Let $F_0$ be a fusion system over a finite $p$-group $S$. A morphism of fusion systems $F_0 \to F_0'$ is a pair $(\Phi, \phi)$ consisting of a group homomorphism $\Phi : S \to S'$ and a covariant functor $\phi : F_0 \to F_0'$ with the following properties:

- for any subgroup $Q$ of $S$, we have $\phi(Q) = \Phi(Q)$;
- for any morphism $\phi : Q \to R$ in $F_0$, we have $\Phi(\phi) = \phi \circ \phi$.

Definition 6 This allows us to define the category of fusion systems over finite $p$-groups: $\text{FUSION}(p)$ with:

- objects: fusion systems over finite $p$-groups and
- morphisms: morphisms between corresponding fusion systems.

Definition 7 Let $G$ be a discrete group. A finite $p$-group $S \leq G$ is called a Sylow $p$-subgroup of $G$ if all finite $p$-groups of $G$ are subconjugate to $S$.

Bemerkungen 1 Infinite discrete groups need not have Sylow $p$-subgroups. An example can be $C_p \times C_p$.

Definition 8 Let $p$ be a prime. Denote by $\text{GROUP}_{p, q}$, the full subcategory of groups which have a Sylow $p$-subgroup.

The cohomology of a fusion system is defined as $H^n(F) := \text{lim}_{K} H^n(K)$, where $K$ runs over $\text{GROUP}_{p, q}$.

This generalizes the classical Theorem of Cartan and Eilenberg, see [Cartan-Eilenberg], that the cohomology of a finite group is given as the taking of the cohomology of the Sylow $p$-subgroup.

Not every fusion system is the fusion system of a finite group. However, in 2007 G. Robinson and I. Leary together with R. Stancu independently constructed groups realizing arbitrary fusion systems, see [Leary-Stancu]. Their models are iterated HNN constructions while Robinson’s models are iterated amalgams of finite groups. [Robinson].

Since it was our goal to associate a classifying space to a saturated fusion system, at least up to $F_2$-homology, it is a natural question to compare the cohomology of the group models realizing a given fusion system to the cohomology of the fusion system. This will be done by constructing homology decompositions.

Definition 9 A ring homomorphism $\gamma : A \to B$ is called an $F$-homomorphism in the sense of Quillen, see [Quillen], if every element in the kernel is adjusted and for every element $b \in B$ there exist $k > 0$ such that $b^k \in \text{Im}(\gamma)$.

Throughout this entire discussion we omit the notion of saturation which is a technical condition modelled on the way a Sylow $p$-subgroup is embedded in a finite group.