An introduction to E_∞-Homology

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Overview

The notion of an operad emerged in algebraic topology in May’s study of iterated loop spaces [May]. Operads model properties of operations such as associativity and commutativity. The operations encoded in a given operad are realized by algebras over this operad.

Let dg-mod denote the category of differential graded modules over a fixed ground ring k. Examples of algebras over operads in dg-mod include A_n-algebras and E_∞-algebras, generalizing the concept of associativity and commutativity:

- An A_n-algebra is a differential graded module equipped with a multiplication that is associative up to homotopies of all possible higher degrees.
- An E_∞-algebra is a differential graded module with a multiplication which is associative and commutative up to all higher homotopies.

These algebras are algebras over so-called E_n-operads, the notion of A_n-algebras corresponding to the case n = 1, the example of E_∞-algebras corresponding to n = ∞.

To every (sufficiently good) operad O we can associate a homology theory H_O and a cohomology theory H_O^∗ defined for algebras over the operad O. In particular, we can associate a homology and a cohomology theory to E_∞-operads.

By neglect of structure every ordinary commutative associative algebra A can be thought of as an algebra over an E_∞-operad. In particular, the homology and cohomology theories associated to the operad are defined for A.

The cases n = 1 and n = ∞ are once again provide familiar examples.

For n = 1 we retrieve Hochschuld homology and cohomology, the classical theory associated to associative and commutative up to all higher homotopies.

The project described here is concerned with gaining knowledge about the intermediate cases 1 < n < ∞. For example, there is an operad A∞ in the category of modules over a ground ring k, so that being an associative k-algebra is equivalent to being an algebra over A∞. Similarly, there is an algebra C∞ encoding commutative algebras.

One aim of this project is to investigate if the approaches to E∞-cohomology by utilizing [F]

References


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E∞-Homology

Operads

E∞-Homology

E∞-Homology

Commutative Algebras as E∞-Algebras and their Homology

Commutative Algebras as E∞-Algebras

There is a variant of the notions of E_∞-operad in the category dg-mod of chain complexes. For n = 1 and n = ∞ one recovers the notions of A_∞ and E_∞-algebras, generalizing the notions of associativity and commutativity.

An obvious example of an E_∞-algebra is given by any operad defined for associative and commutative up to all higher homotopies.

One can think of algebras over E_n-operads with arbitrary n as some kind of interpolation between these two cases.

In particular, every ordinary commutative associative algebra is an E_∞-algebra and therefore an algebra over some E_∞-operad for any other n as well.

E∞-Homology

To every E_∞-algebra one can define a homology and a cohomology theory. Considering a commutative associative algebra as an algebra over an E_∞-operad, this particularly defines a homology and a cohomology theory for commutative and associative algebras, retrieving familiar theories in the cases n = 1 and n = ∞, as described above.

Recent developments provide new approaches to E∞-homology. In [F] Fresse generalizes the construction of the classical bar complex of differential graded algebras to E∞-algebras.

He finds that one can iterate this construction n times for a given E∞-algebra and that the iterated bar complex serves as some sort of delooping.

**Theorem 2** (Fresse) If A is a (sufficiently good) algebra over an (sufficiently good) E_∞-algebra, the E∞-homology of A equals the homology of the n-th iterated bar complex of the chain complex A:

$$H^n_{E∞}(A) ∼= H^n_{A∞}(B^n(A)).$$

He also shows that in the case of an ordinary commutative associative algebra the operadic bar construction coincides with the classical bar construction of Eilenberg and Mac Lane [EM].

[32] Livernet and Richter observe that E_∞-homology of commutative associative algebras may be interpreted as the homology of certain functors. More precisely, they define the E_∞-homology of functors F: Ep∞ → k-mod from the category Ep∞ of planar trees with n leaves to the category of k-modules by constructing an explicit multicategory associated to F.

They further show that one can calculate the homology of these functors via derived functors.

**Theorem 3** (Livernet-Richter) There exists a functor H^n_{E∞}(F) → k-mod so that for every functor F: Ep∞ → k-mod one has

$$H^n_{E∞}(F) ∼= H^n_{E∞}(A^n,F).$$

Additional Structure

The project described is mainly concerned with investigating two possible additional structures on E∞-cohomology:

- For n = 1 and n = ∞, the associated cohomology theories, i.e. Hochschuld and André-Quillen cohomology, are equipped with a well known graded lie bracket.

- Objects expressed via Ext functors naturally allow the definition of cohomology operations. One aim of this project is to construct and understand cohomology operations on E∞-cohomology by utilizing an interpretation of E∞-cohomology in terms of Ext functors similar to the results in [LR].