For any discrete commutative ring $R$, one studies the category of its modules, $R$-Mod, the associated chain complex or monoids, the differential graded $R$-algebras. Since the structure of $R$-Mod is often too rigid to effectively work in, it becomes more appropriate to operate in the derived category $D(R)$, which is the homotopy category of differential graded $R$-modules, $Ho(DGR-Mod)$. A generalization of homological algebra is offered by algebra of symmetric spectra. In this frame, symmetric spectra take the place of abelian groups, and the analog in terms of symmetric monoidal structures goes as follows:

**Homological algebra**

- $A_b \otimes \mathbb{Z}$
- $(S^p)^{\otimes n} \otimes S$

**Algebra of spectra**

- $HR$, R. ring
- $HR$, Eilenberg - Mac Lane ring spectrum.

A well-behaved symmetric smash product $\wedge$ on spectra took time to be elaborated, but, in return, it made possible clear categorical definitions of ring, module and algebra spectra. Before, the notions of spectra and ring spectra already existed, but all algebraic structures had complex, up to homotopy, properties.

In the older context, Alan Robinson [Rob] established a connection between rings and ring spectra. He defined a notion of $A_b$-modules over the ring spectrum $HR$, and showed that, up to a suitable notion of homotopy, the category of $A_b$-modules over $HR$ is equivalent to $D(R)$. Nevertheless, it was difficult to obtain a similar result for algebras because of involved definitions. However, this became achievable in the medium setting of algebra, where the homotopy theory is encoded in Quillen model structures.

After strengthening the result of Robinson by showing that the category of $HR$-module spectra is equivalent to $DGR-Mod$

$$D(R) \cong Ho(DGR-Mod) \sim Ho(HR-module spectra).$$

Brooke Shipley [Shi, Theorem 1.1] showed that the $HR$-algebra spectra capture the same “up to homotopy” information as differential graded $R$-algebras:

**Theorem 1 (Shipley)** For any discrete commutative ring $R$, the model categories of unbounded differential graded $R$-algebras and $HR$-algebra spectra are Quillen equivalent. The associated composite derived functors are denoted

$$\mathcal{N} : DG-Alg \Rightarrow HR-\text{Alg}_{gr}$$

and

$$\Theta : HR-\text{Alg}_{gr} \Rightarrow DG-Alg.$$

In other words, this theorem states an equivalence of categories

$$Ho(DG-Alg) \sim Ho(HR-\text{algebra spectra}).$$

The aim of this Master thesis was to acquire a sufficient knowledge of algebra of spectra and of model category theory in order to understand the above result, and to explain the essential arguments employed in the proof. For simplicity, we concentrated on the case $R = \mathbb{Z}$.

**Strategy of the Proof**

The strategy Shipley uses to prove the Quillen equivalence in Theorem 1 is the following:

1. study first the relation between $H$-module spectra and differential graded modules;
2. use then the fact that $H$-module spectra and DG-modules are the categories of monoids in $H$-Mod$_{gr}$ and $DG-Mod_{gr}$ respectively.

To make a connection between $H$-Mod$_{gr}$ and $DG-Mod_{gr}$, we will need to consider two intermediate categories, the category $Sp(C(\mathbb{Z}))$ of symmetric spectra over simplicial abelian groups and the category $Sp_i(C_b)$ of symmetric spectra over non-negative chain complexes.

We start with a chain of three more or less “classical” adjoint pairs, defined on the underlying categories

$$sSet \lora \mathbb{Z} \lora C_b \lora DGR-Mod.$$

Lifting these adjunctions to the categories of corresponding symmetric sequences, and then to spectra, gives the following zig-zag of adjoint pairs

$$DG-Mod_{gr} \lora Sp_i(\mathbb{Z}) \lora Sp(C_b) \lora DGR-Mod;$$

which lies at the heart of the proof.

For this zig-zag to be coherent, two types of conditions must be satisfied:

- **Conditions on the categories**: they summarize in demanding compatibility between the model and the monoidal structure;
- **Conditions on the adjoint functor pairs**: they require the monoidal-model category structures be appropriately transported via the zig-zag.

The desired Quillen equivalence will follow from the fact that the four categories and their adjoint pairs in the main zig-zag satisfy key Theorems 2 and 3 opposite. This will imply that the corresponding categories of monoids - the $H$-algebra spectra and the differential graded algebra - are equipped with induced model structures.

The three adjoint pairs $(Z, U)$, $(L, C(\mathbb{Z}))$, $(D, R)$ from the zig-zag induce Quillen equivalences on the categories of monoids as follows:

$$HAlg_{gr} \lora Sp_i(\mathbb{Z}) \lora Sp(C_b) \lora DAlg;$$

and the desired functors $N : DG-Alg \Rightarrow HR-\text{Alg}_{gr}$ and $\Theta : HR-\text{Alg}_{gr} \Rightarrow DG-Alg$ emerge as the composite total derived functors induced on the correspondent homotopy categories.

**Key Theorems**

In [SS90, Theorem 4.1], Schwede and Shipley establish what are the necessary conditions on a category $C$ to define the induced model structures on the associated categories of monoids, $R$-modules and $R$-algebras. We mention here only the part concerning algebras, where taking $R = \mathbb{Z}$ will give the result for $Mon_C$.

**Theorem 2 (Schwede and Shipley)** Let $C$ be a cofibrantly generated, model monoidal category, satisfying the monoid axiom and such that every object in $C$ is small. Let $R$ be a cofibrant monoid in $C$. Then the category $R$-Alg$_{gr}$ of $R$-algebras over $C$ is cofibrantly generated model category.

Sufficient conditions for extending Quillen equivalences between two monoidal model categories on the associated categories of monoids are given by Schwede and Shipley in [SS93, Theorem 3.12].

**Theorem 3 (Schwede and Shipley)** Let $C$ and $D$ be monoidal model categories and $R : C \Rightarrow D$ be the right adjoint of a weak monoidal Quillen pair. Suppose that the unit objects in $C$ and $D$ are cofibrant. If the forgetful functor creates model structure for monoids in $C$ and $D$, then the adjoint pair

$$L^{cof} : Mon_C \Rightarrow Mon_D : R^{cof}$$

is a Quillen equivalence.

References


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