Overview

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Motivation from physics: Several identical particles

Assumption: the state of one particle is given by an element in a Hilbert space $\mathcal{H}$.

Aim: Description of systems consisting of several identical, non-interacting particles.

**Fermions** are a class of particles (e.g. including electrons $\odot$, protons $\ominus$) with the following properties:

- Particles of the same sort are indistinguishable.
- No two particles can be in the same state.

Describing a system of fermions

- Description of a system of $k$ particles: specify an element of $\mathcal{H}$ for each particle.
- States can be entangled $\Rightarrow$ more accurate description is given by $\mathcal{H} \otimes \mathcal{H} \otimes \cdots \otimes \mathcal{H}$.
- By the properties above, $v \otimes v = 0$ for all $v \in \mathcal{H}$, hence we get $\Lambda^k \mathcal{H}$.
- Interpretation of $v_1 \wedge v_2 \wedge \cdots \wedge v_k \in \mathcal{H}$ as $k$ particles occupying the states $v_1, \ldots, v_k$.
- For an unknown number of identical fermions: elements of the exterior algebra $\Lambda \mathcal{H}$.

This vector space will become the (algebraic) **Fock space**.
Creation and annihilation

On $\Lambda \mathcal{H}$, we have the operations of creation and annihilation of particles:

- For each $v \in \mathcal{H}$: **Creation operator** (also denoted by $v$), given by
  \[ v(\xi) = v \wedge \xi \]
  for $\xi \in \Lambda \mathcal{H}$.
  
  “Create a particle in the state $v$”

- **Annihilation operator** $D_v$ for each $v \in \mathcal{H}$ (here: norm 1), given by
  \[ D_v(v \wedge w_1 \wedge w_2 \wedge \cdots \wedge w_n) = w_1 \wedge w_2 \wedge \cdots \wedge w_n \]
  if all $w_i \perp v$.

The unit $1 \in \Lambda \mathcal{H}$ is called the vacuum state.

Creation and annihilation algebras

Algebra generated by **creation** operators: Exterior algebra $\Lambda \mathcal{H}$, acting on the Fock space by left multiplication.

Definition of the **annihilation** operator $D_w$ for $w \in \mathcal{H}$ on the exterior algebra:

- $v \in \mathcal{H} = \Lambda^1 \mathcal{H} \Rightarrow D_w(v) = \langle w, v \rangle \cdot 1 \in \Lambda^0 \mathcal{H}$.

- $\xi, \eta \in \Lambda \mathcal{H} \Rightarrow D_w(\xi \wedge \eta) = D_w(\xi) \wedge \eta + (-1)^{|\xi|}\xi \wedge D_w(\eta)$.
  (graded derivation)

In the complex case: conjugate $\mathbb{C}$-action

Relations between annihilation operators: $D_v D_v = 0$ and $D_w D_v + D_v D_w = 0$

The annihilation operators also generate an exterior algebra $(\Lambda \overline{\mathcal{H}})^{op}$.

($D_v D_w$ annihilates $w \wedge v^*$)
The Clifford algebra action on $\Lambda \mathcal{H}$

Combining the creation and annihilation operators into one algebra acting on $\Lambda \mathcal{H}$

Denote by

$\mathcal{H}$ the vector space of annihilation operators $D_w$ (from now on also written as $\overline{w}$)

$\alpha : \mathcal{H} \oplus \overline{\mathcal{H}} \rightarrow \mathcal{H} \oplus \overline{\mathcal{H}}$ the isometric involution given on $\mathcal{H}$ by $v \mapsto \overline{v}$.

Let $b$ be the symmetric bilinear form on $V = \mathcal{H} \oplus \overline{\mathcal{H}}$ given by

$$b(x, y) = \langle \alpha(x), y \rangle.$$

**Proposition** The creation and annihilation operators define an action of the Clifford algebra $\mathcal{C}(V)$ (with respect to the bilinear form $b$) on the Fock space $\Lambda \mathcal{H}$.

Proof of the Proposition

$\mathcal{H}$ and $\overline{\mathcal{H}}$ are isotropic with respect to $b \Rightarrow$ they generate the exterior algebras $\Lambda \mathcal{H}$ and $\Lambda \overline{\mathcal{H}}^{op}$.

Relations between creation and annihilation operators

Let $v, w \in \mathcal{H}$. Write any element of $\Lambda \mathcal{H}$ as a sum of elements of the form $x \wedge w_1 \wedge \cdots \wedge w_k$ with $w_i \perp v$.

$\Rightarrow$ enough to check relations on these. Let $\xi = w_1 \wedge \cdots \wedge w_k$. 
Then we get
\[
(D_v w + w D_v)(x \wedge \xi)
\]
\[
= D_v(w \wedge x \wedge \xi) + w \wedge \langle v, x \rangle \xi
\]
\[
= D_v(-x \wedge w \wedge \xi) + \langle v, x \rangle w \wedge \xi
\]
\[
= -\langle v, x \rangle w \wedge \xi + x D_v(w) \wedge \xi + \langle v, x \rangle w \wedge \xi
\]
\[
= \langle v, w \rangle x \wedge \xi.
\]
This implies the relation
\[
\overline{v} w + w \overline{v} = D_v w + w D_v
\]
\[
= \langle v, w \rangle = \langle \alpha(\overline{v}), w \rangle = b(\overline{v}, w).
\]

**Lagrangians and algebraic Fock spaces**

**Definition**  Let $V$ be a Hilbert space with isometric involution $\alpha$

$(\alpha(v) = \overline{v}, \mathbb{C}$-antilinear in the complex case), $b(v, w) = \langle \alpha(v), w \rangle$.

A **Lagrangian** of $V$ is a closed subspace $L$ which is isotropic with respect to $b$
and for which $L \oplus \overline{L} = V$.

**Example**  For $V = \mathcal{H} \oplus \overline{\mathcal{H}}$, the subspace $\mathcal{H}$ is Lagrangian.

Constructing algebraic Fock spaces from these data:

**Definition**  For a Hilbert space $V$ with involution $\alpha$ and a choice of Lagrangian $L$,
the associated algebraic Fock space is the $\mathcal{O}(V)$-module $F_{alg}(L) = \Lambda L$. 

Graded modules

- $\mathcal{O}(V)$ is $\mathbb{F}_2$-graded
- Decompose $\Lambda(L)$ as
  \[ \Lambda L = \bigoplus_{n \text{ even}} \Lambda^n L \oplus \bigoplus_{n \text{ odd}} \Lambda^n L \]
- $\mathbb{F}_2$-grading on the Fock space

Compatibility $\Rightarrow F_{alg}(L)$ becomes a graded module over $\mathcal{O}(V)$.

Properties of Clifford algebras

- For a Hilbert space $V$ with isometric involution $\alpha$, denote by $-V$ the same space equipped with the involution $-\alpha$.
- Convention: if no involution is specified, always assume $\alpha = id$.

Natural isomorphisms:
- $\mathcal{O}(V \oplus W) = \mathcal{O}(V) \otimes \mathcal{O}(W)$ (graded tensor product)
- $\mathcal{O}(-V) = \mathcal{O}(V)^{op}$

A $\mathcal{O}(V \oplus (-W))$-module structure can also be viewed as a $\mathcal{O}(V) - \mathcal{O}(W)$-bimodule structure.
Inner product and completion

For a Hilbert space $V$, define an inner product on $\Lambda^k V$ by

$$\langle v_1 \wedge v_2 \wedge \ldots \wedge v_k, w_1 \wedge w_2 \wedge \ldots \wedge w_k \rangle = \det(\langle v_i, w_j \rangle).$$

Remark Creation and annihilation are adjoint:

$$w \in \mathcal{H}, \eta, \xi \in F_{alg}(L) \Rightarrow \langle D_w(\xi), \eta \rangle = \langle \xi, w \wedge \eta \rangle.$$ 

The fermionic Fock space is the completion of the algebraic Fock space with respect to this inner product.

The Segal-Shale equivalence criterion

Changing the Lagrangian

Let $V$ be a Hilbert space with orthonormal basis $\{e_i\}$, and $\phi : V \to W$ an operator into a second Hilbert space $W$.

Recall: $\phi$ is called Hilbert-Schmidt if

$$\sum_i \|\phi(e_i)\|^2 < \infty.$$ 

Theorem (Segal-Shale) Let $L$ and $L'$ be two Lagrangians of $V$. The corresponding Fock spaces $F(L)$ and $F(L')$ are isomorphic if and only if the composition $L' \to V \to L$ is a Hilbert-Schmidt operator.

The grading is preserved if and only if $\dim(L \cap L')$ is even.
The finite-dimensional case

Proposition Let $V$ be an $n$-dimensional real inner product space. Then there is a bijection between the set of linear isometries $f : \mathbb{R}^n \to V$ and the set $\mathcal{L}$ of Lagrangian subspaces of $V \oplus (-\mathbb{R}^n)$, given by $f \mapsto \Gamma_f$ (the graph of $f$).

Proof (isotropic) Let $v, w \in \mathbb{R}^n$; then
\[
b(v + f(v), w + f(w)) = \langle -v + f(v), w + f(w) \rangle \\
= \langle -v, w \rangle + \langle f(v), f(w) \rangle = 0.
\]

With the usual topology on both sets, the bijection becomes a homeomorphism.

Orientations

- Linear isometries $f : \mathbb{R}^n \to V$ correspond to elements of $O(n)$; connected components of $O(n)$ to orientations of $V$.
- Application of $\pi_0$ to the homeomorphism above $\Rightarrow$
  Orientations of $V \cong \pi_0(\mathcal{L})$.

- **Fact:** In the finite-dimensional case, any irreducible module over $\mathcal{O}(V \oplus (-\mathbb{R}^n))$ is isomorphic to some Fock space.
- Segal-Shale for finite-dimensional spaces: Any operator is Hilbert-Schmidt, hence all irreducible modules are isomorphic.
- Isomorphism classes of graded irreducible modules over $\mathcal{O}(V \oplus (-\mathbb{R}^n))$ correspond to orientations of $V$.  

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Functoriality

In which sense are the constructions $V \mapsto \mathcal{C}(V)$ and $L \mapsto F_{alg}(L)$ functorial?

Let $\mathcal{C}$ be the category with

- objects $(V, \alpha)$ (Hilbert spaces with isometric involution)
- morphisms $\mathcal{C}(V, V')$ given by the set of Lagrangians of $V' \oplus (-V)$.

Composition of morphisms $L_1 \subset V_2 \oplus (-V_1)$ and $L_2 \subset V_3 \oplus (-V_2)$: Let

$$L_3 = (L_2 \oplus L_1) \cap U\perp / (L_2 \oplus L_1) \cap U \subset (V_3 \oplus -V_2 \oplus V_2 \oplus -V_1) \cap U\perp / U,$$

where $U = \{0, v_2, v_2, 0\}$ and $U\perp$ is the annihilator with respect to $b$.

For trivial involutions:

$$L_3 = \{v_3 + v_1 \in V_3 \oplus (-V_1) \mid \exists v_2 \in V_2 : v_3 + v_2 \in L_2 \land v_2 + v_1 \in L_1\}.$$
**Generalised Lagrangians**

Let $V$ be a Hilbert space with involution.

**Definition** A generalised Lagrangian of $V$ is a homomorphism $L : W \to V$ with

- $\dim \ker(L) < \infty$
- such that the closure $\overline{L_W}$ of $\text{im}(L)$ is a Lagrangian of $V$.

Associated algebraic Fock space:

$$\mathcal{F}_{alg}(L) = \Lambda^{top}(\ker L)^* \otimes \Lambda(\overline{L_W}).$$

- $(\ker L)^*$ dual space; $top = \dim(\ker L)$
- $\mathcal{Cl}(V)$-action on the second factor.

**Some points to remember**

- Lagrangian subspace of $(V, \alpha)$: $L \oplus \alpha(L) = V$ and $b|_{V \times V} = 0$
- Algebraic Fock space associated to a Lagrangian $L$:
  $\Lambda L$ with action of the Clifford algebra $\mathcal{Cl}(V)$, induced by creation and annihilation operations
- Fock space: given by completing $\Lambda L$
- Isomorphism classes of graded $\mathcal{Cl}(V) - \mathcal{Cl}_n$-bimodules correspond to orientations of $V$.
- In special cases: functoriality
Problem session

Exercise 1 (One possible state). Suppose that $\mathcal{H} = \mathbb{R}$ is the Hilbert space describing a system of one particle and only one possible state $s$. Then the corresponding Fock space $\mathcal{H} = \Lambda^0\mathcal{H} \oplus \Lambda^1\mathcal{H}$ is two-dimensional with basis $\{1, s\}$. By writing the creation and annihilation operators as matrices with respect to this basis, it is easy to see that they generate the whole endomorphism algebra $\text{End}_\mathbb{R}(\Lambda\mathcal{H})$.

Exercise 2. This holds more generally: Let $\mathcal{H}$ be an $n$-dimensional Hilbert space. Then $\mathcal{O}(\mathcal{H} \oplus \overline{\mathcal{H}}) \cong \text{End}(\Lambda\mathcal{H})$.

Solution. Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis for $\mathcal{H}$. Denote the annihilation operators corresponding to $e_i$ by $D_{e_i}$ and choose the set of all elements $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_r}$ with $0 < r \leq n$ and $i_1 < i_2 < \cdots < i_r$ as a basis for the Fock space.

Comparison of the dimensions yields $2^{2n}$ for both algebras, hence it is enough to show that for each two basis elements $x = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_r}$ and $y = e_{j_1} \wedge \cdots \wedge e_{j_s}$, we can find an operator mapping $x$ to $y$ and all other basis elements to 0. An operator satisfying these conditions (up to a sign) is given by the composition

$$e_{j_1} \circ \cdots \circ e_{j_s} \circ D_{e_1} \circ \cdots \circ D_{e_n} \circ e_{k_1} \circ \cdots \circ e_{k_{n-r}},$$

where $e_{k_1}, \ldots, e_{k_{n-r}}$ denote the elements not occurring in $x$.  

\[\square\]