Young Women in Topology
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Constructing Generalised Leray Spectral Sequences
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Variations on the theme
Bousfield-Wisniewski cohomology is in a sense a generalization of various other notions of cohomology, such as Hochschild-Mitchell cohomology, or cohomology with local coefficients. In general, consider

\[ FE \xrightarrow{\delta} B \xrightarrow{\theta} R \xrightarrow{\phi} \mathbb{B} \]

and write \( C \) for any of the categories in this diagram and \( \alpha^\ast : FE \to C \) for the composite functor. As in the previous theorem we have in each case a diagram

\[ \begin{array}{ccc}
[FU, \mathbb{B}] & \xrightarrow{\beta} & [C, \mathbb{B}] \\
\downarrow \alpha \downarrow & \xrightarrow{\beta \circ \phi} & \downarrow \alpha \downarrow \\
[BU, \mathbb{B}] & \xrightarrow{\beta' \circ \phi} & [D, \mathbb{B}]
\end{array} \]

where \( \alpha \) denotes the constant diagram functors, \( \alpha^\ast \) is precomposition with \( \alpha \), and the other functors in the diagram are the right adjoints of those, given by the limits \( \lim_{\mathbb{B}} \) and \( \lim a_{\mathbb{B}} \). The spectral sequence for the derived functors of the composite

\[ \lim a_{\mathbb{B}} \circ \lim_{\mathbb{B}} \circ \lim a_{\mathbb{B}} \circ \lim_{\mathbb{B}} \]

converges to the Bousfield-Wisniewski cohomology of \( E \) with coefficients in \( D \in [FE, \mathbb{B}] \).

In certain cases the \( \mathbb{K} \) extension and the \( E_2 \) term can be identified explicitly and simplified:

Theorem 2 (Case \( C = FB, \alpha = Fu \) for \( E \) and \( B \) above) Given a functor \( u : E \to B \) and a natural system \( D : FE \to \mathbb{B} \), there is a first quadrant cohomology spectral sequence

\[ E_2^{pq} \approx \lim_{\mathbb{B}}(B, \lim_{\mathbb{B}}(D^q(Fu)_p)) \Rightarrow \lim_{\mathbb{B}}(E, D) \]

where \( F \cdot \mathbb{B} \) is the forgetful functor.

Theorem 3 (Case \( C = B, \alpha : FE \to B \) for \( E \) and \( B \) above) Given a functor \( u : E \to B \) and a \( D : FE \to B \) as above, and \( \beta : \mathbb{B} \to \mathbb{B} \) an object of the facilitated category \( FB \), there is an isomorphism of categories \( \beta(u) \approx F(B(u)) \).

Definition 2 Let \( M \to \mathbb{S}p \) be the closed model category of fibrant spectra and let \( E : E \to M \to \mathbb{B} \) be a functor. The null generalised cohomology group of the category \( E \) with coefficients \( B \) is defined as

\[ H^p(\mathbb{E}, \pi_n B) \approx \tau_{B_n} E \mathbb{C} = \pi_n \lim_{\mathbb{B}}(E, \mathbb{C}) \]

Bousfield-Wisniewski cohomology yields a generalised Leray spectral sequence for the derived functor of the constant diagram functor.

Theorem 5 (Generalised Leray spectral sequence) For \( u : E \to B \) and \( E \to M \) we have

\[ E_2^{pq} \approx \lim_{\mathbb{B}}(B, \pi_n E_{pq}) \Rightarrow \pi_n \lim_{\mathbb{B}}(E, \mathbb{C}) \]

This follows from a kind of Grothendieck spectral sequence for the composition of right derived functors:

\[ E_2^{pq} \approx \pi_n r_{B_n} E \mathbb{C} \Rightarrow \pi_n \lim_{\mathbb{B}}(E, \mathbb{C}) \]

Thomason’s spectral sequence \( [Th] \) arises as an absolute case: taking \( u = 1_{\mathbb{C}} \) we have

\[ E_2^{pq} \approx H^p(E, \pi_n C) \Rightarrow \pi_n \lim_{\mathbb{B}}(E, \mathbb{C}) \]

References


