Secondary Operations in the Cohomology of Harish-Chandra Modules

G. Harder

April 6, 2013

Contents

Contents

1 Introduction

In this note we study a special case of a problem that can be formulated in a much more general context. We consider induced Harish-Chandra modules $\mathrm{Ind}_P^G \mathcal{D}_{\chi_\mu}$ which have non trivial cohomology with coefficients in a finite dimensional highest weight module $\mathcal{M}_\lambda$. This cohomology is the cohomology of a complex $\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathrm{Ind}_P^G \mathcal{D}_{\chi_\mu} \otimes \mathcal{M}_{\lambda,F})$ and it can be computed by a theorem of Delorme. The theory of these induced modules provides intertwining operators $T_\chi : \mathrm{Ind}_P^G \mathcal{D}_{\chi_\mu} \to \mathrm{Ind}_P^G \mathcal{D}_{\chi_{\mu'}}$. These intertwining operators may have a non trivial kernel $\mathbb{D}_\chi$ (sometimes this will be a discrete series representation). Then we may have a non trivial kernel of the linear map

$$H^q(\mathfrak{g}, K_\infty, \mathbb{D}_\chi \otimes \mathcal{M}_\lambda) \to H^q(\mathfrak{g}, K_\infty, \mathrm{Ind}_P^G \mathcal{D}_{\chi_\mu} \otimes \mathcal{M}_\lambda)$$

Under certain conditions we find a natural element $\omega \in \mathrm{Hom}_{K_\infty}(\Lambda^q(\mathfrak{g}/\mathfrak{k}), \mathbb{D}_\chi \otimes \mathcal{M}_\lambda)$ representing a given cohomology class $\xi = [\omega]$ in the kernel. Then we find a form $\psi \in \mathrm{Hom}_{K_\infty}(\Lambda^q(\mathfrak{g}/\mathfrak{k}), \mathrm{Ind}_P^G \mathcal{D}_{\chi_\mu} \otimes \mathcal{M}_\lambda)$ which bounds ω, i.e. $d\psi = \omega$. The intertwining operator induces a homomorphism $T^\bullet$ between the complexes and hence we get $d(T^{q-1}(\psi)) = T^q(\omega) = 0$. Hence we get a closed form and therefore a cohomology class

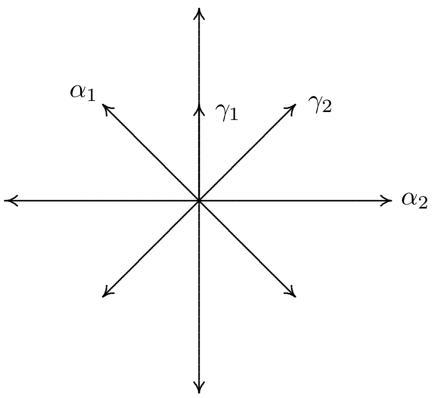
$$\kappa_T(\xi) = [T^{q-1}(\psi)] \in H^{q-1}(\mathfrak{g}, K_\infty, \mathrm{Ind}_P^G \mathcal{D}_{\chi_{\mu'}} \otimes \mathcal{M}_\lambda)$$

The problem is to compute this class.

We will solve this problem in a special case. This will allow us to give an explicit formula for the constant c which appears in the formula on p. 258 in [1-2-3].

2 The example $G = \mathrm{Sp}_2/\mathbb{Z}$

2.1 Some notations and structural data



The maximal torus is

$$T_0/\mathbb{Z} = t = \left\{ \begin{pmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 & t_1^{-1} \end{pmatrix} \right\}$$

the simple roots are

$$\alpha_1(t) = t_1/t_2, \alpha_2(t) = t_2^2$$

and the fundamental dominant weights are

$$\gamma_1(t) = t_1, \gamma_2(t) = t_1 t_2$$

and finally we have

$$2\gamma_1^M = t_1/t_2$$

The torus $T_0/\mathbb{Z}$ contains the subtorus

$$Z_0/\mathbb{Z} = \left(\begin{matrix} z & 0 & 0 & 0 \\ 0 & z & 0 & 0 \\ 0 & 0 & z^{-1} & 0 \\ 0 & 0 & 0 & z^{-1} \end{matrix} \right),$$

the centralizer M of Z_0 is a Levi subgroup of the standard maximal Siegel parabolic subgroup $P/\mathbb{Z}$, let $U_0/\mathbb{Z}$ be its unipotent radical. The roots in this unipotent radical are $\{\alpha_2, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2\}$. The torus T_0 has the subtorus $T_0^{(1)}$ where the entries satisfy $t_1t_2 = 1$ The character module $X^*(T_0^{(1)})$ is generated by $\gamma_1^M(t) = t_1$ the character $2\gamma_1^M$ extends to a character on T_0 and we have

$$2\gamma_1^M(t) = t_1/t_2$$

and this character is trivial on Z_0.

In the semi simple part of M we have a maximal torus

$$T_1/\mathbb{Q} = \begin{pmatrix} a & b & 0 & 0 \\ -b & a & 0 & 0 \\ 0 & 0 & a & -b \\ 0 & 0 & b & a \end{pmatrix}, a^2 + b^2 = 1.$$

The torus $\tilde{T}_1 = T_1 \cdot Z_0$ is a maximal torus in $\mathrm{Sp}_2/\mathbb{Q}$. This torus is not split over $\mathbb{Q}$, it splits over the quadratic extension $F = \mathbb{Q}(i)$. It acts by the adjoint action on the Lie algebra $\mathfrak{u}_0 = \mathrm{Lie}(U_0)$. The sub torus $Z_0/\mathbb{Q}$ acts by the character $z^2 = \gamma_2|Z_0$. Under the action of $T_1/\mathbb{Q}$ the module $\mathfrak{u}_0 \otimes F$ decomposes into weight eigenspaces

$$\mathfrak{u}_0 \otimes F = F\ E_+ \oplus F\ E_0 \oplus F\ E_-$$

where

$$\mathrm{Ad}(t)E_+ = (a+bi)^2 E_+, \mathrm{Ad}(t)E_0 = E_0,\ \mathrm{Ad}(t)E_- = (a-bi)^2 E_-, .$$

For the basis of the eigenspaces we choose the specific elements

$$E_+ = \begin{pmatrix} 0 & 0 & 1 & -i \\ 0 & 0 & i & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ E_0 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ E_- = \begin{pmatrix} 0 & 0 & 1 & i \\ 0 & 0 & -i & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We also have the subgroup $K/\mathbb{Q}$, up to isogeny this is $K^{(1)}/\mathbb{Q}$ times $Z_c/\mathbb{Q}$ where

$$K^{(1)} = \begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix},\ a^2 + b^2 + c^2 + d^2 = 1$$

and

$$Z_c = \begin{pmatrix} x & 0 & 0 & y \\ 0 & x & y & 0 \\ 0 & -y & x & 0 \\ -y & 0 & 0 & x \end{pmatrix},\ x^2 + y^2 = 1$$

Then $K/\mathbb{Q}$ is a reductive subgroup, which over $F = \mathbb{Q}(i)$ is conjugate to M/F. The group $K(\mathbb{R}) = K_\infty$ is a maximal compact subgroup in $G(\mathbb{R})$.

We introduce the group K^M it is the intersection

$$K^M = M \cap K \supset T_1(\mathbb{R}).$$

We have the element

$$\epsilon = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in K_\infty$$

and

$$K^M = T_1 \rtimes \{\epsilon\}.$$

So we have three maximal tori: The standard split maximal torus $T_0 = T_0^{(1)} \cdot Z_0$ the above torus $\tilde{T}_1 = T_1 \cdot Z_0$ and the torus $T_c = T_1 \cdot Z_c$. If we base change to F they become conjugate and we we identify the character modules. To be more precise we choose one of the two possible identifications

$$X^*(T_1 \times F) \xrightarrow{\sim} X^*(T_0^{(1)})$$

and using this identification we define $\tilde{\gamma}_1^M$. (We choose the identification such that $\tilde{\gamma}_1^M(t) = (a + bi)$.

By the same token we get an identification $X^*(Z_c \times F) \xrightarrow{\sim} X^*(Z_0)$ it sends $\tilde{\gamma}_2$ to γ_2. We normalize the identification such that for $z \in Z_c$ we get $\tilde{\gamma}_2(z) = (x + iy)^2$. Since all three tori are written as products of one dimensional tori, we get an identification of the character modules of these tori.

Let $\mathfrak{g}, \mathfrak{k}, \mathfrak{k}^M = \mathfrak{t}_1$ be the Lie-algebras. Then we have the isomorphism

$$\mathfrak{g}/\mathfrak{k} = \mathfrak{m}/\mathfrak{k}^M \oplus \mathfrak{u}_0.$$

The direct sum on the right hand side has a distinguished basis if we tensorize by F

$$\mathfrak{m}/\mathfrak{k}^M \otimes F \oplus \mathfrak{u}_0 \otimes F = FP_+ \oplus FP_- \oplus FH_0 \oplus FE_+ \oplus FE_0 \oplus FE_-$$

where

$$P_+ = \begin{pmatrix} 1 & i & 0 & 0 \\ i & -1 & 0 & 0 \\ 0 & 0 & 1 & -i \\ 0 & 0 & -i & -1 \end{pmatrix}, \; P_- = \begin{pmatrix} 1 & -i & 0 & 0 \\ -i & -1 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & i & -1 \end{pmatrix}, \; H_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Both summands are modules under the adjoint action of M and all summands are eigenspaces under the action of T_1 :

$$\mathrm{Ad}(t)P_+ = (a + bi)^2 P_+, \; \mathrm{Ad}(t)P_- = (a - bi)^2 P_-, \mathrm{Ad}(t)H_0 = H_0$$

The Lie algebra of $Z_0/\mathbb{Q}$ is $\mathbb{Q}H_0$, the center Z_0 acts on $\mathfrak{u}_0$ by the character $\mathrm{Ad}(z)V = z^2 V$. From this we get certain Lie brackets which will play a role. The E-s commute, we get

$$[H_0, V] = 2V, [P_+, E_0] = -2E_+, [P_-, E_0] = 2E_-.$$

We choose a highest weight $\lambda = n_1\gamma_1 + n_2\gamma_2$ we assume $n_1 > 0$ and even. Let $\mathcal{M}_\lambda$ be a corresponding highest weight module. It is a free $\mathbb{Z}$ module with an action of the group scheme $G/\mathbb{Z}$. Under the action of $T_0/\mathbb{Z}$ it decomposes into weight spaces

$$\mathcal{M}_\lambda = \bigoplus_\mu \mathcal{M}_{\lambda,\mu}$$

and $\mathcal{M}_{\lambda,\lambda} = \mathbb{Z}e_\lambda$ is a highest weight submodule.

We get the following list of Kostant representatives for the Siegel parabolic subgroup P they provide the following list of weights.

$$\begin{aligned}
1\lambda &= \lambda = \tfrac{1}{2}(2n_2+n_1)\gamma_2 + n_1\gamma_1^{M_1} \\
s_2\lambda &= \tfrac{1}{2}n_1\gamma_2 + (2n_2+n_1)\gamma_1^{M_1} \\
s_2s_1\lambda &= -\tfrac{1}{2}n_1\gamma_2 + (2n_2+n_1)\gamma_1^{M_1} \\
s_2s_1s_2\lambda &= \tfrac{1}{2}(-2n_2-n_1)\gamma_2 + n_1\gamma_1^{M_1},
\end{aligned}$$

We consider the rectangle spanned by the four weights

$$\begin{aligned}
s_2\lambda &= \tfrac{1}{2}n_1\gamma_2 + (2n_2+n_1)\gamma_1^{M} \\
s_1s_2\lambda &= \tfrac{1}{2}n_1\gamma_2 - (2n_2+n_1)\gamma_1^{M} \\
s_2s_1\lambda &= -\tfrac{1}{2}n_1\gamma_2 + (2n_2+n_1)\gamma_1^{M} \\
s_1s_2s_1\lambda &= -\tfrac{1}{2}n_1\gamma_2 - (2n_2+n_1)\gamma_1^{M}
\end{aligned}$$

Here the characters are characters on the torus T_0. We get a corresponding list of weights for the other two tori if we place a tilda on γ_2 or γ_1^M. We introduce numbers

$$m = \frac{1}{2}n_1, l = 2n_2 + n_1, k = 4 + l.$$

Some facts:

The centralizer of T_1 is reductive a split group, its semi simple part M_1' is isomorphic to $\mathrm{Sl}_2/\mathbb{Q}$, the torus $Z_0 \subset M_1'$ is a maximal torus. We have

$$\mathfrak{m}_1' \otimes F = FH_0 \oplus FE_0 \oplus FE_0^*.$$

where $E_0^* = \mathrm{Ad}(s_1s_2s_1)(E_0)$. We see that for any weight $\mu = a\tilde{\gamma}_1^M + b\gamma_2$ occurring in the set of weights in $\mathcal{M}_\lambda$ we always have $|a| \leq l$. On the extremal line joining $m\gamma_2 + l\tilde{\gamma}_1^M$ and $-m\gamma_2 + l\tilde{\gamma}_1^M$ we have the weights $\nu\gamma_2 + l\tilde{\gamma}_1^M$. These are weights occurring in an irreducible M_1' module $\mathcal{N}_+$

$$\mathcal{N}_+ = \oplus Fe_{l\tilde{\gamma}^M + \nu\gamma_2}.$$

(The choice of the generators will be discussed later, in any case they can be fixed up to a unit in $\mathbb{Z}[i]$ The same procedure allows us to define

$$\mathcal{N}_- = \oplus F e_{-l\tilde{\gamma}^M + \nu\gamma_2}$$

The modules $\mathcal{N}_\pm \subset \mathcal{M}_{\lambda,F}$ are also modules under the action of Z_c. Hence we get a second decomposition

$$\mathcal{N}_\pm = \oplus F \; e_{\pm l\tilde{\gamma}_1^{M_1} + \nu\tilde{\gamma}_2}$$

For any $-m \leq \nu \leq m$ the vector $e_{l\tilde{\gamma}_1^{M_1}+\nu\tilde{\gamma}_2}$ is the highest weight vectors of an irreducible K module $\mathcal{M}^K_{l\gamma_1^{M_1}+\nu\tilde{\gamma}_2} = \mathcal{M}^K(\nu)$. We put $\mathcal{M}^K_{l\gamma_1^{M_1}} = \oplus \mathcal{M}^K(\nu)$

Another element of importance is

$$c_\infty = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \in \mathrm{GSp}_2(\mathbb{Q}).$$

Since we assumed that n_1 is even, our representation $\mathcal{M}_\lambda$ can be viewed as a GSp_2-module and where the centre acts trivially. Hence the action of c_∞ on $\mathcal{M}_\lambda$ is the same as the action of

$$c'_\infty = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} \in \mathrm{Sp}_2(F)$$

2.2 The induced Harish-Chandra modules

We consider the standard Borel subgroup $B \subset M$ the standard split torus $T_0 \subset B$ it contains our torus Z_0. We consider a character (s is a complex variable)

$$\chi = (k, s) : B(\mathbb{R}) \to \mathbb{C}^\times, \; \chi(t) = \gamma_1^M(t)^k |\gamma_2|^s$$

We define the induced Harish-Chandra module $I_{B(\mathbb{R})}^{M(\mathbb{R})}\chi$: We consider the functions

$$f : M(\mathbb{R}) \to \mathbb{C}; f(bg) = \chi(b)f(g); f|T_1 \text{ is of finite type .}$$

This is in fact a $\mathfrak{g}, K_\infty^M$ -module. The module contains the discrete representation $\mathcal{D}_{\chi_\mu}$. We have the decomposition

$$\mathcal{D}_{\chi_\mu} = \bigoplus_{\nu \equiv 0(2), |\nu| \geq k} F\phi_{\chi,\nu}$$

where

$$\phi_{\chi,\nu}(g) = \phi_{\chi,\nu}(b \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi \end{pmatrix}) = \chi(b) e^{2\pi m i \phi}.$$

Of course $K_\infty^{M,0} = T_1(\mathbb{R}) = \{e(\phi) = \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi \end{pmatrix}\}$ and we can write $e(\phi)^m = e^{2\pi m i \phi}$.

We can define the induced module $\mathrm{Ind}_P^G \mathcal{D}_{\chi_\mu}$, it is again a Harish-Chandra module. We have a decomposition of $\mathrm{Ind}_P^G \mathcal{D}_{\chi_\mu}$ into K_∞ types. We put $V_k = \{\sum a_\nu v_+^\nu v_-^{k-\nu}\}$ on this module we have the standard action of $K_\infty^{(1)} = SU(2)$. For integer m we have the action of Z_c on the module given by scalar multiplication by $(x+iy)^m$. For $k \equiv m \mod 2$ this gives us an explicitly constructed module $V_{k,m}$ with an irreducible action of K_∞. We have a positive definite hermitian scalar product which is invariant and normalized by

$$< v_+^k, v_+^k >= 1$$

The lowest $K_\infty^{(1)}$ *type in* $\mathrm{Ind}_P^G \mathcal{D}_{\chi_\mu}$ *is* $k\gamma_1^M$ *, i.e. it is the module* V_k*. It occurs with multiplicity infinity. On this lowest* $K_\infty^{(1)}$ *-type we have the action of* Z_c*. Under this action the lowest* $K_\infty^{(1)}$ *type decomposes into*

$$\mathrm{Ind}_P^G \mathcal{D}_{\chi_\mu}^{\mathrm{low}} = \bigoplus_{m \equiv (2)} \mathrm{Ind}_P^G \mathcal{D}_{\chi_\mu}(V_{k,m})$$

We will construct an explicit isomorphism

$$i_{k,\nu} : V_{k,\nu} \xrightarrow{\sim} \mathrm{Ind}_P^G \mathcal{D}_{\chi_\mu}(V_{k,\nu})$$

Now we define the characters $\chi = (k, m+2)$ and $\chi' = (k, -m+1)$. We have a non trivial intertwining operator

$$T(\chi, \chi') : \mathrm{Ind}_P^G \mathcal{D}_{\chi_\mu} \to \mathrm{Ind}_P^G \mathcal{D}_{\chi_{\mu'}} \tag{1}$$

which is unique up to a scalar. The kernel of this this operator is a discrete series representation $\mathbb{D}_\chi$ whose lowest K_∞ type is $(k, m+1)$. At this moment we normalize $T(\chi, \chi')$ such that it is the identity on the K_∞-type $V_{k,0}$.

We have

$$\mathrm{Hom}_{K_\infty}(\Lambda^3(\mathfrak{g}/\mathfrak{k}), \mathbb{D}_\chi \otimes \mathcal{M}_\lambda) = F\Omega_{2,1} \oplus F\Omega_{1,2} = H^3(\mathfrak{g}, K_\infty, \mathbb{D}_\chi \otimes \mathcal{M}_\lambda)$$

where the Ω-s are specific differential forms (See next section). We consider the map

$$H^3(\mathfrak{g}, K_\infty, \mathbb{D}_\chi \otimes \mathcal{M}_\lambda) \to H^3(\mathfrak{g}, K_\infty, \mathrm{Ind}_P^G \mathcal{D}_{\chi_\mu} \otimes \mathcal{M}_\lambda).$$

The theorem of Delorme implies (see next section) that $H^3(\mathfrak{g}, K_\infty, \mathrm{Ind}_P^G \mathcal{D}_{\chi_\mu} \otimes \mathcal{M}_\lambda)$ is one dimensional, the element c_∞ acts by $(-1)^m$. The two forms Ω_{21}, Ω_{12} yield non zero classes in $H^3(\mathfrak{g}, K_\infty, \mathrm{Ind}_P^G \mathcal{D}_{\chi_\mu} \otimes \mathcal{M}_\lambda)$ and we have $c_\infty(\Omega_{21}) = (-1)^m \Omega_{12}$ (i.e. we normalize the choice of Ω_{12} in terms of the choice of Ω_{21}. We could also say that we adapt the choices of Ω_{21} and Ω_{12} such that they yield the same class in $H^3(\mathfrak{g}, K_\infty, \mathrm{Ind}_P^G \mathcal{D}_{\chi_\mu} \otimes \mathcal{M}_\lambda)$) and therefore the difference of the two classes in $H^3(\mathfrak{g}, K_\infty, \mathrm{Ind}_P^G \mathcal{D}_{\chi_\mu} \otimes \mathcal{M}_\lambda)$ is zero. We find a form $\Psi \in \mathrm{Hom}_{K_\infty}(\Lambda^2(\mathfrak{g}/\mathfrak{k}), \mathbb{D}_\chi \otimes \mathcal{M}_\lambda)$ which bounds this difference:

$$d(\Psi) = \Omega_{21} - \Omega_{12}$$

Since the Ω-s go to zero under the intertwining operator T the form $T(\Psi)$ in $\mathrm{Hom}_{K_\infty}(\Lambda^2(\mathfrak{g}/\mathfrak{k}), \mathrm{Ind}_P^G \mathcal{D}_{\chi_{\mu'}} \otimes \mathcal{M}_\lambda)$ is closed.

Our aim is to compute the class

$$\kappa_T([\Omega_{21} - \Omega_{12}]) = [T(\Psi)].$$

2.3 The calculation

In the following we evaluate at $s=m+2$ or at $s=-m+1$, i.e. the character is χ or χ'.T hen our induced modules are in fact defined over $\mathbb{Q}$ and over F. We get a commutative diagram

$$\begin{array}{ccc}\mathrm{Hom}_{K_\infty}(\Lambda^q(\mathfrak{g}/\mathfrak{k}),\mathrm{Ind}_P^G\mathcal{D}_{\chi_\mu}\otimes\mathcal{M}_{\lambda,F}) & \xrightarrow{j^{(q)}} & \mathrm{Hom}_{K^M}(\Lambda^\bullet((\mathfrak{m}/\mathfrak{k}^M\oplus\mathfrak{u}_0)\otimes F,\mathcal{D}_{\chi_\mu}\otimes\mathcal{M}_{\lambda,F}) \\ \downarrow T^\bullet & & \downarrow t^\bullet \\ \mathrm{Hom}_{K_\infty}(\Lambda^q(\mathfrak{g}/\mathfrak{k}),\mathrm{Ind}_P^G\mathcal{D}_{\chi_{\mu'}}\otimes\mathcal{M}_{\lambda,F}) & \xrightarrow{j^{(q)}} & \mathrm{Hom}_{K^M}(\Lambda^\bullet((\mathfrak{m}/\mathfrak{k}^M\oplus\mathfrak{u}_0)\otimes F,\mathcal{D}_{\chi_{\mu'}}\otimes\mathcal{M}_{\lambda,F})\end{array}$$

The horizontal arrows are isomorphisms. The operator $T^\bullet$ is the intertwining operator which is normalized to be the identity on the K_∞ type $\mathrm{Ind}_P^G\mathcal{D}_{\chi_\mu}(k,0)$ and $t^\bullet$ is defined via the diagram.

Firstly we look at the right hand side, we consider the complex

$$\mathrm{Hom}_{K^M}(\Lambda^\bullet((\mathfrak{m}/\mathfrak{k}^M\oplus\mathfrak{u}_0)\otimes F,\mathcal{D}_{\chi_\mu}\otimes\mathcal{M}_{\lambda,F}).$$

here $\chi=(k,s)$. We look at the $K_\infty^{M,0}$-types. The discrete series representation has only $e(\phi)^m$ with $|m|\geq k$. The highest weight for T_1 in $\mathcal{M}_\lambda$ sits in the modules $\mathcal{N}_\pm$ and is $e(\phi)^{\pm l}$. The gap between these weights is 4 and this can only be bridged if we have weight ± 4 in $(\mathfrak{m}/\mathfrak{k}^M\oplus\mathfrak{u}_0)\otimes F$. This implies that our complex looks like (we restrict to $K_\infty^{M,0}=T_1(\mathbb{R})$)

$$0\to\mathrm{Hom}_{T_1(\mathbb{R})}(\Lambda^2(\mathfrak{m}/\mathfrak{k}^M\oplus\mathfrak{u}_0)\otimes F,\mathcal{D}_{\chi_\mu}\otimes\mathcal{M}_{\lambda,F})\to\mathrm{Hom}_{T_1(\mathbb{R})}(\Lambda^3(\mathfrak{m}/\mathfrak{k}^M\oplus\mathfrak{u}_0)\otimes F,\mathcal{D}_{\chi_\mu}\otimes\mathcal{M}_{\lambda,F})\to$$
$$\mathrm{Hom}_{T_1(\mathbb{R})}(\Lambda^4(\mathfrak{m}/\mathfrak{k}^M\oplus\mathfrak{u}_0)\otimes F,\mathcal{D}_{\chi_\mu}\otimes\mathcal{M}_{\lambda,F})\to 0$$

The tensor product $\mathcal{D}_{\chi_\mu}\otimes\mathcal{M}_{\lambda,F}$ is in fact a M module, hence we have an action of $\mathfrak{m}\oplus\mathfrak{u}_0$ the unipotent radical acts trivially. The module $\mathcal{D}_{\chi_\mu}$ decomposes into $K_\infty^{M,0}$ types and $P_+\phi_{\chi,k}=c\phi_{k+2}, P_-\phi_{\chi,k}=0$. Hence we see that for $q=2,3,4$ the forms in degree q are given by

$$\mathrm{Hom}_{T_1(\mathbb{R})}(\Lambda^q(\mathfrak{m}/\mathfrak{k}^M\oplus\mathfrak{u}_0)\otimes F,\mathcal{D}_{\chi_\mu}(k)\otimes\mathcal{N}_-\oplus\mathcal{D}_{\chi_\mu}(-k)\otimes\mathcal{N}_+).$$

Now we can determine bases for these spaces. We decompose $\mathcal{N}_\pm$ with respect to the action of Z_0 :

$$\mathcal{N}_+=\bigoplus_{\nu=-m}^{\nu=m}\mathcal{N}_+(\nu\gamma_2)$$

we choose generators $e_{+l\tilde{\gamma}_1^{M_1}+\nu\gamma_2}\in\mathcal{N}_+(\nu\gamma_2)$. We observe that $E_0\mathcal{N}_+(\nu\gamma_2)\subset\mathcal{N}_+((\nu+1)\gamma_2)$ hence we can define numbers $E_0e_{+l\tilde{\gamma}_1^{M_1}+\nu\gamma_2}=c(\nu,m)e_{l\tilde{\gamma}_1^{M}+(\nu+1)\gamma_2}$. These numbers depend of course on the choice of the generators. Later we will see that we can control the choice of the generators and hence the numbers $c(|nu,m)$ by representation theory of M'. In any case we have $c(\nu,m)\neq 0$ for $-m\leq\nu\leq m-1$ and $c(m,m)=0$.

We define forms in degree 2

$$\omega_\nu^{(2,+)}=P_+^\vee\wedge E_+^\vee\otimes\phi_{\chi,k}\otimes e_{-l\tilde{\gamma}_1^{M_1}+\nu\gamma_2},\ \omega_\nu^{(2,-)}=P_-^\vee\wedge E_-^\vee\otimes\phi_{\chi,-k}\otimes e_{+l\tilde{\gamma}_1^{M_1}+\nu\gamma_2}$$

these forms form a basis for the forms in degree 2. In degree 3 we have twice as many basis elements

$$\omega^{(3,+)}_{H_0,\nu} = P_+^\vee \wedge E_+^\vee \wedge H_0^\vee \otimes \phi_{\chi,k} \otimes e_{-l\tilde{\gamma}_1^{M_1}+\nu\gamma_2},\ \omega^{(3,-)}_{H_0,\nu} = P_-^\vee \wedge E_-^\vee \wedge H_0^\vee \otimes \phi_{\chi,-k} \otimes e_{+l\tilde{\gamma}_1^{M_1}+\nu\gamma_2}$$

$$\omega^{(3,+)}_{E_0,\nu} = P_+^\vee \wedge E_+^\vee \wedge E_0^\vee \otimes \phi_{\chi,k} \otimes e_{-l\tilde{\gamma}_1^{M_1}+\nu\gamma_2},\ \omega^{(3,-)}_{E_0,\nu} = P_-^\vee \wedge E_-^\vee \wedge H_0^\vee \otimes \phi_{\chi,-k} \otimes e_{+l\tilde{\gamma}_1^{M_1}+\nu\gamma_2}$$

and in degree 4 we have

$$\omega^{(4,+)}_{\nu} = P_+^\vee \wedge E_+^\vee \wedge H_0^\vee \wedge E_0^\vee \otimes \phi_{\chi,k} \otimes \tilde{e}, n,\ \omega^{(4,-)}_{\nu} = P_-^\vee \wedge E_-^\vee \wedge H_0^\vee \wedge E_0^\vee \otimes \phi_{\chi,-k} \otimes e_{+l\tilde{\gamma}_1^{M_1}+\nu\gamma_2}$$

We have to compute the differentials. In this process we observe that E_0 sends $\mathcal{N}_\pm(\nu\gamma_2)$ to $\mathcal{N}_\pm((\nu+1)\gamma_2)$ and more precisely we have $E_0 e_{-l\tilde{\gamma}_1^{M_1}+\nu\gamma_2} = c(\nu,m)e_{-l\tilde{\gamma}_1+(\nu+1)}$ where $c(\nu,m)\neq 0$ for $\nu=-m,\dots \nu=m-1$ and clearly $c(m,m)=0$. The values of these numbers depend on the choice of the generators and will be specified later.

$$d\,\omega^{(2,+)}_{\nu}(P_+,E_+,H_0) = H_0(\omega^{(2,+)}_{\nu}(P_+,E_+)) - \omega^{(2,+)}_{\nu}([E_+,H_0],P_+) =$$

$$H_0(\omega^{(2,+)}_{\nu}(P_+,E_+)) - \omega^{(2,+)}_{\nu}(P_+,[H_0,E_+]) = 2(z+\nu-1)\phi_{\chi,k}\otimes e_{-l\tilde{\gamma}_1^{M_1}+\nu\gamma_2}$$

$$d\,\omega^{(2,+)}_{\nu}(P_+,E_+,E_0) = \phi_{\chi,k}\otimes E_0 e_{-l\tilde{\gamma}_1^{M_1}+\nu\gamma_2} = c(\nu,m)\phi_{\chi,k}\otimes e_{-l\tilde{\gamma}_1^{M}+(\nu+1)\gamma_2}$$

$$d\,\omega^{(3,+)}_{H_0,\nu}(P_+,E_+,H_0,E_0) = -E_0\phi_{\chi,k}\otimes e_{-l\tilde{\gamma}_1^{M_1}+\nu\gamma_2} = -c(\nu,m)\phi_{\chi,k}\otimes e_{-l\tilde{\gamma}_1^{M}+(\nu+1)\gamma_2}$$

$$d\,\omega^{(3,+)}_{E_0,\nu}(P_+,E_+,H_0,E_0) =$$

$$H_0\omega^{(3,+)}_{E_0,\nu}(P_+,E_+,E_0) - \omega^{(3,+)}_{E_0,\nu})([E_+,H_0],P_+,E_0) - \omega^{(3,+)}_{E_0,\nu}([H_0,E_0],P_+,E_+) =$$

$$2(z+\nu-2)\phi_{\chi,k}\otimes e_{-l\tilde{\gamma}_1^{M_1}+\nu\gamma_2}$$

We need to understand the subcomplex $\mathrm{Hom}_{K^M}()$, we write a differential form in degree 2 (or 4) as an array

$$\omega = \sum_{\nu=-m}^{\nu=m} a_\nu(\omega^{(d,+)}_{\nu} \pm \omega^{(d,-)}_{\nu}) = \{a_{-m},\dots,a_\nu,\dots,a_m\} \tag{2}$$

and in degree 3 as

$$\sum_{\nu=-m}^{\nu=m} b_\nu\omega^{(3,+)}_{H_0,\nu} + c_\nu\omega^{(3,-)}_{E_0,\nu} = \{b_{-m},\dots,b_\nu,\dots,b_m;c_{-m},\dots,c_\nu,\dots,c_m\} \tag{3}$$

then our formulae above say that the differential in degree 2 does

$$d:\{a_{-m},\dots,a_\nu,\dots,a_m\}\mapsto$$

$$\{(z-m-1)a_{-m},\dots,(z+\nu-1)a_\nu,\dots(z+m-1)a_m;0,c(-m,m)a_{-m},\dots,c(\nu,m)a_\nu,\dots,c(m-1,m)a_{m-1}\}$$

and the differential in degree 3

$$d:\{b_{-m},\dots,b_\nu,\dots,b_m;c_{-m}\ \dots,c_\nu,\dots,c_m\}\mapsto$$
$$\{(z-m-2)c_{-m},(z-m-1)c_{-m+1}-c(-m,m)b_m,\dots,(z+m-2)c_m-c(m-1,m)b_{m-1}\}$$

Clearly the cohomology of this complex is trivial unless z assumes the special values $z=-m+1$ or $z=m+2$.

If $z=-m+1$ we see that $d\ \omega^{(2,+)}_m=0$ and $d\ \omega^{(3,+)}_{H_0,m}=0$. Then it is clear that these two classes represent generators for the cohomology $H^\bullet(\mathfrak{m}/\mathfrak{k}^M\oplus\mathfrak{u}_0,K^M_\infty,\mathcal{D}_{\chi_\mu}\otimes\mathcal{M}_{\lambda,F})$ in degree 2 and 3.

If $z=m+2$ then $d(\omega^{(3,+)}_{E_0,-m})=0$ represents a non zero class in H^3. The class $\omega^{(4,+)}_{-m}$ is not in the image of d hence we find a non zero class in H^4. This is all the cohomology in this case.

We have still the action of c_∞ on our complex $\mathrm{Hom}_{K^M}(\Lambda^\bullet((\mathfrak{m}/\mathfrak{k}^M\oplus\mathfrak{u}_0)\otimes F,\mathcal{D}_{\chi_\mu}\otimes\mathcal{M}_{\lambda,F})$. It acts on the Lie-algebra via the adjoint action hence trivially on $\mathfrak{m}/\mathfrak{k}^M$ and by -1 on $\mathfrak{u}_0$). It acts trivially on the $\mathcal{D}_{\chi_\mu}$ and the the $e_{-l\tilde\gamma_1^{M_1}+\nu\gamma_2}$ are eigenvectors with eigenvalue $(-1)^\nu$.

We conclude that

a)the forms $\omega^{(2,+)}_\nu$ are eigenclasses with eigenvalue $(-1)^{\nu+1}$

b) the forms $\omega^{(3,+)}_{H_0,\nu}$ are eigenclasses with eigenvalue $(-1)^{\nu+1}$.

b1) the forms $\omega^{(3,+)}_{E_0,\nu}$ are eigenclasses with eigenvalue $(-1)^\nu$.

c) the forms $\omega^{(4,+)}_{,\nu}$ are eigenclasses with eigenvalue $(-1)^\nu$.

If we now choose $z=m+2$ then the cohomology in degree 3 and 4 has eigenvalue $(-1)^m$. If $z=-m+1$ then the cohomology in degree 2,3 is in the eigenspace $(-1)^{m+1}$.

This computation of the cohomology is of course in accordance with Delormes theorem.

We introduced the submodule $\mathcal{N}_+$, this was the submodule on which $K^{M,1}_\infty$ acts by $l\tilde\gamma_1^M$ it is only defined over F. On this module we have the action of our group M_1'. This group contains Z_c as a maximal torus, hence

$$\mathcal{N}_+=\bigoplus_{\nu=-m}^{\nu=m}Fe_{+l\tilde\gamma_1^{M_1}+\nu\tilde\gamma_2},$$

we specify a choice of the generators later. Now it is clear that the $e_{+l\tilde\gamma_1^{M_1}+\nu\tilde\gamma_2}$, are highest weight vectors for K_∞ of weight $+l\tilde\gamma_1^{M_1}+\nu\tilde\gamma_2$ Hence it defines an irreducible K_∞ module $\tilde{\mathcal{N}}_\nu\subset\mathcal{M}_\lambda$, here $\nu=-m,\dots,\nu=m$. Let $\tilde{\mathcal{N}}=\oplus\tilde{\mathcal{N}}_\nu$.

We have

$$\mathrm{Hom}_{K_\infty}(\Lambda^q(\mathfrak{g}/\mathfrak{k}),\mathrm{Ind}_P^G\mathcal{D}_{\chi_\mu}\otimes\mathcal{M}_{\lambda,F})\xrightarrow{\sim}\mathrm{Hom}_{K_\infty}(\Lambda^q(\mathfrak{g}/\mathfrak{k}),\mathrm{Ind}_P^G\mathcal{D}^{\mathrm{low}}_{\chi_\mu}\otimes\tilde{\mathcal{N}})$$

We have the standard decomposition

$$\mathfrak{g}/\mathfrak{k}\otimes F=\mathfrak{u}_+\oplus\mathfrak{u}_-$$

where $\mathfrak{u}_+,\mathfrak{u}_-$ are the unipotent radicals of the two parabolic subgroups containing K_∞. These unipotent radicals are $K_\infty-$ modules and isomorphic to $V_{2,1}$ (resp.) $V_{2,-1}$.

Then

$$\Lambda^2(\mathfrak{g}/\mathfrak{k}) \supset V_{2,1} \otimes V_{2,-1} \;\supset V_{4,0}$$

$$\Lambda^3(\mathfrak{g}/\mathfrak{k}) \supset \Lambda^2(V_{2,1}) \otimes V_{2,-1} \oplus V_{2,1} \otimes \Lambda^2(V_{2,-1}) \supset V_{4,1} \oplus V_{4,-1}$$

and (summation always $\nu = -m, \nu = m$)

$$\mathrm{Hom}_{K_\infty}(\Lambda^2(\mathfrak{g}/\mathfrak{k}), \mathrm{Ind}_P^G \mathcal{D}_{\chi_\mu} \otimes \mathcal{M}_{\lambda,F}) = \mathrm{Hom}_{K_\infty}(V_{4,0}, \oplus_\nu \mathrm{Ind}_P^G \mathcal{D}_{\chi_\mu}(V_{k,\nu}) \otimes \mathcal{M}^K(-\nu))$$

$$\mathrm{Hom}_{K_\infty}(\Lambda^3(\mathfrak{g}/\mathfrak{k}), \mathrm{Ind}_P^G \mathcal{D}_{\chi_\mu} \otimes \mathcal{M}_{\lambda,F}) =$$

$$\mathrm{Hom}_{K_\infty}(V_{4,1}, \oplus_\nu \mathrm{Ind}_P^G \mathcal{D}_{\chi_\mu}(V_{k,\nu+1}) \otimes \mathcal{M}^K(-\nu)) \bigoplus \mathrm{Hom}_{K_\infty}(V_{4,-1}, \oplus_\nu \mathrm{Ind}_P^G \mathcal{D}_{\chi_\mu}(V_{k,-\nu-1}) \otimes \mathcal{M}^K(\nu))$$

The individual summands are of dimension one. We can exhibit explicit generators, i.e. we write

$$\mathrm{Hom}_{K_\infty}(\Lambda^2(\mathfrak{g}/\mathfrak{k}), \mathrm{Ind}_P^G \mathcal{D}_{\chi_\mu} \otimes \mathcal{M}_{\lambda,F}) = \bigoplus F\Omega_\nu^{(2,\chi)}$$

$$\mathrm{Hom}_{K_\infty}(\Lambda^3(\mathfrak{g}/\mathfrak{k}), \mathrm{Ind}_P^G \mathcal{D}_{\chi_\mu} \otimes \mathcal{M}_{\lambda,F}) = \bigoplus (F\Omega_{\nu+1}^{(3,\chi,o)} \oplus F\Omega_{-\nu-1}^{(3,\chi,*)})$$

We will give formulae for the generators. We have a basis for

$$\mathfrak{g}/\mathfrak{k} \otimes F = \mathfrak{u}_+ \oplus \mathfrak{u}_- = F\ E_p \oplus F\ E_{null} \oplus F\ E_m \bigoplus F\ E_p^* \oplus F\ E_{null}^* \oplus F\ E_m^*$$

where

$$E_p = \begin{pmatrix} -\frac{i}{2} & \frac{1}{2} & \frac{i}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{i}{2} & -\frac{1}{2} & \frac{i}{2} \\ \frac{i}{2} & -\frac{1}{2} & -\frac{i}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{i}{2} & -\frac{1}{2} & \frac{i}{2} \end{pmatrix}, E_{null} = \begin{pmatrix} i & 0 & 0 & -1 \\ 0 & i & -1 & 0 \\ 0 & -1 & -i & 0 \\ -1 & 0 & 0 & -i \end{pmatrix}, E_m = \begin{pmatrix} \frac{i}{2} & \frac{1}{2} & \frac{i}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{i}{2} & \frac{1}{2} & \frac{i}{2} \\ \frac{i}{2} & \frac{1}{2} & \frac{i}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{i}{2} & -\frac{1}{2} & -\frac{i}{2} \end{pmatrix}$$

and

$$E_p^* = \begin{pmatrix} -\frac{i}{2} & \frac{1}{2} & -\frac{i}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{i}{2} & \frac{1}{2} & -\frac{i}{2} \\ -\frac{i}{2} & \frac{1}{2} & -\frac{i}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{i}{2} & -\frac{1}{2} & \frac{i}{2} \end{pmatrix} E_{null}^* = \begin{pmatrix} i & 0 & 0 & 1 \\ 0 & i & 1 & 0 \\ 0 & 1 & -i & 0 \\ 1 & 0 & 0 & -i \end{pmatrix} E_m^* = \begin{pmatrix} \frac{i}{2} & \frac{1}{2} & -\frac{i}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{i}{2} & -\frac{1}{2} & -\frac{i}{2} \\ -\frac{i}{2} & -\frac{1}{2} & \frac{i}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{i}{2} & -\frac{1}{2} & -\frac{i}{2} \end{pmatrix}$$

The vectors E_p, E_{null}, E_m are eigenvalues for the action of $T_1(\mathbb{R})$ an element of $T_1(\mathbb{R})$ acts by the eigenvalues $(a+bi)^2, 1, (a-bi)^2$ respectively. The same is true for E_p^*, E_{null}^*, E_m^*. The torus Z_c acts by $(x+iy)^2$ on the first three vectors and by $(x-iy)^2$ on the second group. For $t \in T_1(\mathbb{R})$ we have

$$\tilde{\gamma}_1^M(t) = (a+bi)^2, \tilde{\gamma}_2(z) = (x+iy)^2.$$

We take the χ with parameter $z = m+2$. To identify $\Omega_{\nu+1}^{(3,\chi,o)}$ we give its value

$$\Omega_{\nu+1}^{(3,\chi,o)}(E_p, E_{null}, E_p^*) = e_{k,\nu+1} \otimes e_{-l\tilde{\gamma}_1^{M_1} - \nu\tilde{\gamma}_2} + \cdots \in \mathrm{Ind}_P^G \mathcal{D}_{\chi_\mu}(V_{k,\nu+1}) \otimes \mathcal{M}^K(-\nu)))$$

$$\Omega_{\nu+1}^{(3,\chi,o)}(E_p^*, E_{null}^*, E_p) = 0$$

For the "conjugate" we get

$$\Omega^{(3,\chi,*)}_{-\nu-1}(E_p, E_{null}, E^*_p) = 0$$

$$\Omega^{(3,\chi,*)}_{-\nu-1}(E^*_p, E^*_{null}, E_p) = e_{k,-\nu-1}\otimes e_{-l\tilde{\gamma}_1^{M_1}+\nu\tilde{\gamma}_2}+\cdots \in \mathrm{Ind}_P^G \mathcal{D}_{\chi_\mu}(V_{k,-\nu-1})\otimes\mathcal{M}^K(\nu))))$$

We want to determine the image of these differential forms under the Delorme-Frobenius isomorphism. The interpretation of this isomorphism yields that

$$\Omega^{(3,\chi,?)}_{\nu,1} \mapsto a_0^? \tilde{\omega}^{(3,+,?)}_{H_0,\nu} + b_0^? \tilde{\omega}^{(3,+,?)}_{E_0,\nu}$$

with some coefficients a_0, b_0. Here we define these $\tilde{\omega}$ with respect to the tilde basis:

$$\tilde{\omega}^{(3,+)}_{H_0,\nu}(P_+, E_+, H_0) = \phi_{\chi,k}\otimes e_{-l\tilde{\gamma}_1^{M_1}+\nu\tilde{\gamma}_2}, \tilde{\omega}^{(3,+)}_{E_0,\nu}(P_+, E_+, E_0) = \phi_{\chi,k}\otimes e_{-l\tilde{\gamma}_1^{M_1}+\nu\tilde{\gamma}_2}$$

To determine these coefficients a_0, b_0 we recall that $\mathfrak{g}/\mathfrak{k} = \mathfrak{m}/\mathfrak{k}^M \oplus \mathfrak{u}_0$ and we have

$$P_+ \wedge E_+ \wedge H_0 + 2P_+ \wedge E_+ \wedge E_0 = E_p \wedge E_{null} \wedge E^*_p$$
$$P_+ \wedge E_+ \wedge H_0 - 2P_+ \wedge E_+ \wedge E_0 = E^*_p \wedge E^*_{null} \wedge E_p$$

and hence $a_0 + 2b_0 = 1, a_0 - 2b_0 = 0$ hence $b_0 = 1/4, a_0 = 1/2$.

We are especially interested in the two elements

$$\Omega^{(3,\chi,o)}_{m+1} = \frac{1}{2}P^\vee_+\wedge E^\vee_+\wedge H^\vee_0\otimes\phi_{\chi,k}\otimes e_{-l\tilde{\gamma}_1^{M_1}-m\tilde{\gamma}_2}+\frac{1}{4}P^\vee_+\wedge E^\vee_+\wedge E^\vee_0\phi_{\chi,k}\otimes e_{-l\tilde{\gamma}_1^{M_1}-m\tilde{\gamma}_2}$$

$$c_\infty(\Omega^{(3,\chi,o)}_{m+1}) = -\frac{1}{2}P^\vee_+\wedge E^\vee_+\wedge H^\vee_0\otimes\phi_{\chi,k}\otimes c_\infty(e_{-l\tilde{\gamma}_1^{M_1}-m\tilde{\gamma}_2})+\frac{1}{4}P^\vee_+\wedge E^\vee_+\wedge E^\vee_0\phi_{\chi,k}\otimes c_\infty(e_{-l\tilde{\gamma}_1^{M_1}-m\tilde{\gamma}_2}),$$

we have the minus sign because c_∞ acts on P_+, H_0 trivially and on $E_+, E-0$ by -1.

These classes are closed and hence they define cohomology classes. Again we study the action of c_∞ on the complex. The adjoint action on the Lie-algebra sends E_p, E_{null}, E_m to E^*_p, E^*_{null}, E^*_m

$$c_\infty(\Omega^{(3,\chi,o)}_{m+1})(E^*_p, E^*_{null}, E_p) = c_\infty(\Omega^{(3,\chi,o)}_{m+1}(E_p, E_{null}, E^*_p)) = e_{k,\nu+1}\otimes c_\infty(\tilde{e}_{-l\gamma_1^{M_1}-\nu\tilde{\gamma}_2})+\ldots$$

The element c_∞ induces the automorphism -1 on Z_c and hence we see that

$$c_\infty(\tilde{e}_{-l\gamma_1^{M_1}-\nu\tilde{\gamma}_2}) = i^{w_\nu}\tilde{e}_{-l\gamma_1^{M_1}+\nu\tilde{\gamma}_2}$$

the unit in front depends on the choice of the basis vectors.

We know already that c_∞ acts by the eigenvalue $(-1)^m$ on the cohomology

$$H^3(\mathfrak{g}, K_\infty, \mathrm{Ind}_P^G\mathcal{D}_{\chi_\mu}\otimes\mathcal{M}_\lambda) = H^3(\mathfrak{m}/\mathfrak{k}^M\oplus\mathfrak{u}_0, K^M_\infty, \mathcal{D}_{\chi_\mu}\otimes\mathcal{M}_\lambda) = F[\omega^{(3,+)}_{E_0,-m}]$$

Hence we conclude that $\Omega^{(3,\chi,o)}_{m+1} + (-1)^m c_\infty(\Omega^{(3,\chi,o)}_{m+1})$ provides a non zero class in the H^3 we will compute its value on $[\omega^{(3,+)}_{E_0,-m}]$, i.e. we will compute

$$[\Omega^{(3,\chi,o)}_{m+1} + (-1)^m c_\infty(\Omega^{(3,\chi,o)}_{m+1})] = a(k,m)[\omega^{(3,+)}_{E_0,-m}]$$

Then $\Omega^{(3,\chi,o)}_{m+1} - (-1)^m c_\infty(\Omega^{(3,\chi,o)}_{m+1})$ yields the trivial class and hence we find a $\Psi \in \mathrm{Hom}_{K_\infty}(\Lambda^2(\mathfrak{g}/\mathfrak{k}), \mathrm{Ind}_P^G \mathcal{D}_{\chi_\mu} \otimes \mathcal{M}_{\lambda,F})$ which satisfies

$$d\Psi = \Omega^{(3,\chi,o)}_{m+1} - (-1)^m c_\infty(\Omega^{(3,\chi,o)}_{m+1})$$

If we apply the intertwining operator

$$T^\bullet : \mathrm{Hom}_{K_\infty}(\Lambda^2(\mathfrak{g}/\mathfrak{k}), \mathrm{Ind}_P^G \mathcal{D}_{\chi_\mu} \otimes \mathcal{M}_{\lambda,F}) \to \mathrm{Hom}_{K_\infty}(\Lambda^2(\mathfrak{g}/\mathfrak{k}), \mathrm{Ind}_P^G \mathcal{D}_{\chi_{\mu'}} \otimes \mathcal{M}_{\lambda,F})$$

and then $T^\bullet(\Psi)$ will be a closed form. We get a cohomology class and we can compute the class

$$[T^\bullet(\Psi)] = \kappa([\Omega^{(3,\chi,o)}_{m+1} - (-1)^m c_\infty(\Omega^{(3,\chi,o)}_{m+1})]) = b(k,m)[\omega^{(2,+)}_m]$$

This is now hopefully a routine calculation.

We find that

$$\Omega^{(3,\chi,o)}_{m+1} \pm (-1)^m c_\infty(\Omega^{(3,\chi,o)}_{m+1}) =$$

$$\frac{1}{2}(P_+^\vee \wedge E_+^\vee \wedge H_0^\vee \otimes \phi_{\chi,k} \otimes (e_{-l\tilde{\gamma}_1^{M_1} - m\tilde{\gamma}_2} \pm (-1)^{m+1} c_\infty(e_{-l\tilde{\gamma}_1^{M_1} - m\tilde{\gamma}_2}))$$
$$+\frac{1}{4} P_+^\vee \wedge E_+^\vee \wedge E_0^\vee \phi_{\chi,k} \otimes (e_{-l\tilde{\gamma}_1^{M_1} - m\tilde{\gamma}_2} \pm (-1)^m c_\infty(e_{-l\tilde{\gamma}_1^{M_1} - m\tilde{\gamma}_2}))$$

We have identified $\mathrm{Sl}_2 \xrightarrow{\sim} M_1'$ such that

$$Z_0 \xrightarrow{\sim} \{\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}\};\ E_0 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and an isomorphism of M_1'-modules

$$\iota : \mathcal{N}_+ \otimes_{\mathcal{O}_F} F \xrightarrow{\sim} \mathrm{Sym}^{2m}(\mathcal{O}_F X \oplus \mathcal{O}_F Y) \otimes_{\mathcal{O}_F} F = \{\sum a_\mu X^{2m-\mu} Y^\mu | a_\mu \in \mathcal{O}_F\} \otimes F$$

We want to stress that both modules sides have a given structure as an $\mathcal{O}_F$ module and have a highest weight vector $e_{l\tilde{\gamma}_1^M + m\gamma_2}, X^{2m}$ respectively. These highest weight vectors are unique up to a power of i, we make a choice for the first one. Then we require that $\iota : e_{l\tilde{\gamma}_1^M + m\gamma_2} \mapsto X^{2m}$.

Under this isomorphism $\iota : e_{l\tilde{\gamma}_1^M + \nu\gamma_2} \mapsto X^{m-\nu} Y^{m+\nu}$ we use this to normalize the generators.

We define two new variables

$$U = \frac{1}{\sqrt{2}}(X + iY);\ V = \frac{1}{\sqrt{2}}(X - iY)$$

Clearly these variables are eigenfunctions under the action of Z_c. We have

$$zU = (x + iy)U, zV = (x - iy)V.$$

We have to consider the two expansions

$$U^{2m} + V^{2m} = \frac{1}{2^m}((X + iY)^{2m} + (X - iY)^{2m}) =$$

$$\frac{1}{2^{m-1}}(X^{2m}-\binom{2m}{2}X^{2m-2}Y^2+\binom{2m}{4}X^{2m-4}Y^4\cdots+(-1)^mY^{2m})$$

and

$$U^{2m}-V^{2m}=\frac{1}{2^m}((X+iY)^{2m}-(X-iY)^{2m})=$$

$$\frac{i}{2^{m-1}}(\binom{2m}{1}X^{2m-1}Y-\binom{2m}{3}X^{2m-3}Y^3\cdots+(-1)^m\binom{2m}{2m-1}XY^{2m-1})$$

If we now write these forms in terms of our description in formula 3 , i.e if we write

$$\Omega^{(3,\chi,o)}_{m+1}\pm(-1)^m c_\infty(\Omega^{(3,\chi,o)}_{m+1})=\{b_{-m},\dots,b_\nu,\dots,b_m;c_{-m},\dots,c_\nu,\dots,c_m\}$$

then the b_ν, c_ν will be the binomial coefficients occurring in these polynomials (in opposite order)

$$\Omega^{(3,\chi,o)}_{m+1}+(-1)^m c_\infty(\Omega^{(3,\chi,o)}_{m+1})=\frac{1}{2^{m-1}}\{0,(-1)^{m-1}\frac{i}{2}\binom{2m}{1},0,-\dots;\frac{(-1)^m}{4},0,\frac{(-1)^{m-1}}{4}\binom{2m}{2},0,\dots,\frac{1}{4}\}$$

Then non trivial cohomology class is given by the first entry after the semicolon, i.e.

$$\Omega^{(3,\chi,o)}_{m+1}+(-1)^m c_\infty(\Omega^{(3,\chi,o)}_{m+1})=\frac{(-1)^m}{2^{m+1}}[\omega^{3,+}_{E_0,-m}].$$

If we now consider the -1 class then

$$\Omega^{(3,\chi,o)}_{m+1}-(-1)^m c_\infty(\Omega^{(3,\chi,o)}_{m+1})=\frac{1}{2^m}\{(-1)^m,0,(-1)^{m-1}\binom{2m}{2},0,\dots,1;0,\dots\}$$

The first entry after the ; is zero hence this is the trivial class, we can bound it. Already the first half of the array (up to the ;) determines the bounding form we get (recall $s=m+2$)

$$\frac{1}{2^m}d(\frac{(-1)^m}{2\cdot 1},0,-\frac{(-1)^{m-1}}{2\cdot 3}\binom{2m}{2},0,\frac{(-1)^{m-2}}{2\cdot 5}\binom{2m}{4},\dots,\frac{1}{2(2m+1)})=$$

$$\frac{1}{2^m}\{(-1)^m,0,(-1)^{m-1}\binom{2m}{2},0,\dots,1\}$$

This corresponds to the polynomial

$$\frac{1}{2^m}(\frac{(-1)^m}{2\cdot 1}Y^{2m}+\frac{(-1)^{m-1}}{2\cdot 3}\binom{2m}{2}Y^{2m-2}X^2+\frac{(-1)^{m-2}}{2\cdot 5}\binom{2m}{4}Y^{2m-4}X^4+\dots+\frac{1}{2(2m+1)}X^{2m})$$

We have to compute the effect of $t^\bullet$ on this class, to do so we have to rewrite this in the U,V variables. We have $X=\frac{U+V}{\sqrt{2}}, Y=\frac{U-V}{\sqrt{2}i}$. Now a miracle occurs: Substituting this we get

$$\frac{1}{2m+1}(U^{2m}-U^{2m-1}V+U^{2m-2}V^2-\dots+(-1)^mV^{2m})$$

The analysis of the differential operators in $\mathrm{Ind}_P^G\mathcal{D}_{\chi_\mu}$ yields that t^2 maps this to

$$\frac{1}{\binom{2m}{m}}\frac{1}{2m+1}(U^{2m}+\binom{2m}{2m-1}U^{2m-1}V+\binom{2m}{2m-2}U^{2m-2}V^2-\cdots+V^{2m})=$$

$$\frac{1}{\binom{2m}{m}}\frac{1}{2m+1}(U+V)^{2m}=\frac{(-1)^m 2^m}{\binom{2m}{m}}\frac{1}{2m+1}X^{2m}.$$

This yields

$$2^m[\Omega^{(3,\chi,o)}_{m+1}+(-1)^m c_\infty(\Omega^{(3,\chi,o)}_{m+1})]=\frac{(-1)^m}{2}[\omega^{3,+}_{E_0,-m}]. \tag{4}$$

$$\kappa_T(2^m[\Omega^{(3,\chi,o)}_{m+1}-(-1)^m c_\infty(\Omega^{(3,\chi,o)}_{m+1})])=\frac{2^{2m}}{\binom{2m}{m}}\frac{1}{2m+1}[\omega^{(2,+)}_m] \tag{5}$$

2.4 The other intertwining operator

We claimed the existence of an intertwining operator $T(\chi,\chi'):\mathrm{Ind}_P^G\mathcal{D}_{\chi_\mu}\to\mathrm{Ind}_P^G\mathcal{D}_{\chi_{\mu'}}$ (see (1)) without any justification. It was normalized to be the identity on $V_{k,0}$. We recall the standard construction of such an operator. At this moment we assume that $\chi=(s,k),\chi'=(3-s,k)$. Then we can write down an intertwining operator which is given by an integral

$$T^{\mathrm{int}}(\chi,\chi'):f\mapsto\int f(gu)du \tag{6}$$

and it is well known that this operator restricted to $V_{k,0}$ is a scalar and this scalar is

$$\frac{\Gamma(n_1+n_2+2+s)/(2\pi)^{n_1+n_2+2+s}}{\Gamma(n_1+n_2+3+s)/(2\pi)^{n_1+n_2+3+s}}\frac{\Gamma(m+1/2+s)/\pi^{m+1/2+s}}{\Gamma(m+1+s)/\pi^{m+1+s}}=\frac{2\pi^{3/2}}{n_1+n_2+2}\frac{\Gamma(m+1/2+s)}{\Gamma(m+1+s)}$$

We evaluate at $s=0$ and get

$$\frac{2\pi^2}{n_1+n_2+2}\frac{\binom{2m}{m}}{2^{2m}}$$

This yields

$$\kappa_{T^{\mathrm{int}}}(2^m[\Omega^{(3,\chi,o)}_{m+1}-(-1)^m c_\infty(\Omega^{(3,\chi,o)}_{m+1})])=\frac{1}{2m+1}\frac{2\pi^2}{n_1+n_2+2}[\omega^{(2,+)}_m]$$

2.5 The mixed Anderson motives.(sketch)

In my lecture Notes volume "Eisensteinkohomologie....." I discuss the construction of certain mixed Tate motives which are visible in the cohomology of the Shimura variety attached to $\mathrm{Gl}_2/\mathbb{Z}$. I computed the Hodge-de Rham extension class of these motives. (It is given by the formula in the middle of p.138, actually there the exposition and the computation is much too complicated an improved version will be contained in the final version of this paper.).

In my manuscript [M-M] I discuss the analogous construction of mixed Tate motives obtained from the symplectic group $\mathrm{GSp}_2/\mathrm{Spec}(\mathbb{Z})$. These mixed Tate-motives are labelled by classical modular forms f of weight $k = 4 + 2n_2 + n_1$ They are also discussed in 1-2-3. They also depend on the choice of an auxiliary prime p_0. (This aspect is not mentioned in [1-2-3]) Such a motive is an a certain sense an exact sequence

$$0 \to \mathbb{Z}(-1-n_2) \to \mathcal{X}_{p_0}(f) \to \mathbb{Z}(-2-n_1-n_2) \to 0.$$

We see that the same modular form f can give rise to several mixed motives, these motives are labelled by f, n_1, p_0.

For the moment we assume that f is unramified everywhere hence it corresponds to an embedding of $\pi_\infty \times \prod_p \sigma_p \hookrightarrow \mathcal{A}_0(\mathrm{Gl}_2(\mathbb{Q})\backslash\mathrm{Gl}_2(\mathbb{A}))$, where π_∞ is the discrete series representation and where σ_p is an unramified principal series for all p.

For our construction we have to choose an auxiliary prime p_0. We choose $\psi_f = \prod_{p\neq p_0} \psi_p \times \psi_{p_0}$ where for $p \neq p_0$ ψ_p is the normalized spherical function and where ψ_{p_0} is only invariant under a suitable parahori subgroup and satisfies a certain support condition. (This support condition corresponds to a certain sheaf with support conditions and our mixed Tate motive is visible in the cohomology of this sheaf.)

The computation of the Hodge-de-Rham extension class (See [M-I], [M-M]) comes down to the evaluation of an integral

$$\int_{\mathfrak{z}(f)} \mathrm{Eis}((\Omega_{m+1}^{(3,\chi,o)} - (-1)^m c_\infty(\Omega_{m+1}^{(3,\chi,o)})) \times \psi_f) \tag{7}$$

where $\mathfrak{z}(f)$ is a certain relative cycle which depends on the modular form f.(For more details I refer to [M-M], 2.2, the existence of this cycle follows from the known results on Eisenstein cohomology and the results of Lan and Suh on the vanishing of torsion. This has to be discussed in more detail. It seems that we have to assume that we may have to invert the primes $\ell < \lambda$ in some sense. On the other hand the results in [H-p] assert show that under certain conditions we can interpolate the ordinary part of the torsion.)

We apply Stokes theorem and get for the integral

$$\frac{1}{2m+1}\frac{2\pi^2}{n_1+n_2+2}\frac{L(f,n_1+n_2+2)}{L(f,n_1+n_2+3)}\frac{\zeta(n_1+1)}{\zeta(n_1+2)}L(p_0)T_{p_0}(\psi_{p_0})(e)\int_{\partial(\mathfrak{z})}[\omega_m^{(2,+)}] \times \psi_f \tag{8}$$

Here $L(p_0)$ is the product of the inverses of the local Euler factors at p_0 . The factor $T_{p_0}(\psi_{p_0})(e)$ is the evaluation of the local intertwining operator. Let us abbreviate this product by $C(\sigma_{p_0}, \lambda)$.(See [H-E], p. 127 and 139). We rewrite the first ratio in terms of the $\Lambda(f, *)$ and get

$$\frac{\pi}{2m+1}\frac{\Lambda(f,n_1+n_2+2)}{\Lambda(f,n_1+n_2+3)}\frac{\zeta(n_1+1)}{\zeta(n_1+2)}C(\sigma_{p_0},\lambda)\int_{\partial(\mathfrak{z})}[\omega_m^{(2,+)}] \times \psi_f \tag{9}$$

We have a look at the last factor. This factor is equal to the period $\Omega(\sigma_f)^\pm$ defined in my paper [H-A], it compares the $\pm$ parts provided by σ_f in the cohomology of the boundary. We get for our integral

$$\frac{1}{\Omega(\sigma_f)^{\epsilon(k,m)}}\frac{\pi}{2m+1}\frac{\Lambda(f,n_1+n_2+2)}{\Lambda(f,n_1+n_2+3)}\frac{\zeta(n_1+1)}{\zeta(n_1+2)}C(\sigma_{p_0},\lambda)(\psi_{p_0})(e) \tag{10}$$

Finally we look at the ratio of the two ζ values.

We apply the functional equation and get for $n_1>0$ and even

$$\frac{\zeta(n_1+1)}{\zeta(n_1+2)}=-\frac{n_1+1}{\pi^2}\frac{\zeta'(-n_1)}{\zeta(-1-n_1)}$$

Recall that $2m=n_1$, we get the final formula for the Hodge-de Rham extension class

$$\mathcal{X}(f)_{H-dRh}=-C(\sigma_{p_0},\lambda)\Big(\frac{1}{\Omega(\sigma_f)^{\epsilon(k,m)}}\frac{\Lambda(f,n_1+n_2+2)}{\Lambda(f,n_1+n_2+3)}\Big)\frac{1}{\zeta(-1-n_1)}\frac{\zeta'(-n_1)}{\pi} \tag{11}$$

I have not yet carried out the computation of $C(\sigma_{p_0},\lambda)$ but in analogy to the computation in [H-E] 4.3.3 one might guess that up to a power of p_0 it is equal to the inverse of the local Euler factor at p_0 in the ratio of L-values. If a_{p_0} is the p_0-th Fourier coefficient, i.e. the eigenvalue of T_{p_0} then $a_{p_0}=\alpha_{p_0}+\beta_{p_0}, \alpha_{p_0}\beta_{p_0}=p^{k-1}$ and we should have

$$\begin{aligned}C(\sigma_{p_0},\lambda)=\frac{(1-\alpha_{p_0}p_0^{-n_1-n_2-2})(1-\beta_{p_0}p_0^{-n_1-n_2-2})}{(1-\alpha_{p_0}p_0^{-n_1-n_2-3})(1-\beta_{p_0}p_0^{-n_1-n_2-3})}\frac{1}{p_0}\frac{1-p_0^{-n_1-1}}{1-p_0^{-n_1-2}}=\\ \frac{1-a_{p_0}p_0^{-n_1-n_2-2}+p_0^{-n_1-1}}{1-a_{p_0}p_0^{-n_1-n_2-3}+p_0^{-n_1-3}}\frac{1}{p_0}\frac{1-p_0^{-n_1-1}}{1-p_0^{-n_1-2}}\end{aligned} \tag{12}$$

2.6 The essence of this note

This is now an almost perfect formula for the Hodge-de-Rham extension class of the mixed motive $\mathcal{X}_{p_0}(f)$. I say "almost" because I lost a factor i somewhere, I have to look for the lost i. I have not given the explicit form of the factor $C(\sigma_{p_0},\lambda)$, it has to be computed by a local computation at p_0 (Comp. [H-E]), this should not become difficult anymore. (See the remark above, here I see some problems related to the ramification of our motive at p_0, comp. [H-E], p.139))

This formula is in many respects better than the formula in 1-2-3 on p. 258. In [1-2-3] is a factor c in front and I assert that this number is a rational number which contains only "small" primes as factors. This number is a product $c=C(\sigma_{p_0},\lambda)c_\infty(\lambda)$ where the first factor is a local contribution at the auxiliary prime p_0 and the second factor a contribution at ∞.

The consequence of our formula above is that $c_\infty(\lambda)$ only depends very mildly on λ it is equal to a sign (or a power of i). The second consequence is that the periods $\Omega_\pm$ disappear they are replaced by the ratio $\Omega(\sigma_f)^{\epsilon(k,m)}$. The period $\Omega(\sigma_f)$ is well defined up to some units in $\mathcal{O}[1/N]^\times$. The primes dividing N can be computed from the cohomology of the boundary stratum corresponding to the Siegel parabolic viewed as module under the Hecke algebra. (Compare [p-adic]) The primes not dividing N will be called large.

The formula should allow us to make the conjectures on congruences more precise.

In the case that σ_f is the modular form of weight 22 (as in 1-2-3) the number N should be the product $131 \cdot 593$, this is the numerator of $\zeta(-21)$. We do have congruences between the cusp form and the Eisenstein cohomology but no congruences between cusp forms.

2.7 Arithmetic implications, still very imprecise

If we now believe that there are no exotic mixed Tate motives, then we can give a much more precise formulation of the conjecture on congruences in 1-2-3. In 1-2-3 we assume that the prime ℓ should be large without specifying what that means.

Here we consider primes ℓ which are prime to N and we look at the power of ℓ (or perhaps better $\mathfrak{l}$) occurring in

$$(\frac{1}{\Omega(\sigma_f)^{\epsilon(k,m)}}\frac{\Lambda(f,n_1+n_2+2)}{\Lambda(f,n_1+n_2+3)}) = \mathfrak{l}^{\delta(n_1,n_2,k)}\dots .$$

Assume that $\delta(n_1,n_2,k) < 0$, i.e. $\mathfrak{l}$ occurs in the denominator, then we expect a congruence.

More precisely: If in addition the following three assumptions are satisfied

(0) The factor $C(p_0,\lambda)$ is coprime to $\mathfrak{l}$,

(1) The modular form σ_f is ordinary at $\mathfrak{l}$,

(2) The Soule element $c_{\mathfrak{l}}(n_1) \in H^1(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell, \mathbb{Z}_\ell(n_1+1))$ is a generator,

then I have an argument which implies (assuming that the mixed Tate motive is not exotic) that we should have a congruence between the Hecke-eigenvalues of a cohomology eigenclass for the Siegel modular group and the eigenvalues a_p for σ_f

$$\lambda(p) \equiv p^{n_2+1} + a_p + p^{n_1+n_2+2} \mod \mathfrak{l}^{-\delta(n_1,n_2,k)}.$$

The assumption (0) is harmless, we still have another variable p_0 and we can try to minimize the $\mathfrak{l}$ contribution. This adds a finite number of primes which should be inverted in the beginning.

The assumptions (1) and (2) have to be verified in any specific case, they are always "probably" true but it is difficult to prove general theorems which assert their validity.

The case of the modular form of weight 22 is already interesting in this respect. Looking at the tables in [1-2-3] we see the relatively small primes 2,3,5,7,13,17,19 occurring in the L-values but none of them is ordinary and hence we should not expect a congruence (but there may still be one). On the other hand the prime 11 is ordinary but it also does not occur in the L-values. We do not have a congruence $\mod 11$.

2.8 p-adic interpolation

An important consequence of of our formula is that the product $C(p_0,\lambda)c_\infty(\lambda)$ is a ℓ-adic meromorphic function in the variable λ. Since we assume that our modular form $\sigma_f(=f)$ should be ℓ-ordinary we can put σ_f into an ℓ-adic family $\tilde{\sigma}_f$. Under certain assumptions we can show that the factor

$$(\frac{1}{\Omega(\sigma_f)^{\epsilon(k,m)}}\frac{\Lambda(f,n_1+n_2+2)}{\Lambda(f,n_1+n_2+3)})$$

also is a ℓ-adic analytic function in λ. (Project with Mahnkopf). If the derivative of this ℓ-adic L-function is not zero then we can produce congruences of arbitrary high order.

2.9 The local factor at p_0

When writing this note I realized that there are interesting aspects of the factor $C(\sigma_{p_0},\lambda)$. To discuss these aspects I go back to the group Gl_2 and look again on the Anderson motives considered in [H-E]. In section 4.3.4 on p. 139 I discuss the corresponding term

$$C(\phi_{p_0},\lambda)=\frac{p_0^{n+1}-1}{p_0^{n+2}-1}$$

where ϕ is the character on p. 107, we also require $\chi_1=\chi_2=1$. In [H-E] I observe that this factor will introduce denominators for the Eisenstein classes and therefore it should be a source of congruences. The congruences which we considered so far where congruences originating from the divisibility of a L-value by a prime and we found congruences between unramified modular forms(For instance forms on $\mathrm{Gl}_2(\mathbb{Z})$ and $\mathrm{GSp}_2(\mathbb{Z})$.)These congruences are of *global origin.*

If we have a prime ℓ dividing the denominator $p_0^{n+2}-1$ then we expect a congruence between the Eisenstein class $\mathrm{Eis}(\phi)$ which provides a cohomology class for $\mathrm{Sl}_2(\mathbb{Z})$ and a cohomology class for $\Gamma(p_0)$, the cohomology class is ramified at p_0 and corresponds to a new form.

These congruences should be known. I give an example. We consider the case $n=8$ so we are talking about modular forms of weight 10. We take $p_0=3$. Then

$$3^{10}-1=2^3\cdot 11^2\cdot 61$$

The prime factors 3 and 11 are of "von Staudt-Clausen" type, they occur in the denominator of $\zeta(-9)$ and hence cancel. The interesting prime is 61.

From W. Stein's tables we get that there is a modular form of weight 10 for $\Gamma_0(3)$ and its eigenvalues a_p for $T_p, p=2,3,5,7,\dots,37$ are

$$\{-36,-81,-1314,-4480,1476,-151522,108162,593084,-969480,-6642522,7070600,-7472410\}$$

One checks easily that for $p=2,5,7,\dots,37$ we have the Ramanujan type congruence

$$a_p\equiv p^9+1 \mod 61$$

This is a congruence of *local origin.*

Of course we expect a similar thing for the pair of groups $\mathrm{GSp}_2(\mathbb{Z}), \mathrm{Gl}_2(\mathbb{Z})$. For a prime p_0 we define $\Gamma_0(p_0) \subset \mathrm{GSp}_2(\mathbb{Z})$ as the inverse image of $P(\mathbb{F}_{p_0})$ where P is of course the Siegel parabolic.

Let us start from Δ. We compute the denominator of $C(\sigma_{p_0})$ we look only at the local factor of the L- factor for Δ. Now we consider integers $n_1, n_2,\ n_1$ even which satisfy $n_1 + 2n_2 = 8$ these are $\{0,4\},\{2,3\},\{4,2\},\{6,1\},\{8,0\}$ perhaps we should stick to the regular weights and leave out the two extremal ones. We factorize

$$1 - a_{p_0} p_0^{-n_1-n_2-3} + p_0^{-n_1-3}$$

and look for "large" prime factors. For $p_0 = 2,3$ we do not find large prime factors. For $p_0 = 5$ we find a 61 for $\{2,3\}$ a 41 for $\{4,2\}$ and a 83 for $\{6,1\}$. Hence we should find Siegel modular eigenforms of weights $\{2,6\},\{4,5\},\{6,4\}$ with systems of eigenvalues $\lambda_1(p), \lambda_2(p), \lambda_3(p)$ respectively and look for congruences of local origin

$$\begin{aligned} \lambda_1(p) &= p^4 + a_p + p^7 \mod 61 \\ \lambda_2(p) &= p^3 + a_p + p^8 \mod 41 \\ \lambda_3(p) &= p^2 + a_p + p^9 \mod 83 \end{aligned} \tag{13}$$

for $p \neq 5$

I think it is better to stick to $p_0 = 2$ and go up with the weight because in this case there some data are available (See [B-F-G]). For weight 16 we get the pairs $\{2,5\},\{4,4\},\{6,3\},\{8,2\},\{10,1\}$ and find a 19 for $\{8,2\}$ hence we may try to find a congruence $\mod 19$ between a Siegel modular form on $\Gamma_0(2)$ of weight $\{8,5\}$ and the modular form of weight 16.

We try the modular form of weight 18 and $p_0 = 2$.

$\{4,5\}$ large prime $\{97\}$

$\{6,4\}$ large prime $\{13\}$

$\{8,3\}$ large prime $\{47\}$

$\{10,2\}$ large prime $\{37\}$

$\{12,1\}$ large prime $\{13\}$

We try the modular form of weight 20 and $p_0 = 2$.

$\{2,7\}$ large prime $\{157\}$

$\{6,5\}$ large prime $\{19?\}$

$\{8,4\}$ large prime $\{499\}$

$\{10,3\}$ large prime $\{113\}$

$\{12,2\}$ large prime $\{311\}$

References

[1-2-3] B-G-H-Z *the 1-2-3 of Modular Forms* Universitext Springer

[B-F-G] Bergström, J., Faber, C., van der Geer: Siegel modular forms of genus 2 and level 2, IMRN, (2008)

[H-E] Harder, G. *Eisensteinkohomologie und die Konstruktion gemischter Motive*, SLN 1562

[H-A] Harder, G.*Arithmetic aspects of rank one Eisenstein cohomology.* In Cycles, motives and Shimura varieties, 131-190, Tata Inst. Fund. Res. Stud. Math., Tata Inst. Fund. Res., Mumbai, 2010.

[H-p] Harder, G. *Interpolating coefficient systems and p-ordinary cohomology of arithmetic groups.* Groups Geom. Dyn. 5 (2011), no. 2, 393-444

[L-S] Kai-Wen Lan and Junecue Suh *Vanishing theorems for torsion automorphic sheaves on general PEL-type Shimura varieties.* submitted

The following manuscripts are available on my home page

`http://www.math.uni-bonn.de/people/harder/Manuscripts`
in the folder /Eisenstein (resp. /buch)

[M-M] *ModularConstruction of mixed motives* (MixMot-2013.pdf)

[H-book] G. Harder, *Cohomology of arithmetic groups,* Book in preparation. Preliminary version available at
`http://www.math.uni-bonn.de/people/harder/Manuscripts/buch`

Günter Harder, Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany. E-mail address: `harder@mpim-bonn.mpg.de`