HOMOTOPY & COHOMOLOGY

GK 1150

Young Women in Topology

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Injective-braid spaces

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Motivation

Braid groups

Let B_n be the n^{th} braid group. Consider the discrete category $\mathcal{B}$ with objects the natural numbers $0, 1, 2, \ldots$ and $End_{\mathcal{B}}(n, n) := B_n$. There is a connection between braided monoidal categories and 2-fold loop spaces. For instance it has been shown (see for instance [Ber, remark 1.9])

$$\overline{N\mathcal{B}} \simeq \Omega^2 S^2$$

i.e. that after group completion the nerve of this category is weakly equivalent to the double loop space of the 2-sphere. In fact this holds more generally, the group completion of the nerve of any braided monoidal category is weakly equivalent to a double loop space [F, theorem 2].

$\mathcal{I}$-spaces

Let $\mathcal{I}$ be the category of finite sets $[n] = \{1, \ldots, n\}$, including the empty set [0], and injective maps. The category of $\mathcal{I}$-spaces, that is functors from $\mathcal{I}$ to topological spaces, has been well studied for example by Steffen Sagave and Christian Schlichtkrull.

An example of an $\mathcal{I}$-space is $B\Sigma$ where $B\Sigma(n)$ is the classifying space of the permutation group Σ_n. This is in fact also a commutative $\mathcal{I}$-space monoid see [S, example 8.2] .

If we restrict to the subcategory $\mathcal{M}$ of injective order preserving maps in $\mathcal{I}$ then we get an $\mathcal{M}$-space monoid $B\mathcal{B}$. The space $B\mathcal{B}[n]$ is the classifying space of the n^{th} braid group. Example 8.9 in [S] shows

$$B\mathcal{B}_{h\mathcal{M}} \simeq \Omega^2 S^2$$

i.e. that the homotopy colimit of this $\mathcal{M}$-space is homotopy equivalent to the double loop space of the 3-sphere.

There is a theorem by Sagave and Schlichtkrull stating that the category of commutative $\mathcal{I}$-space monoids is Quillen equivalent to the category of E_∞ spaces. A grouplike E_∞ space has the weak homotopy type of an infinite loop space, see [M74].

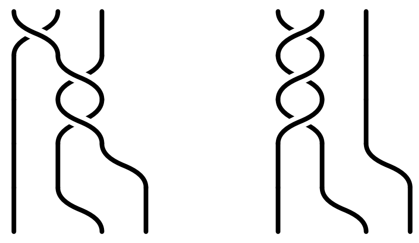
My project

There is a connection between braided monoidal categories and double loop spaces, and commutative $\mathcal{I}$-space monoids are related to infinite loop spaces. The purpose of my project is to see what happens with a braided version of $\mathcal{I}$-spaces. We are hoping to relate commutative monoids to double loop spaces.

Definitions

Injective braids

Intuitively we can think of an injective braid as a braiding of an injective map. The following illustration shows two different braidings of the injective map $1 \mapsto 3$, $2 \mapsto 1$, $3 \mapsto 4$:

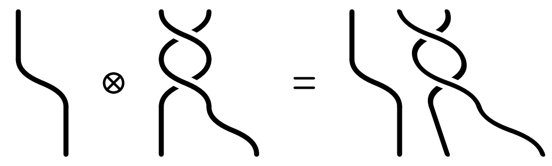


For a formal definition we can generalize the description of the braid groups as fundamental groups given in [Bir]. Let $[n]$ denote the set $\{1, \ldots, n\}$. We define an injective braid α from $[m]$ to $[n]$ as the homotopy class of an m-tuple of paths in $\mathbb{R}^2$. The i^{th} path should start in $(i, 0)$ and end in one of the points $(1, 0), \ldots, (n, 0)$. At any time neither the different paths nor the different homotopies intersect.

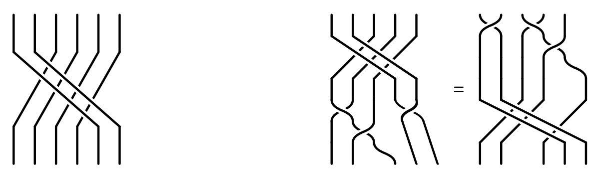
Let $\mathfrak{B}$ be the category of finite sets $[n]$, including the empty set [0], and injective braids.

Braided monoidal structure

The category $\mathfrak{B}$ has a strict monoidal structure where the tensor product of $[m]$ and $[n]$ is $[m+n]$ and on morphisms it is given by concatenation of injective braids. But we may have to slant the latter map as seen in the illustration:



The braiding $[m] \otimes [n] \to [n] \otimes [m]$ is given by moving the first m strings over the n strings preserving the order in m and n respectively. To the left there is an illustration of the braiding $[2] \otimes [4] \to [4] \otimes [2]$, and to the right an illustration of the naturality of the braiding.



$\mathfrak{B}$-spaces

The category of $\mathfrak{B}$-spaces inherits a braided monoidal structure from $\mathfrak{B}$. For two $\mathfrak{B}$-spaces X and Y the product is defined by $X \boxtimes Y(i) = \operatorname*{colim}_{j \otimes k \to i} X(j) \times Y(k)$. And the result follows from the functoriality of the Kan extension.

A non example

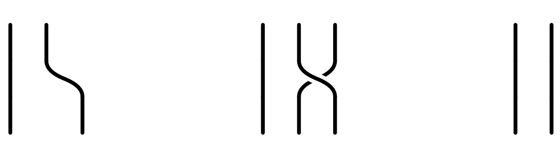
The endomorphisms of $[n]$ in $\mathcal{I}$ are the permutation groups Σ_n. These groups give rise to an $\mathcal{I}$-space $B\Sigma$ with $B\Sigma[n]$ the classifying space of Σ_n in the following way: For an injective map $\alpha : [m] \to [n]$, write $[n] - \alpha$ for the complement of $\alpha[m]$ in $[n]$. We can then define a group homomorphism $\Sigma_m \to \Sigma_n$ by ϕ in Σ_m maps to

$$[n] \cong [m] \otimes ([n] - \alpha) \xrightarrow{\phi \otimes id} [m] \otimes ([n] - \alpha) \cong [n]$$

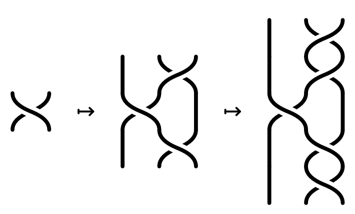
The isomorphism $[m] \otimes ([n] - \alpha) \cong [n]$ is really just extending α to $\overline{\alpha} : [n] \to [n]$ by the order preserving map $\{m+1, m+2, \ldots, n\} \to [n] - \alpha$. The homomorphism can then be written $\phi \mapsto \overline{\alpha}(\phi \otimes id)\overline{\alpha}^{-1}$.

The natural question now is if we can construct a $\mathfrak{B}$-space in an analogous way. The endomorphisms of $[n]$ in $\mathfrak{B}$ are the braid groups $\mathcal{B}_n$. We try to make a functor from $\mathfrak{B}$ to groups in the same way as above. But when we want to extend an injective braid α we also have to choose a braiding on the extended map. We can for instance choose to let all the new strings go under the ones that where there before. Since we have chosen a order preserving function $\{m+1, m+2, \ldots, n\} \to [n] - \alpha$ none of the new strings need to cross over or under each other.

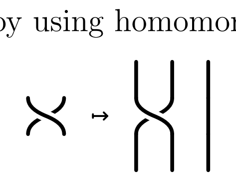
Looking at an example shows that this will not preserve composition, so we do not get a functor in this way. The illustration shows two injective braids $\alpha : [2] \to [3]$ and $\beta : [3] \to [3]$ and the composition $\beta \circ \alpha : [2] \to [3]$:



We chose an element in $\mathcal{B}_2$ and map it with the homomorphism induced by α and then map the result by the homomorphism induced by β:



We see that this is not the same as we get by using homomorphism induced by the composition $\beta \circ \alpha$:



If we choose another braiding on $\overline{\alpha}$ we wil have the same problem.

Further work

The first thing we want to do is to see if the homotopy colimit of a commutative $\mathfrak{B}$-space monoid has an action of a C_2 operad. If that works out we want to find a model structure on the subcategory of commutative monoids in the category of injective braid spaces. Hopefully we will be able to show that the associated homotopy category is equivalent to the homotopy category of C_2-spaces. A connected C_2-space is weakly equivalent to a double loop space [M72, theorem 1.3].

References

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