Graduate Student Seminar Winter Term 2008/09: K-Theory and Derived Equivalences

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The program for the graduate student seminar next term is the paper "K-Theory and derived equivalences" by Daniel Dugger and Brooke Shipley [DS04].

The main result is the following theorem.

Theorem 1. *Let R, S be [discrete] rings such that their derived categories of unbounded chain complexes are equivalent as triangulated categories. Then R and S have equivalent algebraic K-theory (in particular they have isomorphic algebraic K-groups).*

The derived category of a ring is formed by taking the category unbounded chain complexes of modules over this ring and formally invert all homology isomorphism. The algebraic K-theory of a ring is build out of the category of finitely generated projective modules, as we have learned in detail last term. The point is that by forming the homotopy category all "second-order homotopy information" is lost whereas algebraic K-Theory seems to detect some of them.

Maybe as interesting as the result itself are the intermediate steps of the proof. It uses model categories and goes in two steps:

1. If two rings have equivalent derived categories, than the categories on chain complexes are equivalent as model categories.
2. Any equivalence of model categories yields an equivalence of K-theory.

The first step is surprising, since there are examples of model categories with equivalent homotopy categories but not equivalent K-theory [Sch02]. (For that one has to know how one does "K-theory of model categories", but that is possible, though not completely straightforward.) The second step is not surprising, although it needs some work to work this out.

I refer to the nice introduction of the paper for a more coherent overview. The paper itself seems not be difficult but it needs some "classical" results. ("Classical" in the sense that they are established, well-known results of independent interest. I put this in quotes because some of them are younger the ten years.) So we can use the opportunity to discuss them.

Topics which are to be discussed are: model categories, model category structures on chain complexes and on modules over differential graded algebras, algebraic K-Theory of model categories – and hence of rings – via the $S_{\cdot}$-construction, G-Theory, classical Morita Theory of rings, triangulated categories and Gabriel's and Freyd's Theorem.

See the next pages for the list of the talks. You can find the bibliography at the end of this document.

Details of the talks

This is a list of the topics which are to be discussed in the respective talks. The reference I give refers only to [DS04], usually you will need other references which can be found in the paper, or just ask around.

Talk 1: Model categories (Sebastian Thomas) Introduce model categories. Define the homotopy category. Talk about Quillen equivalences. (2.1 - 2.5, the chain complex examples are done later)

Talk 2: The model structure on $\mathrm{Ch}(R)$ (Jasmin Matz) Prove the various model structure on chain complexes and compare them. (2.2, 2.4)

Talk 3: Stable model categories (Fabian Lenhardt) Introduce stable model categories and discuss that the homotopy category is triangulated. Discuss that for stable model categories Quillen equivalences preserve the triangulated structure. Maybe only explain it in the case of the examples (chain complexes over a ring, maybe mention spectra).

Talk 4: The model structure on dga-modules (Boryana Dimitrova / Irakli Patchkoria) Introduce the notions of differential graded algebra (dga) and the category of differential graded modules over an dga. Discuss the enrichment and closed symmetric monoidal structure. Prove the model structure on dga-mod. Maybe do the small Lemma 6.6. (6.2 - 6.3 [, 6.6])

For this present the results of the paper [SS00] which we need. (This is the second talk.)

Talk 5: K-Theory of rings after Waldhausen (Katja Hutschenreuter) Define the algebraic K-Theory of a ring via projective modules (not in the text) and via chain complexes. Introduce the $\mathcal{S}_{\cdot}$-construction. Proof that both definitions agree up to homotopy. Define the algebraic K-Theory of the compact objects of a model category. Use Theorem 3.7 to prove that a $*$Quillen-equivalence of model categories induce an equivalence on the K-Theories. Don't prove Theorem 3.7. (3.1-3.12) (this may be split into two talks)

Talk 6: Proof of Theorem 3.7 (Mark Ullmann) Prove Theorem 3.7. (3.7, Appendix A)

Talk 7: Classical Morita Theory and the Theorem (Saeid Hamzeh Zarghani) Discuss classical Morita Theory, namely Theorem 4.1. Define the notions of localizing subcategory and (weak) generator. Explain the Tilting Theorem and prove the easier implications. (4.1-4.5)

Talk 8: The Generalization of Gabriel's Theorem (Irakli Patchkoria) Prove the Quillen-equivalence $\mathrm{Mod} - \mathrm{End}_R(P) \simeq \mathrm{Ch}(R)$ (Theorem 6.4). (6.1, 6.4-6.7)

Talk 9: Finish the proof of the Theorem (Fabian Lenhardt) Complete the proof of Theorem 4.2. Proof the main results Theorem A and B. (after 6.7 - 6.8, 5 - 5.1)

Talk 10: The G-Theory version (Achim Beckers) Introduce and discuss G-Theory. Prove the results for G-Theory. (3.4, 5.1-5.2)

Talk 11: The many generator version (Moritz Groth) Discuss the version for many generators. (Section 7)

References

[DS04] Daniel Dugger and Brooke Shipley. K-theory and derived equivalences. *Duke Math. J.*, 124(3):587–617, 2004.

[Sch02] Marco Schlichting. A note on K-theory and triangulated categories. *Invent. Math.*, 150(1):111–116, 2002.

[SS00] Stefan Schwede and Brooke E. Shipley. Algebras and modules in monoidal model categories. *Proc. London Math. Soc. (3)*, 80(2):491–511, 2000.