Rational global homotopy theory and geometric fixed points

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1 Introduction

The main purpose of this thesis is to give an algebraic model for rational global stable homotopy theory in terms of geometric fixed points. Before further explaining this statement, we wish to put it into context by reviewing the classical situation.

In the early second half of the 20th century stable homotopy theory emerged as an important subject of great computational complexity in the field of algebraic topology. Loosely speaking, one can say that it takes place in the *stable homotopy category $\mathcal{SHC}$*, the objects of which are *spectra* and can be thought of as 'stabilized spaces' or as corresponding to generalized cohomology theories. The first interpretation is nicely illustrated by the Freudenthal suspension theorem, which says that for finite CW-complexes X and Y the sequence

$$[X,Y] \xrightarrow{\Sigma} [\Sigma X, \Sigma Y] \to \cdots \to [\Sigma^n X, \Sigma^n Y] \to \cdots$$

of homotopy classes consists of isomorphisms for large enough n, where Σ denotes the suspension of a space. The second point of view is essentially the Brown representability theorem, which roughly states that every cohomology theory is represented by a spectrum.

While there is no hope of ever fully 'computing' the stable homotopy category, drastic simplifications occur if we consider it up to rational equivalence (this corresponds to cohomology theories with values in $\mathbb{Q}$-vector spaces). It is a classical fact (going back to Serre's computation of the rational homotopy groups of spheres) that there is an equivalence

$$\pi_*(-) \otimes \mathbb{Q} : \mathcal{SHC}_{\mathbb{Q}} \xrightarrow{\simeq} \mathrm{gr.}\,\mathbb{Q}\text{-}\mathrm{mod}$$

between the rational stable homotopy category and the category of graded $\mathbb{Q}$-vector spaces.

The situation is analogous in the G-equivariant setting for a finite group G, but the bookkeeping becomes more involved. There are several variants of G-equivariant stable homotopy theory and here we talk about the most sophisticated one, called *genuine*, which has recently seen spectacular applications in the solution of the Kervaire Invariant One problem ([HHR16]). Here the cohomology theories modelled are the 'RO(G)-graded' ones, that is, cohomology theories for G-spaces, indexed by (virtual) G-representations with suspension isomorphisms for the associated representation spheres. Associated to a genuine G-spectrum X we now have equivariant homotopy groups $\pi_*^H X$ for each subgroup $H \leq G$ and these values are related by certain restriction, transfer, and conjugation morphisms giving the entire collection the algebraic structure of a *Mackey functor* ([Dre73]). The category $\mathcal{MF}$ of Mackey functors is abelian and all rational Mackey functors (those taking value in $\mathbb{Q}$-vector spaces) are projective. This implies that the derived category of Mackey functors over $\mathbb{Q}$ is equivalent to graded Mackey functors via taking homology.

Theorem (cf. [GM95])**.** *Taking homotopy groups induces an equivalence*

$$\{\pi_*^H(-)\}_{H\leq G} : G\text{-}\mathcal{SHC}_{\mathbb{Q}} \xrightarrow{\simeq} \mathrm{gr.}\, G\text{-}\mathcal{MF}_{\mathbb{Q}} \simeq \mathcal{D}(G\text{-}\mathcal{MF}_{\mathbb{Q}}),$$

between the genuine G-equivariant stable homotopy category and the category of graded rational G-Mackey functors. Moreover, this equivalence is monoidal with respect to the smash product of spectra and the graded tensor product of Mackey functors.

Global homotopy theory

In global (stable) homotopy theory one tries to make precise the idea of encoding 'compatible' actions by all (say finite) groups in a single homotopy type. This is based on the observation that there are equivariant (co-)homology theories such as K-theory and bordism which make sense for all these groups. Moreover, their values are related by *transfer maps* along subgroup inclusions and *restriction maps* along arbitrary group homomorphisms. In addition to the 'Mackey functor relations' between them, the restrictions along surjective homomorphisms also commute with transfers in an appropriate sense. For K-theory these are given by induction of equivariant vector bundles and restricting the action on them. Guided by the existence of such examples, one is lead to organize them into a homotopy theory of their own, a *global* homotopy theory, and one approach to this using orthogonal spectra has recently been extensively developed by Schwede in [Sch17b]. This global point of view has already shown itself to be useful in the equivariant study of the symmetric product filtration ([Sch17a], [Hau16]) and filtrations of K-theory [HO15]. The results have uniform global descriptions that are entirely lost if one would just consider a single group G, but for every 'concrete' group they still provide a concrete answer.

Now, in the global setting the rational comparison question is more subtle. As the group G varies, the equivariant homotopy groups $\pi_*^G X$ of an orthogonal spectrum are related by restriction maps along arbitrary group homomorphisms and transfer maps along subgroup inclusions. The resulting structure is called a *global functor* (or inflation functor, cf. [Web93]) and we denote the abelian category of global functors by $\mathcal{GF}$. Rational global functors are in general not projective and just taking homotopy groups will not even produce a faithful functor:

$$\{\pi_*^G(-)\}_G : \mathcal{GH}_{\mathbb{Q}} \xrightarrow{\not\simeq} \mathrm{gr.}\, \mathcal{GF}_{\mathbb{Q}}.$$

The correct target is the derived category $\mathcal{D}(\mathcal{GF}_{\mathbb{Q}})$ of global functors and one can use stable Morita theory to produce a zigzag of (Quillen-)equivalences (cf. [Sch17b, IV.6]). It would of course be very desirable to construct a more explicit, direct functor in order to have better control over it and to be able to investigate its multiplicative properties. But it does not seems likely that one could just produce a 'spectral global functor' without moving to an ∞-categorical setting (as for Mackey functors [Bar17]), at the very least it would involve serious coherence issues that are usually dealt with by introducing appropriate zigzags.

Instead it turns out be much more convenient to use *geometric fixed points* to map to an equivalent, but algebraically simpler target. Geometric fixed points have long been an important and useful tool in equivariant stable homotopy theory (e.g. see [May96]), satisfying the design criteria of commuting with suspension spectra

$$\Phi^G \Sigma_+^\infty A \simeq \Sigma_+^\infty A^G,$$

and preserving homotopy colimits. Moreover, there is a natural comparison map $\pi_*^G X \to \Phi_*^G X$ of homotopy groups. The (global) geometric fixed point homotopy groups $\underline{\Phi}_* X = \{\Phi_*^G X\}$ of an orthogonal spectrum X are analyzed in [Sch17b]. It turns out that they admit restriction maps along surjective group homomorphisms, also called *inflations*, which only depend on the conjugacy class of the morphism. This can be rephrased by saying that they form $\mathrm{Out}^{\mathrm{op}}$-diagrams of abelian groups, where Out is the category of finite groups and conjugacy classes of epimorphisms. The following two facts show why it is reasonable to expect geometric fixed points to be useful in the task of identifying rational global homotopy theory with an algebraic model:

- There is a rational equivalence

  $$\tau : \mathcal{GF}_\mathbb{Q} \simeq \mathrm{Out}^{\mathrm{op}}\text{-}\mathrm{mod}_\mathbb{Q}$$

  between the abelian categories of global functors and $\mathrm{Out}^{\mathrm{op}}$-modules.

- Under this equivalence the homotopy groups of a spectrum X are identified with the geometric fixed point homotopy groups:

  $$\tau(\underline{\pi}_* X) \cong_\mathbb{Q} \underline{\Phi}_* X$$

Statement of results

We now summarize the results of this thesis.

Rational global homotopy theory

We carefully investigate an appropriate model of geometric fixed point spectra from the global perspective to produce a functor

$$\Phi : \mathrm{Sp}^\mathrm{O} \to \mathrm{Epi}^{\mathrm{op}}\text{-}\mathrm{Sp}^\mathrm{O}$$

to the diagram category of $\mathrm{Epi}^{\mathrm{op}}$-orthogonal spectra, where Epi is the category of finite groups and surjective homomorphisms. These are not quite the correct diagrams, but after moving to the algebraic world and 'adjusting the functoriality' we show

Theorem. *Geometric fixed points induce a symmetric monoidal equivalence*

$$\mathcal{GH}_\mathbb{Q} \simeq \mathcal{D}(\mathrm{Out}^{\mathrm{op}}\text{-}\mathrm{mod}_\mathbb{Q})$$

between the rational global homotopy category and the derived category of rational $\mathrm{Out}^{\mathrm{op}}$-*modules.*

Ring spectra

The geometric fixed point functor is lax symmetric monoidal and can thus be used to analyze categories of ring spectra in the global setting.

Theorem. *Geometric fixed points restrict to an equivalence*

$$\mathrm{Ho}(\mathrm{Ass}(\mathrm{Sp}^{\mathrm{O}})_{\mathrm{gl},\mathbb{Q}}) \simeq \mathrm{Ho}(\mathrm{Out}^{\mathrm{op}}\text{-}\mathrm{DGA}_{\mathbb{Q}})$$

identifying the homotopy theories of global rational associative ring spectra and $\mathrm{Out}^{\mathrm{op}}$*-diagrams in rational differential graded algebras.*

The commutative case turns out to be more involved. We first describe an algebraic result: The homotopy group global functors of a commutative ring spectrum R carry the extra structure of power operations or equivalently norm maps

$$N_H^G : \pi_0^H R \to \pi_0^G R$$

associated to subgroup inclusions $H \leq G$. This turns them into *global power functors*, a global analogue of a Tambara functors.

Theorem. *The functor τ restricts to an equivalence*

$$\mathcal{GPF}_{\mathbb{Q}} \simeq \mathrm{Out}^{\mathrm{op}}_{\mathrm{norm}}\text{-}\mathrm{mod}_{\mathbb{Q}}$$

between the category of rational global power functors and $\mathrm{Out}^{\mathrm{op}}$*-modules with norm maps.*

This can be thought of as the 'π_0-shadow' of a topological comparison result: For commutative ring spectra R we construct natural norm maps

$$N_H^G : \Phi^H R \to \Phi^G R$$

on the geometric fixed points as morphisms of ring spectra and these are closely related to the Hill-Hopkins-Ravenel norm used in the solution of the Kervaire invariant one problem [HHR16].

Theorem. *Let $\mathbb{P}X$ be the free commutative ring on a spectrum X. Then $\underline{\Phi}_*\mathbb{P}X$ is the free commutative* $\mathrm{Out}^{\mathrm{op}}_{\mathrm{norm}}$*-algebra on $\underline{\Phi}_*X$.*

For technical reasons (up to a bar resolution having the correct homotopy type) we cannot yet show an analogous comparison result. We are however confident about the following

Conjecture. *Geometric fixed points restrict to an equivalence*

$$\mathrm{Ho}(\mathrm{Com}(\mathrm{Sp}^{\mathrm{O}})_{\mathrm{gl},\mathbb{Q}}) \simeq \mathrm{Ho}(\mathrm{Out}^{\mathrm{op}}_{\mathrm{norm}}\text{-}\mathrm{CDGA}_{\mathbb{Q}})$$

identifying the homotopy theories of global rational commutative ring spectra and $\mathrm{Out}^{\mathrm{op}}$*-diagrams in rational commutative differential graded algebras.*

Rational splitting of global K-theory

As mentioned above, not every rational global functor (or $\mathrm{Out}^{\mathrm{op}}$-module) is projective. So the derived category does not split as in the case of rational Mackey functors. On the topological side this means that rational global homotopy types need not decompose into sums of global Eilenberg-MacLane spectra, i.e. spectra with homotopy group global functors concentrated in a single degree. Naturally occuring examples are symmetric products of the sphere spectrum (cf. [Hau16]). In general it is a very subtle question if this happens or not. In the case of *global K-theory* one can give a positive answer. Here global K-theory means a global refinement of K-theory (based on the model given in [Joa04]) such that the underlying G-spectra represent equivariant K-theory [Seg68]. The homotopy groups of the global complex K-theory spectrum $\mathbf{KU}$ are given by the *representation ring global functor* $\mathbf{RU}$ in even degrees and they vanish in odd degrees. The splitting follows from the following algebraic result:

Theorem. *Let F be a rational global functor. There are natural isomorphisms*

$$\mathrm{Ext}^k_{\mathcal{GF}_{\mathbb{Q}}}(F, \mathbf{RU}_{\mathbb{Q}}) \cong \varprojlim_{(\mathbb{N},|)}{}^k (F(C_n)^{\vee}_{\mathrm{tr}})$$

for all $k \geq 0$, identifying the higher extensions of $\mathbf{RU}_{\mathbb{Q}}$ as derived limits over the poset of natural numbers with respect to the divisibility relation.

Here $F(C_n)^{\vee}_{\mathrm{tr}}$ denotes a subgroup of the $\mathbb{Q}$-linear forms on the value $F(C_n)$ of F at a cyclic group of order n, namely those that vanish on the image of all transfer maps from proper subgroups. After a choice of generators, the inverse system can be defined by restriction along those epimorphisms preserving the preferred generators. It turns out that the Ext-groups $\mathrm{Ext}^n_{\mathcal{GF}_{\mathbb{Q}}}(F, \mathbf{RU}_{\mathbb{Q}}) = 0$ vanish in all higher degrees $n \geq 2$ and a closer inspection also shows $\mathrm{Ext}^1_{\mathcal{GF}_{\mathbb{Q}}}(\mathbf{RU}_{\mathbb{Q}}, \mathbf{RU}_{\mathbb{Q}}) = 0$. The following is then a formal consequence:

Corollary. *The rationalized global complex K-theory spectrum $\mathbf{KU}_{\mathbb{Q}}$ with respect to finite groups (canonically) splits as a wedge of Eilenberg-MacLane spectra of global functors. The same holds for $\mathbf{KO}$ as a rational retract of $\mathbf{KU}$.*

Real-global homotopy theory

There is also a C_2-equivariant refinement of global homotopy theory, called *Real-global* homotopy theory. If a global homotopy type encodes compatible G-actions for all finite groups G, then a Real-global homotopy type does so for all augmented groups $G \to C_2$, also taking 'twisting by the augmentation' into account. Motivating examples to keep in mind are Atiyah's Real K-theory $K\mathbb{R}$ [Ati66] and *Real*-bordism $M\mathbb{R}$ (studied by Landweber [Lan68] and Fujii [Fuj76]).

Theorem. *Geometric fixed points induce an equivalence*

$$(\mathbb{R}\,\mathcal{GH})_{\mathbb{Q}} \xrightarrow{\simeq} \mathcal{D}(\mathrm{Out}^{\mathrm{op}}_{\mathbb{R}}\text{-}\mathrm{mod}_{\mathbb{Q}})$$

between the rationalized Real-global homotopy category and the derived category of rational $\mathrm{Out}^{\mathrm{op}}_{\mathbb{R}}$-modules, where $\mathrm{Out}^{\mathrm{op}}_{\mathbb{R}}$ is a Real version of the category Out.

Organization

We now outline the structure of this thesis. Chapter 2 is about setting the stage for our later comparison work. In the first half we review global (stable) homotopy theory, mainly recalling the necessary material from [Sch17b]. The second half is concerned with a more detailed discussion of the geometric fixed points functor, especially from the 'global' perspective, and its multiplicative properties. We conclude with a construction of norm maps for commutative ring spectra.

After this technical setup, Chapter 3 contains the main results. We start by recalling generalities on equivalences of triangulated categories and a chain functor enabling the passage from topology to algebra. With the necessary foundations laid, we put everything together in the second section to construct a multiplicative equivalence with the algebraic model (Theorems 3.2.14 and 3.2.20). We also offer a $\mathbb{Z}[\frac{1}{p}]$-local description of global homotopy theory with respect to p-groups as a digram category (Theorem 3.2.30). This is followed by a proof of the rational splitting of global K-theory (Theorem 3.3.1). In the last section we turn our attention towards ring spectra and give rational models for associative (Theorem 3.4.4) and E_∞-ring spectra (Theorem 3.4.7).

Finally, in the last chapter we give a brief introduction to *Real-global* homotopy theory. It is meant as an addendum to the main text, in particular we omit many details. Technically, one now uses *Real unitary spectra*, a unitary version of orthogonal spectra taking complex conjugation into account, and we explain how they can be used to present such a homotopy theory. We then define and discuss a geometric fixed point functor in this setting and give a sketch of the analogous rational comparison program.

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2 Global homotopy theory

2.1 Preliminaries

We recall some background material and fix notation. We work in the category of compactly generated, weak hausdorff spaces and denote it Top (resp. Top_* for pointed/based spaces). Given a finite group G and based G-spaces (i.e. G-objects in Top_*), we write $[X,Y]^G$ for the G-equivariant homotopy classes of maps between X and Y. The *Weyl group* $W_G H$ of a subgroup $H \leq G$ is the quotient $W_G H = N_G H/H$ of the normalizer of H in G by H and we will just write WH if the ambient group G is understood. We will often encounter sums indexed by conjugacy classes (of subgroups or morphisms) where the individual components make explicit reference to representatives. This will usually be written in the form $\oplus_{(H\leq G)} X(H)$, meaning that H runs over a complete set of representatives and the choice of these is understood.

For a ring R, we usually write $R\{X\}$ for the R-linearization of a set X, the free R-module on X. In the case of a group G the more usual notation $R[G]$ for the group algebra will also be used.

The following well-known decomposition formula for the fixed points of a quotient is often used in equivariant homotopy theory (see the proof of [Sch17b, I.2.23] for the argument and further references).

Lemma 2.1.1. *Let X be a $(K \times G^{\mathrm{op}})$-space with free right G-action. Then the projection $X \to X/G$ induces a homeomorphism*

$$(X/G)^K \cong \coprod_{(\alpha: K\to G)} (\alpha^* X)^K / C(\alpha)$$

from a disjoint union indexed by the conjugacy classes $\mathrm{Rep}(K,G)$ of group homomorphisms, where $C(\alpha)$ denotes the centraliser of the image of α.

Remark 2.1.2. The map $(X/G)^K \to \mathrm{Rep}(K,G)$ projecting to the indexing set is independent of the choice of representatives above. It sends the orbit xG to the conjugacy class of the homomorphism $\alpha : K \to G$ determined by $k.x = x.\alpha(k)$ for $k \in K$.

Homotopical algebra

The language of homotopical algebra will be freely used throughout. Here we just recall a few notions. A *homotopical category* ([DHKS04]) is a category $\mathcal{C}$ equipped with a class of morphisms $\mathcal{W}$, the *weak equivalences* ($\xrightarrow{\simeq}$), satisfying the '2 out of 6' property (this is a slight refinement of the notion of a relative category, also see [HHR16, B.1] for an overview). By the usual abuse of notation, the weak equivalence will often be suppressed. A functor between homotopical categories is *homotopical* if it preserves weak equivalences. We write $\mathrm{Ho}(\mathcal{C})$ for the *homotopy category*, which comes with a (localization) functor $\gamma : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ that sends weak equivalences to isomorphisms, and is characterised as the initial such example (if it exists). By the universal property, a homotopical functor $F : \mathcal{C} \to \mathcal{D}$ descends to a unique functor

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d] & \mathcal{D} \arrow[d] \\ \mathrm{Ho}(\mathcal{C}) \arrow[r, "\mathrm{Ho}(F)"] & \mathrm{Ho}(\mathcal{D}) \end{tikzcd}$$

between homotopy categories such that the square commutes.

Model categories (introduced by Quillen in [Qui67]) come with the additional structure of two distinguished auxiliary classes of morphisms, the cofibrations ($\hookrightarrow$) and fibrations ($\twoheadrightarrow$), which ensures that the homotopy theory presented by the underlying homotopical category is well-behaved. In particular, this guarantees the existence of the homotopy category and provides means to compute the morphisms in it. If $\mathcal{M}$ is a model category, then $\mathrm{Ho}(\mathcal{M})$ has the same objects as $\mathcal{M}$ and the localization functor γ induces a bijection

$$[X, Y] \xrightarrow{\cong} \mathrm{Ho}(\mathcal{M})(X, Y)$$

from the homotopy classes of maps between objects X and Y such that X is cofibrant and Y is fibrant. We refer to [DS95] for the axioms and a basic outline of the theory, and to [Hov99] for a more comprehensive treatment.

A model category $\mathcal{M}$ equipped with a compatible symmetric monoidal product $\wedge : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ is said to be *monoidal*. Here compatible means that a pushout product axiom is satisfied (see [Hov99, 4.2.6]). In that case the symmetric monoidal structure descends to a product

$$\wedge^L : \mathrm{Ho}(M) \times \mathrm{Ho}(M) \to \mathrm{Ho}(M)$$

on the level of homotopy categories ([Hov99, 4.3.2]). It is computed by cofibrantly replacing both factors: $X \wedge^L Y \simeq X_c \wedge Y_c$. We note that in the example of rational chain complexes this is not necessary as the tensor product in homotopical in both variables.

A pointed model category $\mathcal{M}$ (i.e. $\mathcal{M}$ has a zero object) admits a *suspension functor* $\Sigma : \mathrm{Ho}\,\mathcal{M} \to \mathrm{Ho}\,\mathcal{M}$ on the level of homotopy categories ([Hov99, 6.1]). In the simplicially/topologically enriched setting this is modelled by smashing with S^1, e.g. in various categories of spectra. For chain complexes, one may use the simplicial tensor $NS^1 \otimes -$, which amounts to the classical shift of complexes. A pointed model category is called *stable* if the suspension functor is an equivalence. One of the most important consequences of stability is the fact that the homotopy category $\mathrm{Ho}(\mathcal{M})$ canonically carries the structure of a *triangulated category* with the suspension functor as autoequivalence ([Hov99, Chapter 7]). A triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

is declared to be distinguished (or exact) if it is isomorphic in $\mathrm{Ho}(\mathcal{C})$ to an elementary triangle arising from the cofiber sequence $A \hookrightarrow B \to B/A \to \Sigma A$ associated to a cofibration $A \hookrightarrow B$ in $\mathcal{C}$. In the topologically enriched setting one can also use the mapping cone sequences

$$X \xrightarrow{f} Y \xrightarrow{i} C(f) \xrightarrow{p} S^1 \wedge X,$$

and analogously for chain complexes.

We also recall the projective model structure on diagram categories (e.g. see [Hir03, 11.6]).

Proposition 2.1.3. *Let $\mathcal{C}$ be a (skeletally) small indexing category and $\mathcal{M}$ a cofibrantly generated model category with generating (acyclic) cofibrations I and J. Then the functor category $\mathrm{F}(\mathcal{C}, M)$ admits the* projective model structure *determined by declaring the weak equivalences and fibrations level-wise. This model structure is again cofibrantly generated with generating sets*

$$FI = \bigcup_{[c]\in \mathrm{Ob}\,\mathcal{C}/\simeq} F_c I \text{ and } FJ = \bigcup_{[c]\in \mathrm{Ob}\,\mathcal{C}/\simeq} F_c J,$$

where c runs over a set of representatives for the isomorphism classes of objects of $\mathcal{C}$

Remark 2.1.4. We note that precomposition $F^* : F(\mathcal{D}, \mathcal{M}) \to F(\mathcal{C}, \mathcal{M})$ with a functor $F : \mathcal{C} \to \mathcal{D}$ preserves (acyclic) fibration since they are defined level-wise. So F^* is automatically a right Quillen functor with left adjoint the left Kan extension (which exists because $\mathcal{M}$ admits all colimits).

The main examples will be orthogonal spectra $\mathcal{M} = \mathrm{Sp}^{\mathrm{O}}$ (equipped with the model structure of [MMSS01, II.9]) and chain complexes $\mathcal{M} = \mathrm{Ch}$. Here chain complexes are endowed with the projective model structure ([Hov99, 2.3]) with weak equivalences the quasi-isomorphisms and fibrations the degree-wise surjections. If the model structure on $\mathcal{M}$ is stable as in these cases then so is the projective model structure because the suspension functor is defined level-wise and hence again an equivalence.

2.2 Global stable homotopy theory

In this section we review global homotopy theory, mainly by recalling the necessary material from [Sch17b], to which we refer for a detailed treatment.

2.2.1 Orthogonal spectra

Let $\mathbf{L}$ be the topological category of finite dimensional real inner product spaces together with linear isometric embeddings. The indexing category $\mathbf{O}$ for orthogonal spectra is obtained from this by passing to certain Thom spaces: Let V, W be inner product spaces. The 'orthogonal complement bundle' is the vector bundle over $\mathbf{L}(V, W)$ with total space $E(V, W) \subset \mathbf{L}(V, W) \times W$ consisting of those pairs (ψ, w) such that $w \in \psi(V)^{\perp}$ lies in the orthogonal complement of the image of ψ. These bundles come with composition pairings $(\psi, w)\circ(\phi, u) = (\psi\circ\phi, w+\psi(u))$ defining the composition in the based topological category $\mathbf{O}$ where $\mathbf{O}(V, W)$ is the associated Thom space of $E(V, W)$ (i.e. the one-point compactification).

Definition 2.2.1. The category Sp^{O} of *orthogonal spectra* is the category of continuous, based functors from $\mathbf{O}$ to pointed spaces.

More concretely, an orthogonal spectrum consists of a collection of pointed spaces $\{X(V)\}$ indexed by finite dimensional inner product spaces together with *action maps* $\alpha_{V,W} : \mathbf{L}(V, W)_+ \wedge X(V) \to X(W)$ and *structure maps* $\sigma_{V,W} : X(V) \wedge S^W \to X(V \oplus W)$ that are unital, associative, and transitive. It actually suffices to specify the action of linear isometric isomorphisms (cf. [MM02, II.4.3]) and we will often describe orthogonal spectra in this more explicit, reduced form.

We now recall some standard constructions. The category of orthogonal spectra is naturally tensored, cotensored, and enriched over pointed spaces. Tensors and cotensors are defined level-wise: The spectra $A \wedge X$ and $\mathrm{map}(A, X)$ for $X \in \mathrm{Sp}^{\mathrm{O}}$ an orthogonal spectrum and A a pointed space are given by

$$(A \wedge X)(V) = A \wedge X(V),\ \mathrm{map}(A, X)(V) = \mathrm{map}(A, X(V))$$

In particular for $A = S^1$ we obtain the suspension and loop functors on orthogonal spectra, denoted by $\Sigma X = S^1 \wedge X$ and $\Omega X = \mathrm{map}(S^1, X)$. The mapping space $\mathrm{map}(X, Y)$ of orthogonal spectra X and Y has as underlying set the collection of morphisms from X to Y. It is topologized as a subset of the product of mapping spaces $\mathrm{map}(X, Y) \subset \prod_V \mathrm{map}(X(V), Y(V))$ and we have adjunction homeomorphisms

$$\mathrm{map}(A, \mathrm{map}(X, Y)) \cong \mathrm{map}(A \wedge X, Y) \cong \mathrm{map}(X, \mathrm{map}(A, Y)).$$

The *shift* $\mathrm{sh}\, X$ of X is the orthogonal spectrum defined in level V by

$$(\mathrm{sh}\, X)(V) = X(\mathbb{R} \oplus V).$$

The structure maps of X define a natural comparison map

$$\lambda_X : \Sigma X \to \mathrm{sh}\, X$$

given in level V by the composition

$$S^1 \wedge X(V) \cong X(V) \wedge S^1 \xrightarrow{\sigma} X(V \oplus \mathbb{R}) \cong X(\mathbb{R} \oplus V).$$

The diagrammatic description of orthogonal spectra is convenient for the description of free spectra: Given a G-representation V, the associated free spectrum on a G-space A is the orthogonal spectrum $F_{G,V}A = \mathbf{O}(V, -) \wedge_G A$. It is free in the sense that $F_{G,V}$ corepresents evaluation at V:

$$\mathrm{Sp}^{\mathrm{O}}(F_{G,V}A, X) \cong \mathrm{Top}_*(A, X(V))^G$$

The unstable version of an orthogonal spectrum is an *orthogonal space*, which is defined to be a continuous functor from $\mathbf{L}$ to spaces. There is a generalised suspension-loops adjunction

$$\Sigma^\infty_+ : \mathbf{L}\text{-}\mathrm{Top} \leftrightarrows \mathrm{Sp}^{\mathrm{O}} : \Omega^\bullet$$

Here the suspension spectrum $\Sigma^\infty_+ X$ of an orthogonal space X is given at the inner product space V by $(\Sigma^\infty_+ X)(V) = X(V)_+ \wedge S^V$ and the orthogonal space $\Omega^\bullet Y$ is defined by $(\Omega^\bullet Y)(V) = \Omega^V Y(V)$.

Example 2.2.2. Let G be a finite group. The free orthogonal space $\mathbf{L}_{G,V}$ generated by a point at the G-representation V is given by $\mathbf{L}_{G,V} = \mathbf{L}(V, -)/G$. The Yoneda element $\mathrm{Id}_V \in \mathbf{L}_{G,V}(V)$ determines a natural isomorphism $\mathrm{Hom}(\mathbf{L}_{G,V}, X) \cong X(V)^G$ for all orthogonal spaces X.

For a detailed discussion of the following multiplicative notions see [Sch17b, I.4, III.5]. The Day convolution product with respect to the orthogonal sum of inner product spaces $\oplus : \mathbf{O} \wedge \mathbf{O} \to \mathbf{O}$ defines a symmetric monoidal product

$$- \wedge - : \mathrm{Sp}^{\mathrm{O}} \times \mathrm{Sp}^{\mathrm{O}} \longrightarrow \mathrm{Sp}^{\mathrm{O}}$$

on orthogonal spectra, the *smash product*. We recall that this means the following: Let X and Y be orthogonal spectra. The external smash product $X \bar{\wedge} Y$ is the functor on $\mathbf{O} \wedge \mathbf{O}$ defined by $(X \bar{\wedge} Y)(V, W) = X(V) \wedge Y(W)$. A *bimorphism* $b : (X, Y) \to Z$ is a natural transformation $X \bar{\wedge} Y \to Z \circ \oplus$ and the smash product of X and Y is characterised as an orthogonal spectrum $X \wedge Y$ together with an initial bimorphism $\iota : (X, Y) \to X \wedge Y$. Analogously the *box product* $X \boxtimes Y$ of orthogonal spaces is defined with respect to the orthogonal sum pairing on the linear isometries category $\mathbf{L}$. Both products are related via a natural transformation

$$\Sigma^\infty_+ X \wedge \Sigma^\infty_+ Y \longrightarrow \Sigma^\infty_+ (X \boxtimes Y),$$

which is determined by the bimorphism

$$X(V)_+ \wedge S^V \wedge Y(W)_+ \wedge S^W \cong (X(V) \times Y(W))_+ \wedge S^{V \oplus W} \xrightarrow{\iota \wedge S^{V \oplus W}} (X \boxtimes Y)(V \oplus W)_+ \wedge S^{V \oplus W}$$

of orthogonal spectra, where $\iota : (X, Y) \to X \boxtimes Y$ is the universal bimorphism associated to the box product of the orthogonal spaces X and Y.

Proposition 2.2.3 ([Sch17b, IV.1.18]). *The suspension spectrum functor $\Sigma^\infty_+ : \mathbf{L}\text{-}\mathrm{Top} \to \mathrm{Sp}^{\mathrm{O}}$ is a strong symmetric monoidal functor.*

Example 2.2.4. It is a formal fact that the box product of free orthogonal spaces is again free: There is a canonical isomorphism $\mathbf{L}_{G,V} \boxtimes \mathbf{L}_{K,W} \cong \mathbf{L}_{G \times K, V \times W}$ corresponding to the bimorphism

$$\mathbf{L}(V, U)/G \times \mathbf{L}(W, Z)/K \xrightarrow{\times} \mathbf{L}(V \times W, U \times Z)/G \times K.$$

By combining this with the previous Proposition we obtain the analogous statement for suspension spectra:

$$\Sigma^\infty_+ \mathbf{L}_{G,V} \wedge \Sigma^\infty_+ \mathbf{L}_{K,W} \cong \Sigma^\infty_+ \mathbf{L}_{G \times K, V \times W}$$

2.2.2 Equivariant homotopy groups

Now that the relevant categories have been set up at the point-set level, they can be utilised from the global equivariant perspective. We begin with the unstable case. Let G be a finite group. An orthogonal G-representation V is a finite dimensional inner product space V together with a continuos G-action through linear isometries, i.e. a morphism $G \to O(V)$. A *G-universe* $\mathcal{U}_G$ is a countably infinite dimensional G-inner product space such that the countable sum of every G-representation embeds into it, and with non-trivial G-fixed points $(\mathcal{U}_G)^G \neq 0$. It is a classical fact that the space of linear isometries $\mathbf{L}(V,\mathcal{U}_G)$ is equivariantly contractible for any G-inner product space V and in this sense universes are determined up to 'contractible' choice. This can be found in [LMSM86, Lemma II.1.5]. However, the proof given there seems to be incomplete as it uses a 'normalized' linear path to connect the identity to a linear isometry. But this will not take values in linear isometries, even up to scaling (see [Sch17b, Proposition I.2.4] for a different argument). We only deal with finite groups and in that case the *regular representation* $\rho_G = \mathbb{R}\{G\}$ provides a canonical universe $\mathcal{U}_G = (\rho_G)^\infty = \bigoplus_{\mathbb{N}} \rho_G$.

The 0-th *G-equivariant homotopy set* of an orthogonal space X is the colimit

$$\pi_0^G X = \operatorname{colim}_{V \subset \mathcal{U}_G} \pi_0(X(V)^G)$$

of the individual path components over the poset of all finite dimensional subrepresentations of the G-universe $\mathcal{U}_G$. As the group varies, the different equivariant homotopy sets are related by *restriction maps* $\alpha^* : \pi_0^K X \to \pi_0^G X$ along group homomorphisms $\alpha : G \to K$. Here α^* takes a representative $x \in X(V)^K \subset X(\alpha^* V)^G$, $V \subseteq \mathcal{U}_K$ and maps it to the class $[X(j)(x)] \in \pi_0 X(j(\alpha^* V))^G$, where $j : \alpha^* V \hookrightarrow \mathcal{U}_G$ is some chosen embedding.

Proposition 2.2.5 (see the discussion around [Sch17b, I.5.12])**.** *Let* Rep *denote the category of finite groups together with conjugacy classes of group homomorphisms. With respect to the restrictions maps the collection $\underline{\pi}_0 X = \{\pi_0^G X\}_G$ of equivariant homotopy sets of an orthogonal space X naturally forms a* $\mathrm{Rep}^{\mathrm{op}}$*-set, that is, a contravariant functor from the category* Rep *to sets.*

We say that X is *closed* if all the structure maps are closed embeddings. In that case the canonical colimit interchange map $\pi_0^G X \xrightarrow{\cong} \pi_0(X(\mathcal{U}_G)^G)$ is a bijection, where we define the value $X(\mathcal{U}_G) = \operatorname{colim}_{V \subset \mathcal{U}_G} X(V)$ of X at the universe $\mathcal{U}_G$ as the colimit over the finite dimensional subrepresentations. A map $f : X \to Y$ between closed orthogonal spaces is a *global equivalence* if it induces a weak equivalence $f(\mathcal{U}_G) : X(\mathcal{U}_G)^G \xrightarrow{\simeq} Y(\mathcal{U}_G)^G$ of spaces for every finite group G.

Example 2.2.6. Let G be a finite group. The *global classifying space* of G is defined as the free orthogonal space $B_{\mathrm{gl}} G = \mathbf{L}_{V,G}$ for a faithful G-representation V. This is well-defined up to the preferred zig-zag

$$\mathbf{L}_{V,G} \xleftarrow{\simeq} \mathbf{L}_{V \oplus W,G} \xrightarrow{\simeq} \mathbf{L}_{W,G}$$

of global equivalences ([Sch17b, I.2.9(ii)]) between closed orthogonal spaces. The identity $[\mathrm{Id}_V] \in (\mathbf{L}_{V,G}(V))^G$ defines a *tautological class* in $\pi_0^G \mathbf{L}_{V,G}$. Moreover, these tautological classes are invariant under the above zigzag and by abuse of notation we just write $u_G \in \pi_0^G B_{\mathrm{gl}}G$. Evaluation at u_G determines an isomorphism ([Sch17b, I.5.16])

$$\mathrm{Rep}(-,G) \xrightarrow{\cong} \underline{\pi}_0 B_{\mathrm{gl}}G,$$

of $\mathrm{Rep}^{\mathrm{op}}$-sets.

Moving on to the stable case, we recall the definition of the equivariant homotopy groups of orthogonal spectra.

Definition 2.2.7. Let X be an orthogonal spectrum and G a finite group. The 0-th *equivariant homotopy group* of X is defined by

$$\pi_0^G X = \pi_0^G \Omega^\bullet X \cong \mathrm{colim}_{V \subset \mathcal{U}_G} [S^V, X(V)]_G.$$

Spelling this out, the colimit system is taken along the *stabilization maps* $\iota_{V,W}$ for $V \subset W$ sending $[S^V \xrightarrow{f} X(V)]$ to the class represented by the composite

$$S^W \cong S^V \wedge S^{W-V} \xrightarrow{f \wedge S^{W-V}} X(V) \wedge S^{W-V} \xrightarrow{\sigma_{V,W-V}} X(V \oplus (W-V)) \cong X(W),$$

where $W - V = V^\perp$ denotes the orthogonal complement of V in W. For $k \in \mathbb{Z}$ this definition is extended by looping or shifting spectra:

$$\pi_k^G X = \begin{cases} \pi_0^G \Omega^k X, & \text{if } k \geq 0 \\ \pi_0^G \operatorname{sh}^{-k} X, & \text{if } k < 0 \end{cases}$$

We will often just write π_* if we do not want to specify the degree.

Proposition 2.2.8 ([Sch17b, III.1.30, III.1.36, III.1.37])**.**

1. *Smashing representatives with S^1 defines the suspension isomorphism*

$$S^1 \wedge - : \pi_*^G X \xrightarrow{\cong} \pi_{1+*}^G (S^1 \wedge X).$$

2. *Let $f : X \to Y$ be a morphism of orthogonal spectra. Associated to the mapping cone sequence $X \xrightarrow{f} Y \xrightarrow{\iota} C(f) \to \Sigma X$ is a long exact sequence*

$$\cdots \longrightarrow \pi_*^G X \xrightarrow{f} \pi_*^G Y \xrightarrow{\iota} \pi_*^G C(f) \longrightarrow \pi_{*-1}^G X \longrightarrow \cdots$$

of homotopy groups.

3. *Let $(X_i)_{i \in I}$ be a family of orthogonal spectra. The natural map*

$$\bigoplus_{i \in I} \pi_*^G X_i \xrightarrow{\cong} \pi_*^G (\bigvee_{i \in I} X_i)$$

induced by the wedge summand inclusions is an isomorphism.

Definition 2.2.9. A morphism $f : X \to Y$ of orthogonal spectra is a (fin-)*global equivalence* if it induces isomorphisms on equivariant homotopy groups for all finite groups G.

Since the equivariant homotopy groups are defined via orthogonal spaces, they automatically come with restriction maps. Additionally, there are also *transfer homomorphisms* $\mathrm{tr}_H^G : \pi_0^H \to \pi_0^G$ associated to subgroup inclusions $H \leq G$. These are already present in classical equivariant stable homotopy theory (e.g. see [May96, IX.3]), so their existence is not a new global phenomenon. A geometric construction can be given via equivariant Pontryagin-Thom collaps maps, alternatively one may use the so called Wirthmüller isomorphism ([Wir74, Thm 2.1]) and this is explained in [Sch17b, III.2]. Since we will not use them directly and one of the points in the later use of geometric fixed points is to forget about them, we do not spell out the construction here.

Example 2.2.10. For closed orthogonal spaces X, the natural zigzags

$$X(V) \to X(V \oplus \mathcal{U}_G) \xleftarrow{\simeq} X(\mathcal{U}_G)$$

define an equivalence $(\Sigma_+^\infty X)_G \simeq \Sigma_+^\infty X(\mathcal{U}_G)$ of G-spectra. Here the right map is a homotopy equivalence since the space of linear isometries $\mathbf{L}(\mathcal{U}_G, V \oplus \mathcal{U}_G)$ is contractible. On stable homotopy groups, the induced map

$$\pi_k^G \Sigma_+^\infty X \xrightarrow{\cong} \pi_k^G \Sigma_+^\infty X(\mathcal{U}_G)$$

is described by composing a representative $S^V \xrightarrow{f} X(V)_+ \wedge S^V$ (say in degree 0) indexed by $V \subseteq \mathcal{U}_G$ with the map $X(V)_+ \wedge S^V \to X(\mathcal{U}_G)_+ \wedge S^V$, and by cofinality this is an isomorphism (cf. [Sch17b, IV.1.8] for the general case). In particular, global equivalences between closed orthogonal spaces are preserved under taking suspension spectra.

The unit $\eta : Y \to \Omega^\bullet \Sigma_+^\infty Y$ of the loop-suspension adjunction induces a natural *stabilisation map*

$$\sigma : \underline{\pi}_0 Y \longrightarrow \underline{\pi}_0 \Sigma_+^\infty Y$$

of $\mathrm{Rep}^{\mathrm{op}}$-sets. As the following fundamental computation shows, this map 'freely builds in transfers' in degree 0. It is a consequence of the tom Dieck splitting [tD75, Satz 2], but the 'π_0-statement' can also be obtained by a direct argument with the so called isotropy separation sequence.

Proposition 2.2.11 ([Sch17b, IV.1.13])**.** *There is a natural isomorphism*

$$\bigoplus_{(H \leq G)} \mathbb{Z}\{\pi_0^H X\}/W_G H \xrightarrow{\cong} \pi_0^G \Sigma_+^\infty X.$$

identifying the 0*-th equivariant stable homotopy groups of an orthogonal space* X*. On the summand indexed by the subgroup* $H \leq G$ *it is given as the composition* $\mathrm{tr}_H^G \circ \sigma$ *of the associated transfer with the stabilisation map.*

We conclude by recalling a natural pairing ([Sch17b, IV.1.20])

$$\pi_k^G X \otimes \pi_l^K Y \xrightarrow{\wedge} \pi_{k+l}^{G\times K}(X \wedge Y)$$

on equivariant homotopy groups which is unital, associative, and commutative. Explicitly, it is defined in degree 0 by smashing together representatives: Elements $[f] \in \pi_0^G X$ and $[g] \in \pi_0^K Y$ are sent to the class $[f]\wedge[g] \in \pi_0^{G\times K}(X\wedge Y)$ represented by the composite

$$S^{V\oplus W} \cong S^V \wedge S^W \xrightarrow{f\wedge g} X(V) \wedge Y(W) \xrightarrow{\iota} X(V \oplus W)$$

and the identity $[\mathrm{Id} : S^0 \to S^0] \in \pi_0\mathbb{S}$ serves as a two-sided unit.

Example 2.2.12. We define the *stable tautological class* $e_G = \sigma(u_G) \in \pi_0^G(\Sigma_+^\infty B_{\mathrm{gl}}G)$ of a global classifying space as the image of the unstable class u_G under the stabilization map. From the explicit description above one sees that under the identification

$$(\Sigma_+^\infty B_{\mathrm{gl}}G) \wedge (\Sigma_+^\infty B_{\mathrm{gl}}K) \simeq \Sigma_+^\infty B_{\mathrm{gl}}(G \times K)$$

the external product $e_G \wedge e_K$ of the stable tautological classes corresponds to the tautological class $e_{G\times K}$.

2.2.3 Global functors

The algebraic structure carried by the collection of equivariant homotopy groups is encoded in the notion of a global functor (also called inflation functor in the algebraic literature, e.g. see [Web93]) and this is discussed in detail in [Sch17b, IV.2].

Definition 2.2.13. The *Burnside category* $\mathbb{A}$ is the pre-additive category with objects the finite groups and morphisms $\mathbb{A}(G, K) = \mathrm{Nat}(\pi_0^G, \pi_0^K)$ the natural transformations. A *global functor* F is an additive functor $\mathbb{A} \to \mathrm{Ab}$ from the Burnside category to abelian groups.

As a diagram category, global functors form an abelian category $\mathcal{GF}$ with enough injectives and projectives. The definition is made so that the equivariant homotopy groups tautologically form global functors. To give it more content we review the close relation of the Burnside category with global classifying spaces. By [Sch17b, IV.2.5] evaluation at the tautological class $e_G \in \pi_0^G(\Sigma_+^\infty B_{\mathrm{gl}}G)$ yields an isomorphism

$$\mathbb{A}(G, -) \xrightarrow{\cong} (\underline{\pi}_0\Sigma_+^\infty B_{\mathrm{gl}}G).$$

We recall that we have already mentioned two examples of morphisms in the Burnside category, namely restrictions and transfers. These are actually the basic building blocks of the Burnside category:

Proposition 2.2.14. *Let G and L be finite groups. The morphisms from G to L in the Burnside category form a free abelian group with basis given by the elements $\mathrm{tr}_K^L \circ\alpha^*$, where the pair $(K \leq L,\ \alpha : K \to G)$ runs over a complete set of representatives of $G \times L$-conjugacy classes.*

Proof. Using the identification $\mathbb{A}(G,L) \cong \pi_0^L \Sigma_+^\infty B_{\mathrm{gl}} G$, the result follows from Proposition 2.2.11. $\square$

For a full 'calculation' of the Burnside category it remains to describe composition in these terms, and this amounts to the following two basic relations:

- Transfers commute with inflations: For $H \leq G$ and surjective $\alpha : L \twoheadrightarrow G$, the equality $\alpha^* \circ \mathrm{tr}_H^G = \mathrm{tr}_K^L \circ (\alpha|_K)^*$ holds, where $K = \alpha^{-1}(H)$ is the preimage of H.
- Double coset formula: The composition of a transfer with a restriction is expressed via the formula

$$\mathrm{res}_K^G \circ \mathrm{tr}_H^G = \sum_{[g] \in K\backslash G/H} \mathrm{tr}_{K \cap {}^g H}^G \circ (c_g)^* \circ \mathrm{res}_{K^g \cap H}^H .$$

Remark 2.2.15. The above definition of the Burnside category is valid in the generality of compact Lie groups, but for finite groups there is a more combinatorial description. Given G and K, the morphisms $\mathbb{A}(G,K)$ can identified with the Grothendieck group of isomorphism classes of finite (K,G)-bisets that are G-free. Composition of morphisms is then the balanced product of bisets:

$$\mathbb{A}(K,L) \times \mathbb{A}(G,K) \to \mathbb{A}(G,L), \quad (N,M) \mapsto N \times_K M$$

The restriction α^* along a group homomorphism $\alpha : K \to G$ corresponds to the (K,G)-biset $\alpha^* G$, the transfer tr_H^G along a subgroup inclusion $H \leq G$ to the (G,H)-biset G, and this fully describes how to pass between these different descriptions.

Using the pairings on homotopy groups, the cartesian product of groups can be turned into a biadditive functor

$$- \times - : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$$

giving the Burnside category a symmetric monoidal structure ([Sch17b, IV.2.15]). In the biset-description this amounts on morphisms to the product of bisets. The category of global functors thus inherits a symmetric monoidal product via Day convolution, the *box product*. Explicitly, this means the following (cf. [Sch17b, IV.2.17]): Let F,F', and F'' be global functors. Then a bimorphism $b : (F,F') \to F''$ consists of a collection of additive maps

$$b_{G,K} : F(G) \otimes F'(K) \to F''(G \times K)$$

for every pair of finite groups K and G, and these commute with transfers and restrictions in each variable. The box product $F \square F'$ is determined up to canonical isomorphism as the target of a universal bimorphism $(F,F') \to F \square F'$, that is, morphisms out of it are in one-to-one correspondence with bimorphisms from (F,F').

Definition 2.2.16. A *global Green functor* is a commutative monoid with respect to the box-product of global functors.

2.2.4 Global model structure

We briefly recall the global model structure and review arithmetic localizations. Model structures will later mainly lurk in the background since we mostly deal with homotopical functors.

A (fin-)*global family* is a collection of finite groups that is closed under isomorphism, taking subgroups, and quotients. It is called *multiplicative* if it is closed under products of groups. There is an obvious notion of $\mathcal{F}$*-global equivalence* for a family $\mathcal{F}$, one only demands isomorphisms at the groups in the family $\mathcal{F}$. A (positive) $\mathcal{F}$-Ω*-spectrum* is a spectrum X such that the adjoint structure maps $\widetilde{\sigma_{V,W}} : X(V) \to \Omega^W X(V \oplus W)$ are G-weak equivalences for G-representation V and W with V faithful (and $V \neq 0$). More generally, there is the notion of a (positive) $\mathcal{F}$-global fibration, but we will not spell this out here.

Theorem 2.2.17 ($\mathcal{F}$-global model structure, [Sch17b, IV.3.17])**.** *Let $\mathcal{F}$ be a global family. The $\mathcal{F}$-global equivalences and the $\mathcal{F}$-global fibrations form the weak equivalences and fibrations of a model structure on the category of orthogonal spectra with fibrant objects the $\mathcal{F}$-Ω-spectra. It is stable, topological, proper, and cofibrantly generated. Furthermore, it is monoidal if the family $\mathcal{F}$ is multiplicative.*

If no family is mentioned, we will implicitly use the global family of all finite groups.

Definition 2.2.18. The *global homotopy category*

$$\mathcal{GH} = \mathrm{Ho}(\mathrm{Sp}^{\mathrm{O}}_{\mathrm{gl}}) \simeq \mathrm{Sp}^{\mathrm{O}}[(\text{gl. equiv.})^{-1}]$$

is the homotopy category of the category of orthogonal spectra equipped with the global model structure.

The following representability result shows the importance of global classifying spaces.

Proposition 2.2.19 ([Sch17b, VI.4.3])**.** *Let G be a finite group. The global classifying spaces corepresent the equivariant homotopy groups*

$$\pi_k^G X \cong [\Sigma^k B_{\mathrm{gl}} G_+, X]_{\mathcal{GH}}.$$

The isomorphism corresponds to the tautological class $e_G \in \pi_0^G \Sigma^\infty_+ B_{\mathrm{gl}}$.

We will also consider the homotopy theories of global (commutative) ring spectra. The existence of the global model structure on associative ring spectra follows from [SS00] and the case of commutative ring spectra is dealt with in [Sch17b, V.4]. As usual, the later requires the positive model structures.

Theorem 2.2.20. *Let $\mathcal{F}$ be a multiplicative global family. The $\mathcal{F}$-global model structure lifts to a model structure on associative ring spectra with weak equivalences* (*resp. fibrations*) *the underlying $\mathcal{F}$-global equivalences* (*resp. fibrations*)*. The positive $\mathcal{F}$-global model structure lifts to a model structure on commutative ring spectra.*

Our main concern in this thesis is rational global homotopy theory, i.e. spectra up to rational global equivalence. More generally, we call arithmetic localisations: Let $R \subset \mathbb{Q}$ be a subring of the rationals. A morphism $f : X \to Y$ of orthogonal spectra is a *R-local global equivalence* if it induces isomorphisms on $\underline{\pi}_*(-) \otimes R$. The *$R$-local global model structure* is the Bousfield localisation of the global model structure at the R-local weak equivalences, which means that the cofibrations remain the same and the class of weak equivalence is enlarged to the R-local ones. Its existence follows for example from the general machinery of [Hir03], but this is not needed in the concrete case at hand. As in [SS02, Section 4] (also see [Pat13] for a treatment of G-orthogonal spectra), one adds certain explicit R-local equivalences to the generating acyclic cofibrations (which we have not specified here). In more detail, we denote by $M(l)$ a mod l Moore space and define the set $J_{R\text{-loc}}$ to consist of the cone inclusions

$$B_{\mathrm{gl}}G_+ \wedge F_n\Sigma^m M(l) \longrightarrow B_{\mathrm{gl}}G_+ \wedge F_n\Sigma^m C(M(l))$$

for $n, m \geq 0$, $l \geq 0$ invertible in R, and where G ranges over the set of isomorphism classes of finite groups. Now, a fibrant spectrum X has the right lifting property with respect to $J_{R\text{-loc}}$ iff all the homotopy classes $[B_{\mathrm{gl}}G_+ \wedge F_n\Sigma^m M(l), X]_{\mathcal{GH}} = 0$ vanish, and the cofiber sequence

$$S^1 \xrightarrow{l} S^1 \longrightarrow M(l)$$

implies that this is equivalent to the homotopy groups of X being R-local.

Theorem 2.2.21 (R-local $\mathcal{F}$-global model structure)**.** *Let $\mathcal{F}$ be a global family. The R-local $\mathcal{F}$-global equivalences and the cofibrations of the $\mathcal{F}$-global model structure form a model structure on the category of orthogonal spectra with fibrant objects the $\mathcal{F}$-Ω-spectra that have R-local homotopy groups. It is stable, topological, proper, and cofibrantly generated.*

Remark 2.2.22. The R-local global homotopy category can also be regarded as the (smashing) localisation (in the sense of [Bou79]) of the global homotopy category at a Moore spectrum $\mathbb{S}R$ for the ring R. The localization functor $\mathcal{GH} \to \mathcal{GH}_R$ restricts to an equivalence from the full subcategory of R-local spectra.

2.3 Geometric fixed points

In this section we introduce the specific model of the geometric fixed points functor we use. We begin with a quick review of the geometric fixed point homotopy groups (cf. [Sch17b, III.3]).

2.3.1 Geometric fixed point homotopy groups

We recall that $\rho_G = \mathbb{R}[G]$ denotes the regular representation of a finite group G. The norm element of unit length

$$N_G = \frac{1}{\sqrt{|G|}} \cdot \sum_{g \in G} g$$

linearly spans the G-fixed points of the regular representation and thus determines a preferred isometric identification $\mathbb{R} \cong (\rho_G)^G$, $t \mapsto t \cdot N_G$ which we will repeatedly use (sometimes implicit) throughout this section.

Definition 2.3.1. Let $X \in \mathrm{Sp}^{\mathrm{O}}$ be an orthogonal spectrum and G a finite group. The 0-th *geometric fixed point homotopy group* of X is defined as the directed colimit

$$\Phi_0^G X = \mathrm{colim}_{V \subset \mathcal{U}_G}[S^{V^G}, X(V)^G]$$

of based homotopy classes between fixed points. The structure maps in the colimit system are 'the fixed points' of those for the stable homotopy groups (Definition 2.2.7): Given an inclusion $V \subset W$ of indexing representations, an element $[f : S^{V^G} \to X(V)^G]$ is sent to the class represented by the composite

$$S^{W^G} \cong S^{V^G} \wedge S^{(W-V)^G} \xrightarrow{f \wedge \mathrm{Id}} X(V)^G \wedge S^{(W-V)^G} \xrightarrow{\sigma^G_{V,W-V}} X(V \oplus (W-V))^G \cong X(W)^G,$$

where we use that taking fixed points commutes with orthogonal complements. This is again extended to integer grading by applying the above construction to appropriately looped and shifted spectra:

$$\Phi_k^G X = \begin{cases} \Phi_0^G \Omega^k X & \text{if } k \geq 0 \\ \Phi_0^G \,\mathrm{sh}^{-k} X, & \text{if } k < 0 \end{cases}$$

The geometric fixed point homotopy groups come with a natural comparison map, the *geometric fixed points map*

$$\phi^G : \pi_0^G X \longrightarrow \Phi_0^G X,$$

defined on representatives by taking fixed points:

$$[f : S^V \to X(V)] \longmapsto [f^G : S^{V^G} \to X(V)^G]$$

Remark 2.3.2. As in the case of the ordinary homotopy groups, the geometric fixed point groups can be written as a sequential colimit by cofinality (and using the identification $(\rho_G)^G \cong \mathbb{R}$) :

$$\Phi_k^G X \cong \mathrm{colim}_{n \in \mathbb{N}}[S^{k+n}, X(n \cdot \rho_G)^G]$$

While there are no longer transfer maps, the geometric fixed point homotopy groups still admit restriction maps along surjective group homomorphisms, also called *inflations*. These are defined in the following way: Let $\alpha : K \twoheadrightarrow G$ be an epimorphism and choose an isometric embedding $\psi : \alpha^* \mathcal{U}_G \hookrightarrow \mathcal{U}_K$. The restriction map along α

$$\alpha^* : \Phi_0^G X \longrightarrow \Phi_0^K X$$

sends $[f : S^{V^G} \to X(V)^G]$ to the class represented by the composite

$$S^{\psi(\alpha^* V)^K} \xleftarrow{\cong} S^{(\alpha^* V)^K} \xrightarrow{f} X(\alpha^* V)^K \xrightarrow{\cong} X(\psi(\alpha^* V))^K,$$

where both homeomorphisms are induced by ψ. This is well defined and only depends on the conjugacy class of α. We summarize the functoriality:

Definition 2.3.3. Let Out denote the category of finite groups together with conjugacy classes of epimorphisms. An $\mathrm{Out}^{\mathrm{op}}$-module (set) is a contravariant functor from Out to abelian groups (sets).

Proposition 2.3.4 ([Sch17b, IV.1.23])**.** *Let X be an orthogonal spectrum. Then the inflation maps endow the collection of geometric fixed point homotopy groups $\underline{\Phi}_* X$ with the structure of an* $\mathrm{Out}^{\mathrm{op}}$*-module, natural in X and independent of all the choices.*

Example 2.3.5. We write $\mathrm{Rep}(-, G)$ for the $\mathrm{Out}^{\mathrm{op}}$-set of conjugacy classes of group homomorphism (not necessarily surjective) into G. Choosing representatives for the conjugacy classes of subgroups of G determines an identification

$$\coprod_{(H \leq G)} WH \backslash \mathrm{Out}(-, H) \xrightarrow{\cong} \mathrm{Rep}(-, G).$$

The Weyl group $WH = N_G H / H$ acts on the represented functor $\mathrm{Out}(-, H)$ by post-composition and the map on the corresponding summand is classified by the subgroup inclusion $H \leq G$.

Proposition 2.3.6. *Let X be an orthogonal space. The composition*

$$\underline{\pi}_0 X \xrightarrow{\sigma} \underline{\pi}_0 \Sigma^\infty_+ X \xrightarrow{\phi} \underline{\Phi}_0 \Sigma^\infty_+ X$$

of the stabilization map with the geometric fixed point map identifies the 0*-th geometric fixed point homotopy group as the linearization of the* 0*-th unstable homotopy group of X: It induces an isomorphism*

$$\mathbb{Z}\{\underline{\pi}_0 X\} \cong \underline{\Phi}_0 \Sigma^\infty_+ X$$

of $\mathrm{Out}^{\mathrm{op}}$*-modules.*

Proof. The composition is induced by the map between the defining colimit systems which at the G-representation V sends $[x] \in \pi_0(X(V)^G)$ to the homotopy class of

$$S^{V^G} \xrightarrow{x \wedge \mathrm{Id}} X(V)^G_+ \wedge S^{V^G}.$$

This is the unstable suspension map $\pi_0(-) \to \pi_n((-)_+ \wedge S^n)$, where $n = \dim V^G$. By the Hurewicz theorem its extension to the linearization of π_0 is an isomorphism for $\dim V^G \geq 2$. □

Corollary 2.3.7. *There is an isomorphism*

$$\mathbb{Z}\{\mathrm{Rep}(-, G)\} \xrightarrow{\cong} \underline{\Phi}_0(\Sigma^\infty_+ B_{\mathrm{gl}} G)$$

of $\mathrm{Out}^{\mathrm{op}}$*-modules which sends $(\alpha : K \to G)$ to $\phi(\alpha^* e_G)$, where $e_G \in \pi_0^G \Sigma^\infty_+ B_{\mathrm{gl}} G$ is the stable tautological class.*

We conclude with a review of the algebraic relation of global functors to $\mathrm{Out}^{\mathrm{op}}$-modules.

Definition 2.3.8. The functor

$$\tau : \mathcal{GF} \longrightarrow \mathrm{Out}^{\mathrm{op}}\text{- mod}$$

if defined at the finite group G by dividing the image of the tranfer homomorphisms from all proper subgroups:

$$(\tau F)(G) = F(G)/(\sum_{H<G} \mathrm{tr}_H^G(F(H)))$$

Example 2.3.9. There is a canonical map

$$\mathbb{Z}\{\mathrm{Rep}(-, G)\} \xrightarrow{\cong} \tau(\mathbb{A}(G, -)),\ (\alpha) \mapsto \alpha^*$$

sending the conjugacy class of a morphisms to its associated restriction and by Proposition 2.2.14 this is an isomorphism.

We then have the following result, showing that rationally, geometric fixed point and ordinary homotopy groups determine each other.

Proposition 2.3.10 ([Sch17b, IV.6])**.** (i) *The functor τ restricts to an equivalence*

$$\mathcal{GF}_{\mathbb{Q}} \simeq \mathrm{Out}^{\mathrm{op}}\text{- mod}_{\mathbb{Q}}$$

of abelian categories.

(ii) *The geometric fixed point map annihilates transfers from proper subgroups and factors over a rational isomorphism*

$$\phi : \tau(\underline{\pi}_* X)_{\mathbb{Q}} \xrightarrow{\cong} (\underline{\Phi}_* X)_{\mathbb{Q}}$$

of $\mathrm{Out}^{\mathrm{op}}$*-modules.*

In fact, part 2 follows from the stronger statement:

Proposition 2.3.11 ([Sch17b, III.4.30])**.** *Let X be a G-orthogonal spectrum. After inverting the order of the group G, the geometric fixed point map induces an isomorphism*

$$\prod(\phi^H \circ \mathrm{res}_H^G) : \pi_k^G \xrightarrow{\cong} \prod_{(H \leq G)} (\Phi_k^H X)^{WH}.$$

We remark that the right-hand side contains the formula for the inverse R (right adjoint over the integers) of the functor τ. If X is an $\mathrm{Out}^{\mathrm{op}}$-module, then the global functor RX takes the value

$$(RX)(G) = \prod_{(H \leq G)} X(H)^{WH}$$

at the finite group G.

2.3.2 Geometric fixed point spectra

We now come to the spectral refinement of the previous construction.

Definition 2.3.12. Let $X \in \mathrm{Sp}^{\mathrm{O}}$ be an orthogonal spectrum. For every finite group G the *geometric fixed point spectrum* $\Phi^G X \in \mathrm{Sp}^{\mathrm{O}}$ is defined at an inner product space V by

$$(\Phi^G X)(V) = X(\rho_G \otimes V)^G.$$

Functoriality in linear isometries is induced from X and the structure maps $\sigma^{\Phi X}_{V,W}$ are given by the composite

$$\begin{aligned} X(\rho_G \otimes V)^G \wedge S^W \cong (X(\rho_G \otimes V) \wedge S^{\rho_G \otimes W})^G \xrightarrow{\sigma^G} \; & X((\rho_G \otimes V) \oplus (\rho_G \otimes W))^G \\ \cong \; & X(\rho_G \otimes (V \oplus W))^G, \end{aligned}$$

where the first isomorphism uses the preferred identification $(\rho_G)^G = \mathbb{R}\{N_G\} \cong \mathbb{R}$ from the previous section.

Remark 2.3.13. More generally, this definition makes sense for all G-spectra (i.e. G-objects in Sp^{O}). In that case one takes fixed points with respect to the diagonal G-action on the $(G \times G)$-spaces $X(\rho_G \otimes V)$.

Remark 2.3.14. We can also express this in a more diagrammatic fashion. Tensoring with the regular representation defines a topological functor

$$\rho_G \otimes - : O \to O_G$$

from the orthogonal indexing category to the subcategory of G-representations and G-fixed morphism spaces (i.e. pairs of equivariant isometries and points lying in the fixed points of the orthogonal complement). On morphisms the functor $(\rho_G \otimes -)$ induces the map

$$O(V, W) \to O_G(\rho_G \otimes V, \rho_G \otimes W) = (O(\rho_G \otimes V, \rho_G \otimes W))^G$$

sending $[\phi, w]$ to $[\rho_G \otimes \phi, N_G \otimes w]$, where $N_G \in (\rho_G)^G$ is the norm element spanning the fixed points. The geometric fixed points of an orthogonal G-spectrum $X : O \to G\text{-}\mathrm{Top}_*$ are then given by the composite in the upper row of the diagram

$$\begin{tikzcd} O \arrow[r, "\rho_G \otimes -"] \arrow[dr, "\cong Id"'] & O_G \arrow[rr, "X^G"] \arrow[d, "(-)^G"] & & \mathrm{Top}_*, \\ & O \arrow[urr, "\Phi^G_M"'] & & \end{tikzcd}$$

where X^G is the functor obtained by restricing to O_G taking fixed points at each value. This also shows the relation to the construction discussed by Mandell-May in [MM02, V.4], which is referred to as the *monoidal geometric fixed point functor* and denoted Φ^G_M in [HHR16, B.10]. It is defined as the (topological) left Kan extension of X^G along the functor $(-)^G : O_G \to O$, $V \mapsto V^G$. The composition with $\rho_G \otimes -$ is canonically isomorphic to the identity. The left Kan extension comes with a natural transformation $X^G \Rightarrow \Phi^G_M \circ (-)^G$ and so we obtain a natural transformation (now with respect to the spectrum X)

$$\Phi^G X \to \Phi^G_M X.$$

As explained in [HHR16, B.10.5], $\Phi^G_M X$ has the correct homotopy type if X is cofibrant in the *complete model structure* used there. In that case a comparison of homotopy groups shows that the above map is a π_*-isomorphism of G-spectra.

Example 2.3.15. (i) Geometric fixed points commute with suspension spectra in the following sense: Let Y be an orthogonal space. Then there is a canonical identification

$$\Phi^G \Sigma^\infty_+ Y = (S^{\rho_G \otimes -} \wedge Y(\rho_G \otimes -))^G \cong S^{(-)} \wedge Y(\rho_G \otimes -)^G = \Sigma^\infty_+ Y(\rho_G \otimes -)^G$$

with the suspension spectrum of $\Phi^G Y = Y(\rho_G \otimes -)^G$, the *$G$-geometric fixed point* orthogonal space of Y. If Y is closed, we can further identify the homotopy groups. Namely, there is a canonical map (induced by a zigzag, compare Example 2.2.10)

$$\pi_k \Sigma^\infty_+ Y(\rho_G \otimes -)^G \xrightarrow{\cong} \pi_k \Sigma^\infty_+ Y(\mathcal{U}_G)^G$$

is an isomorphism.

(ii) We specialise to the case $Y = B_{\mathrm{gl}}G$ and use the decomposition formula 2.1.1 to identify the geometric fixed points of global classifying spaces:

$$\Phi^K B_{\mathrm{gl}}G = (\mathbf{L}(V_G, \rho_K \otimes -)/G)^K \cong \coprod_{(\alpha:K\to G)} \mathbf{L}_K(\alpha^* V_G, \rho_k \otimes -)/C(\alpha).$$

Now $\mathbf{L}_K(\alpha^* V_G, \mathcal{U}_K)$ is a contractible, free right $C(\alpha)$-space, i.e. a model for $EC(\alpha)$. Hence there is an equivalence

$$\Phi^K \Sigma^\infty_+ B_{\mathrm{gl}}G \simeq \bigvee_{(\alpha:K\to G)} \Sigma^\infty_+ BC(\alpha)$$

and mapping the spaces in each summand to the point gives a preferred map

$$\Phi^K \Sigma^\infty_+ B_{\mathrm{gl}}G \longrightarrow \mathbb{S}\{\mathrm{Rep}(K,G)\}.$$

Here $\mathbb{S}\{M\} = \bigvee_{m\in M} \mathbb{S}$ denotes the $\mathbb{S}$-linearization of a set M.

We now discuss the functoriality of the geometric fixed points as the group G varies.

Definition 2.3.16. Let Epi denote the category of finite groups and surjective group homomorphisms.

Given a surjective homomorphism $\alpha : K \twoheadrightarrow G$, we define a K-equivariant linear isometric embedding of regular representations in the other direction:

$$\alpha_! : \alpha^*\rho_G \hookrightarrow \rho_K, \quad g \mapsto \frac{1}{\sqrt{|\ker \alpha|}} \sum_{k \in \alpha^{-1}(g)} k.$$

These 'lower shriek' maps between regular representations preserve the norm elements and hence induce isomorphisms on fixed points that commute with their canonical identifications:

$$\begin{tikzcd}
(\alpha^*\rho_G)^K \arrow[rr, "(\alpha_!)^K"] & & (\rho_K)^K \\
& \mathbb{R} \arrow[ul, "\cong"] \arrow[ur, "\cong"'] &
\end{tikzcd}$$

This allows us to define natural *restriction maps* or *inflation maps*

$$\alpha^* = \mathrm{res}_\alpha : \Phi^G X \longrightarrow \Phi^K X$$

between geometric fixed points of orthogonal spectra. At an inner product space V the restriction along α is the composite

$$(\Phi^G X)(V) = X(\rho_G \otimes V)^G = X((\alpha^*\rho_G) \otimes V)^K \xrightarrow{(\alpha_! \otimes V)_*} X(\rho_K \otimes V)^K = (\Phi^K X)(V),$$

where the second equality uses the surjectivity of α. This is clearly $O(V)$-equivariant and compatibility with the structure maps follows after applying K-fixed points to the commutative square

$$\begin{tikzcd}
X(\alpha^*\rho_G \otimes V) \wedge S^{(\alpha^*\rho_G \otimes W)} \arrow[r, "\sigma"] \arrow[d, "(\alpha_! \otimes V)_* \wedge (\alpha_! \otimes W)_*"] & X(\alpha^*\rho_G \otimes (V \oplus W)) \arrow[d, "(\alpha_! \otimes (V \oplus W))_*"] \\
X(\rho_K \otimes V) \wedge S^{(\rho_K \otimes W)} \arrow[r, "\sigma"] & X(\rho_K \otimes (V \oplus W)).
\end{tikzcd}$$

Now let $\beta : L \twoheadrightarrow K$ be another surjective group homomorphism. By inspection of the formulas defining the embeddings of regular representations one sees that the diagram

$$\begin{tikzcd}
(\alpha\beta)^*\rho_G = \beta^*(\alpha^*\rho_G) \arrow[rr, "(\alpha\beta)_!"] \arrow[dr, "\beta^*\alpha_!"] & & \rho_L \\
& \beta^*\rho_K \arrow[ur, "\beta_!"] &
\end{tikzcd}$$

commutes. This implies that the restriction maps are compatible with composition: $\mathrm{res}_{\alpha\beta} = \mathrm{res}_\beta \circ \mathrm{res}_\alpha$. To summarize, we have obtained a functor

$$\Phi = \{\Phi^G\}_G : \mathrm{Sp}^{\mathrm{O}} \longrightarrow \mathrm{Epi}^{\mathrm{op}}\text{-}\mathrm{Sp}^{\mathrm{O}}.$$

Remark 2.3.17. More generally, the above construction produces restriction maps $\Phi^G X \to \Phi^K(\alpha^* X)$ for all G-orthogonal spectra X.

Proposition 2.3.18. *The geometric fixed point construction commutes with tensors and cotensors over pointed spaces, preserves all limits and those colimits that are preserved by fixed points on spaces.*

Proof. Tensors and cotensors are defined Epi-levelwise and the fact that Φ commutes with them follows at each group from the identities $(A \wedge X(\rho_G \otimes V))^G \cong A \wedge X(\rho_G \otimes V)^G$ and $\mathrm{map}(A, X(\rho_G \otimes V))^G \cong \mathrm{map}(A, X(\rho_G \otimes V)^G)$ for based spaces A and orthogonal spectra X. Since limits and colimits are computed pointwise, the statement about they're preservation is also clear. $\square$

Now we relate this to the geometric fixed point homotopy groups. To deal with negative degrees we will need to use the following natural comparison map

$$\Phi \operatorname{sh}^k X \to \operatorname{sh}^k \Phi X.$$

At the group G it is induced in level V by the isometry

$$\mathbb{R}^k \oplus (\rho_G \otimes V) \hookrightarrow (\rho_G \otimes \mathbb{R}^k) \oplus (\rho_G \otimes V) \cong \rho_G \otimes (\mathbb{R}^k \oplus V)$$

which includes the left summand into the fixed points $(\rho_G \otimes \mathbb{R}^k)^G$. As G varies, these commute with the 'lower shriek' maps between regular representations and so the map is compatible with inflations.

Lemma 2.3.19. *The comparison map*

$$\Phi \operatorname{sh}^k X \xrightarrow{\simeq} \operatorname{sh}^k \Phi X$$

defined above is a π_*-isomorphism of $\mathrm{Epi}^{\mathrm{op}}$*-spectra.*

Proof. Unraveling the definitions one sees that the square

$$\begin{tikzcd}
\operatorname{sh}(\Phi^G X) & \Phi^G(\operatorname{sh} X) \arrow[l] \\
\Sigma(\Phi^G X) \arrow[u, "\lambda_{(\Phi^G X)}"] \arrow[r, "\cong"] & \Phi^G(\Sigma X) \arrow[u, "\Phi^G(\lambda_X)"']
\end{tikzcd}$$

commutes (the comparison maps λ were introduced below Definition 2.2.1). The left vertical map is a stable equivalence ([Sch17b, III.1.25]) and it follows that the comparison map between shifted spectra is surjective on homotopy. In the following we abbreviate notation and write $k + l\rho_G$ for the representation $\mathbb{R}^k \oplus (\mathbb{R}^l \otimes \rho_G)$. To see injectivity we start with an element $[f : S^{l+n} \to X(1+n\rho_G)^G] \in \pi_l \Phi^G(\operatorname{sh} X)$ in the kernel and consider the diagram

$$\begin{array}{ccccccc}
S^{l+n+1} & \xrightarrow{f\wedge S^1} & X(1+n\rho_G)^G \wedge S^1 & \xrightarrow{\sigma} & X(1+n\rho_G+1)^G & \longrightarrow & X(1+(n+1)\rho_G)^G \\
\cong & & \cong & & \cong & & \cong \\
S^{l+n+1} & \xrightarrow{f\wedge S^1} & X(1+n\rho_G)^G \wedge S^1 & \xrightarrow{\sigma} & X(1+n\rho_G+1)^G & \longrightarrow & X((1+n)\rho_G+1)^G \\
 & & & \searrow & & \nearrow & \\
 & & & & X((1+n)\rho_G)^G \wedge S^1 & &
\end{array}$$

The two vertical isomorphisms on the right side are induced by the isometry that interchanges the outer $\mathbb{R}$ summands, the other two are given by a degree -1 map in the right $\mathbb{R}$ coordinate. This makes the outer squares commute and the middle square commute up to homotopy. The upper row is the stabilization of f and thus represents the same class in the stable homotopy group. But the lower row null-homotopic by assumption, showing that $[f]$ vanishes. □

There is a canonical map

$$\pi_0 \Phi^G X \longrightarrow \Phi_0^G X, \quad [f] \longmapsto [S^{(n\cdot\rho_G)^G} \cong S^n \xrightarrow{f} X(n\cdot\rho_G)^G]$$

and by Remark 2.3.2 this is an isomorphism. Using the identification $\Phi(\Omega X) \cong \Omega\Phi X$ (cf. Proposition 2.3.18) and the above stable equivalence $\Phi \operatorname{sh} X \simeq \operatorname{sh}(\Phi X)$ we can prolong this to an isomorphism

$$\pi_* \Phi^G X \xrightarrow{\cong} \Phi_*^G X$$

in all degrees. In particular, the geometric fixed point functor is homotopical.

Proposition 2.3.20. *The isomorphism commutes with inflations, that is, we have an identification* $\pi_*\Phi X \xrightarrow{\cong} \Phi_* X$ *of* $\mathrm{Epi}^{\mathrm{op}}$*-modules.*

Proof. It suffices to check this in degree 0. Let $\alpha : K \twoheadrightarrow G$ be a surjective homomorphism of finite groups. By taking a countable sum we can use the isometry $\alpha_!$ to define an embedding $\psi = (\alpha_!)^\infty : \alpha^*\mathcal{U}_G \hookrightarrow \mathcal{U}_K$ of universes. By spelling out the definitions it follows that the restriction along α in $\underline{\Phi}_0 X$ of an element $[f] \in \pi_0\Phi^G X$ is represented by the composite

$$S^{(n\cdot\alpha_!(\rho_G))^K} \xrightarrow{\cong} S^{(n\cdot\alpha^*\rho_G)^K} \cong S^n \xrightarrow{f} X(n\cdot\alpha^*\rho_G)^K \xrightarrow{\cong} X(n\cdot\alpha_!(\rho_G))^K.$$

Stabilizing along the inclusion $\iota : n\cdot\alpha_!(\rho_G) \subset n\cdot\rho_K$ (which induces an isometric isomorphism on fixed points) amounts to postcomposing with $X(\iota)^K$ and precomposing with the inverse of the isomorphism $(S^\iota)^K$. This is just the composite

$$S^{(\rho_K\otimes V)^K} \cong S^n \xrightarrow{f} X(n\cdot(\alpha^*\rho_G))^K \xrightarrow{X(n\cdot\alpha_!)^G} X(n\cdot\rho_K)^K,$$

and by construction it represents the restriction of $[f]$ in $\pi_0\Phi X$, considered as an element in $\Phi_0^K X$. □

Corollary 2.3.21. *The homotopy groups $\pi_*\Phi X$ of geometric fixed point spectra are* $\mathrm{Out}^{\mathrm{op}}$*-modules.*

Proposition 2.3.22. *The geometric fixed point functor preserves mapping cone sequences.*

Proof. Since mapping cone sequences are defined level-wise, this again follows from the analogous statement for the fixed points in (compactly generated) pointed spaces. □

2.3.3 Monoidal structure

In this subsection we endow the geometric fixed points with the structure of a lax symmetric monoidal functor. Let X and Y be orthogonal spectra. We recall that the smash product $X \wedge Y$ comes with a universal bimorphism $\iota : (X, Y) \to X \wedge Y$. A new natural bimorphism $(\Phi^G X, \Phi^G X) \longrightarrow \Phi^G(X \wedge Y)$ at the group G is obtained by applying fixed points:

Definition 2.3.23. The *lax monoidal structure map* is the natural transformation

$$\mu^{\phi G}_{X,Y} : (\Phi^G X) \wedge (\Phi^G X) \longrightarrow \Phi^G(X \wedge Y)$$

corresponding to the bimorphism defined at the inner product spaces V, W by

$$\begin{aligned} X(\rho_G \otimes V)^G \wedge Y(\rho_G \otimes W)^G \xrightarrow{\iota^G} &(X \wedge Y)((\rho_G \otimes V) \oplus (\rho_G \otimes W))^G \\ &\cong (X \wedge Y)(\rho_G \otimes (V \oplus W))^G. \end{aligned}$$

The inclusion of fixed points $S^V \cong S^{(\rho_G \otimes V)^G}$ defines the *unit map*

$$\eta^{\phi G} : \mathbb{S} \to \Phi^G \mathbb{S}$$

at each level V.

We remark that in the above definition one again obtains a bimorphism because the necessary compatibility diagrams arise from those for the bimorphism ι after applying fixed points. The transformation μ commutes with inflations because bimorphisms are natural in linear isometric embeddings.

Proposition 2.3.24. *The maps η^ϕ, μ^ϕ constructed above define a lax symmetric monoidal structure on the geometric fixed point functor*

$$\Phi : \mathrm{Sp}^{\mathrm{O}} \longrightarrow \mathrm{Epi}^{\mathrm{op}}\text{-}\mathrm{Sp}^{\mathrm{O}}.$$

Here the diagram category $\mathrm{Epi}^{\mathrm{op}}\text{-}\mathrm{Sp}^{\mathrm{O}}$ *is equipped with the group-wise symmetric monoidal structure, i.e. $(X \wedge Y)(G) = X(G) \wedge Y(G)$ for* $\mathrm{Epi}^{\mathrm{op}}$*-spectra X and Y.*

Proof. We check the commutativity of the unitality square

$$\begin{tikzcd}
\Phi^G X \wedge \mathbb{S} \arrow[r, "\Phi^G X \wedge \eta"] \arrow[d, "\cong"] & \Phi^G X \wedge \Phi^G \mathbb{S} \arrow[d, "\mu"] \\
\Phi^G X & \Phi^G(X \wedge \mathbb{S}). \arrow[l, "\cong"]
\end{tikzcd}$$

Rephrased in terms of bimorphisms this is equivalent to the commutativity of the outer square in the following diagram

$$\begin{tikzcd}
X(\rho_G \otimes V)^G \wedge S^W \arrow[rr] \arrow[dd, "\sigma^{\phi G}"] & & X(\rho_G \otimes V)^G \wedge S^{(\rho_G \otimes V)^G} \arrow[dl, "\sigma"] \arrow[d] \\
& X((\rho_G \otimes V) \oplus (\rho_G \otimes W))^G \arrow[dl, "\cong"] & (X \wedge \mathbb{S})((\rho_G \otimes V) \oplus (\rho_G \otimes W))^G \arrow[l] \arrow[d, "\cong"] \\
X(\rho_G \otimes (V \oplus W))^G & & (X \wedge \mathbb{S})(\rho_G \otimes (V \oplus W))^G \arrow[ll]
\end{tikzcd}$$

Since for any orthogonal spectrum Y the universal bimorphism $(Y, \mathbb{S}) \to Y \wedge \mathbb{S}$ is just the structure map of Y the triangle on the right commutes. So does the lower part by functoriality of X. Finally the upper left triangle commutes by definition of the structure maps for geometric fixed point spectra.

Expressing associativity and the compatibility with the symmetry in terms of bimorphisms leads to slightly larger diagrams and one checks that their commutativity in the end amounts to the fact that the identification $(\rho_G \otimes V) \oplus (\rho_G \otimes W) \cong \rho_G \otimes (V \oplus W)$ is associative and symmetric in V and W. We omit the details.

□

Corollary 2.3.25. *Let R be a (commutative) orthogonal ring spectrum. Then the geometric fixed points ΦR of R form an* $\mathrm{Epi}^{\mathrm{op}}$*-diagram of (commutative) orthogonal ring spectra. The unit and multiplication map obtained from those of R at each group G and in level V by passage to fixed points: Explicitly, they are given by*

$$\iota^{\phi G} : S^V \cong (S^{\rho_G \otimes V})^G \xrightarrow{\iota^G} R(\rho_G \otimes V)^G$$

and

$$\begin{aligned}
\mu^{\phi G}_{V,W} : R(\rho_G \otimes V)^G \wedge R(\rho_G \otimes W)^G &\xrightarrow{\mu^G} R((\rho_G \otimes V) \oplus (\rho_G \otimes W))^G \\
&\cong R(\rho_G \otimes (V \oplus W))^G.
\end{aligned}$$

We will later also need the following

Proposition 2.3.26. *The monoidal structure on geometric fixed points is compatible with the suspension isomorphism in the sense that the square*

$$\begin{tikzcd} \Phi(\Sigma X) \wedge \Phi Y \arrow[r, "\mu"] \arrow[d, "\cong"] & \Phi(\Sigma X \wedge Y) \arrow[d, "\cong"] \\ \Sigma(\Phi X \wedge \Phi Y) \arrow[r, "\Sigma\mu"] & \Sigma\Phi(X \wedge Y) \end{tikzcd}$$

commutes.

The lax monoidal structure on geometric fixed points induces a natural pairing

$$\begin{tikzcd} \Phi_k^G X \otimes \Phi_l^G Y \arrow[rr] \arrow[d, phantom, "\cong"] & & \Phi_{k+l}^G(X \wedge Y) \arrow[d, phantom, "\cong"] \\ \pi_k \Phi^G X \otimes \pi_l \Phi^G Y \arrow[r] & \pi_{k+l}(\Phi^G X \wedge \Phi^G Y) \arrow[r] & \pi_{k+l}\Phi^G(X \wedge Y) \end{tikzcd}$$

Spelling out the definitions one sees that this has the following explicit description in degree 0: Let $[f : S^{V^G} \to X(V)^G] \in \Phi_0^G X$ and $[g : S^{V^G} \to Y(V)^G] \in \Phi_0^G Y$. Then $[f] \otimes [g]$ is mapped to the class represented by the composite

$$S^{(V \oplus W)^G} \xrightarrow{f \wedge g} X(V)^G \wedge Y(W)^G \cong (X(V) \wedge Y(W))^G \xrightarrow{\iota} ((X \wedge Y)(V \oplus W))^G$$

and this agrees with the product described in [Sch17b, III.5]. From this we can also see how the geometric fixed point map interacts with the pairings:

Proposition 2.3.27. *The geometric fixed points map commutes with the pairings on equivariant homotopy groups and geometric fixed point homotopy groups:*

$$\begin{tikzcd} \pi_k^G X \otimes \pi_l^G Y \arrow[r] \arrow[d, "\phi \otimes \phi"] & \pi_{k+l}^{G \times G}(X \wedge Y) \arrow[r, "\Delta^*"] & \pi_{k+l}^G(X \wedge Y) \arrow[d, "\phi"] \\ \Phi_k^G X \otimes \Phi_l^G Y \arrow[rr] & & \Phi_{k+l}^G(X \wedge Y) \end{tikzcd}$$

Corollary 2.3.28. *Under the equivalence*

$$(\Sigma_+^\infty B_{\mathrm{gl}} G) \wedge (\Sigma_+^\infty B_{\mathrm{gl}} K) \simeq \Sigma_+^\infty B_{\mathrm{gl}}(G \times K)$$

the pairing on $\underline{\Phi}_0$ *corresponds to the isomorphism*

$$\mathbb{Z}\{\mathrm{Rep}(-, G)\} \otimes \mathbb{Z}\{\mathrm{Rep}(-, K)\} \xrightarrow{\times} \mathbb{Z}\{\mathrm{Rep}(-, G \times K)\}$$

of $\mathrm{Out}^{\mathrm{op}}$*-modules.*

Proof. This now follows from the identification of $\underline{\Phi}_0$ in Corollary 2.3.7 and the behaviour of tautological classes (Example 2.2.12). $\square$

We also discuss the multiplicative properties of the functor $\tau : \mathcal{GF} \to \mathrm{Out}^{\mathrm{op}}\text{-}\mathrm{mod}_{\mathbb{Q}}$. Let F and F' be global functors. For every finite group G, the composite

$$F(G) \otimes F'(G) \to (F \Box F')(G \times G) \xrightarrow{\Delta^*} (F \Box F')(G)$$

descends to the quotients by proper transfers (cf. [Sch17b, IV.2.23]) and this defines a lax symmetric monoidal transformation

$$\tau(F) \otimes \tau(F') \to \tau(F \Box F')$$

with unit (isomorphism) $\mathbb{Z} \cong \mathbb{Z}\{\mathrm{Rep}(-, e)\} \cong \tau(\mathbb{A}(e, -))$.

Proposition 2.3.29. *The functor $\tau : \mathcal{GF} \to \mathrm{Out}^{\mathrm{op}}\text{-}\mathrm{mod}$ is strong monoidal with respect to the box product of global functors and the group-wise tensor product.*

Proof. Both sides of the lax monoidal transformation preserve colimits in each variable and so it suffices to consider corepresented global functors of the form $\mathbb{A}(G, -)$ for G a finite group since every global functor is a colimit of these. As explained in [Sch17b, IV.2.17], there is an isomorphism $\mathbb{A}(G, -) \Box \mathbb{A}(K, -) \cong \mathbb{A}(G \times K, -)$ and one checks that under the identification $\tau(\mathbb{A}(L, -)) \cong \mathbb{Z}\{\mathrm{Rep}(-, L)\}$ of Example 2.3.9 the lax monoidal transformation turns into the isomorphism

$$\mathbb{Z}\{\mathrm{Rep}(-, G)\} \otimes \mathbb{Z}\{\mathrm{Rep}(-, K)\} \xrightarrow{\times} \mathbb{Z}\{\mathrm{Rep}(-, G \times K)\}.$$

□

2.3.4 Norm maps and power operations

We have seen that geometric fixed points are lax symmetric monoidal and hence send commutativ ring spectra to $\mathrm{Epi}^{\mathrm{op}}$-diagrams of commutative ring spectra. But there is more structure available on the resulting $\mathrm{Epi}^{\mathrm{op}}$-diagram. To put this into context, we recall the existence of the Hill-Hopkins-Ravenel norms, a 'multiplicative form of induction' introduced by these authors in [HHR16]. Given a subgroup inclusion $H \leq G$ and a H-spectrum X, the norm $N_H^G X$ is a G-spectrum and there is an equivalence $\Phi^G N_H^G X \simeq \Phi^H X$ (suitably derived, see Remark 2.3.33). Furthermore, the norm construction restricts to a functor between commutative ring spectra that is left adjoint to the restriction functor from G-rings to H-rings. Combining the equivalence above with the counit $N_H^G R_{|H} \to R$ yields a multiplicative *norm map*

$$N_H^G R : \Phi^H R \longrightarrow \Phi^G R.$$

We will now construct this map directly in the model of geometric fixed point spectra that is used here. We begin by recalling the n-fold *wreath product* $\Sigma_n \wr G$ of a finite group G for $n \in \mathbb{N}$. It is the semi-direct product of the n-fold product $G^{\times n}$ and the symmetric group Σ_n with respect to the right action of Σ_n by permuting the factors. Concretely, elements are given by tuples $(\sigma; g_1, \ldots, g_n) \in \Sigma_n \times G^{\times n}$ with multiplication

$$(\sigma; g_1, \ldots, g_n) \cdot (\tau; k_1, \ldots, k_n) = (\sigma \circ \tau; g_{\tau(1)} \cdot k_1, \ldots, g_{\tau(n)} \cdot k_n).$$

Wreath products naturally act from the left on coproducts of G-objects in a category and in the case of G-sets this gives an identification

$$\Sigma_n \wr G \xrightarrow{\cong} \mathrm{Aut}_G(G \sqcup \ldots \sqcup G)$$

as the group of right G-equivariant automorphisms.

Definition 2.3.30. Let R be a commutative ring spectrum and $H \leq G$ a subgroup inclusion of index $m = (G : H)$. The *norm map* $N_H^G R : \Phi^H R \to \Phi^G R$ is defined in level V as the following composite

$$R(\rho_H \otimes V)^H \longrightarrow (R(\rho_H \otimes V)^{\wedge m})^{\Sigma_m \wr H} \longrightarrow R((m \cdot \rho_H) \otimes V)^{\Sigma_m \wr H} \longrightarrow R(\rho_G \otimes V)^G$$

which we explain below.

The first map is simply the diagonal inclusion into the m-fold smash product. The iterated multiplication map

$$R(\rho_H \otimes V) \wedge \ldots \wedge R(\rho_H \otimes V) \xrightarrow{mult} R((\rho_H \otimes V) \oplus \cdots \oplus (\rho_H \otimes V)).$$

is $\Sigma_m \wr H$-equivariant by commutativity and restricts on fixed points to the second map. To define the last map we choose an (ordered) H-basis $b = (g_1, \ldots, g_m)$ of G, that is, the g_i form a complete set of representatives for the right H-cosets in G. This is the same as the choice of a right H-equivariant isomorphism $\coprod_{i=1}^m H \xrightarrow{\cong} G$. As the H-automorphism group of $\sqcup H$ the wreath product $\Sigma_m \wr H$ acts freely and transitively from the right on the set of H-bases and this commutes with the G-action by left translation. So the above choice of an H-basis determines a group homomorphism $\Psi_b : G \to \Sigma_m \wr H$ such that the square

$$\begin{tikzcd}
\coprod_{i=1}^m H \arrow[rr, "{(g_1,\ldots,g_n)}"] \arrow[d, "\Psi_b(g)"] & & G \arrow[d, "g\cdot"] \\
\coprod_{i=1}^m H \arrow[rr, "{(g_1,\ldots,g_n)}"] & & G
\end{tikzcd}$$

commutes. In particular the linear isometric isomorphism $m \cdot \rho_H \xrightarrow{\cong} \rho_G$ obtained by linearizing induces a map which restricts to fixed points

$$R((m \cdot \rho_H) \otimes V)^{\Sigma_m \wr H} \longrightarrow R(\rho_G \otimes V)^G$$

and is independent of the choice of H-basis.

Example 2.3.31. For the sphere spectrum $\mathbb{S}$ all involved maps are isomorphisms commuting with the inclusion of fixed points. So under the unit isomorphism $\mathbb{S} \cong \Phi\mathbb{S}$ the norms $N_H^G\mathbb{S}$ just correspond to the identity.

Proposition 2.3.32. *The level-wise maps defined above assemble to a natural morphism*

$$N_H^G R : \Phi^H R \longrightarrow \Phi^G R$$

of commutative ring spectra.

Proof. As a composition of natural maps the norms are level-wise natural in the ring spectrum R as well as in linear isometric isomorphisms. We check compatibility of N_H^G with the unit maps: The unit map of ΦR factors as the composition

$$\mathbb{S} \to \Phi\mathbb{S} \xrightarrow{\Phi\iota_R} \Phi R$$

of the unit map of $\Phi\mathbb{S}$ with the map induced by the unit of R on geometric fixed points. By naturality it thus suffices to check for the sphere spectrum where it true by inspection as remarked in the example above. It remains to show that the norm maps commute with multiplication in the sense that the equality $\mu^{\phi G} \circ (N_H^G(V) \wedge N_H^G(W)) = N_H^G(V \oplus W) \circ \mu_{V,W}^{\phi H}$ holds for inner product spaces V and W. This will follow from the commutative diagrams (using abbreviated notation $mhV = m \cdot \rho_H \otimes V$)

$$\begin{tikzcd}
R(hV) \wedge R(hW) \arrow[r] \arrow[dr] \arrow[dd, "\mu"] & R(hV)^{\wedge m} \wedge R(hW)^{\wedge m} \arrow[r, "\mu\wedge\mu"] \arrow[d, "\cong"] & R(mhV) \wedge R(mhW) \arrow[d, "\mu"] \\
& (R(hV) \wedge R(hW))^{\wedge m} \arrow[d, "(\mu)^{\wedge m}"] & R(mhV + mhW) \arrow[d, "\cong"] \\
R(h(V \oplus W)) \arrow[r] & R(h(V+W))^{\wedge m} \arrow[r] & R(mh(V+W))
\end{tikzcd}$$

and

$$\begin{tikzcd}
R(mhV) \wedge R(mhW) \arrow[r, "H\text{-basis}"] \arrow[d, "\mu"] & R(gV) \wedge R(gW) \arrow[dd, "\mu"] \\
R(mhV + mhW) \arrow[d, "\cong"] & \\
R(mh(V+W)) \arrow[r, "H\text{-basis}"] & R(g(V+W))
\end{tikzcd}$$

The desired equality is then expressed by the commutative square obtained after passing to fixed points in the outer square formed by the attaching the lower square to right of the first diagram. ☐

Remark 2.3.33. For later use we clarify the relation to the norm construction, that was alluded to earlier: The norm of an H-spectrum X (i.e. an H-object in Sp^{O}) is the 'indexed smash product' $\bigwedge_{G/H} X$ with induced G-action (cf. [HHR16, 2.2.3]). More concretely, after a choice of H-basis b this is modelled by the m-fold smash power of X

$$N_H^G X = \Psi_b^*(X^{\wedge m})$$

with G-action obtained by restriction along the associated homomorphism $\Psi_b : G \to \Sigma_m \wr H$. For an orthogonal spectrum X the composite defining the norm map still makes sense if we replace the iterated multiplication by the iterated bimorphism (or multimorphism) of the m-fold smash power. This yields a more general norm map

$$N_H^G : \Phi^H X \to \Phi^G N_H^G X,$$

and for a commutative ring spectrum R the composition with the multiplication map $N_H^G R \to R$ is exactly the previously defined map. By [HHR16, B.11] and Remark 2.3.14, this norm map is a weak equivalence if X is cofibrant in the complete model structure on G-spectra.

Proposition 2.3.34. (i) *Norm maps are transitive in subgroup inclusions: The equality*

$$N_K^G \circ N_H^K = N_H^G$$

holds for all nested subgroup inclusions $H \leq K \leq G$.

(ii) *Norm maps commute with inflations: Let $\alpha : G \twoheadrightarrow K$ be a surjective group homomorphism and $H \leq K$ a subgroup. We set $L = \alpha^{-1}(H)$, so that there is the commutative square:*

$$\begin{tikzcd} G \arrow[r, two heads, "\alpha"] & K \\ L \arrow[u, phantom, "\vee\!|"] \arrow[r, two heads, "\alpha_{|L}"] & H \arrow[u, phantom, "\vee\!|"] \end{tikzcd}$$

Then the equality

$$\alpha^* \circ N_H^K = N_L^G \circ (\alpha_{|L})^*$$

holds.

Proof. We set $m = (K : H)$, $n = (G : K)$ and choose coset representatives $\underline{k} = (k_1, \ldots, k_m)$, $\underline{g} = (g_1, \ldots, g_n)$. Spelling out the definitions, one sees that after taking fixed points $N_K^G \circ N_H^K$ agrees with the composition in the upper row of the diagram

$$\begin{tikzcd} R(\rho_H \otimes V) \arrow[r] \arrow[dr] & (R(\rho_H \otimes V)^{\wedge m})^{\wedge n} \arrow[r] \arrow[d, phantom, "\cong"] & R(m(n\rho_H) \otimes V) \arrow[r] \arrow[d, phantom, "\cong"] & R(\rho_G \otimes V) \\ & R(\rho_H \otimes V)^{\wedge nm} \arrow[r, "\mu"] & R((mn)\rho_H \otimes V) \arrow[ur] & \end{tikzcd}$$

Here the last map is induced by the isometry $n(m\rho_H) \xrightarrow{n\underline{k}} n\rho_K \overset{\underline{g}}{\hookrightarrow} \rho_G$ and the vertical maps are induced by a choice of enumeration of the set $\{1, \ldots, n\} \times \{1, \ldots, m\}$. The products $g_i \cdot k_j$ form a complete set of representatives for the cosets of H in G and thus determine an H-basis of G under the enumeration. The lower right map is induced by the isometric isomorphism corresponding to that basis and hence the lower composite of the diagram restricts to N_H^G on fixed points.

The second part reduces to the commutativity of the square

$$\begin{tikzcd} m \cdot \rho_H \arrow[r, hook, "(k_1,\ldots,k_m)"] \arrow[d, hook, "(\alpha|_L)_!"] & \rho_K \arrow[d, hook, "\alpha_!"] \\ m \cdot \rho_L \arrow[r, hook, "(g_1,\ldots,g_m)"] & \rho_G \end{tikzcd}$$

of isometric embeddings. Here (k_i) is a H-basis of K and the g_i are choosen preimages under α of the k_i, forming an L-basis of G. This allows one to commute $(\alpha_!)_*$ with the last map in the composite defining the norm. By functoriality in linear isometries it can be further commuted with other maps to arrive at the composition defining $\alpha^* \circ N_H^K$. $\square$

Biset description

Once we pass to homotopy groups the extra algebraic structure of norm maps can be encoded via bisets as before in the case of global functors, see Remark 2.2.15. Thinking of norms as multiplicative transfers, the situation is analogous. There are just fewer restriction maps, namely those along surjections and this explains the transitivity condition in the following

Definition 2.3.35. We denote by $\mathbb{A}_{tr}^+$ the category with objects all finite groups and morphism sets $\mathbb{A}_{tr}^+(G,K)$ the isomorphism classes of finite (K,G)-bisets that are G-free and K-transitive. Composition is the balanced product of bisets.

As in the case of the Burnside category, the (K,G)-biset $\alpha^*G \in \mathbb{A}_{tr}^+(G,K)$ corresponds to restriction along $\alpha : G \twoheadrightarrow K$ and the (G,H)-biset G to the norm N_H^G. There is a canonical functor $\mathrm{Out}^{\mathrm{op}} \to \mathbb{A}_{\mathrm{tr}}^+$, which is the identity on objects and sends α to α^*K as above.

Proposition 2.3.36. *Extending an* $\mathrm{Out}^{\mathrm{op}}$*-diagram* X *with values in an arbitrary category to an* $\mathbb{A}_{tr}^+$*-diagram is equivalent to equipping* X *with the extra structure of norm maps* $N_H^G : X(H) \to X(G)$ *that are transitive in subgroup inclusions and commute with inflations as in Proposition 2.3.34.*

Proof. There is a canonical identification

$$\left(\coprod_{K \leq L} \mathrm{Out}(K,-)\right)/L \xrightarrow{\cong} \mathbb{A}_{\mathrm{tr}}^+(-,L), \quad \alpha \longmapsto L \times_{K,\alpha} H = N_K^L \circ \alpha^*,$$

where L acts by precomposing with conjugation homomorphisms and permuting the summands. After choosing representatives for conjugacy classes and rewriting the left side as $\coprod_{(K \leq L)} \mathrm{Out}(K,-)/WK$, this can be seen by inspecting the combinatorics of bisets. From this it follows that the category $\mathbb{A}_{\mathrm{tr}}^+$ is generated by inflations and norm maps subject to desired relations. $\square$

In light of this we will also denote this Burnside category $\mathbb{A}_{tr}^{+}$ by $\mathrm{Out}_{\mathrm{norm}}^{\mathrm{op}}$.

Corollary 2.3.37. *The geometric fixed point homotopy groups of a commutative ring spectrum naturally take value* $\mathrm{Out}_{\mathrm{norm}}^{\mathrm{op}}$*-graded commutative rings.*

Power operations

We also wish to relate this to power operations, as defined in [Sch17b, V.1]. The 0-th homotopy group global functor $\underline{\pi}_0 R$ of a commutative orthogonal ring spectrum R is not only a global Green functor, but also comes with the extra structure of *power operations*

$$P^m : \pi_0^G R \longrightarrow \pi_0^{\Sigma_m \wr G} R$$

for every finite group G and integer $m \geq 1$. They are defined by sending a G-equivariant map $f : S^V \to R(V)$ to the class represented by the $(\Sigma_m \wr G)$-equivariant composition

$$S^{mV} = (S^V)^{\wedge m} \xrightarrow{f^{\wedge m}} R(V)^{\wedge m} \longrightarrow R(mV)$$

of the m-fold smash power of f with the commutative multiplication of R. We remark that these are not additive maps.

Definition 2.3.38. A *global power functor* is a global Green functor F together with power operations $P^m : F(G) \to F(\Sigma_m \wr G)$ for every integer $m \geq 0$ and finite group G subject to the relations as stated in [Sch17b, V.1.6]. A morphism of global power functors is a morphism of underlying Green functors that commutes with the power operations.

We will only spell out the relations as needed in the proof of the proposition below. The structure of power operations on a global Green functor can be recast in the form of norm maps which is better suited for our purposes. Given a subgroup inclusion $H \leq G$ of index m, we choose a morphism $\alpha : G \to \Sigma_m \wr H$ corresponding to a decomposition of G into right H-orbits. Then the norm map $N_H^G : F(H) \to F(G)$ is the composition

$$F(H) \xrightarrow{P^m} F(\Sigma_m \wr H) \xrightarrow{\alpha^*} F(G)$$

of the m-th power operation with the restriction along α. The functor $\tau : \mathcal{GF} \to \mathrm{Out}^{\mathrm{op}}\text{-}\,\mathrm{mod}$ preserves this structure.

Proposition 2.3.39. *Let F be a global power functor. The norms of F descend to additive and multiplicative norm maps on the commutative* $\mathrm{Out}^{\mathrm{op}}$*-ring τF, giving it the canonical structure of a commutative* $\mathrm{Out}_{\mathrm{norm}}^{\mathrm{op}}$*-ring.*

Proof. Power operations are multiplicative, and additive up to a sum of proper transfers ([Sch17b, V.1.6.(iv), V.1.6.(vii)]). This together with the commutation formula $P^m \circ \mathrm{tr}_H^G = \mathrm{tr}_{\Sigma_m \wr H}^{\Sigma_m \wr G} \circ P^m$ shows that for a proper subgroup inclusion $H \lneq G$ the evaluation $P^m(x + \mathrm{tr}_H^G y)$ differs from $P^m(x)$ by proper transfers. By the double coset formula the same is true for the norm maps since they are obtained by composing power operations with a restriction. Hence they pass to well-defined ring maps. As explained in [Sch17b, V.1.7], they are transitive in subgroup inclusions and commute with inflations. □

Finally, we record the expected

Proposition 2.3.40. *The geometric fixed point map commutes with norms. In other words, for every commutative ring spectrum R the induced map*

$$\tau(\underline{\pi}_0 R) \xrightarrow{\phi} \underline{\Phi}_0 R$$

is a morphisms of commutative $\mathrm{Out}^{\mathrm{op}}_{\mathrm{norm}}$*-rings.*

Proof. Suppose that $H \leq G$ is a subgroup inclusion of index m and let $f : S^{\rho_H \otimes V} \to R(\rho_H \otimes V)$ (without loss of generality) represent an element in $\pi_0^H R$. After the choice of an H-basis of G the norm is obtained from the composition

$$S^{m\rho_H \otimes V} \xrightarrow{f^m} R(\rho_H \otimes V)^{\wedge m} \longrightarrow R(m\rho_H \otimes V)$$

by conjugating with the associated isometric isomorphism $m\rho_H \cong \rho_G$. Passage to G-fixed points yields the composition of $\phi(f) \in \Phi_0^H R$ with the norm map $N_H^G(V)$ of ΦR in level V. $\square$

3 Rational global homotopy theory

With most of the foundational work behind us we come to the main results of this thesis in this chapter.

3.1 Preliminiaries

As a final piece of preparation we recall some generalities on equivalences of triangulated categories and review the rational chain functor we will use to move from topology to algebra.

3.1.1 Equivalences of triangulated categories

Let $\mathcal{T}$ be a triangulated category admitting arbitrary sums. We write $[X, Y]_{\mathcal{T}}$ for the abelian group of morphisms between objects X and Y in $\mathcal{T}$. An object $A \in \mathcal{T}$ is called *compact* if mapping out of it preserves sums, meaning that the natural comparison map

$$\bigoplus_{i \in I} [A, X_i]_{\mathcal{T}} \xrightarrow{\cong} [A, \bigoplus_{i \in I} X_i]_{\mathcal{T}}$$

is an isomorphism for all small collections of objects $X_i \in \mathcal{T}$. A set $\mathcal{C}$ of (compact) objects in $\mathcal{T}$ is a *set of (compact) generators* if it detects isomorphisms or equivalently 0-objects: For every $X \in \mathcal{T}$ we have

$$X \cong 0 \text{ if and only if } [\Sigma^k A, X]_{\mathcal{T}} = 0$$

for all $k \in \mathbb{Z}$ and $A \in \mathcal{C}$. The following shows the usefulness of this notion.

Proposition 3.1.1 (e.g. see [SS03, 2.1.1])**.** *Let $\mathcal{T}$ be a triangulated category admitting arbitrary sums and $\mathcal{C}$ a set of compact generators. Let $\mathcal{X}$ be a* localizing *subcategory, i.e. it is non-empty, closed under sums, and if in two objects of a distinguished triangle are contained in $\mathcal{X}$, then so is the third. Then if $\mathcal{X}$ contains the set $\mathcal{C}$, it must contain all objects: $\mathcal{X} = \mathcal{T}$.*

An *exact* functor $F : \mathcal{T} \to \mathcal{T}'$ between triangulated categories is a functor that preserves exact triangles. More precisely, it is equipped with the extra structure of a natural isomorphism F $\circ \Sigma \cong \Sigma \circ F$ such that the triangle

$$FX \to FY \to FZ \to F\Sigma X \cong \Sigma FX$$

is a distinguished triangle for every distinguished triangle $X \to Y \to Z \to \Sigma X$.

The following well-known statement will serve as our fundamental tool to recognise equivalences. Because of its importance and for convenience we sketch the argument.

Proposition 3.1.2. *Let $F : \mathcal{T} \to \mathcal{T}'$ be an exact functor between triangulated categories that preserves sums and such that $\mathcal{T}$ is compactly generated. If F restricts to a fully faithful functor on the suspensions of a set of generators and preserves their compactness, then F is fully faithful.*

Additionally, if the essential image of F contains a set of compact generators, then F is essentially surjective.

Proof. Let $\mathcal{C}$ be set of compact generators of $\mathcal{T}$ as above. We fix an element $c \in \mathcal{C}$ and consider the full subcategory of $\mathcal{T}$ such that for all $i \in \mathbb{Z}$ the map $[\Sigma^i c, -] \xrightarrow{F} [F\Sigma^i c, F-]$ is an isomorphism. This is a localizing subcategory of $\mathcal{T}$: It is closed under sums since F commutes with these and preserves compactness. It is also closed under extensions by the 5-Lemma and because F is exact. But it contains a set of generators by assumption and hence must be all of $\mathcal{T}$. Similarly it now follows that for a fixed object $X \in \mathcal{T}$ the full subcategory on which $[\Sigma^i -, X] \xrightarrow{F} [F\Sigma^i -, FX]$ restricts to an isomorphism for all $i \in \mathbb{Z}$ is localizing and contains a set of generators. Hence F is fully faithful.

The essential image of F is clearly closed under sums by assumption. It is also closed under extensions: We verify one case. In the following diagram let the upper row be a given distinguished triangle with X and Y in the essential image of F:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\cong"] & Y \arrow[r, "g"] \arrow[d, "\cong"] & Z \arrow[r, "h"] \arrow[d, dashed, "\cong"] & \Sigma X \arrow[d, "\cong"] \\ FX' \arrow[r, "Ff'"] & FY' \arrow[r] & FZ' \arrow[r] & \Sigma FX' \end{tikzcd}$$

Since F is full there exists an arrow f' such that the first square commutes. Choosing a cone Z' for f' we obtain a distinguished triangle in the lower row and the dashed arrow making the rest of the diagram commute. By the 2 out of 3 property for maps between distinguished triangles this must be an isomorphism. The other cases are similar and it follows that the essential image is again a localizing subcategory, containing a set of generators by assumption. This shows that F is essentially surjective. □

3.1.2 A rational chain functor

In order to move from the topological to the algebraic world we will need a *rational chain functor*. By this we mean a functor from spectra to chain complexes such that its homology computes rationalized homotopy groups and which descends to an exact functor on homotopy categories. In unpublished work Schwede and Strickland constructed such a functor for symmetric spectra (of simplicial sets) that is also lax monoidal and we review it here. We will define everything directly for orthogonal spectra since the exposition simplifies significantly in that case. But it will be quite apparent that the constructions only use the underlying symmetric spectrum (of simplicial sets).

Definition 3.1.3. Let I be the small skeleton of the category finite sets and injection with objects the standard sets $\mathbf{n} = \{1, \dots, n\}$, $n \geq 0$.

The chain functor will be obtained by first constructing an I-diagram of chain complexes and then passing to the colimit. We denote by NA the *reduced normalized* singular chain complex of a based topological space A. The *shuffle map* (introduced in [EML53, (5.3)], also see [Dol80, VI.12])

$$\nabla : NA \otimes NB \longrightarrow N(A \wedge B)$$

is a lax symmetric monoidal transformation that induces the reduced cross product on homology. We fix a homeomorphism $\tau : |\Delta^1/\partial\Delta^1| \cong S^1$ from the topological 1-simplex with collapsed boundary to the sphere. By precomposing with the projection from the 1-simplex this defines a *fundamental cycle* $e_1 \in NS^1$ representing a generator of $\tilde{H}_1 S^1$. Applying the iterated shuffle map to the m-fold product $e_1 \otimes \cdots \otimes e_1$ gives fundamental cycles $e_m \in NS^m$ in all dimensions and such that $\nabla(e_n \otimes e_m) = e_{n+m}$ holds.

Now let X be an orthogonal spectrum. The associated I-diagram of rational chain complexes

$$\tilde{C}X : I \longrightarrow \mathrm{Ch}_{\mathbb{Q}}$$

is defined on objects by $(\tilde{C}X)(\mathbf{n}) = NX_n[n]$, the n-fold *negative* shift ($C[n]_k = C_{k+n}$) of the (rational) singular chains on the n-th level of X. The symmetric group Σ_n acts by functoriality with an additional sign: $\gamma.x = \mathrm{sgn}(\gamma)\cdot(N\gamma)(x)$ for $\gamma \in \Sigma_n$ and $x \in (NX_n)_*$. The standard inclusion $\iota : \mathbf{n} \hookrightarrow \mathbf{n+1}$ induces the map

$$(\tilde{C}X)(\iota) : (\tilde{C}X)(\mathbf{n}) \longrightarrow (\tilde{C}X)(\mathbf{n+1})$$

defined as the composite

$$NX_n[n] \xrightarrow{-\otimes e_1} (NX_n \otimes NS^1)[n+1] \xrightarrow{\nabla} N(X_n \wedge S^1)[n+1] \xrightarrow{\sigma_n} NX_{n+1}[n+1].$$

This is Σ_n-equivariant with respect to the action on the right side obtained by restriction along $\Sigma_n \times 1 \leq \Sigma_{n+1}$. Furthermore, the image of the m-fold iterate $\tilde{C}(\iota)^m : \tilde{C}(\mathbf{n}) \to \tilde{C}(\mathbf{n+m})$ is invariant under the action of all permutations in Σ_{n+m} that fix the first n elements $\{1, \ldots, n\}$. It is then a combinatorial exercise to check that this uniquely specifies a functor on the index category I.

Definition 3.1.4. The *chain functor* C is defined as the colimit over the I-diagram $\tilde{C}$. Equivalently, we set

$$CX = \mathrm{colim}_I\, \tilde{C}X$$

for every orthogonal spectrum X.

Remark 3.1.5. One could also define the I-diagram $\tilde{C}$ using integral chains and this would still yield rational complex C. For our purposes however this is more of a curious fact and will not be needed.

We now compute the homology of C. The choice of fundamental cycles e_k pins down the Hurewicz homomorphisms $h_k : \pi_k A \to \tilde{H}_k A$, $[f] \mapsto f_*[e_k]$, $k \geq 1$. For every orthogonal spectrum X the diagrams

$$\begin{array}{ccc} \pi_{k+n}X_n & \xrightarrow{\quad h \quad} & \tilde{H}_{k+n}X_n = H_k(\tilde{C}X)(\mathbf{n}) \\ \Big\downarrow{\scriptstyle \sigma_n\circ(-\wedge S^1)} & & \Big\downarrow{\scriptstyle (\iota)_*} \\ \pi_{k+n+1}X_{n+1} & \xrightarrow{h} & \tilde{H}_{k+n+1}X_{n+1} = H_k(\tilde{C}X)(\mathbf{n}+1) \end{array}$$

commute, where the left vertical map is the stabilization map used to compute stable homotopy groups. This follows from the identity

$$\begin{aligned}(\sigma_n \circ (f \wedge S^1))_*[e_{k+n+1}] = (\sigma_n \circ (f \wedge S^1))_*([e_{k+n}] \times [e_1]) &= (\sigma_n)_*(f_*[e_{k+n}] \times [e_1]) \\ &= H_k((\tilde{C}X)(\iota_n))(f_*[e_{k+n}]).\end{aligned}$$

Passing to colimits we obtain a natural transformation

$$\pi_k X \longrightarrow \operatorname{colim}_{n\in\mathbb{N}} H_k((\tilde{C}X)(\mathbf{n})) \longrightarrow \operatorname{colim}_I H_k\tilde{C}X \longrightarrow H_k CX$$

which we will also call Hurewicz map.

Lemma 3.1.6. *Let $F : I \to \mathcal{C}$ be a functor to a cocomplete category.*

(i) *The colimit of F over I can be computed by first dividing out the symmetric group actions and then taking the colimit over the remaining sequential diagram*

$$F(\mathbf{0}) \xrightarrow{\iota_0} F(\mathbf{1})/\Sigma_1 \xrightarrow{\iota_1} F(\mathbf{2})/\Sigma_2 \xrightarrow{\iota_2} \cdots \longrightarrow F(\mathbf{n})/\Sigma_n \longrightarrow \cdots.$$

(ii) *If all even permutations act trivially on the values of F, it suffices to take the sequential colimit of the underlying diagram*

$$F(\mathbf{0}) \xrightarrow{\iota_0} F(\mathbf{1}) \xrightarrow{\iota_1} F(\mathbf{2}) \xrightarrow{\iota_2} \cdots \longrightarrow F(\mathbf{n}) \longrightarrow \cdots.$$

Proof. Every injection $\mathbf{n} \hookrightarrow \mathbf{m}$ differs from the standard inclusion $\iota : \mathbf{n} \hookrightarrow \mathbf{m}$ by a permutation $\tau \in \Sigma_m$ and so a colimit of the sequential diagram in part one also satisfies the universal property of an I-colimit. The second part follows from the observation that for every permutation $\sigma \in \Sigma_n$ there is an even permutation $\sigma' \in \Sigma_m$, $m \geq n$, such that $\sigma' \circ \iota = \iota \circ \sigma$ (e.g. extend an odd σ to a permutation of $\mathbf{n}+\mathbf{2}$ which interchanges the two additional coordinates). □

Proposition 3.1.7. *The Hurewicz map induces a natural isomorphism*

$$h : (\pi_* X)_{\mathbb{Q}} \xrightarrow{\cong} H_* CX$$

between the rationalized homotopy groups of the orthogonal spectrum X and the homology groups of CX.

Proof. Homology commutes with sequential colimits and rationally also with quotients by finite group actions. So by the first part of Lemma 3.1.6 the last map in the composite defining the stable Hurewicz map is a rational isomorphism. Since even permutations are path-connected to the identity in the ambient orthogonal group, they have to act trivially on homology and so the middle map is a rational isomorphism by the second part. It is well known that the first map is a rational isomorphism. □

Proposition 3.1.8. *Let A be a pointed space. Then the natural morphism*

$$NA \otimes \mathbb{Q} \longrightarrow C(\Sigma^\infty A)$$

mapping the 0-th term in the I-diagram $\tilde{C}(\Sigma^\infty A)$ to the colimit is a quasi-isomorphism.

Proof. In this case the sequential colimit computing the homology is taken along the suspension isomorphisms. □

We now turn attention towards monoidal properties. Disjoint union of finite sets gives the category I a symmetric monoidal structure (on objects this amounts to addition). A natural transformation

$$\tilde{C}X \otimes \tilde{C}Y \longrightarrow \tilde{C}(X \wedge Y) \circ +$$

of functors $I \times I \to \mathrm{Ch}_{\mathbb{Q}}$ is then defined at $(\mathbf{n}, \mathbf{m})$ by composing the shuffle map, multiplied by a suitable sign, with the universal bimorphism for $X \wedge Y$:

$$\begin{tikzcd}
\tilde{C}X(\mathbf{n}) \otimes \tilde{C}Y(\mathbf{m}) \arrow[rr] \arrow[d, equal] & & \tilde{C}(X \wedge Y)(\mathbf{n}+\mathbf{m}) \arrow[d, equal] \\
NX_n[n] \otimes NY_m[m] \arrow[r] & N(X_n \wedge Y_m)[n+m] \arrow[r] & N(X \wedge Y)_{n+m}[n+m] \\
x \otimes y \arrow[r, mapsto] & (-1)^{n|y|} \cdot \nabla(x \otimes y) &
\end{tikzcd}$$

On colimits this yields a natural map

$$\Delta : CX \otimes CY \longrightarrow C(X \wedge Y)$$

and a morphism relating the units is defined by 'inclusion at the non-basepoint'

$$\iota : \mathbb{Q}[0] \overset{\cong}{\to} NS^0 \to C\mathbb{S}.$$

Proposition 3.1.9. *The above structure maps define a symmetric monoidal structure on the chain functor C.*

Remark 3.1.10. Additively, the chain functor could be defined by just using the underlying $\mathbb{N}$-diagram, but here one uses the I-functoriality.

Next, we recall the natural pairing

$$\pi_k X \otimes \pi_l Y \longrightarrow \pi_{k+l} X \wedge Y$$

on stable homotopy groups (this is specialization of the pairing on equivariant homotopy groups to the trivial groups). It sends the pair $([f : S^{k+n} \to X_n], [g : S^{l+m} \to Y_m])$ to the class represented by the composite

$$S^{k+l+n+m} \cong S^{k+n+l+m} \xrightarrow{f \wedge g} X_m \wedge Y_n \longrightarrow (X \wedge Y)_{n+m},$$

where the first isomorphism permutes the coordinate blocks corresponding to l and n.

Proposition 3.1.11. *The Hurewicz map is a lax monoidal transformation, that is, the diagram*

$$\begin{tikzcd}
\pi_k X \otimes \pi_l Y \arrow[rr, "- \wedge -"] \arrow[d, "h \otimes h"] & & \pi_{k+l}(X \wedge Y) \arrow[d, "h"] \\
H_k CX \otimes H_l CY \arrow[r, "- \otimes -"] & H_{k+l}(CX \otimes CY) \arrow[r, "H_{k+l}\Delta"] & H_{k+l} C(X \wedge Y)
\end{tikzcd}$$

commutes for all orthogonal spectra X and Y.

Proof. This follows from the identity $\nabla(f_*[e_{k+n}] \otimes g_*[e_{l+m}]) = (f \wedge g)_*[e_{k+n+l+m}]$. We note that the sign in the definition of Δ cancels the sign coming from the permutation isomorphism $S^{k+l+n+m} \cong S^{k+n+l+m}$. □

We conclude our discussion of the chain functor C by sketching how it preserves cofiber sequences. This will follow from the existence of the natural quasi-isomorphism

$$NA \otimes CX \xrightarrow{\simeq} C(A \wedge X)$$

for based spaces A and orthogonal spectra X. In particular there is a natural quasi-isomorphism

$$CX[1] \xrightarrow{e_1 \otimes -} \tilde{N}S^1 \otimes CX \xrightarrow{\simeq} C(\Sigma X)$$

relating shift and suspension.

Proposition 3.1.12. *The chain functor preserves mapping cone sequences up to natural weak equivalence. In particular it induces an exact functor.*

Proof. Let $f : X \to Y$ be a morphism of orthogonal spectra. Applying C to the mapping cone sequence of f yields the upper row in the diagram

$$\begin{tikzcd}
CX \arrow[r, "Cf"] \arrow[d, equal] & CY \arrow[r] \arrow[d, equal] & C(\mathrm{Cone}\, f) \arrow[r] & C(\Sigma X) \\
CX \arrow[r] & CY \arrow[r] & \mathrm{Cone}\, Cf \arrow[r] \arrow[u, dashed] & CX[1] \arrow[u, dashed]
\end{tikzcd}$$

The lower row is the mapping cone sequence in chain complexes of the morphism $Cf : CX \to CY$. The functor C naturally preserves homotopies since the map

$$\tilde{N}\Delta^1 \otimes CX \longrightarrow C(\Delta^1 \wedge X)$$

relates the canonical cylinder objects. Hence the dashed arrow exists making the middle square commute. Moreover C sends the trivial spectrum $*$ to the 0-complex. Since the shift is the cokernel of the lower middle map the right dashed arrow exists. By inspection this is the map relating shift and suspension defined above, in particular a quasi-isomorphism. Using the natural isomorphism $H_*CX \cong (\pi_* X)_{\mathbb{Q}}$ the 5-Lemma now implies that the map relating the cones is a quasi-isomorphism. □

3.2 An algebraic model for rational global homotopy theory

With the necessary foundations set up we are now in a position to identify rational global homotopy theory via geometric fixed points with the algebraic model of $\mathrm{Out}^{\mathrm{op}}$-diagrams in chain complexes.

3.2.1 $\mathrm{Out}^{\mathrm{op}}$-chain complexes

We recall that the category of rational $\mathrm{Out}^{\mathrm{op}}$-chain complexes is endowed with the *projective* model structure: Weak equivalences are the groupwise quasi-isomorphisms and fibrations the groupwise surjections, in particular every object is fibrant. The homotopy category is the derived category $\mathcal{D}(\mathrm{Out}^{\mathrm{op}}\text{-}\mathrm{mod}_{\mathbb{Q}})$ of the abelian category of rational $\mathrm{Out}^{\mathrm{op}}$-modules. This is a triangulated category with suspension functor the groupwise shift of chain complexes. A triangle is distinguished if it is isomorphic to the mapping cone sequence of a morphism of chain complexes. The free diagrams $\mathbb{Q}\{\mathrm{Out}(-,G)\}$ corepresent the homology groups in the derived category and thus form a set of compact generators. The following is a standard formal consequence of t-structures in triangulated categories:

Lemma 3.2.1 ([GM03, IV.4.3])**.** *Let X and Y be $\mathrm{Out}^{\mathrm{op}}$-complexes such that X is $(n-1)$-connected and Y is $(n+1)$-coconnected, i.e. $H_iX = 0$ for $i < n$ and $H_iY = 0$ for $i > n$. Then taking n-th homology induces an isomorphism*

$$[X, Y]_{\mathcal{D}(\mathrm{Out})} \xrightarrow{H_n} \mathrm{Hom}_{\mathrm{Out}}(H_nX, H_nY).$$

We also recall the well known relation of the derived category to Ext-groups. From the viewpoint of model categories this amounts to the fact that projective resolutions of X are cofibrant replacements in the category of chain complexes.

Proposition 3.2.2 ([Wei94, 10.7.5])**.** *Let X and Y be $\mathrm{Out}^{\mathrm{op}}$-modules. There is a natural isomorphism*

$$[X, Y]^*_{\mathcal{D}(\mathrm{Out})} \cong \mathrm{Ext}^*_{\mathrm{Out}}(X, Y)$$

We will use another set of generators that comes up when working with geometric fixed points. Even though the following is homological algebra, we have introduced derived categories as homotopy categories of model categories and hence give a proof in that language.

Lemma 3.2.3. *In the derived category the $\mathrm{Out}^{\mathrm{op}}$-module $\mathbb{Q}\{\mathrm{Rep}(-,G)\}$ corepresents the functor*

$$X \longmapsto \bigoplus_{(H\leq G)} (H_iX(H))^{WH}$$

where the sum is indexed by the conjugacy classes of subgroups of G.

Proof. Linearizing the disjoint union of Example 2.3.5 yields the direct sum decomposition

$$\mathbb{Q}\{\mathrm{Rep}(-,G)\} \cong \bigoplus_{(H\leq G)} \mathbb{Q}\{\mathrm{Out}(-,H)\}/WH.$$

The summand $\mathbb{Q}\{\mathrm{Out}(-,H)\}/WH$ corepresents the right Quillen functor (taking invariants with respect to a finite group is rationally exact)

$$\mathrm{Out}^{\mathrm{op}}\text{-}\mathrm{Ch}_{\mathbb{Q}} \longrightarrow \mathrm{Ch}_{\mathbb{Q}}\,, \quad X \longmapsto X(H)^{WH}$$

and so it is cofibrant as the image of the cofibrant chain complex $\mathbb{Q}$ under the left adjoint. We can thus identify the derived homomorphisms as

$$\begin{aligned}[][\Sigma^i\mathbb{Q}\{\mathrm{Out}(-,H)\}/WH, X]_{\mathcal{D}(\mathrm{Out})} &\cong [\mathbb{Q}[i], X(H)^{WH}]_{\mathcal{D}(\mathbb{Q})}\\ &\cong H_i(X(H)^{WH})\\ &\cong (H_iX(H))^{WH}.\end{aligned}$$

□

Corollary 3.2.4. *The* $\mathrm{Out}^{\mathrm{op}}$*-modules* $\mathbb{Q}\{\mathrm{Rep}(-,G)\}$ *form a set of compact generators for the derived category of* $\mathrm{Out}^{\mathrm{op}}$*-modules. Moreover, for all* G *and* K *the graded abelian group*

$$[\mathbb{Q}\{\mathrm{Rep}(-,G)\}, \mathbb{Q}\{\mathrm{Rep}(-,K)\}]^*_{\mathcal{D}(\mathrm{Out})}$$

of derived morphisms is concentrated in degree 0*.*

Remark 3.2.5. While the represented $\mathrm{Out}^{\mathrm{op}}$-modules are already generators integrally, the above used that we are working rationally.

Remark 3.2.6. Even though we work rationally, the derived category does *not* split and all higher extensions can occur. For example, the $\mathrm{Out}^{\mathrm{op}}$-module $R_e\mathbb{Q}$ which consists of a copy of $\mathbb{Q}$ at the trivial group does not admit a finite projective resolution. It suffices to show this over cyclic groups since the restriction functor along the inclusion of the full subcategory $\mathrm{Out}_{\mathrm{cyc}} \subset \mathrm{Out}$ (see Definition 3.3.5) preserves projective resolutions (Lemma 3.3.8). Let $F_n = \mathbb{Q}\{\mathrm{Out}(-,C_n)\}/\mathrm{Out}(C_n)$ be the 'semi-free' $\mathrm{Out}_{\mathrm{cyc}}^{\mathrm{op}}$-module generated by $\mathbb{Q}$ at the cyclic group C_n, characterized by $\mathrm{Hom}_{\mathrm{Out}}(F_n, X) \cong X(C_n)^{\mathrm{Out}(C_n)}$. It consists of single copy of $\mathbb{Q}$ at every C_m such that n divides m, with identities as structure maps, and vanishes otherwise. A projective resolution of $R_e\mathbb{Q}$ is then defined as follows:

$$R_e\mathbb{Q} \twoheadleftarrow F_e \longleftarrow \bigoplus_p F_p \longleftarrow \bigoplus_{p_1<p_2} F_{p_1p_2} \longleftarrow \cdots \longleftarrow \bigoplus_{p_1<p_2<\cdots<p_n} F_{p_1p_2\cdots p_n} \longleftarrow \cdots$$

Here the sum $P_n = \oplus_{p_1<\cdots<p_n} F_{p_1\cdots p_n}$ is indexed by all n-element sets of primes and the differential $d_n : P_n \to P_{n-1}$ is determined by the formula

$$d(e^{p_1\cdots p_n}_{p_1<\cdots<p_n}) = \sum_{i=1}^n (-1)^i e^{p_1\cdots\widehat{p_i}\cdots p_n}_{p1<\cdots<\widehat{p_i}<\cdots<p_n},$$

where $e^m_{p_1<\cdots<p_n} \in P_n(C_m)$ denotes the generator of the summand indexed by $p_1, \ldots, p_n$, which is given by the class of (any) surjection $[C_m \twoheadrightarrow C_{p_1\cdots p_n}] \in F_{p_1\cdots p_n}(C_m)$. For every integer $m \geq 1$, we construct a chain contraction h of the complex $P_*(C_m)$ of $\mathbb{Q}$-vector spaces: Let $p_1 < \cdots < p_{\omega(m)}$ be the prime factors of m. Given a sequence

$$1 \leq \alpha(1) < \cdots < \alpha(n) \leq \omega(m),$$

we set $e^m_\alpha = e^m_{p_{\alpha(1)}<\cdots<p_{\alpha(n)}}$, and for $k \notin \mathrm{im}(\alpha)$ we let $e^m_{\alpha,k} = e^m_{\tilde{\alpha}}$ be the element corresponding to the sequence $\tilde{\alpha} = (\cdots < \alpha(j-1) < k < \alpha(j) < \cdots)$ obtained by adding k to it. The maps $h_n : P_n(C_m) \to P_{n+1}(C_m)$ are then defined by

$$h_n(e^m_\alpha) = \frac{1}{\omega(m)} \sum_{k \notin \mathrm{im}(\alpha)} (-1)^{j(\alpha,k)} e^m_{\alpha,k}$$

and one checks that this indeed yields a chain contraction, showing that $P_* \twoheadrightarrow R_e\mathbb{Q}$ is a resolution. We claim that the projections $\xi_n : P_n \twoheadrightarrow \mathrm{coker}(d_{n+1})$ define non-trivial elements $[\xi_n] \in \mathrm{Ext}^n_{\mathrm{Out}^{\mathrm{op}}_{\mathrm{cyc}}\text{-}\mathrm{mod}_{\mathbb{Q}}}(R_e\mathbb{Q}, \mathrm{coker}(d_{n+1}))$: The 'universal' cocycle ξ_n is non-trivial by a dimension count and cannot be a coboundary since there are no non-trivial morphisms: $\mathrm{Hom}_{\mathrm{Out}^{\mathrm{op}}_{\mathrm{cyc}}\text{-}\mathrm{mod}_{\mathbb{Q}}}(P_{n-1}, \mathrm{coker}(d_{n+1})) = 0$. This is true because P_n (and hence the cokernel) vanishes at all groups at which P_{n-1} is generated as an $\mathrm{Out}^{\mathrm{op}}_{\mathrm{cyc}}$-module.

3.2.2 Comparison with $\mathrm{Out}^{\mathrm{op}}$-chain complexes

As we explained in the previous chapter, geometric fixed points take values in $\mathrm{Epi}^{\mathrm{op}}$-diagrams of spectra. To obtain a comparison functor to $\mathrm{Out}^{\mathrm{op}}$-chain complexes we will proceed in two steps. First we move to the algebraic setting by prolonging the chain functor C group-wise to a functor

$$\mathrm{Epi}^{\mathrm{op}}\text{-}\mathrm{Sp}^{\mathrm{O}} \xrightarrow{C} \mathrm{Epi}^{\mathrm{op}}\text{-}\mathrm{Ch}_{\mathbb{Q}}$$

on diagram categories which is again denoted by C.

Lemma 3.2.7. *On* $\mathrm{Epi}^{\mathrm{op}}$*-spectra the Hurewicz map prolongs to a natural isomorphism*

$$h : (\pi_* X)_{\mathbb{Q}} \xrightarrow{\cong} H_* CX$$

of $\mathrm{Epi}^{\mathrm{op}}$*-modules and C induces an exact functor on homotopy categories that preserves sums.*

Proof. The first part is clear by naturality of h. Weak equivalences are defined group-wise and so C descends to a functor on homotopy categories as a homotopical functor. Mapping cone sequences are also defined group-wise and the comparison of Proposition 3.1.12 is natural in the spectrum. So C preserves distinguished triangles. □

In the second step we pass to $\mathrm{Out}^{\mathrm{op}}$-chain complexes via left Kan extension

$$\mathrm{Epi}^{\mathrm{op}}\text{-}\mathrm{Ch}_{\mathbb{Q}} \xrightarrow{\mathrm{Lan}} \mathrm{Out}^{\mathrm{op}}\text{-}\mathrm{Ch}_{\mathbb{Q}}$$

along the canonical projection functor $\pi : \mathrm{Epi} \to \mathrm{Out}$. This amounts to dividing out the conjugation actions at each finite group G: If D is an $\mathrm{Epi}^{\mathrm{op}}$-complex, then the inner automorphisms $\mathrm{Inn}(G)$ act via functoriality from the right on $D(G)$ and the value of $\mathrm{Lan}\, D$ evaluated at the finite group G is given by the quotient

$$(\mathrm{Lan}\, D)(G) = D(G)/\,\mathrm{Inn}(G).$$

Lemma 3.2.8. *As a left Quillen functor between stable model categories* Lan *induces an exact functor on homotopy categories that preserves sums. Moreover, it commutes with homology.*

Proof. The right adjoint is by definition the restriction functor and hence a right Quillen functor with respect to the projective model structures. Since rational homology commutes with finite group quotients, the explicit description of the left adjoint above implies the homology statement. □

Remark 3.2.9. The Quillen adjunction

$$\mathrm{Lan} : \mathrm{Epi}^{\mathrm{op}}\text{-}\mathrm{Ch}_{\mathbb{Q}} \rightleftarrows \mathrm{Out}^{\mathrm{op}}\text{-}\mathrm{Ch}_{\mathbb{Q}} : \mathrm{Res}$$

expresses $\mathrm{Out}^{\mathrm{op}}$-$\mathrm{Ch}_{\mathbb{Q}}$ as a homotopical localization of $\mathrm{Epi}^{\mathrm{op}}$-$\mathrm{Ch}_{\mathbb{Q}}$ with local objects the $\mathrm{Epi}^{\mathrm{op}}$-complexes whose homology groups are $\mathrm{Out}^{\mathrm{op}}$-modules. By this we mean that on homotopy categories the right adjoint is fully faithful with essential image these *homology* $\mathrm{Out}^{\mathrm{op}}$*-complexes.* To see this we consider unit and counit of the adjunction, which actually model the derived ones since both functors are homotopical. Now the counit is just the identity transformation and hence always an isomorphism. The unit projects to the quotients by the conjugation actions and rationally this commutes with taking homology. So it is a quasi-isomorphism on the above full subcategory.

Definition 3.2.10. The comparison functor to $\mathrm{Out}^{\mathrm{op}}$-complexes is defined as the composite

$$\Gamma : \mathrm{Sp}^{\mathrm{O}} \xrightarrow{\Phi} \mathrm{Epi}^{\mathrm{op}}\text{ - }\mathrm{Sp}^{\mathrm{O}} \xrightarrow{C} \mathrm{Epi}^{\mathrm{op}}\text{-}\mathrm{Ch}_{\mathbb{Q}} \xrightarrow{\mathrm{Lan}} \mathrm{Out}^{\mathrm{op}}\text{-}\mathrm{Ch}_{\mathbb{Q}}\,.$$

Proposition 3.2.11. *The functor* Γ *comes with a natural isomorphism*

$$H_*\Gamma X \cong (\underline{\Phi}_* X) \otimes \mathbb{Q}$$

of $\mathrm{Out}^{\mathrm{op}}$*-modules. It induces an exact and sum preserving functor on homotopy categories.*

Proof. Spelling out the composite defining Γ we get the following chain of natural identifications:

$$H_*\Gamma X = H_*\,\mathrm{Lan}\, C\Phi X \cong \mathrm{Lan}\, H_* C\Phi X \cong \mathrm{Lan}(\underline{\Phi}_* X)_{\mathbb{Q}} = (\underline{\Phi}_* X)_{\mathbb{Q}}$$

The first isomorphism is the interchange map between left Kan extension and homology (Lemma 3.2.8), the second one is the Hurewicz transformation. Since the geometric fixed point homotopy groups are already $\mathrm{Out}^{\mathrm{op}}$-modules, dividing out the conjugation actions has no effect which is expressed by the last equality. The second part follows since Γ descends to a composition of exact and sum preserving functors. □

Now that the comparison functor Γ is in place, we turn towards the task of showing that it yields an equivalence.

Lemma 3.2.12. *The geometric fixed points of global classifying spaces are rationally concentrated in degree* 0. *In particular there is a preferred equivalence*

$$\Gamma\Sigma^\infty_+ B_{\mathrm{gl}}G \simeq \mathbb{Q}\{\mathrm{Rep}(-,G)\}[0]$$

and hence Γ *sends a set of compact generators to a set of compact generators.*

Proof. This follows from the decomposition (see Example 2.3.15)

$$\Phi^K B_{\mathrm{gl}}G \simeq \coprod_{\alpha\in\mathrm{Rep}(K,G)} BC(\alpha)$$

and the classical fact that the rational homology of finite groups vanishes in positive degrees. ☐

Corollary 3.2.13. *The rational stable homotopy groups of global classifying spaces are concentrated in degree* 0.

Proof. Rational homotopy groups are determined by the geometric fixed point homotopy groups via the isomorphism $\tau(\underline{\pi}_*X) \cong_{\mathbb{Q}} \underline{\Phi}_*X$. Since τ is an equivalence, the statement follows from the previous lemma. ☐

Theorem 3.2.14. *The functor* Γ *induces an equivalence of triangulated categories*

$$\Gamma : \mathcal{GH}_{\mathbb{Q}} \xrightarrow{\simeq} \mathcal{D}(\mathrm{Out}^{\mathrm{op}}\text{-}\mathrm{mod}_{\mathbb{Q}})$$

between the rational global homotopy category and the derived category of $\mathrm{Out}^{\mathrm{op}}$*-modules.*

Proof. We just saw that Γ induces an exact and sum preserving functor on homotopy categories which sends a set of compact generators to a set of compact generators. By the recognition theorem 3.1.2 for equivalences between compactly generated triangulated categories it remains to be checked that Γ is fully faithful on them. Let X and Y be orthogonal spectra. The square

$$\begin{tikzcd}
{[X,Y]_{\mathcal{GH}}} \arrow[r, "\Gamma"] \arrow[d, "\pi_0"] & {[\Gamma X,\Gamma Y]_{\mathcal{D}(\mathrm{Out}^{\mathrm{op}}\text{-}\mathrm{mod}_{\mathbb{Q}})}} \arrow[d, "H_0"] \\
\mathcal{GF}_{\mathbb{Q}}(\underline{\pi}_0X,\underline{\pi}_0Y) \arrow[r, "\tau"] & \mathrm{Out}^{\mathrm{op}}\text{-}\mathrm{mod}_{\mathbb{Q}}(\underline{\Phi}_0X,\underline{\Phi}_0Y)
\end{tikzcd}$$

commutes up to isomorphism (3.2.11). The functor τ is an equivalence (2.3.10) and so the lower horizontal map is an isomorphism. We consider the case in which both X and Y are shifted suspension spectra of global classifying spaces. If X and Y are concentrated in different degrees, then both sides of the upper map are trivial by Corollary 3.2.13 and Corollary 3.2.4. Without loss of generality we may assume that both X and Y

are concentrated in degree 0. Then both vertical maps isomorphisms: For the left map this follows from the fact that homotopy groups are corepresented by global classifying spaces (Proposition 2.2.19) and the statement about the right map follows from Lemma 3.2.1. □

Remark 3.2.15. While the above result was stated and naturally proven on the level of homotopy categories, we remark that this actually yields an equivalence of homotopy theories: Homotopical functors induce maps on derived mapping spaces and their homotopy groups are computed in the homotopy category: $\pi_k \operatorname{map}^L(X,Y) \cong [\Sigma^k X, Y]$. Since Γ commutes with suspensions, it follows that it induces weak equivalences on derived mapping spaces.

Monoidal comparison

Now that we have obtained an algebraic model for rational global homotopy theory, we investigate the multiplicative properties of the comparison. The main computational input will be the following result about geometric fixed point homotopy groups.

Proposition 3.2.16. *At every finite group G the pairing (see the end of Section 2.3.3)*

$$\Phi_*^G X \otimes \Phi_*^G Y \xrightarrow{\cong_{\mathbb{Q}}} \Phi_*^G (X \wedge^L Y)$$

of graded $\mathrm{Out}^{\mathrm{op}}$*-modules is a rational isomorphism.*

Proof. Both sides preserve sums in each variable. Furthermore, if we fix one of the variables then both sides send triangles to long exact sequences, compatible with the natural transformation (cf. Proposition 2.3.26). By a localizing subcategory argument as before (for each variable) it thus suffices to check the statement on compact generators. We recall that

$$\Phi_*^G \Sigma_+^\infty B_{\mathrm{gl}} K \cong \mathbb{Q}\{\mathrm{Rep}(G,K)\}[0]$$

concentrated in degree 0. By Corollary 2.3.28 the external product on $\underline{\Phi}_0$ is an isomorphism in that case. □

Remark 3.2.17. It is known that already in the integral case geometric fixed point spectra are homotopically strong symmetric monoidal ([MM02, V.4.7]): $\Phi^G X \wedge^L \Phi^G Y \simeq \Phi^G (X \wedge^L Y)$. However, we only need the above algebraic consequence and in our setup it is technically more convenient to just show it directly.

As we have seen, the composite functor

$$\mathrm{Sp}^{\mathrm{O}} \xrightarrow{\Phi} \mathrm{Epi}^{\mathrm{op}}\text{-}\mathrm{Sp}^{\mathrm{O}} \xrightarrow{C} \mathrm{Epi}^{\mathrm{op}}\text{-}\mathrm{Ch}_{\mathbb{Q}}$$

is lax symmetric monoidal. The tensor product of rational chain complexes is homotopical in each variable and so the zigzag

$$C\Phi X \otimes C\Phi Y \xleftarrow{\simeq} C\Phi X_c \otimes C\Phi Y_c \longrightarrow C\Phi(X_c \wedge Y_c) = C\Phi(X \wedge^L Y)$$

descends to a natural transformation on homotopy categories.

Corollary 3.2.18. *The above transformation turns the composite*

$$C \circ \Phi : \mathcal{GH} \longrightarrow \mathrm{Ho}(\mathrm{Epi}^{\mathrm{op}}\text{-}\mathrm{Ch}_{\mathbb{Q}})_{H_*\text{-}\mathrm{Out}}$$

into a strong monoidal functor with respect to the group-wise tensor product, where $(-)_{H_*\text{-}\mathrm{Out}}$ *denotes the full subcategory of homology* $\mathrm{Out}^{\mathrm{op}}$*-complexes.*

Proof. Using Proposition 3.1.11, this follows by applying Proposition 3.2.16 group-wise. □

A slight technical complication arises because the left Quillen functor Lan is *not* lax monoidal. Instead the canonical map

$$\begin{array}{ccc} \mathrm{Lan}(C\otimes D)(G) & \longrightarrow & (\mathrm{Lan}\,C)(G)\otimes(\mathrm{Lan}\,D)(G) \\ \| & & \| \\ (C(G)\otimes D(G))/\,\mathrm{Inn}(G) & \longrightarrow & (C(G)/\,\mathrm{Inn}(G))\otimes(D(G)/\,\mathrm{Inn}(G)) \end{array}$$

gives it the structure of an oplax symmetric monoidal functor. Both sides are homotopical in each variable and so this descends to the level of homotopy categories:

Lemma 3.2.19. *The Quillen equivalence (see Remark 3.2.9)*

$$\mathrm{Lan} : (\mathrm{Epi}^{\mathrm{op}}\text{-}\mathrm{Ch}_{\mathbb{Q}})_{H_*\text{-}\mathrm{Out}} \rightleftarrows \mathrm{Out}^{\mathrm{op}}\text{-}\mathrm{Ch}_{\mathbb{Q}} : \mathrm{Res}$$

is homotopically strong monoidal, i.e. the natural transformation

$$\mathrm{Lan}(C\otimes D) \xrightarrow{\simeq} (\mathrm{Lan}\,C)\otimes(\mathrm{Lan}\,D)$$

is a quasi-isomorphism when restricted to homology $\mathrm{Out}^{\mathrm{op}}$*-complexes.*

Proof. This follows immediately from the rational Künneth theorem and because rational homology commutes with taking finite group quotients. □

Theorem 3.2.20. *Geometric fixed points induce a strong symmetric monoidal equivalence*

$$\mathcal{GH}_{\mathbb{Q}} \xrightarrow{\simeq} \mathcal{D}(\mathrm{Out}^{\mathrm{op}}\text{-}\,\mathrm{mod}_{\mathbb{Q}})$$

between the global homotopy category and the derived category of $\mathrm{Out}^{\mathrm{op}}$*-modules.*

Proof. Writing Γ as the composition

$$\mathrm{Sp}^{\mathrm{O}} \longrightarrow (\mathrm{Epi}^{\mathrm{op}}\text{-}\mathrm{Ch}_{\mathbb{Q}})_{H_*\text{-}\mathrm{Out}} \longrightarrow \mathrm{Out}^{\mathrm{op}}\text{-}\mathrm{Ch}_{\mathbb{Q}}$$

it follows from Corollary 3.2.18 and Lemma 3.2.19 that on the level of homotopy categories Γ is a composition of strong monoidal functors. □

Remark 3.2.21. Spelling out the above construction one sees that the monoidal structure on Γ is induced by the zig-zag

$$\begin{tikzcd} \Gamma X\otimes\Gamma Y \arrow[rr] & & \Gamma(X\wedge^L Y) \\ \Gamma X_c\otimes\Gamma Y_c \arrow[u, "\simeq"] \arrow[r, equal] & LC\Phi X_c\otimes LC\Phi Y_c & L(C\Phi X_c\otimes C\Phi Y_c) \arrow[l, "\simeq"'] \arrow[u] \end{tikzcd}$$

3.2.3 Global families and a $\mathbb{Z}[\frac{1}{p}]$-local result

So far we have implicitly used the global family of all finite groups in order to keep the exposition simpler. However, the entire discussion applies verbatim when working with a global family $\mathcal{F}$ of finite groups. One just has to restrict all statements to those groups appearing in the family $\mathcal{F}$. The categories $\mathrm{Epi}_{\mathcal{F}}$ and $\mathrm{Out}_{\mathcal{F}}$ are the full subcategories on those groups lying in $\mathcal{F}$. We record the corresponding result:

Theorem 3.2.22. *Let $\mathcal{F}$ be a global family of finite groups. The functor Γ induces an equivalence of triangulated categories*

$$\Gamma : \mathcal{GH}_{\mathcal{F},\mathbb{Q}} \xrightarrow{\simeq} \mathcal{D}(\mathrm{Out}_{\mathcal{F}}^{\mathrm{op}}\text{-}\mathrm{mod}_{\mathbb{Q}})$$

between the rational $\mathcal{F}$-global homotopy category and the derived category of $\mathrm{Out}_{\mathcal{F}}^{\mathrm{op}}$-modules. Furthermore, if the global family $\mathcal{F}$ is multiplicative, then the equivalence is symmetric monoidal.

Working with families also allows us to say something if we only invert a prime p. This will be the global analogue of the following folklore result: After inverting the order of a finite group G, the homotopy theory of genuine G-spectra decomposes

$$\mathrm{Ho}(G\text{-}\mathrm{Sp}^{\mathrm{O}}) \simeq_{\mathbb{Z}[\frac{1}{|G|}]} \prod_{(H\leq G)} \mathrm{Ho}(\mathbb{S}[WH]\text{-}\mathrm{mod})$$

into a product of spectra with (naive) Weyl group action.

In order to be able to invert group orders globally, we need to restrict to a global family $\mathcal{F}_p$ of finite p-groups (e.g. all finite p-groups). The remainder of this section is devoted to showing that $\mathbb{Z}[\frac{1}{p}]$-locally the homotopy theory of $\mathcal{F}_p$-global spectra is equivalent to that of $\mathrm{Out}_{\mathcal{F}_p}^{\mathrm{op}}$-spectra. We recall that these are equipped with the projective model structure, that is, weak equivalence and fibrations are defined group-wise in the $\mathbb{Z}[\frac{1}{p}]$-local model structure on orthogonal spectra. In particular, fibrant objects have $\mathbb{Z}[\frac{1}{p}]$-local homotopy groups. As before, geometric fixed points followed by left Kan extension along the projection $q : \mathrm{Epi} \to \mathrm{Out}$ yield a comparison functor, but in the topological setting this has to be derived.

Definition 3.2.23. The comparison functor Γ is the composite

$$\Gamma : \mathcal{GH}_{\mathcal{F}_p} \xrightarrow{\Phi} \mathrm{Ho}(\mathrm{Epi}_{\mathcal{F}_p}^{\mathrm{op}}\text{-}\mathrm{Sp}^{\mathrm{O}}) \xrightarrow{\mathbb{L}\mathrm{an}} \mathrm{Ho}(\mathrm{Out}_{\mathcal{F}_p}^{\mathrm{op}}\text{-}\mathrm{Sp}^{\mathrm{O}}),$$

where $\mathbb{L}\mathrm{an}$ denotes the left derived functor of the Quillen pair (left Kan extension, restriction).

Lemma 3.2.24. *Away from the prime p, the derived left Kan extension commutes with taking homotopy groups. More concretely, for a cofibrant $\mathrm{Epi}_{\mathcal{F}_p}^{\mathrm{op}}$-spectrum X the canonical map*

$$(\pi_* X(G))/\mathrm{Inn}(G) \longrightarrow \pi_*(\mathrm{Lan}\, X)(G) = \pi_*(X(G)/\mathrm{Inn}(G))$$

is a $\mathbb{Z}[\frac{1}{p}]$-local isomorphism.

Proof. We consider the derived transformation

$$\eta : (\pi_* X(G))/\operatorname{Inn}(G) \longrightarrow \pi_*(\mathbb{L}\mathrm{an}\, X)(G),$$

on the homotopy category. This is a map of homological functors: Away from the prime p both sides are exact and it commutes with connecting homomorphisms since η is induced form a spectrum level transformation. So the subcategory of $\mathrm{Ho}(\mathrm{Epi}^{\mathrm{op}}_{\mathcal{F}_p}\text{-}\mathrm{Sp}^{\mathrm{O}})$ such that η is a $\mathbb{Z}[\frac{1}{p}]$-local isomorphism is localizing and it suffices to show that it contains the compact generators $\mathbb{S}\{\mathrm{Epi}(-,K)\}$. By direct inspection, it contains all diagrams of the form $\mathrm{Epi}(-,K)_+ \wedge Y$ for a finite group K and a cofibrant spectrum Y. □

Corollary 3.2.25. *The derived unit transformation* $\eta : \mathrm{Id} \to q^* \circ \mathbb{L}\mathrm{an}$ *induces a natural isomorphism* $\underline{\Phi}_* X \cong \pi_* \Gamma X$ *for* $\mathbb{Z}[\frac{1}{p}]$*-local spectra* X*.*

Remark 3.2.26. Similar to 3.2.9, this implies that the restricted Quillen adjunction

$$\mathrm{Lan} : (\mathrm{Epi}^{\mathrm{op}}_{\mathcal{F}_p}\text{-}\mathrm{Sp}^{\mathrm{O}})_{\pi_*\text{-}\mathrm{Out}} \rightleftarrows \mathrm{Out}^{\mathrm{op}}_{\mathcal{F}_p}\text{-}\mathrm{Sp}^{\mathrm{O}} : \mathrm{Res}$$

is a Quillen equivalence.

We recall that $\mathbb{S}\{M\} = \bigvee_{m \in M} \mathbb{S}$ denotes the '$\mathbb{S}$-linearization' of a set M.

Lemma 3.2.27. *For a finite p-group G the canonical map* (*cf. Example 2.3.15*)

$$\Phi(\Sigma^\infty_+ B_{\mathrm{gl}} G) \to \mathbb{S}\{\mathrm{Rep}(-,G)\}$$

is a $\mathbb{Z}[\frac{1}{p}]$*-local equivalence of* $\mathrm{Epi}^{\mathrm{op}}$*-spectra.*

Proof. By the example there is an equivalence

$$\Phi^K \Sigma^\infty_+ B_{\mathrm{gl}} G \simeq \bigvee_{(\alpha) \in \mathrm{Rep}(K,G)} \Sigma^\infty_+ BC(\alpha)$$

and the map is obtained by projecting the classifying spaces to a point. Away form the prime p, these are stably contractible since the group homology of p-groups vanishes in that case. □

Lemma 3.2.28. *The canonical map*

$$[\mathbb{S}\{\mathrm{Rep}(-,G)\}, X]_{\mathrm{Out}^{\mathrm{op}}_{\mathcal{F}_p}\text{-}\mathrm{Sp}^{\mathrm{O}}} \xrightarrow{\pi_0} \mathrm{Hom}(\mathbb{Z}\{\mathrm{Rep}(-,G)\}, \pi_0 X) \cong \prod_{(H \leq G)} (\pi_0 X(H))^{WH}$$

is an isomorphism for $\mathbb{Z}[\frac{1}{p}]$*-local* X*.*

Proof. The decompostion $\mathrm{Rep}(-,G) \cong \sqcup WH\backslash\mathrm{Out}(-,H)$ shows that it suffices to consider the evaluation map

$$[\mathbb{S}\{WH\backslash\mathrm{Out}(-,H)\}, X] \longrightarrow (\pi_0 X(H))^{WH}$$

at $\mathrm{Id}_H \in \mathrm{Out}(H,H)$. Since the WH-action on $\mathrm{Out}(-,H)$ is free, the projection

$$EWH_+ \wedge_{WH} \mathbb{S}\{\mathrm{Out}(-,H)\} \xrightarrow{\simeq} \mathbb{S}\{WH\backslash \mathrm{Out}(-,H)\}$$

is an equivalence (even integrally). The homotopy orbits form a left Quillen functor $X \mapsto EWH_+ \wedge_{WH} X\{\mathrm{Out}(-,H)\}$ with right adjoint the homotopy fixed points $Y \mapsto \mathrm{map}_{WH}(EWH_+, Y(H))$. Using the derived adjunction we compute:

$$\begin{aligned}[EWH_+ \wedge_{WH} \mathbb{S}\{\mathrm{Out}(-,H)\}, X)] &\cong [\mathbb{S}, \mathrm{map}_{WH}(EWH_+, X(H))] \\ &\cong \pi_0 \, \mathrm{map}_{WH}(EWH_+, X(H))\end{aligned}$$

Finally, since X is $\mathbb{Z}[\frac{1}{p}]$-local, π_0 of the homotopy fixed points is just given by the fixed points $(\pi_0 X(H))^{WH}$ (if one wants to avoid using the homotopy fixed point spectral sequence, this can be seen from another straightforward localizing subcategory argument). $\square$

Corollary 3.2.29. *The diagrams $\mathbb{S}\{\mathrm{Rep}(-,G)\}$ form a set of compact generators for $\mathrm{Ho}(\mathrm{Out}^{\mathrm{op}}_{\mathcal{F}_p}\text{-}\mathrm{Sp}^{\mathrm{O}})$, where G ranges over the isomorphism classes of groups in $\mathcal{F}_p$.*

Theorem 3.2.30. *Let $\mathcal{F}_p$ be a global family of finite p-groups for some prime p. Geometric fixed points induce a $\mathbb{Z}[\frac{1}{p}]$-local equivalence*

$$\Gamma : \mathcal{GH}_{\mathcal{F}_p} \xrightarrow{\simeq} \mathrm{Ho}(\mathrm{Out}^{\mathrm{op}}_{\mathcal{F}_p}\text{-}\mathrm{Sp}^{\mathrm{O}})$$

between the $\mathcal{F}_p$-global homotopy category and the homotopy category of $\mathrm{Out}^{\mathrm{op}}_{\mathcal{F}_p}$-spectra.

Proof. We have just seen that Γ sends a set of compact generators to a set of compact generators, so it suffices to show that Γ is fully faithful on their suspensions. This follows from the commutative square

$$\begin{tikzcd}[{[\Sigma^\infty_+ B_{\mathrm{gl}}G, X]}] \arrow[r, "\Gamma"] \arrow[d, "ev_G"] & {[\Gamma\Sigma^\infty_+ B_{\mathrm{gl}}G, \Gamma X]} \arrow[d] \\ \pi_0^G X \arrow[r, "\prod \phi^H"] & \prod_{(H \leq G)} (\Phi_0^H X)^{WH},\end{tikzcd}$$

for $\mathbb{Z}[\frac{1}{p}]$-local X, where the left map is the corepresentability isomorphism (Proposition 2.2.19) and the right vertical map is given by evaluating at the classes $\phi((e_G)|_H)$. Up to isomorphism, this agrees with the map of Lemma 3.2.28 and hence it is also bijective. Finally, the lower horizontal map is an isomorphism by Proposition 2.3.11. $\square$

Remark 3.2.31. More generally, the preceding arguments work in the generality of a subring $R \subseteq \mathbb{Q}$ and a global family $\mathcal{F}$ consisting of groups whose order is invertible in R. In particular, for $R = \mathbb{Q}$ this recovers the previous additive results (while staying in the topological setting). However, for the multiplicative comparison of rational global homotopy theory it much more convenient to first move to the algebraic world since the tensor product of rational chain complexes is exact in both variables.

3.3 Rational splitting of global K-theory

Our aim in this section is to show

Theorem 3.3.1. *The global complex K-theory spectrum* $\mathbf{KU}$ *with respect to finite groups splits rationally as a wedge of Eilenberg-MacLane spectra of global functors. This also holds for* $\mathbf{KO}$ *because it is a rational retract of* $\mathbf{KU}$.

Global K-theory, based on Joachim's model ([Joa04]), is defined and discussed in [Sch17b, Chapter VI], but all we will need to know is that its homotopy is given by the *complex representation ring* global functor $\mathbf{RU}$ in even degrees and that it vanishes otherwise. The representation rings $\mathbf{RU}(G)$ of finite groups G form a global functor with induction of representations as transfer homomorphisms, and restriction of scalars. The splitting is then a consequence of the following computational fact:

Theorem 3.3.2. *Let* $\mathbf{RU}_{\mathbb{Q}} = \mathbb{Q} \otimes \mathbf{RU}$ *be the rationalized complex representation ring functor. There is a natural isomorphism*

$$\mathrm{Hom}_{\mathcal{GF}}(F, \mathbf{RU}_{\mathbb{Q}}) \cong \varprojlim_{(\mathbb{N},|)} (F(C_n)^{\vee}_{\mathrm{tr}})$$

identifying morphisms of global functors into $\mathbf{RU}_{\mathbb{Q}}$ *as an inverse limit over the poset of natural numbers with respect to the divisibility relation, where* $F(-)^{\vee}_{\mathrm{tr}}$ *denotes the linear forms on* $F(-)$ *vanishing on proper transfers. Moreover, this isomorphism also passes to the higher derived functors*

$$\mathrm{Ext}^k_{\mathcal{GF}}(F, \mathbf{RU}_{\mathbb{Q}}) \cong \varprojlim_{(\mathbb{N},|)}{}^k (F(C_n)^{\vee}_{\mathrm{tr}}).$$

Remark 3.3.3. The universal linear forms $\mathbf{RU}_{\mathbb{Q}}(C_n) \to \mathbb{Q}$ do not seem to admit a simple description. As will become apparent in the construction of the isomorphism they are determined by a choice of explicit, compatible $\mathbb{Q}$-bases for the representation rings of cyclic p-groups.

By further inspection this will allow us to draw the following

Corollary 3.3.4. *For any rational global functor* F *the higher Ext-groups*

$$\mathrm{Ext}^n_{\mathcal{GF}}(F, \mathbf{RU}_{\mathbb{Q}}) = 0,\ n \geq 2$$

vanish. Furthermore, the Ext-algebra of $\mathbf{RU}_{\mathbb{Q}}$ *vanishes in all positive degrees:*

$$\mathrm{Ext}^1_{\mathcal{GF}}(\mathbf{RU}_{\mathbb{Q}}, \mathbf{RU}_{\mathbb{Q}}) = 0.$$

In light of this corollary, Theorem 3.3.1 is a formal consequence of the fact that every chain complex C with $\mathrm{Ext}^k(H_n C, H_{n+k-1} C) = 0$ for all $n \in \mathbb{Z}$ and $k \geq 2$ splits in the derived category. We briefly sketch the standard argument: For any connected cover of C the assumption implies by an inductive argument over the Postnikov tower that the lowest Postnikov section splits off (non-canonically). This produces for every $n \in \mathbb{Z}$ a map $(H_nC)[n] \to C$ which induces an isomorphism on H_n and summing these up yields an equivalence

$$\bigoplus_{n\in\mathbb{Z}}(H_nC)[n] \xrightarrow{\cong} C.$$

The rest of this section is now devoted to a proof of the algebraic theorem and its corollary, using the equivalence $\tau : \mathcal{GF}_{\mathbb{Q}} \simeq \mathrm{Out}^{\mathrm{op}}\text{-}\mathrm{mod}_{\mathbb{Q}}$ to perform the calculations in the simper category of $\mathrm{Out}^{\mathrm{op}}$-modules.

Definition 3.3.5. For every $n \geq 1$ we fix a cyclic group of order n with a chosen generator τ_n and define $\mathrm{Out}_{\mathrm{cyc}} \subset \mathrm{Out}$ be the full subcategory on these. For definiteness we take $C_n \subset \mathbb{C}^\times$ to be the n-th roots of unity and $\tau_n = e^{2\pi i/n}$.

There are preferred projections $C_m \twoheadrightarrow C_n$ for all integers $n, m \geq 1$ such that n divides m and under the identification $C_k \cong \mathbb{Z}/k\mathbb{Z}$ they correspond to the unique surjections of rings. We recall that these restrict to surjections on units. The canonical identifications $(\mathbb{Z}/k\mathbb{Z})^\times \cong \mathrm{Out}(C_k) = \mathrm{Aut}(C_k),\ l \mapsto (x \mapsto x^l)$ then also provide surjections $\mathrm{Out}(C_m) \twoheadrightarrow \mathrm{Out}(C_n)$ on automorphisms groups. Furthermore, we note that any two epimorphisms between the same cyclic groups differ by a unique automorphism in the target. This implies

Proposition 3.3.6. *Specifying an $\mathrm{Out}^{\mathrm{op}}_{\mathrm{cyc}}$-module X is equivalent to giving $\mathrm{Out}(C_n)$-representations $X(C_n)$ for each integer n together with restriction maps $X(C_n) \to X(C_{nm})$ associated to the preferred projections $C_{nm} \twoheadrightarrow C_n$, compatible with composition. These have to be $\mathrm{Out}(C_{nm})$-equivariant, where $\mathrm{Out}(C_{nm})$ acts on $X(C_n)$ via the canonical map $\mathrm{Out}(C_{nm}) \to \mathrm{Out}(C_n)$ from above. Similarly for morphisms of $\mathrm{Out}^{\mathrm{op}}_{\mathrm{cyc}}$-modules one only has to demand that the individual maps of representations commute with these distinguished restrictions.*

Our example of interest is the complex representation ring functor and we recall the identification of $\tau\,\mathbf{RU}_{\mathbb{Q}}$ from [Sch17b, IV.6.12]. By Artin's theorem (e.g. see [Ser77, II.9 Thm 17]) every virtual representation is rationally induced from cyclic subgroups. Since transfers in $\mathbf{RU}_{\mathbb{Q}}$ are given by induction of representations, $\tau\,\mathbf{RU}_{\mathbb{Q}}$ vanishes at noncyclic groups. The value

$$\mathbf{RU}_{\mathbb{Q}}(C_n) = \mathbb{Q}[z]/(z^n - 1)$$

at the cyclic group C_n of order n is of course well known, z is the one-dimensional tautological representation of C_n. Over the cyclic p-group C_{p^h} a calculation with characters shows that dividing out transfers corresponds to dividing out the ideal generated by the minimal polynomial $1 + z^{p^{h-1}} + \cdots + z^{(p-1)\cdot p^{h-1}}$ of the primitive p^h-th roots. Hence we can identify

$$\tau\,\mathbf{RU}_{\mathbb{Q}}(C_{p^h}) \cong \mathbb{Q}(\zeta_{p^h})$$

as a cyclotomic field extension for all primes p and non-negative integers h. The group of automorphisms $\mathrm{Out}(C_{p^h}) \cong (\mathbb{Z}/p^h\mathbb{Z})^\times$ acts as the Galois group and restriction along the projection $C_{p^h} \twoheadrightarrow C_{p^{h-1}}$ is given by sending $\zeta_{p^{h-1}}$ to $\zeta_{p^h}^p$.

We recall that via the tensor product of representations the representation ring of a product of two groups is identified with the tensor product of the individual representation rings. For coprime integers n, m we can combine this with the Chinese remainder theorem to obtain a canonical isomorphism

$$\mathbf{RU}_{\mathbb{Q}}(C_n) \otimes \mathbf{RU}_{\mathbb{Q}}(C_m) \cong \mathbf{RU}_{\mathbb{Q}}(C_n \times C_m) \cong \mathbf{RU}_{\mathbb{Q}}(C_{nm})$$

and we note that this is the unique map of commutative rings induced by the two preferred restrictions. Since the proper transfers form an ideal, the objectwise ring structure in $\mathbf{RU}_{\mathbb{Q}}$ passes to the quotient and the restrictions of $\tau\,\mathbf{RU}_{\mathbb{Q}}$ become maps of commutative rings. A transfer in one of the tensor factors corresponds to a transfer from a subgroup of the form $H \times C_m$ or $C_n \times H$ for a proper subgroup H. The groups of this form contain all maximal subgroups of the product because n and m are coprime. Hence the induced multiplication map

$$\tau\,\mathbf{RU}_{\mathbb{Q}}(C_n) \otimes \tau\,\mathbf{RU}_{\mathbb{Q}}(C_m) \cong \tau\,\mathbf{RU}_{\mathbb{Q}}(C_{nm})$$

is again an isomorphism.

Remark 3.3.7. The previous isomorphism allows us to identify

$$\tau\,\mathbf{RU}_{\mathbb{Q}}(C_n) \cong \mathbb{Q}(\zeta_n)$$

as a cyclotomic field extension for all non-negative integers n with $\mathrm{Out}(C_n)$ again acting as the Galois group. However, we will only need to know this for cyclic p-groups.

Since $\tau\,\mathbf{RU}$ is concentrated at cyclic groups, we can further simplify the setting:

Lemma 3.3.8. *Restriction along the inclusion* $\iota : \mathrm{Out}_{\mathrm{cyc}} \hookrightarrow \mathrm{Out}$ *induces isomorphisms*

$$\mathrm{Ext}^n_{\mathrm{Out}^{\mathrm{op}}\text{-}\mathrm{mod}_{\mathbb{Q}}}(X, \tau\,\mathbf{RU}_{\mathbb{Q}}) \cong \mathrm{Ext}^n_{\mathrm{Out}^{\mathrm{op}}_{\mathrm{cyc}}\text{-}\mathrm{mod}_{\mathbb{Q}}}(\iota^* X, \iota^*(\tau\,\mathbf{RU}_{\mathbb{Q}}))$$

on all Ext-*groups.*

Proof. The right Kan extension ι_* simply extends by 0 to non-cyclic groups. As we saw in the preceeding discussion, $\tau\,\mathbf{RU}_{\mathbb{Q}}$ is concentrated at cyclic groups and hence it is right-induced in the sense that the unit map $\tau\,\mathbf{RU}_{\mathbb{Q}} \to \iota_*\iota^*(\tau\,\mathbf{RU}_{\mathbb{Q}})$ is an isomorphism. Both restriction and right Kan extension are exact functors, hence ι^* preserves projective resolutions and the claim now follows by adjointness. $\square$

To proceed we need a more convenient description of $\tau\,\mathbf{RU}_{\mathbb{Q}}$. The Normal Basis Theorem in Galois theory states that the extension field in a finite Galois extension is isomorphic as a representation of the Galois group to the regular representation over the subfield. In the case of cyclotomic field extensions these identifications can be made compatible with the $\mathrm{Out}^{\mathrm{op}}_{\mathrm{cyc}}$-functoriality.

Definition 3.3.9. We define $\mathbb{Q}[\mathrm{Out}(-)]$ as the $\mathrm{Out}_{\mathrm{cyc}}^{\mathrm{op}}$-module given by the collection of regular representations. The restriction map associated to the preferred projection $p : C_{nk} \twoheadrightarrow C_n$ is the morphism

$$p^* : \mathbb{Q}[\mathrm{Out}(C_n)] \to \mathbb{Q}[\mathrm{Out}(C_{nk})]$$

obtained from the surjection $\mathrm{Out}(C_{nk}) \twoheadrightarrow \mathrm{Out}(C_n)$ by summation over the fibers. Since $\mathrm{Out}(C_{nk})$ acts via this on $\mathbb{Q}[\mathrm{Out}(C_n)]$, this map is automatically equivariant. In additive notation, it sends a basis element $j \in (\mathbb{Z}/n\mathbb{Z})^\times \cong \mathrm{Out}(C_n)$ to the sum over all basis elements $\tilde{j}$ such that $\tilde{j} \in (\mathbb{Z}/nk\mathbb{Z})^\times$ reduces to j mod n.

Proposition 3.3.10. *There is an isomorphism of* Out_{cyc}^{op}*-modules*

$$\mathbb{Q}[\mathrm{Out}(-)] \cong \tau\, \mathbf{RU}_{\mathbb{Q}}\,.$$

Proof. We first construct the isomorphism over cyclic p-groups where one can write down a normal basis explicitly. The element $X + X^p + \cdots + X^{(p^{k-1})} \in \mathbb{Q}(\zeta_{p^k})$, $X = \zeta_{p^k}$ generates the cyclotomic field extension as a module over $\mathrm{Out}(C_{p^k})$, which one checks by direct computation. It follows that the $\mathrm{Out}(C_{p^k})$-equivariant map

$$\phi_{p^k} : \mathbb{Q}[\mathrm{Out}(C_{p^k})] \xrightarrow{\frac{-1}{p^{k-1}}(X+X^p+\cdots+X^{(p^{k-1})})} \mathbb{Q}(\zeta_{p^k})$$

is an isomorphism because both sides have the same rank. In the degenerate case $k = 0$ we take for ϕ_0 the canonical identification sending the identity to 1.

We now check compatibility with restrictions. Under the summation of fibers map the unit element $1 \in \mathrm{Out}(C_{p^k})$ maps to the sum over the elements $1 + lp^k \in \mathrm{Out}(C_{p^{k+1}})$ for $0 \leq l \leq p-1$, where we have used additive notation. Letting these act as elements in the Galois group of $\mathbb{Q}(\zeta_{p^{k+1}})$ and writing $Y = \zeta_{p^{k+1}}$, we get the equality

$$(1 + lp^k)^*(Y + Y^p + \cdots + Y^{p^k}) = Y \cdot Y^{lp^k} + Y^p + Y^{p^2} + \cdots + Y^{p^k}.$$

After summation over l this becomes $p \cdot (Y^p + Y^{p^2} + \cdots + Y^{p^k})$ because the terms Y^{lp^k} add up to the minimal polynomial and hence do not contribute. But this is just the restriction of $p \cdot (X + X^p + \cdots + X^{(p^{k-1})})$. So the scaling ensures that for varying $k \geq 0$ the ϕ_{p^k} commute with the restriction maps, the sign is needed for $k = 0$.

We will now assemble these partial morphisms into a single isomorphism of Out_{cyc}^{op}-modules. Let us abbreviate $R = \tau\, \mathbf{RU}_{\mathbb{Q}}$. Given coprime integers n and m, suppose that we have already constructed partial morphisms for these, i.e. maps $\phi_k : \mathbb{Q}[\mathrm{Out}(C_k)] \to R(C_k)$ for all k dividing either n or m which commute with restrictions and such that $\phi_1 = 1$ is the inclusion of the unit element in $R(C_1)$. We will only use that the restrictions of R are maps of commutative rings and that the multiplication maps $R(C_n) \otimes R(C_m) \to R(C_{nm})$ are bijective. The isomorphism ϕ_{nm} is now defined as the composite

$$\mathbb{Q}[\mathrm{Out}(C_{nm})] \cong \mathbb{Q}[\mathrm{Out}(C_n)] \otimes \mathbb{Q}[\mathrm{Out}(C_m)] \xrightarrow{\phi_n \otimes \phi_m} R(C_n) \otimes R(C_m) \xrightarrow{\cong} R(C_{nm}),$$

where the first map is induced by the canonical decomposition $\mathbb{Z}/nm\mathbb{Z} \cong \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$. This definition does not depend on the order of n and m because of the commutativity of R. Furthermore the condition $\phi_0 = 1$ ensures that in the case $n = 1$ or $m = 1$ we recover the original map. Replacing n and m by one of their divisors respectively in the above composite then defines ϕ for all divisors of nm. This is compatible with restrictions because the initially defined maps are and so we have defined a partial morphism for nm.

Finally we remark that this construction is independent of the decomposition into a product of two coprime integers. Considering the prime factorization of an integer n we see that ϕ_n is just given as the analogous construction for several tensor factors applied to the initially defined maps for prime powers. □

This proposition allows us to identify maps into $\tau\,\mathbf{RU}_{\mathbb{Q}}$ as a certain inverse limit over the poset of natural numbers with partial order given by the divisibility relation: If X is an $\mathrm{Out}_{\mathrm{cyc}}^{\mathrm{op}}$-module, we denote by $X^{\vee}$ the inverse system obtained after forming $\mathbb{Q}$-linear duals in each level and forgetting all group actions. The structure maps are defined by precomposition with the preferred restriction maps.

Proposition 3.3.11. *There is a natural isomorphism*

$$\mathrm{Hom}_{\mathrm{Out}_{\mathrm{cyc}}^{\mathrm{op}}\text{-}\mathrm{mod}_{\mathbb{Q}}}(X, \mathbb{Q}[\mathrm{Out}(-)]) \cong \varprojlim_{(\mathbb{N},|)}(X^{\vee})$$

which sends $X \to \mathbb{Q}[\mathrm{Out}(-)]$ to the collection of linear forms

$$(\phi_n : X(C_n) \to \mathbb{Q}[\mathrm{Out}(C_n)] \xrightarrow{\mathrm{pr}_e} \mathbb{Q})_{n\in\mathbb{N}}.$$

Proof. For any finite group, giving an equivariant map into the regular representation is equivalent to specifying the linear form obtained by projecting to the summand of the neutral element. Hence a map $\phi : X \to \mathbb{Q}[\mathrm{Out}(-)]$ is uniquely determined by the linear forms $\phi_n \in X(C_n)^{\vee}$. The condition that ϕ is a natural transformation translates into the condition that these linear forms restrict to each other because the summation over the fibers map commutes with the projection to the summand of the neutral element in $\mathbb{Q}[\mathrm{Out}(-)]$. This means that the collection $\{\phi_n\}$ forms an element in the inverse limit. □

The functor $(-)^{\vee}$ is clearly exact and turns sums into products. It sends enough projectives to injectives, e.g. those of the form $P_V = V \otimes_{\mathrm{Out}(C_n)} \mathbb{Q}[\mathrm{Out}(-, C_n)]$ for an $\mathrm{Out}(C_n)$-representation V. Indeed P_V is just the constant functor V at cyclic groups with order divisible by n and it vanishes elsewhere. Hence we can also identify the higher derived functors:

Corollary 3.3.12. *The map of the previous proposition induces natural isomorphisms*

$$\mathrm{Ext}^n_{\mathrm{Out}_{\mathrm{cyc}}^{\mathrm{op}}\text{-}\mathrm{mod}_{\mathbb{Q}}}(X, \mathbb{Q}[\mathrm{Out}(-)]) \cong \varprojlim_{(\mathbb{N},|)}{}^{n}(X^{\vee})$$

for all $n \geq 0$.

By combining this with the equivalence τ and the identification $\tau\,\mathbf{RU}_{\mathbb{Q}} \cong \mathbb{Q}[\mathrm{Out}(-)]$ we obtain the theorem we wished to show:

$$\begin{aligned}\mathrm{Ext}^k_{\mathcal{GF}}(F, \mathbf{RU}_{\mathbb{Q}}) \cong \mathrm{Ext}^k_{\mathrm{Out}^{\mathrm{op}}\text{-}\mathrm{mod}_{\mathbb{Q}}}(\tau F, \tau\,\mathbf{RU}_{\mathbb{Q}}) &\cong \mathrm{Ext}^k_{\mathrm{Out}^{\mathrm{op}}_{\mathrm{cyc}}\text{-}\mathrm{mod}_{\mathbb{Q}}}(\tau F, \tau\,\mathbf{RU}_{\mathbb{Q}})\\ &\cong \mathrm{Ext}^k_{\mathrm{Out}^{\mathrm{op}}_{\mathrm{cyc}}\text{-}\mathrm{mod}_{\mathbb{Q}}}(\tau F, \mathbb{Q}[\mathrm{Out}(-)])\\ &\cong \varprojlim_{(\mathbb{N},|)}{}^k (\tau F)^\vee\\ &\cong \varprojlim_{(\mathbb{N},|)}{}^k (F(C_n)^\vee_{\mathrm{tr}}).\end{aligned}$$

Finally we observe that $(\mathbb{N}, |)$ contains the sequential poset $\mathbb{N}$ as a cofinal subset via the factorials. Restriction to this subset is exact and the left Kan extension takes an inverse system $\{X_{k!}\}_{k\in\mathbb{N}}$ defined over the factorials to the inverse system defined for all integers n by $X_n = X_{k!}$ where k is minimal such that n divides $k!$. Hence it is also exact and thus we actually only have to compute sequential limits. This allows us to deduce the corollary mentioned in the beginning. There is only a potential $\varprojlim^1$-term for sequential systems and this gives the first part

$$\mathrm{Ext}^n_{\mathcal{GF}}(F, \mathbf{RU}_{\mathbb{Q}}) = 0,\ n \geq 2.$$

The inverse system associated to $X = \mathbb{Q}[\mathrm{Out}(-)]$ only consists of surjective maps and this implies the second part

$$\mathrm{Ext}^1_{\mathcal{GF}}(\mathbf{RU}_{\mathbb{Q}}, \mathbf{RU}_{\mathbb{Q}}) = 0.$$

3.4 Comparison of ring spectra

So far we have obtained an identification of rational global homotopy theory with an algebraic model that is multiplicatively well behaved on the level of homotopy categories. In this section will refine this multiplicative comparison, that is, we give algebraic models for the homotopy theory of rational (commutative) ring spectra from the global perspective. Our strategy will be to carry out the homotopy theoretic analogue of the following classical bit of category theory:

We recall that a monad on a category $\mathcal{C}$ consists of a functor $T : \mathcal{C} \to \mathcal{C}$ together with natural transformations $\eta : \mathrm{Id}_{\mathcal{C}} \to T$ and $\mu : T^2 = T \circ T \to T$ making the triple (T, μ, η) a monoid object with respect to composition in the category of endofunctors $F(\mathcal{C}, \mathcal{C})$. Now let $F : \mathcal{C} \to \mathcal{D}$ be a functor between categories, S a monad on $\mathcal{C}$, and T a monad on $\mathcal{D}$ such that both preserve reflexive coequalizers (coequalizers of pairs admitting a common section). Suppose that F comes with a natural transformation $\lambda : T \circ F \to F \circ S$ that is compatible with the units and multiplications (in the example of the tensor algebra F would be a lax monoidal functor). Then F restricts to a functor $\tilde{F} : \mathcal{C}_S \to \mathcal{D}_T$ between algebra objects over these monads. If $F : \mathcal{C} \xrightarrow{\simeq} \mathcal{D}$ is an equivalence, we would also like the restricted functor to be an equivalence. This is the case if the natural transformation relating the monads is an isomorphism. We briefly review the argument:

Under the above assumptions, the composition

$$\mathrm{Hom}_S(SX,Y)\cong \mathrm{Hom}_{\mathcal{C}}(X,Y)\xrightarrow{F}\mathrm{Hom}_{\mathcal{D}}(FX,FY)\cong \mathrm{Hom}_T(TFX,FY)$$

is bijective and agrees up to the isomorphism λ with the map induced by $\tilde{F}$. Now, for any algebra X over a monad S there is a canonical (reflexive) coequalizer presentation $S^2X \rightrightarrows SX \to X$. This allows us to express the set of algebra homomorphisms as an equalizer

$$\mathrm{Hom}_S(X,Y)\to \mathrm{Hom}_S(SX,Y)\rightrightarrows \mathrm{Hom}_S(S^2X,Y)$$

of morphisms out of free algebras. Under the natural isomorphism λ the functor $\tilde{F}$ takes the above free S-algebra presentation of X to the corresponding free T-algebra presentation of FX. Comparing the resulting equalizer diagrams of morphisms sets shows that $\tilde{F}$ is fully faithful. To see that it is essentially surjective we observe that all free algebras are in the essential image of $\tilde{F}$, in fact $\tilde{F}$ restricts to an equivalence between free algebras. So in the canonical coequalizer presentation $T^2Y \rightrightarrows TY \to Y$ of a T-algebra Y we can lift the entire diagram of free algebras with respect to $\tilde{F}$ and write Y as a coequalizer $\tilde{F}X_1 \rightrightarrows \tilde{F}X_2$.

In the setting of homotopical algebra one has to replace morphism sets by mapping spaces and reflexive coequalizers by bar resolutions. We recall that if X is a T-algebra, the associated *bar resolution* $B_\bullet X$ is the augmented simplicial T-algebra with $B_nX = T^{n+1}X$ and simplicial structure maps induced from those of the monad T and the T-algebra structure on X. Moreover, the underlying augmented simplicial object in $\mathcal{C}$ is split (i.e. it admits extra degeneracies). This ensures that the augmentation induces a simplicial homotopy equivalence $B_\bullet X \to cX$ to the constant simplicial object at X and hence a homotopy equivalence $|B_\bullet X| \simeq X$ on geometric realisations, provided that $\mathcal{C}$ is tensored over simplicial sets (so that the geometric realisation can even be defined). We refer to [JN14] for more details.

3.4.1 Associative and E_∞-ring spectra

We now restrict our considerations to more special monads on orthogonal spectra, namely those arising from (topological) operads. We do not recall the full definition here, only that an operad $\mathcal{O}$ consists of a collection of Σ_n-spaces $\mathcal{O}(n)$ for $n \geq 0$ and extra structure maps such that $T_{\mathcal{O}}X = \bigvee_{n\geq 0}\mathcal{O}(n)_+\wedge_{\Sigma_n} X^{\wedge n}$ defines a monad (in our case on orthogonal spectra). The examples of interest in this section are the associative operad $\mathbf{Ass} = \{\Sigma_n\}$ and E_∞-operads.

Lemma 3.4.1. *The monad $T_{\mathcal{O}}$ commutes with geometric realizations. In particular, the geometric realization of a simplicial $T_{\mathcal{O}}$-algebra can be computed in the underlying category of spectra.*

Proof. For every $n \geq 0$ the n-fold smash product $(-)^{\wedge n}$ commutes with geometric realisations. A proof for the analogous statement in orthogonal spaces is given in [Sch17b, II.1] and the same argument applies here (also see [EKMM97, X.1]). Now, by [JN14, 3.11] this already suffices. □

To ensure that bar resolutions are homotopically well-behaved, we now additionally assume that the collection $\{\mathcal{O}(n)\}$ consists of free Σ_n-CW complexes (more generally we could take Σ-cofibrant operads, cf. [BM03]), and that the first two levels $\mathcal{O}(0) = \mathcal{O}(1) = *$ are trivial. The results of [BM03] also show the existence the transferred model structure on $T_{\mathcal{O}}$-algebras.

Proposition 3.4.2. *Let R be a cofibrant $T_{\mathcal{O}}$-algebra. Then R can be written as the homotopy colimit of its bar resolution*:

$$R \simeq \operatorname{hocolim}_{\Delta^{\mathrm{op}}} B_\bullet R$$

Proof. For cofibrant spectra M, the unit $M \to T_{\mathcal{O}} M$ is a cofibration and the inclusion of a summand. By [JN14, 4.10], this implies that $B_\bullet R$ is Reedy-cofibrant, hence its geometric realization (formed in $T_{\mathcal{O}}$-algebras) models the homotopy colimit. On the other hand we can also compute the geometric realization in orthogonal spectra by the previous lemma and this is equivalent to R since $B_\bullet R$ is split augmented. □

For the use of bar resolutions in the setting of chain complexes, we refer to [Har10] (in particular Theorem 1.8, also see [Fre16]). We only remark here that the situation is simpler because homotopy colimits are modelled by totalization of bicomplexes, which is homotopical.

Associative ring spectra

As we have seen, geometric fixed points are lax symmetric monoidal and thus induce a functor

$$C \circ \Phi : \mathrm{Ass}(\mathrm{Sp}^{\mathrm{O}}) \longrightarrow \mathrm{Epi}^{\mathrm{op}}\text{-}\mathrm{DGA}_{\mathbb{Q}}$$

from associative ring spectra to $\mathrm{Epi}^{\mathrm{op}}$-diagrams in rational differential graded algebras. In order to further move to $\mathrm{Out}^{\mathrm{op}}$-diagrams, we again perform a left Kan extension, now with values in DGA's.

Definition 3.4.3. The derived comparison functor Γ_{Ass} is the composite

$$\mathrm{Ho}(\mathrm{Ass}(\mathrm{Sp}^{\mathrm{O}})_{\mathrm{gl},\mathbb{Q}}) \xrightarrow{C\circ\Phi} \mathrm{Ho}(\mathrm{Epi}^{\mathrm{op}}\text{-}\mathrm{DGA}_{\mathbb{Q}}) \xrightarrow{\mathbb{L}\mathrm{an}} \mathrm{Ho}(\mathrm{Out}^{\mathrm{op}}\text{-}\mathrm{DGA}_{\mathbb{Q}}),$$

where $\mathbb{L}\mathrm{an}$ denotes the derived left Kan extension (cf. Remark 2.1.4).

We write $T = T_{\mathbf{Ass}}$ for the tensor algebra on orthogonal spectra $TX = \bigvee_{n\geq 0} X^{\wedge n}$, the monad associated to the associative operad **Ass**. Then T-algebras are the same as associative ring spectra (and similarly for diagrams in chain complexes).

Theorem 3.4.4. *Geometric fixed points induce an equivalence*

$$\Gamma_{\mathrm{Ass}} : \mathrm{Ho}(\mathrm{Ass}(\mathrm{Sp}^{\mathrm{O}})_{\mathrm{gl},\mathbb{Q}}) \xrightarrow{\simeq} \mathrm{Ho}(\mathrm{Out}^{\mathrm{op}}\text{-}\mathrm{DGA}_{\mathbb{Q}})$$

between the homotopy theories of associative global ring spectra and rational $\mathrm{Out}^{\mathrm{op}}$*-differential graded algebras.*

Proof. The functor $C \circ \Phi$ is homotopically strong monoidal, in other words it commutes with tensor algebras $TC\Phi X \xrightarrow{\simeq} C\Phi TX$ up to equivalence on cofibrant spectra X. Hence for free algebras, it induces an equivalence on derived mapping spaces

$$\begin{array}{ccc} \mathrm{map}_{\mathrm{Ass}(\mathrm{Sp}^{\mathrm{O}})_{\mathbb{Q}}}(TX, S) & \xrightarrow{\simeq} & \mathrm{map}_{\mathrm{Epi}^{\mathrm{op}}\text{-}\mathrm{DGA}_{\mathbb{Q}}}(TC\Phi X, C\Phi S) \\ \wr\| & & \wr\| \\ \mathrm{map}_{\mathrm{Sp}^{\mathrm{O}}_{\mathbb{Q}}}(X, S) & \xrightarrow{\simeq} & \mathrm{map}_{\mathrm{Epi}^{\mathrm{op}}\text{-}\mathrm{Ch}_{\mathbb{Q}}}(C\Phi X, C\Phi S) \end{array}$$

By the previous proposition, any associative ring R can be resolved by such and so we can write

$$\begin{aligned} \mathrm{map}_{\mathrm{Ass}(\mathrm{Sp}^{\mathrm{O}})_{\mathbb{Q}}}(R, S) &\simeq \mathrm{holim}_{n\in\Delta} \mathrm{map}_{\mathrm{Ass}(\mathrm{Sp}^{\mathrm{O}})_{\mathbb{Q}}}(T^{n+1}R, S) \\ &\simeq \mathrm{holim}_{n\in\Delta} \mathrm{map}_{\mathrm{Epi}^{\mathrm{op}}\text{-}\mathrm{DGA}_{\mathbb{Q}}}(T^{n+1}C\Phi R, C\Phi S) \\ &\simeq \mathrm{map}_{\mathrm{Epi}^{\mathrm{op}}\text{-}\mathrm{DGA}_{\mathbb{Q}}}(C\Phi R, C\Phi S) \end{aligned}$$

to conclude that $C \circ \Phi$ is homotopically fully faithful.

To show essential surjectivity up to equivalence, we consider a homology $\mathrm{Out}^{\mathrm{op}}$-DGA D and its associated bar resolution. Each term is in the essential image of $C \circ \Phi$ since the underlying functor is essentially surjective and so the entire simplicial diagram $B_\bullet X \simeq (C \circ \Phi)(R_\bullet)$ can be lifted under $C \circ \Phi$ (see the lemma below). This allows us to write D up to equivalence as

$$D \simeq \mathrm{hocolim}_{\Delta^{op}} T^{n+1}D \simeq \mathrm{hocolim}_{\Delta^{op}}(C \circ \Phi)R_\bullet \simeq (C \circ \Phi)(\mathrm{hocolim}_{\Delta^{op}} R_\bullet).$$

Strictly speaking, the homotopy colimit on the right is computed in the underlying category of spectra, but this is enough to recognize D as the image of some ring spectrum (this follows from modelling the homotopy colimit as a geometric realisation).

Finally, we combine this with the Quillen equivalence below to conclude that Γ_{Ass} is an equivalence. $\square$

Proposition 3.4.5 (cf. Remark 3.2.9)**.** *The Quillen-adjunction*

$$\mathrm{Lan} : \mathrm{Epi}^{\mathrm{op}}\text{-}\mathrm{DGA}_{\mathbb{Q}} \rightleftarrows \mathrm{Out}^{\mathrm{op}}\text{-}\mathrm{DGA}_{\mathbb{Q}}$$

restricts to an equivalence with those $\mathrm{Epi}^{\mathrm{op}}$*-DGA's whose homology groups are* $\mathrm{Out}^{\mathrm{op}}$*-modules.*

Proof. We note that the left Kan extension commutes with the tensor algebra:

$$\mathrm{Lan}^{\mathrm{DGA}}\, TC \cong T\,\mathrm{Lan}\, C$$

for $\mathrm{Epi}^{\mathrm{op}}$-complexes C. Now if C is a homology $\mathrm{Out}^{\mathrm{op}}$-complex, then this implies $\mathrm{Lan}^{\mathrm{DGA}}\, TC \simeq TC$. Again using bar resolutions, we can compute the left derived functor on homology $\mathrm{Out}^{\mathrm{op}}$-DGA's

$$\mathbb{L}\mathrm{an}\, X \simeq \mathrm{hocolim}_{\Delta^{\mathrm{op}}} \mathbb{L}\mathrm{an}\, T^{n+1}X \simeq \mathrm{hocolim}_{\Delta^{\mathrm{op}}} T^{n+1}X \simeq X$$

and conclude that derived unit and counit of the adjunction are equivalences. $\square$

Lemma 3.4.6. *Let I be small category, $F : \mathcal{C} \to \mathcal{D}$ a homotopical and fully faithful functor between combinatorial model categories admitting functorial replacements, and $X \in I\text{-}\mathcal{D}$ an I-diagram in $\mathcal{D}$. If all the objects $X_i \simeq FY_i$ are contained in the essential homotopical image of F, then so is the whole diagram.*

Proof. Since we are not aware of a classical reference, we refer to the theory of infinity categories [Lur09]. The corresponding statement in that setting is clear since equivalences are invertible and hence induce equivalences on diagram categories. In our case the underlying infinity categories of the diagram categories in question are equivalent to the diagram categories formed in infinity categories (this follows form [Lur09, Proposition 4.2.4.4], see the proof of [LNS16, Proposition 2] for the argument). □

E_∞-ring spectra

The previous discussion also applies to the case of (naive) E_∞-ring spectra, i.e. algebras over an E_∞-operad E, the levels of which we generically denote $E\Sigma_n$. We identify the geometric fixed points of the associated monad T_E:

The corresponding space-level statement implies that the geometric fixed point functor commutes with homotopy orbits

$$\Phi^G(E\Sigma_{n+} \wedge_{\Sigma_n} X^{\wedge n}) \cong (E\Sigma_n)_+ \wedge_{\Sigma_n} \Phi^G(X^{\wedge n})$$

up to canonical isomorphism (this is a special case of Lemma 2.1.1). For a K-spectrum Y, K a finite group, the inclusion $Y \to EK_+ \wedge_K Y$ into the homotopy orbits associated to a chosen point of EK induces an isomorphism $(\pi_* Y)_{\mathbb{Q}}/G \cong \pi_*(EK_+ \wedge_K Y)_{\mathbb{Q}}$ on rational homotopy groups. Hence the natural zigzag

$$C(Y)/K \leftarrow (C(EK_+) \otimes C(Y))/K \to C(EK_+ \wedge_K Y)$$

consists of quasi-isomorphisms, where the left map collapses EK to a point. Since the composition $C \circ \Phi$ is homotopically monoidal, we can combine this to obtain a natural equivalence

$$(C \circ \Phi) \circ T_E \simeq \mathbb{P} \circ (C \circ \Phi)$$

with the symmetric algebra $\mathbb{P}D = \bigoplus_{n \geq 0} D^{\otimes n}/\Sigma_n$. The category of $\mathbb{P}$-algebras is the category of commutative differential graded algebras. The same proof as in the associative case yields:

Theorem 3.4.7. *Geometric fixed points induce an equivalence*

$$\Gamma_{E_\infty} : \mathrm{Ho}(E_\infty(\mathrm{Sp}^{\mathrm{O}})_{\mathrm{gl},\mathbb{Q}}) \xrightarrow{\simeq} \mathrm{Ho}(\mathrm{Out}^{\mathrm{op}}\text{-}\mathrm{CDGA}_{\mathbb{Q}})$$

between the homotopy theories of rational E_∞-global ring spectra and rational $\mathrm{Out}^{\mathrm{op}}$*-commutative differential graded algebras.*

3.4.2 Commutative ring spectra

Rational global power functors

This subsection is devoted to a proof of the equivalence between rational global power functors and commutative $\mathrm{Out}^{\mathrm{op}}_{\mathrm{norm}}$-rings (both discussed towards the end of Section 2.3.4). These are categories of algebras over certain monads, and as recalled in the introduction of this section it will thus suffice to show that τ (viewed as a functor from global Green functors to commutative $\mathrm{Out}^{\mathrm{op}}$-rings) preserves these.

Proposition 3.4.8 ([Sch17b, V.2.21], [Sch17b, V.2.14.(i)]). (i) *The forgetful functor from the category of global power functors to global Green functors admits a left adjoint L^{pow}. Moreover, the resulting adjunction is monadic.*

(ii) *Colimits are created in the underlying category of global Green functors, that is, the underlying endofunctor L^{pow} of the monad creating global power functors commutes with colimits.*

The analogous statement for $\mathrm{Out}^{\mathrm{op}}_{\mathrm{norm}}$-diagrams follows by inspection. Here the free functor is given by left Kan extension along $\mathrm{Out}^{\mathrm{op}} \to \mathrm{Out}^{\mathrm{op}}_{\mathrm{norm}}$ and monadicity follows because it is essentially surjective:

Proposition 3.4.9. *Let $\mathcal{C}$ be a cocomplete category. The category of $\mathrm{Out}^{\mathrm{op}}_{\mathrm{norm}}$-diagrams in $\mathcal{C}$ is monadic over $\mathrm{Out}^{\mathrm{op}}$-diagrams with respect to the (left Kan extension, restriction)-adjunction and colimits are computed underlying.*

We will only need to explicitly know L^{pow} in the following case:

Example 3.4.10.

- The free global power functor ([Sch17b, V.1.19]) at the group G is the global Green functor

$$C_G = L^{\mathrm{pow}}\mathbb{A}(G,-) \cong \bigoplus_{m\geq 0} \mathbb{A}(\Sigma_m \wr G,-)$$

  with multiplication induced by the block sum inclusions $\Sigma_m\wr G\times\Sigma_n\wr G \hookrightarrow \Sigma_{m+n}\wr G$. The global power structure is uniquely determined by $P^m 1_G = 1_{\Sigma_m\wr G}$, where $1_G \in \mathbb{A}(G,G)$ is the identity element. It is a universal element letting C_G corepresent evaluation at G in the category of global power functors.

- We recall the canonical isomorphism $\mathbb{Z}\{\mathrm{Rep}(-,G)\} \xrightarrow{\cong} \tau(\mathbb{A}(G,-))$, $\alpha \mapsto \alpha^*$ of $\mathrm{Out}^{\mathrm{op}}$-modules. Applied to the summands of the free global power functor C_G, this gives an identification

$$\mathbb{Z}\{\coprod_{m\geq 0} \mathrm{Rep}(-,\Sigma_m \wr G)\} \xrightarrow{\cong} \tau(C_G)$$

  of commutative $\mathrm{Out}^{\mathrm{op}}$-rings and after descending the power structure on C_G to $\tau(C_G)$ one obtains the following uniquely determined norms on the left hand side:

Let $K \leq L$ be a subgroup inclusion of index n and $\alpha : K \to \Sigma_m \wr G$. Then $N_H^G(\alpha)$ is the conjugacy class of the composite

$$L \to \Sigma_n \wr K \xrightarrow{\Sigma_n \wr \alpha} \Sigma_n \wr (\Sigma_m \wr G) \hookrightarrow \Sigma_{nm} \wr G.$$

Here the last map sends an element $(\tau; (\sigma_1; g_1), \dots, (\sigma_n, g_n))$ to $(\tau_* \circ (\sigma_1 + \cdots + \sigma_n), (g_1, \dots, g_n))$, where the g_i are m-tuples of elements in G, $(g_1, \dots, g_n)$ denotes their concatenation, and $\tau_* \circ (\sigma_1 + \cdots + \sigma_n) \in \Sigma_{nm}$ is the block-sum of the σ_i followed by permuting the blocks according to τ (see [Sch17b, II.2.3] for a more detailed discussion).

The next lemma is the main combinatorial input for showing that τ preserves free functors.

Lemma 3.4.11. *Let G be a finite group and $\{H\}_{(H \leq G)}$ a choice of representatives for the conjugacy classes of subgroups of G. There are canonical bijections*

$$\coprod_{m \geq 0} \mathrm{Rep}(-, \Sigma_m \wr G) \xrightarrow{\cong} \mathbb{A}^+(G, -) \xleftarrow{\cong} \mathbb{N}\{\coprod_{(H \leq G)} \mathbb{A}^+_{\mathrm{tr}}(H, -)/WH\}$$

of commutative $\mathbb{A}^+_{\mathrm{tr}}$-monoids. Here the left map sends $\alpha : K \to \Sigma_m \wr G$ to the biset $\alpha^*(G \sqcup \ldots \sqcup G)$ and the right map is the additive extension of the induction maps $M \mapsto M \times_H G$.

Proof. We first consider the left map. It commutes with inflations because precomposing $\alpha : K \to \Sigma_m \wr G$ with a morphism $\beta : H \to K$ corresponds to restricting the biset $\alpha^*(G \sqcup \ldots \sqcup G)$ from the left along β, the result of which is isomorphic to the balanced product over K with $\beta^* K$. Now let $K \leq L$ be a subgroup inclusion. By decomposing L into its right K-orbits one sees that the norm $N_K^L(\alpha^*(\sqcup_{i=1}^m G)) = L \times_{K,\alpha} (\sqcup_{i=1}^m G)$ consists of nm right G-summands such that the left L-action is given by the composite defining $N_K^L(\alpha)$ described above.

To see that the map is bijective we observe that a finite G-free (K, G)-biset M is isomorphic to an m-fold disjoint union $M \cong G \sqcup \ldots \sqcup G$ for a uniquely determined $m \geq 0$. We also recall that the right G-automorphism group of this is the wreath product $\Sigma_m \wr G$. Thus a left K-action that commutes with the right G-action is the same as a group homomorphism $\alpha : K \to \Sigma_m \wr G$ and a different choice of identifying M has the effect of conjugating α. Hence we have obtained a well-defined inverse map.

The right map is defined entirely in terms of bisets and so it is clear that it is a morphism of commutative $\mathbb{A}^+_{\mathrm{tr}}$-monoids. To show that it is bijective one uses the decompositions $\mathbb{A}^+_{\mathrm{tr}}(-, K) \cong \sqcup_{K \leq L} \mathrm{Out}(K, -)/W_L K$ (cf. Proposition 2.3.36) and similarly $\mathbb{A}^+(-, K) \cong \oplus_{K \leq L} \mathbb{N}\{\mathrm{Rep}(K, -)\}/W_L K$ to reduce the statement to the known formula $\mathrm{Rep}(-, G) \cong \coprod WH \backslash \mathrm{Out}(-, H)$. □

Proposition 3.4.12. *The isomorphism of the above example identifies $\tau(C_G)$ as the free commutative $\mathrm{Out}^{\mathrm{op}}_{\mathrm{norm}}$-ring on the $\mathrm{Out}^{\mathrm{op}}$-module $\tau\mathbb{A}(G, -) \cong \mathbb{Z}\{\mathrm{Rep}(-, G)\}$.*

Proof. Left Kan extending $\mathrm{Rep}(-, G) \cong \coprod \mathrm{Out}(-, H)/WH$ to an $\mathbb{A}_{\mathrm{tr}}^{+}$-diagram and forming free commutative monoids gives the right hand side in the above lemma. The left hand side agrees after $\mathbb{Z}$-linearization with the description of $\tau(C_G)$ given above and one checks that under the bijections the canonical inclusion $\mathrm{Rep}(-, G) \to \coprod_{m \geq 0} \mathrm{Rep}(-, \Sigma_m \wr G)$ corresponds to the universal arrow. □

Theorem 3.4.13. *The functor τ induces an equivalence between the category of rational global power functors and commutative $\mathrm{Out}^{\mathrm{op}}_{\mathrm{norm}}$-algebras over $\mathbb{Q}$.*

Proof. We need to show that the map $\mathrm{Lan}^{\mathrm{norm}}\, \tau R \to \tau L^{\mathrm{pow}} R$ is an isomorphism for all global Green functors. Now both sides preserve colimits of global Green functors and every Green functor R has a canonical coequalizer presentation $\mathbb{P}^2 R \rightrightarrows \mathbb{P} R \to R$, where $\mathbb{P}$ denotes the symmetric algebra with respect to the box product of global functors. We are thus reduced to the case where $R = \mathbb{P}F$ is the free Green functor on a global functor F. As in any diagram category, global functors can in turn be written as colimits of free global functors. In that case the statement follow from the previous proposition. □

Free commutative ring spectra

After this purely algebraic discussion, we will now investigate the interaction of geometric fixed points and symmetric powers. For this we need to a more concrete description of the various free functors.

Proposition 3.4.14. (i) *The free extension $\mathrm{Lan}^{\mathrm{norm}} X$ (abbreviated by L) of an $\mathrm{Out}^{\mathrm{op}}$-diagram X can be explicitly described by the formula*

$$(LX)(G) = \left(\bigoplus_{H \leq G} X(H) \right) /G \cong \bigoplus_{(H \leq G)} X(H)/WH,$$

where G acts by conjugating subgroups and restricting along conjugation maps. The universal map $\iota : X \to LX$ corresponds to the inclusion of the summand indexed by G. The inflation associated to a surjective homomorphism $\alpha : K \twoheadrightarrow G$ is defined on each summand by the inflation associated to the restricted map $\alpha^{-1}H \twoheadrightarrow H$. Finally, the norm maps are given by including into a bigger sum where the indexing subgroups are simply regarded as subgroups of the larger group.

(ii) *Let $\mathbb{P}LD$ be the free commutative $\mathrm{Out}^{\mathrm{op}}_{\mathrm{norm}}$ - CDGA on an $\mathrm{Out}^{\mathrm{op}}$-complex D (where $\mathbb{P}$ denotes the symmetric algebra in chain complexes applied group-wise). Its value at the finite group G can be identified as*

$$(\mathbb{P}LD)(G) \cong \bigoplus_{(\alpha: G \to \Sigma_m)} \left(\bigotimes_{Gj \in G \backslash \{1, \ldots, m\}} D(\mathrm{Stab}_G(j)) \right) /C(\alpha).$$

The ring structure on the right side is given by concatenation of tensor factors, using the block sum inclusions $\Sigma_m \times \Sigma_n \leq \Sigma_{n+m}$. The isomorphism is uniquely *determined by sending $N_H^G c$, $c \in C(H)$ to the summand indexed by the well-defined conjugacy class of $G \to \mathrm{Aut}(G/H) \cong \Sigma_m$, $m = (G : H)$. We note that there is only one tensor factor $C(H)$, since $G/H \cong \{1, \dots, m\}$ is a transitive G-set and H is the stabilizer subgroup of itself in G/H.*

Proof. The first part completely describes the right side as an $\mathrm{Out}^{\mathrm{op}}$-module with norms and from this it is straightforward to check the universal property. Alternatively, one can also observe that the category with objects the subgroups $H \leq G$ and morphisms G is a final subcategory of the comma category appearing in the standard formula for the left Kan extension.

For the second part, we recall the distributivity formula for symmetric powers: Suppose that $(X_i)_{i \in I}$ is a collection of objects indexed by a finite set I. Then there is a canonical identification

$$\mathbb{P}\left(\bigoplus_{i \in I} X_i\right) \cong \bigoplus_{\alpha \in \mathbb{N}^I} \left(\bigotimes_{i \in I} X_i^{\otimes \alpha_i} / \Sigma_{\alpha_i}\right)$$

We will now use the conjugacy classes of subgroups as the indexing set I and write M_α for the finite G-set $M_\alpha = \coprod_{(H \leq G)} (G/H)^{\sqcup \alpha_H}$ associated to an I-tuple α. Using part (i) in the above formula, we get the following chain of isomorphisms:

$$\begin{aligned}
(\mathbb{P}LD)(G) \cong \mathbb{P}\left(\bigoplus_{(H \leq G)} D(H)/WH\right) &\cong \bigoplus_{\alpha \in \mathbb{N}^I} \left(\bigotimes_{(H \leq G)} D(H)^{\otimes \alpha_H} / (\Sigma_{\alpha_H} \wr WH)\right) \\
&\cong \bigoplus_{\alpha \in \mathbb{N}^I} \left(\bigotimes_{Gm \in G \backslash M_\alpha} D(\mathrm{Stab}(m))\right) / \mathrm{Aut}_G(M_\alpha) \\
&\cong \bigoplus_{[M]} \left(\bigotimes_{Gm \in G \backslash M} D(\mathrm{Stab}(m))\right) / \mathrm{Aut}_G(M) \\
&\cong \bigoplus_{(\alpha : G \to \Sigma_m)} \left(\bigotimes_{Gj \in G \backslash \{1, \dots, m\}} D(\mathrm{Stab}_G(j))\right) / C(\alpha)
\end{aligned}$$

Going from top to bottom, we have used the following: First, the wreath product $(\Sigma_{\alpha_H} \wr WH) \cong \mathrm{Aut}_G((G/H)^{\sqcup \alpha_H})$ is the appropriate automorphism group and H is the stabilizer of itself in G/H. Secondly, $\alpha \mapsto M_\alpha$ determines a bijection with the isomorphism classes of finite G-sets. Finally, these in turn biject with conjugacy classes of group homomorphisms $\alpha : G \to \Sigma_m$, $m \geq 0$. $\square$

We now come to the computation of the geometric fixed points of free commutative ring spectra. We denote by $E_G\Sigma_m$ a universal space for the family $\mathcal{F}(G; \Sigma_m)$ of *graph subgroups* of $G \times \Sigma_m$. We recall that this means that $E_G\Sigma_m$ is a $(G \times \Sigma_m)$-CW complex such that the fixed points

$$(E_G\Sigma_m)^H \simeq \begin{cases} *, & \text{if } H \cap (e \times \Sigma_m) = (e, \mathrm{Id}) \\ \varnothing, & \text{else} \end{cases}$$

for a subgroup $H \leq G \times \Sigma_m$ are trivial if it lies in the family and vanish otherwise. A $(G \times \Sigma_m)$-space is an $\mathcal{F}(G, \Sigma_m)$-space (i.e. all isotropy is contained in the family) iff the Σ_m-action is free.

The symmetric algebra

$$\mathbb{P}X = \bigvee_{n \geq 0} \mathbb{P}^n X = \bigvee_{n \geq 0} X^n / \Sigma_n$$

in orthogonal spectra will also be denoted by $\mathbb{P}$. It has the correct homotopy type for positively cofibrant spectra. In the following computation we need to cofibrantly replace the underlying G-spectrum of X in the positive model structure of [HHR16]. In that case the projection $E_G\Sigma_m \to *$ induces a G-stable equivalence

$$E_G \Sigma_{m+} \wedge_{\Sigma_m} X^{\wedge m} \xrightarrow{\simeq} X^{\wedge m}/\Sigma_m = \mathbb{P}^m(X)$$

between the genuine homotopy orbits and the symmetric power ([HHR16, B.117]). It also ensures that the norms appearing have the correct homotopy type. Now, the underlying G-spectrum of a flat spectrum X is G-flat, i.e. cofibrant in the $\mathbb{S}$-model structure of [Sto11]. To be able to further cofibrantly replace, we need to know that the symmetric powers $\mathbb{P}^n$ are homotopical on G-flat spectra. This follows from [Sto11, Thm 2.3.37], but there is a mistake going back to Mandell-May ([MM02, Lemma III.8.4], also see the discussion in [HHR16, B.120]). Namely, as part of showing that the positive $\mathbb{S}$-model structure lifts to commutative ring spectra, the 'naive' homotopy orbits $E\Sigma_{m+} \wedge_{\Sigma_m} X^{\wedge m}$ are used instead of the 'genuine' ones above. The error can be traced to the proof of [Sto11, Lemma 2.3.34]. However, we are confident that the arguments work if one uses $E_G\Sigma_m$ instead of $E\Sigma_n$. Alternatively, one can also follow the treatment of the global model structure on commutative ring spectra in [Sch17b, V.4]. This avoids the use of universal spaces, instead one has to verify a certain symmetrizability condition of cofibrations. In particular, it would impliy that symmetric powers are homotopical on positively G-flat spectra.

Proposition 3.4.15. *Let X be a positive-flat orthogonal spectrum. Then the canonical map*

$$\mathbb{P}L\underline{\Phi}_* X \xrightarrow{\cong_{\mathbb{Q}}} \underline{\Phi}_* \mathbb{P}X$$

induced by the inclusion $X \to \mathbb{P}X$ is a rational isomorphism.

Proof. At each finite group G the conclusion only depends on the underlying (G-flat) G-spectrum. We replace X by a cofibrant G-spectrum in the model structure of [HHR16] (see the discussion above) and use the G-stable equivalence

$$E_G \Sigma_{m+} \wedge_{\Sigma_m} X^{\wedge m} \xrightarrow{\simeq} X^{\wedge m}/\Sigma_m = \mathbb{P}^m(X)$$

to compute the geometric fixed points. Applying the decompostion formula 2.1.1 level-wise (cf. [Dot17]), we obtain

$$\Phi^G(E_G \Sigma_{m+} \wedge_{\Sigma_m} X^{\wedge m}) \cong \bigvee_{(\alpha: G \to \Sigma_m)} (E_G \Sigma_m)_+^{\Gamma(\alpha)} \wedge_{C(\alpha)} \Phi^G(\alpha^*(X^{\wedge m})).$$

Here the notation α^* refers to pulling back the Σ_m-action on the $(\Sigma_m \times G)$-spectrum $\alpha^*(X^{\wedge m})$, so G acts diagonally on each factor and by permuting the factors according to α. This term can be described as a product of norms

$$\alpha^* X^{\wedge m} = \bigwedge_{\alpha^*\{1,\ldots,m\}} X \cong \bigwedge_{Gj \in G\backslash\{1,\ldots,m\}} N^G_{\mathrm{Stab}(j)} X,$$

which is just a matter of grouping together the smash factors according to the G-orbits of $\{1,\ldots,m\}$ and applying the lemma below. Finally, the geometric fixed points of these are identified via the norm maps $\Phi^H X \xrightarrow{\simeq} \Phi^G N^G_H X$ of Remark 2.3.33.

Putting this all together, we obtain an equivalence

$$\bigvee_{(\alpha: G \to \Sigma_m)} \left(\bigwedge_{Gj \in G\backslash\{1,\ldots,m\}} \Phi^{\mathrm{Stab}(j)} X \right)_{hC(\alpha)} \xrightarrow{\simeq} \Phi^G(\mathbb{P}^m(X))$$

and on homotopy groups this gives the algebraic description of Proposition 3.4.14. On the summand indexed by the homomorphism $\alpha : G \to \Sigma_m$ associated to a subgroup $H \leq G$ this map is the composition

$$\Phi^H X \to \Phi^G N^G_H \cong \Phi^G(\alpha^*(X^{\wedge m})) \to \Phi^G(\mathbb{P}^m(X))$$

and after mapping further to $\Phi^G(\mathbb{P}X)$ this can also be factored as

$$\Phi^H X \to \Phi^H \mathbb{P}(X) \xrightarrow{N^G_H} \Phi^G \mathbb{P}(X).$$

Hence the canonical map from $\mathbb{P} L \underline{\Phi}^G_* X$ induces an isomorphism. $\square$

Lemma 3.4.16. *Let $\alpha : G \to \Sigma_m$ be a transitive group homomorphism, $H = \mathrm{Stab}(1)$, and X a G-orthogonal spectrum. Then there is an 'untwisting' isomorphism*

$$N^G_H X|_H \cong \alpha^* X^{\wedge m}.$$

Proof. Let $\{g_1, \ldots, g_m\}$ be coset representatives with associated homomorphism $\Psi : G \to \Sigma_m \wr H$. This is explicitely described by $\Psi(g) = (\alpha(g); (h_1, \ldots, h_m))$, where the h_i are determined by the formula $gg_i = g_{\alpha(g)i} h_i$. As recalled in Remark 2.3.33, the norm is obtained from the $(\Sigma_m \wr H)$-spectrum $X^{\wedge m}$ by pulling back the action along Ψ. From the description above it follows that Ψ is conjugate to $g \mapsto (\alpha(g); (g, \ldots, g))$ as a homomorphism to $\Sigma_m \wr G$. This determines the same G-action as on $\alpha^* X^{\wedge m}$. $\square$

In light of the previous discussion for (associative) ring spectra one would of course expect the above result on free commutative ring spectra to lead to an equivalence of homotopy theories with rational $\mathrm{Out}^{\mathrm{op}}_{\mathrm{norm}}$-commutative differential graded algebras. Unfortunately, we cannot yet justify the use of bar resolutions in this case. The sphere spectrum $\mathbb{S}$ is not positively cofibrant and so the argument for the Reedy-cofibrancy of $B_\bullet R$ does not work, even though we strongly believe it to have the correct homotopy type for cofibrant commutative ring spectra R.

4 Real-global homotopy theory

In this final chapter we sketch the analogous rational comparison program for *Real-global* homotopy theory. It is not meant to be read in isolation and we think of it as an addendum to the main text. We have chosen a geodesic path in the exposition (for example we do not mention Real-global functors), there is of course much more one could say. The foundational material on Real-global homotopy theory appeared at one point in an earlier version of [Sch17b].

4.1 Real Unitary spectra

There is an obvious notion of unitary spectra, one simply replaces orthogonal with unitary groups in the definition of orthogonal spectra. We discuss a more refined version taking conjugate linear isometries into account. Let $\widehat{\mathbf{L}}^{\mathbb{C}}$ be the topological category of finite dimensional complex hermitian inner product spaces with morphisms

$$\widehat{\mathbf{L}}^{\mathbb{C}}(V,W) = \mathbf{L}^{\mathbb{C}}(V,W) \sqcup \mathbf{L}^{\mathbb{C}}_{\mathrm{conj}}(V,W)$$

the linear and conjugate linear isometries (i.e. $\mathbf{L}^{\mathbb{C}}_{\mathrm{conj}}(V,W) = \mathbf{L}^{\mathbb{C}}(V,\overline{W})$). A *Real unitary space* is a continuous functor $\widehat{\mathbf{L}}^{\mathbb{C}} \to \mathrm{Top}_*$. The *Real unitary indexing* category $\mathrm{U}^{\mathbb{R}}$ is obtained from this by the standard procedure of forming orthogonal complement bundles and passing to Thom spaces, as for orthogonal spectra.

Definition 4.1.1. The category of *Real unitary spectra* is the topological category of continuous based functors $\mathrm{U}^{\mathbb{R}} \to \mathrm{Top}_*$.

Remark 4.1.2. The indexing category $\mathrm{U}^{\mathbb{R}}$ is equivalent to any full subcategory which contains the skeleton $\{\mathbb{C}^n\}$, in particular to the subcategory of *Real* hermitian inner product spaces (those equipped with an antilinear involution). From this one sees that the above is equivalent to the category of Real spectra defined in [HHR16, Appendix B.12.1].

Real spectra are tensored and cotensored over C_2-spaces via the level-wise constructions $(A \wedge X)(-) = A \wedge X(-)$ and $\mathrm{map}(A,X)(-) = \mathrm{map}(A, X(-))$. Conjugate linear isometries act by applying the involution on A in addition to their induced functoriality. The definition of free spectra is again straightforward in the diagrammatic description: We set $F_{G,V}A = \mathrm{U}^{\mathbb{R}}(V,-) \wedge_G A$ for a G-space A, and this corepresents the functor $X \mapsto \mathrm{map}_G(A, X(V))$.

There are comparison functors

$$C_2\text{-}\mathrm{Sp}^{\mathrm{O}} \xrightarrow{c} \mathrm{Sp}^{\mathbb{R}} \xrightarrow{u} C_2\text{-}\mathrm{Sp}^{\mathrm{O}}$$

relating Real unitary spectra to C_2-equivariant orthogonal spectra. The *complexification* of a C_2-equivariant spectrum X evaluates at underlying orthogonal inner product spaces:

$$(cX)(V) = X(uV)$$

for a hermitian inner product space V. Conjugate linear isometries act by applying the C_2-action in addition to their induced functoriality. The *forgetful functor* evaluates a Real unitary spectrum Y at complexifications. To obtain an orthogonal spectrum one has to loop this:

$$(uY)(W) = \Omega^{iW} Y(W_{\mathbb{C}})$$

for an orthogonal inner product space W. The C_2-action is by complex conjugation on iW and $W_{\mathbb{C}}$. The adjoint structure map $\tilde{\sigma}_{V,iV} : X(V) \to \Omega^{iV} Y(V \oplus iV) \cong \Omega^{iV} Y(uV_{\mathbb{C}})$ defines a natural map $X \to ucX$.

Definition 4.1.3. A *Real group* is a group G together with an *augmentation* homomorphism $\epsilon : G \to C_2$ to the cyclic group of order 2. A morphism of Real groups is a group homomorphism lying over C_2.

The elements in the kernel G_{ev} of the augmentation (resp. the complement G_{odd}) will be referred to as the *even* (resp. *odd*) elements. By abuse of notation we will often only refer to the group G and keep the augmentation implicit.

Example 4.1.4. The *extended unitary group* $\hat{\mathrm{U}}(V) = \mathrm{U}^{\mathbb{R}}(V,V)$ of a complex hermitian inner product space is a naturally occurring example of a Real group. The even elements are the linear isometries and the odd elements the conjugate linear isometries.

Definition 4.1.5. Let $G \to C_2$ be a Real group. A Real unitary G-representation is a complex hermitian inner product space V together with a homomorphism

$$G \longrightarrow \hat{\mathrm{U}}(V)$$

of Real groups. More concretely, the even elements of G act on V by linear isometries and the odd elements by conjugate linear isometries.

Example 4.1.6. The standard example of a Real G-representation is the complex regular representation $\rho_G^{\mathbb{C}} = \mathbb{C}\{G\}$ with G-action by left translation, and such that odd elements additionally act by complex conjugation. More generally, the complexification $V_{\mathbb{C}} = V \otimes \mathbb{C}$ of an orthogonal G-representation V naturally admits the structure of a Real G-representation in this way.

Every Real G-representation W embedds into its complexification via $W \hookrightarrow (uW)_{\mathbb{C}}$, $w \mapsto \frac{1}{\sqrt{2}}(w \otimes 1 - iw \otimes i)$. In particular, $\mathcal{U}_G^{\mathbb{R}} = (\rho_G^{\mathbb{C}})^{\infty}$ is a complete Real G-universe.

The *equivariant homotopy groups* of a Real spectrum X are defined analogously to the orthogonal case:

$$\pi_0^G X = \pi_0^{G,\alpha} X = \mathrm{colim}_{V \subset \mathcal{U}_G^{\mathbb{R}}} [S^V, X(V)]_G,$$

where $G \xrightarrow{\alpha} C_2$ is a Real group and V runs over the subrepresentations of the complete G-universe. This definition is again extended to all integer degrees by looping or shifting the spectrum X. A bit more care has to be taken in negative degrees, because the shift $\operatorname{sh} X = X(\mathbb{C} \oplus -)$ is formed with respect to the complex numbers, hence in an equivariantly non-trivial 'direction'. One corrects for this by looping with the sign representation: $\pi_{-k}^G X = \pi_0^G(\Omega^{i\mathbb{R}^k} \operatorname{sh}^k X)$. We observe that there is a natural isomorphism

$$\pi_0^G \alpha^*(uX) \xrightarrow{\cong} \pi_0^{G,\alpha} X$$

induced by mapping the colimit systems to each other via the adjunction isomorphism $[S^V, \Omega^{iV} X(V_{\mathbb{C}})]_G \cong [S^{V_{\mathbb{C}}}, X(V_{\mathbb{C}})]_G$. Here one uses that the complexifications of orthogonal G-representations are cofinal among Real unitary G-representations. This prolongs to all integer degrees, in negative ones via the canonical identification $\operatorname{sh}(uX) = u(\Omega^{i\mathbb{R}} \operatorname{sh} X)$.

Definition 4.1.7. A morphism $f : X \to Y$ of Real spectra is a *Real-global equivalence* if it induces isomorphisms on equivariant homotopy groups for all finite Real groups.

We now briefly discuss the relevant model structure on Real spectra. While this is of course needed to properly set up the homotopy theory, the detailed arguments would be very disconnected from the rest of this chapter. Since its main purpose is to explain an analogous comparison of the rationalized Real-global homotopy category with an algebraic model, we will therefore not spell out proofs and mostly just give the necessary statements. A detailed treatment for orthogonal spectra, emphasizing the use of latching constructions, can be found in [Sch17b, IV.3].

A morphism $f : X \to Y$ of Real spectra is a *Real-global level equivalence* (respectively fibration) if the maps $f(\mathbb{C}^m) : X(\mathbb{C}^m) \to Y(\mathbb{C}^m)$ are $\hat{\mathrm{U}}(m)_{\mathrm{fin}}$-equivalences (respectively fibrations), i.e. when restricted to fix points for all finite subgroups. This uniquely determines the *global level model structure* on Real spectra. It is proper, topological, and cofibrantly generated with generating cofibrations and acyclic cofibrations $I_{\mathrm{lvl}}^{\mathbb{R}} = \{G_m(i)\}$ and $J_{\mathrm{lvl}}^{\mathbb{R}} = \{G_m(j)\}$, where i and j range over the generating cofibrations for the $\hat{\mathrm{U}}(m)_{\mathrm{fin}}$-model structure. From this one obtains the stable model structure as a left Bousfield localization. Following the usual approach for spectra ([MMSS01], [MM02]), this can be performed quite explicitely by adding certain generating acyclic cofibrations. Their choice is dictated by the observation that global equivalences coincide with global level equivalences on the class of Real-global Ω-spectra, i.e. those Real spectra X such that the adjoint structure map $X(V) \to \Omega^W X(V \oplus W)$ is a G-equivalence for all Real G-representations V and W with V faithful. Adjoint structure maps are represented by morphisms

$$\lambda_{G;V,W} : F_{G,V\oplus W} S^W \to F_{G,V}$$

of Real spectra and the key observation is that these are global equivalence for faithful V (this can be shown by carrying out the argument of [Sch17b, IV.1.30] in the Real context). By factoring $\lambda_{G;V,W}$ over its mapping cylinder inclusion $\kappa_{G;V,W} : F_{G,V\oplus W} S^W \to Z(\lambda_{G;V,W})$ we replace it up to actual homotopy equivalence with a cofibration in the level model structure. The new candidate for the generating acyclic cofibrations is now defined as $J^{\mathbb{R}}_{\text{stable}} = J^{\mathbb{R}}_{\text{lvl}} \cup \{i \square \kappa_{G,V,W}\}$

Theorem 4.1.8. *There exists a* global model structure *on the category of Real spectra with weak equivalences (respectively fibrations) the global equivalences (respectively fibrations). It is stable, proper, topological, and cofibrantly generated with generating sets* $I^{\mathbb{R}}_{\text{stable}} = I^{\mathbb{R}}_{\text{lvl}}$ *and* $J^{\mathbb{R}}_{\text{stable}}$.

The *Real-global homotopy category*

$$\mathbb{R}\,\mathcal{GH} = \mathrm{Ho}(\mathrm{Sp}^{\mathbb{R}}_{\mathrm{gl}}) \simeq \mathrm{Sp}^{\mathbb{R}}[(\text{Real-gl. eq.})^{-1}]$$

is the homotopy category of the category of Real spectra with respect to the global model structure. Homotopy groups again become representable on the level of homotopy categories, now via the Real version of global classifying spaces. They are defined analogously up to preferred zigzag by

$$B^{\mathbb{R}}_{\mathrm{gl}}G = \widehat{\mathbf{L}}^{\mathbb{C}}(V_G, -)/G,$$

where V_G is a faithful Real G-representation. Assuming the existence of the Real-global model structure, one shows the following representability result exactly as for orthogonal spectra:

Proposition 4.1.9. *The tautological class* $\mathrm{Id}_{V_G} \in \pi_0^G B^{\mathbb{R}}_{\mathrm{gl}}G$ *determines an isomorphism*

$$[\Sigma^\infty_+ B^{\mathbb{R}}_{\mathrm{gl}}G, X]_{\mathbb{R}\,\mathcal{GH}} \cong \pi_0^G X.$$

In particular, the Real global classifying spaces form a set of compact generators for the Real-global homotopy category.

Remark 4.1.10. As indicated in [HHR16, Appendix B.12.4], Real spectra with respect to C_2-equivalences are Quillen equivalent to C_2-orthogonal spectra. The global analogue of this is also true in the following sense: The complexification functor comes with a natural isomorphism $\pi_*^{G,\alpha}(cX) \xrightarrow{\cong} \pi_*^G \alpha^* X$ and using this one checks that both composites $u \circ c$ and $c \circ u$ are naturally weakly equivalent to the respective identity functors. This implies that the Real-global homotopy category is equivalent to the localization of C_2-orthogonal spectra at the Real-global equivalences (i.e. isomorphisms on all $\pi_*^G \alpha^* X$):

$$\mathrm{Ho}(\mathrm{Sp}^{\mathbb{R}}_{\mathrm{gl}}) \simeq C_2\text{-}\mathrm{Sp}^{\mathrm{O}}[(\text{Real-gl. eq.})^{-1}]$$

4.2 Geometric fixed points

We now discuss geometric fixed points of Real unitary spectra. The construction is of course based on the orthogonal case, now using the Real regular representation $\rho_G^{\mathbb{C}}$. Its fixed points are canonically identified (as $\mathbb{R}$-vector spaces) via

$$\mathbb{C} \cong (\rho_G^{\mathbb{C}})^G,\ z \mapsto \frac{1}{\sqrt{|G|}} \left(z \cdot (\sum_{g \in G_{\mathrm{ev}}} g) + \overline{z} \cdot (\sum_{g \in G_{\mathrm{odd}}} g) \right)$$

Definition 4.2.1. Let Y be a Real spectrum and G a finite Real group. The *geometric fixed point spectrum* $\Phi^G Y \in \mathrm{Sp}^{\mathrm{O}}$ of Y with respect to G is defined at the orthogonal vector space V by

$$\Phi^G X(V) = \Omega^{iV} X(\rho_G^{\mathbb{C}} \otimes V)^G.$$

The structure map $\sigma_{V,W}$ is the composition

$$\begin{aligned}(\Omega^{iV} X(\rho_G^{\mathbb{C}} \otimes V)^G) \wedge S^W &\longrightarrow \Omega^{i(V+W)}(X(\rho_G^{\mathbb{C}} \otimes V)^G \wedge S^{\mathbb{C}\otimes W}) \\ &\cong \Omega^{i(V+W)}(X(\rho_G^{\mathbb{C}} \otimes V) \wedge S^{\rho_G^{\mathbb{C}}\otimes W})^G \\ &\xrightarrow{\sigma} \Omega^{i(V+W)} X(\rho_G^{\mathbb{C}} \otimes (V+W))^G,\end{aligned}$$

where the first map is the assembly map moving S^W into $\Omega(-)$, followed by smashing functions with with S^{iW}.

Remark 4.2.2. Despite appearances, this construction does not seem to arise as the underlying orthogonal spectrum of a Real spectrum.

To justify this definition, we consider the homotopy groups:

$$\begin{aligned}\pi_0 \Phi^G X = \mathrm{colim}_{V\subset\mathbb{R}^\infty}[S^V, \Omega^{iV} X(\rho_G^{\mathbb{C}} \otimes V)^G] &\cong \mathrm{colim}_{V\subset\mathbb{R}^\infty}[S^{V_{\mathbb{C}}}, X(\rho_G^{\mathbb{C}} \otimes V)^G] \\ &\cong \mathrm{colim}_{W\subset\mathcal{U}_G^{\mathbb{R}}}[S^{W^G}, X(W)^G] \\ &= \Phi_0^G X.\end{aligned}$$

Here the last isomorphism follows by cofinality and the geometric fixed point homotopy groups are defined by the last equality, which is clearly the direct analogue of the ordinary version for orthogonal spectra. They also come with a natural comparison map

$$\phi : \pi_0^G \to \Phi_0^G.$$

Definition 4.2.3. The category $\mathrm{Epi}_{\mathbb{R}}$ is the category of finite Real groups together with surjective homomorphisms (augmentation preserving).

Restriction maps are defined as before, now using the complexified embeddings

$$\alpha_! \otimes \mathbb{C} : \alpha^* \rho_G^{\mathbb{C}} \hookrightarrow \rho_K^{\mathbb{C}}$$

for $\alpha : G \twoheadrightarrow K$. Here one uses that α respects augmentations in order to ensure that $\alpha_! \otimes \mathbb{C}$ respects the identification of the fixed points with $\mathbb{C}$ and hence induces a morphism

$$\alpha^* : \Phi^K \to \Phi^G$$

of orthogonal spectra. In addition $\Phi^G X$ also comes with an *involution* ι_G defined in each level by complex conjugation and this commutes with the restriction maps. To summarise, the geometric fixed point construction defines a functor

$$\mathrm{Sp}^{\mathbb{R}} \xrightarrow{\Phi} \mathrm{Epi}_{\mathbb{R}}^{\mathrm{op}}\text{-}C_2\text{-}\mathrm{Sp}^{\mathrm{O}}$$

There is also another way to define geometric fixed point spectra by first passing to underlying C_2-orthogonal spectra, pulling back along the augmentation of the Real group G, and then applying Φ^G. Taking fixed points $(\Omega^{i\rho_G \otimes V} X(\rho_G^{\mathbb{C}} \otimes V))^G \xrightarrow{\text{fix}} \Omega^{iV} X(\rho_G^{\mathbb{C}} \otimes V)^G$ in each level V, where we identify $(i\rho_G)^G = \mathbb{R}\{i \cdot \widetilde{\mathrm{N}_G}\}$, defines a canonical comparison map

$$\Phi^G \alpha^*(uX) \to \Phi^{G,\alpha} X$$

from the geometric fixed points of the underlying C_2-orthogonal spectrum.

Proposition 4.2.4. *The above maps assemble to a π_*-isomorphism*

$$\Phi(uX) \xrightarrow{\simeq} \Phi X$$

of $\mathrm{Epi}_{\mathbb{R}}^{\mathrm{op}}$ - C_2*-orthogonal spectra.*

Proof. The left hand side forms an $\mathrm{Epi}_{\mathbb{R}}^{\mathrm{op}} \times C_2$-diagram because inflations are only taken along augmentation preserving group homomorphisms (cf. Remark 2.3.17). On both sides these are defined via the same isometric embeddings of regular representations, which implies that the comparison map commutes with them.

A map on homotopy groups (say in degree 0) in the other direction is given by first using the unit map $A \to \Omega^W(S^W \wedge A)$, and then applying the structure map of X:

$$\begin{aligned}
[S^{V_{\mathbb{C}}}, X(\rho_G^{\mathbb{C}} \otimes V)^G] &\longrightarrow [S^{V_{\mathbb{C}}}, (\Omega^{\rho_G \otimes 2V}(S^{\rho_G \otimes 2V} \wedge X(\rho_G^{\mathbb{C}} \otimes V)))^G] \\
&\longrightarrow [S^{2V}, (\Omega^{i\rho_G \otimes 2V} X(\rho_G^{\mathbb{C}} \otimes 2V))^G].
\end{aligned}$$

The notation is a bit suggestive and $2V$ has to be suitably interpreted as either $i2V$ or $V_{\mathbb{C}}$, and one checks that both composites are isomorphic to stabilization in the colimit systems. $\square$

Remark 4.2.5. While both constructions are equivalent, we have taken the version as definition that has the a priori 'correct' homotopy groups.

Corollary 4.2.6. *Geometric fixed points of Real spectra are homotopical, and preserve mapping cone sequences and wedges up to weak equivalence.*

The inflations on geometric fixed point homotopy groups of Real spectra again only depend on conjugacy classes, but now in a twisted sense. Conjugation with an element g is either trivial or the involution ι_G of Φ_*^G, depending on the degree of G:

$$c_g^* = \begin{cases} \mathrm{Id}, & g \text{ even} \\ \iota_G, & g \text{ odd} \end{cases}$$

Definition 4.2.7. The category $\mathrm{Out}_{\mathbb{R}}$ is obtained from the category $\mathrm{Epi}_{\mathbb{R}} \times C_2$ by dividing out conjugacy classes of homomorphisms in the above sense.

Corollary 4.2.8. *The geometric fixed point homotopy groups of Real spectra are* $\mathrm{Out}_{\mathbb{R}}^{\mathrm{op}}$*-modules.*

Proof. By [Sch17b, III.3.5.(ii)] the composite

$$\Phi^G Y \xrightarrow{(c_g)^*} \Phi^G(c_g^* Y) \xrightarrow{(l_g)_*} \Phi^G Y$$

is the identity for any G-orthogonal spectrum Y. Setting $Y = \alpha^*(uX)$, one obtains the desired relation. $\square$

Geometric fixed points of Real spectra still commute with suspension spectra in the following sense: By inspection of homotopy groups the canonical map

$$\begin{array}{ccc} \Sigma^\infty_+ \Phi^G X & \xrightarrow{\qquad\simeq\qquad} & \Phi^G \Sigma^\infty_+ X \\ \| & & \| \\ X(\rho_G^{\mathbb{C}} \otimes -)^G \wedge S^{(-)} & \longrightarrow & \Omega^{i-}(X(\rho_G^{\mathbb{C}} \otimes -) \wedge S^{\rho_G^{\mathbb{C}} \otimes -})^G \end{array}$$

is a weak equivalence of orthogonal spectra for every Real unitary space X, where we take the left equality as a definition.

Example 4.2.9. We will again need to know the effect of geometric fixed points on Real global classifying spaces. In this case the decomposition formula 2.1.1 yields

$$\begin{aligned} (\widehat{\mathbf{L}}^{\mathbb{C}}(V_G, \rho_K^{\mathbb{C}} \otimes -)/G)^K &\cong \coprod_{(\alpha: K \to G)} \widehat{\mathbf{L}}^{\mathbb{C}}_K(\alpha^* V_G, \rho_K^{\mathbb{C}} \otimes -)/C(\alpha) \\ &\cong \coprod_{(\alpha: K \to G)} \mathbf{L}^{\mathbb{C}}_K(\alpha^* V_G, \rho_K^{\mathbb{C}} \otimes -)/C(\alpha) \times C_2/C(\alpha), \end{aligned}$$

where the last step uses the $(O(W) \times C_2)$-equivariant decomposition $\widehat{\mathbf{L}}^{\mathbb{C}}(V, W_{\mathbb{C}}) \cong \mathbf{L}^{\mathbb{C}}(V, W_{\mathbb{C}}) \times C_2$. A priori, the sum would be indexed by the conjugacy classes of all group homomorphisms, but the spaces of equivariant isometries are non-empty only for augmentation preserving ones. We note that $\coprod_{(\alpha: K \to G)} C_2/C(\alpha) = \mathrm{Rep}_{\mathbb{R}}(K, G)$ is a decomposition of the Real analogue of the $\mathrm{Out}^{\mathrm{op}}$-functor $\mathrm{Rep}(-, G)$. Passing to suspension spectra, we conclude (cf. Example 2.3.15) that there is an equivalence

$$\Phi^K \Sigma^\infty_+ B^{\mathbb{R}}_{\mathrm{gl}} G \simeq \bigvee_{(\alpha, \epsilon) \in \mathrm{Rep}_{\mathbb{R}}(K, G)} \Sigma^\infty_+ BC(\alpha)$$

inducing a preferred (up to zigzag) identification $\Phi_0 \Sigma^\infty_+ B^{\mathbb{R}}_{\mathrm{gl}} G \cong \mathbb{Z}\{\mathrm{Rep}_{\mathbb{R}}(-, G)\}$ of $\mathrm{Out}^{\mathrm{op}}_{\mathbb{R}}$-modules.

4.3 Rationalized Real-global homotopy theory

We shall now identify rationalized Real-global spectra with $\mathrm{Out}^{\mathrm{op}}_{\mathbb{R}}$-complexes. The comparison functor Γ is defined as the expected homotopical composition

$$\mathrm{Sp}^{\mathbb{R}} \xrightarrow{\Phi} \mathrm{Epi}^{\mathrm{op}}_{\mathbb{R}}\text{-}C_2\text{-}\mathrm{Sp}^{\mathrm{O}} \xrightarrow{C} \mathrm{Epi}^{\mathrm{op}}_{\mathbb{R}}\text{-}C_2\text{-}\mathrm{Ch}_{\mathbb{Q}} \xrightarrow{\mathrm{Lan}} \mathrm{Out}^{\mathrm{op}}_{\mathbb{R}}\text{-}\mathrm{Ch}_{\mathbb{Q}},$$

where the diagram categories are equipped with projective model structures. This descends to an exact, sum-preserving functor on homotopy categories. As before, the left Kan extension commutes rationally with homology and thus has no effect on it for the $\mathrm{Epi}_{\mathbb{R}}^{\mathrm{op}}$-diagrams arising from geometric fixed point spectra. In other words, Γ comes with a natural isomorphism

$$H_* \Gamma X \cong \underline{\Phi}_* X$$

of $\mathrm{Out}_{\mathbb{R}}^{\mathrm{op}}$-modules. From Example 4.2.9 we conclude the following

Lemma 4.3.1. *There is a preferred equivalence*

$$\Gamma \Sigma_+^\infty B_{\mathrm{gl}}^{\mathbb{R}} G \xrightarrow{\simeq_{\mathbb{Q}}} \mathbb{Q}\{\mathrm{Rep}_{\mathbb{R}}(-,G)\}[0]$$

of $\mathrm{Out}_{\mathbb{R}}^{\mathrm{op}}$*-complexes.*

We also have a decomposition $\mathrm{Rep}_{\mathbb{R}}(-,G) \cong \coprod_{(H \leq G)} WH \backslash \mathrm{Out}_{\mathbb{R}}(-,H)$ of $\mathrm{Out}_{\mathbb{R}}^{\mathrm{op}}$-modules indexed by (ordinary) conjugacy classes of subgroups and this gives the Real version of Lemma 3.2.3:

Lemma 4.3.2. *In the derived category the* $\mathrm{Out}_{\mathbb{R}}^{\mathrm{op}}$*-module* $\mathbb{Q}\{\mathrm{Rep}_{\mathbb{R}}(-,G)\}$ *corepresents the functor*

$$X \longmapsto \bigoplus_{(H \leq G)} (H_i X(H))^{WH}.$$

As the Real group G varies (over a small skeleton), these form a set of compact generators.

We can now show the main result of this section:

Theorem 4.3.3. *Geometric fixed points induce an equivalence*

$$(\mathbb{R}\,\mathcal{GH})_{\mathbb{Q}} \xrightarrow{\simeq} \mathcal{D}(\mathrm{Out}_{\mathbb{R}}^{\mathrm{op}}\text{-}\mathrm{mod}_{\mathbb{Q}})$$

between the rationalized Real-global homotopy category and the derived category of rational $\mathrm{Out}_{\mathbb{R}}^{\mathrm{op}}$*-modules.*

Proof. By the usual localizing subcategory argument it suffices to show that the map

$$\pi_0^G X \longrightarrow \prod_{(H \leq G)} (\Phi_0^H X)^{WH}$$

is a rational isomorphism for all Real groups $G \xrightarrow{\alpha} C_2$ and Real spectra X. This is isomorphic to

$$\pi_0^G \alpha^*(uX) \longrightarrow \prod_{(H \leq G)} (\Phi_0^H \alpha^*(uX))^{WH}$$

and so we are reduced to the known statement (Proposition 2.3.11) for orthogonal G-spectra. □

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Summary

This thesis deals with global homotopy theory, a form of equivariant homotopy theory in which simultaneous actions of all (in our case finite) groups are considered. The main goal is to give an algebraic model for rational global stable homotopy theory using geometric fixed points.

We investigate a known model of geometric fixed point spectra from the global perspective and based on it we construct an equivalence

$$\mathcal{GH}_{\mathbb{Q}} \simeq \mathcal{D}(\mathrm{Out}^{\mathrm{op}}\text{-}\mathrm{mod}_{\mathbb{Q}})$$

between the rational global homotopy category and the derived category of rational $\mathrm{Out}^{\mathrm{op}}$-modules, where Out is the category of finite groups together with conjugacy classes of surjective group homomorphisms. This equivalence is shown to be multiplicative with respect to the smash product of orthogonal spectra and the group-wise tensor product of chain complexes. We also discuss a $\mathbb{Z}[\frac{1}{p}]$-local version of this, where one has to restrict to p-groups.

The abelian category of rational $\mathrm{Out}^{\mathrm{op}}$-modules has infinite homological dimension. In contrast to the classical situation the question arises if homotopy types rationally decompose into products of Eilenberg-MacLane objects. We answer this question positively in the case of the global equivariant K-theory spectrum $\mathbf{KU}$ by showing that the necessary Ext-groups vanish.

Our comparison functor can also be modified to study highly structured ring spectra and we use it to give rational algebraic models for associative and E_∞-ring spectra in the global setting. These turn out to be equivalent to $\mathrm{Out}^{\mathrm{op}}$-diagrams in rational differential graded algebras respectively commutative differential graded algebras. Furthermore, we also provide strong evidence for the conjectural result that commutative ring spectra are modelled by $\mathrm{Out}^{\mathrm{op}}$-CDGA's with additional norm maps related to the Hill-Hopkins-Ravenel norms.

We conclude with a sketch of the analogous comparison program for Real-global homotopy theory, where one takes twisted actions by finite Real groups (i.e. groups G with an augmentation morphism $G \to C_2$ to the cyclic group of order 2) into account.

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