In 1960, Nakada showed the homological stability for symmetric groups. Later Kerz has given a simple proof for Nakada’s theorem and we generalize it to twisted coefficients.

**Theorem 1** Let $V$ be a coefficient system of degree $k$ (see below). Then the inclusion map $ε: Σ_m → Σ_n$ induces an isomorphism on the homology for $m ≥ n(k−1)$.

$$H_n(Σ_n, V; G) ≅ H_n(Σ_m, V; G)$$

(1)

**Definition 1** Let $F$ be the category of finite sets and injective maps. We define a functor $D: F → F$ which is a disjoint union with one point. For an arbitrary finite set $T$, the image $D(T)$ will be denoted by $T$ in order to simplify the notation.

**Definition 2** Suppose a coefficient system $V$ is a function $V: Σ → Ab$ where $Ab$ is the category of abelian groups. The suspension $sV$ of coefficient system $V$ is defined by $sV = V + D$, i.e., $sV(T) = V(T)$. For any finite set $T$ the inclusion $T → T$ induces a natural transformation $p^*: (T → T) → (V → AV)$.

We denote the coherer of the natural transformation $p^*$ by $ΔV$.

**Example 1** Let $F$ be a field and $T$ be an arbitrary finite set. If we define a coefficient system $V(T) = FT$ where $FT$ is the free abelian group generated by the elements of $T$, then this coefficient system has degree 1.

**Definition 3** For a finite set $T$, let $X_T$ be a semi-small simplicial set with all injective maps $[0,1] → T$ as p-simplices. We denote the augmented chain complex of $X_T$ by $C_*(X_T)$.

1. The symmetric group $Σ_T$ acts transitively on the simplices.
2. In $Σ_T$, the stabilizer of a p-simplex $σ$ is $Σ_{σ_1}$ where $σ_1 = T − σ(0),...σ(p)$. Therefore one gets an isomorphism $C_*(X_T) ≅ Σ_TΣ_{σ_1}Z$.

Since the spectral sequence converges to zero, nothing surviving to the $E^n$-term. Moreover the groups $H_n(Σ_{σ_1}, V; G) ≅ H_n(Σ_T, V; G)$ and $E^1_{n,m}(Σ_T, V; G)$ are zero by induction hypothesis ($r^1$ has the same degree with $V$). Hence one can easily see that $H_{n,m}(Σ_T, V; G)$ is an isomorphism.

**Lemma 2** Suppose the induction assumption holds and $V$ is a coefficient system of degree $k$. Let $S ⊂ T$ be objects in $F$ with $|S| = n ≥ 1$. Then for $m < |T|/2 + k + 1$, $H_n(Σ_T, V; G) → H_n(Σ_S, V; G)$ is an isomorphism.

The following lemma follows from a diagram chasing and the fact that the conjugation maps of permutation groups induces identity in homology.

**Lemma 3** Consider an inclusion $Q → R$ in the category $F$ where $|R| = |Q| + 1$ and let $Q = S, R = T$. If the homomorphisms $q^*: H_n(Σ_S, V) → H_n(Σ_T, V)$ and $q^*: H_{n+1}(Σ_S, V) → H_{n+1}(Σ_T, V)$ are surjective, then the relative group homology $H_n(Σ_R, V; G)$ is zero.

**Lemma 4** Suppose the induction assumption holds, then the morphisms $q^{n+1}$ and $q^n$ are surjective for $m < |T|/2 + k + 1$. We have a short exact sequence of coefficient systems: $0 → V → AV → ΔV → 0$ which leads to a long exact sequence of relative homology: $⋯ → H_n(Σ_S, V; G) → H_n(Σ_T, V; G) → H_n(Σ_R, V; G) → ⋯$

By the induction assumption, the last group is zero. $μ$ is surjective. If we compose $μ$, with the surjective map $R_0(Σ_T, V; G) → H_n(Σ_T, V; G)$ in Lemma(2), the composition $q^*: H_n(Σ_T, V) → H_n(Σ_S, V)$ will be surjective for $m < |T|/2 + k + 1$. We use a similar approach for $q^{n+1}$.

**Proof of the Stability Theorem 1** The natural map $μ_0(Σ_T, V; G) → R_0(Σ_T, V; G)$ is injective because $μ$ is splitting as in equation(2). Moreover from Lemma(2) for $m < |T|/2 + k + 1$, $R_0(Σ_T, V; G) → H_n(Σ_T, V; G)$ is injective as well. Moreover $H_n(Σ_T, V; G) = 0$, as a corollary of Lemmas(3) and (4). Therefore considering the composition of those two injective maps, $μ_0(Σ_T, V; G)$ is.

**Sketch of the theorem part I**

**Theorem 2** (Kerz [Ke]) The homology of $C(T)$ vanishes except in degree $|T|−1$. We can extend this complex to an acyclic one, $C(T)$.

$$0 → ⋯ → C_{|T|−1}(T) → C_{|T|−2}(T) → ⋯ → C_0(T) → C_{−1}(T) → 0$$

For the proof of the Stability Theorem(1) we replace our permutation group $Σ_n$ with permutation group $Σ_n$ of an arbitrary set $T$ in order to use category theory.

Let $S ⊂ T$ be objects in $F$ with $|S| = |T| + k$. We denote the relative group homology $H_n(Σ_T, V; G) → H_n(Σ_S, V; G)$. Because of the long exact sequence of relative group homology $⋯ → H_n(Σ_T, V; G) → H_n(Σ_S, V; G) → H_n(Σ_R, V; G) → ⋯$

the theorem reduces to prove $H_n(Σ_T, V; G) = 0$ for $m < |T|/2 + k + 1$.

We prove this by induction on the degree of the coefficient system and homological degree.

Induction beginning: degree $−1$ is obvious.

**Inductive assumption** Let $V$ be a coefficient system of degree $k$, then:

1. $H_m(Σ_T, V; G) = 0$, for $m < |T|/2 + k + 1$ with $k + 1 < 4$.
2. $H_m(Σ_T, V; G) = 0$, for $m < |T|/2 + 1 + k + 1$ with $k + 1 = 4$.

For the sets $S ⊂ T$, the inclusion map $Σ_T → Σ_S$ leads to a $p$–linear map on the coefficient modules $V(S) → V(T)$. For the semi-simplipical sets $X_T$ and $X_S$, we have a bijection between the set of $Σ_T$-orbits of p-simplices in $X_T$ and the set of $Σ_S$-orbits of p-simplices in $X_S$ for $p ≤ |S| − 1$. Choose representatives of $p$-simplices. Then for the acyclic augmented chain complexes $(C(T), d(T))$, there is a spectral sequence which in our case is:

$$E^2_{p,q} = H_p(Σ_T, V; G) → H_p(Σ_S, V; G) → H_p(Σ_R, V; G)$$

where $σ_p$ is the representative of $p$-simplices. The spectral sequence converges to zero since $C(T)$ and $C(S)$ are acyclic. We can also compute that the differential $d_p: E^2_{p,q} → E^2_{p,q+1}$

$$H_p(Σ_T, V; G) → H_p(Σ_S, V; G)$$

is the stabilizer of the $(−1)$-simplex in a group by itself. Hence the differential $d_p: E_p = E^2_p → E^2_{p−1}$ is.

The stabilizer of a $0$-simplex in $X_T (Σ_T)$ is $Σ_T$ and we take the stabilizer of the $(−1)$-simplex in a group by itself. Hence the differential of $E_p = E^2_p → E^1_p$ is.

**Examples and Applications**

**Lemma 5** Let $V$ and $W$ be coefficient systems of degree $k$ and $m$, respectively. Then $V ⊗ W$ is a coefficient system of degree $m ≤ k + m$ and $V ⊗ W$ is a coefficient system of degree $≤ k + m$.

**Lemma 6** Suppose $F$ is a field and $X$ is a connected pointed space and let $H_∗(X) = H_∗(X; F)$, $m ≥ 0$ then the degree of $V_n = m < 0$.

As an application we study the Borel construction $B(Σ_n(X)) = EN_n X$, $N$ is a Lens-0-collared spectral sequence of $B(Σ_n(X))$.

**Theorem 3** $H_n(Σ_n(X)) = H_n(N_n(X))$ is an isomorphism if $m < 2$.

Furthermore we hope to prove the homological stability of configuration spaces of smooth, connected, noncompact manifolds. Let $F_0(M)$ be the space of ordered configurations of $n$ points. There is a free action of $Σ_n$ on $F_0(M)$, permits the points $C_n(M) = F_0(M)/Σ_n$. There is a map $C_n(M) → C_{n−1}(M)$ adding a point “near infinity” that is well-defined up to homology. Segal has shown that $C_n(M)$ satisfies homological stability but can one show it via Theorem(1)?

**References**


