Braid groups

Let \( \mathcal{B}_n \) be the \( n \)-th braid group. Consider the discrete category \( \mathcal{B} \) with objects the natural numbers \( 0, 1, 2, \ldots \) and \( \text{End}(n, n) = \mathcal{B}_n \). There is a connection between braided monoidal categories and 2-fold loop spaces. For instance it has been shown (see for instance [Ber, remark 1.9])

\[
\mathcal{B}_n \cong \Omega S^2
\]

i.e. that after group completion the nerve of this category is weakly equivalent to the double loop space of the \( n \)-sphere. In fact this holds more generally, the group completion of the nerve of any braided monoidal category is weakly equivalent to a double loop space [F, Theorem 2].

\( \mathcal{B} \)-spaces

The category of \( \mathcal{B} \)-spaces inherits a braided monoidal structure from \( \mathcal{B} \). For two \( \mathcal{B} \)-spaces \( X \) and \( Y \) the product is defined by \( X \times Y = \text{colim}_j (X_j) \times Y(1, j) \). And the result follows from the functoriality of the Kan extension.

A non example

The endomorphisms of \([n] \in \mathcal{I} \) are the permutation groups \( \Sigma_n \). These groups give rise to an \( \mathcal{I} \)-space \( \Sigma \mathcal{I} \) with \( \Sigma \mathcal{I}(n) \) the classifying space of the permutation group \( \Sigma_n \). This is in fact also a commutative \( \mathcal{I} \)-space monoid see [S, example 8.2].

If we restrict to the subcategory \( \mathcal{M} \) of injective order preserving maps in \( \mathcal{I} \) then we get an \( \mathcal{M} \)-space monoid \( \mathcal{B} \mathcal{B} \mathcal{M} \). The space \( \mathcal{B} \mathcal{B} \mathcal{M}(n) \) is the classifying space of the \( n \)-th braid group. Example 8.9 in [S] shows

\[
\mathcal{B} \mathcal{B} \mathcal{M}(n) \cong \Omega^2 S^2
\]

i.e. that the homotopy colimit of this \( \mathcal{M} \)-space is homotopy equivalent to the double loop space of the 3-sphere.

There is a theorem by Sagave and Schlichtkrull stating that the category of commutative \( \mathcal{I} \)-space monoids is Quillen equivalent to the category of \( \mathcal{B} \mathcal{M} \)-spaces. A groupoid \( E \mathcal{M} \mathcal{I} \)-space has the weak homotopy type of an infinite loop space, see [MT4].

My project

There is a connection between braided monoidal categories and double loop spaces, and commutative \( \mathcal{I} \)-space monoids are related to infinite loop spaces. The purpose of my project is to see what happens with a braided version of \( \mathcal{I} \)-spaces. We are hoping to relate commutative monoids to double loop spaces.

Injective braids

Intuitively we can think of an injective braid as a braiding of an injective map. The following illustration shows two different braidings of the injective map \( 1 \to 3, 2 \to 1, 3 \to 4 \):

![Injective braid illustration](image)

For a formal definition we can generalize the description of the braid groups as fundamental groups given in [F]. Let \( [n] \) denote the set \( \{1, \ldots, n\} \). We define an injective braid \( \alpha \) from \([m] \) to \([n] \) as the homotopy class of an \( m \)-tuple of paths in \( \mathbb{R}^2 \). The \( i \)-th path should start in \((i, 0)\) and end in one of the points \((1, 0), \ldots, (n, 0)\). At any time neither the different paths nor the different homotopies intersect.

Let \( \mathcal{B} \) be the category of finite sets \([n] \), including the empty set \([0] \), and injective braids.

Braided monoidal structure

The category \( \mathcal{B} \) has a strict monoidal structure where the tensor product of \([m] \) and \([n] \) is \([m+n] \) and on morphisms it is given by concatenation of injective braids. But we may have to slant the latter map as seen in the illustration:

![Injective braid tensor product illustration](image)

The braiding \([m] \otimes [n] \to [m+n] \) is given by moving the first \( m \) strings over the \( n \) strings preserving the order in \( m \) and \( n \) respectively. To the left there is an illustration of the braiding \( [2] \otimes [4] \to [2] \), and to the right an illustration of the naturality of the braiding.

Further work

The first thing we want to do is to see if the homotopy colimit of a commutative \( \mathcal{B} \)-space monoid has an action of a \( \mathcal{C} \)-operad. If that works out we want to find a model structure on the subcategory of commutative monoids in the category of injective braids. Hopefully we will be able to show that the associated homotopy category is equivalent to the homotopy category of \( \mathcal{C} \)-operads. A connected \( \mathcal{C} \)-operad is weakly equivalent to a double loop space [MT2, theorem 1.3].

References

[S] C. Schlichtkrull, Thom spectra that are symmetric spectra, Documenta Mathematica, 14, (2009), 8. 695-748.