Atithmetic Aspects of Eisenstein Cohomology

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1 Eisenstein cohomology

Our starting point is a smooth group scheme $\mathcal{G}/\mathrm{Spec}(\mathbb{Z})$ whose generic fiber $G = \mathcal{G}\times_{\mathbb{Z}}\mathbb{Q}$ is reductive and quasisplit. We assume the group scheme is reductive over the largest possible open subset of $\mathrm{Spec}(\mathbb{Z})$ and at the remaining places it is given by a maximal parahoric group scheme structure. If G is split, then we assume that $\mathcal{G}$ is split. We define $K_f = \mathcal{G}(\hat{\mathbb{Z}}) = \prod_p \mathcal{G}(\mathbb{Z}_p) \subset G(\mathbb{A}_f)$

We choose a Borel subgroup $B/\mathbb{Q}$ and a torus $T/\mathbb{Q} \subset B/\mathbb{Q}$. We assume that $T(\mathbb{A}_f) \cap K_f = T/\hat{\mathbb{Z}})$ is maximal compact in $T(\mathbb{A}_f)$. Let $\lambda \in X^*(T)$ be a highest weight, let $\mathcal{M}_\lambda$ be a highest weight module attached to this weight. It is a $\mathbb{Z}$-module, the module $\mathcal{M}_\lambda \otimes \mathbb{Q}$ is a highest weight module for the group $G/\mathbb{Q}$.

1.1 The Borel-Serre compactification

We consider our space

$$\mathcal{S}^G_{K_f} = G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K_f$$

and its Borel-Serre compactification

$$i : \mathcal{S}^G_{K_f} \to \bar{\mathcal{S}}_K.$$

Our highest weight module $\mathcal{M}_\lambda$ provides a sheaf $\tilde{\mathcal{M}}_\lambda$ on these spaces.

We have an isomorphism

$$H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) \xrightarrow{\tilde{}} H^\bullet(\mathcal{S}^{\bar{G}}_{K_f}, \tilde{\mathcal{M}}_\lambda)$$

for any coefficient system $\tilde{\mathcal{M}}_\lambda$ coming from a rational representation $\mathcal{M}$ of $G(\mathbb{Q})$. The boundary $\partial\bar{\mathcal{S}}_K$ is a manifold with corners. It is stratified by submanifolds

$$\partial\bar{\mathcal{S}}_K = \bigcup_P \partial_P \mathcal{S}^G_{K_f},$$

where P runs over the $G(\mathbb{Q})$ conjugacy classes of proper parabolic subgroups defined over $\mathbb{Q}$. We identify the set of conjugacy classes of parabolic subgroups with the set of representatives given by the parabolic subgroups that contain our standard Borel subgroup $B/\mathbb{Q}$. Then we have

$$H^\bullet(\partial_P \mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda) = H^\bullet(P(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K_f, \tilde{\mathcal{M}}_\lambda)$$

We have a finite coset decomposition

$$G(\mathbb{A}_f) = \bigcup_{\xi_f} P(\mathbb{A}_f)\xi_f K_f,$$

for any $\underline{\xi_f}$ put $K^P_f(\xi_f) = P(\mathbb{A})_f \cap \xi_f K_f \xi_f^{-1}$. Then we have

$$P(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K_f = \bigcup_{\xi_f} P(\mathbb{Q})\backslash X \times P(\mathbb{A}_f)/K^P_f(\xi_f)\xi_f,$$

If $R_u(P) \subset P$ is the unipotent radical, then

$$M = P/R_u(P)$$

is a reductive group. For any open compact subgroup $K_f \subset G(\mathbb{A}_f)$(resp. for $K_\infty \subset G_\infty$) we define $K^M_f(\xi_f) \subset M(\mathbb{A}_f)$(resp. $K^M_\infty \subset M_\infty$) to be the image of $K^P(\xi_f)$ in $M(\mathbb{A}_f)$ (resp. M_∞). We put

$$\mathcal{S}^M_{K_f(\xi_f)} = M(\mathbb{Q})\backslash M(\mathbb{A})/K^M_\infty K^M_f(\xi_f)$$

and get a fibration

$$\pi_P : P(\mathbb{Q})\backslash X \times P(\mathbb{A}_f)/K^P_f(\xi_f) \to M(\mathbb{Q})\backslash M(\mathbb{A})/M(\mathbb{Q})\backslash K^M_\infty \times K^M_f(\xi_f)$$

where the fibers are of the form $\Gamma_U\backslash R_u(P)(\mathbb{R})$ and where $\Gamma_U \subset U(\mathbb{Z})$ is of finite index and defined by some congruence condition dictated by $K^P_f(\xi_f)$. The Lie-algebra $\mathfrak{u}$ of $R_u(P)$ is a free $\mathbb{Z}$-module and it is clear that we have an integral version of the van -Est theorem which says:

If $R = \mathbb{Z}[\frac{1}{N}]$ where a suitable set of primes has been inverted then

$$H^\bullet(\Gamma_U\backslash R_u(P)(\mathbb{R}), \tilde{\mathcal{M}}_R) \xrightarrow{\sim} H^\bullet(\mathfrak{u}, \tilde{\mathcal{M}}_R).$$

More precisely we know that the local coefficient system $R^\bullet\pi_{P*}(\tilde{\mathcal{M}})$ is obtained from the rational representation of M on $H^\bullet(\mathfrak{u}, \mathcal{M})$.

Hence we get

$$H^\bullet(\partial_P\mathcal{S},\tilde{\mathcal{M}}_R)=\bigcup_{\xi_f} H^\bullet(\mathcal{S}^M_{K_f^M(\xi_f)},\widetilde{H^\bullet(\mathfrak{u},\mathcal{M})}_R),$$

and

$$H^\bullet(\mathfrak{u},\mathcal{M}_R)=\bigoplus_{w\in W^P} H^{l(w)}(\mathfrak{u},\mathcal{M}_R)(w\cdot\lambda),$$

where W^P is the set of Kostant representatives of W/W^M and where $w\cdot\lambda=(\lambda+\rho)^w-\rho$ and ρ is the half sum of positive roots.

The primes which we have to be inverted should be those which are smaller than the coefficients of the dominant weights in the highest weight of $\mathcal{M}$. But at this point we may have to enlarge the set of small primes.

We conclude

The cohomology of the boundary strata $\partial_P\mathcal{S}^G_{K_f}$ with coefficients in $\mathcal{M}$ can be computed in terms of the cohomology of the reductive quotient, where we have coefficients in the cohomology of the Lie algebra of the unipotent radical with coefficients in $\mathcal{M}$

In the following considerations we sometimes suppress the subscripts $K_f, K^M_{K_f^M}$ and so on. Then we mean that the considerations are valid for a fixed level or that we have taken the limit over the K_f. (See the remarks below concerning induction)

1.1.1 The two spectral sequences

The covering of the boundary by the strata $\partial_P\mathcal{S}$ provides a spectral sequence, which converges to te cohomology of the boundary. We can introduce the simplex Δ of types of parabolic subgroups, the vertices correspond to the maximal ones and the full simplex corresponds to the minimal parabolic. To any type of a parabolic P let $d(P)$ its rank, we make the convention that $d(P)-1$ is equal to the dimension of the corresponding face in the simplex. Let $M=M_P=P/R_u(P)$ be the reductive quotient (the Levi quotient). If $Z_M/\mathbb{Q}$ is the connected component of the identity of the center of $M/\mathbb{Q}$ then $d(P)$ is also the dimension of the maximal split subtorus of $Z_M/\mathbb{Q}$ minus the dimension of the maximal split subtorus of $Z_G/\mathbb{Q}$. The covering yields a spectral sequence whose $E_1^{\bullet,\bullet}$ term together with the differentials of our spectral sequence is given by

$$0\to E_1^{0,q}=\bigoplus_{P,d(P)=1} H^q(\partial_P\mathcal{S},\mathcal{M})\xrightarrow{d_1^{0,q}}\cdots\to\bigoplus_{P,d(P)=p+1} H^q(\partial_P\mathcal{S},\mathcal{M})\xrightarrow{d_1^{p,q}} \quad (1)$$

where the boundary map $d_1^{p,q}$ is obtained from the restriction maps (See [Gln]). There is also a homological spectral sequence which converges to the cohomology of the boundary. It can be written as a spectral sequence for the cohomology with compact supports. Let d be the dimension of $\mathcal{S}$ then we have a complex

$$\to\bigoplus_{P,d(P)=p+1} H_c^{d-1-p-q-1}(\partial_P\mathcal{S},\mathcal{M})\xrightarrow{\delta_1}\bigoplus_{P,d(P)=p} H_c^{d-1-p-q}(\partial_P\mathcal{S},\mathcal{M})\to \quad (2)$$

and therefore the $E^1_{\bullet,\bullet}$ term is

$$E^1_{p,q} = \bigoplus_{P, d(P)=p} H_c^{d-1-p-q}(\partial_P \mathcal{S}, \mathcal{M})$$

the (higher) differential go from (p,q) to $(p-r, q+1-r)$.

1.1.2 Induction

The description of the cohomology of a boundary stratum is a little bit clumsy, since we are working with the coset decomposition. The reason is that we are working on a fixed level, if we consider cohomology with integral coefficients. If we have rational coefficients then we can pass to the limit. Then

$$H^\bullet(\partial_P \mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}) = \lim_{K_f} H^\bullet(P(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K_f, \tilde{\mathcal{M}}) =$$

$$\mathrm{Ind}_{\pi_0(M(\mathbb{R})\times P(\mathbb{A}_f)}^{\pi_0(G(\mathbb{R})\times G(\mathbb{A}_f)} \lim_{K_f^M} H^\bullet(\mathcal{S}^M_{K_f^M}, \widetilde{H^\bullet(\mathfrak{u}, \mathcal{M})}) = \mathrm{Ind}_{\pi_0(M(\mathbb{R}))\times P(\mathbb{A}_f)}^{\pi_0(G(\mathbb{R}))\times G(\mathbb{A}_f)} H^\bullet(\mathcal{S}^M, \widetilde{H^\bullet(\mathfrak{u}, \mathcal{M})}),$$

where the induction is ordinary group theoretic induction. We should keep in our mind that the $\pi_0(M(\mathbb{R})) \times P(\mathbb{A}_f)$ -modules are in fact $\pi_0(M(\mathbb{R})) \times M(\mathbb{A}_f)$-modules. We need some simplification in the notation and we will write for any such $\pi_0(M(\mathbb{R})) \times M(\mathbb{A}_f)$ module H

$$\mathrm{Ind}_{\pi_0(M(\mathbb{R}))\times P(\mathbb{A}_f)}^{\pi_0(G(\mathbb{R}))\times G(\mathbb{A}_f)} H = I_M^G H$$

We will use the same notation for an induction from the torus T to M.

Under certain conditions we also have the notion of induction for Hecke - modules and we can work with integral coefficient systems. This will be discussed at another occasion.

But I want to mention that in the case that K_f is a hyperspecial maximal compact subgroup (in the cases where we are dealing with a split semisimple group scheme over $\mathrm{Spec}(\mathbb{Z})$ we can take $K_f = \prod \mathcal{G}(\mathbb{Z}_p)$ (see 1.1)) then $G(\mathbb{Q}_p) = P(\mathbb{Z}_p)K_p = B(\mathbb{Z}_p)K_p$ the group theoretic induction followed by taking K_f invariants gives back the original module. In this case we do not have to induce!

Of course we have to understand the coefficient systems $H^\bullet(\mathfrak{u}, \mathcal{M})$, for this we need the theorem of Kostant which will be discussed in the next section.

1.1.3 A review of Kostants theorem

At this point we can make the assumption that our group $G/\mathbb{Q}$ is quasisplit, we also assume that $G^{(1)}/\mathbb{Q}$ is simply connected. Then we may assume that $\mathcal{M}_\mathbb{Z}$ is irreducible and of highest weight λ. Let $B/\mathbb{Q}$ be a Borel subgroup, we choose a torus $T/\mathbb{Q} \subset B/\mathbb{Q}$. Let $X^*(T) = \mathrm{Hom}(T \times_\mathbb{Q} \bar{\mathbb{Q}}, \mathbb{G}_m \times_\mathbb{Q} \bar{\mathbb{Q}}$ be the character module, it comes with an action of a finite Galois group $\mathrm{Gal}(F/\mathbb{Q})$, here F is the smallest sub field of $\bar{\mathbb{Q}}$ over which $G/\mathbb{Q}$ splits. Let $T^{(1)}/\mathbb{Q} \subset T/\mathbb{Q}$ the maximal torus in $G^{(1)}/\mathbb{Q}$, then $X^*(T^{(1)})$ contains the set Δ of roots, the subset Δ^+ of positive roots (with respect to B.) The set of simple roots is identified to a finite index set $I = \{1, 2, \ldots, r\}$, i.e we write the set of simple roots as $\pi = \{\alpha_1, \ldots, \alpha_i, \ldots, \alpha_r\} \subset \Delta^+$. We assume that the numeration is somehow adapted the Dynkin diagram. The finite Galois group $\mathrm{Gal}(F/\mathbb{Q})$ acts on I and π by permutations. Attached to the simple roots we have the dominant fundamental weights $\{, \ldots, \gamma_i, \ldots, \gamma_j, \ldots\}$ they are related to the simple roots by the rule

$$2\frac{<\gamma_i, \beta_j>}{<\beta_j, \beta_j>} = \delta_{i,j}.$$

The dominant fundamental weights form a basis of $X^*(T^{(1)})$.

Our maximal torus $T/\mathbb{Q}$ is up to isogeny the product of $T^{(1)}$ and the central torus $C/\mathbb{Q}$, i.e. $T = T^{(1)} \cdot C$ and the restriction of characters yields an injection

$$j : X^*(T) \to X^*(T^{(1)}) \oplus X^*(C),$$

this becomes an isomorphism if we tensorize by the rationals

$$X^*_{\mathbb{Q}}(T) = X^*(T) \otimes \mathbb{Q} \xrightarrow{\sim} X^*_{\mathbb{Q}}(T^{(1)}) \oplus X^*_{\mathbb{Q}}(C).$$

This isomorphism gives us canonical lifts of elements in $X^*(T^{(1)})$ or $X^*(C)$ to elements in $X^*_{\mathbb{Q}}(T)$ which will be denoted by the same letter. Especially the fundamental weights $\gamma_i, \ldots$ are elements in $X^*_{\mathbb{Q}}(T)$.

Let $\lambda \in X^*(T)$ be a dominant weight, our decomposition allows us to write it as

$$\lambda = \sum_{i \in I} n_i \gamma_i + \delta = \lambda^{(1)} + \delta$$

we have $n_i \in \mathbb{Z}, n_i \geq 0$ and $\delta \in X^*(C)$. To such a dominant weight λ we have an absolutely irreducible $G \times F$ -module $\mathcal{M}_\lambda$.

We consider maximal parabolic subgroups $P/\mathbb{Q} \supset B/\mathbb{Q}$. These parabolic subgroups are given by the choice of a $\mathrm{Gal}(F/\mathbb{Q})$ orbit $\tilde{i} = J \subset I$ Such an orbit yields a character $\gamma_J = \sum_{i \in J} \gamma_i$ The parabolic subgroup $P/\mathbb{Q}$ provided by this datum is determined by its root system $\Delta^P = \{\beta \in \Delta | <\beta, \gamma_J> \geq 0\}$. The choice of the maximal torus $T \subset P$ also provides a Levi subgroup $M \subset P$ but actually it is better to consider M as the quotient P/U_P.

The set of simple roots of $M^{(1)}$ is the subset $\pi_M = \{\ldots, \alpha_i, \ldots\}_{i \in I_M}$, where of course $I_M = I \setminus J$. We also consider the group $G^{(1)} \cap M = M_1$. It is a reductive group, it has $T^{(1)}$ as its maximal torus. We apply our previous considerations to this group M_1. It has a non trivial central torus $C_1/\mathbb{Q}$. This torus has a simple description, we pick a root $\alpha_i, i \in J$, we know that J is an orbit under $\mathrm{Gal}(F/\mathbb{Q})$. We have the subfield $F_{\alpha_i} \subset F$ such that $\mathrm{Gal}(F/F_{\alpha_i})$ is the stabilizer of α_i. Then it is clear that

$$C_1 \xrightarrow{\sim} R_{F_{\alpha_i}/\mathbb{Q}}(\mathbb{G}_m/F_{\alpha_i}),$$

up to isogeny it is a product of an anisotropic torus $C_1^{(1)}/\mathbb{Q}$ and a copy of $\mathbb{G}_m$. The character module $X^*_{\mathbb{Q}}(C_1)$ is a direct sum

$$X^*_{\mathbb{Q}}(C_1) = X^*_{\mathbb{Q}}(C_1^{(1)}) \oplus \mathbb{Q}\gamma_J. \tag{3}$$

Here $X^*_{\mathbb{Q}}(C_1^{(1)}) = \{\gamma \in X^*_{\mathbb{Q}}(C_1) \mid \ <\gamma, \sum_{i\in J}\alpha_i >= 0\}$. The half sum of positive roots in the unipotent radical is

$$\rho_U = f_P\gamma_J \tag{4}$$

where $2f_P > 0$ is an integer.

We also have the semi simple part $T^{(1,M)} \subset M^{(1)}$ and again we get the orthogonal decomposition

$$X^*_{\mathbb{Q}}(T^{(1)}) = X^*_{\mathbb{Q}}(T^{(1,M)}) \oplus X^*_{\mathbb{Q}}(C_1) = \bigoplus_{i\in I_M}\mathbb{Q}\alpha_i \oplus \bigoplus_{i\in J}\mathbb{Q}\gamma_i = \bigoplus_{i\in I_M}\mathbb{Q}\gamma_i^M \oplus \bigoplus_{i\in J}\mathbb{Q}\gamma_i.$$

Here we have to observe that the $\gamma_i^M, i \in I_M$ are the dominant fundamental weights for the group $M^{(1)}$, they are the orthogonal projections of the γ_i to the first summand in the above decomposition. We have a relation

$$\gamma_j = \gamma_j^M + \sum_{i\in\tilde{i}} c(j,i)\gamma_i, \text{ for } j \in I_M$$

and we have $c(j,i) \geq 0$ for all $i \in J$.

Let W be absolute Weylgroup and subgroup $W_M \subset W$ the Weyl group of M. For the quotient $W_M\backslash W$ we have a canonical system of representatives

$$W^P = \{w \in W \mid w^{-1}(\pi_M) \subset \Delta^+\}.$$

To any $w \in W$ we define $w\cdot\lambda = w(\lambda+\rho)-\rho$ where ρ us the half sum of positive roots. If we do this with an element $w \in W^P$ then $\mu = w\cdot\lambda$ is a highest weight for $M^{(1)}$ and $w\cdot\lambda$ defines us a module for M. Then Kostants theorem says

$$H^\bullet(\mathfrak{u}_P, \mathcal{M}_\lambda) = \bigoplus_{w\in W^P} H^{\ell(w)}(\mathfrak{u}_P, \mathcal{M})(w\cdot\lambda),$$

the summands on the right hand side are the irreducible modules attached to $w\cdot\lambda$, they sit in degree

$$l(w) = \#\{\alpha \in \Delta^+ | w^{-1}\alpha \in \Delta^-\} \tag{5}$$

Each isomorphism class occurs only once.

We write

$$\begin{array}{rccc} w\cdot(\lambda+\rho)-\rho = & \underbrace{\mu^{(1,M)}+\delta_1} & & +\delta \\ & \in X^*_{\mathbb{Q}}(T^{(1,M)}) \oplus X^*_{\mathbb{Q}}(C_1) & & \oplus X^*(C) \end{array} \tag{6}$$

We decompose δ_1 and define the numbers $a(w,\lambda)$ (see (3))

$$\delta_1 = \delta_1' - (a(w,\lambda)+f_P)\gamma_J.$$

In (6) we move the $-\rho = -\rho_G$ to the right and split it into $\rho = \rho_M + f_P\gamma_J$ then we get

$$w(\lambda+\rho) = \mu^{(1,M)} + \rho_M - a(w,\lambda)\gamma_J \tag{7}$$

We also consider the extended Weyl group $\tilde{W}$, this is the group of automorphisms of the root system. Let $w_0 \in W$ be the element sending all positive roots into negative ones. We have an automorphism $\Theta_- \in \tilde{W}$ inducing $t \mapsto t^{-1}$ on the torus. Let $\Theta = w_0 \circ \Theta_-$. This element induces a permutation on the set π of positive roots, which may be the identity and induces -1 on the determinant. Then

$$\Theta\lambda = \sum_{i \in I} n_{\Theta i}\gamma_i - \delta$$

is a dominant weight and the resulting highest weight module is dual module to $\mathcal{M}_\lambda$. Therefore we get a non degenerate pairing

$$H^\bullet(\mathfrak{u}_P, \mathcal{M}_\lambda) \times H^\bullet(\mathfrak{u}_P, \mathcal{M}_{\Theta\lambda}) \to H^{d_{U_P}}(\mathfrak{u}_P, F) = F(-2\rho_U),$$

which respects the decomposition, i.e. we get a bijection $w \mapsto w'$ such that $l(w) + l(w') = d_{U_P}$ and such

$$H^{l(w)}(\mathfrak{u}_P, \mathcal{M}_\lambda)(w \cdot \lambda) \times H^{l(w')}(\mathfrak{u}_P, \mathcal{M}_{\Theta\lambda})(w' \cdot \Theta\lambda) \to H^{d_{U_P}}(\mathfrak{u}_P, F) \qquad (8)$$

is non degenerate. We conclude

$$a(w, \lambda) + a(w', \Theta\lambda) = 0.$$

The element Θ conjugates the parabolic subgroup P into the parabolic subgroup Q, which may be equal to P or not. If $P = Q$ resp. $P \neq Q$ then we say that P is (resp. not) conjugate to its opposite parabolic. If Θ_- is in the Weyl group then all parabolic subgroups are conjugate to their opposite. In this case we have $\Theta = 1$.

Conjugating by the element Θ provides an identification $\theta_{P,Q} : W^P \xrightarrow{\sim} W^{\mathbb{Q}}$. We have two specific Kostant representatives, namely the identity $e \in W^P$ and the element $w_P \in W^P$, this is the element which sends all the roots in U_P to negative roots (the longest element). Its length $l(w_P)$ is equal to the dimension $d_P = \dim(U_P)$.

Any element in $w \in W^P$ can be written as product of reflections

$$w = s_{\alpha_{i_1}} \dots s_{\alpha_{i_\nu}} \qquad (9)$$

where $\nu = l(w)$ and the first factor $\alpha_{i_1} \in J$. We always can complement this product to a product giving the longest element

$$s_{\alpha_{i_1}} \dots s_{\alpha_{i_\nu}} s_{\alpha_{i_{\nu+1}}} \dots s_{\alpha_{i_{d_P}}} = w s_{\alpha_{i_{\nu+1}}} \dots s_{\alpha_{i_{d_P}}} = w_P, \qquad (10)$$

The inverse of the element $s_{\alpha_{i_{\nu+1}}} \dots s_{\alpha_{i_{d_P}}}$ is

$$w' = s_{\alpha_{i_{d_P}}} \dots s_{\alpha_{i_{\nu+1}}} \in W^Q$$

This defines a second bijection $i_{P,Q} : W^P \xrightarrow{\sim} W^Q$ which is defined by the relation

$$w = w_P \cdot i_{P,Q}(w) = w_P \cdot w', \;\; l(w) + l(w') = d_P \qquad (11)$$

The composition $\theta_{P,Q}^{-1} \circ I_{P,Q} : W^P \to W^P$ is the bijection provided by duality.

The element w_P conjugates the Levi subgroup M of P into the Levi subgroup of $Q = w_P P w_P^{-1}$, the element $\tilde{w}_P = \Theta w_P$ conjugates the parabolic subgroup P into its opposite (which is conjugate to Q) and induces an automorphism on the subgroup M which is a common Levi-subgroup of P and its opposite.

If we choose $w = e$ then

$$\sum_{i\in I} n_i\gamma_i + \delta = \sum_{i\in I_M} n_i\gamma_i^M + \sum_{j\in J}(\sum_{i\in I_M} n_i c(i,j) + n_j)\gamma_j + \delta.$$

Since J is the orbit of an element $i \in I$ we see that $< \gamma_j, \alpha_j >$ is independent of j and hence we get easily

$$\sum_{j\in J}(\sum_{i\in I_M} n_i c(i,j) + n_j)\gamma_j = \frac{1}{\#J}(\sum_{j\in J}(\sum_{i\in I_M} n_i c(i,j) + n_j))\gamma_J + \delta'$$

and hence

$$a(e,\lambda) = \frac{1}{\#J}(\sum_{j\in J}(\sum_{i\in I_M} n_i c(i,j) + n_j))$$

If we choose Θ_P then as an M-module $\mathcal{M}_{\Theta_P\cdot\lambda}$ is dual to $\mathcal{M}_{\Theta\lambda}(-2f_J\gamma_J)$. We write $\Theta\lambda + \rho = \sum_{i\in I} n_{\Theta i}\gamma_i - \delta$ and then

$$w_P(\sum_{i\in I} n_i\gamma_i + \delta) = \sum_{i\in I_M} n_{\Theta i}\gamma_i^M - \sum_{j\in J}(\sum_{\Theta i\in I_M} n_{\Theta i} c(\Theta i, \Theta j) + n_{\Theta j})\gamma_j - 2f_J\gamma_J - \delta.$$

and especially we find

$$a(w_P,\lambda) = -(\frac{1}{\#J}(\sum_{j\in J}(\sum_{i\in I_M} n_{\Theta i} c(\Theta i, \Theta j) + n_{\Theta j})) + 2f_J)\gamma_J$$

In general we have the inequalities

$$a(\Theta_P,\lambda) \leq a(w,\lambda) \leq a(e,\lambda).$$

1.1.4 The inverse problem

Later we will encounter the following problem. Our data are as above and we start from a highest weight for M, we write

$$\mu = \mu^{(1)} + \delta_1 + a\gamma_J + \delta = \sum_{i\in I_M} n_{\Theta i}\gamma_i^M + \delta_1 + a\gamma_J + \delta.$$

We ask whether we can find a λ such that we can solve the equation $(Kost)$. More precisely: We give ourselves only the semi simple component $\mu^{(1)}$ of μ and we ask for the solutions

$$w(\lambda + \rho) = \mu^{(1)} + \ldots$$

where $w \in W^P$ and λ dominant, i.e. we only care for the semi simple component.

Let us consider the case where $J = \{i_0\}$, i.e. it is just one simple root. Then the term δ_1 disappears and our equation becomes

$$w(\lambda + \rho) = \mu^{(1)} + a\gamma_{i_0} + \delta,$$

of course the δ is irrelevant, but we want to know the range of the values $a = a(\lambda, w)$ when $\mu^{(1)}$ is fixed, but λ, w vary. Of course it may be empty. Let us fix a w and let us assume we have solved $w(\lambda+\rho) = \mu^{(1)} + \dots$. Then it is clear that the other solutions are of the form $\lambda+\rho+\nu$ where $w\nu \in \mathbb{Q}\gamma_{i_0}$. These ν are of the form $\nu = c\nu_0$ with $c \in \mathbb{Z}$. We write $\nu_0 = \sum_{i\in I} b_i\gamma_i$ and it is easy to see that there must be some $b_i > 0$ and some $b_j < 0$. This implies that $\lambda + c\nu_0$ is dominant if and only if $c \in [a, b]$, an interval with integers as boundary point. This of course implies that -still for a given w - the values $a = a(\lambda, w)$ also have to lie in a fixed finite interval

$$a = a(w, \lambda) \subset [a_{\min}(w, \mu^{(1)}), a_{\max}(w, \mu^{(1)}] = I(w, \mu^{(1)}).$$

This will be of importance because these intervals will be related to intervals of critical values of L-functions.

1.2 The goal of Eisenstein cohomology

The goal of the Eisenstein cohomology is to provide an understanding of the restriction map r in theorem (**??**). More precisely we assume that we have an understanding of the cohomology $H^\bullet(\partial\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$ and in terms of the description of the boundary cohomology we want to describe the image $H^\bullet_{\mathrm{Eis}}(\partial\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$ of r. Under certain conditions we will construct a section $\mathrm{Eis} : H^i_{\mathrm{Eis}}(\partial\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,\mathbb{C}}) \to H^i(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,\mathbb{C}})$. It is clear from the previous considerations that understanding of the understanding of $H^\bullet(\partial\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$ requires understanding of the Eisenstein cohomology of the Eisenstein cohomology of $H^\bullet(\mathcal{S}^M_{K^M_f}, \widetilde{H^\bullet(\mathfrak{u}, \mathcal{M})})$.

We consider certain submodules in the cohomology of the Borel-Serre compactification for which we can construct a section as above. We start from a maximal parabolic subgroup $P/\mathbb{Q}$, let $M/\mathbb{Q}$ be its reductive quotient. We define

$$H^\bullet_!(\partial_P(\mathcal{S}^G_{K_f}), \tilde{\mathcal{M}}_\lambda) = \bigoplus_{w\in W^P} H^{\bullet - l(w)}_!(\mathcal{S}^M_{K^M_f}, H^{l(w)}(\mathfrak{u}_P, \tilde{\mathcal{M}})(w\cdot\lambda)) \subset H^\bullet(\partial_P(\mathcal{S}^G_{K_f}), \tilde{\mathcal{M}}_\lambda) \tag{12}$$

We will abbreviate $H^{l(w)}(\mathfrak{u}_P, \tilde{\mathcal{M}})(w\cdot\lambda) = \tilde{\mathcal{M}}(w\cdot\lambda)$ where always keep in mind that the element $w \in W^P$ knows what the actual parabolic subgroup is and that $\tilde{\mathcal{M}}(w\cdot\lambda)$ sits in degree $l(w)$.

By definition the inner cohomology is the image of the cohomology with compact supports. This implies that the submodule

$$\bigoplus_{P:d(P)=1} H^q_!(\partial_P(\mathcal{S}^G_{K_f}), \tilde{\mathcal{M}}_\lambda) \subset \bigoplus_{P:d(P)=1} H^q(\partial_P(\mathcal{S}^G_{K_f}), \tilde{\mathcal{M}}_\lambda) = E_1^{0,q}$$

is annihilated by all differentials $d_\nu^{0,q}$ and hence we get an inclusion

$$i_P : \oplus_{w\in W^P} I_P^G H^{\bullet - l(w)}_!(\mathcal{S}^M_{K^M_f}, \mathcal{M}(w\cdot\lambda)) \to H^\bullet(\partial\mathcal{S}^G_{K_f}, \mathcal{M}_\lambda) \tag{13}$$

Taking the direct sum over the maximal parabolic subgroups yields a submodule

$$H^\bullet_!(\partial(\mathcal{S}^G_{K_f}), \tilde{\mathcal{M}}_\lambda) \hookrightarrow H^\bullet_!(\partial(\mathcal{S}^G_{K_f}), \tilde{\mathcal{M}}_\lambda) \tag{14}$$

The Hecke algebra acts on these two modules. Let us assume that this submodule when tensorized by $\mathbb{Q}$ is isotypical in $H^\bullet_!(\partial(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda\otimes\mathbb{Q})$. Then we get a decomposition

$$H^\bullet_!(\partial(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda\otimes\mathbb{Q})\oplus H^\bullet_{\mathrm{non!}}(\partial(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda\otimes\mathbb{Q})=H^\bullet(\partial(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda\otimes\mathbb{Q}). \quad (15)$$

We formulated the goal of the Eisenstein cohomology, we described an isotypical subspace and we know can ask: What is the intersection of $H^\bullet_{\mathrm{Eis}}(\partial\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda\otimes\mathbb{Q})$ with this subspace, or what amounts to the same, what is $H^\bullet_{!,\mathrm{Eis}}\partial(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda\otimes\mathbb{Q})$.

The element Θ induces an involution on the set of parabolic subgroups containing B (= set of $G(\mathbb{Q})$ conjugacy classes of parabolic subgroups) two parabolic subgroups $P,Q\supset B$ are called associate if $\Theta P=Q$. We can decompose the cohomology $H^\bullet_!(\partial(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda\otimes\mathbb{Q})$ into summands attached to the classes of associated parabolic subgroups

$$H^\bullet_!(\partial(\mathcal{S}^G_{K_f},\tilde{\mathcal{M}}_\lambda\otimes\mathbb{Q})=\bigoplus_{P:P=\Theta P}H^\bullet_!(\partial_P(\mathcal{S}^G_{K_f}),\tilde{\mathcal{M}}_\lambda)\oplus\bigoplus_{[P,Q]}H^\bullet_!(\partial_P(\mathcal{S}^G_{K_f}),\tilde{\mathcal{M}}_\lambda)\oplus H^\bullet_!(\partial_Q(\mathcal{S}^G_{K_f}),\tilde{\mathcal{M}}_\lambda) \quad (16)$$

where in the second sum $Q=\Theta P$. Each summand is a sum over the elements of W^P and then we can decompose under the action of the Hecke algebra. We choose a sufficiently large extension $F/\mathbb{Q}$ and in the case $P=\Theta P$ we get

$$H^\bullet_!(\partial_P(\mathcal{S}^G_{K_f}),\tilde{\mathcal{M}}_\lambda\otimes F)=\bigoplus_{w\in W^P}\bigoplus_{\sigma_f}H^{\bullet-l(w)}_!(\mathcal{S}^M_{K^M_f},\tilde{\mathcal{M}}(w\cdot\lambda)\otimes F)(\sigma_f) \quad (17)$$

In the case $P\neq\Theta P=Q$ we group the contributions from the two parabolic subgroups together. To any $w\in W^P$ we have the element $i_{P,Q}(w)=w'\in W^Q$. We also group the terms corresponding to w and w' together. To any σ_f which occurs in $H^{\bullet-l(w)}_!(\mathcal{S}^M_{K^M_f},H^{l(w)}(\mathfrak{u}_P,\tilde{\mathcal{M}})(w\cdot\lambda)\otimes F)$ we find a $\sigma'_f=\sigma_f^{w_P}|\gamma_{\Theta j}|_f^{2f_Q}$, which occurs in the second summand.

The decomposition into isotypical pieces becomes

$$\bigoplus_{\sigma_f}\Big(H^{\bullet-l(w)}_!(\mathcal{S}^M_{K^M_f},\tilde{\mathcal{M}}(w\cdot\lambda)\otimes F)(\sigma_f)\oplus H^{\bullet-l(w')}_!(\mathcal{S}^{M'}_{K^{M'}_f},\tilde{\mathcal{M}}(w'\cdot\lambda)\otimes F)(\sigma'_f)\Big) \quad (18)$$

1.2.1 Induction and the local intertwining operator at finite places

Our modules σ_f are modules for the Hecke algebras $\mathcal{H}^M_{K^M_f}=\otimes_p\mathcal{H}^M_{K^M_p}$. Therefore we can write them as tensor product $\sigma_f=\otimes_p\sigma_p$. We consider a prime p where σ_f is unramified then we get can give a standard model for this isomorphism class. The module H_{σ_p} is the rank one $\mathcal{O}_F$ -module $\mathcal{O}_F$, i.e. it comes with a distinguished generator 1. The Hecke algebra acts by a homomorphism (See **??**)

$$h(\sigma_p):\mathcal{H}^{(M,w\cdot\lambda)}_{K^M_p,\mathbb{Z}}\to\mathcal{O}_F \quad (19)$$

and gives us the Hecke-module structure on H_{σ_p}. We can induce H_{σ_p} to a $\mathcal{H}^G_{K^G_p}$ module. This is actually the same $\mathcal{O}_F$ module but now with an action of the algebra $\mathcal{H}^{(G,\lambda)}_{K_p^G,\mathbb{Z}}$. We simply observe that we have an inclusion $\mathcal{H}^{(G,\lambda)}_{K_p^G,\mathbb{Z}} \hookrightarrow \mathcal{H}^{(M,w'\cdot\lambda)}_{K_p^M,\mathbb{Z}}$ and induction simply means restriction.

It follows easily from the description of the description of the spherical (unramified) Hecke modules via their Satake-parameters that the induced modules H_{σ_p} and $H_{\sigma'_p}$ are isomorphic as $\mathcal{H}^{(G,\lambda)}_{K_p^G,\mathbb{Z}}$ -modules and hence we get that after induction the two summands in (18) become isomorphic. We choose a local intertwining operator

$$T_p^{\mathrm{loc}} : H_{\sigma_p} \to H_{\sigma'_p} \tag{20}$$

simply the identity.

We postpone the discussion of a local intertwining operator at ramified places.

1.3 The Eisenstein intertwining operator

We start from an irreducible unitary module $H_{\sigma_\infty} \times H_{\sigma_f} = H_\sigma$ and assume that we have an inclusion $\Phi : H_\sigma \hookrightarrow L^2_{\mathrm{cusp}}(M(\mathbb{Q})\backslash M(\mathbb{A}))$. We consider Φ as an element of $W(\sigma)$ and for the moment we identify H_σ to its image under Φ. We stick to our assumption that σ occurs with multiplicity one in the cuspidal spectrum.

Then we we can consider the induced module, recall that this is the space of functions

$$\{f : G(\mathbb{A}) \to H_\sigma | f(\underline{pg}) = \underline{\bar{p}} f(\underline{g})\} \tag{Ind}$$

where $\underline{\bar{p}}$ is the image of $\underline{p}$ in $M(\mathbb{A})$. We can define the subspace $H_\sigma^{(\infty)}$ consisting of those f which satisfy some suitable smoothness conditions and then we can define a submodule $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} H_\sigma^{(\infty)}$ where the $f(g) \in H_\sigma^{(\infty)}$ and the f themselves also satisfy some smoothness conditions.

We embed this space into the space $\mathcal{A}(P(\mathbb{Q})\backslash G(\mathbb{A}))$ by sending

$$f \mapsto \{g \mapsto f(g)(e_M)\},$$

here $\mathcal{A}$ denotes some space of automorphic forms. This an embedding of $G(\mathbb{A})$-modules or an embedding of Hecke modules if we fix a level.

We have the character $\gamma_P : M \to G_m$, for any complex number z this yields a homomorphism $|\gamma_P|^z : M(\mathbb{A}) \to \mathbb{R}^\times$ which is given by $|\gamma_P| : \underline{m} \mapsto |\gamma_P(\underline{m})|^z$. As usual we denote by $\mathbb{C}(|\gamma_P|^z)$ the one dimensional $\mathbb{C}$ vector space on which $M(\mathbb{A})$ acts by the character $|\gamma_P|^z$. Then we may twist the representation H_σ by this character and put $H_\sigma \otimes z = H \otimes \mathbb{C}(|\gamma_P|^z)$. An element $\underline{g} \in G(\mathbb{A})$ can be written as $\underline{g} = \underline{p}\underline{k}, \underline{p} \in P(\mathbb{A}), \underline{k} \in K_f^0$ where $K_f^0 \supset K_f$ is a suitable maximal compact subgroup and now we define $h(\underline{g}) = |\gamma_P|(\underline{p})$.

Eisenstein summation yields embeddings

$$\mathrm{Eis} : \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} H_\sigma^{(\infty)} \otimes z \to \mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A})), \tag{21}$$

where

$$\mathrm{Eis}(f)(\underline{g}) = \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} f(\gamma \underline{g})(e_M) h(\gamma \underline{g})^z,$$

it is well known that this is locally uniformly convergent provided $\Re(z) >> 0$ and it has meromorphic continuation into the entire z plane (See [Ha-Ch]).

We assumed that H_σ is in the cuspidal spectrum. We get important information concerning these Eisenstein series, if we compute their constant Fourier coefficient with respect to parabolic subgroups: For any parabolic subgroup $P_1/\mathbb{Q} \subset G/\mathbb{Q}$ with unipotent radical $U_1 \subset P_1$ we define (See [Ha-Ch], 4)

$$\mathcal{F}^{P_1}(\mathrm{Eis}(f))(\underline{g}) = \int_{U_1(\mathbb{Q})\backslash U_1(\mathbb{A})} \mathrm{Eis}(f)(\underline{u}\underline{g})(e_M)d\underline{u}.$$

This essentially depends only on the $G(\mathbb{Q})$-conjugacy class of $P_1/\mathbb{Q}$. It it also in [Ha-Ch] , 4 that this constant term is zero unless P_1 is maximal and the conjugacy class of P_1 is equal to the conjugacy class of $P/\mathbb{Q}$ or the conjugacy class of $Q/\mathbb{Q}$. (which may or may not be equal to the conjugacy class of $P/\mathbb{Q}$.)

These constant Fourier coefficients have been computed by Langlands, we have to distinguish the two cases:

a) The parabolic subgroup $P/\mathbb{Q}$ is conjugate to an opposite parabolic $Q/\mathbb{Q}$.

In this case we have a Kostant representative $w^P \in W^P$ which conjugates $Q/\mathbb{Q}$ into $P/\mathbb{Q}$ and it induces an automorphism of $M/\mathbb{Q}$. We get a twisted representation $w^P(\sigma)$ of $M(\mathbb{A})$. In the computation of the the constant term we have to exploit that σ is cuspidal and we get two terms:

$$\mathcal{F}^P \circ \mathrm{Eis} : \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} H_\sigma \otimes z \to$$

$$\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} H_\sigma \otimes z \oplus \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} H_{w^P(\sigma)} \otimes (2f_P - z) \subset \mathcal{A}(P(\mathbb{Q})\backslash G(\mathbb{A})).$$

We can describe the image. It is well known, that we can define a holomorphic family

$$T^{\mathrm{loc}}(z) : \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} H_\sigma \otimes z \to \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} H_{\sigma^{w^P}} \otimes (2f_P - z)$$

which is defined in a neighborhood of $z = 0$ and which is nowhere zero. This local intertwining operator is unique up to a nowhere vanishing holomorphic function $h(z)$. It is the tensor product over all places $T^{\mathrm{loc}}(z) = \otimes_v T_v^{\mathrm{loc}}(z)$. For the unramified finite places the local operator is constant, i.e. does not depend on z and is equal to T_p^{loc} in section (1.2.1) and $T^{\mathrm{loc}}(0) = \otimes_p T_p^{\mathrm{loc}}$. At the remaining factors there is a certain arbitrariness for the choice of the local operator and some fine tuning is appropriate.

We also assume that we have chosen nice model spaces $H_{\sigma_\infty}, H_{\sigma'_\infty}$, and an intertwining operator

$$T_\infty^{\mathrm{loc}} : H_{\sigma_\infty} \to H_{\sigma'_\infty} \tag{22}$$

which is normalized by the requirement that it induces the "identity" on a certain fixed K_∞^M type.

Then we get the classical formula of Langlands for the constant term: For $f \in \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} H_\sigma \otimes z$ we get

$$\mathcal{F}^P \circ \mathrm{Eis}(f) = f + C(\sigma, z) T^{\mathrm{loc}}(z)(f),$$

where $C(\sigma, \lambda, z)$ is a product of local factors $C(\sigma_v, z)$ and where $C(\sigma_v, z)$ is a holomorphic function in z which compares our local intertwining operator to an intertwining operator which is defined by the integral.

The computation of this factor is carried out in H. Kims paper in [C-K-M], chap. 6. He expresses the factor in terms of the automorphic L function attached to σ_f. If we translate this into the cohomological L-function we get for the for the local factor at a prime p the following expression

$$C_p(\sigma, z) = \prod_{\chi} \frac{L^{\mathrm{aut}}(\sigma_p, r_\chi, m_{r_\chi}(z - \frac{f_P}{2}))}{L^{\mathrm{aut}}(\sigma_p, r, m_{r_\chi}(z - \frac{f_P}{2}) + 1)} T^{\mathrm{loc_p}}(z)(f) \tag{23}$$

Here χ runs over a set of of cocharacters of our maximal torus. These cocharacters are those roots in the unipotent radical of $U_P^\vee$ which are the dominant weights of the irreducible representations the Langlands dual group $M^\vee$ on the dual Lie-algebra $\mathfrak{u}_P^\vee$. The r_χ is the associated representation attached to χ. The m_χ are the heights of these roots, this is the coefficient of the coroot χ_J in χ.

We do not discuss the ramified finite places, from now on we assume that σ_f is unramified. Then we get

$$C(\sigma, z) = C(\sigma_\infty, z) \prod_p C_p(\sigma_p, z)$$

The local factor at infinity depends on the choice of T_∞^{loc}, in 1.2.4 we gave some rules how to fix it, if it is not zero on cohomology.

b) The opposite group $Q/\mathbb{Q}$ is not conjugate to $P/\mathbb{Q}$, then we have to compute two Fourier coefficients namely $\mathcal{F}^P$ and $\mathcal{F}^Q$ in this case we get

$$\mathcal{F} : \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} H_\sigma \otimes z \overset{\mathcal{F}^P \oplus \mathcal{F}^Q}{\longrightarrow}$$

$$\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} H_\sigma \otimes z \oplus \mathrm{Ind}_{Q(\mathbb{A})}^{G(\mathbb{A})} H_\sigma \otimes (2f_P - z) \subset \mathcal{A}(P(\mathbb{Q})\backslash G(\mathbb{A})) \oplus \mathcal{A}(Q(\mathbb{Q})\backslash G(\mathbb{A})).$$

and again we get

$$\mathcal{F} \circ \mathrm{Eis}(f) = f + C(\sigma_\infty) \prod_r \frac{L(\sigma_f, r, m_r(z - \frac{f_P}{2}))}{L(\sigma_f, r, m_r(z - \frac{f_P}{2}) + 1)} T^{\mathrm{loc}}(z)(f),$$

where now $T^{\mathrm{loc}}(z)$ is a product of local intertwining operators

$$T_v^{\mathrm{loc}} : \mathrm{Ind}_{P(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)} H_{\sigma_v} \otimes z \to \mathrm{Ind}_{Q(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)} H_{\sigma_v^{w_P}} \otimes (2f_P - z).$$

It is also due to Langlands that the Eisenstein intertwining operator is holomorphic at $z = 0$ if the factor in front of the second term is holomorphic at $z = 0$. Up to here σ can be any representation occurring in the cuspidal spectrum of M.

Now we assume that we have a coefficient system $\mathcal{M} = \mathcal{M}_\lambda$ and a $w \in W^P$ such that our σ_f occurs in $H_!^{\bullet - l(w)}(\mathcal{S}_{K_f^M}^M, \tilde{\mathcal{M}}(w \cdot \lambda) \otimes F)$. Then we find a $(\mathfrak{m}, K_\infty^M)-$ module H_{σ_∞} such that $H^\bullet(\mathfrak{m}, K_\infty^M, H_{\sigma_\infty} \otimes \mathcal{M}(w \cdot \lambda)) \neq 0$. We also find an embedding

$$\Phi_\iota : H_{\sigma_\infty} \otimes H_{\sigma_f} \otimes_{F,\iota} \mathbb{C} \hookrightarrow L^2_{\mathrm{cusp}}(M(\mathbb{Q})\backslash M(\mathbb{A}))$$

Let us assume that $w\cdot\lambda$ or equivalently σ_f are in the positive chamber. In case a) we have holomorphicity at $z=0$ if the weight λ is regular (See [Schw]) and in case b) the Eisenstein series is always holomorphic at $z=0$. We that we are in this case and then we can evaluate at $z=0$ in (46) and get an intertwining operator

$$\mathrm{Eis}\circ\Phi_\iota : \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} H_\sigma \to \mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A})).$$

We get a homomorphism on the de-Rham complexes

$$\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} H_\sigma \otimes_{F,\iota} \mathbb{C}\otimes\mathcal{M}_\lambda) \to \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))\otimes\tilde{\mathcal{M}}_\lambda) \tag{24}$$

We introduce the abbreviation $H_{\iota\circ\sigma_f} = H_{\sigma_f}\otimes_{F,\iota}\mathbb{C}$ and decompose $H_{\iota\circ\sigma} = H_{\sigma_\infty}\otimes H_{\iota\circ\sigma_f}$. We compose (24) with the constant term and get

$$\begin{aligned}\mathcal{F}\circ\mathrm{Eis}^\bullet : \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})} H_{\sigma_\infty}\otimes\mathcal{M}_\lambda)\otimes H_{\iota\circ\sigma_f} \to \\ \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})} H_{\sigma_\infty}\otimes\mathcal{M}_\lambda)\otimes H_{\iota\circ\sigma_f})\oplus\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathrm{Ind}_{Q(\mathbb{R})}^{G(\mathbb{R})} H_{\sigma'_\infty}\otimes\mathcal{M}_\lambda)\otimes H_{\iota\circ\sigma'_f})\end{aligned} \tag{25}$$

where $P=Q$ in case a).

We choose an $\omega\in\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}\otimes\mathcal{M}_\lambda)$ and consider classes $\omega\otimes\psi_f$ and map them by the Eisenstein intertwining operator to the cohomology (or the de-Rham complex) on $\mathcal{S}^G_{K_f}$. Then the restriction of of the Eisenstein cohomology to the boundary is given by the classes

$$\Phi_\iota(\omega\otimes\psi_f + C(\sigma_\infty,\lambda)C(\sigma_f,\lambda)T^{\mathrm{loc}}_\infty(\omega)\otimes T^{\mathrm{loc}}_f(\psi_f))$$

The factor $C(\sigma_f,\lambda)$ can be expressed in terms of the cohomological L-function. Translating the formula (23) yields

$$C(\sigma_f,\lambda)=\prod_p\prod_\chi \frac{L^{\mathrm{coh}}(\sigma_p, r_\chi, <\chi,\mu^{(1)}+\rho_M> + a(w,\lambda)<\chi,\gamma_P>)}{L^{\mathrm{coh}}(\sigma_p, r_\chi, <\chi,\mu^{(1)}+\rho_M> + a(w,\lambda)<\chi,\gamma_P>+1)}$$

The factor $C(\sigma_\infty)T^{\mathrm{loc}}_\infty$ is not always easy to understand in SecOps.pdf we discuss the special case of the symplectic group.

We see that the constant term is the sum of two terms. The first term reproduces the original class from which we started. We assumed that it is in the positive chamber. The second term is some kind of scattering term which is the image of the first term under an intertwining operator. In case a) the restriction of the second term gives a class in the same stratum, in case b) the restriction gives a class in a second stratum.

At this point I formulate a general principle

The second term is of fundamental arithmetic interest, it contains relevant arithmetic information. To exploit this information we have to understand the contribution $C(\sigma_\infty,\lambda)T^{\mathrm{loc}}_\infty$.

Here are some comments which give some support to this belief. We start from σ_∞ and consider the intertwining operator

$$T^{\mathrm{loc},\bullet}_\infty : \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})} H_{\sigma_\infty} \otimes \mathcal{M}_\lambda) \to \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})} H_{\sigma'_\infty} \otimes \mathcal{M}_\lambda) \tag{26}$$

The two complexes can be described by the Delorme isomorphism

$$\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})} H_{\sigma_\infty} \otimes \mathcal{M}_\lambda) \xrightarrow{\sim} \bigoplus_{w \in W^P} \mathrm{Hom}_{K^M_\infty}(\Lambda^{\bullet - l(w)}(\mathfrak{m}^{(1)}_{\mathbb{C}}/\mathfrak{k}^M)), H_{\sigma_\infty} \otimes \mathcal{M}(w \cdot \lambda)) \tag{27}$$

Our intertwining operator respects this decomposition and we get

$$T^{\mathrm{loc},\bullet}_\infty(w) : \mathrm{Hom}_{K^M_\infty}(\Lambda^{\bullet - l(w)}(\mathfrak{m}^{(1)}_{\mathbb{C}}/\mathfrak{k}^M)), H_{\sigma_\infty} \otimes \mathcal{M}(w \cdot \lambda)) \to$$
$$\mathrm{Hom}_{K^M_\infty}(\Lambda^{\bullet - l(w')}(\mathfrak{m}^{(1)}_{\mathbb{C}}/\mathfrak{k}^M)), H_{\sigma'_\infty} \otimes \mathcal{M}(w' \cdot \lambda))$$

We assume that $w \cdot \lambda$ is in the positive chamber and $l(w) \geq l(w')$. Now we know that for regular representations $\mathcal{M}_\lambda$ the cohomology $H^\nu(\mathfrak{m}, K^M_\infty, H_{\sigma_\infty} \otimes \mathcal{M}(w \cdot \lambda))$ is non zero only for ν in a very narrow interval around the middle degree (See [Vo-Zu], Thm. 5.5). If the difference $|l(w) - l(w')|$ is greater then the length of this interval, then it is clear that in any degree $T^{\mathrm{loc},\bullet}_\infty(w)$ induces zero on the cohomology. In such cases the Eisenstein intertwining operator gives us a section

$$\mathrm{Eis} H^{q-l(w)})(\mathcal{S}^M_{K^M_f}, \mathcal{M}(w \cdot \lambda) \otimes F)(\sigma_f) \to H^q(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes F) \tag{28}$$

But even in the case where the intertwining operator induces the zero map on cohomology the second term has influence on the structure of the cohomology. It influences the structure of the integral cohomology $H^q_{\mathrm{int}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes F)$ and apparently provides some understanding of the denominators of the Eisenstein classes.

In the next section we discuss such a situation for the group $G = \mathrm{GSp}_2/\mathbb{Z}$ and where our parabolic subgroup is the Siegel parabolic subgroup. In this case the second term does not contribute to the Eisenstein cohomology. In this case we have experimental evidence that "arithmetic" of the ratio of special values of the form

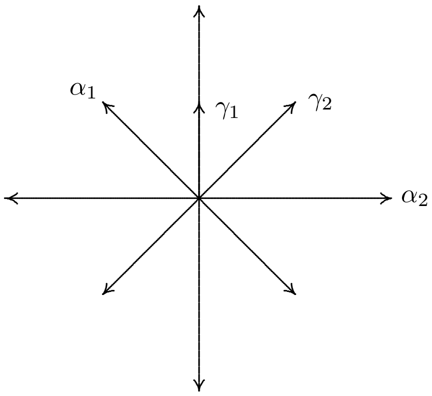
$$\frac{1}{\Omega(\sigma_f, \iota)} \prod_\chi \frac{L^{\mathrm{coh}}(\sigma_f, r_\chi, <\chi, \mu^{(1)} + \rho_M> + a(w, \lambda) <\chi, \gamma_P>)}{L^{\mathrm{coh}}(\sigma_f, r_\chi, <\chi, \mu^{(1)} + \rho_M> + a(w, \lambda) <\chi, \gamma_P> + 1)}$$

has influence on the structure of the cohomology. We will of course also have

cases where the intertwining operator is not zero on the cohomology. In that case we can prove rationality results for these ratios of L-values.

2 The example $G = \mathrm{Sp}_2/\mathbb{Z}$

2.1 Some notations and structural data



The maximal torus is

$$T_0/\mathbb{Z} = t = \left\{ \begin{pmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 & t_1^{-1} \end{pmatrix} \right\}$$

the simple roots are

$$\alpha_1(t) = t_1/t_2, \alpha_2(t) = t_2^2$$

and the fundamental dominant weights are

$$\gamma_1(t) = t_1, \gamma_2(t) = t_1 t_2$$

and finally we have

$$2\gamma_1^M = t_1/t_2$$

We choose a highest weight $\lambda = n_1\gamma_1 + n_2\gamma_2$ let $\mathcal{M}_\lambda$ be a resulting module for $G/\mathrm{Spec}(\mathbb{Z})$. We get the following list of Kostant representatives for the Siegel parabolic subgroup P and they provide the following list of weights.

$$\begin{aligned} 1 \cdot \lambda &= \lambda = \tfrac{1}{2}(2n_2 + n_1)\gamma_2 + n_1\gamma_1^{M_1} \\ s_2 \cdot \lambda &= \tfrac{1}{2}(-2 + n_1)\gamma_2 + (2n_2 + n_1 + 2)\gamma_1^{M_1} \\ s_2 s_1 \cdot \lambda &= \tfrac{1}{2}(-4 - n_1)\gamma_2 + (2 + 2n_2 + n_1)\gamma_1^{M_1} \\ s_2 s_1 s_2 \cdot \lambda &= \tfrac{1}{2}(-6 - 2n_2 - n_1)\gamma_2 + n_1\gamma_1^{M_1}, \end{aligned}$$

We choose for $K_\infty \subset \mathrm{Sp}_2(\mathbb{R})$ the standard maximal compact subgroup $U(2)$, it is the centralizer of the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

which defines a complex structure. With this choice we can define $\mathcal{S}_{K_f}^G = G(\mathbb{Q})\backslash G(\mathbb{R})/K_\infty \times G(\mathbb{A}_f)/K_f$.

2.2 The cuspidal cohomology of the Siegel-stratum

We consider the cohomology groups $H^\bullet(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda)$ and the resulting fundamental exact sequence. We have the boundary stratum $\partial_P(\mathcal{S}^G_{K_f})$ with respect to the Siegel parabolic. Let us assume that we are in the unramified case, then we get

$$H^\bullet(\partial_P(\mathcal{S}^G_{K_f}), \tilde{\mathcal{M}}_\lambda) = \bigoplus_{w \in W^P} H^{\bullet - l(w)}(\mathcal{S}^M_{K_f^M}, H^{l(w)}(\mathfrak{u}_P, \mathcal{M}_\lambda)) \tag{29}$$

We look at the case $w = s_2 s_1$ in this case we know how to describe the corresponding summand in terms of automorphic forms on Gl_2. We introduce the usual abbreviation $H^{l(w)}(\mathfrak{u}_P, \mathcal{M}_\lambda) = \mathcal{M}_\lambda(w \cdot \lambda)$.

Our coefficient modules are the modules attached to the highest weight

$$w \cdot \lambda = \mu = (2 + 2n_2 + n_1)\gamma_1^{M_1} + \frac{1}{2}(-4 - n_1)\gamma_2$$

Let us put $k = 4 + 2n_2 + n_1$ and $m = \frac{1}{2} n_1$. We give the usual concrete realization for these modules as $\mathcal{M}_{2+2n_2+n_1}[n_2 - 3 - k] = \mathcal{M}_{k-2}[n_2 - 3 - k]$

Let us look at the space $\mathcal{S}^M_{K_f^M}$. The group $M/\mathrm{Spec}(\mathbb{Z})$ is isomorphic to Gl_2, it is the Levi-quotient of the Siegel parabolic. The group K_∞^M is the image of $P(\mathbb{R}) \cap K_\infty$ under the projection $P(\mathbb{R}) \to M(\mathbb{R})$. This is the group $\mathbb{O}(2)$ it contains the standard choice $K_\infty^M(1) = \mathrm{SO}(2)$ as a subgroup of index 2. Hence we get a covering of degree 2

$$\mathcal{S}^{\tilde{M}}_{K_f^M} = M(\mathbb{Q}) \backslash M(\mathbb{R}) / K_\infty^M(1) \times M(\mathbb{A}_f)/K_f^M \to \mathcal{S}^M_{K_f^M} \tag{30}$$

We get an inclusion

$$i : H^1(\mathcal{S}^M_{K_f^M}, \mathcal{M}_\lambda(w \cdot \lambda)) \hookrightarrow H^1(\mathcal{S}^{\tilde{M}}_{K_f^M}, \mathcal{M}_\lambda(w \cdot \lambda)). \tag{31}$$

On the cohomology on the right we have the action of $\mathbb{O}(2)/\mathrm{SO}(2) = \mathbb{Z}/2\mathbb{Z}$ and the cohomology decomposes into a $+$ and a $-$ eigenspace. The inclusion i provides an isomorphism of the left hand side and the $+$ eigenspace.

This inclusion is of course compatible with the action of the Hecke algebra. If we pass to a suitable extension $F/\mathbb{Q}$ we get the decompositions into isotypic subspaces if we tensor our coefficient system by F. An isomorphism type σ_f occurs with multiplicity one on the left hand side and with multiplicity two on the right hand side. Over the ring $\mathcal{O}_F$ the modules $H^1_{\pm,\ \mathrm{int}}(\mathcal{S}^M_{K_f^M}, \mathcal{M}_\lambda(w \cdot \lambda)_F)(\sigma_f)$ are of rank one, hence we can find locally in the base $\mathrm{Spec}(\mathcal{O}_F)$ an isomorphism

$$T^{\mathrm{arith}}(\sigma_f) : H^1_+(\mathcal{S}^{\tilde{M}}_{K_f^M}, \mathcal{M}_\lambda(w \cdot \lambda)_F))(\sigma_f) \xrightarrow{\sim} H^1_-(\mathcal{S}^{\tilde{M}}_{K_f^M}, \mathcal{M}_\lambda(w \cdot \lambda)_F))(\sigma_f) \tag{32}$$

The isomorphism given by the fundamental class (see(**??**) interchanges the $+$ and the $-$ eigenspace, hence we can arrange our arithmetic intertwining operator such that it satisfies

$$T^{\mathrm{arith}}(\sigma_f \otimes |\delta_f|) = T^{\mathrm{arith}}(\sigma_f \otimes |\delta_f|)^{-1} \tag{33}$$

We consider the transcendental description of the cohomology groups

$$H^1(\mathcal{S}^{\tilde{M}}_{K_f^M}, \mathcal{M}_\lambda(w\cdot\lambda)_{\mathbb{C}}) = \bigoplus_{\sigma_f} H^1_+(\mathcal{S}^{\tilde{M}}_{K_f^M}, \mathcal{M}_\lambda(w\cdot\lambda)_{\mathbb{C}})(\sigma_f) \oplus H^1_-(\mathcal{S}^{\tilde{M}}_{K_f^M}, \mathcal{M}_\lambda(w\cdot\lambda)_{\mathbb{C}})(\sigma_f) \tag{34}$$

We consider the standard Borel subgroup $B \subset M$ the standard split torus $T_0 \subset B$ it contains our torus Z_0. We define the character

$$\chi_\mu = (k, m+2) : B(\mathbb{R}) \to \mathbb{C}^\times,\ \chi(t) = \gamma_1^M(t)^k |\gamma_2|^{m+2}.$$

It yields the induced Harish-Chandra module $I_{B(\mathbb{R})}^{M(\mathbb{R})}\chi_\mu$: We consider the functions

$$f : M(\mathbb{R}) \to \mathbb{C}; f(bg) = \chi(b)f(g); f|T_1 \text{ is of finite type .}$$

This is in fact a $(\mathfrak{m}, K_\infty^{M,0})$ -module, it contains the discrete representation $\mathcal{D}_{\chi_\mu}$. We have the decomposition

$$\mathcal{D}_{\chi_\mu} = \bigoplus_{\nu\equiv 0(2), |\nu|\geq k} F\phi_{\chi,\nu}$$

where

$$\phi_{\chi,\nu}(g) = \phi_{\chi,\nu}(b\begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi \end{pmatrix}) = \chi(b)e^{2\pi i\nu\phi}.$$

Of course $K_\infty^{M,0} = T_1(\mathbb{R}) = \{e(\phi) = \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi \end{pmatrix}\}$ and we can write $e(\phi)^\nu = e^{2\pi\nu i\phi}$.

We have the well known formula for the $((\mathfrak{m}, K_\infty^{M,0})$ cohomology

$$H^1((\mathfrak{m}, K_\infty^{M,0}), \mathcal{D}_{\chi_\mu} \otimes \mathcal{M}_\lambda(w\cdot\lambda)) = \mathrm{Hom}_{K_\infty^{M,0}}(\Lambda^1(\mathfrak{m}/\mathfrak{k}^M)), \mathcal{D}_{\chi_\mu} \otimes \mathcal{M}_\lambda(w\cdot\lambda)) = \tag{35}$$

$$\mathbb{C}P_+^\vee \otimes \phi_{\chi,-k} \otimes v_{k-2} + \mathbb{C}P_-^\vee \otimes \phi_{\chi,k} \otimes v_{-k+2} = \mathbb{C}\omega_{k,m} + \mathbb{C}\bar{\omega}_{k,m} \tag{36}$$

Here $v_{k-2} = (X+iY)^{k-2}$, resp. $v_{2-k} = (X-iY)^{k-2}$ are two carefully chosen highest (resp. lowest) weight vectors with respect to the action of $K_\infty^{M,0}$. The elements $P_\pm$ are the usual elements in $\mathfrak{m}/\mathfrak{k}$. We choose a model space H_{σ_f} for σ_f i.e. a free rank one $\mathcal{O}_F$ -module on which the Hecke algebra acts by the homomorphism $\sigma_f : \mathcal{H}^M_{K_f^M} \to \mathcal{O}_F$. We also choose and embedding $\iota : F \hookrightarrow \mathbb{C}$ and an $(\mathfrak{m}, K_\infty^{M,0}) \times K_\infty^M \times \mathcal{H}^M_{K_f^M}$- invariant embedding

$$\Phi_\iota : \mathcal{D}_{\chi_\mu} \otimes H_{\sigma_f} \otimes_{F,\iota} \mathbb{C} \to L^2_0(M(\mathbb{Q})\backslash M(\mathbb{A})) \tag{37}$$

this is unique up to a scalar in $\mathbb{C}^\times$ because the representation is irreducible and occurs with multiplicity one in the right hand side. This yields an isomorphism

$$\Phi^1_\iota : H^1((\mathfrak{m}, K_\infty^{M,0}), \mathcal{D}_{\chi_\mu} \otimes \mathcal{M}_\lambda(w\cdot\lambda)) \otimes H_{\sigma_f} \otimes_{F,\iota} \mathbb{C} \xrightarrow{\sim} H^1(\mathcal{S}^{\tilde{M}}_{K_f^M}, \mathcal{M}_\lambda(w\cdot\lambda)_{\mathbb{C}})(\iota\circ\sigma_f)$$

We observe that the element $\epsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in K_\infty^M$ has the following effect

$$\mathrm{Ad}(\epsilon)(P_+) = P_- \ , \epsilon(\phi_{\chi,k}) = \phi_{\chi,-k} \ \text{ and } \epsilon(v_{k-2}) = (-1)^m v_{2-k} \tag{38}$$

Hence we see that

$$\omega^{(+)}_{k,m} = \omega_{k,m} + (-1)^m \bar{\omega}_{k,m} \text{ resp. } \ \omega^{(-)}_{k,m} = \omega_{k,m} - (-1)^m \bar{\omega}_{k,m} \tag{39}$$

are generators of the $+$ and the $-$ eigenspace in $H^1(\mathfrak{m}, K_\infty^{M,0}, \mathcal{D}_{\chi_\mu} \otimes \mathcal{M}_\lambda(w\cdot\lambda))$. Therefore our map Φ and the choice of these generators provide isomorphisms

$$\Phi^{(+)}_\iota : H_{\sigma_f} \otimes_{F,\iota} \mathbb{C} \xrightarrow{\sim} H^1_+(\mathcal{S}^{\tilde{M}}_{K_f^M}, \mathcal{M}_\lambda(w\cdot\lambda)_{\mathbb{C}})(\iota\circ\sigma_f), \tag{40}$$

$$\Phi^{(-)}_\iota : H_{\sigma_f} \otimes_{F,\iota} \mathbb{C} \xrightarrow{\sim} H^1_-(\mathcal{S}^{\tilde{M}}_{K_f^M}, \mathcal{M}_\lambda(w\cdot\lambda)_{\mathbb{C}})(\iota\circ\sigma_f) \tag{41}$$

The choice of P_+, P_- and $\phi_{\chi,-\nu}$ is canonic, hence we see that the identifications depend only on Φ_ι , which is unique up to a scalar. This means that the composition

$$T^{\text{trans}}(\iota\circ\sigma_f) = \Phi^{(-)}_\iota \circ (\Phi^{(+)}_\iota)^{-1}$$
$$: H^1_+(\mathcal{S}^{\tilde{M}}_{K_f^M}, \mathcal{M}_\lambda(w\cdot\lambda)_{\mathbb{C}})(\iota\circ\sigma_f) \xrightarrow{\sim} H^1_-\mathcal{S}^{\tilde{M}}_{K_f^M}, \mathcal{M}_\lambda(w\cdot\lambda)_{\mathbb{C}})(\iota\circ\sigma_f)$$

yields a second (canonical) identification between the $\pm$ eigenspaces in the cohomology. Our arithmetic intertwining operator (See (32) yields an array of intertwining operators

$$T^{\text{arith}}(\sigma_f)\otimes_{F,\iota}\mathbb{C} : H^1_+(\mathcal{S}^{\tilde{M}}_{K_f^M}, \mathcal{M}_\lambda(w\cdot\lambda)_F))(\sigma_f)\otimes_{F,\iota}\mathbb{C} \xrightarrow{\sim} H^1_-(\mathcal{S}^{\tilde{M}}_{K_f^M}, \mathcal{M}_\lambda(w\cdot\lambda)_F))(\sigma_f)\otimes_{F,\iota}\mathbb{C} \tag{42}$$

Hence get an array of periods which compare these two arrays of intertwining operators

$$\Omega(\sigma_f,\iota)T^{\text{trans}}(\iota\circ\sigma_f) = T^{\text{arith}}(\sigma_f)\otimes_{F,\iota}\mathbb{C} \tag{43}$$

Our formula (33) tells us that we can arrange the intertwining operators such that

$$\Omega(\sigma_f\otimes|\delta_f|,\iota) = \Omega(\sigma_f,\iota)^{-1} \tag{44}$$

These periods are uniquely defined up to a unit in $\mathcal{O}_F^\times$.

2.2.1 The Eisenstein intertwining

We pick a σ_f which occurs in $H^1_!(\mathcal{S}^{\tilde{M}}_{K_f^M}, \mathcal{M}_\lambda(w\cdot\lambda)_F))$, we choose a $\iota : F \hookrightarrow \mathbb{C}$ and we choose an embedding

$$\Phi_\iota : \mathcal{D}_{\chi_\mu} \otimes H_{\sigma_f} \otimes_{F,\iota} \mathbb{C} \hookrightarrow L^2_{\text{cusp}}(M(\mathbb{Q})\backslash M(\mathbb{A})) \tag{45}$$

and from this we get the Eisenstein intertwining

$$\mathrm{Eis}\circ\Phi_\iota : \mathrm{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}(\mathcal{D}_{\chi_\mu}) \otimes H_{\sigma_f}\otimes_{F,\iota}\mathbb{C} \to \mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A})) \tag{46}$$

(Here we use that $K_f = \mathrm{GSp}_2(\hat{\mathbb{Z}})$.) Hence we get an intertwining operator

$$\mathrm{Eis}^\bullet : \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), I_P^G(\mathcal{D}_{\chi_\mu}) \otimes \mathcal{M}_\lambda) \otimes H_{\sigma_f} \otimes_{F,\iota} \mathbb{C} \to \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A})) \otimes \mathcal{M}_\lambda) \tag{47}$$

and this induces a homomorphism in cohomology

$$H^3(\mathfrak{g}, K_\infty, I_P^G(\mathcal{D}_{\chi_\mu}) \otimes \mathcal{M}_\lambda) \otimes H_{\sigma_f} \otimes_{F,\iota} \mathbb{C}) \to H^3(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda,\mathbb{C}}) \tag{48}$$

and we want to compose it with the restriction to the cohomology of the boundary. We have to compose it with the the constant Fourier coefficient $\mathcal{F}^P : \mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A})) \to \mathcal{A}(P(\mathbb{Q})U(\mathbb{A})\backslash G(\mathbb{A}))$. We know that $\mathcal{F}^P$ maps into the subspace

$$I_P^G\mathcal{D}_{\chi_\mu} \otimes H_{\sigma_f} \otimes_{F,\iota} \mathbb{C} \xrightarrow{\mathcal{F}^P} I_P^G\mathcal{D}_{\chi_\mu} \otimes H_{\sigma_f} \otimes_{F,\iota} \mathbb{C} \bigoplus I_P^G\mathcal{D}_{\chi_{\mu'}} \otimes H_{\sigma_f^{w_P}|\gamma_{P,f}|^{2f_P}} \otimes_{F,\iota} \mathbb{C} \tag{49}$$

where $\mu' = w_P w \cdot \lambda = s_2 \cdot \lambda = (2+2n_2+n_1)\gamma_1^{M_1} + \frac{1}{2}(-2+n_1)\gamma_2$. More precisely we know that for $h \in I_P^G\mathcal{D}_{\chi_\mu} \otimes H_{\sigma_f} \otimes_{F,\iota} \mathbb{C}$

$$\mathcal{F}^P(h) = h + C(\sigma, 0)T^{\mathrm{loc}}(0)(h) \tag{50}$$

where $T^{\mathrm{loc}}(0) = T^{\mathrm{loc}}_\infty \otimes \otimes_p T^{\mathrm{loc}}_p$. The local intertwining operator at the finite primes is normalized, it maps the standard spherical function into the standard spherical function. The operator T^{loc}_∞ will be discussed below.

Our general formula for the constant term yields for an $h = h_\infty \times h_f$

$$\mathcal{F}^P(h) = h + C(\sigma_\infty, \lambda)T^{\mathrm{loc}}_\infty(h_\infty)\frac{L(f, n_1+n_2+2)}{L(f, n_1+n_2+3)}\frac{\zeta(n_1+1)}{\zeta(n_1+2)} \times T^{\mathrm{loc}}_f(0)(h_f) \tag{51}$$

(For the following compare SecOps.pdf) We analyze the factor $C(\sigma_\infty, \lambda)T^{\mathrm{loc}}_\infty$ more precisely we study the effect of this operator on the cohomology. Let us look at the map between complexes

$$T^{\mathrm{loc},\bullet}_\infty : \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), I_P^G\mathcal{D}_{\chi_\mu} \otimes \mathcal{M}_\lambda) \to \mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), I_P^G\mathcal{D}_{\chi_{\mu'}} \otimes \mathcal{M}_\lambda) \tag{52}$$

The intertwining operator $T^{\mathrm{loc}}_\infty : I_P^G\mathcal{D}_{\chi_\mu} \to I_P^G\mathcal{D}_{\chi_{\mu'}}$ has a kernel $\mathbb{D}_{\chi_\mu}$, this is a discrete series representation. We know that

$$\mathrm{Hom}_{K_\infty}(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k}), \mathbb{D}_{\chi_\mu} \otimes \mathcal{M}_\lambda) = \mathrm{Hom}_{K_\infty}(\Lambda^3(\mathfrak{g}/\mathfrak{k}), \mathbb{D}_{\chi_\mu} \otimes \mathcal{M}_\lambda) = \tag{53}$$

$$H^3(\mathfrak{g}, K_\infty, \mathbb{D}_{\chi_\mu} \otimes \mathcal{M}_\lambda) = \mathbb{C}\Omega_{2,1} \oplus \mathbb{C}\Omega_{1,2} \tag{54}$$

We have the surjective homomorphism

$$H^3(\mathfrak{g}, K_\infty, \mathbb{D}_{\chi_\mu} \otimes \mathcal{M}_\lambda) \to H^3(\Lambda^3(\mathfrak{g}/\mathfrak{k}), I_P^G\mathcal{D}_{\chi_\mu} \otimes \mathcal{M}_\lambda) = H^1(\mathfrak{m}, K^M_\infty, \mathcal{D}_{\chi_\mu} \otimes H^2(\mathfrak{u}_P, \mathcal{M}_\lambda) = \mathbb{C}\omega^{(3)} \tag{55}$$

the differential form $\Omega_{2,1} + \epsilon(\lambda)\Omega_{1,2}$ maps to a non zero multiple $A(\lambda)\omega^{(3)}$. (The factor $\epsilon(\lambda)$ is a sign depending on λ). We can write $\Omega_{2,1} - \epsilon(\lambda)\Omega_{1,2} = d\psi$ where

$$\psi \in \mathrm{Hom}_{K_\infty}(\Lambda^2(\mathfrak{g}/\mathfrak{k}), \mathbb{D}_{\chi_\mu} \otimes \mathcal{M}_\lambda) \tag{56}$$

and $\omega = T_\infty^{\mathrm{loc},2}(\psi) \in \mathrm{Hom}_{K_\infty}(\Lambda^2(\mathfrak{g}/\mathfrak{k}), \mathcal{D}_{\chi_{\mu'}} \otimes \mathcal{M}_\lambda)$ is a closed form, hence it provides a cohomology class. Let us denote this cohomology class by $\kappa(\omega^{(3)})$.

Choosing $\omega^{(3)}$ as a basis element and applying the Eisenstein intertwining operator (47) yields a homomorphism

$$\mathrm{Eis}^{(3)} \circ \Phi_\iota : H^1_!(\mathcal{S}^{\tilde{M}}_{K_f^M}, \mathcal{M}_\lambda(w\cdot\lambda)_F))(\sigma_f \circ \iota) \to H^3(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_\lambda \otimes \mathbb{C}) \tag{57}$$

.

The local intertwining operator T_∞^{loc} maps $\omega^{(3)}$ to zero and hence it follows that the composition $r \circ \mathrm{Eis}^{(3)}$ is the identity, the Eisenstein intertwining operator yields a section on $H^1_!(\mathcal{S}^{\tilde{M}}_{K_f^M}, \mathcal{M}_\lambda(w\cdot\lambda)_F))(\sigma_f)$. (Remember $w = s_2 s_1$). If we define

$$H^3(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda F})(\sigma_f) = r^{-1}(H^1_!(\mathcal{S}^{\tilde{M}}_{K_f^M}, \mathcal{M}_\lambda(w\cdot\lambda)_F))(\sigma_f)) \tag{58}$$

(Induction does not play a role since the level is one) then we get the decomposition

$$H^3_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda F}) \oplus H^3_{\mathrm{Eis}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda F})(\sigma_f) = H^3_!(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda F})(\sigma_f) \tag{59}$$

2.2.2 The denominator of the Eisenstein class

We restrict this decomposition to the integral cohomology (better the image of the integral cohomology in the cohomology with rational coefficients)

$$H^3_{\mathrm{int}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda F})(\sigma_f) \supset H^3_{!,\,\mathrm{int}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda F})(\sigma_f) \oplus H^3_{\mathrm{int,Eis}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda F})(\sigma_f) \tag{60}$$

The image of $H^3_{\mathrm{int,Eis}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda F})(\sigma_f)$ under r is a submodule of finite index in $H^1_{!,\,\mathrm{int}}(\mathcal{S}^{\tilde{M}}_{K_f^M}, \mathcal{M}_\lambda(w\cdot\lambda)_F))(\sigma_f))$ and the quotient is

$$\begin{aligned} H^3_{\mathrm{int}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda F})(\sigma_f)/(H^3_{!,\,\mathrm{int}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda F})(\sigma_f) \oplus H^3_{\mathrm{int,Eis}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda F})(\sigma_f)) = \\ H^1_{!\ \mathrm{int}}(\mathcal{S}^{\tilde{M}}_{K_f^M}, \mathcal{M}_\lambda(w\cdot\lambda)_F))(\sigma_f))/\mathrm{image}(r). \end{aligned} \tag{61}$$

The quotient on the right hand side is $\mathcal{O}_F/\Delta(\sigma_f)$ where $\Delta(\sigma_f)$ is the denominator ideal. Tensoring the exact sequence

$$0 \to H^3_{!,\,\mathrm{int}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda F})(\sigma_f) \oplus H^3_{\mathrm{int,Eis}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda F})(\sigma_f) \to H^1_{\mathrm{int}}(\mathcal{S}^{\tilde{M}}_{K_f^M}, \mathcal{M}_\lambda(w\cdot\lambda)_F))(\sigma_f)) \to \mathcal{O}_F/\Delta(\sigma_f) \to 0 \tag{62}$$

by $\mathcal{O}_F/\Delta(\sigma_f)$ yields an inclusion

$$\mathrm{Tor}^1_{\mathcal{O}_F}(\mathcal{O}_F/\Delta(\sigma_f), \mathcal{O}_F/\Delta(\sigma_f) = \mathcal{O}_F/\Delta(\sigma_f)) \hookrightarrow H^3_{!,\,\mathrm{int}}(\mathcal{S}^G_{K_f}, \tilde{\mathcal{M}}_{\lambda F})(\sigma_f) \otimes \mathcal{O}_F/\Delta(\sigma_f) \tag{63}$$

and this explains the congruences.

2.2.3 The secondary class

We choose generators $\omega^{(3)}(\sigma_f)$(resp. $\omega^{(2)}(\sigma_f^{w_P}|\gamma_{P,f}|^{2f_P})$) for $H^1_{\mathrm{int}}(\mathcal{S}^{\tilde{M}}_{K_f^M},\mathcal{M}_\lambda(w\cdot\lambda)_F))(\sigma_f)$(resp. $H^1_{\mathrm{int}}(\mathcal{S}^{\tilde{M}}_{K_f^M},\mathcal{M}_\lambda(s_2\cdot\lambda))(\sigma_f)$) (Perhaps we can do this only locally on $\mathrm{Spec}(\mathcal{O}_F)$.) We may arrange these generators such that $T^{\mathrm{arith}}(\sigma_f)(\omega^{(3)}(\sigma_f))=\omega^{(2)}(\sigma_f^{w_P}|\gamma_{P,f}|^{2f_P})$. The isomorphism

$$\Phi_\iota^{(3)}: H^3(\mathfrak{g},K_\infty,\mathcal{D}_{\chi_\mu}\otimes\mathcal{M}_\lambda)\otimes H_{\sigma_f}\otimes_{F,\iota}\mathbb{C}\xrightarrow{\sim} H^1_{\mathrm{int}}(\mathcal{S}^{\tilde{M}}_{K_f^M},\mathcal{M}_\lambda(w\cdot\lambda)_F))(\iota\circ\sigma_f) \tag{64}$$

maps

$$(\Omega_{2,1}+\epsilon(\lambda)\Omega_{1,2})\otimes\omega^{(3)}(\iota\circ\sigma_f)\mapsto\Omega_+(\sigma_f,\iota)\omega(\sigma_f)$$

where $\Omega_+(\sigma_f,\iota)$ is a period depending on the choice of Φ_ι. The element

$$(\Omega_{2,1}-\epsilon(\lambda)\Omega_{1,2})\otimes\omega^{(3)}(\iota\circ\sigma_f)=d\psi\otimes\omega^{(3)}(\iota\circ\sigma_f).$$

where $\psi\in\mathrm{Hom}_{K_\infty}(\Lambda^2(\mathfrak{g}/\mathfrak{k}),I_P^G\mathcal{D}_{\chi_\mu}\otimes\mathcal{M}_\lambda)$. The operator $T^{\mathrm{loc}}(0)$ in (50) provides a homomorphism (52)

$$T^{\mathrm{loc},2}\otimes T_f^{\mathrm{loc}}:\mathrm{Hom}_{K_\infty}(\Lambda^2(\mathfrak{g}/\mathfrak{k}),I_P^G\mathcal{D}_{\chi_\mu}\otimes\mathcal{M}_\lambda)\otimes H_{\iota\circ\sigma_f}\to\mathrm{Hom}_{K_\infty}(\Lambda^2(\mathfrak{g}/\mathfrak{k}),I_P^G\mathcal{D}_{\chi_{\mu'}}\otimes\mathcal{M}_\lambda)\otimes H_{\iota\circ\sigma_f^{w_P}}|\gamma_{P,f}|^{2f_P}))$$

Under this homomorphism the class ψ is mapped to a multiple of $\omega^{(2)}(\sigma_f^{w_P}|\gamma_{P,f}|^{2f_P})$). We can calculate this multiple, during this calculation we see a second period $\Omega_-(\sigma_f,\iota)$ depending on Φ_ι and the ratio of these periods will be our period $\Omega(\iota\circ\sigma_f)$ in formula (43) .

This period is independent of Φ_ι. To state the final result we denote by f the modular cusp form attached to σ_f, this is a modular form with coefficients in F, then $\iota\circ f$ is a modular form with coefficients in $\mathbb{C}$. By $\Lambda(f,s)$ we denote the usual completed L -function. We get

$$C(\sigma,0)T^{\mathrm{loc}}(\kappa(\omega^{(3)}(\iota\circ\sigma_f))=$$
$$\Big(\frac{1}{\Omega(\sigma_f,\iota))^{\epsilon(k,m)}}\frac{\Lambda(\iota\circ f,n_1+n_2+2)}{\Lambda(\iota\circ f,n_1+n_2+3)}\frac{1}{\zeta(-1-n_1)}\Big)\frac{\zeta'(-n_1)}{\pi}\omega^{(2)}(\sigma_f^{w_P}|\gamma_{P,f}|^{2f_P}))$$

The factor inside the large brackets is essentially rational (it is in F and behaves invariantly under the action of the Galois group) the factor $\frac{\zeta'(-n_1)}{\pi}$ should viewed as a generator of a group of extension classes of mixed motives.

For me the most difficult part in the calculation is the treatment of the intertwining operator at ∞, this is carried out in SecOps.pdf. At the end of SecOps.pdf. I discuss the arithmetic applications and the conjectural relationship between the primes dividing the denominator of the expression in the large brackets and the denominators of the Eisenstein classes in (2.2.2)