Von Neumann - algebras
and the Connes fusion tensor product

$H$ complex Hilbert space
$B(H)$ bounded op's on $H$

Def.: $M \subseteq B(H)$ is a von Neumann algebra iff it is a unital $\ast$-subalgebra of $B(H)$ that is closed in the weak operator topology.

Note: weak topology is generated by seminorms $\varphi \mapsto |\langle \psi, \varphi \rangle| \quad \forall \psi, \varphi \in H$

Def.: Given $S \subseteq B(H)$, the commutant $S'$ of $S$ is given by
\[ S' = \{ \alpha \in B(H) \mid \forall s \in S \quad \alpha s = s \alpha \} \]

Thm.: (von Neumann - Bicommutant theorem):

$M \subseteq B(H)$ von Neumann algebra $\iff \quad A'' = A$.

("Every von Neumann alg. arises as a commutant of something.")

Ex.: $M = B(H)$ is a von Neumann algebra.

Ex.: Let $G$ be a discrete, countable group. Take $H = L^2(G)$, then
\[ M = (CG)'' \quad \text{is a von Neumann algebra} \]

\[ \quad \uparrow \text{taken in the bounded op's} \quad \text{on } L^2(G) \]

Ex.: $(X, \mu)$ measure space

$H = L^2(X, \mu)$

$M = L^\infty(X, \mu)$ act on $H$ by left multiplication and is a commutative von Neumann algebra

Thm.: Every commutative VN arises in that way for some measure space $X$.

Philosophy: Think of VN's as a non-commutative version of measure theory.
Factors and Type Classification

Def: The center $Z$ of a vNA $M$ is given by $Z = M \cap M'$ and is itself a commutative vNA.

A von Neumann-algebra with trivial center is called a factor.

Why are factors interesting?

- $Z$ is a comm. vNA $\Rightarrow Z = L^\infty (X, \mu)$
  
  If $\mu$ is discrete, then...
  
  $$M = \bigoplus_{i \in I} M_i \quad \text{where all } M_i \text{ are factors}$$

  ... else use direct integral decomposition...
  
  $$M = \int_X M_x \, d\mu(x) \quad \text{where } M_x \text{ is a factor for all } x \in X$$

  $\Rightarrow$ Factors are "building blocks" of general vNA's

- Consider projections $p \in M$ with $p = p^* = p^2$

- Take $x \in M$ self-adjoint element and let
  
  $$x = \int_a^b dE_\lambda \quad \text{be its spectral decomposition, then } E_\lambda \in M \quad \forall \lambda \in [a, b]$$
  
  $\Rightarrow$ $M$ always contains projections

  (Indeed: Every vNA is the norm-closed linear span of its projections.)


equivalence of projections: Let $e, f \in M$, be proj.

$$e \sim f \iff \exists u \in M \text{ with } u^* u = e, uu^* = f$$

$\uparrow$ partial isometry

$\Rightarrow$ induce partial ordering on projections

$$e \prec f \iff \exists e_0 \sim e \text{ s.th. } e_0 \text{ is a subproj. of } f$$

Thm: If $M$ is a factor, this is an ordering.
Def.: A projection \( e \in M \) is called finite, if \( e \) is not equivalent to any proper subprojection of \( e \).

Def.: A factor \( M \) is of

- type \( \text{I} \), if there exists a non-zero minimal projection in \( M \),
- type \( \text{II} \), if \( M \) contains non-zero finite projections and is not of type \( \text{I} \),
- type \( \text{III} \), if no non-zero projection in \( M \) is finite.

A factor \( M \) is called finite, if \( 1 \in M \) is finite.

Thm: \( M \) finite factor \( \Rightarrow \) \( M \) faithful normal (i.e. weakly continuous) tracial state on \( M \) (for short: a trace on \( M \))

- type \( \text{I}_n \): trace takes discrete values on the proj.
  \[ \text{tr} (e) \in \{ 0, \ldots, \dim (e) \} \]
  \( n = \infty \) allowed
- type \( \text{II}_n \): trace takes continuous values on the proj.
  \[ \text{tr} (e) \in [0, 1] \]
- For type \( \text{II}_\infty \), there still is a replacement for \( \text{tr} \), that fulfills
  \[ \text{tr} (e) \in [0, \infty] \]
- For type \( \text{III} \), no trace at all! But finer classification via modular theory \( \longrightarrow \) leads to type \( \text{III}_\lambda \) with \( \lambda \in [0, 1] \)

Def.: A factor \( M \) is called hyperfinite, if

\[ M = \left( \bigcup_{i=1}^{\infty} M_i \right)'' \]

for an increasing sequence \( M_i \subset M_j \subset \ldots \)
of finite dimensional von Neumann algebras

All hyperfinite factors have been classified:

- type \( \text{I}_n \): \( M = B (H) \) with \( n = \dim (H) \)
- type \( \text{II}_n \): Group von Neumann algebras. All isomorphic!
- type \( \text{II}_\infty \): \( \text{I}_\infty \otimes \text{II}_n \)
- type \( \text{III}_\infty \): the Krieger factor
- type \( \text{III}_\lambda \) for \( \lambda \in [0, 1] \): the Powers factor
- type \( \text{III}_n \): The local fermion, defined by Wassermann. All isomorphic
construction of the hyperfinite $\text{II}_1$-factor

start with $M_{4^n \times 4^n}(\mathbb{C})$, embed it into $M_{2^n \times 2^n}(\mathbb{C})$ via
\[ x \mapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \]

s.t.h. the trace on $M_{2^n \times 2^n}(\mathbb{C})$ is given by $\text{tr}_2 \circ \iota = \frac{1}{2} \text{tr}_1$

continue like that embedding $M_{2^n \times 2^n}$ into $M_{2^{n+1} \times 2^{n+1}}$

Take weak closure on direct limit

$\Rightarrow$ trace takes values in $[0,1]$
Modular Theory
(aka: Tomita - Takesaki theory)

Def: \( M = \mathcal{B}(H_0) \) \( vNa \). \( H_0 \) is called vacuum representation or standard form, if \( \exists \Omega \in H_0 \) that is cyclic for \( M \) and for \( M^\prime \).

- Consider the (unbounded), anti-linear op. given by
  \[
  S_0 \ a \Omega = a^* \Omega \Rightarrow \text{closable} \ S = S_0^\dagger
  \]

  Take polar decomposition, since \( S \) is anti-linear, this looks like
  \[
  S = J \Delta^\frac{1}{2}
  \]
  with \( J \) anti-unitary
  and \( \Delta \) positive, self-adjoint.

Ex.: For \( M \) being a type \( II_q \)-factor, \( \exists \text{tr}: M \rightarrow \mathbb{C} \)

\[
\text{tr} \ \text{yields a vacuum repr. via the GNS construction, vacuum vec.} \ \Omega
\]
\[
\Omega \ a, b \in M \quad <b \Omega, S^* S \ a \Omega> = <b^* \Omega, a^* \Omega> = \text{tr}(ba^*) = \text{tr}(a^*b)
\]
\[
= \text{tr}(b^*a) = <b \Omega, a \Omega>
\]
\[
\Rightarrow S \text{ is anti-unitary}
\]
\[
= \Delta = 1 \text{ by uniqueness of polar decomp.}
\]

Rem.: \( J^2 = 1 \), functional calculus lets you define \( \Delta^{it} \) and \( \Delta^{-it} \), \( t \in \mathbb{R} \)

Now...

Thm. (Tomita - Takesaki): \( M \ vNa \) with vac. rep. \((H_0, \Omega)\), then

\[
\text{JMJ} = M^\prime
\]
\[
\Delta^{it} M = M \quad \forall t \in \mathbb{R}
\]

Rem.: \( J \) turns \( H_0 \) into an \( M-M \)-bimodule. Let \( \pi \) be the vac. rep. of \( M \), then:

\[
\pi^{op} (a) = J \pi (a)^* J \in M^{op}
\]

- \( J \Omega = \Delta \Omega = \Omega \)

For a dense subset of \( M \) (the entire analytic elements)

\[
\sigma(a) = \Delta^\frac{1}{2} a \Delta^{-\frac{1}{2}} \in M
\]
Now for \( \varphi_\Omega(a) = \langle \Omega, a \Omega \rangle \) one has

\[
\varphi_\Omega(ba) = \varphi_\Omega(\sigma^{-1}(a) \sigma(b)) \tag{\varphi_\Omega \text{ is called vacuum state}}
\]

Thus the modular operator measures "how much" the vacuum state differs from a trace state.

*Thm:* Iff multiples of \( \Omega \) are the only vectors that are fixed by the modular flow, then \( M \) is a type \( \text{III}_1 \) -factor.

* In analogy to the commutative case, the vac. rep. shall be denoted by \( H_0 = L^2(M) \)

**Tensor products of vNA's**

* Two Hilbert spaces: \( H_1, H_2 \)

Hilbert space tensor product \( H_1 \otimes H_2 \) is the completion of the algebraic tensor product \( H_1 \odot H_2 \) w.r.t. the norm...

\[
\langle \varphi_1 \otimes \varphi_1, \varphi_2 \otimes \varphi_2 \rangle = \langle \varphi_1, \varphi_2 \rangle_{H_1} \cdot \langle \varphi_1, \varphi_2 \rangle_{H_2}, \quad \varphi_i \in H_i, \quad n_i \in H_i
\]

* Consider two vNA's \( N \) and \( M \), now \( N \otimes M \) inherits a \( * \)-algebraic structure

\[
n_1 \otimes m_1 \cdot n_2 \otimes m_2 = n_1 \cdot n_2 \otimes m_1 \cdot m_2 \quad n_i \in N, \quad m_i \in M
\]

\[
(n \otimes m)^* = n^* \otimes m^*
\]

Now consider \( \pi: N \otimes M \longrightarrow B(H_N \otimes H_M) \) with \( N \subset B(H_N) \) and \( M \subset B(H_M) \)

\[
\pi(n \otimes m)(\varphi_1 \otimes \varphi_2) = n \varphi_1 \otimes m \varphi_2
\]

Now define: \( N \otimes M = \pi(N \otimes M)'' \) - the so-called spatial tensor product of vNA's.
Goal: Find the “right tensor product” for vNa-bimodules

Remember: \( H \) is called an \( M-N \)-bimodule (with two vNas \( H, N \)) if it is a left module over \( M \) and a right module over \( N \), where the module actions are given by weakly continuous, \(*\)-preserving, unital homomorphisms:

\[
\begin{align*}
\pi_M &: M \to B(H) \\
\pi_N^{op} &: N^{op} \to B(H)
\end{align*}
\]

\( \uparrow \) vNa with the opposite multiplication

Tensor product of an \( M-N \)-bimod. \( H_1 \) and an \( N-L \)-bimod. \( H_2 \) should have “nice properties”

ex. of a nice property: \( \exists j \in H_1, \eta \in H_2, n \in N \)

\[
\exists j \in H_1, \eta \in H_2, n \in N \quad \exists j \in H_1, \eta \in H_2, n \in N
\]

First construction by Jones et al. for type II_\( \infty \)-factors \( \to \) relative tensor product

Def.: Let \( H_1 \) be an \( M-N \)-bimodule with actions \( \pi_M, \pi_N^{op} \), then \( H_1 = \{ t : L^2(N) \to H_1 \mid t \cdot \pi_M^{op} = \pi_N^{op} \cdot t \} \)

\( \uparrow \) vacuum rep. of \( N^{op} \)

denote the Intertwiners between \( \pi_M^{op} \) and \( \pi_N^{op} \).

Ex.: \( L^2(N) = \{ t : L^2(N) \to L^3(N) \mid t \cdot \pi_M^{op} = \pi_N^{op} \cdot t \} = N^{\infty} = N \)

\( \pi_M^{op}(n) = \int \pi_M(n) \ast j \, j \in N^{\infty} \)

Note that \( H_1 \) can be turned into a right Hilbert module over \( N \) with inner product...

\[
(t, s) = t^*s \in L^2(N) = N
\]

Def.: Given an \( M-N \)-bimod. \( H_1 \) and an \( N-L \)-bimod. \( H_2 \), then the Connes fusion of the two is given by the completion of \( H_1 \otimes H_2 \) w.r.t. the inner product

\[
\langle t \otimes s, r \otimes \eta \rangle = \langle s, (t,s) \cdot \eta \rangle_{H_2}
\]

It is denoted by \( H_1 \boxtimes H_2 \).
Note that:
\[
\langle t n \otimes \xi - t \otimes n \xi, t n \otimes \xi - t \otimes n \xi \rangle = \langle \xi, (t n, t n) \xi \rangle - \langle n \xi, (t, t n) \xi \rangle - \langle \xi, (t n, t) n \xi \rangle + \langle n \xi, (t, t) n \xi \rangle = 0
\]
\[
\Rightarrow t n \otimes \xi = t \otimes n \xi \quad \text{Connes fusion has "nice property".}
\]

**Ex.:** Take weakly cont. unital \(*\)-preserving homomorphism of vNa's
\[
\xi : L \rightarrow N.
\]
\[
L^2(\xi) \text{ is } N-N\text{-bimodule } L^2(N) \text{ considered as } L-N\text{-bimodule with left action...}
\]
\[
\xi(\xi) \cdot z \quad \text{for } \xi \in L, z \in L^2(N).
\]
Let \(H\) be an \(N-M\)-bimodule and \(\hat{H}\) be the corresponding \(L-M\)-bimodule with left action
\[
\xi(\xi) \cdot \eta \quad \text{for } \xi \in L, \eta \in H.
\]

**Thm:** \(L^2(\xi) \boxtimes H \cong \hat{H}.

**Proof:** \(L^2(\xi) = L^2(N) = N\)
\[
\psi : L^2(\xi) \boxtimes H \rightarrow \hat{H}
\]
\[
\forall \eta \rightarrow \eta = n \otimes \eta
\]
\[
\psi : \hat{H} \rightarrow L^2(\xi) \boxtimes H \quad \text{extend to an isomorphism.}
\]
\[
\eta \rightarrow L \otimes n
\]
Therefore:
\[
L^2(\xi) \otimes L^2(\xi) = L^2(\xi \circ \xi) \quad \text{for } \circ : N \rightarrow M
\]
\[
L^2(N) \otimes H \cong \hat{H}
\]
Taking \(\xi = \text{id}\), \(L = N \Rightarrow L^2(N) \otimes H \cong H
\]
for any \(N-M\)-bimodule \(H\).

**Problem:** Identify vectors in \(H_1 \otimes H_2\) in terms of the tensor product \(H_1 \otimes H_2\).
Symmetric form of Connes fusion

$\mathcal{H}_2$ M-N-bimodule

$\mathcal{H}_2$ N-L-bimodule

$\tilde{\mathcal{H}}_2 = \{ s : L^2(N) \to \mathcal{H}_2 \mid s \pi_{N,0} = \pi_{N} s \}$ therefore: $s_1^* s_2 \in \mathcal{N}^{op}$

Now take completion of...

$\mathcal{H}_1 \otimes \tilde{\mathcal{H}}_2$ w.r.t. inner product

$\langle t_1 \otimes s_1, t_2 \otimes s_2 \rangle = \langle t_2^* t_1, \Omega \cdot s_2^* s_1, \Omega \rangle_{L^2(N)}$

...using the inclusion in one get...

$\mathcal{H}_1 \omega \otimes \tilde{\mathcal{H}}_2 \omega$ with symmetric Connes relation...

$\xi \Delta^\frac{1}{4} \eta = \xi \otimes \Delta^\frac{1}{4} \eta \Delta^{-\frac{1}{4}} \eta$ for $\xi \in \mathcal{H}_1 \omega$

$\eta \in \tilde{\mathcal{H}}_2 \omega$

n entire element in $\mathcal{N}$

A. Wassermann "four point-formula"
Connes fusion and the algebraic tensor product

- $H_1$, $M$, $N$, bimodule, left action $\pi^L_M$, right action $\pi^R_N$

Choose cyclic and sep. (vacuum) vector $\Omega \in L^2(N)$

inclusion $i_\Omega : H_1 \rightarrow H_1$

$t \mapsto t \cdot \Omega$ (not canonical, depends on choice of $\Omega$)

Now: $\pi^{\Omega}_N(x) \cdot \Omega = \pi^{\Omega}_N(x) \cdot \Omega = \int \pi^{\Omega}_N(x)^* \cdot \Omega$

intertwining prop.

$= \int \Delta^{\frac{1}{2}} \Delta^{-\frac{1}{2}} \pi^{\Omega}_N(x)^* \cdot \Omega$

invariance of $\Omega$

$= \int \Delta^{\frac{1}{2}} \pi^{\Omega}_N(x)^* \cdot \Omega$

$= \int \Delta^{\frac{1}{2}} \pi^{\Omega}_N(x) \cdot \Omega$

for every entire element $x \in N$

$\Rightarrow i_\Omega(t) \cdot x = i_\Omega(t \cdot \sigma(x))$ with $\sigma(x) = \Delta^{\frac{1}{2}} x \Delta^{-\frac{1}{2}}$

So, instead of above definition, take $H_1 \cdot \Omega \otimes H_2$ with

$H_1 \cdot \Omega \otimes H_2 \cong \mathcal{H}_1 \otimes \mathcal{H}_2$

Connes relation: $\xi \cdot \Omega \otimes \eta = \xi \otimes \sigma(x) \cdot \eta$ $\forall x \in N, \xi \in \text{im } i_\Omega, \eta \in H_2$

Remark: $H_1 \cdot \Omega$ is not $H_1$, but the set of $\omega$-bounded vectors for

$\omega(n) = \langle \Omega, n \cdot \Omega \rangle_{L^2(N)}$

$\xi \in \text{im } i_\Omega$ is $\omega$-bounded iff $\exists C > 0$, s.t. $\|\xi \cdot \eta\|_{H_1} \leq C \cdot \omega(n \cdot \eta)$

If $\Delta = 1$ (for type I or type II$_1$ factors), then $\sigma = \text{id}$

$\Rightarrow$ Connes fusion reduces to an algebraic tensor product of bimodules.
Remember the bi-category $D_n$ (sketchy)

- objects: 0-dim spin mfd's $Z$
- morphisms: spin diffeo $Z_1 \rightarrow Z_2$
  - one-dim. spin mfd $Y$ s.th. $\overline{Y} = Z_1 \sqcup Z_2$
- 2-morphisms: either spin diffeo rel. boundary with element $c \in C(Y_1)^{op}$
  - conf. spin surface $\Sigma$ with $\psi \in F_{alg}(\Sigma)$

enriched elliptic object should functor this to...

The bicategory $\mathcal{V}N$ of von Neumann-algebras

- objects: von Neumann-algebras
- morphisms: A morphism from $N$ to $M$ is an $M-N$-bimodule
  - composition: given by Connes fusion
    \[
    \begin{array}{c}
    M \\ H_1 \\
    \downarrow T \\
    N \\
    \downarrow H_2 \\
    \end{array} = \begin{array}{c}
    M \\
    H_1 \otimes H_2 \\
    \end{array}
    \]
  - trivial element given by $L^2(N)$
  - Connes' fusion is associative up to isomorphisms.

- 2-morphisms:
  Given by Intertwiners: $H, H' \ M-N$-bimodules
  \[T \in \mathcal{B}_{M,N}(H, H') = \{ T : H \rightarrow H' \text{ bounded } | \ T \pi_M = \pi_M^{op} T \text{ and } T \pi_N^{op} = \pi_N^{op} T \} \]

extended gluing lemma: There is a unique unitary isometry of
\[C(Y_3) - C(Y_4) - \text{bimodule}\]
\[F(\Sigma_2) \otimes_{A(\Sigma_2)} F(\Sigma_4) \xrightarrow{\sim} F(\Sigma_3)\]

mapping $\Omega_2 \otimes \Omega_4$ to $\Omega_3$
... further more...

\[ \begin{array}{c}
H_4 \otimes H_2 \\
\downarrow T_4 \otimes T_2 \\
M \otimes L \\
\downarrow T_4 \otimes T_2 \\
H_4' \otimes H_2'
\end{array} =
\begin{array}{c}
H_4 \otimes H_2 \\
\downarrow T_4 \otimes T_2 \\
M \otimes L \\
\downarrow T_4 \otimes T_2 \\
H_4' \otimes H_2'
\end{array}
\]

\[ x \otimes \xi \in H_4 \otimes H_2 \\
(T_4 \otimes T_2)(x \otimes \xi) = T_4 x \otimes T_2 \xi \]

**Adjunction transformations in vN**

- three involutions
  \[ N \mapsto N^{op} \text{ on objects} \]
  \[ H \mapsto \bar{H} \text{ on morphisms, where } \bar{H} \text{ is the conjugate bimodule} \]
- with module actions
  \[ n \cdot \bar{\xi} \cdot m = m^* \xi n^* \quad \xi \in H \]

\[ T \mapsto T^* \text{ on 2-morphisms with the usual adjunction} \]

In view of the adjunctions in the geometric category we would like to have...

\[ vN(C, A_1 \otimes A_2) \to vN(A_1^{op}, A_2) \text{ on morphisms} \]
\[ vN(C, F_2 \otimes_A F_4) \to vN(\bar{F}_2, F_4) \text{ on 2-morphisms} \]

For \( F_4 \) a \( A_1 \otimes A_2 \)-\( C \)-bimodule
\( F_2 \) a \( C \)-\((A_1 \otimes A_2)^{op}\)-bimodule, both lying in the pre-image of the first map

set \( A := A_1 \otimes A_2 \)

Consider:

- intertwiners of the \( A \)-action

\[ \Theta : F_2 \otimes F_4 \to \mathbb{B}_{A}(\bar{F}_2, F_4) \text{ with } \eta_{x, \overline{y}} = (y, x) \eta \]

Take \( x \) that fulfills
\[ x \pi_{A_1}^{op}(a) = \pi_{A_2}^{op}(a) x, \text{ then...} \]
\[ \overline{x} \pi_{A_1}^{op}(a) = x f \cdot \pi_{A_2}^{op}(a) x f = \pi_{A_2}^{op}(a) x f \]

\[ \Rightarrow \overline{x} \pi_{A_1}^{op}(a) = \pi_{A_2}^{op}(a) \overline{x} \Rightarrow (x \in F_2 \Rightarrow \overline{x} \in F_2) \]
$D_{x,y}(\gamma)$ is $A$-linear map (simple comp. using definitions)

$\Rightarrow \Theta$ is well-defined

$\Theta$ is an isometry (shown in Stolz-Teichner for type $\text{III}$ factors)

so...

$\Theta: F_2 \otimes F_\alpha \xrightarrow{\cong} \beta_A(F_2, F_\alpha)$  \[\square]
Interesting subcategories of \( \nu N \)

- Fix an object \( N \in \text{obj}(\nu N) \), type \( III_1 \)-factor

Consider (weakly cont., unital, \(*\)-preserving) endomorphism of \( N \)

\[ \varphi : N \to N \]

Each \( \varphi \) induces via \( L^2(\varphi) \) — another \( N-N \)-bimodule

\[ \text{fusion} \to \text{composition} \to \text{monoidal or tensor categories} \]

\[ \text{direct sum} \to \text{"direct sums" of endomorphisms} \]

leads to fusion rules:

\[ 2 \circ \varphi = \bigoplus_{\lambda \leq \varphi} N^{\varphi}_{\lambda} \cdot \varphi \]

If you take a net of factors instead of a single and demand localizability of endomorphisms you get so-called "fusion rules" of superselection sectors from algebraic quantum field theory.

- Jones extension

Take two factors \( A \subset B \), where \( B \) arises from \( A \) by the "Jones basic construction" all morphisms generated by iterated fusion of \( L^2(B) \), which is an \( A-B \)-bimodule

subfactor has finite Jones index \( \iff \begin{array}{c} F \otimes_B F \text{ and } F \otimes_A F \end{array} \]

contain the vacuum rep. only once.

\[ \text{important for classification of CFTs} \]

\[ \text{invariants of 3-mfds.} \]
Local fermions (sketchy)

- $\mathcal{H}$ complex Hilbert space

$\text{Cliff}(\mathcal{H})$ generated by $a(f), f \in \mathcal{H}$

$$a(f) a(g) + a(g) a(f) = 0$$

$$a(f) a(g)^* + a(g)^* a(f) = (f,g)$$

acts on $\Lambda \mathcal{H} \quad \pi(a(f)) \xi = f \Lambda \xi$

$$c(f) = a(f) + a(f)^*$$

fulfill $c(f)c(g) + c(g)c(f) = 2 \Re(f,g)$

Take projection $p$ into $\mathcal{H}$

representation $\pi_p(a(f)) = \frac{1}{2}(c(f) - i c(i(2P-1)f))$ on $\Lambda \mathcal{H}$

is again irreducible

Now take $\mathcal{H} = L^2(S^1) \otimes V, \quad V = \mathbb{C}^N$

$p$ orthog. proj. onto the Hardy space $H^2(S^1) \otimes V$

$\pi_p$ corr. irr. rep.

... then $\mathcal{M}(\Gamma) = \pi_p(a(f))''$ with $f \in L^2(\Gamma, V)$

is a (net of) von Neumann algebra(s)

properties:

- $\Gamma^c = S^1 \setminus \Gamma$

  - vacuum vector $\Omega$ is cyclic and sep. for each $\mathcal{M}(\Gamma)$
  - modular group acts geometrically

  Let $\Gamma$ be upper semi-circle,

  $$(u_\tau f)(z) = (z \sinh \tau + \cosh \tau)^{-1} \cdot f \left( \frac{z \cosh \tau + \sinh \tau}{z \sinh \tau + \cosh \tau} \right)$$

  "Möbius flow"

  $\Delta^{it} \pi_p(a(f)) \Delta^{-it} = \pi_p(a(u_\tau f)) \quad \forall f \in \mathcal{H}$

- modular conjugation acts geometrically

  - $F$ is "flip" $F(f(z)) = z^{-1} f(z^{-1})$
  - $x$ Klein transformation (?)

  $$J \pi_p(a(f)) J = x^{-1} \pi_p(a(u_\tau f)) x \quad J \mathcal{M}(\Gamma) J = \mathcal{M}(\Gamma^c)$$