Singular bordism

See [CF62] to learn more about singular bordism.

Definition 1 (Unoriented singular bordism) Let \((X, A)\) be a topological pair. A smooth compact oriented manifold \((M, \partial M)\) together with a map \(f : (M, \partial M) \to (X, A)\) is called a singular n-manifold in \((X, A)\). Such a \((M, \partial M, f)\) is said to be bordant if there exists \(F : B \to X\) which satisfies

- \(B\) is a compact oriented \((n + 1)\)-manifold with boundary,
- \(\partial B\) contains \(M\) as a regular submanifold whose orientation is induced by that on \(B\),
- \(F\) restricted to \(M\) is equal to \(f\) and \(F(\partial B) \subset A\).

The oriented singular manifolds \((M, \partial M, f)\) and \((M, \partial M, f')\) are called bordant if the disjoint union \((M \sqcup M, \partial M \sqcup \partial M, f \sqcup f')\) is bordered. Unoriented bordism is defined similarly by dropping the orientability assumptions. "Bordant" is an equivalence relation and the sets of equivalence classes are called \(\Omega^\ast(X, A)\) and \(\mathfrak{A}(X, A)\) respectively.

Čech bordism and the symmetric squaring in bordism

The first idea to transport symmetric squaring to bordism is to just perform the symmetric squaring on the manifold itself. But the quotient by \(\tau\) lacks a canonical smooth structure on the diagonal. There has to be found a way to remove the diagonal from the symmetric square; so that the result is a smooth compact manifold again. It is not desirable, however, to impose a wide range of choices of substratifying the diagonal into our definition. The way out is to look at all such neighbourhoods at the same time:

Definition 2 (Čech bordism) Following the definition of Čech homology as an inverse limit of singular homology groups, Čech bordism is defined as a certain inverse limit of relative bordism groups. More precisely, consider the neighbourhoods \(V \subset Y\) of the subspace \(B \subset Y\) on a topological pair \((Y, B)\) as a quasi-ordered set ordered by inverse inclusion. Then the Čech bordism group of the pair \((Y, B)\) is defined to be the inverse limit of the relative bordism groups of \((Y, V)\) over this quasi-ordered set. Here, this leads to

\[
\Omega^\ast(Y, B) = \lim_{\neg \downarrow V} \Omega^\ast(Y, V, B, \partial V).
\]

Definition 3 (symmetric squaring in bordism) Let \((X, A)\) be a pair of topological spaces. The symmetric squaring in bordism is defined as

\[
\theta^\ast : \Omega^\ast(X, A) \to \Omega^\ast(X, A), \quad [f, M, \partial M] \mapsto \left[\left\{ \left( f(M, \partial M) \cup \{ f(x) \} \right) \right\} \right]_{\partial M}.
\]

It is a rather technical but necessary step to restrict the above limit to those neighbourhoods \(U\) of the diagonal that have a smooth compact bordered manifold \(M \times U\) as their complement. But it is possible and does not change the above definition much.

Compatibility and Čech versions vs. ordinary versions

Putting all given facts together, the following theorem results.

Theorem 3 (compatibility) Let \(k \in \mathbb{N}\) be even and \((X, A)\) a topological pair. Then the diagram

\[
\begin{array}{ccc}
\Omega^k(X, A) & \xrightarrow{\theta^k} & \Omega^{k+1}(X, A) \\
\| & \| & \|
\end{array}
\]

\[
\begin{array}{ccc}
\mathfrak{A}^k(X, A) & \xrightarrow{\mathfrak{A}_{k+1}(X, A) \mathfrak{A}^{k+1}(X, A, Z),}
\end{array}
\]

is commutative.

Using Čech homology and bordism here has been important for the intuition and the proofs. It can be proven that in a lot of interesting cases, the ordinary groups are isomorphic. The statement concerning homology in the following theorem has been shown in [Dold80], proposition 13.17.

Theorem 4 (isomorphy of singular and Čech versions) Let \((X, A)\) be such that \(X\) is an ENR and \(A\subset X\) is an ENR as well. Then

\[
\Omega^k(X, A) \cong \Omega^k(X, A), \quad \mathfrak{A}^k(X, A) \cong \mathfrak{A}^k(X, A, Z).
\]