UNIFORM MODELS FOR RANDOM 3-MANIFOLDS

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Abstract. We construct hyperbolic metrics on random Heegaard splittings and provide uniform bilipschitz models for them.

1. Introduction

Every closed orientable 3-manifold M can be presented as a *Heegaard splitting*. This means that M is diffeomorphic to a 3-manifold M_f obtained by gluing together two handlebodies of the same genus H_g along an orientation preserving diffeomorphism f of their boundaries $\Sigma := \partial H_g$

$$M_f = H_g \cup_{f:\partial H_g \to \partial H_g} H_g.$$

The problem of finding hyperbolic structures on *most* 3-manifolds with a splitting of a fixed *genus* $g \geq 2$ was originally raised by Thurston (as Problem 24 in [30]) and made more precise by Dunfield and Thurston (see Conjecture 2.11 of [13]) via the introduction of the notion of *random Heegaard splittings*.

Such notion is based on the observation that the diffeomorphism type of M_f only depends on the isotopy class of the gluing map f, so it is well-defined for elements in the *mapping class group* $[f] \in \mathrm{Mod}(\Sigma)$. Therefore, Heegaard splittings of genus $g \geq 2$ are naturally parametrized by mapping classes $[f] \in \mathrm{Mod}(\Sigma)$.

A family $(M_n)_{n\in\mathbb{N}}$ of random Heegaard splittings of genus $g \geq 2$, or *random 3-manifolds*, is one of the form $M_n = M_{f_n}$ where $(f_n)_{n\in\mathbb{N}}$ is a random walk on the mapping class group $\mathrm{Mod}(\Sigma)$ driven by some initial probability measure μ with a finite support that generates $\mathrm{Mod}(\Sigma)$. If $(f_n)_{n\in\mathbb{N}}$ is such a random walk, we will denote by $\mathbb{P}_n$ the distribution of the n-th step f_n and by $\mathbb{P}$ the distribution of the path $(f_n)_{n\in\mathbb{N}}$.

Exploiting work of Hempel [14] and the solution of the geometrization conjecture by Perelman, Maher showed in [20] that a random Heegaard splitting M_f of genus $g \geq 2$ admits a hyperbolic metric, thus answering Dunfield and Thurston's conjecture.

Theorem (Maher [20])**.** *A random 3-manifold is hyperbolic.*

The main goal of this article is to provide a *constructive* and *effective* approach to the hyperbolization of random 3-manifolds.

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Our first contribution is a constructive proof of Maher's result

Theorem 1. *There is a Ricci flow free hyperbolization for random 3-manifolds.*

By Ricci flow free hyperbolization we mean that we construct explicitly the hyperbolic metric only using tools from the deformation theory of Kleinian groups. We use the model manifold technology by Minsky [25] and Brock, Canary and Minsky [7], as well as the effective version of Thurston's Hyperbolic Dehn Surgery by Hodgson and Kerckhoff [16] and Brock and Bromberg's Drilling Theorem [5].

We remark that, even though we do not rely on Perelman's solution of the geometrization conjecture, we do use the main result from Maher [20], namely, the fact that the *Hempel distance* of the Heegaard splittings (see Hempel [14]) grows coarsely linearly along the random walk.

Our construction gives new and more refined information than the mere existence of a hyperbolic metric. In fact, we also provide a *model manifold* that captures, up to *uniform bilipschitz distortion*, the geometry of the random 3-manifold and allows the computation of its geometric invariants.

The notion of model manifold that we use is similar to the ones considered by Brock, Minsky, Namazi and Souto in [26], [27], [9] and is depicted in the following definition of *ϵ-model metric*: A Riemannian metric (M_f, ρ) is a ϵ-model metric for $\epsilon < 1/2$ if there is a decomposition into five pieces $M_f = H_1 \cup \Omega_1 \cup Q \cup \Omega_2 \cup H_2$ satisfying the three requirements

(1) Topologically, H_1 and H_2 are homeomorphic to genus g handlebodies, while Q, Ω_1 and Ω_2 are homeomorphic to $\Sigma \times [0,1]$.
(2) Geometrically, ρ has negative curvature sec $\in (-1-\epsilon, -1+\epsilon)$, but outside the region $\Omega = \Omega_1 \cup \Omega_2$ the metric is purely hyperbolic.
(3) The piece Q is almost isometrically embeddable in a complete hyperbolic 3-manifold diffeomorphic to $\Sigma \times \mathbb{R}$.

The importance of the last requirement resides in the fact that we understand explicitly hyperbolic 3-manifolds diffeomorphic to $\Sigma \times \mathbb{R}$ thanks to the work of Minsky [25] and Brock, Canary and Minsky [7] which provides a detailed combinatorial description of their internal geometry.

The following is our more precise version of Theorem 1

Theorem 2. *For every $\epsilon > 0$ and $K > 1$ we have*

$$\mathbb{P}_n[M_f \text{ has a hyperbolic metric } K\text{-bilipschitz to a } \epsilon\text{-model metric}] \overset{n\to\infty}{\longrightarrow} 1.$$

We remark that ϵ-model metrics on random Heegaard splittings, similar to the ones that we build here, are constructed in [15]. There, the existence of a underlying hyperbolic metric is guaranteed by Maher's result and it is unclear whether the ϵ-model metrics are uniformly bilipschitz to it.

However, we should also mention that, using a result claimed by Tian [31], the mere fact that a metric ρ is a ϵ-model metric and that the regions

Ω_1, Ω_2 where it is not hyperbolic have uniformly bounded diameter (as follows from [15]), implies, if $\epsilon > 0$ is sufficiently small, that ρ is uniformly close up to third derivatives to a hyperbolic metric. However, Tian's result is not published. In order to provide a uniform bilipschitz control we exploit, instead, ergodic properties of the random walk and drilling and filling theorems by Hodgson and Kerckhoff [16] and Brock and Bromberg [5].

Our methods follow closely [9] and [8] where uniform ϵ-model metrics are constructed for special classes of 3-manifolds.

The idea is the following: We can obtain a hyperbolic metric on M_f by a hyperbolic cone manifold deformation from a finite volume metric on a *drilled* manifold $\mathbb{M}$ which has the following form: Let $\Sigma \times [1,4]$ be a tubular neighbourhood of $\Sigma \subset M_f$. We consider 3-manifolds diffeomorphic to

$$\mathbb{M} = M_f - (P_1 \times \{1\} \cup P_2 \times \{2\} \cup P_3 \times \{3\} \cup P_4 \times \{4\})$$

where P_j is a pants decomposition of the surface $\Sigma \times \{j\}$. A finite volume hyperbolic metric on such a manifold can be constructed explicitly by gluing together the convex cores of two *maximally cusped handlebodies* H_1, H_2 and three *maximally cusped I-bundles* Ω_1, Q, Ω_2.

$$\mathbb{M} = H_1 \cup \Omega_1 \cup Q \cup \Omega_2 \cup H_2.$$

Most of our work consists of finding suitable pants decompositions for which the Dehn surgery slopes needed to pass from $\mathbb{M}$ to M_f satisfy the assumptions of the effective Hyperbolic Dehn Surgery Theorem [16]. In order to find them we crucially need two major tools: The work of [15] on the geometry of hyperbolic handlebodies and ergodic properties of the random walks proved by Baik, Gekhtman and Hamenstädt [1].

We stress the fact that, for both Theorem 1 and Theorem 2, we assume that the support of μ is finite and generates the *entire* mapping class group.

We describe now some consequences of Theorem 2.

We start with a geometric application: We exploit the geometric control given by the ϵ-model metric to compute the coarse growth or decay rate of the geometric invariants along the family $(M_{f_n})_{n\in\mathbb{N}}$.

The general strategy is very simple: We use the model manifold technology [25], [7] and compute the geometric invariants for the middle piece Q. Then, we argue that the invariants of Q are uniformly comparable with those of M_f.

For example, combined with a result of Brock [4], Theorem 2 allows the computation of the coarse growth rate of the volume, which is well-known to be linear as explained in [20] (see also [15]). Combined with results of Baik, Gekhtman and Hamenstädt [1] it shows that the smallest positive eigenvalue of the Laplacian behaves like $1/n^2$ as computed in [15]. We notice that Theorem 2 allows a uniform approach to those result.

Here we do not carry out those computations because they are already well established. Instead, we have chosen to consider the diameter growth rate, which appears to be not available in the literature

Proposition 3. *There exists $c > 0$ such that*

$$\mathbb{P}_n[\mathrm{diam}(M_f) \in [n/c, cn]] \overset{n\to\infty}{\longrightarrow} 1.$$

The ingredients of the proof are Theorem 2 and a result by White [33].

In a completely different direction we use Theorem 2 to prove the following

Proposition 4. *For $\mathbb{P}$-almost every $(f_n)_{n\in\mathbb{N}}$ the following holds*

(1) *There are at most finitely many 3-manifolds in the family $(M_{f_n})_{n\in\mathbb{N}}$ that are arithmetic.*
(2) *There are at most finitely many 3-manifolds in the family $(M_{f_n})_{n\in\mathbb{N}}$ that are in the same commensurability class.*

The proof combines a study of geometric limits of random 3-manifolds, Proposition 5.1, with arguments from Biringer and Souto [3].

Overview. In Section 2 we outline the construction of the ϵ-model metric. In Section 3 we develop the two main technical tools that we need and use them to build many examples to which the model metric construction applies. In Section 4 we prove Theorem 2 by showing that the examples of Section 3 are generic from the point of view of a random walk. Lastly, in Section 5 we prove Proposition 3 and Proposition 4.

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2. A gluing scheme

Here we outline a construction for the ϵ-model metric which follows closely ideas of Brock and Dunfield [8] and Brock, Minsky, Namazi and Souto [9]. At the end of the discussion we formulate a criterion of applicability.

2.1. **Assembling simple pieces.** The construction is somehow implicit in the description of a ϵ-model metric. It has two steps. We start with five building blocks H_1, H_2 and Q, Ω_1, Ω_2 which are the *convex cores* of *geometrically finite maximally cusped* complete hyperbolic structures on H_g and $\Sigma \times [1,2]$ respectively. The pieces Ω_1 and Ω_2 will play the role of the collars of the other structures as we are going to explain later on.

For convenience of the reader, we briefly describe the geometry of H_1, H_2 and Q, Ω_1, Ω_2. The convex core Q of a geometrically finite maximally cusped structure on $\Sigma \times [1,2]$ is diffeomorphic to the drilled product

$$Q \simeq \Sigma \times [1,2] - (P_1 \times \{1\} \cup P_2 \times \{2\})$$

where P_1, P_2 are pants decompositions of Σ such that no curve in P_1 is isotopic to a curve in P_2. The drilled product is endowed with a complete finite volume hyperbolic metric with totally geodesic boundary

$$\partial Q = \partial_1 Q \sqcup \partial_2 Q = (\Sigma \times \{1\} - P_1 \times \{1\}) \sqcup (\Sigma \times \{2\} - P_2 \times \{2\}).$$

and rank one cusps at $P_1 \cup P_2$. If we fix the isotopy class of the identification of Q with the drilled product, there exists a unique maximally cusped structure with cusp data $P_1 \cup P_2$. We denote it by $Q(P_1, P_2)$.

Analogously, the convex core H of a geometrically finite maximally cusped structure on H_g is diffeomorphic to the drilled handlebody

$$H \simeq H_g - P$$

where P is a pants decomposition of $\partial H_g = \Sigma$ (throughout this article we keep this identification fixed) with the property that every curve in P is not compressible and no two curves in P are isotopic within H_g. Again, H is endowed with a complete finite volume hyperbolic metric with totally geodesic boundary

$$\partial H \simeq \partial H_g - P$$

and rank one cusps at P. If we keep track of the isotopy class of the identification between H and the drilled handlebody, there exists a unique maximally cusped structure with cusp data P. We denote it by $H(P)$.

Each component of the boundaries $\partial Q, \partial H$ is a three punctured sphere. It inherits a complete finite area hyperbolic metric. Such a structure is unique up to isometries isotopic to the identity. Hence, once we decided a pairing of the components of ∂H_1, ∂H_2 with $\partial_1 \Omega_1$, $\partial_2 \Omega_2$ and of $\partial_2 \Omega_1$, $\partial_1 \Omega_2$ with $\partial_1 Q$, $\partial_2 Q$, there is no ambiguity in implementing it to an isometric diffeomorphism. Gluing the pieces together along such a diffeomorphism we get a 3-manifold

$$\mathbb{M} := H_1 \cup_{\partial H_1 \simeq \partial_1 \Omega_1} \Omega_1 \cup_{\partial_2 \Omega_1 \simeq \partial_1 Q} Q \cup_{\partial_2 Q \simeq \partial_1 \Omega_2} \Omega_2 \cup_{\partial_2 \Omega_2 \simeq \partial H_2} H_2$$

which is non-compact and has a naturally defined complete finite volume hyperbolic structure.

In our case the pairing is natural as our structures are of the form

$$\begin{aligned} H_1 &= H(P_1), \\ \Omega_1 &= Q(P_1, P_2), \\ Q &= Q(P_2, P_3), \\ \Omega_2 &= Q(P_3, P_4), \\ H_2 &= H(f^{-1} P_4). \end{aligned}$$

We think of Ω_1 and Ω_2 as the collar structures of the boundaries of the three larger pieces $\mathbb{N}_1 = H_1 \cup \Omega_1$, $\mathbb{Q} = \Omega_1 \cup Q \cup \Omega_2$ and $\mathbb{N}_2 = \Omega_2 \cup H_2$.

Topologically, $\mathbb{M}$ is diffeomorphic to a drilled M_f, namely, let $\Sigma \times [1,4]$ denote a tubular neighbourhood of the Heegaard surface $\Sigma \subset M_f$, then

$$\mathbb{M} \simeq M_f - (P_1 \times \{1\} \cup P_2 \times \{2\} \cup P_3 \times \{3\} \cup P_4 \times \{4\}).$$

The pieces Ω_1, Q, Ω_2 are identified with

$$\begin{aligned} \Omega_1 &= \Sigma \times [1,2] - (P_1 \times \{1\} \sqcup P_2 \times \{2\}), \\ Q &= \Sigma \times [2,3] - (P_2 \times \{2\} \sqcup P_3 \times \{3\}), \\ \Omega_2 &= \Sigma \times [3,4] - (P_3 \times \{3\} \sqcup P_4 \times \{4\}). \end{aligned}$$

The curves in $P_1 \cup P_2 \cup P_3 \cup P_4$ represent the rank two cusps of $\mathbb{M}$.

In order to pass from $\mathbb{M}$ to the closed 3-manifold M_f we have to perform Dehn fillings on each cusp. This is the second step of the construction. The filling slopes are completely determined by the identification of $\mathbb{M}$ with the drilled M_f: They are the meridians γ of small tubular neigbourhoods of the curves in $\alpha \times \{j\} \subset P_j \times \{j\}$ inside $\Sigma \times [1,4]$.

Under such circumstances, the Hodgson and Kerckhoff effective version [16] of Thurston's Hyperbolic Dehn Surgery Theorem gives us sufficient conditions to guarantee that M_f has a hyperbolic metric obtained via a hyperbolic *cone manifold deformation* of the metric $\mathbb{M}$.

The condition is as follows: For every cusp of $\mathbb{M}$ we fix a torus horosection $\mathbb{T} \subset \mathbb{M}$ on the boundary of the η_M-thin part where $\eta_M > 0$ is some fixed Margulis constant. On each such horosection we have the slope $\gamma \subset \mathbb{T}$, determined by the gluing. We represent it as a simple closed geodesic for the intrinsic flat metric of $\mathbb{T}$. Hodgson and Kerckhoff deformation theory requires that the flat geodesic γ has sufficiently large *normalized length*, a quantity defined by

$$\mathrm{nl}(\gamma) := l(\gamma)/\sqrt{\mathrm{Area}(\mathbb{T})}.$$

We have

THEOREM 2.1 (Hodgson-Kerckhoff [16]). *Let $\mathbb{M}$ be a complete finite volume hyperbolic 3-manifold with n cusps. Let γ_j be flat geodesic slopes on torus horosections of the cusps. Suppose that the normalized length of each γ_j is at least $\mathrm{nl}_{HK} = 10.6273$. Then, there is a family $(M_t)_{t\in[0,2\pi]}$ of hyperbolic cone manifold structures on the Dehn filled manifold M whose singular loci are the core curves of the added tori and such that the cone angles of M_t equal t. The final hyperbolic cone manifold $M_{2\pi}$ is non singular. Moreover, the length of the core geodesic α_j is controlled by $l_{M_{2\pi}}(\alpha_j) \leq a/\mathrm{nl}(\gamma_j)^2$ for some universal constant $a > 0$.*

We want to guarantee that these conditions are fulfilled. This is where most of our work lies.

Once we know that $\mathbb{M}$ and M_f are connected by a family of hyperbolic cone manifolds, an application of Brock and Bromberg's Drilling Theorem [5] ensures that $\mathbb{M}$ is K-bilipschitz to M_f away from its cusps. The constant

K only depends on the length $l_{M_f}(\alpha_j)$ which, by Theorem 2.1, is again controlled by the normalized length $\mathrm{nl}(\gamma_j)^2$.

THEOREM 2.2 (Brock-Bromberg [5]). *Let $\eta_M > 0$ be a Margulis constant. For every $n > 0$ and $\xi > 0$ there exists $0 < \eta_B(\xi) < \eta_M$ such that the following holds: Let M be a geometrically finite hyperbolic 3-manifold. Let $\Gamma = \alpha_1 \sqcup \cdots \sqcup \alpha_n \subset M$ be a collection of simple closed geodesics of length $l_M(\alpha_j) < \eta_B(\xi)$ for all $j \leq n$. Let N be the unique geometrically finite hyperbolic structure on $M - \Gamma$ with the same conformal boundary as M. Then, there exists a $(1+\xi)$-bilipschitz diffeomorphism*

$$\left(N - \bigsqcup_{j\leq n} \mathbb{T}_{\eta_M}(\alpha_j), \bigsqcup_{j\leq n} \partial\mathbb{T}_{\eta_M}(\alpha_j)\right) \longrightarrow \left(M - \bigsqcup_{j\leq n} \mathbb{T}_{\eta_M}(\alpha_j), \bigsqcup_{j\leq n} \partial\mathbb{T}_{\eta_M}(\alpha_j)\right)$$

where $\mathbb{T}_{\eta_M}(\alpha)$ denotes a standard η_M-Margulis neighbourhood for α.

2.2. **Two criteria for Dehn filling with long slopes.** Certifying that the filling slopes have large normalized length is the main point that we have to address. We now discuss two criteria to check this condition.

The argument branches in two cases: We consider separately the filling slopes in $\mathbb{Q} = \Omega_1 \cup Q \cup \Omega_2$ and the ones in $\mathbb{N}_j = H_j \cup \Omega_j$. The two cases are similar in spirit, but the second one is technically more involved than the first one. However, the ideas are the same, so we will explain them with more details in the easier setting.

The I-bundle case. Consider first the hyperbolic manifold

$$\begin{aligned}\mathbb{Q} &= \Omega_1 \cup Q \cup \Omega_2 \\ &= Q(P_1, P_2) \cup Q(P_2, P_3) \cup Q(P_3, P_4).\end{aligned}$$

Topologically it is diffeomorphic to

$$\Sigma \times [1,4] - (P_1 \times \{1\} \sqcup P_2 \times \{2\} \sqcup P_3 \times \{3\} \sqcup P_4 \times \{4\})$$

The curves in P_1 and P_4 represent rank one cusps on $\partial\mathbb{Q}$ while the curves in P_2 and P_3 represent rank two cusps. We now try to understand what happens when we Dehn fill only the *rank two* cusps.

The filling slopes are chosen such that after the Dehn surgery, the natural inclusions $\Sigma \hookrightarrow \Omega_1, Q, \Omega_2$ become isotopic in the filled manifold so that it is naturally identified with

$$\mathbb{Q}^{\text{fill}} \simeq \Sigma \times [1,4] - (P_1 \times \{1\} \sqcup P_4 \times \{4\}).$$

We observe that there exists a unique marked maximally cusped structure on $\mathbb{Q}^{\text{fill}}$ where the rank one cusps are precisely given by $P_1 \times \{1\} \sqcup P_4 \times \{4\}$ (we assume that no curve in P_i is isotopic to a curve in P_j if $i \neq j$). We denote such a structure by $Q(P_1, P_4)$.

We are now ready to explain the main idea. Recall that our goal is to show that the filling slopes we singled out on the rank two cusps of $\mathbb{Q}$ have very large normalized length. This can be checked also in $\mathbb{Q}^{\text{fill}}$ once we know that $\mathbb{Q}$ uniformly bilipschitz embeds in $\mathbb{Q}^{\text{fill}}$ away from standard cusp neighbourhoods.

The strategy is as follows: Consider the maximally cusped structure $Q(P_1, P_4)$ and denote by Γ the collection of geodesic representatives of P_2 and P_3. Suppose that the collection Γ consists of extremely short simple closed geodesics, say of length at most $\eta < \eta_B(1/2)$, and that it is isotopic to $P_2 \times \{2\} \cup P_3 \times \{3\}$ under the identification with the drilled product.

Under such assumptions, we have the following.

Topologically, since the diffeomorphism type of $Q(P_1, P_4) - \Gamma$ only depends on the isotopy class of Γ, the manifold $Q(P_1, P_4) - \Gamma$ is diffeomorphic to $\mathbb{Q}$.

Geometrically, by Theorem 2.2, we can replace, up to 3/2-bilipschitz distortion away from standard Margulis neighbourhoods of Γ, the hyperbolic metric on $Q(P_1, P_4) - \Gamma$ with the unique geometrically finite structure with the same conformal boundary and rank two cusps instead of Γ. By uniqueness, such a geometrically finite structure structure on $Q(P_1, P_4) - \Gamma$ is precisely our initial manifold $\mathbb{Q} = \Omega_1 \cup Q \cup \Omega_2$.

In conclusion, $\mathbb{Q} = \Omega_1 \cup Q \cup \Omega_2$ uniformly bilipschitz embeds in $Q(P_1, P_4) - \Gamma$ and the filling slopes are mapped to meridians of large Margulis tubes. Comparing the normalized length of a filling slope γ in the two metrics we deduce that it must be very large because in $Q(P_1, P_4)$ the curve γ is the meridian on the boundary of a very large Margulis tube. In fact

Lemma 2.3. *Let $\mathbb{T}_{\eta_M}(\alpha)$ be a Margulis tube of radius R around a simple closed geodesic α of length $l(\alpha) < \eta_M$. Let γ be the flat geodesic representing the meridian on $\partial\mathbb{T}_{\eta_M}(\alpha)$. Then the normalized length is*

$$\mathrm{nl}(\gamma) = \sqrt{\frac{2\pi \tanh(R)}{l(\alpha)}}.$$

In particular $\mathrm{nl}(\gamma) \to \infty$ *as* $l(\alpha) \to 0$ *independently of the radius* R.

For example, there exists $\eta > 0$ such that if $l(\alpha) < \eta$ then $\mathrm{nl}(\gamma)$ is much bigger than nl_{HK}, the Hodgson-Kerckhoff constant.

Proof. The metric on $\mathbb{T}_{\eta_M}(\alpha)$ can be written in Fermi coordinates as

$$ds^2 = dr^2 + \cosh(r)^2 dl^2 + \sinh(r)^2 d\theta^2$$

where $(r, l, \theta) \in [0, R] \times [0, l(\gamma)] \times [0, 2\pi]$ are, respectively, the distance from α, the length along α and the angle around α parameters. The flat torus on the boundary has area $\mathrm{Area}(\partial\mathbb{T}_{\eta_M}(\alpha)) = 2\pi l(\alpha)\cosh(R)\sinh(R)$. The flat meridian $\gamma \subset \partial\mathbb{T}_{\eta_M}(\alpha)$ is represented by the curve $\theta \to (R, 0, \theta)$ of length $l(\gamma) = 2\pi \sinh(R)$. Hence the formula for the normalized length.

Notice that $\tanh(R)$ is roughly 1 when R is very large so that, in this case, the normalized length is approximately $\mathrm{nl}(\gamma) \approx l(\alpha)^{-1/2}$. It follows from work of Brooks and Matelski [10] that the radius of the Margulis tube

$\mathbb{T}_{\eta_M}(\alpha)$ is at least $R \geq \frac{1}{2}\log(\eta_M/l(\alpha)) - R_0$ where $R_0 > 0$ is some universal constant. Hence the second claim of the lemma when $l(\alpha)$ is very small. □

Applying Lemma 2.3 to the previous situation, we can conclude the following criterion

> **Criterion for I-bundles**: Fix $\mathrm{nl}_0 > \mathrm{nl}_{HK}$. The normalized length of the filling slopes corresponding to P_2 and P_3 is at least nl_0 provided that the collection of geodesic representatives in $Q(P_1, P_4)$ of the curves in $P_2 \cup P_3$ consists of simple geodesics of length at most η, where η only depends on nl_0, and is isotopic to $P_2 \times \{2\} \cup P_3 \times \{3\}$.

This concludes the I-bundle case.

The handlebody case. The second part consists of the same analysis for $\mathbb{N}_j = H_j \cup \Omega_j$ and $j = 1, 2$. The strategy is exactly the same. We only consider $\mathbb{N}_1 = H_1 \cup \Omega_1$ as the case of $\mathbb{N}_2 = \Omega_2 \cup H_2$ is completely analogous.

Parametrize a collar neighbourhood of $\Sigma = \partial H_g$ in H_g as $\Sigma \times [1, 2]$ with $\partial H_g = \Sigma \times \{2\}$. Topologically we have

$$\mathbb{N}_1 = H_g - (P_1 \times \{1\} \sqcup P_2 \times \{2\}).$$

Geometrically, the curves in P_2 correspond to rank one cusps while the one in P_1 correspond to rank two cusps. We are interested in filling in the rank two cusps. As before, the filling slopes are determined by the gluing.

After filling we have

$$\mathbb{N}_1^{\mathrm{fill}} = H_g - P_2.$$

Again, there is a unique maximally cusped structure on $\mathbb{N}_1^{\mathrm{fill}}$ whose cusps are given by P_2. We denote it by $H(P_2)$. We argue as before and assume that the collection Γ of geodesic representatives of P_1 consists of very short curves and is isotopic to $P_1 \times \{1\}$. Using the Drilling Theorem we compare the normalized length in $\mathbb{N}_1$ and $H(P_2)$.

Again, relying on Lemma 2.3, we will use the following criterion.

> **Criterion for handlebodies**: Fix $\mathrm{nl}_0 > \mathrm{nl}_{HK}$. The normalized length of the filling slopes corresponding to P_1 is at least nl_0 provided that the collection of the geodesic representatives in $H(P_2)$ of the curves in P_1 consists of simple closed geodesic of length at most η, where η only depends on nl_0, and is isotopic to $P_1 \times \{1\}$.

When considering $\mathbb{N}_2 = \Omega_2 \cup H_2 = Q(P_3, P_4) \cup H(f^{-1}P_4)$, we ask the same requirements replacing P_1 with $f^{-1}P_4$ and P_2 with $f^{-1}P_3$.

This concludes the handlebody case.

Thus, from the previous discussion we established the following

PROPOSITION 2.4. *Fix $K \in (1, 2)$. Suppose that there are four pants decompositions P_1, P_2, P_3, P_4 such that the* I-bundle *and the* handlebody criteria *are satisfied with parameter η sufficiently small only depending on K. Then,*

M_f admits a hyperbolic metric and a model metric $\mathbb{M}$. Furthermore, $\mathbb{M}$ and M_f can be connected by a family of hyperbolic cone manifolds and we have a K-bilipschitz diffeomorphism

$$\left(\mathbb{M} - \bigsqcup_{\alpha\in P_1\cup P_2\cup P_3\cup P_4} \mathbb{T}_{\eta_M}(\alpha)\right) \simeq \left(M_f - \bigsqcup_{\alpha\in P_1\cup P_2\cup P_3\cup P_4} \mathbb{T}_{\eta_M}(\alpha)\right).$$

We conclude with a small remark. The model manifold technology of Minsky [25] and Brock, Canary and Minsky [7], provides several tools to locate and measure the length of the geodesic representatives of P_2 and P_3 in $Q(P_1, P_4)$. However, the same technology is not available for handlebodies. This is the place where the difficulties arise.

3. A family of examples

In this section we construct many examples satisfying the I-bundle and handlebody criteria. Later, in the next section, we will show that this family is *generic* from the point of view of random walks.

We need two ingredients: The first one is a model for a collar of the boundary of a maximally cusped handlebody H or I-bundle Q. Following [15], we have that, in certain cases, it is possible to force a H and Q to look exactly like a maximally cusped I-bundle Ω near the boundary ∂H and $\partial_1 Q$ or $\partial_2 Q$. This is roughly the content of Propositions 3.1 and 3.2.

The second ingredient is a family of hyperbolic mapping tori T_ψ on which we want to model the collars Ω. These mapping tori have a distinguished fiber $\Sigma \subset T_\psi$ with a pants decomposition P consisting of extremely short geodesics. The collars Ω will look like a large portion of the infinite cyclic covering of T_ψ. See Theorem 3.3 and its corollaries, in particular Corollary 3.7.

In the end we will be able to detect whether M_f can be described as one of the examples we constructed simply by staring at the geometry of the Teichmüller segment $[o, fo]$ where $o \in \mathcal{T}$ is some base point that we will carefully fix once and for all. This is the content of Proposition 3.9.

3.1. **The geometry of the collars.** We discuss now the first main tool, that is, Propositions 3.1 and 3.2. For the statements we need to introduce some terminology and facts from the deformation theory of geometrically finite structures on handlebodies and I-bundles. We also need a suitable definition of collars for the boundary of such structures which is not just purely topological, but also geometrically significant.

We start by describing the deformation spaces of geometrically finite metrics. Even if we are mainly interested in maximal cusps, we begin with the more flexible class of *convex cocompact* structures.

A convex cocompact hyperbolic metric on a handlebodies H_g or an I-bundle $\Sigma \times [1, 2]$ is a complete hyperbolic metric on the interior, $\text{int}(H_g)$

or $\Sigma \times (1,2)$, that has a compact subset which is convex in a strong sense. This means that it contains all the geodesics joining two of its points. The minimal such subset is called the *convex core*. It is always a topological submanifold homeomorphic to the ambient manifold (except in the *fuchsian* case which we ignore). Its boundary is parallel to the boundary of the ambient manifold.

The Ahlfors-Bers theory associates to each convex cocompact metric a conformal structure on each boundary component. The deformation spaces of such metrics are parametrized by those conformal structures. Hence, they are identified with the Teichmüller space of the boundary. For each $Y \in \mathcal{T}(\partial H_g)$ and $(X,Y) \in \mathcal{T}(\Sigma \times \{1\}) \times \mathcal{T}(\Sigma \times \{2\})$ there are convex cocompact structures $H(Y)$ on H_g and $Q(X,Y)$ on $\Sigma \times [1,2]$, unique up to isometries isotopic to the identity, realizing those boundary data.

Geometrically finite *maximally cusped* hyperbolic structures on H_g or $\Sigma \times [1,2]$ can be thought as lying on the boundary of the deformation spaces. For every pair of pants P on ∂H_g such that no curve in P is compressible and no two curves in P are isotopic in H_g there exists a unique maximally cusped handlebody $H(P)$ with rank one cusps at P. Similarly, for every pants decomposition $P_1 \cup P_2$ of $\Sigma \times \{1\} \cup \Sigma \times \{2\}$ such that no curve in P_1 is isotopic to a curve in P_2, there exists a unique maximally cusped structure $Q(P_1, P_2)$ on $\Sigma \times [1,2]$ realizing those cusp data.

With a slight abuse of notations, sometimes we will denote both the complete convex cocompact or maximally cusped structure and the corresponding convex core in the same way. However, it will be clear from the context which one we are using.

The internal geometry of the convex cores of geometrically finite I-bundles has a rich structure. It is captured by the combinatorics and geometry of the *curve graph* $\mathcal{C} = \mathcal{C}(\Sigma)$ by the groundbreaking work of Minsky [25] and Brock, Canary and Minsky [7] with fundamental contributions by Masur and Minsky [22], [23].

This is the second piece of deformation theory that we need, it goes under the name of *model manifold technology*. Our use of this technology will not be heavy as we only need a few concepts and consequences, but we mostly hide the relation between the two. We briefly explain what we need.

The starting point is the following: To every convex cocompact structure Q on $\Sigma \times [1,2]$ we have an associated pair of curve graph invariants P_1 and P_2. They are pants decompositions on $\Sigma \times \{1\}$ and $\Sigma \times \{2\}$ that are the shortest for the conformal structure on the boundary. They might not be uniquely defined, in such case we just pick two. Similarly, for a maximally cusped structure Q we associate to it the cusp data P_1 and P_2. We think of these pants decompositions as subsets of the curve graph $\mathcal{C}$.

Recall now that for every proper essential subsurface $W \subsetneq \Sigma$ which is not a three punctured sphere there is a *subsurface projection*, as defined by

Masur and Minsky in [23]. It associates to each curve $\alpha \in \mathcal{C}$ the subset $\pi_W(\alpha)$ (possibly empty) of the curve graph $\mathcal{C}(W)$ of all possible essential surgeries of $\alpha \cap W$. The definition is slightly different for annuli. We associate to the curve graph invariants P_1 and P_2 the collection of coefficients

$$\{d_W(P_1, P_2) = \mathrm{diam}_{\mathcal{C}(W)}(\pi_W(P_1) \cup \pi_W(P_2))\}_{W \subsetneq \Sigma}.$$

As established by Minsky [25], the pants decompositions P_1 and P_2 together with the list $\{d_W(P_1, P_2)\}_{W \subsetneq \Sigma}$ allow to determine and locate the collection of *short curves* in Q. A special case, which is important for Propositions 3.1 and 3.2, is when the subsurface coefficients are all uniformly bounded. It corresponds to the situation where the only possible very short curves are the geodesic representatives of P_1 and P_2. For each other closed geodesic there is a positive uniform lower bound for the length.

The following notion was introduced by Minsky in [24] (see also [9]).

Definition (Bounded Combinatorics and Height). We say that two pants decompositions P_1, P_2 of Σ have *R-bounded combinatorics* if for every proper subsurface $W \subsetneq \Sigma$ we have $d_W(P_1, P_2) \leq R$. We say that they have *height* at least h if we have $d_{\mathcal{C}}(P_1, P_2) \geq h$.

As for the internal geometry of a geometrically finite handlebody the situation is more complicated as the compressibility of the boundary brings in several issues. We will restrict our attention to the geometry of some *collars* of the the boundary of the convex core.

We still choose for every convex cocompact structure on H_g a curve graph invariant, namely, a pants decomposition P on $\Sigma = \partial H_g$ which is the shortest when measured with the conformal boundary. In a similar way we associate to every maximally cusped structure the cusp data P.

Definition (Disk Set). The *disk set* $\mathcal{D}$ associated to the handlebody H_g is the subset of the curve graph $\mathcal{C}$ of the boundary $\Sigma = \partial H_g$ defined by

$$\mathcal{D} = \{\delta \in \mathcal{C} \mid \delta \text{ compressible in } H_g\}.$$

In order to construct a model for the collar of a geometrically finite handlebody we will have to keep track of how the curve graph invariant P of the geometrically finite structure interacts with the disk set $\mathcal{D}$.

The idea is the following: If P is far away from $\mathcal{D}$ then a large collar of the boundary of the convex core looks like a geometrically finite I-bundle.

We are almost ready for the statements of Propositions 3.1 and 3.2, we only need one last definition, the one of a geometrically controlled collar of the boundary of a geometrically finite structure. For convenient technical simplifications, it will be better for us to work with *quasi collars* (see below for the definition) instead of using directly collars. The reason is that we might wish to allow ourselves to throw away a uniform initial piece from a collar and still call the result a collar.

Let $M = H$ or Q denote the convex core of either a convex cocompact or a maximally cusped structure on either H_g or $\Sigma \times [1,2]$. Consider

$$M^{\rm nc} = M - \bigcup_{\alpha \in {\rm cusp}(M)} \mathbb{T}_{\eta_M}(\alpha).$$

the *non cuspidal part* of M. As before, $\mathbb{T}_{\eta_M}(\alpha)$ denotes a standard η_M-Margulis neighbourhood of the cusp α. We have that $M^{\rm nc}$ is homeomorphic to M. Its boundary $\partial M^{\rm nc}$ is parallel to the boundary of the ambient manifold, that is ∂H_g or $\Sigma \times \{1\} \cup \Sigma \times \{2\}$. Hence, it is naturally identified with it, up to isotopy. In particular, each component of $\partial M^{\rm nc}$ is always naturally identified with Σ.

The definition of quasi collar is analogous to the one of *product region* given in [15]: Consider a component Σ_0 of $\partial M^{\rm nc}$ and identify it with $\Sigma_0 \simeq \Sigma$ as above.

DEFINITION (Quasi Collar). A *quasi collar* of *size* (D, W, K) of the component $\Sigma_0 \subset \partial M^{\rm nc}$, denoted by

$${\rm collar}_{D,W,K}(\Sigma_0),$$

is a subset of a topological collar of Σ_0 in $M^{\rm nc}$, denoted by ${\rm collar}(\Sigma_0)$. We require the following additional geometric properties: There exists a parametrization ${\rm collar}(\Sigma_0) = \Sigma \times [0,3]$ such that Σ_0 is identified with $\Sigma \times \{0\}$ and ${\rm collar}_{D,W,K}(\Sigma_0)$ corresponds to $\Sigma \times [1,2]$. Furthermore we have

- The diameter of $\Sigma \times \{1\}$ and $\Sigma \times \{2\}$, measured with the intrinsic metric, is at most D.
- The *width* of ${\rm collar}_{D,W,K}(\Sigma_0)$, that is the distance between $\Sigma \times \{1\}$ and $\Sigma \times \{2\}$, is at least W and at most $2W + 2D$.
- The distance of $\Sigma \times \{1\}$ from the distinguished boundary $\Sigma \times \{0\} = \Sigma_0$ is at least K and at most $2K + 2D$.

Notice that each quasi collar ${\rm collar}_{D,W,K}(\Sigma_0)$ is *marked* with the isotopy class of an inclusion of Σ. Using this marking we can associate to every homotopy equivalence f between quasi collars a homotopy class $[f] \in {\rm Mod}(\Sigma)$.

We are ready to state Propositions 3.1 and 3.2.

PROPOSITION 3.1 (Propositions 4.1 and 6.1 of [15]). *For every* $R, \epsilon, \xi > 0$ *there exist* $D_0 = D_0(R, \epsilon) > 0$ *and* $K_0, W_0 > 0$ *such that for every* $W \geq W_0$ *there exists* $h = h(\epsilon, R, \xi, W) > 0$ *such that the following holds: Consider* $(Y, Z) \in \mathcal{T} \times \mathcal{T}$ *and* $X \in \mathcal{T}$. *Suppose that* $X, Y \in \mathcal{T}_\epsilon$. *Let* P_X, P_Y *and* P_Z *be short pants decompositions for* X, Y *and* Z *respectively. Consider the convex cores of the convex cocompact structures* $Q(X, Y)$ *and* $H(Y), Q(Z, Y)$. *Suppose that*

- P_X, P_Y *have* R-bounded combinatorics *and* height *at least* h.
- *In the handlebody case* H_g:

$$d_{\mathcal{C}}(P_Y, \mathcal{D}) \geq d_{\mathcal{C}}(P_Y, P_X) + d_{\mathcal{C}}(P_X, \mathcal{D}) - R.$$

- *In the I-bundle case* $\Sigma \times [1,2]$*:*
$$d_{\mathcal{C}}(P_Y, P_Z) \geq d_{\mathcal{C}}(P_Y, P_X) + d_{\mathcal{C}}(P_X, P_Z) - R.$$

Then, there exist $(1+\xi)$*-bilipschitz diffeomorphisms of quasi collars*
$$f : \mathrm{collar}_{D_0,W,K_0}(\partial_2 Q(X,Y)) \to \mathrm{collar}_{D_1,W_1,K_1}(\partial H(Y))$$
and
$$f : \mathrm{collar}_{D_0,W,K_0}(\partial_2 Q(X,Y)) \to \mathrm{collar}_{D_1,W_1,K_1}(\partial_2 Q(Z,Y))$$
for some slightly perturbed parameters D_1, W_1, K_1*. The diffeomorphisms are in the homotopy class of the identity with respect to the natural markings.*

Proposition 3.1 follows from [15].

We will use the following mild variation for maximally cusped structures.

PROPOSITION 3.2. *For every* R, ξ *there exist* $D_0 = D_0(R) > 0$ *and* $W_0, K_0 > 0$ *such that for every* $W \geq W_0$ *there exists* $h = h(\xi, R, W) > 0$ *such that the following holds: Consider pants decompositions* P_Y*,* P_X *and* P_Z *of* Σ*. Consider the convex cores of the maximally cusped structures* $Q(P_X, P_Y)$ *and* $H(P_Y), Q(P_Z, P_Y)$*. Suppose that*

- P_X, P_Y *have* R-bounded combinatorics *and* height *at least* h*.*
- *In the handlebody case* H_g*:*
$$d_{\mathcal{C}}(P_Y, \mathcal{D}) \geq d_{\mathcal{C}}(P_X, P_Y) + d_{\mathcal{C}}(P_1, \mathcal{D}) - R.$$
- *In the I-bundle case* $\Sigma \times [1,2]$*:*
$$d_{\mathcal{C}}(P_Y, P_Z) \geq d_{\mathcal{C}}(P_Y, P_X) + d_{\mathcal{C}}(P_X, P_Z) - R.$$

Then, there exist $(1+\xi)$*-bilipschitz diffeomorphism between quasi collars*
$$f : \mathrm{collar}_{D_0,W,K_0}(\partial_2 Q(P_X,P_Y)) \to \mathrm{collar}_{D_1,W_1,K_1}(\partial H(P_Y))$$
and
$$f : \mathrm{collar}_{D_0,W,K_0}(\partial_2 Q(P_X,P_Y)) \to \mathrm{collar}_{D_1,W_1,K_1}(\partial_2 Q(P_Z,P_Y))$$
for some slightly perturbed parameters D_1, W_1, K_1*. The diffeomorphisms are in the homotopy class of the identity with respect to the natural markings.*

Sketch of proof. Using Theorem 2.2 it is possible to quickly reduce Proposition 3.2 to the previous one. We only sketch the proof. We only treat the handlebody case as the I-bundle case is completely analogous.

First, we approximate P_X, P_Y with hyperbolic surfaces X, Y on which the pair of pants decompositions consist of very short geodesics, say of length contained in the interval $[\epsilon, 2\epsilon]$ with ϵ much smaller than a Margulis constant. Such surfaces are contained in $\mathcal{T}_\epsilon$.

By results of Canary [12] and Otal [28], the collections of geodesic representatives $\Gamma_X \cup \Gamma_Y$ and Γ_Y of the curves $P_X \cup P_Y$ and P_Y in $Q(X,Y)$ and $H(Y)$ have length $O(\epsilon)$ and are isotopic to $P_X \cup P_Y \subset \partial_1 Q(X,Y) \cup \partial_2 Q(X,Y)$ and $P_Y \subset \partial H(Y)$. Hence, by Theorem 2.2, if ϵ is small enough, we have $(1+\xi)$-bilipschitz embeddings of the non cuspidal part of the convex core of the maximally cusped structures in the corresponding complete convex cocompact hyperbolic 3-manifolds

$$\phi_Q : \left(Q(P_X, P_Y) - \bigcup_{\alpha \in P_X \cup P_Y} \mathbb{T}_{\eta_M}(\alpha) \right) \to \left(Q(X, Y) - \bigcup_{\alpha \in P_X \cup P_Y} \mathbb{T}_{\eta_M}(\alpha) \right)$$

and

$$\phi_H : \left(H(P_Y) - \bigcup_{\alpha \in P_Y} \mathbb{T}_{\eta_M}(\alpha) \right) \to \left(H(Y) - \bigcup_{\alpha \in P_Y} \mathbb{T}_{\eta_M}(\alpha) \right).$$

Now, if h is large enough only depending on ϵ, ξ, R, and W, we can apply Proposition 3.1 and find a $(1+\xi)$-bilipschitz diffeomorphism

$$g : \mathrm{collar}_{D_0,W,K}(\partial_2 Q(X,Y)) \to \mathrm{collar}_{D_1,W_1,K_1}(\partial H(Y)).$$

If K, W are sufficiently large, then both quasi collars will be contained in the images of ϕ_Q and ϕ_H. We just compose those with g, that is $f := \phi_H g \phi_Q^{-1}$. □

3.2. **Models for the collars.** As anticipated, we use Proposition 3.2 to construct a very particular class of maximally cusped handlebodies with a simple collar structure. This is our second main ingredient.

Recall that our goal is to construct examples that satisfy the criteria for handlebodies and for I-bundles. Also, recall that these criteria require to control the length and the isotopy class of the collection of geodesic representatives of a pants decomposition of the boundary. The examples we are going to describe are exactly tailored for that goal.

The idea is as follows: We first construct maximally cusped I-bundles Ω for which the length and isotopy class conditions are satisfied almost by definition for many pants decompositions. If a maximally cusped structure on H_g or $\Sigma \times [1,2]$ has a collar that is geometrically very close to Ω, then it also satisfies the criteria.

We now develop the strategy in more details.

The structure of the collar Ω will be modelled on the geometry of a *hyperbolic mapping torus*, or pseudo-Anosov mapping class, *with a short pants decomposition*. Such object is one that is obtained from the following procedure: Let P be a pants decomposition of Σ. Let $\phi \in \mathrm{Mod}(\Sigma)$ be a mapping class such that no curve in P is isotopic to a curve in ϕP. For example, a large power of any pseudo-Anosov suffices. Consider the convex core Q of the maximally cusped structure $Q(P, \phi P)$. The boundary ∂Q consists of totally geodesic hyperbolic three punctured spheres that are paired according to ϕ. We glue them together isometrically as prescribed by the pairing. The glued manifold is a finite volume hyperbolic 3-manifold diffeomorphic to

$$T_\phi - P \times \{0\} = (\Sigma \times [0,1]/(x,0) \sim (\phi x, 1)) - P \times \{0\}.$$

The curves in $P \times \{0\}$ represent rank two cusps. By Thurston's Hyperbolic Dehn Surgery (see Chapter E.6 of [2]) we can do Dehn surgery on the cusps so that the resulting manifold still carries a hyperbolic metric for which the core curves of the added solid tori are very short geodesics.

Furthermore, we can restrict ourselves to Dehn fillings for which the filled manifold still fibers over S^1 in a way compatible with the restriction of the fibering of T_ϕ to $T_\phi - P \times \{0\}$. In fact, observe that for each $\alpha \in P$ corresponding to a boundary torus $\mathbb{T}_\alpha$ we have a preferred meridian m_α and longitude l_α coming from the fibering of T_ϕ. If we perform Dehn surgeries with slopes $m_\alpha + kl_\alpha$, the filled manifold will be diffeomorphic to the mapping torus T_ψ where $\psi = \phi\delta_P^k$ and $\delta_P \in \mathrm{Mod}(\Sigma)$ is a Dehn twist about the pants decomposition P.

Consider the infinite cyclic covering $\hat{T}_\psi$ of T_ψ. Topologically, we can identify it with $\Sigma \times \mathbb{R}$ where the level sets $\Sigma_n := \Sigma \times \{n\}$ correspond to all the lifts of the fiber $\Sigma \times \{0\} \subset T_\psi$ and in such a way that the curves in

$$\bigcup_{n \in \mathbb{Z}} \psi^n P \times \{n\} \subset \bigcup_{n \in \mathbb{Z}} \Sigma \times \{n\}$$

are very short geodesics. A fundamental domain for the deck group action on $\hat{T}_\psi$ is given by the submanifold $[\Sigma_0, \Sigma_1]$ bounded by Σ_0 and Σ_1. The region $[\Sigma_n, \Sigma_m]$ bounded by Σ_n and Σ_m with $n < m$ is a stack of $m - n$ isometric copies of $[\Sigma_0, \Sigma_1]$.

We now approximate $\hat{T}_\psi$ with a maximally cusped I-bundle $Q(P, \psi^n P)$. Our collars will be of the form $\Omega = Q(P, \psi^m P)$ for some suitably chosen m.

We will use the following from [8], see also Figure 3.7 of the same article.

THEOREM 3.3 (Theorem 3.5 of [8]). *Let ψ be a mapping class with a short pants decomposition P. For every $\xi > 0$ there exist $k > 0$ and $d > 0$ such that for every $n > 0$ sufficiently large the non-cuspidal part of $Q_n = Q(P, \psi^n P)$ admits a decomposition*

$$Q_n^{\mathrm{nc}} = A_n \cup B_n \cup C_n$$

where A_n and C_n have diameter bounded by d while B_n is the image of a $(1+\xi)$-bilipschitz embedding with a quasi collar image

$$f : [\Sigma_k, \Sigma_{n-k}] \subset \hat{T}_\psi \to Q_n^{\mathrm{nc}}.$$

The embedding f is in the homotopy class of the identity with respect to the natural markings. Moreover, we can parametrize Q_n^{nc} as $\Sigma \times [0,3]$ in such a way that A_n, B_n and C_n correspond respectively to $\Sigma \times [0,1], \Sigma \times [1,2]$ and $\Sigma \times [2,3]$.

Observe that in the maximally cusped I-bundles $Q(P, \psi^n P)$ we have, by default, many pants decompositions whose length and isotopy class are well controlled.

In order to be able to exploit such control to check I-bundles and handlebody criteria we will need three consequences of Theorem 3.3. For them, we use the following fact proved in the Appendix A.

Lemma 3.4. *For every $\eta < \eta_M/2$ there exists $\xi > 0$ such that the following holds: Let $\mathbb{T}_{\eta_M}(\alpha)$ be a Margulis tube with core geodesic α of length $l(\alpha) \in [\eta, \eta_M/2]$. Suppose that there exists a $(1+\xi)$-bilipschitz embedding of the tube in a hyperbolic 3-manifold $f : \mathbb{T}_{\eta_M}(\alpha) \to M$. Then $f(\alpha)$ is homotopically non-trivial and it is isotopic to its geodesic representative within $f(\mathbb{T}_{\eta_M}(\alpha))$.*

Consider the $(1+\xi)$-bilipschitz embedding given by Theorem 3.3

$$f : [\Sigma_k, \Sigma_{n-k}] \subset \hat{T}_\psi \to Q_n^{\mathrm{nc}}.$$

Recall that $[\Sigma_k, \Sigma_{n-k}] = \Sigma \times [k, n-k]$ and that the curves $\psi^j P \times \{j\} \subset \Sigma_j = \Sigma \times \{j\}$ are short geodesics in the infinite cyclic covering and have length in the interval $[\eta, \eta_M]$. Denote by Γ_j the collection of geodesic representatives of $f(\psi^j P \times \{j\})$ in Q_n. By Lemma 3.4, if ξ is small compared to η, we get

Corollary 3.5. *The collection $\Gamma = \Gamma_k \cup \cdots \cup \Gamma_{n-k}$ is isotopic to*

$$\bigcup_{k<j<n-k} f(\psi^j P \times \{j\}) \subset \bigcup_{k<j<n-k} f(\Sigma_j).$$

via an isotopy supported on $\bigsqcup_{\alpha \in \psi^{k+1}P \cup \cdots \cup \psi^{n-k-1}P} f(\mathbb{T}_{\eta_M}(\alpha))$.

We now locate suitable quasi collars inside $Q(P, \psi^n P)$. First, notice that

$$[\Sigma_k, \Sigma_{n-k}] = \bigcup_{k<j\leq n-k} [\Sigma_{j-1}, \Sigma_j]$$

and each $[\Sigma_{j-1}, \Sigma_j]$ is an isometric copy of the fundamental domain $[\Sigma_0, \Sigma_1]$. Each $f[\Sigma_{j-1}, \Sigma_{j+1}] \subset Q$ is a quasi collar for $\partial_1 Q^{\mathrm{nc}}$ for every $k < j < n-k$. We now estimate the quasi collar size (D, W, K).

By Theorem 3.3, we also have that each component of

$$Q^{\mathrm{nc}} - f[\Sigma_k, \Sigma_{n-k}]$$

has diameter bounded by $d = d(\psi, \xi)$. Denote by $w = w(\psi) > 0$ the width of the fundamental domain $[\Sigma_0, \Sigma_1]$. Denote, instead, by $a = a(\psi) > 0$ the intrinsic diameter of the isometric surfaces Σ_j. Notice that, up to replacing ψ with a power (a change that does not seriously affect any of the arguments), we can as well assume that $2a$ is much smaller than w. Since f is $(1+\xi)$-bilipschitz, up to increasing a little and uniformly a and w, those are also the diameter and width parameters for each $f[\Sigma_{j-1}, \Sigma_j]$. We have for $j \geq k$

$$w(j-k) \leq d_Q(f(\Sigma_j), \partial_1 Q^{\mathrm{nc}}) \leq (w+a)(j-k) + d.$$

Therefore the size of the quasi collar $f[\Sigma_{j-1}, \Sigma_{j+1}]$ can be chosen to be

$$\begin{aligned} D &= a, \\ W &= 2w, \\ K_j &= (w+a)(j-k) + d + 2w. \end{aligned}$$

Analogous estimates hold for $\partial_2 Q^{\mathrm{nc}}$. Hence

Corollary 3.6. *There exists $w, a > 0$ and only depending on ψ such that for every $k < j < n-k$ the surface $f(\Sigma_j)$ is contained in*

$$\mathrm{collar}_{a,2w,K_j}(\partial_1 Q)$$

and, similarly, the surface $f(\Sigma_{n-j})$ is contained in

$$\mathrm{collar}_{a,2w,K_j}(\partial_2 Q).$$

The Corollaries 3.5 and 3.6 combined with Proposition 3.2 help us in checking that the handlebody and I-bundle criteria are satisfied. In fact, we have the following: With the same notation as before, consider again the $(1+\xi)$-bilipschitz embedding as a quasi collar

$$f : [\Sigma_k, \Sigma_{n-k}] \subset \hat{T}_\psi \to Q_n = Q(P, \psi^n P).$$

The bilipschitz parameter ξ can be chosen to be arbitrarily small provided that n is sufficiently large.

Corollary 3.7. *Let ψ be a mapping class with a short pants decomposition P of length η. Consider $Q_n = Q(P, \psi^n P)$. Let Σ_0 be a component of ∂M^{nc} where M is a maximally cusped handlebody or I-bundle. Suppose that we have a $(1+\xi)$-bilipschitz diffeomorphism*

$$g : \mathrm{collar}_{a,2w,K_j}(\partial_1 Q_n) \to \mathrm{collar}_{D,W,K}(\Sigma_0)$$

or

$$g : \mathrm{collar}_{a,2w,K_j}(\partial_2 Q_n) \to \mathrm{collar}_{D,W,K}(\Sigma_0)$$

for some D, W, K. If ξ is small enough (only depending on ψ), n is large enough (only depending on ψ and ξ) and $k < j < n-k$, then the collection of geodesic representatives of

$$gf(\psi^j P \times \{j\}) \subset gf(\Sigma_j)$$

or

$$gf(\psi^{n-j} P \times \{n-j\}) \subset gf(\Sigma_{n-j})$$

has length $O(\eta)$, is contained in the image of g, and is isotopic within it to $gf(\psi^j P \times \{j\})$ or $gf(\psi^{n-j} P \times \{n-j\})$.

Proof. We only treat the first case, the other one is analogous. By Corollary 3.5 and Corollary 3.6, we can assume that the geodesic representatives Γ_j of $\psi^j P$ in Q_n are contained in $\mathrm{collar}_{a,2w,K_j}(\partial_1 Q_n)$ and isotopic within it to $f(\psi^j P \times \{j\}) \subset f(\Sigma_j)$. Their length is $O(\eta)$. Since g is a $(1+\xi)$-bilipschitz diffeomorphism, if ξ is small enough compared to η, by Lemma 3.4, we can assume that the geodesic representatives of $g(\Gamma_j)$ in M are contained in $\mathrm{collar}_{D,W,K}(\partial H)$ and isotopic within it to $g(\Gamma_j)$ which, in turn, is isotopic to $gf(\psi^j P \times \{j\}) \subset gf(\Sigma_j)$. $\square$

3.3. **Criteria for I-bundles and handlebodies revised.** We are now ready to give a more manageable version of the criteria for I-bundles and handlebodies and construct many example that satisfy those conditions. This is the goal of Proposition 3.8.

PROPOSITION 3.8. *Let ψ, ϕ be mapping classes with short pants decompositions P, P' of length in $[\eta, \eta_M/2]$. There exists $j = j(\psi, \phi)$ such that the following holds: Consider*

$$(P_1, P_2) = (\psi^{-j}P, P) \text{ and } (P_3, P_4) = (P', \phi^j P').$$

Suppose that for some n very large we have respectively

(1) $d_{\mathcal{C}}(P, \mathcal{D}) \geq d_{\mathcal{C}}(P, \psi^{-n}P) + d_{\mathcal{C}}(\psi^{-n}P, \mathcal{D}) - R$,
(2) $d_{\mathcal{C}}(\psi^{-j}P, \phi^j P') \geq d_{\mathcal{C}}(\psi^{-j}P, \psi^n P) + d_{\mathcal{C}}(\psi^n P, \phi^j P') - R$,

and

(3) $d_{\mathcal{C}}(P', f\mathcal{D}) \geq d_{\mathcal{C}}(P', \phi^n P') + d_{\mathcal{C}}(\phi^n P', f\mathcal{D}) - R$,
(4) $d_{\mathcal{C}}(\psi^{-j}P, \phi^j P') \geq d_{\mathcal{C}}(\phi^j P', \phi^{-n}P') + d_{\mathcal{C}}(\phi^{-n}P', \psi^{-j}P) - R$.

Then P_1, P_2, P_3, P_4 satisfies the I-bundle *and* handlebody criteria *with parameter $O(\eta)$.*

Proof. We have to check two handlebody and one I-bundle criteria. The arguments for the three different cases follow the same lines. In order to avoid repetitions, we only prove in details that there exists $j = j(\phi, \psi)$ such that the pair $(\psi^{-j}P, P)$ satisfies the handlebody criterion if n is large enough. The other cases are completely analogous and require no new ideas. In the end of the proof, we briefly discuss the adjustments needed for the I-bundle criterion.

The handlebody criterion. In order to check the handlebody criterion for $(\psi^{-j}P, P)$, by Corollary 3.7, we just need to get a $(1 + \xi)$-bilipschitz diffeomorphism

$$g : \text{collar}_{a,2w,K_j}(\partial_2 Q(\psi^{-n}P, P)) \to \text{collar}_{D,W,K}(\partial H(P))$$

in the homotopy class of the identity. Such a diffeomorphism will be provided by Proposition 3.2. Notice at this point that $Q(\psi^{-n}P, P)$ and $Q(P, \psi^n P)$ are isometric as they only differ by the marking.

In order to apply Proposition 3.2, observe that, by work of Minsky [24], the pairs $(\psi^n P, \psi^m P)$ and $(\phi^m P', \phi^n P')$ satisfy for all n, $m \in \mathbb{Z}$ the R-bounded combinatorics condition for some $R = R(\psi, \phi) > 0$. Furthermore, for any fixed h, if $|n - m|$ is large, again, depending only on ψ, ϕ and h, they also satisfy the large height assumption as pseudo-Anosov elements act as loxodromic motions on the curve graph by Masur and Minsky [22]. Property (1) from our assumptions is exactly the last one needed to guarantee that Proposition 3.2 can be applied.

Before applying Proposition 3.2 we have to be a bit careful with the various constants and their dependence. We pause for a moment and discuss this delicate point. The mapping classes ψ and ϕ determine η and R and also the parameters a and w of Corollary 3.7 and D_0 of Proposition 3.2. Furthermore, the mapping classes together with the choice of ξ determine k and d in Proposition 3.3. In turn, k determines the allowable range $k < j < n - k$.

So, we want to choose ξ much smaller than the one, only depending on the mapping classes, required by Corollary 3.7 to hold. Once this is fixed we have a collection of potential candidates for the quasi collars

$$\text{collar}_{a,2w,K_j}(\partial_2 Q(\psi^{-n}P, P))$$

with $k < j < n - k$ for any n very large.

Once we fixed ξ, we have also fixed $K_0, W_0 > 0$ of Proposition 3.2. So, for every $W \geq W_0$ we have a $(1+\xi)$-bilipschitz diffeomorphism of quasi collars

$$f : \text{collar}_{D_0,W,K_0}(\partial_2 Q(\psi^{-n}P, P)) \to \text{collar}_{D_1,W_1,K_1}(\partial H(P)).$$

In order to get the desired embeddings, we just have to choose $k < j < n-k$ and W so that one of our candidate quasi-collars is contained in the domain of definition of f

$$\text{collar}_{a,2w,K_j}(\partial_2 Q(\psi^{-n}P, P)) \subset \text{collar}_{D_0,W,K_0}(\partial_2 Q(\psi^{-n}P, P)).$$

It suffices to do the following: We first choose j so that $K_j - a > K_0 + D_0$. This determines a minimal $j = j(\psi, \phi)$ as required by the statement of Proposition 3.8. Then, we choose W so that $K_0 + W - D_0 > 2K_j + 4a + 2w$. This finally determines a final threshold for h and n.

The I-bundle criterion. The proof is word by word the same as in the handlebody case, one only has to replace the collar of $\partial H(P)$ with $\partial_1 Q(\psi^{-j}P, \phi^j P')$ and $\partial_2 Q(\psi^{-j}P, \phi^j P')$. Again, if n is very large, Proposition 3.2 furnishes $(1+\xi)$-bilipschitz diffeomorphisms

$$\text{collar}_{a,2w,K_j}(\partial_1 Q(\psi^{-j}P, \psi^{n-j}P)) \to \text{collar}_{D_1,W_1,K_1}(\partial_1 Q(\psi^{-j}P, \phi^j P'))$$

and

$$\text{collar}_{a,2w,K_j}(\partial_2 Q(\phi^{j-n}P, \phi^j P)) \to \text{collar}_{D_1,W_1,K_1}(\partial_2 Q(\psi^{-j}P, \phi^j P')).$$

Notice that the constant $j = j(\phi, \psi)$ is the same as before. Corollary 3.7 together with a careful bookkeping of the markings concludes the proof. $\square$

3.4. **From the curve graph to Teichmüller space.** We now translate the curve graph conditions (1) - (4) in terms of Teichmüller geometry in such a way that it will not be hard to check them for a random segment $[o, fo]$.

It is convenient to recall now a few facts due to Masur and Minsky [22], [23] about the relation between Teichmüller space $\mathcal{T}$ endowed with the Teichmüller metric $d_{\mathcal{T}}$ and the curve graph $\mathcal{C}$.

The connection is established via the *shortest curves projection* $\Upsilon : \mathcal{T} \to \mathcal{C}$, a coarsely defined map that associates to every marked hyperbolic surface

$X \in \mathcal{T}$ a shortest geodesic pants decomposition on it $\Upsilon(X)$. We choose $o \in \mathcal{T}$ with the following property

> **Standing assumption**: The base point $o \in \mathcal{T}$ is chosen so that its projection to the curve graph lies on the disk set of the handlebody $\Upsilon(o) \in \mathcal{D}$.

From a geometric point of view, Masur and Minsky proved that the curve graph $\mathcal{C}$ is a Gromov hyperbolic space (see [22]) and the disk set $\mathcal{D} \subset \mathcal{C}$ is a uniformly quasi-convex subspace (see [21]).

It also follows from [22] that Υ is uniformly Lipschitz and sends Teichmüller geodesics to uniform *unparametrized* quasi-geodesics. This means that there is a constant B only depending on Σ such that $d_{\mathcal{C}}(\Upsilon(Y), \Upsilon(Z)) \geq d_{\mathcal{C}}(\Upsilon(Y), \Upsilon(X)) + d_{\mathcal{C}}(\Upsilon(X), \Upsilon(Z)) - B$ for every $Z < X < Y$ aligned on a Teichmüller geodesic. In particular, by hyperbolicity of $\mathcal{C}$, the image $\Upsilon[Z, Y] \subset \mathcal{C}$ is a uniformly quasi-convex subset.

We have the following:

Proposition 3.9. *Let ψ, ϕ be pseudo-Anosov mapping classes with short pants decompositions P, P'. Let $l_\psi, l_\phi : \mathbb{R} \to \mathcal{T}$ be their Teichmüller geodesics. For every $\delta > 0$ there exists $h > 0$ such that the following holds: Suppose that on the segment $[o, fo]$ there are four points $o < S_1 < S_2 < S_3 < S_4 < fo$ with the following properties*

(i) $[S_1, S_2]$ *and* $[S_3, S_4]$ *have length at least h and δ-fellow travel l_ψ and l_ϕ respectively.*
(ii) *We have $d_{\mathcal{C}}(\Upsilon[S_1, S_4], \mathcal{D}) \geq h$ and $d_{\mathcal{C}}(\Upsilon[S_1, S_4], f\mathcal{D}) \geq h$.*

Then, up to perhaps replacing P, P' with $\psi^r P, \phi^{r'} P'$, (1) - (4) *hold.*

The proof uses the arguments from Proposition 7.1 of [15].

Proof. We have to show that (i) and (ii) imply (1) - (4).

We start with Properties (1) and (3). Let us only consider (1) as (3) is completely analogous. Recall that we fixed o so that $\Upsilon(o) \in \mathcal{D}$ and hence $\Upsilon(fo) = f\Upsilon(o) \in f\mathcal{D}$.

As $\Upsilon[o, fo]$ is a uniformly quasi-convex subset of the Gromov hyperbolic space $\mathcal{C}$, there is a coarsely well defined nearest point projection $\pi : \mathcal{C} \to \Upsilon[o, fo]$. Since $\mathcal{D}, f\mathcal{D}$ are also uniformly quasi-convex and $\Upsilon[S_1, S_4]$ is very far from both while the endpoints satisfy $\Upsilon(o) \in \mathcal{D}$ and $f\Upsilon(o) \in f\mathcal{D}$, we conclude that $\pi(\mathcal{D})$ and $\pi(f\mathcal{D})$ lie on opposite sides of $\Upsilon[S_1, S_4]$ and are far from it.

Consider $S_1 \leq a < b \leq S_4$ and the projections $\alpha := \Upsilon(a)$ and $\beta := \Upsilon(b)$. By hyperbolicity of $\mathcal{C}$ and uniform quasi-convexity of $\Upsilon[o, fo]$, any geodesic joining $\delta_0 \in \mathcal{D}$ to β can be broken into two subsegments $[\delta_0, \beta_0] \cup [\beta_0, \beta]$ where β_0 is uniformly close to $\pi(\delta_0)$, the nearest point projection of δ_0, which has the form $\Upsilon(t)$ for some $o < t < S_1$. By the uniform unparametrized quasi-geodesic image property of Υ, the segment $[\beta_0, \beta]$ passes uniformly close to

α. Therefore, we have

$$d_{\mathcal{C}}(\beta,\delta_0) \geq d_{\mathcal{C}}(\beta,\alpha) + d_{\mathcal{C}}(\alpha,\delta_0) - R_0$$

for some uniform R_0. Taking the minimum over all $\delta_0 \in \mathcal{D}$ we get

$$d_{\mathcal{C}}(\Upsilon(b),\mathcal{D}) \geq d_{\mathcal{C}}(\Upsilon(a),\Upsilon(b)) + d_{\mathcal{C}}(\Upsilon(a),\mathcal{D}) - R_0.$$

The last ingredient that we need is the fact that the sequence of curves $\{\psi^n P\}_{n\in\mathbb{Z}}$ lie uniformly close, only depending on ψ, to the uniform quasi-axis of ψ given by the composition Υl_ψ. This follows from work of Minsky [24]. Notice that Υl_ψ lies uniformly close to $\Upsilon[S_1,S_2]$ by fellow traveling assumption. In particular, there are $\psi^{r+n}P$, $\psi^r P$ and $\psi^{r-n}P$ that lie uniformly close, only depending on ψ, to $\Upsilon(S_1)$, $\Upsilon(S_\psi)$ and $\Upsilon(S_2)$ where $S_1 < S_\psi < S_2$. The difference $(r+n)-(r-n) = 2n$ is bounded from below by some linear function of h of the form $\kappa h - \kappa$ with $\kappa > 0$ only depending on ψ.

Therefore, as $S_1 < S_\psi$, we have

$$d_{\mathcal{C}}(\psi^r P,\mathcal{D}) \geq d_{\mathcal{C}}(\psi^r P,\psi^{r-n}P) + d_{\mathcal{C}}(\psi^{r-n}P,\mathcal{D}) - R$$

for some uniform R, only depending on ψ.

For simplicity, we replace P with $\psi^r P$ and still denote it by P.

We now move on to Properties (2) and (4). Again, we consider only (2) as (4) uses the same arguments. Property (2) just follows from the uniform unparametrized quasi-geodesic property of $\Upsilon[S_1,S_4]$.

In more details we proceed as follows: As before, up to replacing P and P' with $\psi^r P$ and $\phi^{r'}P'$, we can assume that $\psi^{-n}P, \psi^n P$ (resp. $\phi^{-n'}P', \phi^{n'}P'$) are uniformly close, only depending on ψ (resp. ϕ), to $\Upsilon(S_1),\Upsilon(S_2)$ (resp. $\Upsilon(S_3),\Upsilon(S_4)$).

Recall that P_1 and P_4 are of the form $P_1 = \psi^{-j}P$ and $P_4 = \phi^j P'$ for some uniform $j = j(\phi,\psi)$. So, up to a uniform error, we can replace them with P and P'. For them we have

$$d_{\mathcal{C}}(\Upsilon(S_\psi),\Upsilon(S_\phi)) \geq d_{\mathcal{C}}(\Upsilon(S_\psi),\Upsilon(S_2)) + d_{\mathcal{C}}(\Upsilon(S_2),\Upsilon(S_\phi)) - R.$$

□

4. The proof of Theorem 1

In this section we prove

Theorem 2. *For every $\epsilon > 0$ and $K > 1$ we have*

$$\mathbb{P}_n[M_f \text{ has a hyperbolic metric } K\text{-bilipschitz to a } \epsilon\text{-model metric}] \stackrel{n\to\infty}{\longrightarrow} 1.$$

The proof of Theorem 2 does not use 3-dimensional hyperbolic geometry anymore. Rather, via Proposition 3.9, we will only have to work with the dynamics of a random walk on Teichmüller space and the curve graph.

Thank to the work done in the previous sections, namely Proposition 2.4, Proposition 3.8 and Proposition 3.9, we only need to check that the

Teichmüller segment $[o, fo]$ contains four points $o < S_1 < S_2 < S_3 < S_4 < fo$ satisfying the conditions (i) and (ii) of Proposition 3.9.

In order to find them we will exploit a useful ergodic property of random walks on Teichmüller space due to Baik, Gekhtman and Hamenstädt [1].

4.1. **Random walks.** We start by recalling some background material on random walks on the mapping class group. We crucially consider only random walks driven by probability measures μ whose support S is a finite symmetric generating set for the *full* mapping class group.

DEFINITION (Random Walk). Let $(s_n)_{n\in\mathbb{N}}$ be a sequence of independent random variables with values in S and distribution μ. The *n-th step of the random walk* is the random variable $f_n := s_1 \dots s_n$. We denote by $\mathbb{P}_n$ its distribution. The *random walk driven by* μ is the process $(f_n = s_1 \dots s_n)_{n\in\mathbb{N}} \in \mathrm{Mod}(\Sigma)^{\mathbb{N}}$. It has a distribution which we denote by $\mathbb{P}$.

The mapping class group acts on Teichmüller space $\mathrm{Mod}(\Sigma) \curvearrowright \mathcal{T}$. If we fix a base point $o \in \mathcal{T}$ we can associate to every random walk $(f_n)_{n\in\mathbb{N}}$ an orbit $\{f_n o\}_{n\in\mathbb{N}} \subset \mathcal{T}$. This sequence of jumps in Teichmüller space has two important properties, *positive* linear drift and *sublinear* geodesic tracking, which we now explain.

It is a standard consequence of the subadditive ergodic theorem that there exists a constant $L \geq 0$, called the *drift* of the random walk on Teichmüller space, such that for $\mathbb{P}$-almost every sample path $(f_n)_{n\in\mathbb{N}}$ we have

$$\frac{d_{\mathcal{T}}(o, f_n o)}{n} \overset{n\to\infty}{\longrightarrow} L.$$

In general, however, the drift might vanish $L = 0$. It has been established by Kaimanovich and Masur [17] that, in our case, $L > 0$ and that the orbit $\{f_n o\}_{n\in\mathbb{N}}$ converges $\mathbb{P}$-almost surely to some point on the Thurston compactification of Teichmüller space $\mathcal{PML}$. Tiozzo [32] then showed that this convergence happens by sublinearly tracking a Teichmüller ray:

THEOREM 4.1 (Tiozzo [32]). *For $\mathbb{P}$-almost every sample path $(f_n)_{n\in\mathbb{N}}$ and for every base point $o \in \mathcal{T}$, there exists a unit speed Teichmüller ray $\tau : [0, +\infty)$ starting at $\tau(0) = o$ and converging to $\mathcal{PML}$ such that*

$$\lim_{n\to\infty} \frac{d_{\mathcal{T}}(f_n o, \tau(Ln))}{n} = 0.$$

More precisely, the sublinear tracking property can be implemented to the following (see, for example, the proof of Proposition 6.11 of [1]): There exists $\delta > 0$ such that for every $\epsilon > 0$ and $\mathbb{P}$-almost every $(f_n)_{n\in\mathbb{N}}$ with tracking ray τ we have that the segment $[o, f_n o]$ δ-fellow travels $\tau[0, (1-\epsilon)Ln]$ for every n sufficiently large.

4.2. **Random Teichmüller rays.** We can now state the result from Baik, Gekhtman and Hamenstädt [1] that we need

THEOREM 4.2 (Proposition 6.9 of [1]). *Let $W \subset \mathcal{T}$ be a $\mathrm{Mod}(\Sigma)$-invariant open subset containing the axis of a pseudo-Anosov element. For every $h > 0$ and every $0 < a < b$ and for $\mathbb{P}$-almost every $(f_n)_{n\in\mathbb{N}}$ with tracking ray τ there exists $n_0 > 0$ such that for every $n \geq n_0$ we have that $\tau[an, bn] \cap W$ contains a segment of length at least h.*

The statement of Theorem 4.2 slightly differs from the one of Proposition 6.9 of [1]. In fact, there it is only considered the case where W is a large metric neighbourghood of the mapping class group orbit of the base point. However, their arguments apply also to the more general setting.

A relevant example of an invariant open set, which is also the smallest possible, is the following: Let ϕ be a pseudo-Anosov mapping class. It determines a closed geodesic in moduli space $\gamma_\phi \subset \mathcal{M} := \mathcal{T}/\mathrm{Mod}(\Sigma)$ and an axis in Teichmüller space $l_\phi \subset \mathcal{T}$.

Fix $\delta > 0$ very small and consider a tiny δ-metric neighbourhood $V_{\phi,\delta}$ of γ_ϕ. Its preimage in Teichmüller space $W_{\phi,\delta} \subset \mathcal{T}$ consists of the union of tiny neighbourhoods of the Teichmüller axes gl_ϕ of the pseudo-Anosov elements $g\phi g^{-1}$ as g varies in the mapping class group.

The set $W_{\phi,\delta}$ has the following useful property

LEMMA 4.3. *For every ϕ and h there exists $\delta = \delta(\phi, h) > 0$ such that if $l : [a, b] \to \mathcal{T}$ is a Teichmüller geodesic segment of length $b - a \geq h$ whose image is entirely contained in $W_{\phi,\delta}$ then it 1-fellow travels one of the lines $gl_\phi \subset W_{\phi,\delta}$.*

Proof. This can be proved by contradiction. Suppose this is not the case and consider a sequence $\delta_n \downarrow 0$ and Teichmüller segments $l_n : [0, h_n] \to W_{\phi,\delta_n}$ of length $h_n \geq h$. Up to the action of the mapping class group, we can assume that $l_n(0)$ lies in a fixed compact set. Up to subsequences, we can then arrange that $l_n[0, h]$ converges uniformly to a Teichmüller geodesic of length h entirely contained in $\bigcap_{n\in\mathbb{N}} W_{\phi,\delta_n}$ which is nothing but the union of the axes gl_ϕ. Hence, l_∞ is a subsegment of one of the axes gl_ϕ and, therefore, at a certain point l_n must have 1-fellow traveled this axis, a contradiction. □

4.3. **The proof of Theorem 2.** Consider the Teichmüller segment $[o, fo]$. Fix $\delta > 0$ large enough. We need to find two pseudo Anosov mapping classes ψ and ψ' with short pants decompositions and four surfaces $o < S_1 < S_2 < S_3 < S_4 < fo$ such that

(i) $[S_1, S_2]$, $[S_3, S_4]$ have length at least h, depending only on δ, ψ, ψ', and δ-fellow travel $l_\psi, l_{\psi'}$.
(ii) $d_{\mathcal{C}}(\Upsilon[S_1, S_4], \mathcal{D}) \geq h$ and $d_{\mathcal{C}}(\Upsilon[S_1, S_4], f\mathcal{D}) \geq h$

We first prove the second property: Recall that we chose $o \in \mathcal{T}$ so that $\delta := \Upsilon(o) \in \mathcal{D}$ (and hence $f\delta = \Upsilon(fo) \in f\mathcal{D}$).

Claim: For every $h > 0$ and $\epsilon > 0$, for $\mathbb{P}$-almost every $(f_n)_{n\in\mathbb{N}}$ with associated tracking ray τ, there exists n_0 and such that for every $n > n_0$

both $d_{\mathcal{C}}(\Upsilon\tau[\epsilon Ln,(1-\epsilon)Ln],\mathcal{D})$ and $d_{\mathcal{C}}(\Upsilon\tau[\epsilon Ln,(1-\epsilon)Ln],f_n\mathcal{D})$ are greater than h.

Proof of the claim. Denote by $\ell = \lim d_{\mathcal{C}}(\delta, f_n\delta)/n > 0$ the drift of the random walk on the curve graph. It is positive by a result of Maher [19]. The claim is a consequence of

THEOREM 4.4 (Maher [20]). *For every $\epsilon > 0$ we have*

$$\mathbb{P}_n\left[d_{\mathcal{C}}(\mathcal{D}, f\mathcal{D}) \in [(\ell-\epsilon)n,(\ell+\epsilon)n]\right] \overset{n\to\infty}{\longrightarrow} 1.$$

Choose $\epsilon_2 > \epsilon_1 > 0$ much smaller than ϵ. By Theorem 4.4 we have $d_{\mathcal{C}}(\mathcal{D}, f_n\delta) \geq (\ell-\epsilon_1)n$ for n large.

By Theorem 4.1, the random orbit $f_n o$ has a positive drift $d_{\mathcal{T}}(o, f_n o)/n \to L > 0$ and is sublinearly tracked by a geodesic ray $\tau : [0,\infty) \to \mathcal{T}$ with $\tau(0) = o$. This means that $d_{\mathcal{T}}(f_n o, \tau(Ln)) = o(n)$. We assume $d_{\mathcal{T}}(f_n o, \tau(Ln)) < \epsilon_1 n$ and $\ell - \epsilon_1 < d_{\mathcal{C}}(\delta, f_n\delta) < \ell + \epsilon_1$ for every n large.

Consider $m \in [\epsilon_1 n, (1-\epsilon_2)n]$.

We have the following estimate on the distance from $\mathcal{D}$: Let $B > 0$ be the Lipschitz constant of $\Upsilon : \mathcal{T} \to \mathcal{C}$

$$\begin{aligned} d_{\mathcal{C}}(\Upsilon\tau(Lm),\mathcal{D}) &\geq d_{\mathcal{C}}(f_m\delta,\mathcal{D}) - d_{\mathcal{C}}(f_m\delta,\Upsilon\tau(Lm)) \\ &\geq (\ell-\epsilon_1)m - B\epsilon_1 m \\ &\geq (\ell-\epsilon_1-B\epsilon_1)\epsilon_1 n. \end{aligned}$$

Notice that if ϵ_1 is small enough, the right hand side increases linearly in n with uniform constants.

As for the other disk set $f_n\mathcal{D}$, we also get

$$\begin{aligned} d_{\mathcal{C}}(\Upsilon\tau(Lm), f_n\mathcal{D}) &\geq d_{\mathcal{C}}(\delta, f_n\mathcal{D}) - d_{\mathcal{C}}(\delta, f_m\delta) - d_{\mathcal{C}}(f_m\delta,\Upsilon\tau(Lm)) \\ &\geq (\ell-\epsilon_1)n - (\ell+\epsilon_1)m - B\epsilon_1 m \\ &\geq [(\ell-\epsilon_1) - (\ell+\epsilon_1)(1-\epsilon_2) - B\epsilon_1(1-\epsilon_2)]n. \end{aligned}$$

As before, if ϵ_1 is very small compared to ϵ_2, the right hand side increases linearly in n with uniform constants. In conclusion, if ϵ_1 is small enough and n is large enough, the claim holds as $[\epsilon Ln,(1-\epsilon)Ln] \subset [\epsilon_1 n,(1-\epsilon_2)n]$.

This settles the proof of property (ii) for the segment $\tau[\epsilon Ln,(1-\epsilon)Ln]$. Observe that any subsegment $[S_1^n, S_4^n]$ will enjoy the same property.

We now take care of (i).

Claim: Let ϕ pseudo-Anosov element with a short pants decomposition. Let $l_\phi : \mathbb{R} \to \mathcal{T}$ be its axis. For every $\epsilon > 0$, for every $h > 0$, for $\mathbb{P}$-almost every $(f_n)_{n\in\mathbb{N}}$ with tracking ray $\tau : [0,\infty) \to \mathcal{T}$ there exists n_0 such that for every $n \geq n_0$ the Teichmüller segments $\tau[\epsilon n, 2\epsilon n]$ and $\tau[(1-2\epsilon)n,(1-\epsilon)n]$ 1-fellow travel along subsegments $\tau[t_1^n, t_2^n]$ and $\tau[t_3^n, t_4^n]$ of length at least h some translates $\psi = g_n l_\phi$ and $\psi' = g_n' l_\phi$ of the axis l_ϕ.

Proof of the claim. Let $\delta = \delta(\phi, h)$ and $W = W_{\phi,\delta}$ be as in Lemma 4.3. We just need to apply Theorem 4.2 to W with parameters $0 < a < b$ given by $0 < L\epsilon < 2\epsilon L$ and $0 < (1-2\epsilon)L < (1-\epsilon)L$ respectively.

Conclusion of the proof: For every fixed $\epsilon > 0$ the Teichmüller segment $[o, f_n o]$ uniformly fellow travels $\tau[\epsilon Ln, (1-\epsilon)Ln]$ we define $o < S_1^n < S_2^n < S_3^n < S_4^n < f_n o$ to be the four surfaces that fellow travel $\tau(t_1^n) < \tau(t_2^n) < \tau(t_3^n) < \tau(t_4^n)$ as given by the second claim.

By construction they satisfy the properties (i) and (ii) of Proposition 3.9 and hence can be used in the model metric construction of Proposition 2.4.

This concludes the proof of Theorem 2.

5. Three applications

We describe three applications of Theorem 2.

5.1. **The model metric.** For convenience of the reader we recall again the description of the model metric $\mathbb{M}_n = H_1^n \cup \Omega_1^n \cup Q_n \cup \Omega_2^n \cup H_2^n$ that comes from the proof of Theorem 2.

For the applications of this section we focus only on the maximally cusped structure $Q_n = Q(P_2^n, P_3^n)$ and recall that it bilipschitz embeds, away from its cusps, into M_{f_n} with bilipschitz constant arbitrarily close to 1 as n goes to ∞.

A careful inspection shows that P_2^n and P_3^n can be chosen to be very short pants decompositions on some hyperbolic surfaces S_2^n and S_3^n that are located in Teichmüller space uniformly close to the segments $\tau[\epsilon Ln, 2\epsilon Ln]$ and $\tau[(1-2\epsilon)Ln, (1-\epsilon)Ln]$ where $\tau : [0, \infty) \to \mathcal{T}$ is the tracking ray of the random walk and ϵ is an arbitrarily small constant.

Theorem 2.2 allows us to replace, up to uniformly bilipschitz distortion $Q(P_2^n, P_3^n)$ with $Q(S_2^n, S_3^n)$ away from the cusps. It will be convenient to work also with $Q(S_2^n, S_3^n)$ instead of Q_n. They are both geometrically very close to M_{f_n}.

5.2. **Geometric limits of random 3-manifolds.** As a first application, we exploit the model metric structure to establish the existence of certain geometric limits (see Chapter E.1 of [2] for the definition of the pointed geometric topology) for families of random 3-manifolds.

Proposition 5.1. *For every pseudo-Anosov mapping class $\phi \in \mathrm{Mod}(\Sigma)$ and $\mathbb{P}$-almost every $(f_n)_{n\in\mathbb{N}}$ there exists a sequence of base points $x_n \in M_{f_n}$ such that the sequence of pointed manifolds (M_{f_n}, x_n) converges to the infinite cyclic covering of T_ϕ in the pointed geometric topology.*

Proof. As in the proof of the second claim of Theorem 2, Proposition 4.2 and Lemma 4.3 imply that the segment $\tau[3\epsilon Ln, (1-3\epsilon)Ln]$, and hence $[S_2^n, S_3^n]$, 1-fellow travels a translate $g_n l_\phi$ of the Teichmüller axis l_ϕ along an arbitrarily long subsegment. Therefore, up to remarking $[S_2^n, S_3^n]$, an operation that does not change the isometry type of $Q(S_2^n, S_3^n)$, we can assume that $[S_2^n, S_3^n]$ 1-fellow travels l_ϕ along the subsegment $l_\phi[-a_n, a_n]$ with $a_n \uparrow \infty$. In particular, the sequence of Teichmüller segments $[S_2^n, S_3^n]$ is converging uniformly on compact subsets to l_ϕ. By Thurston's Double Limit Theorem [29] and the solution of the Ending Lamination Conjecture [25], [7], this implies that, if we take suitable base points, the sequence of convex cocompact manifolds $Q(S_2^n, S_3^n)$ converges in the geometric topology to $\hat{T}_\psi$. As $Q(S_2^n, S_3^n)$ becomes geometrically arbitrarily close to M_{f_n}, the claim follows. ☐

5.3. **Commensurability and arithmeticity.** Dunfield and Thurston, using a simple homology computation, have shown in [13] that their notion of random 3-manifold is not biased towards a certain fixed set of 3-manifolds. This means that for every fixed 3-manifold M, only finitely many elements in the family $(M_{f_n})_{n\in\mathbb{N}}$ can be diffeomorphic to M.

Using geometric tools it is possible to strengthen this conclusions and show that Dunfield and Thurston's notion of random 3-manifolds is also transverse, in a sense made precise in the proposition below, to the class of arithmetic hyperbolic 3-manifolds and to the class of 3-manifolds which are commensurable to a fixed 3-manifold M.

Proposition 4. *For $\mathbb{P}$-almost every $(f_n)_{n\in\mathbb{N}}$ the following holds*

(1) *There are at most finitely many 3-manifolds in the family $(M_{f_n})_{n\in\mathbb{N}}$ that are arithmetic.*
(2) *There are at most finitely many 3-manifolds in the family $(M_{f_n})_{n\in\mathbb{N}}$ that are in the same commensurability class.*

Proof. The argument is mostly borrowed from Biringer-Souto [3].

The proof of both points starts from the following observation: Each M_{f_n} finitely covers a maximal orbifold $M_{f_n} \to \mathcal{O}_n$. By Proposition 5.1 we can choose base points $x_n \in M_{f_n}$ so that the sequence (M_{f_n}, x_n) converges geometrically to (Q_∞, x_∞) where Q_∞ is a doubly degenerate structure on $\Sigma \times \mathbb{R}$ with $\mathrm{inj}(Q_\infty) > 0$.

Suppose that infinitely many M_{f_n} are arithmetic, say all of them. In this case, the orbifolds $\mathcal{O}_n$ are congruence and have $\lambda_1(\mathcal{O}_n) \geq 3/4$ (see [11] or Theorem 7.1 in [3]). By Proposition 4.3 of [3], the orbifolds $\mathcal{O}_n$ cannot be all different, hence we can assume that they are fixed all the time $\mathcal{O}_n = \mathcal{O}$. We get a contradiction by observing that $\mathcal{O}$ is covered by closed 3-manifolds M_{f_n} with arbitrarily small injectivity radius.

Suppose that infinitely many M_{f_n} are commensurable. By the first part we can also assume that they are non-arithmetic. Commensurability and non-arithmeticity imply together that $\mathcal{O}_n = \mathcal{O}$ is fixed all the time: It is the orbifold corresponding to the commensurator $\mathrm{Comm}(\pi_1(M_{f_n}))$, which is a discrete subgroup of $\mathrm{PSL}_2\mathbb{C}$ by Margulis (see Theorem 10.3.5 in [18]) and is an invariant of the commensurability class. We conclude with the same argument as before. □

5.4. **Diameter growth.** As a more geometric application, we compute the coarse growth rate for the diameter of random 3-manifolds.

Proposition 3. *There exists* $c > 0$ *such that*

$$\mathbb{P}_n[\text{diam}(M_f) \in [n/c, cn]] \overset{n\to\infty}{\longrightarrow} 1.$$

The proof of Proposition 3 has two different arguments, one for the coarse upper bound and one for the coarse lower bound. The upper bound comes from a result by White [33] that relates the diameter to the *presentation length* of the fundamental group, a topological and algebraic invariant. Of a different nature is the coarse lower bound where we heavily use the ϵ-model metric structure of Theorem 2 and the relation with the model manifold.

We start with the upper bound. We need the following definition

Definition (Presentation Length). Let G be a finitely presented group. The length of a finite presentation $G = \langle F \,|R\rangle$ is given by

$$l(F, R) = \sum_{r\in R} |r|_F - 2$$

where $|r|_F$ denotes the word length of the relator $r \in R$ with respect to the generating set F. The presentation length of G is defined to be

$$l(G) := \min\{l(F, R) \mid G = \langle F \,|R\rangle \text{ finite presentation}\}.$$

We also recall that a relator $r \in R$ is triangular if $|r|_F \leq 3$.

Theorem 5.2 (White [33]). *There exists* $c > 0$ *such that for every closed hyperbolic 3-manifold* M *we have*

$$\text{diam}(M) \leq c \cdot l(\pi_1 M).$$

Let $S \subset \text{Mod}(\Sigma)$ be the finite support of the probability measure μ.

Lemma 5.3. *There exists* $C(S) > 0$ *such that for every* $f \in \text{Mod}(\Sigma)$ *we have*

$$l(\pi_1(M_f)) \leq C|f|_S.$$

In particular $\text{diam}(M_f) \leq K|f|_S$ *where* $K = c \cdot C$.

Proof. The 3-manifold M_f admits a triangulation T with a number of simplices uniformly proportional, depending on S, to the word length $|f|_S$. We have $\pi_1(M_f) = \pi_1(T_2)$ where T_2 denotes the 2-skeleton of T. By van Kampen, the fundamental group of a 2-dimensional connected simplicial complex X admits a presentation $\pi_1(X) = \langle F \,|R\rangle$ where every relation is triangular and the number of relations $|R|$ is roughly the number of 2-simplices. □

As a corollary, we get

$$\text{diam}(M_{f_n}) \leq K|f_n = s_1 \dots s_n|_S \leq Kn$$

thus proving the upper bound in Proposition 3.

The coarse lower bound follows from the structure of the model metric and the following estimate that comes from the model manifold technology of Minsky [25].

PROPOSITION 5.4 (Theorem 7.16 of [6]). *For every $L > 0$ there exists $A > 0$ such that the following holds: Let Q be a marked hyperbolic structure on $\Sigma \times \mathbb{R}$. Suppose that $\alpha, \beta \in \mathcal{C}$ have length bounded by $l_Q(\alpha), l_Q(\beta) \leq L$. Then*

$$d_Q(\mathbb{T}_{\eta_M}(\alpha), \mathbb{T}_{\eta_M}(\beta)) \geq A d_{\mathcal{C}}(\alpha, \beta) - A.$$

In particular, if $Q = Q(P_2, P_3)$ is a maximal cusp then the distance between the boundary components of its non-cuspidal part Q^{nc} is at least $A d_{\mathcal{C}}(P_2, P_3) - A$. In the case of random 3-manifolds we have

$$\begin{aligned} d_{\mathcal{C}}(P_2^n, P_3^n) &\simeq d_{\mathcal{C}}(\Upsilon(S_2^n), \Upsilon(S_3^n)) \\ &\geq d(o, f_n o) - d_{\mathcal{C}}(\Upsilon(o), \Upsilon(S_2^n)) - d_{\mathcal{C}}(\Upsilon(S_3^n), f_n \Upsilon(o)) \\ &\simeq \ell n - o(n). \end{aligned}$$

APPENDIX A. ISOTOPIES OF MARGULIS TUBES

We prove the following

LEMMA. *For every $\eta < \eta_M/2$ there exists $\xi > 0$ such that the following holds: Let $\mathbb{T}_{\eta_M}(\alpha)$ be a Margulis tube with core geodesic α of length $l(\alpha) \in [\eta, \eta_M/2]$. Suppose that there exists a $(1+\xi)$-bilipschitz embedding of the tube in a hyperbolic 3-manifold $f : \mathbb{T}_{\eta_M}(\alpha) \to M$. Then $f(\alpha)$ is homotopically non-trivial and it is isotopic to its geodesic representative within $f(\mathbb{T}_{\eta_M}(\alpha))$.*

Proof. The universal cover of $\mathbb{T}_{\eta_M}(\alpha)$ is a a-neighbourhood $N_a(l)$ of a geodesic $l \subset \mathbb{H}^3$. Denote by $F : N_a(l) \to \mathbb{H}^3$ the lift of f to the universal coverings.

By basic hyperbolic geometry, we have that for every subsegment $[p, q] \subset l$ of length $l([p, q]) \leq \eta$, the image $F[p, q]$ is contained in the ϵ-neighbourhood of the geodesic $[F(p), F(q)]$ with $\epsilon = O(\xi)$. This implies, if ξ is sufficiently small, that F restricted to l is a uniform quasi-geodesic. As a consequence f is π_1-injective and $f(\alpha)$ is homotopic its geodesic representative β within $N_\epsilon(\beta)$ with $\epsilon = O(\xi)$. We want to show that $f(\alpha)$ is actually isotopic to β.

The proof can now be concluded using topological tools.

Up to a very small isotopy we can assume that $f(\alpha)$ is disjoint from β and still contained in $N_\epsilon(\beta)$. For safety, we assume that an entire metric tubular neighbourhood of $f(\alpha)$ of the form $f(N_\delta(\alpha))$ for some tiny δ is disjoint from β and contained in $N_\epsilon(\beta)$.

Since the radius of the tube $f(\mathbb{T}_{\eta_M}(\alpha))$ is large, we can assume that a metric tubular neighbourhood of β of the form $N_r(\beta)$ with $r > \epsilon$ is contained in $f(\mathbb{T}_{\eta_M}(\alpha))$. Denote by $T_\beta = \partial N_r(\beta)$ its boundary and observe that $T_\beta \subset f(\mathbb{T}_{\eta_M}(\alpha)) - f(N_\delta(\alpha))$. The complementary region $f(\mathbb{T}_{\eta_M}(\alpha)) - f(N_\delta(\alpha))$ is diffeomorphic to $T_\alpha \times [0, 1]$ where T_α is a 2-dimensional torus.

Notice that T_β is incompressible in $T_\alpha \times [0,1]$. In fact, the only possible compressible curve on T_β is the boundary ∂D_β of the compressing disk D_β of the tubular neighbourhood of $N_r(\beta)$. Every other simple closed curve is homotopic in $f(\mathbb{T}_{\eta_M}(\alpha))$ to a multiple of $\beta \simeq f(\alpha)$ and hence it is not trivial. However, the curve ∂D_β cannot be compressible in $T_\alpha \times [0,1]$ otherwise it would bound a disk D'_β with interior disjoint from D_β and together they would give a 2-sphere $S^2 \simeq D_\beta \cup D'_\beta$ intersecting once β. Such a sphere is homologically non trivial in $f(\mathbb{T}_{\eta_M}(\alpha))$, but a solid torus does not contain such an object.

By standard 3-dimensional topology, incompressibility implies that T_β is parallel to $T_\alpha \times \{1\} = f(\mathbb{T}_{\eta_M}(\alpha))$. Therefore, β is the core curve $\beta \simeq 0 \times S^1$ for another product structure $f(\mathbb{T}_{\eta_M}(\alpha)) \simeq D^2 \times S^1$ or, in other words, there exists an orientaion preserving self diffeomorphism of $f(\mathbb{T}_{\eta_M}(\alpha))$ that sends $f(\alpha)$ to β. Such a diffeomorphism is isotopic to a power of the Dehn twist along the meridian disk of the solid torus, hence it does not change the isotopy class of the core curve.

This concludes the proof. □

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