Real Johnson-Wilson theories

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Young Women in Topology
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Suppose $E$ is a complex oriented generalized cohomology theory.

- $E^*(\mathbb{C}P^\infty) \cong E^*[x]$ for $x \in E^2(\mathbb{C}P^\infty)$
  $x$ = the first Chern class (Euler class) of the tautological line bundle
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- Examples: $H^{*}(-; R)$, $KU^{*}(-)$, and $MU^{*}(-)$
Suppose $E$ is a complex oriented generalized cohomology theory.

- $E^*(\mathbb{C}P^\infty) \cong E^*[[x]]$ for $x \in E^2(\mathbb{C}P^\infty)$, where $x$ is the first Chern class (Euler class) of the tautological line bundle.

- The classifying map $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \xrightarrow{\mu} \mathbb{C}P^\infty$ of the tensor product of line bundles gives rise to a power series $F_E$ over $E^*$.

\[
E^*(\mathbb{C}P^\infty) \xrightarrow{\mu^*} E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \\
\| \hspace{1cm} \| \\
E^*[[x]] \xrightarrow{} E^*[[x,y]]
\]
Complex Oriented Theories and Formal Group Laws

Suppose $E$ is a complex oriented generalized cohomology theory.

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\| \quad \|\]
\[
E^*[[x]] \xrightarrow{\mu} E^*[[x, y]]
\]

- $F_E$ is a formal group law over $E^*$
Quillen and Landweber

Theorem (Quillen)

If $F$ is a formal group law over $R$, then there is a unique map $MU^* \to R$ carrying $F_{MU}$ to $F_R$. 

Natural Question: When is this a cohomology theory?

Theorem (Landweber)

Let $v_i$ be the coefficient of $x^{p^i}$ in $\left[ p \right] F(x)$. If the sequence $(v_0, v_1, v_2, ...)$ forms a regular sequence in $R$ for every prime $p$, then $MU^*(X) \otimes MU^* R$ is a cohomology theory.
Theorem (Quillen)

*If* $F$ *is a formal group law over* $R$, *then there is a unique map* $MU^* \to R$ *carrying* $F_{MU}$ *to* $F_R$.

Given a formal group law $F$ over $R$, we can form

$$MU^*(X) \otimes_{MU^*} R$$
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Theorem (Landweber)

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Two cohomology theories

Fix prime \( p = 2 \).

- **Johnson-Wilson theory** \( E(n) \): Landweber exact theory with
  \[
  E(n)_* = \mathbb{Z}_2(v_1, \ldots, v_{n-1}, v_n^\pm), \quad |v_i| = 2(2^i - 1)
  \]

- **Morava \( E \)-theory** \( E_n \): Landweber exact theory with
  \[
  (E_n)_* = W(\mathbb{F}_{2^n})[u_1, \ldots, u_{n-1}][u^\pm], \quad |u_i| = 0, |u| = 2
  \]

- Related by completion and homotopy fixed points:
  \[
  \widehat{E(n)} = L_{K(n)}E(n), \quad E_n(Gal) = E_n^{hG}
  \]
  \[
  \widehat{E(n)} \simeq E_n(Gal)
  \]
  \[
  \widehat{E(n)}_* = (E(n)_*)_{I_n} = \mathbb{Z}_2(v_1, \ldots, v_{n-1})[v_n^\pm]
  \]

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A natural question

We have $\hat{E}(n) \simeq E_n(Gal)$ and...

\[ \mathbb{Z}/2 \text{ acts on } \hat{E}(n) \]

Complex conjugation action

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Action of the subgroup of Morava stabilizer group generated by the formal inverse
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First a little more background...
Real theories

Complex conjugation action on $E(n)$ arises in context of Real theories ($\mathbb{Z}/2$-equivariant $RO(\mathbb{Z}/2)$-graded)

- Atiyah, 1966: Real $K$-theory $\text{KR}$
  
  $$\text{KR}(X) = G \left\{ \text{cplx v.b. } \pi : E \to X \text{ \mid \text{antilin. on fibers, } } \pi \text{ equiv} \right\}$$

- Landweber, 1967: Real cobordism $\text{MR}$
  Uses $\mathbb{Z}/2$-action of complex conjugation on $BU(k)$.

- Araki, 1978: Defined $\text{BPR}$ using a Quillen idempotent argument

- Hu & Kriz, 2001: Defined $\text{KR}(n)$ and $\text{ER}(n)$ as $\text{MR}$-modules
Real theories

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  \hline
  \end{array} \right\}
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Kitchloo and Wilson’s real Johnson-Wilson theory

Real theory $\mathbb{E} \leadsto$ naïve $\mathbb{Z}/2$-equivariant theory

- $KR \leadsto KU$
- $MR \leadsto MU$
- $ER(n) \leadsto E(n)$

Taking homotopy fixed points gives new theories:

- $KU_{\mathbb{Z}/2} = KO$
- $E_{\mathbb{Z}/2} = ER(n)$, Kitchloo and Wilson’s “real Johnson-Wilson”

Moral: complex forget $\leftarrow$ Real fixed pts $\rightarrow$ real
Kitchloo and Wilson’s real Johnson-Wilson theory

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Moral:

$$\text{complex} \xlongleftarrow{\text{forget}} \text{Real} \xrightarrow{\text{fixed pts}} \text{real}$$
Kitchloo and Wilson’s real Johnson-Wilson theory

The \( ER(n) \) are higher real \( K \)-theories.

\[
E(1) = KU(2) \quad \quad ER(1) = KO(2)
\]

Kitchloo-Wilson: There is a fibration

\[
\Sigma^{\lambda(n)} ER(n) \xrightarrow{\times(n)} ER(n) \rightarrow E(n)
\]

that reduces when \( n = 1 \) to the classical fibration

\[
\Sigma KO(2) \xrightarrow{\eta} KO(2) \rightarrow KU(2)
\]

Makes computations feasible (Bockstein spectral sequence).

\[
\lambda(n) = 2^{2n+1} - 2^{n+2} + 1
\]
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The Morava stabilizer group

Morava $E$-theory:

- Landweber exact cohomology theory $E_n$

\[ (E_n)^* = \mathcal{W}(\mathbb{F}_{2^n})[[u_1, \ldots, u_{n-1}]][u^\pm] \]
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Note: \(S_n\) has a subgroup of order two generated by the formal inverse \(i(x)\).
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Morava stabilizer group: $S_n := \text{Aut}(H_n)$

Note: $S_n$ has a subgroup of order two generated by the formal inverse $i(x)$.

Extended Morava stabilizer group: $G_n := \text{Gal}(\mathbb{F}_{2^n}/\mathbb{F}_2) \rtimes S_n$
Lubin-Tate theory $\Rightarrow \mathbb{G}_n$ acts on $(E_n)_*$
Hopkins-Miller-Goerss

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- Hopkins-Miller-Goerss $\Rightarrow \mathbb{G}_n$ acts on $E_n$ by $E_\infty$-maps
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- Get interesting $E_\infty$-ring spectra by $E_n^{hK}$ for $K \subseteq \mathbb{G}_n$
  e.g. $E_n^{h\mathbb{G}_n} = L_{K(n)}S$
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Define $E_n(\text{Gal}) := E_n^{hK}$ for $K = \text{Gal}(\mathbb{F}_{2^n}/\mathbb{F}_2) \times \mathbb{F}_{2^n}^\times$. 
Lubin-Tate theory $\Rightarrow \mathbb{G}_n$ acts on $(E_n)_*$

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Get interesting $E_\infty$-ring spectra by $E_n^{hK}$ for $K \subseteq \mathbb{G}_n$
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Define $E_n(\text{Gal}) := E_n^{hK}$ for $K = \text{Gal}(\mathbb{F}_{2^n}/\mathbb{F}_2) \rtimes \mathbb{F}_{2^n}^\times$.

$E_n(\text{Gal})_* = \hat{\mathbb{Z}}_2[[v_1, \ldots, v_{n-1}]][v_n^\pm]$

Order 2 subgroup generated by $i(x)$ acts on $E_n(\text{Gal})$
A natural question

We have $\widehat{E(n)} \simeq E_n(\text{Gal})$ and...

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Natural question: Are these actions related?

YES
Answer: Yes

**Theorem (A.)**

*There is an equivalence*

\[
\widehat{E(n)}^{h\mathbb{Z}/2} \cong E_n(Gal)^{h\mathbb{Z}/2}
\]

*and the natural map*

\[
ER(n) = E(n)^{h\mathbb{Z}/2} \rightarrow \widehat{E(n)}^{h\mathbb{Z}/2}
\]

*induces an algebraic completion on coefficients.*
Consequences

Corollary

After completion, $ER(n)$ is an $E_\infty$-ring spectrum.
Consequences

Corollary

After completion, \( ER(n) \) is an \( E_\infty \)-ring spectrum.

Corollary

\[
(E_n(\text{Gal})^{h\mathbb{Z}/2})_* = \mathbb{Z}_2[[\hat{\nu}_k(l) \mid 0 \leq k < n, l \in \mathbb{Z}]] [x, \nu_n^{\pm 2^{n+1}}] / J
\]

\( J \) is the ideal generated by the relations

\[
\hat{\nu}_0(0) = 2 \quad x^{2^{k+1} - 1} \hat{\nu}_k(l) = 0
\]

and for \( k \leq m \),

\[
\hat{\nu}_m(l) \hat{\nu}_k(2^{m-k}s) = \hat{\nu}_m(l + s) \hat{\nu}_k(0)
\]

\[
|x| = \lambda(n) = 2^{2n+1} - 2^{n+2} + 1 \quad |\nu_n^{2^{n+1}}| = 2^{n+2}(2^n - 1)^2
\]

\[
|\hat{\nu}_k(l)| = 2(2^k - 1) + l2^{k+2}(2^n - 1)^2 - 2(2^k - 1)(2^n - 1)^2
\]
A bit about the proof

We’d like *equivariant* map \( \varphi : \hat{E}(n) \to E_n(\text{Gal}) \) that is also an equivalence. But we only have a *homotopy equivariant* one.
A bit about the proof

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But we only have a *homotopy equivariant* one.

- Try replacing \( \hat{E}(n) \) by \( \hat{E}(n) \wedge F(\hat{E}(n), E_n(\text{Gal}))_\varphi \)
- \( \varphi \) homotopy equivariant \( \Rightarrow \) conjugation action on \( F(\hat{E}(n), E_n(\text{Gal}))_\varphi \)
- \( ev : \hat{E}(n) \wedge F(\hat{E}(n), E_n(\text{Gal}))_\varphi \to E_n(\text{Gal}) \) is honestly equivariant
A bit about the proof

We’d like *equivariant* map $\varphi : \hat{E}(n) \to E_n(\text{Gal})$ that is also an equivalence.

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- Try replacing $\hat{E}(n)$ by $\hat{E}(n) \wedge F(\hat{E}(n), E_n(\text{Gal})))\varphi$

- $\varphi$ homotopy equivariant $\Rightarrow$ conjugation action on $F(\hat{E}(n), E_n(\text{Gal})))\varphi$

- $ev : \hat{E}(n) \wedge F(\hat{E}(n), E_n(\text{Gal})))\varphi \to E_n(\text{Gal})$ is honestly equivariant

- If $F(\hat{E}(n), E_n(\text{Gal})))\varphi \simeq pt$, then $\hat{E}(n) \wedge F(\hat{E}(n), E_n(\text{Gal})))\varphi \simeq \hat{E}(n)$.

- Need appropriate category so that $F(\hat{E}(n), E_n(\text{Gal})))\varphi \simeq pt$. Try $S$-algebra maps.

- Problem: not known if $\mathbb{Z}/2$-action on $E(n)$ is a $S$-algebra map.
A bit about the proof

- Instead use $F_{S-\text{alg}}(v_n^{-1}\widehat{MU}, E_n(\text{Gal}))$. New problem: not contractible.

- Dirty trick: create $S$-algebra $T$ so that $F_{T-\text{alg}}(v_n^{-1}\widehat{MU}, E_n(\text{Gal}))$ is homotopy discrete.
A bit about the proof

- Instead use $F_{S-alg}(v_n^{-1}\widehat{MU}, E_n(Gal))$. New problem: not contractible.

- Dirty trick: create $S$-algebra $T$ so that $F_{T-alg}(v_n^{-1}\widehat{MU}, E_n(Gal))$ is homotopy discrete.

- $T =$ free $S$-algebra on a bunch of spheres
- $\pi_*(v_n^{-1}\widehat{MU}) = \pi_*(\widehat{E(n)} \wedge T)$
- Compute BKSS for $F_{T-alg}(\cdot, E_n(Gal))$
- A map $\widehat{E(n)} \wedge T \to v_n^{-1}\widehat{MU}$ gives a map of spectral sequences that is an iso on $E_2$
- Since that for $\widehat{E(n)} \wedge T$ collapses, so does that for $v_n^{-1}\widehat{MU}$
A bit about the proof

- Now
  \[
  \nu_n^{-1} \widehat{MU} \wedge F_{T\text{-alg}}(\nu_n^{-1} \widehat{MU}, E_n(\text{Gal}))_\nu \rightarrow E_n(\text{Gal})
  \]
  is equivariant.

- After taking homotopy fixed points, obtain a factorization

\[
\begin{align*}
\nu_n^{-1} \widehat{MU}^{h\mathbb{Z}/2} & \rightarrow E_n(\text{Gal})^{h\mathbb{Z}/2} \\
\widehat{E(n)}^{h\mathbb{Z}/2} & \rightarrow 
\end{align*}
\]
Thank you!