The first approach is to find a map of ring spectra

\[ \Sigma \colon \text{Ho}(M) \rightarrow \text{Ho}(M) \]

where \( \text{Ho}(M) \) is the category of homotopy classes of maps. This approach can be seen as finding some algebraic criteria for detecting rigidity of ring spectra. For example, the 2-local real topological K-theory ring spectra

\[ \Sigma : \text{Ho}(\Sigma(\mathbb{R})) \rightarrow \text{Ho}(\Sigma(\mathbb{R})) \]

are rigid. Furthermore, I would like to find some algebraic criteria for detecting rigidity of ring spectra.

Another approach

(i) The second approach is to consider the two Postnikov towers of the ring spectra

\[ M \quad \text{and} \quad M \]

and to prove inductively that the ring spectra \( P_i(M) \) and \( P_i(N) \) are stably equivalent for every \( i > 0 \).

For each natural number \( n \), one has to show that there exists exactly one ring extension of spectra

\[ v \colon Y \rightarrow P_n(Y) \]

such that \( v_n(Y) \) is an isomorphism for \( i \leq n \) and such that the ring spectra \( Y \) and \( P_i(Y) \) have the same ring of homotopy groups and the same Toda brackets.

One can do this by using a result of Dugger and Shipley: In (DS), they classify the possible extensions.

\[ v \colon Y \rightarrow P_n(Y) \]

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Dugger and Shipley prove that, up to a zig-zag of weak equivalences, all these extensions are classified by

\[ \text{Ho}(B \rightarrow \text{Alg})(C \rightarrow \Sigma^{\infty+1}(M)) \]

where \( \text{Ho}(B \rightarrow \text{Alg})(A \rightarrow \text{Alg}) \) denotes the homotopy mapping space from \( A \rightarrow \text{Alg} \) in the category of \( \Sigma \)-algebras over \( \Sigma \) and \( \Sigma^{\infty+1}(M) \) is the \( \Sigma \)-local homotopical Hochschild cohomology group of \( M \) with coefficients in \( M \) (cf. Examples 8.1.1 and 8.8.1).

To give a vague idea of this classification, we will briefly sketch how Dugger and Shipley obtain the extension \( v \rightarrow P_n(Y) \) corresponding to a homotopy class \( [v] \) in \( \text{Ho}(B \rightarrow \text{Alg})(C \rightarrow \Sigma^{\infty+1}(M)) \).

Unfortunately, this theorem of Dugger and Shipley provides not much information about the product of elements \( x \cdot y \in v_n(Y) \) on Toda brackets which are subsets of \( v_n(Y) \). In general, we only know the product \( x \cdot y \in v_n(Y) \) in trivial cases, for example if \( x \cap y \cap \in v_n(Y) \) or if \( Y \rightarrow M \in \Sigma \).

Remark 1 Using these two methods, one can prove that the ring spectra \( P_1(M) \), \( P_2(M) \), and \( P_3(M) \) are rigid.

If one requires additional technical properties of the model categories, then it is sufficient to consider only model categories of modules in order to answer the question raised above [SS, Theorem 3.1.1].

Examples

Below, we list some examples of rigid and non-rigid ring spectra:

- The sphere spectrum \( S \) is a rigid ring spectrum [Schw].
- Further examples for rigid ring spectra are the Eilenberg-MacLane ring spectra \( H^n(R) \) for any ring \( R \) (SS, Theorem 5.1.1).
- The ring spectrum \( P_2(R) \) is rigid for all \( n \leq 0 \).
- In contrast, the Morava K-theory ring spectra are not rigid.

References


Young Women in Topology

Bonn, June 25 – 27, 2010

Rigid ring spectra

Katja Hutschenreuter

Some known results and one possible approach

In order to prove that a ring spectrum \( R \) is rigid, it thus suffices to show that \( R \) is stably equivalent to the ring spectra \( R = \Sigma^b(\text{K}^{(1)}(R)) \). The ring spectra \( K^{(1)}(R) \) are a triangulated equivalence between the homotopy categories of modules over these ring spectra. Then the following holds:

\[ \nu_1(\Sigma^b(\text{K}^{(1)}(R))) \cong \nu_1(\text{K}^{(1)}(R) \otimes \text{K}^{(1)}(R)) \]

In case that the ring spectrum \( R \) is rigid, it thus suffices to show that the ring spectra \( R = \Sigma^b(\text{K}^{(1)}(R)) \). The ring spectra \( K^{(1)}(R) \) are a triangulated equivalence between the homotopy categories of modules over these ring spectra. Then the following holds:

\[ \nu_1(\Sigma^b(\text{K}^{(1)}(R))) \cong \nu_1(\text{K}^{(1)}(R) \otimes \text{K}^{(1)}(R)) \]