Khovanov-Rozansky homology via stable Hochschild homology of Soergel bimodules

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Since the discovery of Khovanov homology [4] the study of categorified link invariants has become a very active field of research that has been approached by an amazing variety of different perspectives and techniques. Given an n-variable polynomial link invariant $\mathcal{P}$ like the Reshetikhin-Turaev invariant $\mathcal{P}_{\mathfrak{g},V}$ associated with a complex simple Lie algebra $\mathfrak{g}$ and a finite-dimensional simple representation V of its quantum group $\mathcal{U}_q(\mathfrak{g})$, see [10], a *categorification* of $\mathcal{P}$ is an $(n+1)$-variable polynomial link invariant from which $\mathcal{P}$ is obtained by taking Euler characteristic, i.e. by specializing the new variable to -1. In this sense, Khovanov homology is a categorification of the Jones polynomial, and by now categorifications have been found for all $\mathcal{P}_{\mathfrak{g},V}$ as well as for the HOMFLYPT polynomial defined in [3] (subsuming all $\mathcal{P}_{\mathfrak{sl}(k),\mathrm{nat}}$). Still, there are quite a few open questions, some concerning the *existence* of categorifications (e.g. is there a categorification of the Kauffman polynomial?), some concerning the *enhancement* of categorified link invariants by additional structure like differentials or algebra actions, and finally those concerning the *uniqueness* of categorifications of a given link invariant. This last question of uniqueness is already very interesting in the case of $\mathcal{P}_{\mathfrak{sl}(k),\mathrm{nat}}$ for which many categorifications are known, coming from representation theory, algebraic geometry, algebraic topology and commutative algebra. The author's talk was concerned with the comparison of two particular such: on the one hand, Mazorchuk-Stroppel-Sussan's categorifications [13, 8, 14] based on Bernstein-Gelfand-Gelfand category $\mathcal{O}$, and on the other hand, Khovanov-Rozansky's categorification KR_k [5] constructed using matrix factorizations.

The construction of Khovanov-Rozansky homology KR_k goes as follows: Given an oriented link L, one chooses firstly a triple-point free projection of L onto the plane. Secondly, one cuts the projection into pieces each of which looks like an unknotted single strand $\uparrow$ or one of the two crossings ⤧ or ⤨, and assigns a variable to any point where a cut was made. Thirdly, to each of the pieces just obtained one associates a certain fixed complex of $\mathbb{Z}$-graded matrix factorizations, the ground ring being the polynomial ring over $\mathbb{Q}$ over the variables attached to the open ends of the piece. Finally, one takes the tensor product of all these complexes to obtain a complex of matrix factorizations of potential 0. Taking total cohomology in each of its matrix factorization components, one gets a complex of graded vector spaces; $\mathrm{KR}_k(L) \in \mathbb{Z}[a^{\pm 1}, q^{\pm 1}]$ is then defined as the graded Poincaré series of the cohomology of this complex.

Given any ring S and a central element $w \in Z(S)$, the homotopy category $\underline{\mathrm{MF}}(S, w)$ of matrix factorizations of type (S, w) arises as the derived category $\mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(S, w)$ of a suitable model structure on the abelian category $\mathrm{LF}(S, w)$ of all linear factorizations of type (S, w), i.e. diagrams $M^0 \stackrel{\delta}{\to} M^1 \stackrel{\delta}{\to} M^0$ of not necessarily free S-modules M^0, M^1 satisfying $\delta^2 = w \cdot \mathrm{id}$. This category $\mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(S, w)$ is called the *contraderived category* of linear factorizations; it was introduced by

Positselski in [9] and further studied in [1]. The important point is that even though $\delta^2 \neq 0$ there exists a reasonable notion of weak equivalence of linear factorizations which upon localization turns $\mathrm{LF}(S,w)$ into $\underline{\mathrm{MF}}(S,w)$. The canonical functor $S/(w)\text{-Mod} \to \mathrm{LF}(S,w) \to \mathbf{D}^{\mathrm{ctr}}(S,w) \cong \underline{\mathrm{MF}}(S,w)$ is called the *stabilization functor* and denoted $(-)^{\{w\}}$; if S is regular local and $w \in \mathfrak{m}_S \setminus \{0\}$, it coincides with the classical stabilization functor from commutative algebra. Further, there is a derived tensor product on the $\mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(S,-)$ giving rise to the following definition of "stable" Hochschild homology:

Definition. *Let K be a field and A be a commutative K-algebra with enveloping algebra $A^{\mathrm{en}} = A \otimes_K A$. Further, let $w \in A$ and M be an A-bimodule such that $w.m = m.w$ for all $m \in M$. Then the stabilization $M^{\{w^{\mathrm{en}}\}}$ of M with respect to $w^{\mathrm{en}} := w \otimes 1 - 1 \otimes w \in A^{\mathrm{en}}$ is defined, and we call*

$$\mathrm{sHH}^*_w(M) \;:=\; \mathrm{H}^*[\Delta^{\{-w^{\mathrm{en}}\}} \overset{\mathbb{L}}{\otimes}_{A^{\mathrm{en}}} M^{\{w^{\mathrm{en}}\}}]$$

the w-stable Hochschild homology of M.

Khovanov-Rozansky homology now admits the following description:

Theorem. *Let β be a braid with labels x_i resp. y_i on the upper resp. lower ends of its strands. Then the complex of matrix factorizations $\mathrm{KR}_k(\beta)$ is termwise contraderived equivalent to the stabilization, with respect to $\sum x_i^{k+1} - y_i^{k+1}$, of the Rouquier complex of Soergel bimodules $\mathrm{RC}(\beta)$ associated to β,* [11, 12].

In particular, one recovers the following Theorem of Webster [15]:

Corollary. *Given an oriented link L presented as the closure of an n-strand braid word β with writhe $w(\beta)$, its Khovanov-Rozansky homology $\mathrm{KR}_k(L)$ equals*

$$(a^{-1}q^{k+1})^{w(\beta)} \sum_{i,j\in\mathbb{Z}} \dim_{\mathbb{Q}} \mathrm{H}^i\,[\mathrm{sHH}^*_k\,\mathrm{RC}(\beta)_j]\, a^i q^j \in \mathbb{Z}[a^{\pm 1}, q^{\pm 1}], \tag{1}$$

where sHH_k denotes stable Hochschild homology with respect to $\sum x_i^{k+1} - y_i^{k+1}$.

This theorem is analogous to the description [7] of triply graded Khovanov-Rozansky homology [6] (categorifying the HOMFLYPT-polynomial) as ordinary Hochschild homology of Rouquier complexes of Soergel bimodules.

It turns out that the expression (1) defines an invariant of oriented links for any base field K with $\mathrm{char}\, K \nmid k+1$, and that its invariance under the Markov moves can be checked rather quickly working in the contraderived category. More generally, we have the following:

Theorem. *Let R be a Noetherian $\mathbb{Z}[\frac{1}{k+1}]$-algebra. Then, for an n-strand braid word β with writhe $w(\beta)$, the complex*

$$\Sigma^{-w(\beta)}\rho_n\,[\mathrm{sHH}^*_k\;\;\mathrm{RC}_R(\beta)]\,\langle (k+1)\,w(\beta)\rangle$$

has finitely generated cohomology over R. Moreover, its isomorphism class in $\boldsymbol{D}(R[x_1, x_2, ...]\text{-Mod})$ is invariant under the Markov moves, hence an invariant of oriented links. Here, ρ_n denotes the functor induced by the homomorphism

$R[x_1, x_2, ...] \to R[x_1, x_2, ..., x_n]$, given by $x_i \mapsto x_i$ for $i \leq n$ and $x_i \mapsto x_n$ for $i \geq n$, and RC_R denotes the Rouquier complex defined over R.

If $k+1=0$ in R, it turns out that (1) still defines an invariant of oriented links after a suitable renormalization, and that it agrees with Khovanov-Rozansky's triply graded categorification of the HOMFLYPT-polynomial after some specialization; this is because the canonical spectral sequence from ordinary to k-stable Hochschild homology degenerates on the E_1-page in this case.

The above results can be considered a first step in a comparison of Khovanov-Rozansky homology with the categorifications defined by Mazorchuk, Stroppel and Sussan. The latter is based on shuffling functors restricted to certain parabolic versions of Bernstein-Gelfand-Gelfand category $\mathcal{O}$, and it is known that these shuffling functors can be described in terms of Rouquier complexes of Soergel bimodules before restriction to parabolic category $\mathcal{O}$. It therefore remains to be studied whether and in what sense restriction of shuffling functors to parabolic subcategories of $\mathcal{O}$ is equivalent to stabilization of the corresponding Rouquier complexes.

The results described in this abstract will appear in [2].

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