Winterterm 2013/2014
Seminar on Kac-Moody algebras and Ringel-Hall algebras of Quivers

The aim of this seminar is to learn the basic theory of Kac-Moody algebras and their relation to Ringel-Hall algebras of categories of representations of quivers over finite fields.

1. Generalized Cartan Matrices and Kac-Moody Algebras

This talk is based on [2, Chapter 14.1, 14.2]. It begins with the definition of *generalized Cartan Matrices (GCM)* and (minimal) realizations of such. Then we want to see the construction of the "preliminary version" $\tilde{L}(A)$ of the Kac-Moody algebra attached to a GCM A in terms of generators and relations, as well as its triangular and root space decomposition for $\tilde{L}(A)$, analogous to the case of finite-dimensional, semisimple Lie algebras. A short reminder on Lie algebras described by generators and relations and Serre's Theorem might be useful.

Literature: [2, Chapter 14.1, 14.2]

Date: 31th October 2013

Speaker: Britta Kleberger

2. The classification of generalized Cartan matrices

In the first part of this talk the definition of the *Kac-Moody algebra* $L(A)$ associated with a GCM A is presented, along with its *triangular and root space decomposition* inherited from the preliminary version $\tilde{L}(A)$ discussed in talk 1. In the second part we want to get an overview over the *classification of generalized Cartan matrices* as explained in [2, Chapter 15]. The material is far too voluminuous to be explained in all details, so the speaker has to decide on his own what to discuss more and what to discuss less thoroughly. However, we should definitely see the statements of [2, Theorem 15.1, Corollary 15.11, Proposition 15.14] as well as the notion of *symmetrizable GCM*. Finally, the classification of finite resp. affine GCM in terms of Dynkin diagrams of finite resp. affine type [2, Theorems 15.19, 15.23] should be discussed, with at least a sketch of proof. It is helpful to pay attention to [2, Remark 15.12] right from the beginning.

Literature: [2, Chapter 6.3, Chapter 15]

Date: ***Wednesday, 6th November 2013, 2:30pm-4:00pm***

Speaker: Thomas Pickhinke

3. Quivers and their representations

This talk introduces the basics of the representations theory of quivers, following [6]. It starts with the definition of *quivers* and their representations [6, §1.1-1.3]; classical notions of *simple*, *projective*, *injective*, *indecomposable modules* should be recalled only very quickly, if at all, and the speaker should instead focus on the explicit description of simples, injectives and projectives [6, §1.6,1.7], as well as the remark that for the indecomposables, there is no explicit description. The *Krull-Remak-Schmidt Theorem* [6, Theorem 2.4.1] should also only be stated[1]. Next, introduce the *Grothendieck group* of a quiver, the *(symmetrized) Euler form* and the *Weyl group* acting by orthogonal reflections [6, §3.2]. It follows from talk 2 that the symmetrized Euler form is positive definite resp. positive semi-definite if and only if the graph underlying the quiver is of Dynkin resp. Euclidean/affine type, and that in the affine case the radical of the symmetrized Euler form is free of rank 1, spanned by a strictly positive vector; this is also proved directly in [6, Theorem 4.2.1]. From now on, restrict to the Dynkin or Euclidean case, define the notion of a root and at least sketch [6, Proposition 4.31, Lemma 4.3.2]; note in particular that the Weyl group is finite in the Dynkin case.

The second part of the talk is concerned with the proof of *Gabriel's Theorem* [6, Theorem 5.1.1] via reflection functors: Introduce *reflection functors*, state (without proof) their main properties [6, §3.3, in particular Theorem 3.3.5], define the *Coxeter functors resp. transformation* [6, §3.4 resp. §4.4] and finally present the proof of [6, Theorem 5.1.1]. The crucial point here is understanding the effect of reflection functors (and hence also the Coxeter functors) on the dimension vector, as well as [6, Lemma 4.4.4]. Finally, time permitting, describe the situation in the Euclidean case [6, Theorem 5.3.1] and state [6, Corollary 5.3.3].

Literature: [6], and [7, 8] for the curious.

Date: 14th November 2013

Speaker: Ögmundur Eiriksson

4. The invariant bilinear form and the Weyl group

The first part of this talk discusses the existence and uniqueness of an *invariant bilinear form* on the Kac-Moody algebra associated with a symmetrizable GCM [2, Proposition 16.1, Theorem 16.2, Proposition 16.6]. After that, simple reflections acting on the Cartan subalgebra H and its dual [2, Proposition 16.11, Proposition 16.14] and the Weyl group are defined. The talk ends with the description [2, Theorems 16.16, 16.17] of the Weyl group as a Coxeter group.

Literature: [2, Chapter 16.1, 16.2]

Date: 21th November 2013

Speaker: Michael Ehrig

5. The roots of a Kac-Moody algebra and Kac's Theorem

In contrast to the finite-dimensional situation, a general Kac-Moody algebra may contain roots that are not conjugate to a simple root under the Weyl group. This leads to the notion of *real and imaginary roots* this talk begins with. The main results are [2, Therem 16.24, Corollary 16.25, Proposition 16.26, Theorem 16.27];

[1] As it is so ubiquitous it might be interesting to note that it is true (see e.g. [8, Theorem 7.5]; it isn't stated in the desired generality, but the proof applies verbatimly) in any additive category $\mathscr{A}$ where firstly any object decomposes, in *some* way, as a finite direct sum of indecomposable modules, and where secondly the endomorphism ring of any indecomposable object is local; such a category is called *Krull-Remak-Schmidt category*, see [7, §3.1], and finite length abelian categories (such as categories of finite-dimensional representations) provide many examples for this [7, Corollary 3.3.3]

though the time won't suffice to present them in all details, we should at least see sketches of their proofs.

In the second part, *Kac's Theorem* (originally [5, Theorem §1.10], but also see [11, Theorem 3.13] or [3, Theorem 1]) should be carefully stated and illustrated in simple examples: For a finite quiver $\vec{Q}$ without loops, the dimension vectors of the indecomposable finite-dimensional representations of $\vec{Q}$ over an algebraically closed field are in natural bijection with the positive roots of the corresponding Kac-Moody algebra; moreover, under this bijection the real roots correspond to those dimension vectors for which there is, up to isomorphism, a unique indecomposable representation. Note that the definitions of real and imaginary roots in [5] agree with the ones for the corresponding Kac-Moody algebra by [2, Theorem 16.24]. Show how Kac's Theorem implies Gabriel's Theorem discussed in talk 3, and illustrate it for the Kronecker quiver (corresponding to $\widehat{\mathfrak{sl}_2}(\mathbb{C})$) [11, Example 3.11] and the quiver of type $\widetilde{D_4}$ (corresponding to $\widehat{\mathfrak{so}_8}(\mathbb{C})$) [11, Example 3.12]. The first part of the table in [11, End of §3.3] gives a nice summary of the connection between a quiver and the Kac-Moody algebra associated to its underlying graph.

Literature: [2, Chapter 16] for the first part, and [5, 11, 3] for Kac's Theorem.

Date: 28th November 2013

Speaker: Tomasz Przezdziecki

6. The definition of the Ringel-Hall algebra

This talk initiates our study of Ringel-Hall algebras, mainly following [11] and [4]. To begin, quickly recall the notion of an *abelian category*, introduce *finitary abelian categories* and give some examples [11, §1.1]. Then, define the Grothendieck group of an abelian category and introduce the (symmetrized) Euler form on it, provided the category has finite global dimension [11, §1.2]. Show that it agrees with our ad-hoc definition of the Grothendieck group and the (symmetrized) Euler form for categories of finite-dimensional representations of quivers.

Next, define the *untwisted and twisted versions of the Ringel-Hall algebra* $\mathcal{H}_{\mathscr{A}}$ associated to a finitary abelian category $\mathscr{A}$: they are both freely generated by the isomorphism classes of objects, and the product μ captures in how many ways to given objects can be "glued together" to give a prescribed third one. The precise definitions can be found in [11, Proposition 1.1], [4, Formula (6.1)] (twisted case) and [4, Lemma 2.2] (untwisted case). The proof of associativity should be discussed; it doesn't matter if you treat the twisted or the untwisted case. The motivation for introducing the twisted version of the product is that in case of representations of quivers it removes the dependence of the Ringel-Hall algebra on the orientation of the quiver. Note the natural grading of the Ringel-Hall algebra by the Grothendieck group as well as the interpretation of iterated multiplications, see [11, Remark 1.3(ii,v)] or [4, Formulas (1.6), (1.7)]. All these definitions shouldn't take too long, so that instead much time remains for *example calculations*: 1) The Ringel-Hall algebra of a semisimple finitary category like the category of finite-dimensional vector spaces over a finite field; see [11, Remark 1.3(iii), Example 3.14] or [10, Example 3.7] 2) For the Ringel-Hall algebra of a quiver with two vertices, present some calculations in special cases [11, Example 3.15]. 3) Given a discrete valuation ring $(\mathcal{O}, \mathfrak{m})$ with finite residue field k, the category $\mathscr{A}_{\mathcal{O}}$ of finite length $\mathcal{O}$-modules is finitary; e.g., if $\mathcal{O} = \mathbb{Z}_p$, one gets the category of finite abelian p-groups, and for

$\mathcal{O} = \mathbb{F}_q[\![t]\!]$ one gets finite dimensional $\mathbb{F}_q$ vector spaces equipped with a nilpotent endomorphism, i.e. nilpotent representations of the quiver with one vertex and one loop. In any case, the Ringel-Hall algebra of $\mathscr{A}_{\mathcal{O}}$ is called the *classical Hall algebra* [10, §2] and has a natural basis indexed by partitions, and the grading by the Grothendieck group $K_0(\mathscr{A}_{\mathcal{O}}) \cong \mathbb{Z}$ is the grading by the size of the partitions. Present some basic computation in this algebra; if you like, you may restrict to one of the special cases $\mathbb{Z}_p$ or $\mathbb{F}_q[\![t]\!]$ right from the beginning. For $\mathbb{F}_q[\![t]\!]$, you find some calculcations in [11, Example 2.2]. The classical Hall algebra will be studied in detail in talk 9.

Literature: [4, 11, 10]

Date: 5th December 2013

Speaker: Thomas Poguntke

7. Green's coproduct

Introduce *Green's coproduct* on the Ringel-Hall algebra of a finitary abelian category having the finite subobject object property; just as the product in the Ringel-Hall algebra counts the ways in which two given objects can be "glued together" to give a third one, the coproduct counts in how many ways a given objects can be split into two given ones. Again, there is an untwisted and a twisted version: the twisted version and its coassociativity are discussed in [11, Proposition 1.4], while the untwisted version is treated in [4, Lemma 2.4]; as in talk 6, define them both and present the proof of coassociativity for the version you like more. Then carefully state (without proof!) *Green's Theorem*: Twisting the multiplication on $\mathcal{H}_{\mathscr{A}} \otimes \mathcal{H}_{\mathscr{A}}$, Green's coproduct $\Delta : \mathcal{H}_{\mathscr{A}} \to \mathcal{H}_{\mathscr{A}} \otimes \mathcal{H}_{\mathscr{A}}$ is a morphism of algebras. This can either be stated using both the twisted version of μ and Δ and the twisting of multiplication on $\mathcal{H}_{\mathscr{A}} \otimes \mathcal{H}_{\mathscr{A}}$ defined in [11, Beginning of §1.5] or [4, Formula (6.13)], or by using the untwisted vrsions of μ and Δ and by twisting the multiplication on $\mathcal{H}_{\mathscr{A}} \otimes \mathcal{H}_{\mathscr{A}}$ as in [4, Formula (4.1)]; note that using twists everywhere, the twist in $\mathcal{H}_{\mathscr{A}} \otimes \mathcal{H}_{\mathscr{A}}$ is just a Koszul sign rule. Also, using twists everywhere allows for the definition of the extended twisted Ringel-Hall algebra $\widetilde{\mathcal{H}_{\mathscr{A}}}$, which then becomes a genuine bialgebra, and this should be presented in detail; see [11, End of §1.5] or [4, §7]. Next, introduce *Green's scalar product* on $\widetilde{\mathcal{H}_{\mathscr{A}}}$ and show that it is a non-degenerate Hopf-pairing; [11, Proposition 1.12, Corollary 1.13] or [4, Proposition 2.20]. Finally, define *Xiao's antipode* on $\widetilde{\mathcal{H}_{\mathscr{A}}}$ and check that it satisfies the axioms of an antipode, finishing the construction of the Ringel-Hall algebra as a Hopf-algebra; [11, §1.7] or [4, §9].

Literature: [11, 4]

Date: 12th December 2013

Speaker: Marc Sauerwein

8. Proof of Green's Theorem

The bulk of this talk is devoted to the *proof of Green's Theorem*, either following [11, Theorem 1.9] or [4, Theorem 2.6]. After that, provide some *example calculations* of Green's coproduct and Xiao's antipode in the examples from talk 6, e.g. [11, Example 2.5].

Literature: [11, 4].

Date: 19th December 2013

Speaker: Martin Peetz

9. The Jordan quiver and the classical Hall algebra

In this talk we study in detail the *classical Hall algebra*, i.e. the Ringel-Hall algebra of the category of finite length modules over a discrete valuation ring $(\mathcal{O}, \mathfrak{m})$ with finite residue field k; this includes for example the category of finite abelian p-groups and the category of nilpotent representations of the quiver with one vertex and one loop (the *Jordan quiver*) over a finite field; see the description of talk 6.

Firstly, note that the Euler form is trivial, so that twisting is unnecessary and the untwisted Hall algebra is already a genuine bialgebra algebra over $\mathbb{Q}$ [11, Remark following Theorem 2.1 for $\mathcal{O} = k[\![t]\!]$]. Then, use duality[2] to prove that the classical Hall algebra is commutative and cocommutative, show that it is a polynomial ring on countably many generators given by the classes of the k^n, and calculate Green's coproduct for them; [11, Theorem 2.6] for $\mathcal{O} = k[\![t]\!]$ or [10, Lemma 2.3, Theorem 2.4, Lemma 2.6] in general. This shows that the classical Hall algebra is isomorphic to the Hopf algebra of *symmetric polynomials* over $\mathbb{Q}$ in countably many variables; [11, §2.4] or [10, Theorem 2.7]; in fact, the explicit calculations show that everything can be defined and is true already over $\mathbb{Z}[q^{-1}]$ instead of $\mathbb{Q}$ (the multiplication is defined over $\mathbb{Z}$, while Green's comultiplication and Xiao's antipode require q to be invertible).

It is possible to treat the order q of the residue field $k = \mathcal{O}/\mathfrak{m}$ as an indeterminate in the definition of the classical Hall algebra, yielding the *generic classical Hall algebra*, a Hopf algebra over $\mathbb{Z}[t, t^{-1}]$: For this, one first checks that the structure constants in the multiplication of the classical Hall algebra only depend on q, moreover in a *polynomial* way; see [11, Proposition 2.7] or [10, Theorem 2.5]. The resulting integer polynomials are called *Hall polynomials* and they are taken as structure constants for some (a priori not known to be associative) algebra over $\mathbb{Z}[t]$. Specialization of t at arbitrary prime powers shows that one indeed gets an associative algebra over $\mathbb{Z}[t]$ which is again a polynomial ring generated by the countably many classes of the k^n. After base extension to $\mathbb{Z}[t, t^{-1}]$ one gets the generic classical Hall algebra – at this stage only an algebra. The comultiplication is then defined via [11, Formula (2.4) in Theorem 2.6], and the antipode via [11, Above Lemma 2.8]. Finally, this is the generic classical Hall algebra as a Hopf algebra over $\mathbb{Z}[t, t^{-1}]$ (not to be confused with the extended twisted Hall algebra!); again, it is isomorphic as a Hopf algebra over $\mathbb{Z}[t, t^{-1}]$ to the Hopf algebra of symmetric functions in countably many variables over $\mathbb{Z}[t, t^{-1}]$, and under this isomorphism, the canonical basis of the generic classical Hall algebra gives the *Hall-Littlewood polynomials*, symmetric polynomials over $\mathbb{Z}[t]$ in countably many variables deforming the monomial symmetric polynomials for $t = 0$ into the Schur polynomials for $t = 1$ [9, §III.2].

Literature: [11, 10, 9]

Date: 9th January 2013

[2] A special case of Matlis duality [1, Proposition 3.3.12], since the injective hull of k is given by $Q(\mathcal{O})/\mathcal{O} = \varinjlim_n \mathcal{O}/\mathfrak{p}^n$.

Speaker: Charlotte Ricke

10. The Ringel-Hall algebra of quivers

Finally, we study *Ringel-Hall algebras of quivers* and connect them to Kac-Moody algebras. Begin by presenting the example calculations of [11, Example 3.15], emphasizing on the similarity of [11, Formulas (3.6), (3.7)] with the *Serre relations* for $\mathcal{U}(\mathfrak{sl}_3(\mathbb{C}))$. Afterwards, give the direct definition [11, (3.8)-(3.11)] of the *quantum group* $\mathcal{U}_\nu(\mathfrak{b}')$ (of the derived algebra $\mathfrak{b}'$ of the Borel subalgebra $\mathfrak{b}$ of the Kac-Moody algebra associated with the graph underlying the given quiver). Finally, state and prove *Ringel-Green's Theorem* [11, Theorem 3.16], drawing the desired precise connection between (quantum groups of) Kac-Moody algebras and Hall algebras of quivers, summarized in the table on [11, End of §3.3].

Literature: [11, §3.3]

Date: 16th January 2014

Speaker: Joanna Meinel

11. The composition algebra of the projective line

Date: 23th January 2014

Speaker: Christian Reinecke

Literatur

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