathcal{O}) \times \mathrm{Gl}_2(\mathcal{O})$ by the map $g \mapsto (g, \bar{g})$ where $\bar{g}$ is of course the conjugate. If now our $\lambda = n_1\gamma_1 + d_1 \det_1 + n_2\gamma_2 + d_2\det_2 = \lambda_1 + \lambda_2$ then we have our two $\mathrm{Gl}_2(\mathcal{O})$ modules $\mathcal{M}_{\lambda_1,\mathcal{O}}, \mathcal{M}_{\lambda_2,\mathcal{O}}$ and hence the $\mathrm{Gl}_2(\mathcal{O})\times\mathrm{Gl}_2(\mathcal{O})$- module $\mathcal{M}_{\lambda_1,\mathcal{O}}\otimes\mathcal{M}_{\lambda_2,\mathcal{O}}$, is now our $\mathcal{M}_{\lambda,\mathcal{O}}$ is simply the restriction of this tensor product module to $\mathcal{G}(\mathbb{Z})$.

4.4.2 The principal series representations

We consider the two real algebraic groups $G = \mathrm{Gl}_2/\mathbb{R}$ and $G = R_{\mathbb{C}/\mathbb{R}}Gl_2(\mathbb{C})$, Let $T/\mathbb{R}$, resp.$B/\mathbb{R}$ be the standard diagonal torus (resp.) Borel subgroup. Let us put $Z/\mathbb{R} = \mathbb{G}_m$ (resp. $R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$). We have the determinant $\det : G/\mathbb{R} \to Z/\mathbb{R}$ and moreover $Z/\mathbb{R} = \mathit{center}(G/\mathbb{R})$. If we restrict the determinant to the center then this becomes the map $z \mapsto z^2$. Let us denote by $\mathfrak{g}, \mathfrak{t}, \mathfrak{b}, \mathfrak{z}$ the corresponding Lie-algebras.

Our aim is to to construct certain irreducible representations of $G(\mathbb{R})$ and their "algebraic skeleton" the associated Harish-Chandra modules. Of course homomorphism $\eta : Z \to \mathbb{C}^\times$ yields via composition with the determinant a one dimensional $G(\mathbb{R})$ module $\mathbb{C}\eta$. We want to construct infinite dimensional $G(\mathbb{R})$ modules.

We start from a continuous homomorphism (a character) $\chi : T(\mathbb{R}) \to \mathbb{C}^\times$, of course this can also be seen as a character $\chi : B(\mathbb{R}) \to \mathbb{C}^\times$. This allows us to define the induced module

$$I_B^G\chi := \{f : G(\mathbb{R}) \to \mathbb{C} \mid f \in \mathcal{C}_\infty(G(\mathbb{R})), f(bg) = \chi(b)f(g),\ \forall\, b \in B(\mathbb{R}), g \in G(\mathbb{R})\} \tag{159}$$

This space of functions is a $G(\mathbb{R})$ - module, the group $G(\mathbb{R})$ acts by right translations: For $f \in I_B^G\chi, g \in G(\mathbb{R})$ we put

$$R_g(f)(x) = f(xg)$$

Inside $G(\mathbb{R})$ we have the connected component of identity of the standard maximal compact subgroup $K_\infty^0 (= SO(2)$ resp. $U(2))$ and we know that $G(\mathbb{R}) = B(\mathbb{R})\cdot K_\infty$. This implies that a function $f \in I_B^G\chi$ is determined by its restriction to K_∞. In other words we have an identification of vector spaces

$$I_B^G\chi = \{f : K_\infty \to \mathbb{C} \mid f(t_c k) = \chi(t_c)f(k), t_c \in K_\infty \cap B(\mathbb{R}), k \in K_\infty\}. \tag{160}$$

We put $T_c = B(\mathbb{R}) \cap K_\infty$ and define χ_c to be the restriction of χ to T_c. Then the module on the right in the above equation can be written as $I_{T_c}^{K_\infty}\chi_c$. By its very definition $I_{T_c}^{K_\infty}\chi_c$. is only a K_∞ module it can be endowed with the structure of a $G(\mathbb{R})$ module via the above identification.

Inside $I_{T_c}^{K_\infty}\chi_c$ we have the submodule of vectors of finite type

$$^\circ I_{T_c}^{K_\infty}\chi_c := \{f \in I_{T_c}^{K_\infty}\chi_c \mid \text{ the translates } R_k(f) \text{ lie in a finite dimensional subspace}\} \tag{161}$$

The famous Peter-Weyl theorem tells us that all irreducible representations (satisfying some continuity condition) are finite dimensional and occur with finite multiplicity in $I_{T_c}^{K_\infty}\chi_c$ and therefore we get

$$^\circ I_{T_c}^{K_\infty}\chi_c = \bigoplus_{\vartheta \in \hat{K}_\infty} V_\vartheta^{m(\vartheta)} = \bigoplus_{\vartheta \in \hat{K}_\infty} {}^\circ I_{T_c}^{K_\infty}\chi_c[\vartheta] \tag{162}$$

where $\hat{K}_\infty$ is the set of isomorphism classes of irreducible representations of K_∞, where V_ϑ is an irreducible module of type ϑ and where $m(\vartheta)$ is the multiplicity of ϑ in ${}^\circ I_{T_c}^{K_\infty}\chi_c$. Of course ${}^\circ I_{T_c}^{K_\infty}\chi_c$ is a submodule ${}^\circ I_B^G\lambda_R$, this submodule is not invariant invariant under the operation of $G(\mathbb{R})$ in other words if $0 \neq f \in {}^\circ I_{T_c}^{K_\infty}\chi_c$ and $g \in G(\mathbb{R})$ a sufficiently general element then $R_g(f) \notin {}^\circ I_{T_c}^{K_\infty}\chi_c$.

But we can differentiate the action of $G(\mathbb{R})$ on $I_B^G\lambda_{\mathbb{R}}$. We have the well known exponential map $\exp : \mathfrak{g} = \mathrm{Lie}(G/\mathbb{R}) \to G(\mathbb{R})$ and we define for $f \in I_B^G, X \in \mathfrak{g}$

$$Xf(g) = \lim_{t\to 0} \frac{f(g\exp(tX)) - f(g)}{t} \tag{163}$$

and it is well known and also easy to see, that this gives an action of the Lie-algebra on I_B^G, we have $X_1(X_2 f) - X_2(X_1 f) = [X_1, X_2]f$. The Lie-algebra is a K_∞ module under the adjoint action and is obvious that for $f \in {}^\circ I_{T_c}^{K_\infty}\chi_c[\vartheta]$ the element Xf lies in $\bigoplus_{\vartheta\in\hat{K}_\infty} {}^\circ I_{T_c}^{K_\infty}\chi_c[\vartheta']$ where ϑ' runs over the finitely many isomorphism types occurring in $V_\vartheta \otimes \mathfrak{g}$.

Proposition 4.1. *The submodule ${}^\circ I_{T_c}^{K_\infty}\chi_c \subset I_B^G\chi_c$ is invariant under the action of $\mathfrak{g}$.*

We will denote by $\mathfrak{I}_B^G\chi$ the submodule ${}^\circ I_{T_c}^{K_\infty}\chi_c$ together with this action of $\mathfrak{g}$. Such a module will be called a $(\mathfrak{g}, K_\infty)$ - module or a Harish-Chandra module this means that we have an action of the Lie-algebra $\mathfrak{g}$, an action of K_∞ and these two actions satisfy some obvious compatibility conditions.

We also observe that ${}^\circ I_{T_c}^{K_\infty}\chi_c$ is also invariant under right translation R_z for $z \in Z(\mathbb{R})$. Hence we can extend the action of K_∞ to the larger group $\tilde{K}_\infty = K_\infty \cdot Z(\mathbb{R})$. Then $\mathfrak{I}_B^G\chi$ becomes a $(\mathfrak{g}, \tilde{K}_\infty)$ module.

These $(\mathfrak{g}, \tilde{K}_\infty)$ modules $\mathfrak{I}_B^G\chi$ are called the principal series modules.

We denote the restriction of χ to the central torus $Z = \{\begin{pmatrix} t & 0 \\ 0 & t\end{pmatrix}\}$ by ω_χ. Then $Z(\mathbb{R})$ acts on $\mathfrak{I}_B^G\chi$ by the character ω_χ, i.e. $R_z(f) = \omega_\chi(z)f$.

4.4.3 The decomposition into K_∞-types

Kutypes

We look briefly at the K_∞-module ${}^\circ I_{T_c}^{K_\infty}\chi_c$. In case A) the group

$$K_\infty = SO(2) = \{\begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix} = e(\phi)\} \tag{164}$$

and $T_c = T(\mathbb{R}) \cap K_\infty$ is cyclic of order two with generator $e(\pi)$. Then χ_c is given by an integer mod 2, i.e. $\chi_c(e(\pi)) = (-1)^m$. For any $n \equiv m$ mod 2 we define $\psi_n \in \mathfrak{I}_B^G\chi$ by $\psi_n(e(\phi))) = e^{in\phi}$ and then

decoKuA

$$\mathfrak{I}_B^G\chi = \bigoplus_{k\equiv m \mod 2} \mathbb{C}\psi_k \tag{165}$$

In the case B) the maximal compact subgroup is

$$U(2) \subset G(\mathbb{R}) = R_{\mathbb{C}/\mathbb{R}}(\mathrm{Gl}_2/\mathbb{C})(\mathbb{R}) \subset \mathrm{Gl}_2(\mathbb{C}) \times \mathbb{G}_2(\mathbb{C})$$

this is the group of real points of the reductive group $U(2)/\mathbb{R}$. The intersection

$$T_c = T(\mathbb{R}) \cap K_\infty = \{\begin{pmatrix} e^{i\phi_1} & 0 \\ 0 & e^{i\phi_2} \end{pmatrix} = e(\underline{\phi})\}.$$

The base change $U(2) \times \mathbb{C} = \mathrm{Gl}_2/\mathbb{C}$ and $T_c \times \mathbb{C}$ becomes the standard maximal compact torus. The irreducible finite dimensional $U(2)$-modules are labelled by dominant highest weights $\lambda_c = n\gamma_c + d\det \in X^*(T_c \times \mathbb{C})$ (See section (4.4.1), here again $n \geq 0, n \in \mathbb{Z}, n \equiv 2d \mod 2$ and $\gamma_c(e(\underline{\phi})) = e^{i(\phi_1-\phi_2)/2}$.)

We denote these modules by $\mathcal{M}_{\lambda_c}$ after base change to $\mathbb{C}$ they become the modules $\mathcal{M}_{\lambda,\mathbb{C}}$.

As a subgroup of $G(\mathbb{R}) \subset \mathrm{Gl}_2(\mathbb{C}) \times \mathbb{G}_2(\mathbb{C})$ our torus is

$$T_c = \{\begin{pmatrix} e^{i\phi_1} & 0 \\ 0 & e^{i\phi_2} \end{pmatrix} \times \begin{pmatrix} e^{-i\phi_1} & 0 \\ 0 & e^{-i\phi_2} \end{pmatrix}\} \xrightarrow{\sim} \{\begin{pmatrix} e^{i\phi_1} & 0 \\ 0 & e^{i\phi_2} \end{pmatrix}\} \tag{166}$$

and the restriction of χ to T_c is of the form

$$\chi_c(e(\underline{\phi})) = e^{ia\phi_1 + ib\phi_2} = e^{\frac{a-b}{2}(\phi_1-\phi_2)} e^{\frac{a+b}{2}(\phi_1+\phi_2)} \tag{167}$$

and this character is $(a-b)\gamma_c + \frac{a+b}{2}\det$. Then we know

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$$^\circ I_{T_c}^{K_\infty}\chi_c = \mathfrak{I}_B^G\chi = \bigoplus_{\mu_c = k\gamma_c + \frac{a+b}{2}\det; k\equiv(a-b) \mod 2; k\geq|a-b|} \mathcal{M}_{\mu_c} \tag{168}$$

4.4.4 Intertwining operators

Let $N(T)$ the normalizer of $T/\mathbb{R}$, the quotient $W = N(T)/T$ is a finite group scheme. The in our case the group $W(\mathbb{R})$ is cyclic of order 2 and generated by

$$w_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

In case a) we have $W(\mathbb{R}) = W(\mathbb{C})$ in case b) we have

$$G \times_{\mathbb{R}} \mathbb{C} = (\mathrm{Gl}_2 \times \mathrm{Gl}_2)/\mathbb{C}\ ;\ T \times_{\mathbb{R}} \mathbb{C} = T_1 \times T_2\ ;\ \text{ and } W(\mathbb{C}) = \mathbb{Z}/2 \times \mathbb{Z}/2$$

where the two factors are generated by $s_1 = (w_0, 1), s_2 = (1, w_0)$. The group $W(\mathbb{R})$ is the group of real points of the Weyl group, the group $W = W(\mathbb{C})$ is the Weyl group or the absolute Weyl group.

We introduces the special character

$$|\rho| : \begin{pmatrix} t_1 & u \\ 0 & t_2 \end{pmatrix} \to |\frac{t_1}{t_2}|^{\frac{1}{2}}$$

The group $W(\mathbb{R})$ acts on $T(\mathbb{R})$ by conjugation and hence it also acts on the group of characters, we denote this action by $\chi \mapsto \chi^w$. We define the twisted action

$$w \cdot \chi = (\chi|\rho|)^w |\rho|^{-1}$$

We recall some well known facts

i) We have a non degenerate $(\mathfrak{g}, K_\infty)$ invariant pairing

$$\mathfrak{I}_B^G \chi \times \mathfrak{I}_B^G \chi^{w_0} |\rho|^2 \to \mathbb{C}\omega_\chi^2 \text{ given by } (f_1, f_2) \mapsto \int_{K_\infty} f_1(k) f_2(k) dk \quad (169)$$

We define the dual $\mathfrak{I}_B^{G,\vee}\chi$ of a Harish-Chandra as a submodule of $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{I}_B^G\chi, \mathbb{C})$, it consists of those linear maps which vanish on almost all K_∞ types. It is clear that this is again a $(\mathfrak{g}, K_\infty)$-module. The above assertion can be reformulated

ii) We have an isomorphism of $(\mathfrak{g}, K_\infty)$ modules

$$\mathfrak{I}_B^G \chi \delta_\chi \to \mathfrak{I}_B^{G,\vee} \chi^{w_0} |\rho|^2 \quad (170)$$

The group $T(\mathbb{R}) = T_c \times (\mathbb{R}_{>0}^\times)^2$ and hence we can write any character χ in the form

$$\chi(t) = \chi_c(t) |t_1|^{z_1} |t_2|^{z_2} \quad (171)$$

where $z_1, z_2 \in \mathbb{C}$.

For $f \in \mathfrak{I}_B^G \chi, g \in G(\mathbb{R})$ we consider the integral

$$T_\infty^{\mathrm{loc}}(f)(g) = \int_{U(\mathbb{R})} f(w_0 u g) du \quad (172)$$

It is well known and easy to check that these integrals converge absolutely and locally uniformly for $\Re(z_1 - z_2) >> 0$ and it is also not hard to see that they extend to meromorphic functions in the entire $\mathbb{C}^2$. We can "evaluate" them at all (z_1, z_2) by suitably regularizing at poles (for instance taking residues). This needs some explanation. To define the regularized intertwining operator we consider the "deformed" intertwining operator

$$T_\infty^{\mathrm{loc}}(\lambda_{\mathbb{R}}^{w_0} |\gamma|^z) : \mathfrak{I}_B^G \lambda_{\mathbb{R}}^{w_0} |\gamma|^z \to \mathfrak{I}_B^G \lambda_{\mathbb{R}} |\rho|^2 |\gamma|^{-z} \quad (173)$$

(See 172, $\chi = \lambda_{\mathbb{R}}^{w_0} |\gamma|^z$) and this integral converges if $\Re(z) >> 0$. We have decomposed

$$\mathfrak{I}_B^G \lambda_{\mathbb{R}}^{w_0} |\gamma|^z = \bigoplus_{\vartheta \in \hat{K}_\infty} {}^\circ I_{T_c}^{K_\infty} \chi_c[\vartheta] = \bigoplus_{\vartheta \in \hat{K}_\infty} \mathfrak{I}_B^G \lambda_{\mathbb{R}}^{w_0} |\gamma|^z [\vartheta]$$

and our intertwining operator is a direct sum of linear maps between finite dimensional vector spaces

$$c(\lambda_{\mathbb{R}}^{w_0} |\gamma|^z, \vartheta) : \mathfrak{I}_B^G \lambda_{\mathbb{R}}^{w_0} |\gamma|^z [\vartheta] \to \mathfrak{I}_B^G \lambda_{\mathbb{R}} |\rho|^2 |\gamma|^{-z} [\vartheta]$$

The finite dimensional vector spaces do not depend on z and the $c(\lambda_{\mathbb{R}}^{w_0}|\gamma|^z, \vartheta)$ can be expressed in terms of values of the $\Gamma-$ function. Especially they are meromorphic functions in the variable z (See sl2neu.pdf,). Hence we can can find an integer $m \geq 0$ such that

$$z^m \mathfrak{I}_B^G \lambda_{\mathbb{R}}^{w_0}|\gamma|^z|_{z=0} : \mathfrak{I}_B^G \lambda_{\mathbb{R}}^{w_0} \to \mathfrak{I}_B^G \lambda_{\mathbb{R}}|\rho|^2$$

is a non zero intertwining operator and this is now our regularized operator $T_\infty^{\mathrm{loc,reg}}(\lambda_{\mathbb{R}}^{w_0})$.

iii) The regularized values define non zero intertwining operators

$$T_\infty^{\mathrm{loc,reg}}(\chi) : \mathfrak{I}_B^G \chi \to \mathfrak{I}_B^G \chi^{w_0}|\rho|^2 \tag{174}$$

These operators span the one dimensional space of intertwining operators $\mathrm{Hom}_{(\mathfrak{g},K_\infty)}(\mathfrak{I}_B^G\chi, \mathfrak{I}_B^G w_0 \cdot \chi)$

Finally we discuss the question which of these representations are unitary. This means that we have to find a pairing

$$\psi : \mathfrak{I}_B^G\chi \times \mathfrak{I}_B^G\chi \to \mathbb{C} \tag{175}$$

which satisfies

a) it is linear in the first and conjugate linear in the second variable

b) It is positive definite, i.e. $\psi(f,f) > 0 \, \forall f \in \mathfrak{I}_B^G\chi$

c) It is invariant under the action of K_∞ and Lie-algebra invariant under the action of $\mathfrak{g}$, i.e. we have

For $f_1, f_2 \in \mathfrak{I}_B^G\chi$ and $X \in \mathfrak{g}$ we have $\psi(Xf_1, f_2) + \psi(f_1, Xf_2) = 0$.

We are also interested in quasi-unitatry modules. This is notion is perhaps best explained if and instead of c) we require

d) There exists a continuous homomorphism (a character) $\eta : G(\mathbb{R}) \to \mathbb{R}^\times$ such that $\psi(gf_1, gf_2) = \eta(g)\psi(f_1, f_2)$, $\forall g \in G(\mathbb{R}), f_1 f_2 \in \mathfrak{I}_B^G\chi$

It is clear that a non zero pairing ψ which satisfies a) and c) is the same thing as a non zero $(\mathfrak{g}, K_\infty)$-module linear map

$$i_\psi : \mathfrak{I}_B^G\chi \to \overline{(\mathfrak{I}_B^G\chi)^\vee} \tag{176}$$

by definition i_ψ is a conjugate linear map from $\mathfrak{I}_B^G\chi$ to $(\mathfrak{I}_B^G\chi)^\vee$. The map i_ψ and the pairing ψ are related by the formula $\psi(v_1, v_2) = i_\psi(v_2)(v_1)$.

Of course we know that (See (170))

$$\overline{(\mathfrak{I}_B^G\chi)^\vee} \xrightarrow{\sim} \mathfrak{I}_B^G\overline{\chi^{w_0}|\rho|^2\delta_\chi^{-1}} \tag{177}$$

and we find such an i_ψ if

$$\chi = \overline{\chi^{w_0}|\rho|^2\delta_\chi^{-1}} \text{ or } \chi^{w_0}|\rho|^2 = \overline{\chi^{w_0}|\rho|^2\delta_\chi^{-1}} \tag{178}$$

We write our χ in the form (171). A necessary condition for the existence of a hermitian form is of course that all $|\omega_\chi(x)| = 1$ for $x \in Z(\mathbb{R})$ and this means that $\Re(z_1 + z_2) = 0$, hence we write

$$z_1 = \sigma + i\tau_1, z_2 = -\sigma + i\tau_2 \tag{179}$$

Then the two conditions in (178) simply say

$$(\mathrm{un}_1) : \sigma = \frac{1}{2} \text{ or } (\mathrm{un}_2) : \tau_1 = \tau_2 \text{ and } \chi_c = \chi_c^{w_0} \tag{180}$$

In both cases we can write down a pairing which satisfies a) and c). We still have to check b). In the first case, i.e. $\sigma = \frac{1}{2}$ we can take the map $i_\psi = \mathrm{Id}$ and then we get for $f_1, f_2 \in \mathfrak{I}_B^G\chi$ the formula

$$\psi(f_1, f_2) = \int_{K_\infty} f_1(k)\overline{f_2(k)}dk \tag{181}$$

and this is clearly positive definite. These are the representation of the unitary principal series.

In the second case we have to use the intertwining operator in (174) and write

$$\psi(f_1, f_2) = T_\infty^{\mathrm{loc,reg}}(f_2)(f_1) \tag{182}$$

Now it is not clear whether this pairing satisfies b). This will depend on the parameter σ. We can twist by a character $\eta : Z(\mathbb{R}) \to \mathbb{C}^\times$ and achieve that $\chi_c = 1, \tau_1 = \tau_2 = 0$. We know that for $\sigma = \frac{1}{2}$ the intertwining operator T_∞^{loc} is regular at χ and since in addition under these conditions $\mathfrak{I}_B^G\chi$ is irreducible we see that

$$T_\infty^{\mathrm{loc}}(\chi) = \alpha \text{ Id with } \alpha \in \mathbb{R}_{>0}^\times \tag{183}$$

Since we now are in case a) and b) at the same time we see that the two pairings defined by the rule in case (un_1) and (un_2) differ by a positive real number hence the pairing defined in (182) is positive definite if $\sigma = \frac{1}{2}$.

But now we can vary σ. It is well known that $\mathfrak{I}_B^G\chi$ stays irreducible as long as $0 < \sigma < 1$ (See next section) and since $T_\infty^{\mathrm{loc}}(\chi)(f)(f)$ varies continuously we see that (182) defines a positive definite hermitian product on $\mathfrak{I}_B^G\chi$ as long as $0 < \sigma < 1$. This is the supplementary series. What happens if we leave this interval will be discussed in the next section.

4.4.5 Reducibility and representations with non trivial cohomology

As usual we denote by $\rho \in X^*(T) \otimes \mathbb{Q}$ the half sum of positive roots we have $\rho = \gamma$ resp. $\rho = \gamma_1 + \gamma_2 \in X^*(T) \otimes \mathbb{Q}$ in case A) (resp. B)).

For any character $\lambda \in X^*(T \times \mathbb{C})$ we define $\lambda_\mathbb{R}$ to be the restriction (or evaluation)

$$\lambda_\mathbb{R} : T(\mathbb{R}) \to \mathbb{C}^\times.$$

This provides a homomorphism $B(\mathbb{R}) \to T(\mathbb{R})$ and hence we get the Harish-Chandra modules $\mathfrak{I}_B^G\lambda_\mathbb{R}$ are of special interest for our subject namely the cohomology of arithmetic groups.

We just mention the fact that $\mathfrak{I}_B^G\chi$ is always irreducible unless $\chi = \lambda_\mathbb{R}$ (See sl2neu.pdf, Condition (red)).

We return to the situation discussed in section (4.4.1), especially we reintroduce the field $F/\mathbb{Q}$. Then we have $X^*(T \times F) = X^*(T \times \mathbb{C})$ and hence $\lambda \in X^*(T \times F)$. We assume that λ is dominant, i.e. $n \geq 0$ in case a) or $n_1, n_2 \geq 0$ in case b). In this case we realized our modules $\mathcal{M}_\lambda$ as submodules in the algebra of regular functions on $\mathcal{G}/\mathbb{Z}$ and if we look at the definition (See (156)) we see immediately that $\mathcal{M}_{\lambda,\mathbb{C}} \subset \mathfrak{I}_B^G \lambda_{\mathbb{R}}^{w_0}$ and hence we get an exact sequence of $(\mathfrak{g}, K_\infty)$ modules

$$0 \to \mathcal{M}_{\lambda,\mathbb{C}} \to \mathfrak{I}_B^G \lambda_{\mathbb{R}}^{w_0} \to \mathcal{D}_\lambda \to 0 \tag{184}$$

Hence we see that $\mathfrak{I}_B^G \lambda_{\mathbb{R}}^{w_0}$ is not irreducible. We can also look at the dual sequence. Here we recall that we wrote $\lambda = n\gamma + d\det$. Then we will see later that $\mathcal{M}_{\lambda,\mathbb{C}}^\vee = \mathcal{M}_{\lambda-2d\det,\mathbb{C}}$. Hence after twisting the dual sequence becomes

$$0 \to \mathcal{D}_\lambda^\vee \otimes \det_{\mathbb{R}}^{2d} \to \mathfrak{I}_B^{G,\vee} \lambda_{\mathbb{R}}^{w_0} \to \mathcal{M}_{\lambda,\mathbb{C}} \to 0 \tag{185}$$

Equation (170) yields $\mathfrak{I}_B^{G,\vee} \lambda_{\mathbb{R}}^{w_0} \xrightarrow{\sim} \mathfrak{I}_B^G \chi|\rho|^2$ and our second sequence becomes

$$0 \to \mathcal{D}_\lambda^\vee \otimes \det_{\mathbb{R}}^{2d} \to \mathfrak{I}_B^G \lambda_{\mathbb{R}}|\rho|^2 \to \mathcal{M}_{\lambda,\mathbb{C}} \to 0 \tag{186}$$

Now we consider the two middle terms in the two exact sequences (184,186) above. The equation (174) claims that we have two non zero *regularized* intertwining operators

$$T_\infty^{\mathrm{loc,reg}}(\lambda_{\mathbb{R}}^{w_0}) : \mathfrak{I}_B^G \lambda_{\mathbb{R}}^{w_0} \to \mathfrak{I}_B^G \lambda_{\mathbb{R}}|\rho|^2 \quad ; T_\infty^{\mathrm{loc,reg}}(\lambda_{\mathbb{R}}|\rho|^2) : \mathfrak{I}_B^G \lambda_{\mathbb{R}}|\rho|^2 \to \mathfrak{I}_B^G \lambda_{\mathbb{R}}^{w_0} \tag{187}$$

If we now look more carefully at our two regularized intertwining operators above then a simple computation yields (see sl2neu.pdf)

Proposition 4.2. *The kernel of $T_\infty^{loc,\mathrm{reg}}(\lambda_{\mathbb{R}}^{w_0})$ is $\mathcal{M}_{\lambda,\mathbb{C}}$ and this operator induces an isomorphism*

$$\bar{T}(\lambda_R) : \mathcal{D}_\lambda \xrightarrow{\sim} \mathcal{D}_\lambda^\vee \otimes \det_{\mathbb{R}}^{2d}$$

(Remember λ is dominant) The kernel of $T_\infty^{loc,\mathrm{reg}}(\lambda_{\mathbb{R}}|\rho|^2)$ is $\mathcal{D}_\lambda^\vee \otimes \det_{\mathbb{R}}^{2d}$ and it induces an isomorphism of $\mathcal{M}_{\lambda,\mathbb{C}}$.

The module $\mathfrak{I}_B^G \chi$ is reducible if $T_\infty^{loc,\mathrm{reg}}(\chi)$ not an isomorphism and this happens if an only if $\chi = \lambda_{\mathbb{R}}$ or $\lambda_{\mathbb{R}}^{w_0}|\rho|^2$ and λ dominant. (There is one exception to the converse of the above assertion, namely in the case A) and $\sigma = \frac{1}{2}$ and $\chi_c^{w_0} \neq \chi_c$.)

For us of of relevance is to know whether we have a positive definite hermitian form on the $(\mathfrak{g}, K_\infty)$-modules $\mathcal{D}_\lambda$. To discuss this question we treat the cases A) and B) separately.

We look at the decomposition into K_∞-types. (See (165)) In case A) (See (165)) it is clear that $\mathcal{M}_{\lambda,\mathbb{C}}$ is the direct sum of the K_∞ types $\mathbb{C}\psi_l$ with $|l| \leq n$. Hence

$$\mathcal{D}_\lambda = \bigoplus_{k \leq -n-2, k \equiv m(2)} \mathbb{C}\psi_k \oplus \bigoplus_{k \geq n+2, k \equiv m(2)} \mathbb{C}\psi_k = \mathcal{D}_\lambda^- \oplus \mathcal{D}_\lambda^+ \tag{188}$$

Proposition 4.3. *The representations $\mathcal{D}_\lambda^-, \mathcal{D}_\lambda^+$ are irreducible, these are the discrete series representations.*

The operator $\bar{T}(\lambda_R)$ induces a quasi-unitary structure on the $(\mathfrak{g}, \tilde{K}_\infty)$-module $\mathcal{D}_\lambda$. The two sets of K_∞ types occurring in $\mathcal{M}_{\lambda,\mathbb{C}}$ and in $\mathcal{D}_\lambda$ (resp.) are disjoint.

Proof. Remember that as a vector space $\mathcal{D}_\lambda^\vee \otimes \det_{\mathbb{R}}^{2d} = \mathcal{D}_\lambda^\vee$, only the way how $\tilde{K}_\infty$ acts is twisted by $\det_{\mathbb{R}}^{2d}$. Then the form $h_\psi(f_1, f_2) = T_\infty^{\mathrm{loc,reg}}(\lambda_{\mathbb{R}}^{w_0})(f_2)(f_1)$ defines a quasi invariant hermitian form. It is positive definite (for more details see sl2neu.pdf). □

A similar argument works in case B).We restrict the $\mathrm{Gl}_2(\mathbb{C}) \times \mathrm{Gl}_2(\mathbb{C})$ module $\mathcal{M}_{\lambda,\mathbb{C}}$ to $U(2) \times U(2)$ then it becomes the highest weight module $\mathcal{M}_{\lambda_c} = \mathcal{M}_{\lambda_{1,c}} \otimes \mathcal{M}_{\lambda_{2,c}}$. (See4.4.1) Under the action of $U(2) \subset U(2) \times U(2)$ it decomposes into $U(2)$ types according to the Clebsch-Gordan formula

$$\mathcal{M}_{\lambda_c}|_{U(2)} = \bigoplus_{\mu_c = k\gamma_c + \frac{d_1+d_2}{2}\det; k \equiv (n_1 - n_2) \mod 2; n_1+n_2 \geq k \geq |n_1-n_2|} \mathcal{M}_{\mu_c} \tag{189}$$

Hence we get

$$\mathcal{D}_{\lambda_c}|_{U(2)} = \bigoplus_{\mu_c = k\gamma_c + \frac{d_1+d_2}{2}\det; k \equiv (n_1 - n_2) \mod 2; k \geq n_1+n_2+2} \mathcal{M}_{\mu_c} \tag{190}$$

Again we have

Proposition 4.4. *The operator $T_\infty^{loc,\mathrm{reg}}(\lambda_{\mathbb{R}}^{w_0})$ induces an isomorphism*

$$\bar{T}(\lambda_R) : \mathcal{D}_\lambda \xrightarrow{\sim} \mathcal{D}_\lambda^\vee \otimes \det\nolimits_{\mathbb{R}}^{2d}$$

The $(\mathfrak{g}, K_\infty)$ modules are irreducible.

The operator $T_\infty^{loc,\mathrm{reg}}(\lambda_{\mathbb{R}}^{w_0})$ induces the structure of a quasi-unitary module on $\mathcal{D}_\lambda$ if and only if $n_1 = n_2$. This is the only case when we have a quasi-unitary structure on $\mathcal{D}_\lambda$. The two sets of K_∞ types occurring in $\mathcal{M}_{\lambda,\mathbb{C}}$ and in $\mathcal{D}_\lambda$ (resp.) are disjoint.

The Weyl W group acts on T by conjugation, hence on $X^*(T \times \mathbb{C})$ and we define the twisted action by

$$s \cdot \lambda = s(\lambda + \rho) - \rho \tag{191}$$

Given a dominant λ we may consider the four characters $w \cdot \lambda, w \in W(\mathbb{C}) = W$ and the resulting induced modules $\mathfrak{I}_B^G w \cdot \lambda_{\mathbb{R}}$. We observe (notation from (4.4.1))

$$s_1 \cdot (n_1\gamma + d_1 \det + n_2\bar{\gamma} + d_2\overline{\det}) = (-n_1 - 2)\gamma + d_1 \det + n_2\bar{\gamma} + d_2\overline{\det})$$

$$s_2 \cdot (n_1\gamma + d_1 \det + n_2\bar{\gamma} + d_2\overline{\det}) = n_1\gamma + d_1 \det + (-n_2 - 2)\bar{\gamma} + d_2\overline{\det}) \tag{192}$$

Looking closely we see that that the K_∞ types occurring in $\mathfrak{I}_B^G s_1 \cdot \lambda$ or $\mathfrak{I}_B^G s_2 \cdot \lambda$ are exactly those which occur in $\mathcal{D}_\lambda$. This has a simple explanation, we have

Proposition 4.5. *For a dominant character λ we have isomorphisms between the $(\mathfrak{g}, K_\infty)$ modules*

$$\mathcal{D}_\lambda \xrightarrow{\sim} \mathfrak{I}_B^G s_1 \cdot \lambda, \ \mathcal{D}_\lambda \xrightarrow{\sim} \mathfrak{I}_B^G s_2 \cdot \lambda. \tag{193}$$

The resulting isomorphism $\mathfrak{I}_B^G s_1 \cdot \lambda \xrightarrow{\sim} \mathfrak{I}_B^G s_2 \cdot \lambda$ is of course given by $T_\infty^{\mathrm{loc}}(s_1 \cdot \lambda)$.

Interlude: Here we see a fundamental difference between the two cases A) and B). In the second case the infinite dimensional subquotients of the induced representations are again induced representations. In the case A) this is not so, the representations $\mathcal{D}_\lambda^\pm$ are not isomorphic to representations induced from the Borel subgroup.

These representation are called discrete series representations and we want to explain briefly why.

Let G be a semi simple Lie group for example our $G = G(\mathbb{R})$, here we allow both cases. Then we have an action of $G \times G$ on $L^2(G)$ by left and right translations. Then Harish-Chandra has investigated the question how this "decomposes" into irreducible submodules. Let $\hat{G}$ be the set of isomorphism classes of irreducible unitary representations of G.

Then Harish-Chandra shows that there exist a positive measure μ on $\hat{G}$ and a measurable family H_ξ of irreducible unitary representations of G such that

$$L^2(G) = \int_{\hat{G}} H_\xi \otimes \overline{H_\xi} \mu(d\xi) \tag{194}$$

(If instead of a semi simple Lie group we take a finite group G then this is the fundamental theorem of Frobenius that the group ring $\mathbb{C}[G] = \oplus_\theta V_\theta \otimes V_\theta^\vee$ where V_θ are the irreducible representations.)

If we are in the case A) then the sets consisting of just one point $\{\mathcal{D}_\lambda^\pm\}$ have strictly positive measure, i.e. $\mu(\{\mathcal{D}_\lambda^\pm\}) > 0$. This means that the irreducible unitary $G \times G$ modules $\mathcal{D}_\lambda^\pm \otimes \mathcal{D}_{\lambda^\vee}^\pm$ occur as direct summand (i.e. discretely in $L^2(G)$.).

Such irreducible direct summands do not exist in the case B), in this case for any $\xi \in \hat{G}$ we have $\mu(\{\xi\}) = 0$.

We return to the sequences (184),(186). We claim that both sequences do do not split as sequences of $(\mathfrak{g}, K_\infty)$-modules. Of course it follows from the above proposition that these sequences split canonically as sequence of K_∞ modules. But then it follows easily that complementary summand is not invariant under the action of $\mathfrak{g}$. This means that the sequences provide non trivial classes in $\mathrm{Ext}^1_{(\mathfrak{g},K_\infty)}(\mathcal{D}_\lambda, \mathcal{M}_{\lambda,\mathbb{C}})$ and hence these $\mathrm{Ext}^\bullet$ modules are interesting.

The general principles of homological algebra teach us that we can understand these extension groups in terms of relative Lie-algebra cohomology. Let $\mathfrak{k}$ resp. $\tilde{\mathfrak{k}}$ be the Lie-algebras of K_∞ resp. $\tilde{K}_\infty$ the group $\tilde{K}_\infty$ acts on $\mathfrak{g}, \tilde{\mathfrak{k}}$ via the adjoint action (see 1.3) We start from a $(\mathfrak{g}, \tilde{K}_\infty)$ module $\mathfrak{I}_B^G\chi$ and a module $\mathcal{M}_{\lambda,\mathbb{C}}$.

Our goal is to compute the cohomology of the complex (See Chap.III, 4.1.4)

$$\mathrm{Hom}_{\tilde{K}_\infty}(\Lambda^\bullet(\mathfrak{g}/\tilde{\mathfrak{k}}), \mathfrak{I}_B^G\chi \otimes \mathcal{M}_{\lambda,\mathbb{C}}). \tag{195}$$

There is an obvious condition for the complex to be non zero. The group $Z(\mathbb{R}) \subset \tilde{K}_\infty$ acts trivially on $\mathfrak{g}/\mathfrak{k}$ and hence we see that the complex is trivial unless we have

$$\omega_\chi^{-1} = \lambda_\mathbb{R}|_{Z(\mathbb{R})}$$

we assume that this relation holds.

We will derive a formula for these cohomology modules which is a special case of a formula of Delorme, which will also be discussed in Chap. III. An element $\omega \in \mathrm{Hom}_{\tilde{K}_\infty}(\Lambda^n(\mathfrak{g}/\tilde{\mathfrak{k}}), \mathfrak{I}_B^G\chi \otimes \mathcal{M}_{\lambda,\mathbb{C}})$ attaches to any n tuple $v_1, \dots, v_n$ of elements in $\mathfrak{g}/\tilde{\mathfrak{k}}$ an element

$$\omega(v_1, \dots, v_n) \in \mathfrak{I}_B^G\chi \otimes \mathcal{M}_{\lambda,\mathbb{C}} \tag{196}$$

such that $\omega(\mathrm{Ad}(k)v_1, \dots, \mathrm{Ad}(k)v_n) = k\omega(v_1, \dots, v_n)$ for all $k \in \tilde{K}_\infty$.

By construction

$$\omega(v_1,\dots,v_n)=\sum f_\nu\otimes m_\nu \text{ where } f_\nu\in\mathfrak{I}_B^G\chi, m_\nu\in\mathcal{M}_{\lambda,\mathbb{C}}$$

and f_ν is a function in $\mathcal{C}_\infty$ which is determined by its restriction to $\tilde{K}_\infty$ (and this restriction is $\tilde{K}_\infty$ finite). We can evaluate this function at the identity $e_G\in G(\mathbb{R})$ and then

$$\omega(v_1,\dots,v_n)(e_G)=\sum f_\nu(e)\otimes m_\nu\in\mathbb{C}\chi\otimes\mathcal{M}_{\lambda,\mathbb{C}}$$

The $\tilde{K}_\infty$ invariance (196) implies that ω is determined by this evaluation at e_G. Let $\tilde{T}_c=T(\mathbb{R})\cap\tilde{K}_\infty=Z(\mathbb{R})\cdot T_c$. Then it is clear that

$$\omega^*:\{v_1,\dots,v_n\}\mapsto\omega(v_1,\dots,v_n)(e_G) \tag{197}$$

is an element in

$$\omega^*\in\ \mathrm{Hom}_{\tilde{T}_c}(\Lambda^n(\mathfrak{g}/\tilde{\mathfrak{k}}),\mathbb{C}\chi\otimes\mathcal{M}_{\lambda,\mathbb{C}}) \tag{198}$$

and we have: The map $\omega\mapsto\omega^*$ is an isomorphism of complexes

$$\mathrm{Hom}_{\tilde{K}_\infty}(\Lambda^\bullet(\mathfrak{g}/\tilde{\mathfrak{k}}),\mathfrak{I}_B^G\chi\otimes\mathcal{M}_{\lambda,\mathbb{C}})\xrightarrow{\sim}\mathrm{Hom}_{\tilde{T}_c}(\Lambda^\bullet(\mathfrak{g}/\tilde{\mathfrak{k}}),\mathbb{C}\chi\otimes\mathcal{M}_{\lambda,\mathbb{C}}) \tag{199}$$

The Lie algebra $\mathfrak{g}$ can be written as a sum of $\tilde{T}_c$ invariant submodules

$$\mathfrak{g}=\mathfrak{b}+\tilde{\mathfrak{k}}=\mathfrak{t}+\mathfrak{u}+\tilde{\mathfrak{k}} \tag{200}$$

in case B) this sum is not direct, we have $\mathfrak{b}\cap\tilde{\mathfrak{k}}=\mathfrak{t}\cap\tilde{\mathfrak{k}}=\tilde{\mathfrak{t}}_c$ and hence we get the direct sum decomposition into $\tilde{T}_c$-invariant subspaces

$$\mathfrak{g}/\tilde{\mathfrak{k}}=\mathfrak{t}/\tilde{\mathfrak{t}}_c\oplus\mathfrak{u}. \tag{201}$$

We get an isomorphism of complexes

$$\mathrm{Hom}_{\tilde{T}_c}(\Lambda^\bullet(\mathfrak{g}/\tilde{\mathfrak{k}}),\mathbb{C}\chi\otimes\mathcal{M}_{\lambda,\mathbb{C}})\xrightarrow{\sim}\mathrm{Hom}_{\tilde{T}_c}(\Lambda^\bullet(\mathfrak{t}/\tilde{\mathfrak{k}}),\mathbb{C}\chi\otimes\ \mathrm{Hom}(\Lambda^\bullet(\mathfrak{u}),\mathcal{M}_{\lambda,\mathbb{C}})) \tag{202}$$

the complex on the left is isomorphic to the total complex of the double complex on the right.

Intermission, the theorem of Kostant The next step is the computation of the cohomology of the complex $\mathrm{Hom}(\Lambda^\bullet(\mathfrak{u}),\mathcal{M}_{\lambda,\mathbb{C}})$. To do this we work over the rationals and we we study the case A) first. So our group is now $G/\mathbb{Q}=\mathrm{Gl}_2/\mathbb{Q}$ and our module $\mathcal{M}_{\lambda,\mathbb{C}}$ will be replaced by $\mathcal{M}_{\lambda,\mathbb{Q}}$. Then $\mathfrak{u}=\mathbb{Q}E_+$ where $E_+=\begin{pmatrix}0&1\\0&0\end{pmatrix}$. The module has a decomposition into weight spaces

$$\mathcal{M}_{\lambda,\mathbb{Q}}=\bigoplus_{\nu=1}^{\nu=n-\nu}\mathbb{Q}X^{n-\nu}Y^\nu=\bigoplus_{\mu=-n,\mu\equiv n(2)}^{\mu=n}\mathbb{Q}e_\mu. \tag{203}$$

The torus $T^{(1)}=\{\begin{pmatrix}t&0\\0&t^{-1}\end{pmatrix}\}$ acts on $e_\mu=X^{n-\nu}Y^\nu$ by

$$\rho_\lambda(\begin{pmatrix}t&0\\0&t^{-1}\end{pmatrix})e_\mu=t^\mu e_\mu \tag{204}$$

We also have the action of the Lie algebra on $\mathcal{M}_{\lambda,\mathbb{Q}}$ (See section **??**) and by definition we get

$$d(\rho_\lambda)(E_+)e_\mu = E_+ e_\mu = \frac{n-\mu}{2} e_{\mu+2} \tag{205}$$

Now we can write down our complex $\mathrm{Hom}(\Lambda^\bullet(\mathfrak{u}), \mathcal{M}_{\lambda,\mathbb{C}})$ very explicitly. Let $E_+^\vee \in \mathrm{Hom}(\mathfrak{u}, \mathbb{Q})$ be the element $E_+^\vee(E_+) = 1$ then the complex becomes

$$0 \to \bigoplus_{\mu=-n,\mu\equiv n(2)}^{\mu=n} \mathbb{Q}e_\mu \xrightarrow{d} \bigoplus_{\mu=-n,\mu\equiv n(2)}^{\mu=n} \mathbb{Q}E_+^\vee \otimes e_\mu \to 0 \tag{206}$$

where $d(e_\mu) = \frac{n-\mu}{2} E_+^\vee \otimes e_{\mu+2}$. This gives us a decomposition of our complex into two sub complexes

$$\mathrm{Hom}(\Lambda^\bullet(\mathfrak{u}), \mathcal{M}_{\lambda,\mathbb{C}}) = \mathbb{H}^\bullet(\mathfrak{u}, \mathcal{M}_{\lambda,\mathbb{Q}}) \oplus AC^\bullet \tag{207}$$

where $AC^\bullet$ as acyclic (it has no cohomology) and in

$$\mathbb{H}^\bullet(\mathfrak{u}, \mathcal{M}_{\lambda,\mathbb{Q}}) = \{0 \to \mathbb{Q}\ e_n \xrightarrow{d} \mathbb{Q}\ E_+^\vee \otimes e_{-n} \to 0\} \tag{208}$$

the differential is zero. Hence we get

$$H^\bullet(\mathfrak{u}, \mathcal{M}_{\lambda,\mathbb{Q}}) = H^\bullet(\ \mathrm{Hom}(\Lambda^\bullet(\mathfrak{u}), \mathcal{M}_{\lambda,\mathbb{Q}})) = \mathbb{H}^\bullet(\mathfrak{u}, \mathcal{M}_{\lambda,\mathbb{Q}}) \tag{209}$$

We notice that the torus T acts on $H^\bullet(\mathfrak{u}, \mathcal{M}_{\lambda,\mathbb{Q}})$ (The Borel subgroup B acts on the complex $\mathrm{Hom}(\Lambda^\bullet(\mathfrak{u}), \mathcal{M}_{\lambda,\mathbb{Q}})$ but since the Lie algebra cohomology is the derived functor of taking invariants under U (elements annihilated by $\mathfrak{u}$) it follows that this action is trivial on U).

Hence we see that T acts by the character λ on $\mathbb{Q}\ e_n = H^0(\mathfrak{u}, \mathcal{M}_{\lambda,\mathbb{Q}})$ and by the character $\lambda^- - \alpha = w_0 \cdot \lambda = \lambda^{w_0} - 2\rho$ on $\mathbb{Q}\ E_+^\vee \otimes e_{-n} = H^1(\mathfrak{u}, \mathcal{M}_{\lambda,\mathbb{Q}})$.

Here we see the simplest example of the famous theorem of Kostant which will be discussed in Chap. III 6.1.3.

We discuss the case B). Again we want that our group $G/\mathbb{R} = R_{\mathbb{C}/\mathbb{R}}(\mathrm{Gl}_2/\mathbb{C})$ is a base change from a group $G/\mathbb{Q}$ denoted by the same letter. We need an imaginary quadratic extension $F/\mathbb{Q}$ and put $G/\mathbb{Q}R_{F/\mathbb{Q}}(\mathrm{Gl}_2/F)$. We choose a dominant weight $\lambda = \lambda_1 + \lambda_2 = n_1\gamma_1 + d_1 \det_1 + n_2\gamma_2 + d_2 \det_2$ and then $\mathcal{M}_{\lambda,F} = \mathcal{M}_{\lambda_1,F} \otimes \mathcal{M}_{\lambda_2,F}$ is an irreducible representation of $G \times_{\mathbb{Q}} F = \mathrm{Gl}_2 \times \mathrm{Gl}_2/F$. Now we have $\mathfrak{u} \otimes F = FE_+^1 \oplus FE_+^2$. Then basically the same computation yields:

The cohomology $H^\bullet(\mathfrak{u}, \mathcal{M}_{\lambda,F})$ is equal the complex

$$\begin{aligned}\mathbb{H}^\bullet(\mathfrak{u}, \mathcal{M}_{\lambda,F}) = \{0 \to Fe_{n_1}^{(1)} \otimes Fe_{n_2}^{(2)} \xrightarrow{d} FE_+^{1,\vee} \otimes e_{-n_1}^{(1)} \otimes e_{n_2}^{(2)} \oplus FE_+^{1,\vee} \otimes e_{n_1}^{(1)} \otimes E_+^{2,\vee} \otimes e_{-n_2}^{(2)} \\ \xrightarrow{d} FE_+^{1,\vee} \otimes e_{-n_1}^{(1)} \otimes E_+^{2,\vee} \otimes e_{-n_2}^{(2)} \to 0\}\end{aligned} \tag{210}$$

where all the differentials are zero. The torus T acts by the weights

$$\lambda \text{ in degree } 0,\ s_1 \cdot \lambda, s_2 \cdot \lambda \text{ in degree } 1,\ w_0 \cdot \lambda \text{ in degree } 2 \tag{211}$$

We go back to (212) and get a homomorphism of complexes

$$\mathrm{Hom}_{\tilde{T}_c}(\Lambda^\bullet(\mathfrak{g}/\tilde{\mathfrak{k}}), \mathbb{C}\chi \otimes \mathcal{M}_{\lambda,\mathbb{C}}) \to\ \mathrm{Hom}_{\tilde{T}_c}(\Lambda^\bullet(\mathfrak{t}/\tilde{\mathfrak{k}}), \mathbb{C}\chi \otimes \mathbb{H}^\bullet(\mathfrak{u}, \mathcal{M}_{\lambda,\mathbb{C}})) \tag{212}$$

which induces an isomorphism in cohomology so that finally

$$H^\bullet(\mathfrak{g}, K_\infty, \mathfrak{I}_B^G\chi \otimes \mathcal{M}_{\lambda,\mathbb{C}}) \xrightarrow{\sim} H^\bullet(\ \mathrm{Hom}_{\tilde{T}_c}(\Lambda^\bullet(\mathfrak{t}/\tilde{\mathfrak{k}}), \mathbb{C}\chi \otimes H^\bullet(\mathfrak{u}, \mathcal{M}_{\lambda,\mathbb{C}})) \tag{213}$$

and combining this with the results above we get

Theorem 4.4. *If we can find a $w \in W(\mathbb{C})$ such that $\chi^{-1} = w \cdot \lambda_\mathbb{R}$ then*

$$H^\bullet(\mathfrak{g}, K_\infty, \mathfrak{I}_B^G\chi \otimes \mathcal{M}_{\lambda,\mathbb{C}}) \xrightarrow{\sim} H^{l(w)}(\mathfrak{u}, \mathcal{M}_{\lambda,\mathbb{C}})(w \cdot \lambda) \otimes \Lambda^\bullet(\mathfrak{t}/\tilde{\mathfrak{k}})^\vee$$

If there is no such w then the cohomology is zero.

Proof. □

4.4.6 The cohomology of the $\mathcal{D}_\lambda$ the cohomology of unitary modules

4.4.7 The Eichler-Shimura Isomorphism

4.5 Modular symbols, $L-$ values and denominators of Eisenstein classes.

4.5.1 Modular symbols attached to maximal tori in Gl_2.

Compact symbols attached to anisotropic tori, relative symbols attached to split tori....

4.5.2 Evaluation of cuspidal classes on modular symbols

We discuss the results on special values of L-functions attached eigenforms

4.5.3 Evaluation of Eisenstein classes on modular symbols and the determination of the denominator (in certain cases)

Here we (hopefully) prove (141); we discuss the two cases cases $\mathrm{Sl}_2(\mathbb{Z})$ and some special congruence subgroups $\Gamma \subset \mathrm{Sl}_2[\mathbb{Z}[i]]$.

4.5.4 The Deligne-Eichler-Shimura theorem

In this section the material is not presented in a satisfactory form. One reason is that it this point we should start using the language of adeles, but there also other drawbacks. So in a final version of these notes this section probably be removed.

Begin of probably removed section

In this section I try to explain very briefly some results which are specific for Gl_2 and a few other low dimensional algebraic groups. These results concern representations of the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ which can be attached to irreducible constituents Π_f in the cohomology. These results are very deep and reaching a better understanding and more general versions of these results is a fundamental task of the subject treated in these notes. The first cases have been tackled by Eichler and Shimura, then Ihara made some contributions and finally Deligne proved a general result for $\mathrm{Gl}_2/\mathbb{Q}$.

We start from the group $G = \mathrm{Gl}_2/\mathbb{Q}$, this is now only a reductive group and its centre is isomorphic to $\mathbb{G}_m/\mathbb{Q}$. Its group of real points is $\mathrm{Gl}_2(\mathbb{R})$ and the centre $\mathbb{G}_m(\mathbb{R})$ considered as a topological group has two components, the connected component of the identity is $\mathbb{G}_m(\mathbb{R})^{(0)} = \mathbb{R}^\times_{>0}$. Now we enlarge the maximal compact connected subgroup $SO(2) \subset \mathrm{Gl}_2(\mathbb{R})$ to the group $K_\infty = SO(2) \cdot \mathbb{G}_m(\mathbb{R})^{(0)}$. The resulting symmetric space $X = \mathrm{Gl}_2(\mathbb{R})/K_\infty$ is now a union of a upper and a lower half plane: We write $X = \mathbb{H}_+ \cup H_-$.

We choose a positive integer $N > 2$ and consider the congruence subgroup $\Gamma(N) \subset \mathrm{Gl}_2(\mathbb{Q}))$. We modify our symmetric space: This modification may look a little bit artificial at this point, it will be justified in the next chapter and is in fact very natural. (At this point I want to avoid to use the language of adeles.)

We replace the symmetric space by

$$X = (\mathbb{H}_+ \cup \mathbb{H}_-) \times \mathrm{Gl}_2(\mathbb{Z}/N\mathbb{Z}).$$

On this space we have an action of $\Gamma = \mathrm{Gl}_2(\mathbb{Z})$, on the second factor it acts via the homomorphism $\mathrm{Gl}_2(\mathbb{Z}) \to \mathrm{Gl}_2(\mathbb{Z}/N\mathbb{Z})$ by translations from the left. Again we look at the quotient of this space by the action of $\mathrm{Gl}_2(\mathbb{Z})$. This quotient space will have several connected components. The group $\mathrm{Gl}_2(\mathbb{Z})$ contains the group $\mathrm{Sl}_2(\mathbb{Z})$ as a subgroup of index two, because the determinant of an element is ± 1. The element $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ interchanges the upper and the lower half plane and hence we see

$$\mathrm{Gl}_2(\mathbb{Z})\backslash X = \mathrm{Gl}_2(\mathbb{Z})\backslash((\mathbb{H}_+ \cup \mathbb{H}_-) \times \mathrm{Gl}_2(\mathbb{Z}/N\mathbb{Z})) = \mathrm{Sl}_2(\mathbb{Z})\backslash(\mathbb{H}_+ \times \mathrm{Gl}_2(\mathbb{Z}/N\mathbb{Z})),$$

the connected components of $(\mathbb{H}_+ \times \mathrm{Gl}_2(\mathbb{Z}/N\mathbb{Z}))$ are indexed by elements $g \in \mathrm{Gl}_2(\mathbb{Z}/N\mathbb{Z})$. The stabilizer of such a component is the full congruence subgroup

$$\Gamma(N) = \{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a, d \equiv 1 \mod N, b, c \equiv 0 \mod N\}$$

this group is torsion free because we assumed $N > 2$.

The image of the natural homomorphism $\mathrm{Sl}_2(\mathbb{Z}) \to \mathrm{Gl}_2(\mathbb{Z}/N\mathbb{Z})$ is the subgroup $\mathrm{Sl}_2(\mathbb{Z}/N\mathbb{Z})$ (strong approximation), therefore the quotient is by this subgroup is $(\mathbb{Z}/N\mathbb{Z})^\times$.

We choose as system of representatives for the determinant the matrices $t_a = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, a \in (\mathbb{Z}/N\mathbb{Z})^\times$. The stabilizer of then we get an isomorphism

$$S_N = \mathrm{Gl}_2(\mathbb{Z})\backslash(\mathbb{H} \times \mathrm{Gl}_2(\mathbb{Z}/N\mathbb{Z})) \xrightarrow{\sim} (\Gamma(N)\backslash\mathbb{H}) \times (\mathbb{Z}/N\mathbb{Z})^\times.$$

To any prime p, which does not divide N we can again attach Hecke operators. Again we can attach Hecke operators

$$T_{p^r} = T\left(\begin{pmatrix} p^r & 0 \\ 0 & 1 \end{pmatrix}, u_{\begin{pmatrix} p^r & 0 \\ 0 & 1 \end{pmatrix}}\right)$$

to the double cosets and using strong approximation we can prove the recursion formulae.

We consider the cohomology groups $H^\bullet_c(S_N, \tilde{\mathcal{M}}_n), H^\bullet(S_N, \tilde{\mathcal{M}}_n)$ and define $H^\bullet_!(S_N, \tilde{\mathcal{M}}_n)$ as before. This is a semi simple module for the cohomology.

The theorem 3 extends to this situation without change. We have a small addendum: If denote by $Z^{(N,\times)} \in \mathbb{Q}^\times$ the subgroup of those numbers which are units at the primes dividing N. We have the homomorphism $r: \ Z^{(N,\times)} \to (\mathbb{Z}/N\mathbb{Z})^\times$

On each absolutely irreducible component Π_f the Hecke operators $T(z,u_z)$ act by a scalar $\omega(z) \in \mathcal{O}_L$ and the map $z \mapsto \omega(z)$ factors over r and induces a character $\omega(\Pi_f) : (\mathbb{Z}/N\mathbb{Z})^\times \to (\mathcal{O}_L)^\times$. This character is called the central character of Π_f.

The following things will be explained in greater detail in the class

Now we exploit the fact, that the Riemann surface $\Gamma(N)\backslash X$ is in fact the space of complex points of the moduli scheme $M_N \to$ Spec$(\mathbb{Z}[1/N])$. On this moduli scheme we have the universal elliptic curve with N level structure

$$\begin{array}{c} \mathcal{E} \\ \downarrow \pi \\ M_N \end{array}$$

On $\mathcal{E}$ we have the constant ℓ-adic sheaf $\mathbb{Z}_\ell$. For $i = 0,1,2$ we can consider the ℓ- adic sheaves $R^i\pi_*(\mathbb{Z}_\ell)$ on M_N. We have the spectral sequence

$$H^p(M_N \times \bar{\mathbb{Q}}, R^q\pi_*(\mathbb{Z}_\ell)) \Rightarrow H^n(\mathcal{E} \times \bar{\mathbb{Q}}, \mathbb{Z}_\ell).$$

We can take the fibered product of the universal elliptic curve

$$\mathcal{E}^{(n)} = \mathcal{E} \times_{M_N} \mathcal{E} \times \cdots \times_{M_N} \mathcal{E} \xrightarrow{\pi_N} M_N$$

where n is the number of factors. This gives us a more general spectral sequence

$$H^p(M_N \times \bar{\mathbb{Q}}, R^q\pi_{N,*}(\mathbb{Z}_\ell)) \Rightarrow H^n(\mathcal{E}^{(n)} \times \bar{\mathbb{Q}}, \mathbb{Z}_\ell).$$

The stalk $R^q\pi_{N,*}(\mathbb{Z}_\ell)_y$) of the sheaf $R^q\pi_{N,*}(\mathbb{Z}_\ell)$ in a geometric point y of M_N is the q-th cohomology $H^q(\mathcal{E}_y^{(n)}, \mathbb{Z}_\ell)$ and this can be computed using the Kuenneth formula

$$H^q(\mathcal{E}_y^{(n)}, \mathbb{Z}_\ell) \xrightarrow{\sim} \bigoplus_{a_1,a_2\dots,a_n} H^{a_1}(\mathcal{E}_y, \mathbb{Z}_\ell) \otimes H^{a_2}(\mathcal{E}_y, \mathbb{Z}_\ell) \cdots \otimes H^{a_n}(\mathcal{E}_y, \mathbb{Z}_\ell),$$

where the $a_i = 0,1,2$ and sum up to q. We have $H^0(\mathcal{E}_y, \mathbb{Z}_\ell) = \mathbb{Z}_\ell(0), H^2(\mathcal{E}_y, \mathbb{Z}_\ell) = \mathbb{Z}_\ell(-1)$ and the most interesting factor is $H^1(\mathcal{E}_y, \mathbb{Z}_\ell)$ which is a free $\mathbb{Z}_\ell$ module af rank 2.

This tells us that the sheaf decomposes into a direct sum according to the type of Kuenneth summands. We also have an action of the symmetric group S_q which is obtained from the permutations of the factors in $\mathcal{E}^{(n)}$ which also permutes the types. We are mainly interested in the case $q = n$ and then we have the special summand where $a_1 = a_2 \cdots = a_n = 1$. This summand is invariant under S_n and contains a summand on which S_n acts by the signature character $\sigma : S_n \to \{\pm 1\}$. This defines a unique subsheaf $R^n\pi_{*,n}(\mathbb{Z}_\ell)(\sigma) \subset R^n\pi_{*,n}(\mathbb{Z}_\ell)$ and hence we get an inclusion

$$H^1(M_N \times \bar{\mathbb{Q}}, R^n\pi_{*,n}(\mathbb{Z}_\ell)(\sigma) \hookrightarrow H^{n+1}(\mathcal{E}^{(n)} \times \bar{\mathbb{Q}}, \mathbb{Z}_\ell)$$

and we can do the same thing for the cohomology with compact supports.

Now I will explain:

A) If we extend the scalars from $\mathbb{Q}$ to $\mathbb{C}$ then then extension of $R^n\pi_{*,n}(\mathbb{Q}_\ell)(\sigma)$ is isomorphic to the restriction of $\mathcal{M}_n \otimes \mathbb{Q}_\ell$ to the etale topology.

B) The Hecke operators T_p for $p \not| N$ are coming from algebraic correspondences $T_p \subset M_N \times M_N$ and induce endomorphisms $T_p : H^1(M_N \otimes \bar{\mathbb{Q}}, R^n\pi_{*,n}(\mathbb{Z}_\ell)(\sigma)) \to H^1(M_N \otimes \bar{\mathbb{Q}}, R^n\pi_{*,n}(\mathbb{Z}_\ell)(\sigma))$ which commute with the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on the cohomology.

C) This tells us that after extension of the scalars of the coefficient system we get

$$H^1(M_N(\mathbb{C}), \mathcal{M}_n \otimes \mathbb{Q}_\ell) \xrightarrow{\sim} H^1(M_N \times \bar{\mathbb{Q}}, R^n\pi_{*,n}(\mathbb{Q}_\ell)(\sigma))$$

and this gives us the structure of a $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \times \mathcal{H}_\Gamma$ on $H^1(M_N(\mathbb{C}), \mathcal{M}_n \otimes \mathbb{Q}_\ell)$.

D) The operation of the Galois group on $H^1(M_N(\mathbb{C}), \mathcal{M}_n \otimes \mathbb{Q}_\ell)$ is unramified outside N, therefore we have the conjugacy class Φ_p^{-1} for all $p \not| N$ as endomorphism of $H^1(M_N(\mathbb{C}), \mathcal{M}_n \otimes \mathbb{Q}_\ell)$.

Now we use another fact, which will be explained in Chapter III. We also can define a Hecke algebra $\mathcal{H}_p$ for the primes $p|N$, and hence we get an action of a larger Hecke algebra

$$\mathcal{H}_N^{\text{large}} = \bigotimes_p{}' \mathcal{H}_p$$

and this algebra commutes with the action of the Galois group.

We now apply our theorem 2 to the cohomology $H^1_!(M_N(\mathbb{C}), \mathcal{M}_n \otimes \mathbb{Q}_\ell)$, as a module under this large Hecke algebra. Then the isotypical summands will be invariant under the Galois group.

Theorem 4: *a) The multiplicity of an irreducible representation $\Pi_f \in$ Coh$(M_N(\mathbb{C}), \tilde{\mathcal{M}}_{n,L_\mathfrak{l}})$ is two.*

b) This gives a product decomposition

$$H^1_!(M_N(\mathbb{C}), \mathcal{M}_n \otimes L_\mathfrak{l}) \xrightarrow{\sim} H_{\Pi_f} \otimes W(\Pi_f),,$$

where H_{Π_f} is irreducible of type Π_f and where $W(\Pi_f)$ is a two dimensional Gal$(\bar{\mathbb{Q}}/\mathbb{Q})$ module.

The module $W(\Pi_f)$ is unramified outside N and

$$tr(\Phi_p^{-1}|W(\Pi_f)) = \lambda(\pi_p), \det(\Phi_p^{-1}|W(\Pi_f)) = p^{n+1}\omega(\Pi_f)(p)$$

This theorem is much deeper than the previous ones. The assertion a) follows from the theory of automorphic forms on Gl_2 and b) requires some tools from algebraic geometry. We have to consider the reduction $M_N \times \mathrm{Spec}(\mathbb{F}_p)$ and to look at the reduction of the Hecke operator T_p modulo p. I will resume this discussion in Chap. V.

I want to discuss some applications.

A) To any isotypical component Π_f we can attach an (so called automorphic) L function

$$L(\Pi_f, s) = \prod_p L(\pi_p, s)$$

where for $p \not| N$ we define

$$L(\pi_p, s) = \frac{1}{1 - \lambda(\pi_p)p^{-s} + p^{n+1}\omega(\Pi_f)(p)p^{-2s}}$$

and for $p|N$ we have

$$L(\pi_p,s)=\begin{cases}\frac{1}{1-p^{n+1}\omega(\Pi_f)(p)p^{-s}} & \text{if } \pi_p \text{ is a Steinberg module}\\ 1 & \text{else}\end{cases}$$

This L-function, which is defined as an infinite product is holomorphic for $\Re(s) >> 0$ it can written as the Mellin transform of a holomorphic cusp form F of weight $n+2$ and this implies that

$$\Lambda(\Pi,s)=\frac{\Gamma(s)}{2\pi^s}L(\Pi_f,s)$$

has a holomorphic continuation into the entire complex plane and satisfies a funtional equation

$$\Lambda(\Pi_f,s)=W(\Pi_f)(N(\Pi_f))^{s-1-n/2}\Lambda(\Pi_f,n+2-s)$$

Here $W(\Pi_f)$ is the so called root number, it can be computed from the π_p where $p|N$, its value is ± 1, the number $N(\Pi_f)$ is the conductor of Π_f it is a positive integer, whose prime factors are contained in the set of prime divisors of N.

B) But we also can interpret an isotypic component as a submotive in $H^{n+1}(\mathcal{E}^{(n)}\times\bar{\mathbb{Q}},\mathbb{Z})$, this is the so called Scholl motive.

If we apply the results of Deligne in Weil II, which have been proved in the winter term 2003/4, we get the estimate

$$|\iota(\lambda(\pi_p))|\leq 2p^{(n+1)/2}$$

for any embedding ι of L into $\mathbb{C}$.

End of probably removed section

2.2.5 The ℓ-adic Galois representation in the easiest non trivial case

Again we consider the module $\mathcal{M}=\mathcal{M}_{10}[-10]$. We choose a prime ℓ and for some reason let us assume $\ell>7$. Then we can consider the cohomology groups

$$H^1(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}/\ell^n\tilde{\mathcal{M}})$$

and the projective limit

$$H^1(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}\otimes\mathbb{Z}_\ell)=\lim_{\leftarrow}H^1(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}/\ell^n\tilde{\mathcal{M}}).$$

Now it is known that the quotient space is the "moduli space" of elliptic curves, this is an imprecise and even incorrect statement, but it contains a lot of truth. What is true is that we can define the moduli stack $S/\operatorname{Spec}(\mathbb{Z})$ of elliptic curves, this is a smooth stack and it has the universal elliptic curve $\mathcal{E}\xrightarrow{\pi}S$ over it.

We can define etale torsion sheaves $(\mathcal{M}/\ell^n\tilde{\mathcal{M}})_{et}$ on this stack and we know that

$$H^1_{et}(S\times_{\operatorname{Spec}(\mathbb{Z})}\bar{\mathbb{Q}},(\mathcal{M}/\ell^n\tilde{\mathcal{M}})_{et})\xrightarrow{\sim}H^1(\Gamma\backslash\mathbb{H},\tilde{\mathcal{M}}_{10}/\ell^n\tilde{\mathcal{M}}_{10}).$$

On these etale cohomology groups we have an action of the Galois group. Using correspondences we can define Hecke operators T_p for all $p\neq\ell$, they induce endomorphism on the etale cohomology and they commute with the action of the Galois group.

We denote this action of the Galois group as a representation

$$\rho_n : \ \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gl}(H^1_{et}(S \times_{\mathrm{Spec}(\mathbb{Z})} \bar{\mathbb{Q}}, (\mathcal{M}/\ell^n \tilde{\mathcal{M}})_{et})).$$

This representation is unramified outside ℓ, and this means:

The finite extension $K_\ell^{(n)}/\mathbb{Q}$ for which $\mathrm{Gal}(\bar{\mathbb{Q}}/K_\ell^{(n)})$ is the kernel of ρ_n is unramified outside ℓ.

By transport of structure we have the same projective system of Hecke$\times$Galois modules on the right hand side.

We recall our fundamental exact sequence, the Galois groups acts on the individual terms of this sequence, we get projective systems of Galois-modules and passing to the limit yields

$$\rho_! : \ \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gl}(H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{Z}_\ell))$$

and

$$\rho_\partial : \ \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gl}(\mathbb{Z}_\ell e_{10}).$$

The field $K_\ell = \bigcup_n K_\ell^{(n)}$ defines the kernel $\mathrm{Gal}(\bar{\mathbb{Q}}/K_\ell)$, the extension $K_\ell/\mathbb{Q}$ is unramified at all primes $p \neq \ell$. If $\mathfrak{p}$ is a prime in $\mathcal{O}_{K_\ell}$ which lies above then the geometric Frobenius $\Phi_\mathfrak{p}$ is the unique element in $\mathrm{Gal}(K_\ell/\mathbb{Q})$ which fixes $\mathfrak{p}$ and induces $x \mapsto x^{-p}$ on the residue field $\mathcal{O}_{K_\ell}/\mathfrak{p}$. This element defines a unique conjugacy class Φ_p in $\mathrm{Gal}(K_\ell/\mathbb{Q})$.

Theorem(Deligne)*For any prime $p \neq \ell$ we have*

$$\rho_\partial(\Phi_p) = p^{11} Id$$

and

$$\det(Id - \rho(\Phi_p)t | H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{Z}_\ell)) = 1 - \tau(p)t + p^{11}t^2$$

This is a special case of the general theorem stated in the previous section and it one of the aims of the subject treated in this book to generalize this theorem to larger groups.

We conclude by giving a few applications.

A) The function $z \mapsto \Delta(z)$ is a function on the upper half plane $\mathbb{H} = \{z | \Im(z) > 0\}$ and it satisfies

$$\Delta(\frac{az+b}{cz+d}) = (cz+d)^{12}\Delta(z)$$

and this means that it is a modular form of weight 12. Since it goes to zero if $z = iy \to \infty$ it is even a modular cusp form.

For such a modular cusp form we can define the Hecke L-function

$$L(\Delta, s) = \int_0^\infty \Delta(iy) y^s \frac{dy}{y} = \frac{\Gamma(s)}{(2\pi)^s} \sum_{n=1}^\infty \frac{\tau(n)}{n^s} = \frac{\Gamma(s)}{(2\pi)^s} \prod_p \frac{1}{1 - \tau(p)p^{-s} + p^{11-2s}}$$

the product expansion has been discovered by Ramanujan and has been proved by Mordell and Hecke.

Now it is in any textbook on modular forms that the transformation rule

$$\Delta(-\frac{1}{z}) = z^{12}\Delta(z)$$

implies that $L(\Delta, s)$ defines a holomorphic function in the entire s plane and satisfies the functional equation

$$L(\Delta, s) = (-1)^{12/2}L(\Delta, 12-s) = L(\Delta, 12-s).$$

This function $L(\Delta, s)$ is the prototype of an automorphic L-function. The above theorem shows that it is equal to a "motivic" L-function. We gave some vague explanations of what this possibly means: We can interpret the projective system $(\mathcal{M}/\ell^n\tilde{\mathcal{M}})_{et}$ as the $\ell-$adic realization of a motive:

$$\mathcal{M} = \mathrm{Sym}^{10}(R^1(\pi : \mathcal{E} \to S))$$

(All this is a translation of Deligne's reasoning into a more sophisticated language.)

It is a general hope that "motivic" L-functions $L(M, s)$ have nice properties as functions in the variable s (meromorphicity, control of the poles, functional equation). So far the only cases, in which one could prove such nice properties are cases where one could identify the "motivic" L-function to an automorphic L function. The greatest success of this strategy is Wiles' proof of the Shimura-Taniyama-Weil conjecture, but also the Riemann ζ-function is a motivic $L-$ function and Riemann's proof of the functional equation follows exactly this strategy.

B) But we also have a flow of information in the opposite direction. In 1973 Deligne proved the Weil conjectures which in this case say that the two roots of the quadratic equation

$$x^2 - \tau(p)x + p^{11} = 0$$

have absolute value $p^{11/2}$, i.e. they have the same absolute value. This implies the famous Ramanujan- conjecture

$$\tau(p) \leq 2p^{11/2}$$

and for more than 50 years this has been a brain-teaser for mathematicians working in the field of modular forms.

C) We consider the Galois representation

$$\rho : \ \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gl}(H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{Z}_\ell))$$

and and its sub and quotient representations

$$\rho_! : \ \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gl}(H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{Z}_\ell)), \rho_\partial : \ \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gl}(\mathbb{Z}_\ell e_{10}).$$

The representation ρ_∂ is the $\ell-$ adic realization of the Tate-motive $\mathbb{Z}(-11)$ (For a slightly more precise explanation I refer to MixMot.pdf on my home-page). On $\mathbb{Z}_\ell(-1) = H^2(\mathbb{P}^1 \times \bar{\mathbb{Q}}, \mathbb{Z}_\ell)$ the Galois group acts by the Tate-character

$$\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \ \mathrm{Gal}(\mathbb{Q}(\zeta_{\ell^\infty})/\mathbb{Q}) \xrightarrow{\alpha} \mathbb{Z}_\ell^\times$$

where $\mathbb{Q}(\zeta_{\ell^\infty})$ is the cyclotomic field of all ℓ^n-th roots of unity ($n \to \infty$). We identify $\mathrm{Gal}(\mathbb{Q}(\zeta_{\ell^\infty})/\mathbb{Q}) = \mathbb{Z}_\ell^\times$, the identification is given by the map

$x \mapsto (\zeta \mapsto \zeta^x)$ and then $\alpha(x) = x^{-1}$. Hence the first assertion in Delignes theorem simply says:

$$\rho_\partial = \alpha^{11}.$$

We say a few words concerning

$$\rho_! : \operatorname{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \operatorname{Gl}(H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{Z}_\ell)).$$

It is easy to see that the cup product provides a non degenerate alternating pairing

$$< , >: H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{Z}_\ell) \times H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{Z}_\ell) \to \mathbb{Z}_\ell(-11)$$

and clearly for any $\sigma \in \operatorname{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ we must have

$$< \rho(\sigma)u, \rho(\sigma)v >= \alpha^{11}(\sigma) < u, v > .$$

This means we have $\det(\rho(\sigma)) = \alpha^{11}(\sigma)$ and we can ask what is the image of $\operatorname{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ in $\operatorname{Gl}(H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{Z}_\ell) = \operatorname{Gl}_2(\mathbb{Z}_\ell)$. We ask a seemingly simpler question and we want to understand the image of

$$\rho_{!, \mod \ell} \operatorname{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \operatorname{Gl}(H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{F}_\ell) = \operatorname{Gl}_2(\mathbb{F}_\ell).$$

This question is discussed in the paper " On ℓ-adic representations and congruences for coefficients of modular forms," Springer lecture Notes 350, Modular Functions of one Variable III by H.P.F. Swinnerton-Dyer.

Here we can say that the image of this homomorphism composed with the determinant will be $(\mathbb{F}_\ell^\times)^{11} \subset \mathbb{F}_\ell^\times$. It is shown in the above paper that for $\ell \neq 2, 3, 5, 7, 23, 691$ the image of the Galois group will simply be as large as possible, namely it will be the inverse image of $(\mathbb{F}_\ell^\times)^{11}$.

We can apply the Manin-Drinfeld principle and conclude that after tensorization by $\mathbb{Q}_\ell$ the representation $\rho \otimes \mathbb{Q}_\ell$ splits

$$\rho \otimes \mathbb{Q}_\ell = \rho_1 \otimes \mathbb{Q}_\ell \oplus \mathbb{Q}_\ell e_{10}(-11).$$

In section 2.2.3 we have seen that we have such a splitting also for the integral cohomology, i.e. for the module $H^1(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}} \otimes \mathbb{Z}_\ell)$ provided ℓ is not one of the small primes, which have been inverted and $\ell \neq 691$.

But if $\ell = 691$ then we have seen in 2.2.3 that we have a homomorphism

$$j : \mathbb{Z}/(691)(-11) \hookrightarrow H^1_!(\Gamma\backslash\mathbb{H}, \tilde{\mathcal{M}}_{\mathbb{Z}/(691)}),$$

this is a homomorphism of Galois-modules. This means that the representation of the of the Galois group modulo $\ell = 691$ is of the form

$$\rho_{!, \mod 691} : \operatorname{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

$$\rho_{!, \mod 691}(\sigma) \mapsto \begin{pmatrix} \alpha(\sigma)^{11} & u(\sigma) \\ 0 & 1 \end{pmatrix}$$

The field $K^{(1)}_{691}$ contains the $691-$ th roots of unity and is an unramified extension of degree 691, in a sense this extension is now obtained by an explicit construction.