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ON THE THEORY OF DERIVATORS

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Abstract. In Part 1, we develop some aspects of the theory of derivators, pointed derivators, and stable derivators. As a main result, we show that the values of a stable derivator can be canonically endowed with the structure of a triangulated category. Moreover, the functors belonging to the stable derivator can be turned into exact functors with respect to these triangulated structures. Along the way, we give a simplification of the axioms of a pointed derivator and a reformulation of the base change axiom in terms of Grothendieck (op)fibration. Furthermore, we have a new proof that a combinatorial model category has an underlying derivator.

In Part 2, we develop the theory of monoidal derivators and the related notions of derivators being tensored, cotensored, or enriched over a monoidal derivator. The passage from model categories to derivators respects these notions and, hence, gives rise to natural examples. Moreover, we introduce the notion of the center of additive derivators which allows for a convenient formalization of linear structures on additive derivators and graded variants thereof in the stable situation. As an illustration we discuss some derivators related to chain complexes and symmetric spectra.

In the last part, we take a closer look of the derivator associated to a differential-graded algebra over a field. A theorem of Kadeishvili ensures that the homotopy type of such a dga A can be encoded by a minimal A_∞-algebra structure on the homology algebra. Moreover, a result of Renaudin guarantees that the derivator $\mathbb{D}_A$ of differential-graded A-modules essentially captures the homotopy theory associated to A. This motivates that these two structures –the A_∞-algebra and the derivator $\mathbb{D}_A$– should determine each other, and we give a first step towards such a comparison result.

Contents

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Zusammenfassung[1] Diese Arbeit beschäftigt sich mit verschiedenen Aspekten der Theorie der Derivatore. Bei den Derivatoren handelt es sich um einen Zugang zur axiomatischen Homotopietheorie, welcher insofern ein elementarer Zugang ist, als sich die Theorie ganz in der Welt der (2-)Kategorien abspielt. Die Theorie der *stabilen* Derivatore kann dadurch motiviert werden, dass sie uns eine 'Verbesserung' der Theorie der (in den verschiedensten Bereichen der Mathematik auftretenden) triangulierten Kategorien liefert. Typische formale Defekte der triangulierten Kategorien –wie die Nicht-Funktorialität der Kegelkonstruktion– können behoben werden, wenn man beim Übergang vom Modell zu den Homotopiekategorien 'mehr Information mitnimmt'.

Die Grundidee besteht darin, dass man z.B. im Kontext einer abelschen Kategorie nicht nur die derivierte Kategorie bilden sollte. Zeitgleich sollte man auch verschiedene Funktorkategorien mit Werten in dieser abelschen Kategorien betrachten. Bei diesen Diagrammkategorien handelt es sich wieder um abelsche Kategorien, so dass wir auch zu diesen die derivierten Kategorien assoziieren können. So entsteht ein System von derivierten Kategorien, welche mit diversen Einschränkungsfunktoren verbunden sind, und ähnliche Beobachtungen lassen sich auf natürlichen Transformationen übertragen. Des weiteren haben die Einschränkungsfunktoren oftmals Adjungierte auf beiden Seiten, welche wir dann als Homotopie(ko)limiten im absoluten Fall bzw. als Homotopie-Kan-Erweiterungen im relativen Fall interpretieren. Spezialfälle dieser Homotopie(ko)limiten liefern uns eine funktorielle Variante der Kegelkonstruktion.

Eine Axiomatisierung einer solcher Situation führt zu dem Begriff des (stabilen) Derivators. Die Theorie der Derivatore ist allerdings mehr als 'lediglich' eine Verbesserung der Theorie der triangulierten Kategorien. Es gibt eine ganze Hierarchie solcher Stukturen von Derivatoren, punktierten Derivatoren über additive Derivatore zu stabilen Derivatoren, und die Theorie der Derivatore ist somit auch in wesentlich allgemeineren, insbesondere auch in nicht-stabilen, Situationen interessant.

Das Hauptziel des ersten Teils der Arbeit besteht darin, die Theorie der stabilen Derivatore soweit zu entwickeln, dass wir –aufbauend auf Ideen von Franke– einen Beweis liefern können, dass die Werte eines stabilen Derivators kanonisch mit triangulierten Strukturen versehen werden können. Desweiteren zeigen wir, dass die zu einem stabilen Derivator gehörenden Einschränkungsfunktoren und Homotopie-Kan-Erweiterungen kanonisch exakte Strukturen tragen. Auf dem Weg zu diesem zentralen Satz vereinfachen wir die Axiomatik der punktierten Derivatore, liefern eine Charakterisierung von Derivatoren über ein gewisses Verhalten bei Basiswechseln und geben einen neuen Beweis, dass kombinatorische Modellkategorien einen zugrundeliegenden Derivator besitzen.

Im zweiten Teil entwickeln wir die Theorie der monoidalen Derivatore und die verwandten Theorien der tensorierten, kotensorierten oder angereicherten Derivatore. Der Übergang von Modellkategorien zu Derivatoren erhält diese Strukturen und liefert so kanonische Beispiele. Des weiteren führen wir das Zentrum von additiven Derivatoren ein, welches es uns erlaubt, über lineare Strukturen auf Derivatoren zu sprechen. Wir illustrieren diese

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Begriffsbildungen an Beispielen im Kontext der Kettenkomplexe und symmetrischen Spektren.

Im letzten Teil studieren wir den einer differentiell-graduierten Algebra über einem Körper zugeordneten Derivator etwas genauer. Nach einem Satz von Kadeishvili kann man den Homotopietyp einer solchen differentiell-graduierten Algebra über das sogenannte minimale Modell auf der Homologie kodieren. Des weiteren motiviert ein Resultat von Renaudin, dass der assoziierte Derivator ebenfalls die Homotopietheorie speichern sollte. Wir liefern einen ersten Schritt, wie man diese beiden Wege des Kodierens in Bezug zueinander setzen kann.

Die einzelnen Teile der Arbeit haben separate, ausführliche Einführungen. Insbesondere finden sich dort auch die Referenzen auf die jeweilige Literatur.

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Part 1. Derivators, pointed derivators, and stable derivators

0. Introduction and plan

The theory of stable derivators as initiated by Heller [Hel88, Hel97] and Grothendieck [Gro] and studied, at least in similar settings, among others, by Franke [Fra96], Keller [Kel91] and Maltsiniotis [Mal07a], can be motivated by saying that it provides an enhancement of triangulated categories. Triangulated categories suffer the well-known defect that the cone construction is not functorial. A consequence of this non-functoriality of the cone construction is the fact that there is no good theory of homotopy (co)limits for triangulated categories. One can still define these notions, at least in some situations where the functors are defined on categories which are freely generated by a graph. This is the case e.g. for the cone construction itself, the homotopy pushout, and the homotopy colimit of a sequence of morphisms. But in all these situations, the 'universal objects' are only unique up to *non*-canonical isomorphism. The slogan used to describe this situation is the following one: diagrams in a triangulated category do not carry sufficient information to define their homotopy (co)limits in a *canonical* way.

But in the typical situations, as in the case of the derived category of an abelian category or in the case of the homotopy category of a stable model resp. $\infty-$category, the 'model in the background' allows for such constructions in a functorial manner. So, the passage from the model to the derived resp. homotopy category truncates the available information too strongly. To be more specific, let $\mathcal{A}$ be an abelian category such that the derived categories which occur in the following discussion exist. Moreover, let us denote by $C(\mathcal{A})$ the category of chain complexes in $\mathcal{A}$. As usual, let $[1]$ be the ordinal $0 \leq 1$ considered as a category $(0 \longrightarrow 1)$. Hence, for an arbitrary category $\mathcal{C}$, the functor category $\mathcal{C}^{[1]}$ of functors from $[1]$ to $\mathcal{C}$ is the arrow category of $\mathcal{C}$. With this notation, the cone functor at the level of abelian categories is a functor $\mathsf{C}\colon C(\mathcal{A}^{[1]}) \cong C(\mathcal{A})^{[1]} \longrightarrow C(\mathcal{A})$. But to give a *construction* of the cone functor in terms of homotopical algebra only, one has to consider more general diagrams. For this purpose, let $f\colon X \longrightarrow Y$ be a morphism of chain complexes in $\mathcal{A}$. Then the cone $\mathsf{C}f$ of f is the *homotopy* pushout of the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \\ 0 & & \end{array}$$

At the level of derived categories, the cone construction is again functorial *when considered as a functor* $D(\mathcal{A}^{[1]}) \longrightarrow D(\mathcal{A})$. The important point is that one forms the arrow categories *before* passage to the derived categories. Said differently, at the level of derived categories, we have, in general, $D(\mathcal{A}^{[1]}) \ncong D(\mathcal{A})^{[1]}$. Moreover, as we have mentioned, to actually give a construction of this functor one needs apparently also the derived category of diagrams in $\mathcal{A}$ of the above shape and a homotopy pushout functor. More systematically, one should not only consider the derived category of an abelian category but also the derived categories of diagram categories and restriction and homotopy Kan extension functors between them. This is the basic idea behind the notion of a derivator.

But the theory of derivators is more than 'only an enhancement of triangulated categories'. In fact, they give us an alternative axiomatic approach to an abstract homotopy theory (cf. Remark 1.25). As in the theory of model categories and $\infty-$categories, there is a certain hierarchy of such structures: the unpointed situation, the pointed situation, and the stable situation. In the classical situation of topology, this hierarchy corresponds to the passage from spaces to pointed spaces and then to spectra. In classical homological algebra, the passage from the derived category of non-negatively graded chain complexes to the unbounded derived category can be seen as a second example for passing from the pointed to the stable situation. In the theory of derivators this threefold hierarchy of structures is also present, and the corresponding notions are then derivators, pointed derivators, and stable derivators. Franke has introduced in [Fra96] a theory of systems of triangulated diagram categories which is similar to the notion of a stable derivator. The fact that the theory of derivators admits the mentioned threefold hierarchy of structures is one main advantage over the approach of Franke.

In this part, we give a complete and self-contained proof that the values of a stable derivator can be *canonically* endowed with the structure of a triangulated category. Similarly, we show that the functors which are part of the derivator can be *canonically* turned into exact functors with respect to these structures. This is in a sense the main work and will occupy the bulk of this part. We build on ideas of Franke [Fra96] from his theory of systems of triangulated diagram categories and adapt them to this alternative set of axioms. Similar ideas are used in Lurie's [Lur11] on the theory of stable $\infty-$categories. The main reason responsible for the length of this paper is the following: In many propositions we have to show that a certain canonical morphism (often a base change morphism) is an isomorphism. In most cases, it is quite easy to construct an abstract isomorphism between the relevant gadgets. The main work is then to identify this abstract isomorphism as the canonical one.

Along the way we give a simplification of the axioms of a pointed derivator. The usual definition of a pointed derivator, here called a strongly pointed derivator, is formulated using the notion of closed and open immersions of categories. Given a closed immersion i resp. an open immersion j, one usually demands that the homotopy left Kan extension functor $i_!$ along i has itself a left adjoint $i^?$ resp. that the homotopy right Kan extension functor j_* along j has in turn a right adjoint $j^!$. Motivated by algebraic geometry, these functors are then called exceptional resp. coexceptional inverse image functors. We show that this definition can be simplified. It suffices to ask that the underlying category of the derivator is pointed, i.e., has a zero object. This definition is more easily motivated, more intuitive for topologists, and, of course, simpler to check in examples. We give a direct proof of the equivalence of these two notions at the end of Section 1. A second proof of this is given in the stable setting using the fact that recollements of triangulated categories are overdetermined (cf. Subsection 4.3).

The author is aware of the fact that there will be a written up version of a proof of the existence of these canonical triangulated structures in a future paper by Maltsiniotis. In fact, Maltsiniotis presented an alternative, unpublished variant of Franke's theorem in a seminar in Paris in 2001. He showed that this notion of stable derivators is equivalent to a variant thereof (as used in the thesis of Ayoub [Ayo07a, Ayo07b]) where the triangulations are part of the notion. But, since the author needs this theory in the third part of his thesis, in particular, the existence and also the construction of the canonical triangulated structures, we give this account. Moreover, the construction of the suspension functor in [CN08] and the axioms in [Mal07a] indicate that that proof will use the (co)exceptional inverse image functors. But one point here is to show that these functors are not needed for these purposes.

We now turn to a short description of the content of the paper. In Section 1, we give the central definitions and deduce some immediate consequences of the axioms. We show in Proposition 1.18, that with a derivator $\mathbb{D}$ also the prederivator $\mathbb{D}_M$ (cf. Example 1.7) is a derivator. As main class of examples, we give a simple, i.e., completely formal, proof that combinatorial model categories have underlying derivators. We then shortly discuss pointed and 'strongly pointed' derivators. In the last subsection, we deduce some elementary results on homotopy Kan extensions which will be used throughout the paper. We finish the section by showing that the notion of pointed derivators and 'strongly pointed' derivators coincide.

In Section 2, we discuss morphisms of (pre)derivators. We introduce the important classes of morphisms preserving (certain) homotopy Kan extensions. Proposition 2.12 allows us to give minimal such definitions in three relevant cases. We show that homotopy Kan extensions in derivators of the form $\mathbb{D}_M$ are calculated pointwise (Proposition 2.9). An important consequence thereof is the fact that the essential image of homotopy Kan extensions in $\mathbb{D}_M$ along fully-faithful functors can be described pointwise (Corollary 2.11). These results are used to give a proof of the existence of the canonical triangulated structures on the values of a stable derivator in Section 4.

In Section 3, we introduce Cartesian and coCartesian squares and develop some important properties of them. An important example of this kind of results is the composition and cancellation property of (co)Cartesian squares (Proposition 3.9). Another one is a 'detection result' for (co)Cartesian squares in larger diagrams (Proposition 3.5) which is due to Franke [Fra96]. We finish the section by defining left exact, right exact, and exact morphisms of pointed derivators and give a relevant example.

In Section 4, we stick to the stable situation. We introduce the suspension, loop, cone, and fiber functors for pointed derivators. We then deduce that these functors are equivalences in the case of a stable derivator. The main aim of the section is to establish the canonical triangulated structures on the values of a stable derivator (Theorem 4.20). These are preserved by exact morphisms of stable derivators (Proposition 4.23) and, in particular, by the functors belonging to the stable derivator itself (Corollary 4.24). In the last subsection, we remark that, given a stable derivator and a closed or an open immersion of categories, we obtain a recollement of triangulated categories. This reproves, in the stable case, that pointed derivators are 'strongly pointed'.

In Appendix A we give an alternative characterization of derivators using the theory of Grothendieck (op)fibrations. We show that the usual axiom expressing that homotopy Kan extensions in derivators can be calculated using homotopy (co)limits (Kan's formulas) can be replaced by the assumption that the prederivator satisfies base change for Grothendieck (op)fibrations. This is used to conclude the proof that with a derivator $\mathbb{D}$ also $\mathbb{D}_M$ is a derivator.

Finally, in Appendix B, we give some lengthy proofs which we did not wish to give in the body of the text. These are mainly proofs which are not particularly enlightening and where the understanding of the proof is not necessary for the rest of the paper.

There are three more remarks in order before we begin with the paper. First, we do not develop the general theory of derivators for its own sake and also not in the broadest generality. In this part, we only develop as much of the general theory as is needed to give complete, self-contained proofs of the mentioned results. Nevertheless, this paper may serve as an introduction to many central ideas in the theory of derivators and no prior knowledge is assumed.

The second remark concerns duality. Many of the statements in this paper have dual statements which also hold true by the dual proof (the reason for this is Example 1.17). In most cases, we will not make these statements explicit and we will hardly ever give a proof of both statements. Nevertheless, we allow ourselves to refer to a statement also in cases where, strictly speaking, the dual statement is needed.

The last remark concerns the terminology employed here. In the existing literature on derivators, the term 'triangulated derivator' is used instead of 'stable derivator'. We preferred to use this different terminology for two reasons: First, the terminology 'triangulated derivator' (introduced by Maltsiniotis in [Mal07a]) is a bit misleading in that no triangulations are part of the initial data. One main point of this paper is to give a proof that these triangulations can be canonically constructed. Thus, from the perspective of the typical distinction between *structures* and *properties* the author does not like the former terminology too much. Second, in the related theories of model categories resp. $\infty-$categories, corresponding notions exist and are called *stable* model categories resp. *stable* $\infty-$categories. So, the terminology stable derivator reminds us of the related theories.

1. Derivators and pointed derivators

1.1. **Basic notions.** As we mentioned in the introduction, the basic idea behind a derivator is to consider simultaneously derived or homotopy categories of diagram categories of different shapes. So, the most basic notion in this business is the following one.

Definition 1.1. A *prederivator* $\mathbb{D}$ is a strict 2-functor $\mathbb{D}\colon \mathsf{Cat}^{\mathrm{op}} \longrightarrow \mathsf{CAT}$.

Here, Cat denotes the 2-category of small categories, $\mathsf{Cat}^{\mathrm{op}}$ is obtained from Cat by reversing the direction of the functors, while CAT denotes the '2-category' of not necessarily small categories. There are the usual set-theoretical problems with the notion of the '2-category' CAT in that this will not be a category enriched over Cat. Since we will never need this non-fact in this paper, we use slogans as the '2-category CAT' as a convenient parlance and think instead of a prederivator as a function $\mathbb{D}$ as we describe it now. Given a prederivator $\mathbb{D}$ and a functor $u\colon J \longrightarrow K$, an application of $\mathbb{D}$ to u gives us two categories $\mathbb{D}(J)$, $\mathbb{D}(K)$, and a functor

$$\mathbb{D}(u) = u^*\colon \mathbb{D}(K) \longrightarrow \mathbb{D}(J).$$

Similarly, given two functors $u,\, v\colon J \longrightarrow K$ and a natural transformation $\alpha\colon u \longrightarrow v$, we obtain an induced natural transformation α^* as depicted in the next diagram:

$$\begin{tikzcd} J \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] \arrow[r, phantom, "\Downarrow\alpha"] & K \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathbb{D}(K) \arrow[r, bend left, "u^*"] \arrow[r, bend right, "v^*"'] \arrow[r, phantom, "\Downarrow\alpha^*"] & \mathbb{D}(J) \end{tikzcd}$$

This datum is compatible with compositions of functors and natural transformations and respects identity functors and identity transformations in a strict sense, i.e., we have equalities of the respective expressions and not only coherent natural isomorphisms between them. For the relevant basic 2-categorical notions, which were introduced by Ehresmann in [Ehr63], we refer to [Kel05b] or to [Bor94a, Chapter 7], but nothing deep from that theory is needed here.
The following examples give an idea of how such prederivators arise. Among these probably the second, third, and forth one are the examples to have in mind in later sections.

Example 1.2. Let $\mathcal{C}$ be a category. Then we can consider the following prederivator, again denoted by $\mathcal{C}$:

$$\mathcal{C}\colon J \longmapsto \mathsf{Fun}(J, \mathcal{C})$$

Here, $\mathsf{Fun}(J, \mathcal{C})$ denotes the functor category of functors from J to $\mathcal{C}$. A functor $u\colon J \longrightarrow K$ gives rise to a precomposition functor $\mathcal{C}(u) = u^*\colon \mathsf{Fun}(K, \mathcal{C}) \longrightarrow \mathsf{Fun}(J, \mathcal{C})$. Similarly, for a natural transformation $\alpha\colon u \longrightarrow v$ between parallel functors $u,\, v\colon J \longrightarrow K$, we obtain a natural transformation $\alpha^*\colon u^* \longrightarrow v^*$. Thus, setting $\mathcal{C}(\alpha) = \alpha^*$ concludes the definition of the *prederivator represented by* $\mathcal{C}$.

Although we have not yet introduced the notion of a morphism of prederivators (cf. Section 2) and natural transformations, we want to mention that the above example can be extended to give an embedding of the 2-category of categories into the 2-category of prederivators. Natural transformations are defined in [Gro11] as modifications where we discuss

2-categorical aspects of the theory more systematically. In this thesis we introduce many notions for derivators which are analogs of well-known notions from category theory. Then it will be important to see that these notions are extensions of the classical ones in that both notions coincide on the represented (pre)derivators.

Example 1.3. Let $\mathcal{A}$ be a sufficiently nice abelian category, i.e., such that we can form the derived categories occurring in this example. Recall that, by definition, the derived category $D(\mathcal{A})$ is the localization of the category of chain complexes at the class of quasi-isomorphisms. For a category J, the functor category $\mathsf{Fun}(J,\mathcal{A})$ is again an abelian category. In the associated category of chain complexes $C(\mathsf{Fun}(J,\mathcal{A}))\cong \mathsf{Fun}(J,C(\mathcal{A}))$, quasi-isomorphisms are defined pointwise, so that restriction of diagram functors induce functors on the level of derived categories. Thus, we have the *prederivator* $\mathbb{D}_{\mathcal{A}}$ *associated to an abelian category* $\mathcal{A}$:

$$\mathbb{D}_{\mathcal{A}}\colon J\longmapsto \mathbb{D}_{\mathcal{A}}(J)=D\big(\mathsf{Fun}(J,\mathcal{A})\big)$$

The next example assumes some knowledge of model categories. The original reference is [Qui67] while a well written, leisure introduction to the theory can be found in [DS95]. Much more material is treated in the monographs [Hov99] and [Hir03].

Convention 1.4. In this paper model categories are assumed to have limits and colimits of all *small* (as opposed to only finite) diagrams. Furthermore, we do *not* take functorial factorizations as part of the notion of a model category. First, this would be an additional structure on the model categories which is anyhow not respected by the morphisms, i.e., by Quillen functors. Second, this assumption would be a bit in conflict with the philosophy of higher category theory. The category of -say- cofibrant replacements of a given object in a model category is contractible so that any choice is equally good and there is no essential difference once one passes to homotopy categories. For many results along these lines cf. to [Hir03, Part 2].

Example 1.5. Let $\mathcal{M}$ be a cofibrantly-generated model category. Recall that one of the good things about cofibrantly-generated model categories is that diagram categories $\mathsf{Fun}(J,\mathcal{M})$ can be endowed with the so-called *projective model structure.* In more detail, let us call a natural transformation of $\mathcal{M}-$valued functors a projective fibration resp. projective weak equivalence if all components are fibrations resp. weak equivalences in $\mathcal{M}$. A projective cofibration is a map which has the left-lifting-property with respect to all maps which are simultaneously projective fibrations and projective weak equivalences. With these definitions, $\mathsf{Fun}(J,\mathcal{M})$ is again a model category and we can thus consider the associated homotopy category. Recall that the canonical functor $\gamma\colon\mathcal{M}\longrightarrow\mathsf{Ho}(\mathcal{M})$ from $\mathcal{M}$ to its homotopy category is a 2-localization. This means, that γ induces for every category $\mathcal{C}$ an *isomorphism of categories*

$$\gamma^{*}\colon \mathcal{C}^{\mathsf{Ho}(\mathcal{M})}\longrightarrow \mathcal{C}^{(\mathcal{M},W)}$$

where the right-hand-side denotes the full subcategory of $\mathcal{C}^{\mathcal{M}}$ spanned by the functors which send weak equivalences to isomorphisms. Moreover, since projective weak equivalences are defined as levelwise weak equivalences, these are preserved by restriction of diagram functors. By the universal property of the localization functors the restriction of diagram functors descend uniquely to the homotopy categories. Thus, given such a cofibrantly-generated model category $\mathcal{M}$, we can form the *prederivator* $\mathbb{D}_{\mathcal{M}}$ *associated to* $\mathcal{M}$ if we set

$$\mathbb{D}_{\mathcal{M}}\colon J \longmapsto \mathbb{D}_{\mathcal{M}}(J) = \mathsf{Ho}\left(\mathsf{Fun}(J,\mathcal{M})\right).$$

A similar example can be given using the theory of $\infty-$categories (aka. quasi-categories, weak Kan complexes), i.e., of simplicial sets satisfying the inner horn extension property. These were originally introduced by Boardman and Vogt in their work [BV73] on homotopy invariant algebraic structures. Detailed accounts of this theory are given in the tomes due to Joyal [Joy08b, Joy08c, Joy08a, Joy] and Lurie [Lur09, Lur11]. A short exposition of many of the central ideas and also of the philosophy of this theory can be found in [Gro10b].

Example 1.6. Let $\mathcal{C}$ be an $\infty-$category and let $K \in \mathsf{Set}_\Delta$ be a simplicial set. Then one can show that the simplicial mapping space $\mathcal{C}^K_\bullet = \mathsf{hom}_{\mathsf{Set}_\Delta}(\Delta^\bullet \times K, \mathcal{C})$ is again an $\infty-$category (as opposed to a more general simplicial set). This follows from the fact that the Joyal model structure ([Joy08b]) on the category of simplicial sets is Cartesian. We can hence vary the simplicial set K and consider the associated homotopy categories $\mathsf{Ho}(\mathcal{C}^K)$. Using the nerve functor N which gives us a fully-faithful embedding of the category Cat in the category Set_Δ of simplicial sets, we thus obtain the *prederivator* $\mathbb{D}_{\mathcal{C}}$ *associated to the* $\infty-$*category* $\mathcal{C}$:

$$\mathbb{D}_{\mathcal{C}}\colon J \longmapsto \mathbb{D}_{\mathcal{C}}(J) = \mathsf{Ho}\left(\mathcal{C}^{N(J)}\right)$$

The functoriality of this construction follows from Theorem 5.14 of [Joy08a].

The last example which we are about to mention now does not seem to be too interesting in its own right. But as we will see later it largely reduces the amount of work in many proofs.

Example 1.7. Let $\mathbb{D}$ be a prederivator and let M be a fixed category. Then the association

$$\mathbb{D}_M\colon\ \mathsf{Cat}^{\mathrm{op}} \longrightarrow \mathsf{CAT}\colon J \longmapsto \mathbb{D}_M(J) = \mathbb{D}(M \times J)$$

is again a prederivator. Similarly, given a functor $u\colon L \longrightarrow M$ we obtain a morphism of 2-functors $u^*\colon\ \mathbb{D}_M \longrightarrow \mathbb{D}_L$. There is a notion of morphisms of prederivators (cf. Section 2) and it is easy to see that the pairing $(\mathbb{D}, M) \longmapsto \mathbb{D}_M$ is actually functorial in both variables. It follows almost immediately that the category of prederivators is (right-)tensored over $\mathsf{Cat}^{\mathrm{op}}$. Thus, we have coherent isomorphisms $(\mathbb{D}_L)_M \cong \mathbb{D}_{L\times M}$ and $\mathbb{D}_e \cong \mathbb{D}$ for the terminal category e.

Remark 1.8. i) There is the following remark concerning the shapes of 'admissible diagrams'. In some situations, in particular under certain finiteness conditions, one does not wish to consider diagrams of arbitrary shapes but only of a certain kind (e.g. finite, finite-dimensional, posets). There is a notion of a *diagram category* Dia which is a 2-subcategory of Cat having certain closure properties. Correspondingly, there is then the associated notion of a *prederivator of type* Dia. We preferred to not give these definitions at the very beginning since we wanted to start immediately with the development of the theory. Once the main results are established we check which properties have been used and come back

to this point (cf. the discussion before Definition 4.26). So, the reader is invited to replace 'a (pre-)derivator' by 'a (pre-)derivator *of type* Dia' throughout this thesis. An example of the usefulness of this more flexible notion is given by Keller in [Kel07] where he shows that there is a stable derivator associated to an exact category in the sense of Quillen [Qui73] if one restricts to finite directed diagrams.
ii) There is an additional remark concerning the definition of a prederivator. In our setup a prederivator is a 2-functor $\mathbb{D}\colon \mathsf{Cat}^{\mathrm{op}} \longrightarrow \mathsf{CAT}$ as opposed to a more general pseudo-functor (which is for example done in [Fra96]). More specifically, we insisted on the fact that $\mathbb{D}$ preserves identities and compositions in a strict sense and not only up to coherent natural isomorphisms. Since all examples showing up in nature have this stronger functoriality we are fine with this notion. However, from the perspective of 'homotopical invariance of structures', a definition based on pseudo-functors would be better: let $\mathbb{D}$ be a prederivator and let us be given a category $\mathcal{E}_J$ for each small category J. Let us moreover assume that we are given equivalences $\mathbb{D}(J) \longrightarrow \mathcal{E}_J$. Then, in general, we cannot use the equivalences to obtain a prederivator $\mathbb{E}$ with $\mathbb{E}(J) = \mathcal{E}_J$ such that the equivalences of categories assemble to an equivalence of prederivators. This would only be the case if the equivalences are, in fact, isomorphisms which –by the basic philosophy of category theory– is a too strong notion. Nevertheless, for the sake of a simplification of the exposition we preferred to stick to 2-functors but want to mention that everything we do here can also be done with pseudo-functors.

Let now $\mathbb{D}$ be a prederivator and let $u\colon J \longrightarrow K$ be a functor. Motivated by the above examples we call the induced functor $\mathbb{D}(u) = u^*\colon \mathbb{D}(K) \longrightarrow \mathbb{D}(J)$ a *restriction of diagram functor* or *precomposition functor*. As a special case of this, let $J = e$ be the terminal category, i.e., the category with one object and identity morphism only. For an object k of K, we denote by $k\colon e \longrightarrow K$ the unique functor sending the unique object of e to k. Given a prederivator $\mathbb{D}$, we obtain, in particular, for each object $k \in K$ an associated functor $k^*\colon \mathbb{D}(K) \longrightarrow \mathbb{D}(e)$ which takes values in the *underlying category* $\mathbb{D}(e)$. Let us call such a functor an *evaluation functor*. For a morphism $f\colon X \longrightarrow Y$ in $\mathbb{D}(K)$, let us write $f_k\colon X_k \longrightarrow Y_k$ for its image in $\mathbb{D}(e)$ under k^*. Similarly, a morphism $\alpha\colon k_1 \longrightarrow k_2$ in K can be considered as a natural transformation of the corresponding classifying functors and thus gives rise to

$$\begin{tikzcd} e \arrow[r, bend left, "k_1"] \arrow[r, bend right, "k_2"'] \arrow[r, phantom, "\Downarrow\alpha"] & K, \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathbb{D}(K) \arrow[r, bend left, "k_1^*"] \arrow[r, bend right, "k_2^*"'] \arrow[r, phantom, "\Downarrow\alpha^*"] & \mathbb{D}(e). \end{tikzcd}$$

Under the categorical exponential law which can be written in a suggestive form as

$$\Big(\mathbb{D}(e)^{\mathbb{D}(K)}\Big)^K \cong \mathbb{D}(e)^{K\times\mathbb{D}(K)} \cong \Big(\mathbb{D}(e)^K\Big)^{\mathbb{D}(K)},$$

we obtain hence an *underlying diagram functor*

$$\mathsf{dia}_K\colon \mathbb{D}(K) \longrightarrow \mathsf{Fun}(K,\mathbb{D}(e)) = \mathbb{D}(e)^K.$$

Similarly, given a product $J \times K$ of two categories and $j \in J$, we can consider the corresponding functor

$$j \times \mathrm{id}_K \colon K \cong e \times K \longrightarrow J \times K.$$

Following the same arguments as above, we obtain a *partial underlying diagram functor*

$$\mathsf{dia}_{J,K} \colon \mathbb{D}(J \times K) \longrightarrow \mathbb{D}(K)^J.$$

These diagram functors will not be used until Section 4 where we discuss stable derivators. Nevertheless, we wanted to mention them at the very beginning to emphasize the importance of the difference between the categories $\mathbb{D}(K)$ and $\mathbb{D}(e)^K$.

Example 1.9. Let $\mathcal{M}$ be a model category. One can show that the prederivator $\mathbb{D}_{\mathcal{M}}$ of the above example makes sense without any assumption on $\mathcal{M}$. Let us believe this for the moment but see also the discussion around Theorem 1.24. Then $\mathbb{D}_{\mathcal{M}}([1])$ is the homotopy category $\mathsf{Ho}(\mathcal{M}^{[1]})$ of the arrow category $\mathcal{M}^{[1]}$ of $\mathcal{M}$ while $\mathbb{D}_{\mathcal{M}}(e)^{[1]}$ is the arrow category $\mathsf{Ho}(\mathcal{M})^{[1]}$ of the homotopy category of $\mathcal{M}$. These are, in general, non-equivalent categories, but they are connected by the underlying diagram functor

$$\mathsf{dia}_{[1]} \colon \mathsf{Ho}(\mathcal{M}^{[1]}) \longrightarrow \mathsf{Ho}(\mathcal{M})^{[1]}.$$

One can check in this case that $\mathsf{dia}_{[1]}$ is full and essentially surjective.

Definition 1.10. Let $\mathbb{D}$ be a prederivator and let $u \colon J \longrightarrow K$ be a functor.
i) The prederivator $\mathbb{D}$ admits *homotopy left Kan extensions along* u if the induced functor u^* has a left adjoint:

$$(u_! = \mathrm{HoLKan}_u, u^*) \colon \mathbb{D}(J) \rightharpoonup \mathbb{D}(K)$$

The prederivator $\mathbb{D}$ admits *homotopy colimits of shape* J if the functor p_J^* induced by $p_J \colon J \longrightarrow e$ has a left adjoint:

$$(p_{J!} = \mathrm{Hocolim}_J, p_J^*) \colon \mathbb{D}(J) \rightharpoonup \mathbb{D}(e)$$

ii) The prederivator $\mathbb{D}$ admits *homotopy right Kan extensions along* u if the induced functor u^* has a right adjoint:

$$(u^*, u_* = \mathrm{HoRKan}_u) \colon \mathbb{D}(K) \rightharpoonup \mathbb{D}(J)$$

The prederivator $\mathbb{D}$ admits *homotopy limits of shape* J if the functor p_J^* induced by $p_J \colon J \longrightarrow e$ has a right adjoint:

$$(p_J^*, p_{J*} = \mathrm{Holim}_J) \colon \mathbb{D}(e) \rightharpoonup \mathbb{D}(J)$$

Recall from classical category theory, that under cocompleteness resp. completeness assumptions left resp. right Kan extensions can be calculated pointwise by certain colimits resp. limits [ML98, p. 237]. More precisely, consider $u \colon J \longrightarrow K$ and $F \colon J \longrightarrow \mathcal{C}$ where $\mathcal{C}$ is a complete category:

$$\begin{tikzcd} J \arrow[r, "F"] \arrow[d, "u"'] & \mathcal{C} \\ K \arrow[ur, dashed, "\mathrm{RKan}_u(F)"'] & \end{tikzcd}$$

Then the right Kan extension $\mathrm{RKan}_u(F)$ of F along u exists and can be described using *Kan's formula* [Kan58] as

$$\mathrm{RKan}_u(F)_k \cong \lim_{J_{k/}} \mathrm{pr}^*(F) = \lim_{J_{k/}} F \circ \mathrm{pr}, \qquad k \in K.$$

In the above formula, we have used the following notation. Let $u\colon J \longrightarrow K$ be a functor and let k be an object of K. Then one can form the *slice category $J_{k/}$ of objects u-under k*. An object in this category is a pair (j, f) consisting of an object $j \in J$ together with a morphism $f\colon k \longrightarrow u(j)$ in K. Given two such objects (j_1, f_1) and (j_2, f_2), a morphism $g\colon (j_1, f_1) \longrightarrow (j_2, f_2)$ is a morphism $g\colon j_1 \longrightarrow j_2$ in J such that the obvious triangle in K commutes. Dually, one can form the *slice category $J_{/k}$ of objects u-over k*. In both cases, there are canonical functors

$$\mathrm{pr}\colon J_{k/} \longrightarrow J \qquad \text{and} \qquad \mathrm{pr}\colon J_{/k} \longrightarrow J$$

forgetting the morphism component. We will not distinguish these projection morphisms notationally but it will always be clear from the context which projection morphism we are considering. A dual formula holds for left Kan extension in the case of a cocomplete target category $\mathcal{C}$ and will not be made explicit.

The corresponding property for *homotopy* Kan extensions holds in the case of model categories (cf. the proof of Proposition 1.22) and will be demanded axiomatically for a derivator. In order to be able to formulate this axiom, we have to talk about base change morphisms. For this purpose, let $\mathbb{D}$ be a prederivator and consider a natural transformation of functors $\alpha\colon w \circ u \longrightarrow u' \circ v$. By an application of $\mathbb{D}$, we thus have the following diagram:

$$\begin{tikzcd} J \arrow[r, "v"] \arrow[d, "u"'] \arrow[dr, phantom, "\nearrow"] & J' \arrow[d, "u'"] \\ K \arrow[r, "w"'] & K' \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathbb{D}(J) & \mathbb{D}(J') \arrow[l, "v^*"'] \\ \mathbb{D}(K) \arrow[u, "u^*"] \arrow[ur, phantom, "\nearrow"] & \mathbb{D}(K') \arrow[l, "w^*"] \arrow[u, "u'^*"'] \end{tikzcd}$$

Let us assume that $\mathbb{D}$ admits homotopy right Kan extensions along u and u'. We denote any chosen adjoints and the corresponding adjunction morphisms by

$$(u^*, u_*), \qquad \eta\colon \mathrm{Id} \longrightarrow u_* \circ u^*, \qquad \text{and} \qquad \epsilon\colon u^* \circ u_* \longrightarrow \mathrm{Id}$$

in the case of u and similarly in the case of u'. We can thus define the natural transformation β as the following sequence of natural transformations:

$$\begin{tikzcd} w^* \circ u'_* \arrow[r, "\beta"] \arrow[d, "\eta"'] & u_* \circ v^* \\ u_* \circ u^* \circ w^* \circ u'_* \arrow[r, "\alpha^*"'] & u_* \circ v^* \circ u'^* \circ u'_* \arrow[u, "\epsilon'"'] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{D}(J) \arrow[d, "u_*"'] & \mathbb{D}(J') \arrow[l, "v^*"'] \arrow[d, "u'_*"] \arrow[dl, phantom, "\nwarrow"] \\ \mathbb{D}(K) & \mathbb{D}(K') \arrow[l, "w^*"] \end{tikzcd}$$

This natural transformation β is called 'the' *base change morphism associated to* α. Dually, consider a natural transformation $\alpha\colon u'\circ v\longrightarrow w\circ u$ as in:

$$\begin{tikzcd} J \arrow[r, "v"] \arrow[d, "u"'] \arrow[dr, phantom, "\swArrow"] & J' \arrow[d, "u'"] \\ K \arrow[r, "w"'] & K' \end{tikzcd} \qquad \begin{tikzcd} \mathbb{D}(J) \arrow[dr, phantom, "\swArrow"] & \mathbb{D}(J') \arrow[l, "v^*"'] \\ \mathbb{D}(K) \arrow[u, "u^*"] & \mathbb{D}(K') \arrow[l, "w^*"] \arrow[u, "u'^*"'] \end{tikzcd}$$

Let us now assume that the prederivator admits homotopy left Kan extensions along u and u' and let us use similar notation as above. We can then define the natural transformation β as follows:

$$\begin{tikzcd} u_!\circ v^* \arrow[r, "\beta"] \arrow[d, "\eta'"'] & w^*\circ u'_! \\ u_!\circ v^*\circ u'^*\circ u'_! \arrow[r, "\alpha^*"'] & u_!\circ u^*\circ w^*\circ u'_! \arrow[u, "\epsilon"'] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{D}(J) \arrow[d, "u_!"'] \arrow[dr, phantom, "\seArrow"] & \mathbb{D}(J') \arrow[l, "v^*"'] \arrow[d, "u'_!"] \\ \mathbb{D}(K) & \mathbb{D}(K') \arrow[l, "w^*"] \end{tikzcd}$$

This natural transformation β is again called 'the' *base change morphism associated to* α.

In both cases, we constructed the base change morphism by choosing certain adjoints to given precomposition functors. We then precomposed the given natural transformation with the corresponding unit and postcomposed it with the corresponding counit. So, it is obvious that the resulting base change morphism depends on the actual choice of the adjunction. But, by a straightforward application of Lemma 1.26, one can check that base change morphisms obtained from different choices of adjunctions differ only by natural isomorphisms. Thus, in particular, the fact that the base change morphism is an isomorphism is independent of the choice of adjunctions.

We will mainly need the base change morphisms in the following situation. Let $u\colon J\longrightarrow K$ be a functor and $k\in K$ an object. Identifying k again with the corresponding functor $k: e\longrightarrow K$, we have the following two natural transformations α in the context of the slice constructions:

$$\begin{tikzcd} J_{k/} \arrow[r, "\mathrm{pr}"] \arrow[d, "p_{J_{k/}}"'] \arrow[dr, phantom, "\neArrow"] & J \arrow[d, "u"] \\ e \arrow[r, "k"'] & K \end{tikzcd} \qquad \begin{tikzcd} J_{/k} \arrow[r, "\mathrm{pr}"] \arrow[d, "p_{J_{/k}}"'] \arrow[dr, phantom, "\swArrow"] & J \arrow[d, "u"] \\ e \arrow[r, "k"'] & K \end{tikzcd}$$

The components of α at $(j,f\colon k\longrightarrow u(j))$ resp. $(j,f\colon u(j)\longrightarrow k)$ are f in both cases. Assuming $\mathbb{D}$ to be a prederivator admitting the necessary homotopy Kan extensions, we thus obtain the following base change morphisms:

$$\begin{tikzcd} \mathbb{D}(J_{k/}) \arrow[d, "\mathrm{Holim}_{J_{k/}}"'] \arrow[dr, phantom, "\nwArrow"] & \mathbb{D}(J) \arrow[l, "\mathrm{pr}^*"'] \arrow[d, "u_*"] \\ \mathbb{D}(e) & \mathbb{D}(K) \arrow[l, "k^*"] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{D}(J_{/k}) \arrow[d, "\mathrm{Hocolim}_{J_{/k}}"'] \arrow[dr, phantom, "\seArrow"] & \mathbb{D}(J) \arrow[l, "\mathrm{pr}^*"'] \arrow[d, "u_!"] \\ \mathbb{D}(e) & \mathbb{D}(K) \arrow[l, "k^*"] \end{tikzcd}$$

Asking these base change morphisms to be natural isomorphisms is a convenient way to axiomatize Kan's formulas. With these preparations we can give the central definition of a derivator.

Definition 1.11. A prederivator $\mathbb{D}$ is called a *derivator* if it satisfies the following axioms:
(Der1) For two categories J_1 and J_2, the functor $\mathbb{D}(J_1 \sqcup J_2) \longrightarrow \mathbb{D}(J_1) \times \mathbb{D}(J_2)$ induced by the inclusions is an equivalence of categories. Moreover, the category $\mathbb{D}(\emptyset)$ is not the empty category.
(Der2) A morphism $f\colon X \longrightarrow Y$ in $\mathbb{D}(J)$ is an isomorphism if and only if $f_j\colon X_j \longrightarrow Y_j$ is an isomorphism in $\mathbb{D}(e)$ for every object $j \in J$.
(Der3) For every functor $u\colon J \longrightarrow K$, there are homotopy left and right Kan extensions along u:

$$(u_!, u^*)\colon\ \mathbb{D}(J) \rightharpoonup \mathbb{D}(K) \qquad \text{and} \qquad (u^*, u_*)\colon\ \mathbb{D}(K) \rightharpoonup \mathbb{D}(J).$$

(Der4) For every functor $u\colon J \longrightarrow K$ and every $k \in K$, the base change morphisms

$$\mathrm{Hocolim}_{J_{/k}}\, \mathrm{pr}^*(X) \xrightarrow{\beta} u_!(X)_k \qquad \text{and} \qquad u_*(X)_k \xrightarrow{\beta} \mathrm{Holim}_{J_{k/}}\, \mathrm{pr}^*(X)$$

are isomorphisms for all $X \in \mathbb{D}(J)$.

A few remarks on the axioms are in order. The first axiom says of course that a diagram on a disjoint union is completely determined by its restrictions to the direct summands. The second part of the first axiom is included in order to exclude the 'empty derivator' as an example. The second axiom can be motivated from the examples as follows. A natural transformation is an isomorphism if and only if it is pointwise an isomorphism. Similarly, in the context of an abelian category, there is the easy fact that a morphism of chain complexes in a functor category is a quasi-isomorphism if and only if it is a quasi-isomorphism at each object. Moreover, in the context of model categories, whatever model structure one establishes on a diagram category with values in a model category, one certainly wants the class of weak equivalences to be defined pointwise. Finally, the corresponding result for $\infty-$categories is established by Joyal as Theorem 5.14 in [Joy08a]. The last two axioms of course encode a 'homotopical bicompleteness property' together with Kan's formulas. One could easily develop a more general theory of prederivators which are only homotopy (co)complete or even only have a certain class of homotopy (co)limits.

Example 1.12. Let $\mathcal{C}$ be a category. The represented prederivator $y(\mathcal{C})\colon J \longmapsto \mathsf{Fun}(J, \mathcal{C})$ is a derivator if and only if $\mathcal{C}$ is bicomplete. Thus, the 2-category of bicomplete categories is embedded into the 2-category of derivators.

Definition 1.13. A derivator $\mathbb{D}$ is called *strong* if the partial underlying diagram functor

$$\mathsf{dia}_{[1],K}\colon\ \mathbb{D}([1] \times K) \longrightarrow \mathbb{D}(K)^{[1]}$$

is full and essentially surjective for each category K.

Remark 1.14. The strongness property of a derivator is a bit harder to motivate. Above we mentioned that derivators associated to model categories are examples of strong derivators. Moreover, since the partial underlying diagram functors are isomorphisms for represented derivators, these are certainly also strong. Thus, the strongness property is satisfied by the naturally occurring derivators.

In this paper, the strongness will play a key role in the construction of the triangulated structures on the values of a stable derivator. The point is that the strongness property allows one to lift morphisms in the underlying category $\mathbb{D}(e)$ to the category $\mathbb{D}([1])$ where we can apply certain constructions to it. Similarly, it allows us to lift morphisms between morphisms in $\mathbb{D}(e)$ to morphisms in $\mathbb{D}([1])$ or even to objects in $\mathbb{D}([1]\times[1])$.

But it is not only the case for the stable context that the strongness property is convenient. Already in the context of pointed derivators it is very helpful. This property allows the construction of fiber and cofiber sequences associated to a morphism in the underlying category of the derivator. Similarly, one might expect that in later developments of the theory this property might also be useful in the unpointed context. Nevertheless, we follow Maltsiniotis in not including the strongness as an axiom of the basic notion of a derivator.

Moreover, it might be helpful to consider variants of the definition. Given a family $\mathcal{F}$ of small categories, let us call a derivator $\mathbb{D}$ to be $\mathcal{F}$-*strong* if the partial underlying diagram functors $\mathsf{dia}_{J,K}$ are full and essentially surjective for all $J\in\mathcal{F}$ and all categories K. Heller considered in [Hel88] the case where $\mathcal{F}$ consists of all finite, free categories.

The idea is of course that the derivator encodes additional structure on its values. One nice feature of this approach is that this structure does not have to be chosen but its existence can be deduced from the axioms. Note that all axioms are of the form that they demand a *property*; the only actual *structure* is the given prederivator. This is similar to the situation of additive categories where the enrichment in abelian groups can uniquely be deduced from the fact that the underlying category has certain exactness properties. We will come back to this point later in the context of stable derivators (cf. Remark 4.25).

As a first example of this 'higher structure' we give the following example. We will pursue this more systematically from Subsection 1.3 on. Let J be a category and consider the coproduct $J\sqcup J$ together with the codiagonal and the inclusion functors:

$$\begin{tikzcd} J \arrow[r, "i_1"] \arrow[dr, "\mathrm{id}_J"'] & J\sqcup J \arrow[d, "\nabla_J"] & J \arrow[l, "i_2"'] \arrow[dl, "\mathrm{id}_J"] \\ & J & \end{tikzcd}$$

Proposition 1.15. *Let $\mathbb{D}$ be a derivator and let J be a category.*
i) *The value of $\mathbb{D}$ at the empty category $\emptyset$ is trivial, i.e., $\mathbb{D}(\emptyset)$ is equivalent to e.*
ii) *The category $\mathbb{D}(J)$ admits an initial object $\emptyset$ and a terminal object $*$.*
iii) *The category $\mathbb{D}(J)$ admits finite coproducts and finite products.*

Proof. i) Considering the disjoint union $\emptyset=\emptyset\sqcup\emptyset$, (Der1) implies that we have an equivalence given by the diagonal functor $\mathbb{D}(\emptyset)\longrightarrow\mathbb{D}(\emptyset)\times\mathbb{D}(\emptyset)$. Thus, all morphism sets are

singletons and the category $\mathbb{D}(\emptyset)$ is trivial. We denote any object of $\mathbb{D}(\emptyset)$ by 0.
ii) Consider the unique empty functor $\emptyset_J : \emptyset \longrightarrow J$ and apply (Der3) in order to obtain left resp. right adjoints

$$\emptyset_{J!} \colon \mathbb{D}(\emptyset) \longrightarrow \mathbb{D}(J), \qquad \emptyset_{J*} \colon \mathbb{D}(\emptyset) \longrightarrow \mathbb{D}(J).$$

Since a left (right) adjoint preserves initial (final) objects, the image of any object 0 under $\emptyset_{J!}$ ($\emptyset_{J*}$) is an initial (terminal) object in $\mathbb{D}(J)$. Let us denote any such image by $\emptyset$ resp. $*$.
iii) By (Der1), we have an equivalence of categories $(i_1^*, i_2^*) : \mathbb{D}(J \sqcup J) \xrightarrow{\simeq} \mathbb{D}(J) \times \mathbb{D}(J)$. Choose an inverse equivalence k and set for $X, Y \in \mathbb{D}(J)$:

$$X \sqcup Y = \nabla_{J!} \circ k(X, Y) \in \mathbb{D}(J)$$

One can now easily check directly the universal property. Alternatively, note that we have the following composite adjunction

$$\begin{tikzcd} \mathbb{D}(J) \times \mathbb{D}(J) \arrow[r, shift left, "k"] & \mathbb{D}(J \sqcup J) \arrow[l, shift left, "(i_1^*, i_2^*)"] \arrow[r, shift left, "\nabla_{J!}"] & \mathbb{D}(J) \arrow[l, shift left, "\nabla_J^*"] \end{tikzcd}$$

where the right adjoint is the diagonal functor $(i_1^*, i_2^*)\nabla_J^* = \Delta_{\mathbb{D}(J)}$. Similarly, $\nabla_{J*} \circ k$ will define a product functor on $\mathbb{D}_J$. □

Let us quickly recall the dualization process for derivators. As the author was confused for a while about the different dualizations for 2-categories we will give some details. First, we can consider Cat as a *symmetric* monoidal category. The formation of opposite categories can always be performed in the context of enriched categories as soon as the enrichment level is symmetric monoidal. Thus, we can form the dual of a 2-category $\mathcal{C}$ as a category enriched over Cat . The result of this dualization is the 2-category $\mathcal{C}^{\mathrm{op}}$ in which the 1-morphisms have changed the direction. Alternatively, since the Cartesian monoidal structure on the 1-category Cat behaves well with dualization, there is a second way of dualizing a general 2-category. More precisely, we can consider the dualization of small categories as a monoidal functor $(-)^{\mathrm{op}} \colon \mathsf{Cat} \longrightarrow \mathsf{Cat}$ with respect to the Cartesian structures. Since any monoidal functor induces a base change functor at the level of enriched categories (cf. to Section 3 of [Gro11]), we obtain a 2-category $\mathcal{C}^{\mathrm{co}}$. This 2-category is obtained from $\mathcal{C}$ by inverting the direction of 2-morphisms. Finally, applying both dualizations to $\mathcal{C}$ we obtain the 2-category $\mathcal{C}^{\mathrm{co,op}} = \mathcal{C}^{\mathrm{op,co}}$. Thus, given a 2-category using the various dualizations we obtain 4 different 2-categories. More generally, an n-category has 2^n different dualizations.

Remarking that the dualization on the 2-category Cat inverts the direction of the 2-morphisms but keeps the direction of the 1-morphisms we thus can make the following definition.

Definition 1.16. Let $\mathbb{D}$ be a prederivator, then we define the dual prederivator $\mathbb{D}^{\mathrm{op}}$ by the following diagram:

$$\begin{tikzcd} \mathsf{Cat}^{\mathrm{op}} \arrow[r, "\mathbb{D}^{\mathrm{op}}"] \arrow[d, "(-)^{\mathrm{op}}"'] & \mathsf{CAT} \\ \mathsf{Cat}^{\mathrm{co,op}} \arrow[r, "\mathbb{D}"'] & \mathsf{CAT}^{\mathrm{co}} \arrow[u, "(-)^{\mathrm{op}}"'] \end{tikzcd}$$

Example 1.17. A prederivator $\mathbb{D}$ is a derivator if and only if its dual $\mathbb{D}^{\mathrm{op}}$ is a derivator.

Recall the notation of Example 1.7. The following result is quite important and will be used again and again. It allows, in particular, to prove a property of an arbitrary value of a derivator by considering only its underlying category.

Proposition 1.18. *Let $\mathbb{D}$ be a derivator and let M be a category. Then the prederivator $\mathbb{D}_M$ is again a derivator.*

Proof. The verification of (Der1) is straightforward. Axiom (Der2) follows immediately from the following: for a morphism $f\colon X \longrightarrow Y$ in $\mathbb{D}_M(J) = \mathbb{D}(M \times J)$ we have

$$\begin{aligned} & f\colon X \longrightarrow Y \text{ is an isomorphism} \\ \Longleftrightarrow \quad & f_{m,j}\colon X_{m,j} \longrightarrow Y_{m,j} \text{ is an isomorphism for all } m, j \\ \Longleftrightarrow \quad & f_j\colon X_j \longrightarrow Y_j \text{ is an isomorphism for all } j. \end{aligned}$$

Axiom (Der3) is again immediate from the corresponding property for $\mathbb{D}$. The remaining and the longest part of this proof is (Der4) which can be found as Lemma A.6 in Appendix A. □

1.2. **Model categories give rise to derivators.** Before we begin with the development of the theory of derivators we quickly pause to give the important example of derivators associated to nice model categories. We include this here not only for the sake of completeness but also because our proof differs from the one given in [Cis03]. Our proof is completely self-dual and is simpler in that it does not make use of the explicit description of the generating (acyclic) projective cofibrations of a diagram category associated to a cofibrantly-generated model category. We restrict attention to the following situation.

Definition 1.19. A model category $\mathcal{M}$ is called *combinatorial* if it is cofibrantly-generated and if the underlying category is presentable.

This class of model categories was introduced by Smith and is studied e.g. in [Lur09, Ros09, Bek00, Dug01a]. For background on cofibrantly-generated model categories we refer to [Hov99]. The theory of presentable categories was initiated by Gabriel and Ulmer in [GU71]. Further references to this theory are [Bor94b, AR94]. One basic idea of the presentability assumption is the following one. The presentability imposes beyond the bicompleteness a certain 'smallness condition' on a category which has at least two important consequences. The first one is that the usual set-theoretic problems occurring when one considers functor categories disappear at least if one restricts attention to colimit-preserving functors. But this is anyhow the adapted class of morphism to be studied in this context. Moreover, in the world of presentable categories one can focus more on conceptual ideas than on technical points of certain arguments: a functor between presentable categories is a left adjoint if and only if it is colimit-preserving. The usual 'solution set condition' of Freyd's adjoint functor theorem is automatically fulfilled in this context. For more comments in this direction see Subsection 2.6 in [Gro10b], where these ideas are discussed in the context of presentable $\infty-$categories. Important examples of presentable categories are the categories of sets, simplicial sets, all presheaf categories (and, more generally, all Grothendieck toposes), algebraic categories as well as the Grothendieck abelian categories and the category Cat. A non-example is the category of topological spaces although this can be repaired if one sticks to the 'really convenient category' (Smith) of Δ-generated spaces ([FR08]). The slogan is that 'presentable categories are small enough so that certain set-theoretical problems disappear but are still large enough to include many important examples'.

Anyhow, for this subsection all we need from the theory of combinatorial model categories is the validity of the next proposition. Many of the results which follow will not use the presentability assumption but could be deduced axiomatically from the conclusion of this proposition.

Proposition 1.20. *Let $\mathcal{M}$ be a combinatorial model category and let J be a small category. The category $\mathsf{Fun}(J, \mathcal{M})$ can be endowed with the projective and with the injective model structure.*

Recall that the *projective* model structure is determined by the fact that the weak equivalences and the fibrations are defined levelwise. In the *injective* model structure this is the case for the weak equivalences and the cofibrations. We will denote the functor categories $\mathcal{M}^J$ endowed with the corresponding model structures by $\mathcal{M}^J_{proj}$ resp. $\mathcal{M}^J_{inj}$. In the special case where the combinatorial model category we start with is the category of simplicial sets endowed with the homotopy-theoretic Kan model structure, the projective model structure on a diagram category is the Bousfield-Kan structure of [BK72] while the injective model structure is the Heller structure of [Hel88]. One point of these model structures is that certain adjunctions are now Quillen adjunctions for trivial reasons.

Lemma 1.21. *Let $\mathcal{M}$ be a combinatorial model category and let $u\colon J \longrightarrow K$ be a functor. Then we have the following Quillen adjunctions*

$$(u_!, u^*)\colon \mathcal{M}^J_{proj} \longrightarrow \mathcal{M}^K_{proj} \qquad \text{and} \qquad (u^*, u_*)\colon \mathcal{M}^K_{inj} \longrightarrow \mathcal{M}^J_{inj}.$$

We now have almost everything at our disposal needed to establish the following result.

Proposition 1.22. *Let $\mathcal{M}$ be a combinatorial model category. Then the assignment*

$$\mathbb{D}_{\mathcal{M}}\colon \mathsf{Cat}^{\mathrm{op}} \longrightarrow \mathsf{CAT}\colon J \longmapsto \mathsf{Ho}(\mathcal{M}^J)$$

defines a derivator.

Proof. The first axiom (Der1) is immediate. (Der2) holds in this case since the weak equivalences are precisely the morphisms which are inverted by the formation of homotopy categories and since the weak equivalences are defined levelwise. It is thus enough to consider the two axioms on homotopy Kan extensions. We treat only the case of homotopy right Kan extensions. The other case follows by duality. Axiom (Der3) on the existence of homotopy Kan extension functors follows easily from the last lemma since one only has to consider the associated derived adjunctions at the level of homotopy categories. So it remains to establish Kan's formula. For this purpose, let $u\colon J \longrightarrow K$ be a functor and let $k \in K$ be an object. Consider the following diagram, which commutes up to natural isomorphism by the usual base change morphism from classical category theory:

$$\begin{tikzcd} \mathcal{M}^{J_{k/}} \arrow[d, "\lim"'] \arrow[dr, phantom, "\cong"] & \mathcal{M}^J \arrow[l, "\mathrm{pr}^*"'] \arrow[d, "u_*"] \\ \mathcal{M} & \mathcal{M}^K \arrow[l, "k^*"] \end{tikzcd}$$

By the last lemma, the functors lim and u_* are right Quillen functors with respect to the injective model structures. If we can show that also the functors k^* and pr^* are right Quillen functors with respect to the injective model structures, then we are done. In fact, in that case the two compositions of derived right Quillen functors are canonically isomorphic and this in turn shows that the base change morphism is an isomorphism. So let us show that k^* is a right Quillen functor. By definition of the injective model structures, k^* preserves weak equivalences. Hence it is enough to show that k^* preserves fibrations. Using the adjunction $(k_!, k^*)$ it is enough to show that $k_!\colon \mathcal{M} \longrightarrow \mathcal{M}^K$ preserves acyclic cofibrations. But an easy calculation with left Kan extension shows that we have $k_!(X)_l \cong \coprod_{\mathsf{hom}_K(k,l)} X$. From this description it is immediate that $k_!$ preserves acyclic cofibrations. Finally, we will show in Lemma 1.23 that also pr^* is a right Quillen functor with respect to the injective model structure. □

To conclude the proof of Proposition 1.22 we have to show that the functor $\mathrm{pr}^*\colon \mathcal{M}^J \longrightarrow \mathcal{M}^{J_{k/}}$ is a right Quillen functor with respect to the injective model structures. It is again immediate that pr^* preserves injective weak equivalences. Hence it suffices to show that pr^* preserves injective fibrations. We will prove such a result for arbitrary Grothendieck opfibrations with discrete fibers (cf. Appendix A for a very short review of this notion). By Example A.2 this applies to our situation.

Lemma 1.23. *Let $u\colon J \longrightarrow K$ be a Grothendieck opfibration with discrete fibers and let $\mathcal{M}$ be a combinatorial model category. Then the functor $u^*\colon \mathcal{M}^K \longrightarrow \mathcal{M}^J$ preserves injective fibrations.*

Proof. By adjointness, it is enough to show that the left adjoint $u_!\colon \mathcal{M}^J \longrightarrow \mathcal{M}^K$ preserves acyclic injective cofibrations. For this purpose, let $X \in \mathcal{M}^J$ and let $k \in K$. Then we make the following calculation:

$$u_!(X)_k \;\cong\; \mathrm{colim}_{J_{/k}}\, X \circ \mathrm{pr} \;\cong\; \mathrm{colim}_{J_k}\, X \circ \mathrm{pr} \circ c \;\cong\; \coprod_{j \in J_k} X_j$$

The first isomorphism is again Kan's formula for Kan extensions. The second isomorphism comes from the characterization of Grothendieck opfibration in Proposition A.1 together with the cofinality of right adjoints applied to the canonical functor $c\colon J_k \longrightarrow J_{/k}$. Finally, the last isomorphism uses the fact that the Grothendieck opfibration has discrete fibers. From this explicit description of $u_!$ the claim follows immediately. $\square$

The proof of the above theorem actually shows a bit more. Given a cofibrantly-generated model category $\mathcal{M}$, the prederivator $\mathbb{D}_{\mathcal{M}}$ is a what could be called *cocomplete prederivator* (with the obvious meaning). But by far more is true. There is the following more general result which is due to Cisinski [Cis03].

Theorem 1.24. *Let $\mathcal{M}$ be a model category and let J be a small category. Denote by W_J the class of levelwise weak equivalences in $\mathsf{Fun}(J,\mathcal{M})$. Then the assignment*

$$\mathbb{D}_{\mathcal{M}}\colon\ \mathsf{Cat}^{\mathrm{op}} \longrightarrow \mathsf{CAT}\colon J \longmapsto \mathsf{Fun}(J,\mathcal{M})[W_J^{-1}]$$

defines a derivator.

The basic idea is to reduce the situation of an arbitrary diagram category using certain cofinality arguments to the situation where the indexing categories are so-called Reedy categories ([Hov99]). The proof can be found in [Cis03]. From the proof it will, in particular, follow that the above localizations make sense (i.e., that no change of universe is necessary!) although, in general, there is no model structure on $\mathsf{Fun}(J,\mathcal{M})$ with the levelwise weak equivalences as weak equivalences.

Remark 1.25. • It can be shown that the above assignment $\mathcal{M} \longmapsto \mathbb{D}_{\mathcal{M}}$ suitably restricted defines a bi-equivalence of theories. Loosely speaking this says that nice model categories and nice derivators do the same job. More precisely, Renaudin has shown such a result in [Ren09] by establishing the following two steps: Let ModQ denote the 2-category of combinatorial model categories with Quillen adjunctions $(F,U)\colon \mathcal{M} \rightharpoonup \mathcal{N}$ as morphisms and natural transformations of left adjoints as 2-morphisms. Renaudin shows that there is a pseudo-localization $\mathsf{ModQ}[W^{-1}]$ of the combinatorial model categories at the class W of Quillen equivalences. Moreover, let $\mathsf{Der}^{\mathsf{Pr}}$ denote the 2-category of derivators of small presentation together with adjunctions as morphisms. A derivator is said to be of small presentation if it can be obtained as a 'nice' localization of the derivator associated to simplicial presheaves. The assignment $\mathbb{D}_{(-)}\colon\ \mathsf{ModQ} \longrightarrow \mathsf{Der}^{\mathsf{Pr}}\colon \mathcal{M} \longmapsto \mathbb{D}_{\mathcal{M}}$ factors then up to natural isomorphism over the pseudo-localization $\mathsf{ModQ}[W^{-1}]$ as indicated in:

$$\begin{tikzcd}
\mathsf{ModQ} \arrow[r, "\mathbb{D}_{(-)}"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\cong"] & \mathsf{Der}^{\mathsf{Pr}} \\
\mathsf{ModQ}[W^{-1}] \arrow[ur, bend right, "\mathbb{D}_{(-)}"'] &
\end{tikzcd}$$

Renaudin showed that the induced 2-functor $\mathbb{D}_{(-)}\colon\ \mathsf{ModQ}[W^{-1}] \longrightarrow \mathsf{Der}^{\mathsf{Pr}}$ is a biequivalence, i.e., a 2-functor which is biessentially surjective and fully-faithful in the sense that it induces *equivalences* of morphism categories (for biequivalences cf. e.g. to [Str96, Lac10] and to [Lac02, Lac04] for their more conceptual role).
• A combination of Theorem 1.24 and Example 1.12 thus shows that derivators form quite

a general framework. Among others they subsume bicomplete categories and model categories. Thus, this framework allows us to treat categorical limits and colimits and the homotopical variants on an equal footing. This is similar to the case of $\infty-$categories. In that theory, the notion of limits and colimits also subsumes the two variants. In the case of nerves of categories, the notion reduces to the classical notion of (co)limits, while when applied to coherent nerves of simplicial model categories it coincides with the notion of homotopy (co)limits.

• We want to include a remark on different approaches to a theory of $(\infty, 1)-$categories. There are by now many different ways to axiomatize such a theory. Among these are the model categories, the $\infty-$categories, and the derivators. These theories are interrelated by various constructions. For a simplicial model category, one can use the coherent nerve construction of Cordier [Cor82] to obtain an underlying $\infty-$category. Moreover, given a bicomplete $\infty-$category or a model category, by forming systematically homotopy categories one obtains an associated derivator. These three theories are in fact all 'equivalent in a certain sense' if one is willing to restrict to nice subclasses. These comparison results rely heavily on the following 'two-step hierarchy'. In classical category theory there are the presheaf categories which can be considered as universal cocompletions. More precisely, the fact that every contravariant set-valued functor on a small category is canonically a colimit of representable ones can be used to prove such a result. Nice localizations of these presheaf categories (the so-called accessible, reflective localizations) give us precisely the presentable categories. These two main steps, namely to establish the universal property of presheaf categories and to characterize presentable categories as nice localizations of presheaf categories, can be redone for all the different theories. To achieve this one has to replace presheaf categories by *simplicial presheaf categories* which is fine with the basic philosophy of higher category theory. Moreover, the classical localization theory is replaced by a suitable Bousfield localization theory [Bou75, Hir03]. For model categories, this was done by Dugger in [Dug01b, Dug01a], while the corresponding results for $\infty-$categories can be found in Lurie's [Lur09]. The characterization of presentable $\infty-$categories as being precisely the accessible, reflective localizations of simplicial presheaf categories is therein credited to [Sim07]. For derivators, the free generation property of the derivator associated to simplicial presheaves can be found in [Cis08]. Note however that the basic model used in the background is not the category of simplicial sets but the category of small categories. It can be shown that this way also all 'homotopy types are modeled'. Finally, until now, the 'normal form theorem' for derivators of small presentation is only turned into a definition in [Ren09]. The author plans to come back to this point in a later project. Having established these similar theories at all different levels one can then establish the comparison results if one restricts to the subclasses of (simplicial) combinatorial model categories, presentable $\infty-$categories, and derivators of small presentation.

1.3. **Some properties of homotopy Kan extensions.** Before we deduce some properties of derivators, we recall the following elementary fact from category theory. For simplicity, we will only formulate one half of the lemma. Every statement has a dual statement which is also true (by the dual proof).

Lemma 1.26. i) *Let* $(L,R)\colon \mathcal{C} \rightharpoonup \mathcal{D}$ *be a pair of adjoint functors and let* η *resp.* ϵ *be the unit resp. counit. Then* L *is fully-faithful if and only if* $\eta\colon \mathrm{id} \longrightarrow RL$ *is an isomorphism. Moreover, in this case,* $d \in \mathcal{D}$ *lies in the essential image of* L *if and only if* $\epsilon_d\colon LRd \longrightarrow d$ *is an isomorphism.*
ii) *Let* $(L,R)\colon \mathcal{C} \rightharpoonup \mathcal{D}$ *be a pair of adjoint functors such that* L *is fully-faithful and* $RL = \mathrm{id}$. *Then we can find a possibly different adjunction isomorphism such that the adjunction unit* η *is the identity, i.e.,* $\eta = \mathrm{id}\colon \mathrm{id} \longrightarrow RL = \mathrm{id}$.
iii) *Let* $(L,R,\eta,\epsilon),\ (L,R',\eta',\epsilon')\colon \mathcal{C} \rightharpoonup \mathcal{D}$ *be two adjunctions with the same left adjoint, then there is a natural isomorphism* $\phi\colon R \xrightarrow{\cong} R'$ *which is compatible with the units and the counits. Thus,* ϕ *makes the two following diagrams commutative:*

$$
\begin{tikzcd}
\mathrm{id}_{\mathcal{C}} \arrow[r, "\eta"] \arrow[dr, bend right, "\eta'"'] & RL \arrow[d, "\phi"] \\
& R'L
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
LR \arrow[r, "\epsilon"] \arrow[d, "\phi"'] & \mathrm{id}_{\mathcal{D}} \\
LR' \arrow[ur, bend right, "\epsilon'"'] &
\end{tikzcd}
$$

Using the 2-functoriality of prederivators, the following is immediate.

Lemma 1.27. *Let* $\mathbb{D}$ *be a prederivator and let* $(u,v)\colon J \rightharpoonup K$ *be an adjunction. Then we obtain an adjunction*

$$(v^*,u^*)\colon\ \mathbb{D}(J) \rightharpoonup \mathbb{D}(K).$$

Moreover, if u *resp.* v *is fully-faithfully, then so is* v^* *resp.* u^*.

Proof. An adjunction is alternatively given by a quadruple (u,v,η,ϵ) where

$$\eta\colon\ \mathrm{id} \longrightarrow vu \qquad \text{and} \qquad \epsilon\colon uv \longrightarrow \mathrm{id}$$

are the unit resp. counit transformations subject to the so-called triangular identities:

$$
\begin{tikzcd}
u \arrow[r, "\eta"] \arrow[dr, "\mathrm{id}"'] & uvu \arrow[d, "\epsilon"] \\
& u
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
v \arrow[r, "\eta"] \arrow[dr, "\mathrm{id}"'] & vuv \arrow[d, "\epsilon"] \\
& v
\end{tikzcd}
$$

An application of $\mathbb{D}$ to such a quadruple yields a quadruple $(v^*,u^*,\eta^*,\epsilon^*)$ where

$$\eta^*\colon\ \mathrm{id} \longrightarrow u^*v^* \qquad \text{and} \qquad \epsilon^*\colon v^*u^* \longrightarrow \mathrm{id},$$

and which satisfies the following:

$$
\begin{tikzcd}
v^* \arrow[r, "\eta^*"] \arrow[dr, "\mathrm{id}"'] & v^*u^*v^* \arrow[d, "\epsilon^*"] \\
& u^*
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
u^* \arrow[r, "\eta^*"] \arrow[dr, "\mathrm{id}"'] & u^*v^*u^* \arrow[d, "\epsilon^*"] \\
& u^*
\end{tikzcd}
$$

We thus have an adjunction $(v^*,u^*,\eta^*,\epsilon^*)$. The second part follows directly by part i) of the last lemma. $\square$

Thus, for an adjunction $(u,v)\colon J \rightharpoonup K$, the homotopy Kan extension functors $u_!$ and v_* exist for every prederivator and we have natural isomorphisms

$$v^* \cong u_! \qquad \text{and} \qquad u^* \cong v_*.$$

Moreover, these are compatible with the unit resp. counit transformation by the previous lemma. Taken together with the homotopy (co)limit functors on J and K they imply the following result on the cofinality of right adjoints and dually for left adjoints.

Lemma 1.28. *Let $\mathbb{D}$ be a derivator and consider an adjunction $(u,v)\colon J \rightharpoonup K$. Then there are natural isomorphisms $\mathrm{Hocolim}_J \cong \mathrm{Hocolim}_K v^*$ and $\mathrm{Holim}_K \cong \mathrm{Holim}_J u^*$ such that the following two diagrams commute:*

$$\begin{tikzcd} \mathrm{Hocolim}_K v^* p_J^* \arrow[r, equal] & \mathrm{Hocolim}_K p_K^* \arrow[d, "\epsilon"] \\ \mathrm{Hocolim}_J p_J^* \arrow[u] \arrow[r, "\epsilon"'] & \mathrm{id} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{id} \arrow[r, "\eta"] \arrow[d, "\eta"'] & \mathrm{Holim}_K p_K^* \arrow[d] \\ \mathrm{Holim}_J p_J^* \arrow[r, equal] & \mathrm{Holim}_J u^* p_K^* \end{tikzcd}$$

An important special case is treated in the following lemma.

Lemma 1.29. *Let $\mathbb{D}$ be a derivator and let J be a category admitting a terminal object t.*
i) *The functor p_J induces a fully-faithful functor $p_J^*\colon \mathbb{D}(e) \longrightarrow \mathbb{D}(J)$. Moreover, for $X \in \mathbb{D}(J)$ we have a natural isomorphism $\mathrm{Hocolim}_J X \cong X_t$ which is compatible with the units.*
ii) *We have an isomorphism of functors $t_* \cong p_J^*$. In particular, the objects in the image of t_* have the property that all structure maps in the underlying diagram are isomorphisms.*

Proof. From the adjunction $(p_J, t, \eta, \epsilon)\colon J \rightharpoonup e$ given by the terminal element t we obtain an adjunction $(t^*, p_J^*, \eta^*, \epsilon^*)$ with a fully-faithful right adjoint. Moreover, by the uniqueness of left adjoints we have a natural isomorphism $\mathrm{Hocolim}_J \cong t^*$, which gives i). Similarly, we obtain a natural isomorphism $t_* \cong p_J^*$. Thus, an object $X \in \mathbb{D}(J)$ lies in the essential image of t_* if and only if it lies in the essential image of p_J^*. Since we have an adjunction (t^*, p_J^*) with a fully-faithful right adjoint this is the case if and only if the adjunction unit $X \longrightarrow p_J^* t^* X$ is an isomorphism. Because isomorphisms are in particular pointwise isomorphisms it suffices now to consider an arbitrary morphism $f\colon j_1 \longrightarrow j_2$ in J. Since all structure maps of $p_J^* t^* X$ are identities we obtain the following commutative diagram:

$$\begin{tikzcd} X_{j_1} \arrow[r, "\eta", "\cong"'] \arrow[d] & p_J^* t^*(X)_{j_1} \arrow[r, equal] & X_t \arrow[d, "\mathrm{id}"] \\ X_{j_2} \arrow[r, "\cong", "\eta"'] & p_J^* t^*(X)_{j_2} \arrow[r, equal] & X_t \end{tikzcd}$$

This diagram reveals that the structure maps of X are isomorphisms. □

We will very soon see that the essential image of t_* is actually characterized by the fact that all structure maps in the underlying diagram are isomorphisms (cf. Lemma 1.32). This last lemma can be used to show that homotopy Kan extensions along fully-faithful functors are 'honest extensions'. More precisely, we have the following result.

Proposition 1.30. *Let $\mathbb{D}$ be a derivator and let $u\colon J \longrightarrow K$ be a fully-faithful functor. The homotopy Kan extension functors $u_!$, $u_*\colon \mathbb{D}(J) \longrightarrow \mathbb{D}(K)$ are then fully-faithful, i.e., for $X \in \mathbb{D}(J)$, the adjunction morphisms $\eta_X\colon X \longrightarrow u^*u_!X$ and $\epsilon_X\colon u^*u_*X \longrightarrow X$ are natural isomorphisms.*

Proof. We give the proof for the case of homotopy right Kan extensions. Consider the adjunction $(u^*, u_*)\colon \mathbb{D}(K) \rightharpoonup \mathbb{D}(J)$ and $X \in \mathbb{D}(J)$. For $j \in J$, we calculate:

$$(u_*X)_{u(j)} \quad \overset{\beta}{\cong} \quad \mathrm{Holim}_{J_{u(j)/}} \mathrm{pr}^*(X) \quad \cong \quad \mathrm{pr}^*(X)_{(j,u(j)=u(j))} \quad = \quad X_j,$$

where the first isomorphism is given by a base change isomorphism via (Der4). The second isomorphism is given by the last lemma. This lemma applies in our situation since the slice category has an initial object $(j, u(j) = u(j))$ by the fully-faithfulness of u. The remaining task is to identify the resulting isomorphism as the adjunction counit. For this purpose, we explicitly write down the composite isomorphism:

$$\begin{tikzcd}
u^*u_*(X)_j = u(j)^*u_*(X) \arrow[d, "\eta"'] & \\
\mathrm{Holim}_{J_{u(j)/}} p^*_{J_{u(j)/}} u(j)^*u_*(X) \arrow[d, "\alpha^*"'] & \\
\mathrm{Holim}_{J_{u(j)/}} \mathrm{pr}^*\, u^*u_*(X) \arrow[d, "\epsilon"'] & \\
\mathrm{Holim}_{J_{u(j)/}} \mathrm{pr}^*\, X \arrow[r, "\phi"'] & (j, u(j) = u(j))^*\, \mathrm{pr}^*\, X
\end{tikzcd}$$

In this diagram, η and ϵ are the adjunction morphisms and $\phi\colon \mathrm{Holim}_{J_{u(j)/}} \xrightarrow{\cong} (j, u(j) = u(j))^*$ is the natural isomorphism guaranteed by the last lemma. Finally, α is the natural transformation as depicted in:

$$\begin{tikzcd}
J_{u(j)/} \arrow[r, "\mathrm{pr}"] \arrow[d, "p = p_{J_{u(j)/}}"'] \arrow[dr, phantom, "\nearrow\!\!\!\nearrow"] & J \arrow[d, "u"] \\
e \arrow[r, "u(j)"'] & K
\end{tikzcd}$$

The diagram can be extended to the following one:

$$\begin{tikzcd}
u^*u_*(X)_j = u(j)^*u_*(X) \arrow[d, "\eta"'] \arrow[dr, bend left, "\mathrm{id}"] & & \\
\mathrm{Holim}_{J_{u(j)/}} p^*u(j)^*u_*(X) \arrow[r, "\phi"'] \arrow[d, "\alpha^*"'] & (j, u(j) = u(j))^* p^* u(j)^* u_*(X) \arrow[r, "="] \arrow[d, "\alpha^*"'] & u(j)^*u_*X \arrow[d, "="] \\
\mathrm{Holim}_{J_{u(j)/}} \mathrm{pr}^*\, u^*u_*(X) \arrow[r, "\phi"'] \arrow[d, "\epsilon"'] & (j, u(j) = u(j))^* \,\mathrm{pr}^*\, u^*u_*(X) \arrow[r, "="'] \arrow[d, "\epsilon"'] & u^*u_*(X)_j \arrow[d, "\epsilon_j"] \\
\mathrm{Holim}_{J_{u(j)/}} \mathrm{pr}^*\, X \arrow[r, "\phi"'] & (j, u(j) = u(j))^* \,\mathrm{pr}^*\, X \arrow[r, "="'] & X_j
\end{tikzcd}$$

In this diagram, the two squares on the left and the bottom square on the right commute by naturality. Moreover, we know that the isomorphism ϕ is compatible with the units. Hence, by part ii) of Lemma 1.26, the triangle also commutes since the unit of the adjunction $(p^*, (j, u(j) = u(j))^*)$ can be chosen to be the identity. It remains to check that the upper right square commutes. But this is true since we have $(j, u(j) = u(j))^*\alpha^* = \mathrm{id}$. Thus, we have identified the above isomorphism with the adjunction counit ϵ_j and can conclude by axiom (Der2). $\square$

Since we know now that, for fully-faithful $u\colon J \longrightarrow K$, the homotopy Kan extension functors $u_!, u_*\colon \mathbb{D}(J) \longrightarrow \mathbb{D}(K)$ are fully-faithful, we would like to obtain a characterization of the objects in the essential images. The point of the next lemma is that one only has to control the adjunction morphisms at arguments $k \in K - u(J)$.

Lemma 1.31. *Let $\mathbb{D}$ be a derivator, $u\colon J \longrightarrow K$ a fully-faithful functor, and $X \in \mathbb{D}(K)$.*
i) X lies in the essential image of $u_!$ if and only if the adjunction counit $\epsilon\colon u_!u^* \longrightarrow \mathrm{id}$ induces an isomorphism $\epsilon_k\colon u_!u^*(X)_k \longrightarrow X_k$ for all $k \in K - u(J)$.
ii) X lies in the essential image of u_* if and only if the adjunction unit $\eta\colon \mathrm{id} \longrightarrow u_*u^*$ induces an isomorphism $\eta_k\colon X_k \longrightarrow u_*u^*(X)_k$ for all $k \in K - u(J)$.

Proof. We give a proof of ii), so let us consider the adjunction $(u^*, u_*)\colon \mathbb{D}(K) \rightharpoonup \mathbb{D}(J)$. By Proposition 1.30, u_* is fully-faithful. Thus, $X \in \mathbb{D}(K)$ lies in the essential image of u_* if and only if the adjunction unit $\eta\colon X \longrightarrow u_*u^*X$ is an isomorphism. Since isomorphisms can be tested pointwise, this is the case if and only if we have an isomorphism $\eta_k\colon X_k \longrightarrow u_*u^*(X)_k$ for all $k \in K$. For the converse direction, consider an arbitrary element $u(j) \in K$ in the image of u. Then, by the triangular identities for an adjunction, we have:

$$\begin{tikzcd}
u^* \arrow[r, "\eta"] \arrow[dr, "\mathrm{id}"'] & u^*u_*u^* \arrow[d, "\epsilon"] \\
& u^*
\end{tikzcd}$$

Since the adjunction counit ϵ is an isomorphism we deduce that also $\eta_{u(j)} = u^*(\eta)_j$ is an isomorphism. $\square$

This result allows us to give the following additional information in the situation of Lemma 1.29.

Lemma 1.32. *Let $\mathbb{D}$ be a derivator and let J be a category admitting a terminal object t. The essential image of the fully-faithful functor $t_*\colon \mathbb{D}(e) \longrightarrow \mathbb{D}(J)$ consists of precisely those objects for which all structure maps in the underlying diagram are isomorphisms.*

Proof. By Lemma 1.29 it suffices to show that an object $X \in \mathbb{D}(J)$ with all structure maps isomorphisms lies in the essential image of t_*. Since isomorphisms can be detected pointwise Lemma 1.31 shows us that we only have to verify that the unit $\eta\colon X \longrightarrow t_*t^*X$ is an isomorphism at all points different from t. The proof of that lemma shows that this is always the case at t. So, let $j \in J$, let $f\colon j \longrightarrow t$ be the unique morphism and let us consider the following commutative diagram:

$$\begin{tikzcd}
X_j \arrow[r, "\eta"] \arrow[d, "f"', "\cong"] & t_*(X_t)_j \arrow[d, "\cong"] \\
X_t \arrow[r, "\cong", "\eta"'] & t_*(X_t)_t
\end{tikzcd}$$

By our assumption the left vertical morphism is an isomorphism, the right one is by Lemma 1.29 and the morphism on the bottom by our previous discussion. Hence, we can conclude that η is an isomorphism. $\square$

There are two important classes of fully-faithful functors where the essential image of homotopy Kan extensions can be characterized more easily. So let us give their definition.

Definition 1.33. Let $u\colon J \longrightarrow K$ be a fully-faithful functor which is injective on objects.
i) The functor u is called a *closed immersion* or a *cosieve* if whenever we have a morphism $u(j) \longrightarrow k$ in K then k lies in the image of u.
ii) The functor u is called an *open immersion* or a *sieve* if whenever we have a morphism $k \longrightarrow u(j)$ in K then k lies in the image of u.

The following proposition and a variant for the case of pointed derivators (cf. Proposition 1.40) will be frequently used throughout this paper.

Proposition 1.34. *Let $\mathbb{D}$ be a derivator.*
i) *Let $u\colon J \longrightarrow K$ be a closed immersion, then the homotopy left Kan extension $u_!$ is fully-faithful and $X \in \mathbb{D}(K)$ lies in the essential image of $u_!$ if and only if $X_k \cong \emptyset$ for all $k \in K - u(J)$.*
ii) *Let $u\colon J \longrightarrow K$ be an open immersion, then the homotopy right Kan extension u_* is fully-faithful and $X \in \mathbb{D}(K)$ lies in the essential image of u_* if and only if $X_k \cong *$ for all $k \in K - u(J)$.*

Proof. We give a proof of i). The statement about the fully-faithfulness of $u_!$ follows from the fully-faithfulness of closed immersions and Proposition 1.30. So we only have to check the criterion of Lemma 1.31. But for $k \in K - u(J)$ we have

$$u_! u^*(X)_k \cong \mathrm{Hocolim}_{J_{/k}} \mathrm{pr}^* u^*(X) = \mathrm{Hocolim}_{\emptyset} \mathrm{pr}^* u^*(X) = \emptyset.$$

In this sequence, the isomorphism is a base change isomorphism, the first equality follows from the definition of closed immersions and the second equality follows from the description of initial objects. Thus $\epsilon_k \colon u_! u^*(X)_k \longrightarrow X_k$ is an isomorphism for all $k \in K - u(J)$ if and only if $X_k \cong \emptyset$ for all $k \in K - u(J)$. □

1.4. **Pointed derivators.** Since we are mainly interested in stable derivators, we turn immediately to the next richer structure, namely the pointed derivators. There are at least two ways to axiomatize a notion of a pointed derivator. From these two notions, we turn the 'weaker one' into a definition. The 'stronger one' will be referred to as a strongly pointed derivator, but we will show that these two notions coincide.

Definition 1.35. A derivator $\mathbb{D}$ is *pointed* if the underlying category $\mathbb{D}(e)$ of $\mathbb{D}$ is pointed, i.e., admits a zero object $0 \in \mathbb{D}(e)$.

Note that the pointedness is again only a property and not an additional structure. For a prederivator one would impose a slightly stronger condition: a prederivator is pointed if and only if all of its values and all restriction of diagram functors are pointed. In the case of a derivator it is immediate that these stronger pointedness assumptions follow from the weaker one. Recall that we denote the unique functor from a category J to the terminal category e by $p_J \colon J \longrightarrow e$. The functor p_J^* is both a left adjoint and a right adjoint; so it preserves initial and final objects, hence also zero objects. Thus, for a pointed derivator $\mathbb{D}$ and a category J, the following objects

$$(\emptyset_J)_!(0), \qquad p_J^*(0), \qquad \text{and} \qquad (\emptyset_J)_*(0)$$

are zero objects in $\mathbb{D}(J)$. Similarly, for every functor $u \colon J \longrightarrow K$ the induced restriction functor u^* and also the homotopy Kan extension functors $u_!$, u_* have an adjoint on at least one side and are hence pointed, i.e., send zero objects to zero objects.

Proposition 1.36. *Let $\mathbb{D}$ be a pointed derivator and let J be a category. Then $\mathbb{D}_J$ is also pointed.*

Example 1.37. i) Let $\mathcal{C}$ be a category. Then the represented prederivator $\mathcal{C}$ is pointed if and only if the category $\mathcal{C}$ is pointed.
ii) The derivator $\mathbb{D}_{\mathcal{M}}$ associated to a pointed combinatorial model category $\mathcal{M}$ is pointed.
iii) A derivator $\mathbb{D}$ is pointed if and only if its dual $\mathbb{D}^{\mathrm{op}}$ is pointed.

We now want to give the stronger axiom as used by Maltsiniotis in [Mal07a].

Definition 1.38. A derivator $\mathbb{D}$ is *strongly pointed* if it has the following two properties:
i) For every open immersion $j \colon J \longrightarrow K$, the homotopy right Kan extension functor j_* has a right adjoint $j^!$:

$$(j_*, j^!) \colon \mathbb{D}(J) \rightharpoonup \mathbb{D}(K)$$

ii) For every closed immersion $i\colon J \longrightarrow K$, the homotopy left Kan extension functor $i_!$ has a left adjoint $i^?$:

$$(i^?, i_!)\colon \mathbb{D}(K) \rightharpoonup \mathbb{D}(J)$$

It is an immediate corollary of the definition that a strongly pointed derivator is pointed. In fact, one of the two additional properties is enough to ensure this.

Corollary 1.39. *If $\mathbb{D}$ is a strongly pointed derivator, then $\mathbb{D}$ is pointed.*

Proof. It is enough to consider the closed immersion $\emptyset_e\colon \emptyset \longrightarrow e$. For an initial object $\emptyset_{e!}(0) \in \mathbb{D}(e)$ and an arbitrary $X \in \mathbb{D}(e)$, we then deduce $\mathsf{hom}_{\mathbb{D}(e)}(X, \emptyset_{e!}(0)) \cong \mathsf{hom}_{\mathbb{D}(\emptyset)}(\emptyset_e^? X, 0) = *$, so that $\emptyset_{e!}(0)$ is also terminal. $\square$

At the end of this subsection, we will prove the converse to this (cf. Corollary 1.42). A further proof of that converse will be given in the stable situation, i.e., for stable derivators. That second proof is quite an indirect one. It relies on the fact that recollements of triangulated categories are overdetermined and will be given in Subsection 4.3. As a preparation, for the direct proof, we mention the following immediate consequence of Proposition 1.34. It states that the homotopy left Kan extension along a closed immersion and the homotopy right Kan extension along an open immersion are given by 'extension by zero functors'.

Proposition 1.40. *Let $\mathbb{D}$ be a pointed derivator.*
i) *Let $i\colon J \longrightarrow K$ be a closed immersion, then the homotopy left Kan extension $i_!$ is fully-faithful and $X \in \mathbb{D}(K)$ lies in the essential image of $i_!$ if and only if $X_k \cong 0$ for all $k \in K - i(J)$.*
ii) *Let $j\colon J \longrightarrow K$ be an open immersion, then the homotopy right Kan extension j_* is fully-faithful and $X \in \mathbb{D}(K)$ lies in the essential image of j_* if and only if $X_k \cong 0$ for all $k \in K - j(J)$.*

Following Heller [Hel97], we introduce the following notation. Let $\mathbb{D}$ be a pointed derivator and let $u\colon J \longrightarrow K$ be the inclusion of a full subcategory. Then we denote by $\mathbb{D}(K, J) \subseteq \mathbb{D}(K)$ the full, replete subcategory spanned by the objects X which vanish on J, i.e., such that $u^*(X) = 0$. If i resp. j is now a closed resp. an open immersion, we obtain the following equivalences of categories:

$$(i_!, i^*)\colon \mathbb{D}(J) \xrightarrow{\simeq} \mathbb{D}(K, K - J) \qquad \text{and} \qquad (j^*, j_*)\colon \mathbb{D}(K, K - J) \xrightarrow{\simeq} \mathbb{D}(J)$$

This proposition, although easily proved, will be of central importance in all what follows. It plays a similar role in the development of the theory as Proposition 4.3.2.15 of [Lur09] does in the theory of stable ∞-categories.

The above proposition will be of constant use in the study of the important cone, fiber, suspension and loop functors and also in the proof that the values of a stable derivator can be canonically endowed with the structure of a triangulated category. However, we first have to establish some properties of coCartesian and Cartesian squares and this is the aim of Section 3.

To conclude this section we will now give the proof that pointed derivators are actually strongly pointed. The constructions involved in the proof are motivated by the paper of

Rezk [Rez] in which he gives a nice construction of the 'natural model structure' on Cat. This model structure is due to Joyal and Tierney [JT91] and the adjective 'natural' refers to the fact that the weak equivalences in that model structure are precisely the equivalences in the 2-category Cat. We use a minor modification of the mapping cylinder categories used in [Rez]. Instead of forming the product with the groupoid generated by [1] we use the category [1] itself. This leads to two 'differently oriented versions' of the mapping cylinder and both of them will be needed in the proof.

Lemma 1.41. i) *Let $u\colon J \longrightarrow K$ be a closed immersion. Then the subcategory $\mathbb{D}(K,J) \subseteq \mathbb{D}(K)$ is coreflective, i.e., the inclusion functor ι admits a right adjoint.*
ii) *Let $u\colon J \longrightarrow K$ be an open immersion. Then the subcategory $\mathbb{D}(K,J) \subseteq \mathbb{D}(K)$ is reflective, i.e., the inclusion functor ι admits a left adjoint.*

Proof. We will give the details for the proof of ii) and mention the necessary modifications for i). So, let $u\colon J \longrightarrow K$ be an open immersion and let us construct the mapping cylinder category $\mathsf{cyl}(u)$. By definition, $\mathsf{cyl}(u)$ is the full subcategory of $K \times [1]$ spanned by the objects $(u(j),1)$ and $(k,0)$. Thus, it is defined by the following pushout diagram where i_0 is the inclusion at 0:

$$\begin{tikzcd} J \arrow[r, "u"] \arrow[d, "i_0"'] & K \arrow[d] \\ J \times [1] \arrow[r] & \mathsf{cyl}(u) \end{tikzcd}$$

There are the natural functors $i\colon J \longrightarrow \mathsf{cyl}(u)\colon j \longmapsto (u(j),1)$ and $s\colon K \longrightarrow \mathsf{cyl}(u)\colon k \longmapsto (k,0)$. Moreover, $\mathrm{id}\colon K \longrightarrow K$ and $J \times [1] \xrightarrow{\mathrm{pr}} J \xrightarrow{u} K$ induce a unique functor $p\colon \mathsf{cyl}(u) \longrightarrow K$. These functors satisfy the relations $p \circ i = u,\ p \circ s = \mathrm{id}_K$. Consider now an object $X \in \mathbb{D}(\mathsf{cyl}(u), i(J))$ and let us calculate the value of $p_!(X)$ at some $u(j) \in K$. By the base change formulas, we obtain $p_!(X)_{u(j)} \cong \mathrm{Hocolim}_{\mathsf{cyl}(u)_{/u(j)}}\,\mathrm{pr}^*(X)$. But, since u is open, we obtain the isomorphism in the following diagram:

$$\begin{tikzcd} i(J)_{/i(j)} \arrow[r] \arrow[d, "\mathrm{pr}"'] & i(J)_{/i(j)} \times [1] \arrow[r, "\cong"] \arrow[d, "\mathrm{pr}"'] & \mathsf{cyl}(u)_{/u(j)} \arrow[d, "\mathrm{pr}"'] \\ i(J) \arrow[r] & i(J) \times [1] \arrow[r, "\subseteq"] & \mathsf{cyl}(u) \end{tikzcd}$$

In this diagram, the left horizontal arrows are induced by the inclusion of 1 and are hence right adjoints. Using Lemma 1.28 on the cofinality of right adjoints, we thus have

$$p_!(X)_{u(j)} \cong \mathrm{Hocolim}_{\mathsf{cyl}(u)_{/u(j)}}\,\mathrm{pr}^*(X) \cong \mathrm{Hocolim}_{i(J)_{/i(j)}}\,\mathrm{pr}^*(X\mid_{i(J)}) \cong 0.$$

The adjunction $(p_!, p^*)$ restricts to an adjunction $(p_!, p^*)\colon \mathbb{D}(\mathsf{cyl}(u), i(J)) \rightharpoonup \mathbb{D}(K, u(J))$. Moreover, since we defined the mapping cylinder forming the product with [1] as opposed to the groupoid generated by it, $s\colon K \longrightarrow \mathsf{cyl}(u)$ is an open immersion. Hence, by Proposition 1.40, we have an induced equivalence $(s_*, s^*)\colon \mathbb{D}(K) \xrightarrow{\simeq} \mathbb{D}(\mathsf{cyl}(u), \mathsf{cyl}(u) - s(K)) =$

$\mathbb{D}(\mathsf{cyl}(u), i(J))$. Putting these together we obtain the adjunction

$$(p_! \circ s_*, s^* \circ p^*)\colon\ \mathbb{D}(K) \rightharpoonup \mathbb{D}(\mathsf{cyl}(u), i(J)) \rightharpoonup \mathbb{D}(K, u(J)).$$

The relation $p \circ s = \mathrm{id}$ implies that the right adjoint of this adjunction is the inclusion ι as intended and the reflection is given by $r = p_! \circ s_*$.

The proof of i) is similar. Instead of using $\mathsf{cyl}(u)$ one uses this time the mapping cylinder category $\mathsf{cyl}'(u)$ which is obtained by a similar pushout but using the inclusion i_1. Let us denote the corresponding functors again by i, p, and s. Using a similar calculation of p_* and the fact that s is now a closed immersion, we can construct a coreflection c. □

Corollary 1.42. *Let $\mathbb{D}$ be a pointed derivator, then $\mathbb{D}$ is also strongly pointed.*

Proof. Let us construct a right adjoint to u_* in the case of an open immersion $u\colon J \longrightarrow K$. The inclusion $v\colon K - u(J) \longrightarrow K$ of the complement is a closed immersion. The above lemma applied to v thus gives us a coreflection $c\colon \mathbb{D}(K, K - u(J)) \rightharpoonup \mathbb{D}(K)$. Putting this together with the equivalence induced by u_* we obtain the desired adjunction:

$$(u_*, u^!)\colon\ \mathbb{D}(J) \underset{u^*}{\overset{u_*}{\rightleftarrows}} \mathbb{D}(K, K - u(J)) \underset{c}{\overset{\iota}{\rightleftarrows}} \mathbb{D}(K)$$

The proof in the case of a closed immersion is, of course, the dual one. □

The proofs of the last two results were constructive proofs. So, for later reference, let us give precise formulas for these additional adjoint functors. Let $\mathbb{D}$ be a pointed derivator and let $u\colon J \longrightarrow K$ be a closed immersion. Let us denote by $v\colon J' = K - u(J) \longrightarrow K$ the open inclusion of the complement. The adjunction $(u^?, u_!)\colon\ \mathbb{D}(K) \rightharpoonup \mathbb{D}(J)$ is given by the following composite adjunction:

$$u^?\colon\ \mathbb{D}(K) \underset{s^*}{\overset{s_*}{\rightleftarrows}} \mathbb{D}(\mathsf{cyl}(v), i(J')) \underset{p^*}{\overset{p_!}{\rightleftarrows}} \mathbb{D}(K, v(J')) \underset{u_!}{\overset{u^*}{\rightleftarrows}} \mathbb{D}(J)\ :u_!$$

Here, $\mathsf{cyl}(v)$ is the mapping cylinder obtained from identifying the bottom $J' \times \{0\}$ of $J' \times [1]$ with the image of v, i is the inclusion in the cylinder, p is the projection and s is the canonical section of p. There is a similar decomposition for $v^!$ which we do not make explicit.

2. Morphisms of derivators

Let $\mathbb{D}$ and $\mathbb{D}'$ be prederivators. A *morphism of prederivators* $F\colon \mathbb{D} \longrightarrow \mathbb{D}'$ is a pseudo-natural transformation between the 2-functors $\mathbb{D}$ and $\mathbb{D}'$ (cf. to Definition 7.5.2 of [Bor94a]). Spelling out this definition such a morphism is a pair $(F_\bullet, \gamma_\bullet)$ consisting of a collection of functors

$$F_J\colon \mathbb{D}(J) \longrightarrow \mathbb{D}'(J), \qquad J \in \mathsf{Cat},$$

and a family of natural isomorphisms $\gamma_u\colon u^* \circ F_K \longrightarrow F_J \circ u^*,\ u\colon J \longrightarrow K$ as indicated in

$$\begin{tikzcd} \mathbb{D}(K) \arrow[r, "F_K"] \arrow[d, "u^*"'] \arrow[dr, phantom, "\cong"] & \mathbb{D}'(K) \arrow[d, "u^*"] \\ \mathbb{D}(J) \arrow[r, "F_J"'] & \mathbb{D}'(J). \end{tikzcd}$$

This datum is subject to the following coherence properties. Given a pair of composable functors $J \xrightarrow{u} K \xrightarrow{v} L$ and a natural transformation $\alpha\colon u_1 \longrightarrow u_2\colon J \longrightarrow K$, we then have the following relation resp. commutative diagrams:

$$\gamma_{\mathrm{id}_J} = \mathrm{id}_{F_J} \qquad \begin{tikzcd} u^*v^*F \arrow[r, "\gamma_v"] \arrow[dr, bend right, "\gamma_{vu}"'] & u^*Fv^* \arrow[d, "\gamma_u"] \\ & Fu^*v^* \end{tikzcd} \qquad \begin{tikzcd} u_1^*F \arrow[r, "\alpha^*"] \arrow[d, "\gamma_{u_1}"'] & u_2^*F \arrow[d, "\gamma_{u_2}"] \\ Fu_1^* \arrow[r, "\alpha^*"'] & Fu_2^* \end{tikzcd}$$

Here, we suppressed the indices of F (as we will frequently do in the sequel) to avoid awkward notation.

As usual the notion of a pseudo-natural transformation can be relaxed or can be strengthened. In the more relaxed situation there would be two versions of such morphisms, the lax ones and the colax ones, but we do not need this more general class. Strictly speaking, also in our situation there are two notions depending on the direction of the natural transformations γ. Since one can always pass to the inverse natural transformations these notions are equivalent. In what follows, we will be a bit sloppy in notation in that we will not distinguish notationally between the natural isomorphisms γ belonging to such a morphism and their inverses γ^{-1}. In the case of a 2-natural transformation, i.e., if all natural transformations γ are given by identities, we speak of a *strict morphism*. The class of strict morphisms is too narrow in that many examples will only be pseudo-natural transformation but it is conceptually easier. This becomes manifest, for example, in the 2-categorical Yoneda lemma as opposed to the more general bi-categorical Yoneda lemma.

A *(strict) morphism of derivators* is just a (strict) morphism of underlying prederivators.

Example 2.1. i) Let $\mathcal{C}$ resp. $\mathcal{D}$ be categories and let us consider the associated represented prederivators. A functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ induces a morphism of prederivators again denoted by $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ in the following way: For a category J, let the component of F at J be given by postcomposition with F:

$$F_J\colon \mathsf{Fun}(J, \mathcal{C}) \longrightarrow \mathsf{Fun}(J, \mathcal{D})\colon X \longmapsto F \circ X$$

The associativity of composition of functors implies that this defines a morphism of prederivators if we choose all components of γ to be the identity. This assignment is faithful and the morphisms in the image are precisely the strict morphisms, i.e., the 2-natural transformations.
ii) Let $(F,U)\colon \mathcal{M} \longrightarrow \mathcal{N}$ be a Quillen adjunction between combinatorial model categories. Then the formation of derived Quillen functors gives us two (in general non-strict) morphisms of derivators $\mathbb{L}F\colon \mathbb{D}_{\mathcal{M}} \longrightarrow \mathbb{D}_{\mathcal{N}}$ and $\mathbb{R}U\colon \mathbb{D}_{\mathcal{N}} \longrightarrow \mathbb{D}_{\mathcal{M}}$. These are part of an adjunction of derivators $(\mathbb{L}F,\mathbb{R}U)\colon \mathbb{D}_{\mathcal{M}} \rightharpoonup \mathbb{D}_{\mathcal{N}}$ (cf. to the first section of [Gro11]).

We want to have a notion of morphisms of derivators which preserve homotopy left or right Kan extensions. For this, we proceed as in the construction of the base change morphisms showing up in Kan's formulas. So, let $F\colon \mathbb{D} \longrightarrow \mathbb{D}'$ be a morphism of derivators and let $u\colon J \longrightarrow K$ be a functor. By definition, we then have a natural isomorphism $\gamma\colon u^*F \longrightarrow Fu^*$, which allows us to construct two *canonical morphisms* as the following compositions:

$$\begin{tikzcd} u_!F \arrow[r, "\beta"] \arrow[d, "\eta"'] & Fu_! \\ u_!Fu^*u_! \arrow[r, "\gamma"'] & u_!u^*Fu_! \arrow[u, "\epsilon"'] \end{tikzcd} \qquad \begin{tikzcd} Fu_* \arrow[r, "\beta"] \arrow[d, "\eta"'] & u_*F \\ u_*u^*Fu_* \arrow[r, "\gamma"'] & u_*Fu^*u_* \arrow[u, "\epsilon"'] \end{tikzcd}$$

Again, these canonical morphisms depend on the choice of adjunctions. But since left or right adjoint functors are unique up to an isomorphism which is compatible with the units and counits, the answer to the question whether such a canonical morphism is an isomorphism is independent of the choices.

Definition 2.2. Let $F\colon \mathbb{D} \longrightarrow \mathbb{D}'$ be a morphism of derivators.
i) The morphism F *preserves homotopy left Kan extensions* if the canonical morphism $\beta\colon u_!F \longrightarrow Fu_!$ is a natural isomorphism for all $u\colon J \longrightarrow K$.
ii) The morphism F *preserves homotopy right Kan extensions* if the canonical morphism $\beta\colon Fu_* \longrightarrow u_*F$ is a natural isomorphism for all $u\colon J \longrightarrow K$.

Example 2.3. i) Let $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a functor between bicomplete categories and let us consider the above canonical morphisms in the absolute case. So, let J be a category and $u = p_J\colon J \longrightarrow e$ be the unique functor to the terminal category. The above canonical morphisms then take the form:

$$\begin{tikzcd} \operatorname{colim}_J F \arrow[r, "\beta"] \arrow[d, "\eta"'] & F\operatorname{colim}_J \\ \operatorname{colim}_J Fu^*\operatorname{colim}_J \arrow[r, equal] & \operatorname{colim}_J u^*F\operatorname{colim}_J \arrow[u, "\epsilon"'] \end{tikzcd} \qquad \begin{tikzcd} F\lim_J \arrow[r, "\beta"] \arrow[d, "\eta"'] & \lim_J F \\ \lim_J u^*F\lim_J \arrow[r, equal] & \lim_J Fu^*\lim_J \arrow[u, "\epsilon"'] \end{tikzcd}$$

In the left diagram, β evaluated at $X \in \mathsf{Fun}(J,\mathcal{C})$ is the canonical map from the colimit of $F \circ X$ to the image of $\operatorname{colim} X$ under F. Thus, we recover the usual notion of a colimit preserving functor, i.e., the functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ preserves colimits if and only if the induced

morphism of derivators $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ preserves homotopy colimits. Similar comments apply to the right diagram and (homotopy) limits.
ii) Let $(F,U)\colon \mathbb{D} \longrightarrow \mathbb{D}'$ be an adjunction of derivators. Then the left adjoint F preserves homotopy colimits and the right adjoint U preserves homotopy limits (cf. to the first section of [Gro11]). This applies, in particular, to derived Quillen adjunctions $(\mathbb{L}F, \mathbb{R}U)\colon \mathbb{D}_{\mathcal{M}} \rightharpoonup \mathbb{D}_{\mathcal{N}}$ between derivators associated to combinatorial model categories.

Using the fact that isomorphisms in a derivator are detected pointwise and that we have Kan's formula available one establishes the following lemma.

Lemma 2.4. *Let $F\colon \mathbb{D} \longrightarrow \mathbb{D}'$ be a morphism of derivators. Then F preserves homotopy left resp. homotopy right Kan extensions if and only if F preserves homotopy colimits resp. homotopy limits.*

There is an obvious analogous notion of morphisms which preserve only (certain types of) homotopy (co)limits. We give *minimal* definitions in two special cases.

Definition 2.5. Let $F\colon \mathbb{D} \longrightarrow \mathbb{D}'$ be a morphism of derivators. *F preserves initial objects* if the underlying functor $F\colon \mathbb{D}(e) \longrightarrow \mathbb{D}'(e)$ preserves initial objects.

This definition does fit into the general pattern of morphisms of derivators which preserve certain homotopy colimits: F preserves initial objects in our sense if and only if the following natural transformation is an isomorphism:

$$\begin{tikzcd}
\emptyset_{e!}F_{\emptyset} \arrow[rr, "\beta"] \arrow[d, "\eta"'] & & F_e\emptyset_{e!} \\
\emptyset_{e!}F_{\emptyset}\emptyset_e^*\emptyset_{e!} \arrow[rr, "\gamma"'] & & \emptyset_{e!}\emptyset_e^*F_e\emptyset_{e!} \arrow[u, "\epsilon"']
\end{tikzcd}$$

As always η and ϵ denote the adjunction morphisms while γ denotes the natural transformation belonging to the morphism F. In fact, in that diagram η and γ are isomorphisms. Thus, β is an isomorphism if and only if ϵ is an isomorphism. And this latter is the fact if and only if the image of any initial object under F_e is again initial.

This definition is actually the definition we want to have as we see by the next lemma.

Lemma 2.6. *Let $F\colon \mathbb{D} \longrightarrow \mathbb{D}'$ be a morphism of derivators. Then F preserves initial objects if and only if $F\colon \mathbb{D}_M \longrightarrow \mathbb{D}'_M$ preserves initial objects for all categories M.*

Proof. We have to check that $F_M\colon \mathbb{D}(M) \longrightarrow \mathbb{D}'(M)$ preserves initial objects if F_e does. But this is obvious since an initial object $\emptyset_M$ of $\mathbb{D}(M)$ is given by $p_M^*(\emptyset)$ and similarly for $\mathbb{D}'$. We can thus conclude by the following chain of isomorphisms: $F_M(\emptyset_M) \cong F_M(p_M^*(\emptyset)) \cong p_M^*(F_e(\emptyset)) \cong \emptyset_M$ □

Of course, this can be dualized to obtain a corresponding statement about morphisms of derivators which preserve terminal objects. There is a similar *minimal* definition of a coproduct preserving morphism of derivators. Again, we will not make explicit the dual part of the story.

Definition 2.7. Let $F\colon \mathbb{D} \longrightarrow \mathbb{D}'$ be a morphism of derivators. We say that F *preserves coproducts* if the underlying functor $F_e\colon \mathbb{D}(e) \longrightarrow \mathbb{D}'(e)$ preserves coproducts.

It is easy to check that this is equivalent to the statement that the following natural transformation is a natural isomorphism:

$$\begin{tikzcd}
\nabla_{e!}F_{e\sqcup e} \arrow[r, "\beta"] \arrow[d, "\eta"'] & F_e\nabla_{e!} \\
\nabla_{e!}F_{e\sqcup e}\nabla_e^*\nabla_{e!} \arrow[r, "\gamma"'] & \nabla_{e!}\nabla_e^*F_e\nabla_{e!} \arrow[u, "\epsilon"']
\end{tikzcd}$$

The following proposition shows that this definition is the one we had in mind: it suffices to ask for the above compatibility for e to guarantee that all $F_M\colon \mathbb{D}(M) \longrightarrow \mathbb{D}'(M)$ preserve coproducts.

Proposition 2.8. *Let $F\colon \mathbb{D} \longrightarrow \mathbb{D}'$ be a morphism of derivators. Then F preserves coproducts if and only if $F\colon \mathbb{D}_M \longrightarrow \mathbb{D}'_M$ preserves coproducts for all categories M.*

A direct proof of this case is already quite lengthy so we refrain from giving it. Instead, we will deduce this proposition as a special case from the more general Proposition 2.12. For that result we first have to establish that homotopy Kan extensions in $\mathbb{D}_M$ are calculated pointwise. Note moreover that the last two definitions were that simple because in that two cases the homotopy colimits under consideration were (up to equivalence) honest categorical colimits. In more general situations, we have to refer to base change morphisms.

Proposition 2.9. *Let $\mathbb{D}$ be a derivator and let $u\colon L \longrightarrow M$ be a functor. The induced functor $u^*\colon \mathbb{D}_M \longrightarrow \mathbb{D}_L$ preserves homotopy Kan extensions.*

Proof. By Lemma 2.4 and duality it is enough to treat the case of homotopy limits. So, let J be a category and let us consider the following diagram

$$\begin{tikzcd}
L \times J \arrow[r, "u\times \mathrm{id}_J"] \arrow[d, "\mathrm{id}_L \times p_J"'] & M \times J \arrow[d, "\mathrm{id}_M \times p_J"] \\
L \times e \arrow[r, "u\times \mathrm{id}_e"'] & M \times e.
\end{tikzcd}$$

This diagram is a pullback diagram and the projection functor on the right is a Grothendieck fibration. Hence, by Proposition A.3, we obtain a base change isomorphism

$$u_e^* \circ \mathrm{Holim}_J^{\mathbb{D}_M} \xrightarrow{\cong} \mathrm{Holim}_J^{\mathbb{D}_L} \circ u_J^*.$$

Now, one only has to remark that this base change isomorphism is precisely the canonical morphism occurring in the definition of a morphism of derivators which preserves homotopy limits. $\square$

Alternatively, this follows from the first section of [Gro11]. An application of this proposition to the special situation of functors of the form $m\colon e \longrightarrow M$ shows that homotopy Kan extensions in $\mathbb{D}_M$ are formed pointwise. More precisely, for $X \in \mathbb{D}_M(J)$ and $u\colon J \longrightarrow K$ we have canonical isomorphisms

$$\beta\colon u_!(X_m) \xrightarrow{\cong} (u_!X)_m \qquad \text{and} \qquad \beta\colon (u_*X)_m \xrightarrow{\cong} u_*(X_m).$$

Similarly, in the absolute case, i.e., in the case of $u = p_J\colon J \longrightarrow e$, we obtain canonical isomorphisms

$$\beta\colon \operatorname{Hocolim}_J(X_m) \xrightarrow{\cong} (\operatorname{Hocolim}_J X)_m \qquad \text{and} \qquad \beta\colon (\operatorname{Holim}_J X)_m \xrightarrow{\cong} \operatorname{Holim}_J(X_m).$$

These isomorphisms are well-behaved as described in the following lemma which we mention for later reference.

Lemma 2.10. *Let $\mathbb{D}$ be a derivator, let M be a category and let $u\colon J \longrightarrow K$ be a functor. The above canonical isomorphism $\beta\colon u_!(m\times \mathrm{id}_J)^* \longrightarrow (m\times\mathrm{id}_K)^*(\mathrm{id}_M \times u)_!$ is compatible with the counits in the sense that the following diagram commutes:*

$$\begin{tikzcd}
u_!(m\times \mathrm{id}_J)^*(\mathrm{id}_M\times u)^* \arrow[r, "\beta"] \arrow[d, "="'] & (m\times\mathrm{id}_K)^*(\mathrm{id}_M\times u)_!(\mathrm{id}_M\times u)^* \arrow[d, "\epsilon"] \\
u_!u^*(m\times\mathrm{id}_K)^* \arrow[r, "\epsilon"'] & (m\times \mathrm{id}_K)^*
\end{tikzcd}$$

Proof. This is straightforward. Unraveling the definition of the canonical morphism β and using a slight abuse of notation, we obtain that our diagram is the following one:

$$\begin{tikzcd}
u_!m^*u^* \arrow[r, "\eta"] \arrow[dr, bend right, "\mathrm{id}"'] & u_!m^*u^*u_!u^* \arrow[r, equal] \arrow[d, "\epsilon"'] & u_!u^*m^*u_!u^* \arrow[r, "\epsilon"] \arrow[d, "\epsilon"] & m^*u_!u^* \arrow[d, "\epsilon"] \\
 & u_!m^*u^* \arrow[r, equal] & u_!u^*m^* \arrow[r, "\epsilon"'] & m^*
\end{tikzcd}$$

But this diagram commutes, since the triangle does by the triangular identities for adjunctions and the square does by naturality. □

This lemma implies, in particular, that, for $X \in \mathbb{D}_M(K)$, the counit $\epsilon\colon u_!u^*(X) \longrightarrow X$ is an isomorphism in $\mathbb{D}_M(K)$ if and only if the counit $\epsilon\colon u_!u^*(X_m) \longrightarrow X_m$ is an isomorphism in $\mathbb{D}(K)$ for all objects $m \in M$. For later reference, we collect the following convenient consequence for the case of a fully-faithful $u\colon J \longrightarrow K$.

Corollary 2.11. *Let $\mathbb{D}$ be a derivator, M a category and $u\colon J \longrightarrow K$ a fully-faithful functor. An object $X \in \mathbb{D}_M(K)$ lies in the essential image of $u_!\colon \mathbb{D}_M(J) \longrightarrow \mathbb{D}_M(K)$ if and only if X_m lies in the essential image of $u_!\colon \mathbb{D}(J) \longrightarrow \mathbb{D}(K)$ for all $m \in M$.*

With these preparations we can now deduce the following nice result.

Proposition 2.12. *Let $F\colon \mathbb{D} \longrightarrow \mathbb{D}'$ be a morphism of derivators and let $u\colon J \longrightarrow K$ be a functor. F preserves homotopy left Kan extensions along u if and only if $F\colon \mathbb{D}_M \longrightarrow \mathbb{D}'_M$ preserves homotopy left Kan extensions along u for all categories M.*

This result justifies the above minimal definition of a coproduct-preserving morphism of derivators. Namely, applied to the special case where $u = \nabla_e \colon e \sqcup e \longrightarrow e$ this result gives us precisely the situation of Proposition 2.8. Similarly, Lemma 2.6 on morphisms which preserve initial objects is a special case of our proposition when applied to the situation of $u = \emptyset_e \colon \emptyset \longrightarrow e$. But Proposition 2.12 obviously covers by far more such situations. The ingredients for the proof are by now all established but we preferred to give the still somewhat lengthy proof in Appendix B.1.

3. Cartesian and coCartesian squares

In this section, we will mainly prepare the ground for the construction of the important suspension, loop, cone, and fiber functors for pointed and stable derivators. This will be done in the next section using certain properties of coCartesian resp. Cartesian squares. The corresponding properties are well-known in classical category theory and will be reproved here in the setting of derivators. The main results are the behavior of (co)Cartesian squares under cancellation and composition (Proposition 3.9) and a 'detection result' (due to Franke [Fra96]) for (co)Cartesian squares (Proposition 3.5). The lengthy proofs of these two results will be deferred to Appendix B.

We denote the category $[1]\times[1]$ by $\square$, i.e., $\square$ is the following poset considered as a category:

$$\begin{tikzcd} (0,0) \arrow[r] \arrow[d] & (1,0) \arrow[d] \\ (0,1) \arrow[r] & (1,1) \end{tikzcd}$$

For the treatment of Cartesian and coCartesian squares, it is important to consider the following two inclusions of subcategories $\mathrm{i}_\ulcorner\colon\ulcorner\longrightarrow\square$ resp. $\mathrm{i}_\lrcorner\colon\lrcorner\longrightarrow\square$ which are given by the subposets:

$$\begin{tikzcd} (0,0) \arrow[r] \arrow[d] & (1,0) \\ (0,1) & \end{tikzcd} \qquad \text{resp.} \qquad \begin{tikzcd} & (1,0) \arrow[d] \\ (0,1) \arrow[r] & (1,1) \end{tikzcd}$$

Definition 3.1. Let $\mathbb{D}$ be a derivator and let $X\in\mathbb{D}(\square)$.
i) The square X is *coCartesian* if it lies in the essential image of $\mathrm{i}_{\ulcorner !}\colon\mathbb{D}(\ulcorner)\longrightarrow\mathbb{D}(\square)$.
ii) The square X is *Cartesian* if it lies in the essential image of $\mathrm{i}_{\lrcorner *}\colon\mathbb{D}(\lrcorner)\longrightarrow\mathbb{D}(\square)$.

The following characterization of (co)Cartesian squares is an immediate consequence of the fully-faithfulness of homotopy Kan extensions along fully-faithful functors (Proposition 1.30) and Lemma 1.31.

Proposition 3.2. *Let $\mathbb{D}$ be a derivator and let $X\in\mathbb{D}(\square)$.*
i) *The square X is coCartesian*
if and only if the canonical morphism $\epsilon_X\colon \mathrm{i}_{\ulcorner !}\,\mathrm{i}_\ulcorner^*\,X\longrightarrow X$ is an isomorphism
if and only if the canonical morphism $\epsilon_{(1,1)}\colon \mathrm{i}_{\ulcorner !}\,\mathrm{i}_\ulcorner^*(X)_{(1,1)}\longrightarrow X_{(1,1)}$ is an isomorphism
if and only if the map $\mathsf{hom}_{\mathbb{D}(\square)}(X,Y)\longrightarrow\mathsf{hom}_{\mathbb{D}(\ulcorner)}(\mathrm{i}_\ulcorner^*\,X,\mathrm{i}_\ulcorner^*\,Y)$ is an isomorphism for all $Y\in\mathbb{D}(\square)$. ii) *The square X is Cartesian*
if and only if the canonical morphism $\eta_X\colon X\longrightarrow \mathrm{i}_{\lrcorner *}\,\mathrm{i}_\lrcorner^*\,X$ is an isomorphism
if and only if the canonical morphism $\eta_{(0,0)}\colon X_{(0,0)}\longrightarrow \mathrm{i}_{\lrcorner *}\,\mathrm{i}_\lrcorner^*(X)_{(0,0)}$ is an isomorphism
if and only if the map $\mathsf{hom}_{\mathbb{D}(\square)}(Y,X)\longrightarrow\mathsf{hom}_{\mathbb{D}(\lrcorner)}(\mathrm{i}_\lrcorner^*\,Y,\mathrm{i}_\lrcorner^*\,X)$ is an isomorphism for all $Y\in\mathbb{D}(\square)$.

Our first aim in this section is to establish a 'detection result' for (co)Cartesian squares in larger diagrams which will be used frequently later on. So, let us quickly give the notion of a square.

Definition 3.3. Let J be a category. A *square* in J is a functor $i\colon \square \longrightarrow J$ which is injective on objects.

We split off an innocent looking, technical lemma which is easily proved. The point with this lemma is that it reduces considerably the size of diagrams to be considered in later proofs. In the statement and also in the proof, η and ϵ denote adjunction morphisms while α and β are as in the base change axiom.

Lemma 3.4. *Let $\mathbb{D}$ be a derivator and let $u\colon J \longrightarrow K$ be fully-faithful. Moreover, let $X \in \mathbb{D}(K)$ and let $k \in K$. The composition* $\mathrm{Hocolim}_{J_{/k}} \mathrm{pr}^* u^*(X) \xrightarrow{\beta} u_! u^*(X)_k \xrightarrow{\epsilon} X_k$ *is given by*

$$\mathrm{Hocolim}_{J_{/k}} \mathrm{pr}^* u^*(X) \xrightarrow{\alpha^*} \mathrm{Hocolim}_{J_{/k}} p^*_{J_{/k}} X_k \xrightarrow{\epsilon} X_k.$$

Proof. Recall that the base change morphism β is the natural transformation:

$$\begin{tikzcd}
\mathrm{Hocolim}_{J_{/k}} \mathrm{pr}^* \arrow[r, "\beta"] \arrow[d, "\eta"'] & k^* u_! \\
\mathrm{Hocolim}_{J_{/k}} \mathrm{pr}^* u^* u_! \arrow[r, "\alpha^*"'] & \mathrm{Hocolim}_{J_{/k}} p^*_{J_{/k}} k^* u_! \arrow[u, "\epsilon"']
\end{tikzcd}$$

An application of this to $u^*(X)$ and a postcomposition with $\epsilon\colon u_! u^*(X)_k \longrightarrow X_k$ gives us the following diagram:

$$\begin{tikzcd}
\mathrm{Hocolim}_{J_{/k}} \mathrm{pr}^* u^*(X) \arrow[d, "\eta"'] \arrow[dr, bend left, "\mathrm{id}"] & \\
\mathrm{Hocolim}_{J_{/k}} \mathrm{pr}^* u^* u_! u^*(X) \arrow[r, "\epsilon"'] \arrow[d, "\alpha^*"'] & \mathrm{Hocolim}_{J_{/k}} \mathrm{pr}^* u^*(X) \arrow[d, "\alpha^*"] \\
\mathrm{Hocolim}_{J_{/k}} p^*_{J_{/k}} k^* u_! u^*(X) \arrow[r, "\epsilon"'] \arrow[d, "\epsilon"'] & \mathrm{Hocolim}_{J_{/k}} p^*_{J_{/k}} X_k \arrow[d, "\epsilon"] \\
k^* u_! u^*(X) \arrow[r, "\epsilon"'] & X_k
\end{tikzcd}$$

The two squares commute by naturality while the triangle is commutative by one of the triangular identities. Thus, the composition is as claimed. $\square$

Here is now the intended 'detection result' which will be proved in Appendix B.2.

Proposition 3.5. *Let $i\colon \square \longrightarrow J$ be a square in J and let $f\colon K \longrightarrow J$ be a functor.*

i) *Assume that the induced functor $\ulcorner \xrightarrow{\tilde{i}} \big(J - i(1,1)\big)_{/i(1,1)}$ has a left adjoint and that $i(1,1)$ does not lie in the image of f. Then for all $X = f_!(Y) \in \mathbb{D}(J),\ Y \in \mathbb{D}(K)$, the induced square $i^*(X)$ is coCartesian.*

ii) *Assume that the induced functor $\lrcorner \xrightarrow{\tilde{i}} \big(J - i(0,0)\big)_{i(0,0)/}$ has a right adjoint and that $i(0,0)$ does not lie in the image of f. Then for all $X = f_*(Y) \in \mathbb{D}(J),\ Y \in \mathbb{D}(K)$, the induced square $i^*(X)$ is Cartesian.*

Typical applications of this proposition will be given when the categories under consideration are posets. Let J and K be posets considered as categories. Recall that a functor $u\colon J \longrightarrow K$ is the same as an order-preserving map. Moreover, an adjunction $(u,v)\colon J \rightharpoonup K$ is equivalently given by two order-preserving maps $u\colon J \longrightarrow K$ and $v\colon K \longrightarrow J$ such that $j \leq vu(j),\ j \in J$, and $uv(k) \leq k,\ k \in K$. In fact, in this case the triangular identities are automatically satisfied.

For $n \geq 0$, we denote by $[n]$ the ordinal number $0 < \ldots < n$ considered as a category. Moreover, let us denote the standard cosimplicial face resp. degeneracy maps by $d^i\colon [n-1] \longrightarrow [n],\ 0 \leq i \leq n$, resp. $s^j\colon [n+1] \longrightarrow [n],\ 0 \leq j \leq n$. Here, d^i is the unique monotone injection omitting i while s^j is the unique monotone surjection hitting j twice. The images of these cosimplicial structure maps under a contravariant functor will, as usual, be written as d_i resp. s_j.

Lemma 3.6. *For every $0 \leq i \leq n-1$ we have an adjunction $(s^i, d^i)\colon [n] \rightharpoonup [n-1]$. In particular, we thus obtain the adjunctions $(s^0, d^0)\colon [2]\times[1] \rightharpoonup [1]\times[1]$ and $(s^1, d^1)\colon [2]\times[1] \rightharpoonup [1] \times [1]$.*

Before we come to the nice cancellation and composition property of (co)Cartesian squares we include two more lemmas. These will be needed later when we show that the values of a stable derivator are preadditive categories. The first lemma gives an alternative description of coproducts in $\mathbb{D}(e)$ using homotopy left Kan extensions. This is well-known in classical category theory: if a category admits an initial object $\emptyset$ then the coproduct of two objects X, Y is the pushout of the two unique maps $\emptyset \longrightarrow X$ and $\emptyset \longrightarrow Y$.

To formulate this in the language of derivators, consider the functor $j\colon e \sqcup e \longrightarrow \square$ which classifies the objects $(1,0)$ and $(0,1)$. This functor admits two factorizations

$$\begin{tikzcd} e \sqcup e \arrow[r, "j_2"] \arrow[d, "j_1"'] & \lrcorner \arrow[d, "i_{\lrcorner}"] \\ \ulcorner \arrow[r, "i_{\ulcorner}"'] & \square \end{tikzcd}$$

such that j_1 is a closed immersion and j_2 an open immersion. The homotopy left Kan extension $j_{1!}\colon \mathbb{D}(e \sqcup e) \longrightarrow \mathbb{D}(\ulcorner)$ ‘extends diagrams by an initial object’ (cf. Proposition 1.34). Moreover, let $k\colon \mathbb{D}(e) \times \mathbb{D}(e) \longrightarrow \mathbb{D}(e \sqcup e)$ be an inverse equivalence to $(i_1^*, i_2^*)\colon \mathbb{D}(e \sqcup e) \longrightarrow \mathbb{D}(e) \times \mathbb{D}(e)$. For two objects $X,\ Y \in \mathbb{D}(e)$ the analogue of the pushout diagram from category theory is given by $j_!k(X,Y) \in \mathbb{D}(\square)$.

Lemma 3.7. *Let $\mathbb{D}$ be a derivator and let X, Y be objects of $\mathbb{D}(e)$. The underlying diagram of $\mathrm{i}_{\lrcorner}{}^{*}\, j_! k(X,Y)$ is a coproduct diagram for X and Y.*

Proof. The fully-faithfulness of $\mathrm{i}_{\lrcorner !}$ gives us a natural isomorphism

$$\mathrm{i}_{\lrcorner}{}^{*}\, j_! k(X,Y) \cong \mathrm{i}_{\lrcorner}{}^{*}\, \mathrm{i}_{\lrcorner !}\, j_{2!} k(X,Y) \cong j_{2!} k(X,Y).$$

Hence, it suffices to establish the claimed property for $j_{2!}k(X,Y)$. But using Lemma 1.32 it suffices to have a close look at the following chain of natural isomorphisms:

$$\begin{aligned}
\mathsf{hom}_{\mathbb{D}(e)}(j_{2!}k(X,Y)_{(1,1)}, Z) &\cong \mathsf{hom}_{\mathbb{D}(\lrcorner)}(j_{2!}k(X,Y), (1,1)_* Z)\\
&\cong \mathsf{hom}_{\mathbb{D}(e\sqcup e)}(k(X,Y), j_2^*(1,1)_* Z)\\
&\cong \mathsf{hom}_{\mathbb{D}(e)}(X, (1,0)^*(1,1)_* Z) \times \mathsf{hom}_{\mathbb{D}(e)}(X, (0,1)^*(1,1)_* Z)\\
&\cong \mathsf{hom}_{\mathbb{D}(e)}(X, Z) \times \mathsf{hom}_{\mathbb{D}(e)}(Y, Z)
\end{aligned}$$

$\square$

Thus, the underlying diagram of $j_! k(X,Y)$ looks like:

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & X \arrow[d] \\
Y \arrow[r] & X \sqcup Y
\end{tikzcd}$$

In the next proposition, we will consider squares $X \in \mathbb{D}(\square)$ in a derivator and its associated subdiagrams. To establish some short hand notation let us denote the face maps $\mathrm{id} \times d^i \colon [1] \longrightarrow [1] \times [1] = \square$ giving rise to 'horizontal faces' by d_h^i and similarly the ones giving rise to 'vertical faces' by $d_v^i = d^i \times \mathrm{id}$.

Proposition 3.8. *Let $\mathbb{D}$ be a derivator.*
i) *An object $f \in \mathbb{D}([1])$ is an isomorphism if and only if f lies in the essential image of $0_!$.*
ii) *Let $X \in \mathbb{D}(\square)$ be a square such that ${d_h^1}^* X$ is an isomorphism, i.e., we have $X_{0,0} \xrightarrow{\cong} X_{1,0}$. The square X is coCartesian if and only if also ${d_h^0}^* X$ is an isomorphism.*

Proof. i) This is a special case of Lemma 1.29.
ii) By i) our assumption on X is equivalent to the adjunction counit $0_! 0^* {d_h^1}^* X \longrightarrow {d_h^1}^* X$ being an isomorphism. Let us reformulate this in a way which is more convenient for this proof. For this purpose let us consider the following commutative diagram:

$$\begin{tikzcd}
{[1]} \arrow[r, "j"] \arrow[dr, bend right, "d_v^1"'] & \ulcorner \arrow[d, "\mathrm{i}_{\ulcorner}"] \\
 & \square
\end{tikzcd}$$

Claim: The counit $0_! 0^* {d_h^1}^* X \longrightarrow {d_h^1}^* X$ is an isomorphism iff the counit $j_! j^*\, \mathrm{i}_{\ulcorner}{}^{*}\, X \longrightarrow \mathrm{i}_{\ulcorner}{}^{*}\, X$ is an isomorphism.
Let us prove this claim first. By Lemma 1.31 it is enough to show that the first counit is an isomorphism when evaluated at 1 if and only if the second counit is an isomorphism at

$(1,0)$. Since we want to calculate the above homotopy Kan extensions at these respective values using Kan's formula let us consider the respective diagrams:

$$\begin{tikzcd}
{[1]}_{/(1,0)} \arrow[r, "\mathrm{pr}"] \arrow[d, "p"'] \arrow[dr, phantom, "\swArrow"] & {[1]} \arrow[d, "j"] \\
e \arrow[r, "{(1,0)}"'] & \ulcorner
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
e_{/1} \arrow[r, "\mathrm{pr}"] \arrow[d, "p"'] \arrow[dr, phantom, "\swArrow"] & e \arrow[d, "0"] \\
e \arrow[r, "1"'] & {[1]}
\end{tikzcd}$$

Using the unique isomorphism $c\colon e_{/1} \cong [1]_{/(1,0)}$ we obtain the following two commutative diagrams where i is defined such that the lower triangle commute:

$$\begin{tikzcd}
e_{/1} \arrow[r, "c"] \arrow[d, "\mathrm{pr}"'] & {[1]}_{/(1,0)} \arrow[d, "\mathrm{pr}"] \\
e \arrow[r, "0"] \arrow[d, "0"'] & {[1]} \arrow[d, "j"] \\
{[1]} \arrow[r, "i"] \arrow[dr, bend right, "d_h^1"'] & \ulcorner \arrow[d, "\mathrm{i}_\ulcorner"] \\
& \square
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
e_{/1} \arrow[r, "c"] \arrow[d, "p"'] & {[1]}_{/(1,0)} \arrow[d, "p"] \\
e \arrow[r, "\mathrm{id}"] \arrow[d, "1"'] & e \arrow[d, "{(1,0)}"] \\
{[1]} \arrow[r, "i"] \arrow[dr, bend right, "d_h^1"'] & \ulcorner \arrow[d, "\mathrm{i}_\ulcorner"] \\
& \square
\end{tikzcd}$$

As a final ingredient let us denote the natural transformation in the square related to the Kan extension $j_!$ by α' and the natural transformation occurring in the other square by α. These two transformations are related by the formula $i\circ\alpha = \alpha'\circ c$. With these preparations we can now prove that the following diagram commutes:

$$\begin{tikzcd}
j_! j^* \,\mathrm{i}_\ulcorner{}^*(X)_{(1,0)} \arrow[r, "\epsilon"] & \mathrm{i}_\ulcorner{}^*(X)_{(1,0)} \arrow[r, "="] & d_h^{1\,*}(X)_1 \\
\mathrm{Hocolim}_{[1]_{/(1,0)}} \mathrm{pr}^*\, j^* \,\mathrm{i}_\ulcorner{}^*\, X \arrow[u, "\beta'", "\cong"'] \arrow[dr, bend right, "\cong"'] & & 0_! 0^* d_h^{1\,*}(X)_1 \arrow[u, "\epsilon"'] \\
& \mathrm{Hocolim}_{e_{/1}} c^*\,\mathrm{pr}^*\, j^*\, \mathrm{i}_\ulcorner{}^*\, X \arrow[r, "="'] & \mathrm{Hocolim}_{e_{/1}} \mathrm{pr}^*\, 0^* d_h^{1\,*} X \arrow[u, "\cong", "\beta"']
\end{tikzcd}$$

Here, the undecorated isomorphism is induced by Lemma 1.28 applied to c and the morphisms decorated by β and β' are base change isomorphisms coming from Kan's formula. Once we have shown that this diagram commutes the claim follows immediately. So, let us check the commutativity of this diagram which will be done by applying Lemma 3.4 to both base change morphisms. That lemma yields that the two respective composites of the base change morphism and the adjunction counit can be written as the top and the bottom row of the next diagram respectively:

$$
\begin{tikzcd}
\mathrm{Hocolim}_{[1]_{/(1,0)}} \mathrm{pr}^* j^* \mathrm{i}_{\ulcorner}{}^* X \arrow[r, "\alpha'^*"] \arrow[d, "\cong"'] & \mathrm{Hocolim}_{[1]_{/(1,0)}} p^*(1,0)^* \mathrm{i}_{\ulcorner}{}^* X \arrow[r, "\epsilon"] \arrow[d, "\cong"'] & \mathrm{i}_{\ulcorner}{}^*(X)_{(1,0)} \arrow[d, "="] \\
\mathrm{Hocolim}_{e_{/1}} c^* \mathrm{pr}^* j^* \mathrm{i}_{\ulcorner}{}^* X \arrow[r, "c^*\alpha'^*"] \arrow[d, "="'] & \mathrm{Hocolim}_{e_{/1}} c^* p^*(1,0)^* \mathrm{i}_{\ulcorner}{}^* X \arrow[r, "\epsilon"] \arrow[d, "="] & \mathrm{i}_{\ulcorner}{}^*(X)_{(1,0)} \arrow[d, "="] \\
\mathrm{Hocolim}_{e_{/1}} \mathrm{pr}^* 0^* d_h^{1*} X \arrow[r, "\alpha^* d_h^{1*}"'] & \mathrm{Hocolim}_{e_{/1}} p^* 1^* d_h^{1*} X \arrow[r, "\epsilon"'] & 1^* d_h^{1*} X
\end{tikzcd}
$$

The upper left and the lower right squares commute by naturality while the upper right square commutes because the isomorphism expressing the cofinality of c is compatible with the adjunction counits (Lemma 1.28). Finally, the bottom left square commutes by the relations $i \circ \alpha = \alpha' \circ c$ and $d_h^1 = \mathrm{i}_{\ulcorner} \circ i$.

Now, using our claim, we can reformulate the condition on X being coCartesian by drawing the following diagram:

$$
\begin{tikzcd}
\mathrm{i}_{\ulcorner}{}_! j_! j^* \mathrm{i}_{\ulcorner}{}^* X \arrow[r, "\epsilon", "\cong"'] \arrow[d, "\cong"'] & \mathrm{i}_{\ulcorner}{}_! \mathrm{i}_{\ulcorner}{}^* X \arrow[d, "\epsilon"] \\
d^1_{v!} d^{1*}_v X \arrow[r, "\epsilon"'] & X
\end{tikzcd}
$$

Thus, applying Lemma 1.31 again, X is coCartesian iff the counit $d^1_{v!} d^{1*}_v (X)_{1,1} \longrightarrow X_{1,1}$ is an isomorphism. We claim that this equivalent condition can be reformulated in a convenient way.

Claim: The counit $d^1_{v!} d^{1*}_v (X)_{1,1} \longrightarrow X_{1,1}$ is an isomorphism if and only if the counit $0_! 0^* d^{0*}_h (X)_1 \longrightarrow d^{0*}_h (X)_1$ is an isomorphism.

Once we have established this claim we are done because we can then conclude that X is coCartesian if and only if $0_! 0^* d^{0*}_h (X)_1 \longrightarrow d^{0*}_h (X)_1$ which by i) is equivalent to $d^{0*}_h X$ being an isomorphism. So let us establish this second claim. The diagrams involved in calculating the two homotopy Kan extensions using Kan's formula are as follows:

$$
\begin{tikzcd}
{[1]_{/(1,1)}} \arrow[r, "\mathrm{pr}"] \arrow[d, "p"'] \arrow[dr, phantom, "\swarrow"] & {[1]} \arrow[d, "d^1_v"] \\
e \arrow[r, "{(1,1)}"'] & \square
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
e_{/1} \arrow[r, "\mathrm{pr}"] \arrow[d, "p"'] \arrow[dr, phantom, "\swarrow"] & e \arrow[d, "0"] \\
e \arrow[r, "1"'] & {[1]}
\end{tikzcd}
$$

Let us denote the 2-cell on the left by α' and the one on the right by α. A good way to relate diagrams is to observe that the functor $r\colon e_{/1} \longrightarrow [1]_{/(1,1)}\colon (e, 0 \longrightarrow 1) \longmapsto (1, (0,1) \longrightarrow (1,1))$ classifies the terminal object, i.e., is a right adjoint. The compatibility is described by the commutativity of the following two diagrams:

$$\begin{tikzcd}
e_{/1} \arrow[r, "r"] \arrow[d, "\mathrm{pr}"'] & {[1]_{/(1,1)}} \arrow[d, "\mathrm{pr}"] \\
e \arrow[r, "1"] \arrow[d, "0"'] & {[1]} \arrow[d, "d_v^1"] \\
{[1]} \arrow[r, "d_h^0"'] & \square
\end{tikzcd}
\qquad
\begin{tikzcd}
e_{/1} \arrow[r, "r"] \arrow[d, "p"'] & {[1]_{/(1,1)}} \arrow[d, "p"] \\
e \arrow[r, "="] \arrow[d, "1"'] & e \arrow[d, "{(1,1)}"] \\
{[1]} \arrow[r, "d_h^0"'] & \square
\end{tikzcd}$$

Let us check that the following diagram commutes:

$$\begin{tikzcd}
d_{v!}^1 d_v^{1*}(X)_{1,1} \arrow[r, "\epsilon"] & X_{1,1} \arrow[r, "="] & (d_h^0)^*(X)_1 \\
\mathrm{Hocolim}_{[1]_{/(1,1)}} \mathrm{pr}^* d_v^{1*} X \arrow[u, "\beta", "\cong"'] \arrow[dr, bend right, "\cong"'] & & 0_! 0^* d_h^{0*}(X)_1 \arrow[u, "\epsilon"'] \\
& \mathrm{Hocolim}_{e_{/1}} r^* \mathrm{pr}^* d_v^{1*} X \arrow[r, "="'] & \mathrm{Hocolim}_{e_{/1}} \mathrm{pr}^* 0^* d_h^{0*} X \arrow[u, "\cong", "\beta"']
\end{tikzcd}$$

Here, the isomorphisms denoted by β are again base change isomorphisms and the unlabeled isomorphism is induced by the cofinality of the right adjoint r (cf. Lemma 1.28). Expanding this diagram similarly as in the case of the previous claim one can check that it commutes. In fact, applying Lemma 3.4 twice we see that the commutativity of our diagram is equivalent to the commutativity of the following one:

$$\begin{tikzcd}
\mathrm{Hocolim}_{[1]_{/(1,1)}} \mathrm{pr}^* d_v^{1*} X \arrow[r, "\alpha'^*"] \arrow[d, "\cong"'] & \mathrm{Hocolim}_{[1]_{/(1,1)}} p^*(1,1)^* X \arrow[r, "\epsilon"] \arrow[d, "\cong"'] & X_{(1,1)} \arrow[d, "="] \\
\mathrm{Hocolim}_{e_{/1}} r^* \mathrm{pr}^* d_v^{1*} X \arrow[r, "r^*\alpha'^*"] \arrow[d, "="'] & \mathrm{Hocolim}_{e_{/1}} r^* p^*(1,1)^* X \arrow[r, "\epsilon"] \arrow[d, "="] & X_{(1,1)} \arrow[d, "="] \\
\mathrm{Hocolim}_{e_{/1}} \mathrm{pr}^* 0^* d_h^{0*} X \arrow[r, "\alpha^* d_h^{0*}"'] & \mathrm{Hocolim}_{e_{/1}} p^* 1^* d_h^{0*} X \arrow[r, "\epsilon"'] & d_h^{0*}(X)_1
\end{tikzcd}$$

Here, all squares but the lower left one commute by the same reasons as in the previous claim. For the last square, one uses the relation $\alpha' \circ r = d_h^0 \circ \alpha$. $\square$

We now discuss the composition and cancellation property of (co)Cartesian squares. Recall from classical category theory that for a diagram in a category of the shape

$$\begin{tikzcd} X_{0,0} \arrow[r] \arrow[d] & X_{1,0} \arrow[r] \arrow[d] & X_{2,0} \arrow[d] \\ X_{0,1} \arrow[r] & X_{1,1} \arrow[r] & X_{2,1} \end{tikzcd}$$

the following holds: if the square on the left is a pushout, then the square on the right is a pushout if and only if the composite square is. The corresponding result in the theory of derivators is the content of the next proposition. Since the proof is quite lengthy we have deferred it to Appendix B.3. The methods are similar to the ones used in the proof of Proposition 3.8.

Proposition 3.9. *Let $\mathbb{D}$ be a derivator and let $X \in \mathbb{D}([2] \times [1])$.*
i) *If $d_2(X) \in \mathbb{D}(\square)$ is coCartesian, then $d_0(X)$ is coCartesian if and only if $d_1(X)$ is coCartesian.*
ii) *If $d_0(X) \in \mathbb{D}(\square)$ is Cartesian, then $d_2(X)$ is Cartesian if and only if $d_1(X)$ is Cartesian.*

Now, that we have established the properties of (co)Cartesian squares necessary for our purposes, we will quickly define left exact, right exact, and exact morphisms of derivators. As an intermediate step there is the following *minimal* definition.

Definition 3.10. Let $F\colon \mathbb{D} \longrightarrow \mathbb{D}'$ be a morphism of derivators. $\mathbb{D}$ *preserves coCartesian squares* if the following natural transformation is an isomorphism:

$$\begin{tikzcd} \mathrm{i}_{\ulcorner !} F_{\ulcorner} \arrow[rr, "\beta"] \arrow[d, "\eta"'] & & F_{\square} \mathrm{i}_{\ulcorner !} \\ \mathrm{i}_{\ulcorner !} F_{\ulcorner} \mathrm{i}_{\ulcorner}^{*} \mathrm{i}_{\ulcorner !} \arrow[rr, "\gamma"'] & & \mathrm{i}_{\ulcorner !} \mathrm{i}_{\ulcorner}^{*} F_{\square} \mathrm{i}_{\ulcorner !} \arrow[u, "\epsilon"'] \end{tikzcd}$$

As expected by now this suffices to obtain the same behavior on all derivators obtained by tensoring the given ones with a category M. In fact, we have the next result which is an immediate consequence of Proposition 2.12.

Proposition 3.11. *Let $F\colon \mathbb{D} \longrightarrow \mathbb{D}'$ be a morphism of derivators. Then F preserves coCartesian squares if and only if $F\colon \mathbb{D}_M \longrightarrow \mathbb{D}'_M$ preserves coCartesian squares for all categories M.*

Moreover, as an immediate consequence of Corollary 2.11 we have the following result.

Corollary 3.12. *Let $\mathbb{D}$ be a derivator and let M be a category. An object $X \in \mathbb{D}_M(\square)$ is coCartesian if and only if the squares $X_m \in \mathbb{D}(\square)$ are coCartesian for all objects $m \in M$.*

Definition 3.13. Let $F\colon \mathbb{D} \longrightarrow \mathbb{D}'$ be a morphism of derivators.
i) The morphism F is *left exact* if it preserves Cartesian squares and final objects.
ii) The morphism F is *right exact* if it preserves coCartesian squares and initial objects.
iii) The morphism F is *exact* if it is left exact and right exact.

Example 3.14. i) Let $(F,U)\colon \mathcal{M} \longrightarrow \mathcal{N}$ be a Quillen adjunction between combinatorial model categories. The morphism $\mathbb{L}F\colon \mathbb{D}_{\mathcal{M}} \longrightarrow \mathbb{D}_{\mathcal{N}}$ is right exact and the morphism $\mathbb{R}U\colon \mathbb{D}_{\mathcal{N}} \longrightarrow \mathbb{D}_{\mathcal{M}}$ is left exact. This holds more generally for an arbitrary adjunction of derivators.
ii) Let $\mathbb{D}$ be a derivator and let $u\colon L \longrightarrow M$ be a functor. The induced morphism of derivators $u^{*}\colon \mathbb{D}_M \longrightarrow \mathbb{D}_L$ is exact.

4. Stable derivators

4.1. **Suspensions, loops, cones, and fibers in the pointed setting.** Let $\mathbb{D}$ be a pointed derivator and let J be a category. In this subsection we want to construct the suspension and loop functors on $\mathbb{D}(J)$ and the cone and fiber functors on $\mathbb{D}(J\times[1])$. By Proposition 1.36, we can assume $J=e$.

Let us begin with the suspension functor Σ and the loop functor Ω. The 'extension by zero functors' as given by Proposition 1.40 will again be crucial. Let us consider the following sequences of functors:

$$e \xrightarrow{(0,0)} \ulcorner \xrightarrow{i_\ulcorner} \square \xleftarrow{(1,1)} e, \qquad\qquad e \xrightarrow{(1,1)} \lrcorner \xrightarrow{i_\lrcorner} \square \xleftarrow{(0,0)} e.$$

Since $(0,0)$ resp. $(1,1)$ is an open resp. a closed immersion the homotopy Kan extension functors $(0,0)_*$ resp. $(1,1)_!$ give us 'extension by zero functors' by Proposition 1.40.

Definition 4.1. Let $\mathbb{D}$ be a pointed derivator.

i) The *suspension functor* Σ is given by $\Sigma\colon\ \mathbb{D}(e)\xrightarrow{(0,0)_*}\mathbb{D}(\ulcorner)\xrightarrow{i_{\ulcorner !}}\mathbb{D}(\square)\xrightarrow{(1,1)^*}\mathbb{D}(e).$

ii) The *loop functor* Ω is given by $\Omega\colon\ \mathbb{D}(e)\xrightarrow{(1,1)_!}\mathbb{D}(\lrcorner)\xrightarrow{i_{\lrcorner *}}\mathbb{D}(\square)\xrightarrow{(0,0)^*}\mathbb{D}(e).$

The motivation for these definitions should be clear from topology. Recall that given a pointed topological space X, the suspension ΣX is constructed by first taking two instances of the canonical inclusion into the (contractible!) cone $\mathsf{C}X$ and then forming the pushout:

$$\begin{array}{ccc} X & \longrightarrow & \mathsf{C}X \\ \downarrow & & \\ \mathsf{C}X & & \end{array} \qquad\qquad \begin{array}{ccc} X & \longrightarrow & \mathsf{C}X \\ \downarrow & & \downarrow \\ \mathsf{C}X & \longrightarrow & \Sigma X \end{array}$$

We can consider this diagram as a homotopy pushout. The above definition abstracts precisely this construction. Of course, we want to show that these functors define an adjoint pair $(\Sigma,\Omega)\colon\ \mathbb{D}(e)\rightharpoonup\mathbb{D}(e)$. For this purpose, let us denote by $\mathcal{M}\subset\mathbb{D}(\square)$, $\mathcal{M}^{\ulcorner}\subset\mathbb{D}(\ulcorner)$, and $\mathcal{M}^{\lrcorner}\subset\mathbb{D}(\lrcorner)$ the respective full subcategories spanned by the objects X with $X_{1,0}\cong 0\cong X_{0,1}$.

Proposition 4.2. *If* $\mathbb{D}$ *is a pointed derivator, then we have an adjunction* $(\Sigma,\Omega)\colon\ \mathbb{D}(e)\rightharpoonup\mathbb{D}(e)$.

Proof. With the notation established above, the suspension and the loop functor can be factored as follows:

$$\begin{array}{rccccccccccl} \Sigma\colon & \mathbb{D}(e) & \xrightarrow[\simeq]{(0,0)_*} & \mathcal{M}^{\ulcorner} & \xrightarrow{i_{\ulcorner !}} & \mathcal{M} & \xrightarrow{i_{\lrcorner}^*} & \mathcal{M}^{\lrcorner} & \xrightarrow[\simeq]{(1,1)^*} & \mathbb{D}(e) & & \\ & \| & & \| & & \| & & \| & & \| & & \\ & \mathbb{D}(e) & \xleftarrow[\simeq]{(0,0)^*} & \mathcal{M}^{\ulcorner} & \xleftarrow{i_{\ulcorner}^*} & \mathcal{M} & \xleftarrow{i_{\lrcorner *}} & \mathcal{M}^{\lrcorner} & \xleftarrow[\simeq]{(1,1)_!} & \mathbb{D}(e) & & :\Omega \end{array}$$

The existence of the factorization is clear and the fact that the functors $(0,0)_*$ and $(1,1)_!$ restricted this way are equivalences follows from there fully-faithfulness and Proposition

1.40. From this description, one sees immediately that we have an adjunction (Σ, Ω) which is, in fact, given as a composite adjunction of four adjunctions among which two are equivalences. $\square$

Using similar constructions, one can introduce *cone* and *fiber functors* for pointed derivators. Again, the definition is easily motivated from topology. If we consider a map of spaces $f\colon X \longrightarrow Y$ then the mapping cone $\mathsf{C}f$ of f is constructed in two steps by forming a pushout as indicated in the next diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \\ \mathsf{C}X & \end{tikzcd} \qquad\qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ \mathsf{C}X \arrow[r] & \mathsf{C}f \end{tikzcd}$$

To axiomatize this in the context of a pointed derivator, let us consider the following morphisms of posets:

$$[1] \xrightarrow{i} \ulcorner \xrightarrow{i_\ulcorner} \square \xleftarrow{i_\lrcorner} \lrcorner \xleftarrow{j} [1]$$

Here, i is the open immersion classifying the horizontal arrow while j is the closed immersion classifying the vertical arrow. In particular, by Proposition 1.40, we have again extension by zero functors i_* and $j_!$.

Definition 4.3. Let $\mathbb{D}$ be a pointed derivator.
i) The *cone functor* $\mathsf{Cone}\colon \mathbb{D}([1]) \longrightarrow \mathbb{D}([1])$ is defined as the composition:

$$\mathsf{Cone}\colon\ \mathbb{D}([1]) \xrightarrow{i_*} \mathbb{D}(\ulcorner) \xrightarrow{i_\ulcorner !} \mathbb{D}(\square) \xrightarrow{j^*} \mathbb{D}([1])$$

ii) The *fiber functor* $\mathsf{Fiber}\colon \mathbb{D}([1]) \longrightarrow \mathbb{D}([1])$ is defined as the composition:

$$\mathsf{Fiber}\colon\ \mathbb{D}([1]) \xrightarrow{j_!} \mathbb{D}(\lrcorner) \xrightarrow{i_\lrcorner *} \mathbb{D}(\square) \xrightarrow{i^*} \mathbb{D}([1])$$

Moreover, let $\mathsf{C}\colon \mathbb{D}([1]) \longrightarrow \mathbb{D}(e)$ resp. $\mathsf{F}\colon \mathbb{D}([1]) \longrightarrow \mathbb{D}(e)$ be the functors obtained from the cone resp. fiber functors by evaluation at 1 resp. 0.

Proposition 3.8 shows that the cone $\mathsf{C}f$ of an isomorphism f is the zero object 0. The converse is only true in the stable situation (cf. Lemma 4.9). For a counterexample to the converse in the unstable situation one can consider the following situation. Let $\mathcal{E}$ be an exact category in the sense of Quillen (cf. [Qui73]). Moreover, let us assume $\mathcal{E}$ to have enough injectives but also that $\mathcal{E}$ is not Frobenius, i.e., the classes of injectives and projectives do not coincide. The stable category $\underline{\mathcal{E}}$ which is obtained from $\mathcal{E}$ by dividing out the maps factoring over injectives is a 'suspended category' in the sense of [KV87]. Let now X be an object of $\mathcal{E}$ of injective dimension 1 and let $0 \longrightarrow X \longrightarrow I^0 = I \longrightarrow I^1 = \Sigma X \longrightarrow 0$ be an injective resolution of X. By definition of the suspended structure on $\underline{\mathcal{E}}$ (cf. [KV87] or [Hap88, Chapter I]) the diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "\mathrm{id}"'] & I \arrow[r, "v"] \arrow[d, "\mathrm{id}"] & \Sigma X \arrow[d, "\mathrm{id}"] \\ X \arrow[r] & I \arrow[r] & \Sigma X \end{tikzcd}$$

gives rise to the distinguished triangle $X \xrightarrow{u} I \xrightarrow{v} \Sigma X \xrightarrow{\mathrm{id}} \Sigma X$. Since ΣX is trivial in the stable category $\underline{\mathcal{E}}$ the morphism u is an example of a morphism which is not an isomorphism but still has a vanishing cone. In the stable situation, i.e., in the Frobenius case, this counterexample cannot exist. In fact, the above resolution of X would split because ΣX is by assumption injective, hence projective, showing that the injective dimension of X is zero. This example can be made into an example about pointed derivators by using [Kel07].

As a preparation for the next proof, let us denote by $\mathcal{N} \subset \mathbb{D}(\square)$, $\mathcal{N}^{\ulcorner} \subset \mathbb{D}(\ulcorner)$, and $\mathcal{N}^{\lrcorner} \subset \mathbb{D}(\lrcorner)$ the respective full subcategories spanned by the objects X with $X_{0,1} \cong 0$.

Proposition 4.4. *Let $\mathbb{D}$ be a pointed derivator, then we have an adjunction:*

$$(\mathsf{Cone}, \mathsf{Fiber})\colon\ \mathbb{D}([1]) \rightharpoonup \mathbb{D}([1])$$

Proof. There are the following factorizations of the cone and fiber functors:

$$\begin{tikzcd} \mathsf{Cone}\colon & \mathbb{D}([1]) \arrow[r, "i_*", "\simeq"'] \arrow[d, equal] & \mathcal{N}^{\ulcorner} \arrow[r, "i_{\ulcorner !}"] \arrow[d, equal] & \mathcal{N} \arrow[r, "i_{\lrcorner}^*"] \arrow[d, equal] & \mathcal{N}^{\lrcorner} \arrow[r, "j^*", "\simeq"'] \arrow[d, equal] & \mathbb{D}([1]) \arrow[d, equal] & \\ & \mathbb{D}([1]) & \mathcal{N}^{\ulcorner} \arrow[l, "i^*"', "\simeq"] & \mathcal{N} \arrow[l, "i_{\ulcorner}^*"'] & \mathcal{N}^{\lrcorner} \arrow[l, "i_{\lrcorner *}"'] & \mathbb{D}([1]) \arrow[l, "j_!"', "\simeq"] & \colon \mathsf{Fiber} \end{tikzcd}$$

The existence of these factorizations is again obvious and the fact that the outer functors are equivalences follows again from Proposition 1.40. Thus, this shows that the pair $(\mathsf{Cone}, \mathsf{Fiber})$ is the composition of four adjunctions among which two are equivalences. □

We just want to mention that in the third part the authors thesis there is an alternative description of the functors introduced in this subsection. This alternative description will be helpful in the understanding of morphisms in $\mathbb{D}([1])$ which induce zero morphisms on underlying diagrams.

4.2. **Stable derivators and the canonical triangulated structures.** In this subsection, we come to the central notion of a stable derivator. Similarly to the situation of a stable model category or a stable ∞-category, one adds a 'linearity condition' to the pointed situation. This will ensure, in particular, that the suspension and the loop functor define a pair of inverse equivalences

$$(\Sigma, \Omega)\colon\ \mathbb{D}(e) \xrightarrow{\simeq} \mathbb{D}(e).$$

This notion was introduced by Maltsiniotis in [Mal07a] by forming a combination of the axioms of Grothendiecks derivators [Gro] and Franke's systems of triangulated diagram categories [Fra96]. More details on the history can be found in the paper [CN08] by Cisinski and Neeman.

Definition 4.5. A strong derivator $\mathbb{D}$ is *stable* if it is pointed and if an object of $\mathbb{D}(\square)$ is coCartesian if and only if it is Cartesian.

The strongness property will be crucial in two situations in the construction of the canonical triangulated structures. Let us call a square *biCartesian* if it satisfies the equivalent conditions of being Cartesian or coCartesian.

Example 4.6. i) Let $\mathcal{M}$ be a stable combinatorial model category then the associated derivator $\mathbb{D}_{\mathcal{M}}$ is stable. Thus, we have, in particular, the stable derivator associated to unbounded chain complexes, modules over a differential graded algebra, spectra based on simplicial sets and module spectra over a given symmetric ring spectrum. These derivators can be endowed with some additional structure: they are examples of monoidal derivators resp. derivators tensored over a monoidal derivator as discussed in [Gro11].
ii) A derivator $\mathbb{D}$ is stable if and only if the dual derivator $\mathbb{D}^{\mathrm{op}}$ is stable.

Let us begin by the following convenient result. This will allow us to reduce the situation in many later proofs from an arbitrary category J to the special case $J = e$.

Proposition 4.7. *Let $\mathbb{D}$ be a stable derivator and let J be a category. Then $\mathbb{D}_J$ is again stable.*

Proof. It is immediate that a derivator $\mathbb{D}$ is strong if and only if $\mathbb{D}_J$ is strong for all categories J. The corresponding statement is true for pointed derivators by Proposition 1.36. Thus, let us consider the (co)Cartesian squares. For an object $X \in \mathbb{D}_J(\square)$, using Corollary 3.12, we have that X is coCartesian if and only if $X_j \in \mathbb{D}(\square)$ is coCartesian for all $j \in J$. Using the stability of $\mathbb{D}$ and the same result for Cartesian squares in $\mathbb{D}_J(\square)$ we are done. $\square$

We give immediately the expected result on the suspension and loop functors in this stable situation. Recall the definition of the categories $\mathcal{M}$, $\mathcal{M}^{\lrcorner}$, $\mathcal{M}^{\ulcorner}$, and the factorization of (Σ, Ω) in the case of a pointed derivator. Let us denote, in addition, by $\mathcal{M}^{\Sigma} \subset \mathcal{M}$ resp. $\mathcal{M}^{\Omega} \subset \mathcal{M}$ the full subcategories which are spanned by the squares which are coCartesian resp. Cartesian. With this notation, in the case of a pointed derivator, there is the following additional factorization of (Σ, Ω) :

$$
\begin{tikzcd}
\Sigma\colon & \mathbb{D}(e) \arrow[r, "(0,0)_*", "\simeq"'] \arrow[d, equal] & \mathcal{M}^{\ulcorner} \arrow[r, "i_{\ulcorner !}", "\simeq"'] \arrow[d, equal] & \mathcal{M}^{\Sigma} \arrow[r, "i_{\lrcorner}^*"] & \mathcal{M}^{\lrcorner} \arrow[r, "(1,1)^*", "\simeq"'] \arrow[d, equal] & \mathbb{D}(e) \arrow[d, equal] & \\
& \mathbb{D}(e) & \mathcal{M}^{\ulcorner} \arrow[l, "(0,0)^*"', "\simeq"] & \mathcal{M}^{\Omega} \arrow[l, "i_{\ulcorner}^*"'] & \mathcal{M}^{\lrcorner} \arrow[l, "i_{\lrcorner *}"', "\simeq"] & \mathbb{D}(e) \arrow[l, "(1,1)_!"', "\simeq"] & :\Omega
\end{tikzcd}
$$

In this diagram, all but possibly the two restriction functors in the middle are equivalences. In the case of a stable derivator, we have $\mathcal{M}^{\Sigma} = \mathcal{M}^{\Omega}$ and these two restriction functors are also equivalences:

$$
\begin{tikzcd}
\Sigma\colon & \mathbb{D}(e) \arrow[r, "(0,0)_*", "\simeq"'] \arrow[d, equal] & \mathcal{M}^{\ulcorner} \arrow[r, "i_{\ulcorner}{}_!", "\simeq"'] \arrow[d, equal] & \mathcal{M}^{\Sigma} \arrow[r, "i_{\lrcorner}{}^*", "\simeq"'] \arrow[d, equal] & \mathcal{M}^{\lrcorner} \arrow[r, "(1,1)^*", "\simeq"'] \arrow[d, equal] & \mathbb{D}(e) \arrow[d, equal] & \\
& \mathbb{D}(e) & \mathcal{M}^{\ulcorner} \arrow[l, "(0,0)^*"', "\simeq"] & \mathcal{M}^{\Omega} \arrow[l, "i_{\ulcorner}{}^*"', "\simeq"] & \mathcal{M}^{\lrcorner} \arrow[l, "i_{\lrcorner}{}_*"', "\simeq"] & \mathbb{D}(e) \arrow[l, "(1,1)_!"', "\simeq"] & :\Omega
\end{tikzcd}
$$

This proves the first half of the next result. The second half can be proved in a similar way.

Proposition 4.8. *Let $\mathbb{D}$ be a stable derivator, then we have the equivalences of categories*

$$(\Sigma, \Omega)\colon\ \mathbb{D}(e) \xrightarrow{\simeq} \mathbb{D}(e) \qquad \textit{and} \qquad (\mathsf{Cone}, \mathsf{Fiber})\colon\ \mathbb{D}([1]) \xrightarrow{\simeq} \mathbb{D}([1]).$$

Let us mention the following result which shows that in the stable situations isomorphisms can be characterized by the vanishing of the cone. We use the same notation as in Proposition 3.8.

Lemma 4.9. *Let $\mathbb{D}$ be a stable derivator and let $X \in \mathbb{D}(\square)$. If two of the three following statements hold for the square X then so does the third one:*
i) *the square X is coCartesian,*
ii) *the arrow $d_h^{0\,*}X$ is an isomorphism,*
iii) *the arrow $d_h^{1\,*}X$ is an isomorphism.*
In particular, an object $f \in \mathbb{D}([1])$ is an isomorphism if and only if the cone $\mathsf{C}f$ is zero.

Proof. For the first part we can apply Proposition 3.8 to see that we only have to show that i) and ii) imply iii). But this statement follows from the dual of Proposition 3.8 which can be applied because every coCartesian square is also Cartesian in the stable situation. Finally, the second part follows from the first part when applied to the special case of the defining square of the cone. $\square$

The next aim is to show that, in the stable case, finite coproducts and finite products in $\mathbb{D}(J)$ are canonically isomorphic. By Proposition 4.7, we can assume that $J = e$. But let us first mention the following result which is immediate from Proposition 3.9 on the composition and the cancellation properties of (co)Cartesian squares. That result is crucial in order to establish the preadditivity.

Proposition 4.10. *Let $\mathbb{D}$ be a stable derivator and let $X \in \mathbb{D}([2] \times [1])$. If two of the squares $d_0(X), d_1(X)$, and $d_2(X)$ are biCartesian, then so is the third one.*

We now give the result on the preadditivity of the values of a stable derivator.

Proposition 4.11. *Let $\mathbb{D}$ be a stable derivator and consider a functor $u\colon J \longrightarrow K$. Then, in $\mathbb{D}(J)$, finite coproducts and finite products exist and are canonically isomorphic. Moreover, these are preserved by $u^*, u_!$, and u_*.*

Proof. For the first part, it is again enough to show the result for the case $J = e$. Let us consider the inclusion $j_2\colon L_2 \longrightarrow L_3$ of the left poset L_2 in the right poset L_3:

$$\begin{tikzcd} & (1,0) \arrow[r] & (2,0) \\ (0,1) \arrow[d] & & \\ (0,2) & & \end{tikzcd} \qquad \begin{tikzcd} (0,0) \arrow[r] \arrow[d] & (1,0) \arrow[r] & (2,0) \\ (0,1) \arrow[d] & & \\ (0,2) & & \end{tikzcd}$$

Moreover, let $j_1\colon e \sqcup e \longrightarrow L_2$ be the map $(1,0) \sqcup (0,1)$ and let $j_3\colon L_3 \longrightarrow [2] \times [2] = L$ be the obvious inclusion. Since j_1 resp. j_2 is an open resp. a closed immersion the homotopy Kan extension functors j_{1*} resp. $j_{2!}$ are 'extension by zero functors' by Proposition 1.40. Let us consider the functor:

$$\mathbb{D}(e) \times \mathbb{D}(e) \simeq \mathbb{D}(e \sqcup e) \xrightarrow{j_{1*}} \mathbb{D}(L_2) \xrightarrow{j_{2!}} \mathbb{D}(L_3) \xrightarrow{j_{3!}} \mathbb{D}(L)$$

The image $Q \in \mathbb{D}(L)$ of a pair $(X, Y) \in \mathbb{D}(e) \times \mathbb{D}(e)$ under this functor has as underlying diagram:

$$\mathsf{dia}_L(Q): \qquad \begin{tikzcd} 0 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0 \arrow[d] \\ Y \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & Y' \arrow[d] \\ 0 \arrow[r] & X' \arrow[r] & Z \end{tikzcd}$$

Let us denote the four inclusions of the smaller squares in L by $i_k,\ k = 1, \ldots, 4$, i.e., let us set

$$i_1 = d^2 \times d^2, \qquad i_2 = d^0 \times d^2, \qquad i_3 = d^2 \times d^0, \qquad \text{and} \qquad i_4 = d^0 \times d^0.$$

An application of Proposition 3.5 to these inclusions $i_k\colon \square \longrightarrow L,\ k = 1, \ldots, 4$, and $f = j_3$ allows us to deduce that all squares are biCartesian. In fact, in all four cases, $i_k(1,1) \notin \mathrm{Im}(j_3)$ and we only have to check that the induced functors $\tilde{i}_k\colon \ulcorner \longrightarrow L - i_k(1,1)_{/i_k(1,1)}$ are right adjoints. For $k = 1$, this functor is an isomorphism while in the other three cases Lemma 3.6 applies. By Proposition 4.10, also the composite squares $(d_2 \times d_1)(Q)$ and $(d_1 \times d_2)(Q)$ are biCartesian. Hence, Proposition 3.8 ensures that we have isomorphisms $X \cong X'$ and $Y \cong Y'$. Similarly, the square $(d_1 \times d_1)(Q)$ is biCartesian and we obtain an isomorphism $Z \cong 0$. Thus, using the alternative description of coproducts via homotopy left Kan extensions (Lemma 3.7) and its dual, we see that B is simultaneously a coproduct of X and Y and a product of $X' \cong X$ and $Y' \cong Y$.

The fact that these biproducts are preserved by $u^*, u_!$ and u_* follows immediately since each of the three functors has an adjoint functor on at least one side. $\square$

Corollary 4.12. *Let $\mathbb{D}$ be a stable derivator and let J be a category. Every object of $\mathbb{D}(J)$ is canonically a commutative monoid object and a cocommutative comonoid object. In particular, the morphism set $\mathsf{hom}_{\mathbb{D}(J)}(X,Y),\ X,Y \in \mathbb{D}(J)$, carries canonically the structure of an abelian monoid.*

Proof. For $X \in \mathbb{D}(J)$, the diagonal map $\Delta_X \colon X \longrightarrow X \times X \cong X \sqcup X$ is counital, coassociative and cocommutative. Dually, the codiagonal $\nabla_X \colon X \times X \cong X \sqcup X \longrightarrow X$ is unital, associative and commutative. These can be used to define the sum of two morphism $f,g \colon X \longrightarrow Y$ using the usual convolution or cup product, i.e., as:

$$f+g \colon X \xrightarrow{\Delta_X} X \times X \xrightarrow{f\times g} Y \times Y \cong Y \sqcup Y \xrightarrow{\nabla_Y} Y$$

□

We will from now on use the standard notation $\oplus$ for the biproduct. The next aim is to show that objects of the form ΩX resp. ΣX are even abelian *group* resp. *cogroup* objects. We give the proof in the case of ΩX in which case the constructions can be motivated by the process of concatenation of loops in topology. We begin with some preparations. Since the aim is to 'model categorically' the concatenation and inversion of loops we have to consider finite direct sums of 'loop objects'. For the construction of the finite sums of loop objects there is the following conceptual approach which admits an obvious dualization. Let $\lrcorner_n$ be the poset with objects $e_0,\ldots,e_n$ and $(1,1)$ and with ordering generated by $e_i \leq (1,1),\ i=0,\ldots,n$. The pictures of $\lrcorner_n$ for $n=1$ and $n=2$ are:

$$\begin{tikzcd} & e_1 \arrow[d] \\ e_0 \arrow[r] & (1,1) \end{tikzcd} \qquad \begin{tikzcd} e_1 \arrow[dr] & e_2 \arrow[d] \\ e_0 \arrow[r] & (1,1) \end{tikzcd}$$

Let $\mathcal{F}in$ denote the category of the finite sets $\langle n\rangle = \{0,\ldots,n\}$ with all set-theoretic maps as morphisms between them. The association $\langle n\rangle \longmapsto \lrcorner_n$ can be made into a functor $\mathcal{F}in \longrightarrow \mathsf{Cat}$ if we send $f \colon \langle k\rangle \longrightarrow \langle n\rangle$ to $\lrcorner_f \colon \lrcorner_k \longrightarrow \lrcorner_n \colon e_i \longmapsto e_{f(i)}$. Since $(1,1)\colon e \longrightarrow \lrcorner_n$ is a closed immersion, $(1,1)_! \colon \mathbb{D}(e) \longrightarrow \mathbb{D}(\lrcorner_n)$ gives us an 'extension by zero functor'. Define P_n as

$$P_n \colon\ \mathbb{D}(e) \xrightarrow{\quad (1,1)_! \quad} \mathbb{D}(\lrcorner_n) \xrightarrow{\quad \mathrm{Holim}_{\lrcorner_n} \quad} \mathbb{D}(e)$$

and note that we have a canonical isomorphism $P_1X \cong \Omega X$. The definition of $P_\bullet$ on morphisms and the proof of its functoriality will be given in Appendix B.4.

Lemma 4.13. *Let $\mathbb{D}$ be a stable derivator. The above construction defines a bifunctor:*

$$P \colon \mathcal{F}in^{\mathrm{op}} \times \mathbb{D}(e) \longrightarrow \mathbb{D}(e) \colon (\langle n\rangle, X) \longmapsto P_nX$$

We introduce notations for some morphisms in $\mathcal{F}in$. Given a $(k+1)$-tupel $(i_0,i_1,\ldots,i_k)$ of elements of $\langle n\rangle$ let us denote the corresponding morphism $\langle k\rangle \longrightarrow \langle n\rangle$ which sends j to i_j by $(i_0 i_1 \ldots i_k)$. For $n \geq 1$ and $1 \leq k \leq n$, we have thus the morphism $(k-1,k)\colon \langle 1\rangle \longrightarrow \langle n\rangle$.

So, for a stable derivator $\mathbb{D}$ and an object $X \in \mathbb{D}(e)$, we obtain by the last lemma induced maps:

$$(k-1,k)^* = P((k-1,k), \mathrm{id}_X) \colon P_nX \longrightarrow P_1X \cong \Omega X$$

These maps taken together define the following *Segal maps* and satisfy the 'usual' Segal condition ([Seg74]) which will also be shown in Appendix B.4.

Lemma 4.14. *Let $\mathbb{D}$ be a stable derivator and let $X \in \mathbb{D}(e)$. For $n \geq 1$ and $1 \leq k \leq n$, the $(k-1,k)^*$ together define a natural isomorphism in $\mathbb{D}(e)$:*

$$s = s_n \colon P_nX \xrightarrow{\cong} \prod_{k=1}^{n} P_1(X) \cong \bigoplus_{k=1}^{n} \Omega X$$

Having the functorial construction of finite direct sums of *loop objects* at our disposal, we want to show now that ΩX is always canonically an abelian group object. As an intermediate step, let us construct a pairing $\star\colon \Omega X \oplus \Omega X \longrightarrow \Omega X$ which will be called the *concatenation map.* Using the last lemma, this pairing can be defined as the composite

$$\star\colon \Omega X \oplus \Omega X \xrightarrow{\cong} P_2(X) \xrightarrow{(02)^*} \Omega X$$

where the first arrow is the inverse of the Segal map.

Lemma 4.15. *Let $\mathbb{D}$ be a stable derivator and let X be an object of $\mathbb{D}(e)$. The concatenation map $\star\colon \Omega X \oplus \Omega X \longrightarrow \Omega X$ is an associative pairing on ΩX.*

Proof. Let U be a further object of $\mathbb{D}(e)$ and consider three morphisms $f, g, h\colon U \longrightarrow \Omega X$ in $\mathbb{D}(e)$. In the following diagram, all maps labeled by s are Segal maps:

$$\begin{tikzcd}
 & \Omega X & & \\
 & & P_2X \arrow[ul, "(02)^*"'] \arrow[dl, "s"'] & \\
U \arrow[uur, bend left, "f\star(g\star h)"] \arrow[r, "{f,g\star h}"] \arrow[ddr, bend right, "{f,g,h}"'] & \Omega X \oplus \Omega X & & P_3X \arrow[ul, "(013)^*"'] \arrow[dl, "s"] \\
 & & \Omega X \oplus P_2X \arrow[ul, "\mathrm{id}\oplus(02)^*"] \arrow[dl, "\mathrm{id}\oplus s"] & \\
 & \Omega X \oplus \Omega X \oplus \Omega X & &
\end{tikzcd}$$

The two quadrilaterals on the left commute by definition of the concatenation and the right one commutes by functoriality of $P_\bullet$. We can thus deduce the relation $f \star (g \star h) = (03)^*m(f,g,h)$ where $m(f,g,h)\colon U \longrightarrow P_3X$ is the unique map such that $s \circ m(f,g,h) = (f,g,h)$. This 'associative description' of $f \star (g \star h)$ together with the Yoneda lemma implies the associativity of the concatenation map. $\square$

Heading for the additive inverse of the identity on loop objects, let us consider the only non-trivial automorphism $\sigma\colon \langle 1\rangle \longrightarrow \langle 1\rangle$ in $\mathcal{F}in$. Then $\lrcorner_\sigma\colon \lrcorner \longrightarrow \lrcorner$ is the isomorphism interchanging the vertices $(1,0)$ and $(0,1)$. There is thus an induced automorphism $\sigma^* = (10)^*\colon \Omega X \longrightarrow \Omega X$ which we call the *inversion of loops*.

Proposition 4.16. *Let $\mathbb{D}$ be a stable derivator and let $X \in \mathbb{D}(e)$. The inversion of loops map $\sigma^*\colon \Omega X \longrightarrow \Omega X$ is an additive inverse to $\mathrm{id}_{\Omega X}$. In particular, $\Omega X \in \mathbb{D}(e)$ is an abelian* group *object.*

Proof. By functoriality of the construction $P_\bullet X$, there is a right action of the symmetric group on three letters on P_2X. We want to describe the corresponding action on $\Omega X \oplus \Omega X$ obtained by conjugation with the Segal map s. The strategy of the proof is then to use this action in order to relate the concatenation product and the addition of morphisms.

For different elements $i, j \in \langle 2\rangle$ let us denote by σ_{ij} the associated transposition. One checks that the following diagram commutes

$$\begin{tikzcd} P_2X \arrow[r, "\sigma_{02}^*"] \arrow[d, "s"'] & P_2X \arrow[d, "s"] \\ \Omega X \oplus \Omega X \arrow[r, "\begin{pmatrix} 0 & \sigma^* \\ \sigma^* & 0 \end{pmatrix}"'] & \Omega X \oplus \Omega X \end{tikzcd}$$

where the arrows labeled by s are again Segal maps. From the equality of the maps

$$\sigma_{01} \circ (01) = (01) \circ \sigma\colon \langle 1\rangle \longrightarrow \langle 2\rangle$$

we conclude that the endomorphism of $\Omega X \oplus \Omega X$ corresponding to σ_{01} is a lower triangular matrix

$$s \circ \sigma_{01}^* \circ s^{-1} = \begin{pmatrix} \sigma^* & 0 \\ \alpha & \beta \end{pmatrix}\colon \Omega X \oplus \Omega X \longrightarrow \Omega X \oplus \Omega X$$

for some maps $\alpha, \beta\colon \Omega X \longrightarrow \Omega X$. The fact that σ_{01} is an involution implies the relations:

$$\alpha\sigma^* + \alpha = 0 \qquad \text{and} \qquad \beta^2 = \mathrm{id}$$

The aim is now to show that both maps α and β are identities which would in particular imply that σ^* is an additive inverse of $\mathrm{id}_{\Omega X}$. From the relation $(02) = \sigma_{01} \circ (12)$ we immediately get $(02)^* = (12)^* \circ \sigma_{01}^*\colon P_2X \longrightarrow \Omega X$. Using the matrix description of the map induced by σ_{01} we see that for two maps $f, g\colon U \longrightarrow \Omega X$ there is the following formula for the concatenation product:

$$f \star g = \alpha f + \beta g\colon U \longrightarrow \Omega X$$

By the last lemma we know that the concatenation pairing is associative which already gives the first relation $\beta = \mathrm{id}_{\Omega X}$. Instead of using $(02) = \sigma_{01} \circ (12)$, we can also use the relation $(02) = \sigma_{12} \circ (01)\colon \langle 1\rangle \longrightarrow \langle 2\rangle$ to obtain a further description of the concatenation product. First, since

$$\sigma_{12} = \sigma_{02} \circ \sigma_{01} \circ \sigma_{02}\colon \langle 2\rangle \longrightarrow \langle 2\rangle$$

we obtain that the endomorphism on $\Omega X \oplus \Omega X$ induced by σ_{12}^* has the following matrix description:

$$s \circ \sigma_{12}^* \circ s^{-1} = \begin{pmatrix} \sigma^*\beta\sigma^* & \sigma^*\alpha\sigma^* \\ 0 & \sigma^* \end{pmatrix} : \Omega X \oplus \Omega X \longrightarrow \Omega X \oplus \Omega X$$

From this and the formula $(02)^* = (12)^*\sigma_{01}^*$ we see that the concatenation product can also be written as:

$$f \star g = \sigma^*\beta\sigma^* f + \sigma^*\alpha\sigma^* g\colon U \longrightarrow \Omega X$$

Comparing these two descriptions concludes the proof since we obtain $\alpha = \sigma^*\beta\sigma^* = \mathrm{id}_{\Omega X}$.

□

Remark 4.17. Although we will not make use of this remark we want to emphasize the following. The proof of the last proposition shows that the addition on mapping spaces into loop objects coincides with the pairing induced by the concatenation of loops. Similarly, additive inverses are given by the inversion of loops. Thus for maps $f, g\colon U \longrightarrow \Omega X$ we have:

$$f + g = f \star g \qquad \text{and} \qquad -f = (01)^* f$$

A combination of this proposition, the result on the preadditivity of $\mathbb{D}(J)$ (Proposition 4.11) and the fact that (Σ, Ω) is a pair of inverse equivalences in the stable situation gives us immediately the following corollary.

Corollary 4.18. *If $\mathbb{D}$ is a stable derivator then $\mathbb{D}(J)$ is an additive category for an arbitrary J. Moreover, for an arbitrary functor $u\colon J \longrightarrow K$, the induced functors $u^*, u_!$, and u_* are additive.*

With this preparation we can now attack the main result of this section, namely, that given a stable derivator $\mathbb{D}$ then the categories $\mathbb{D}(J)$ are canonically triangulated categories. Using Proposition 4.7, we can again assume without loss of generality that we are in the case $J = e$. The suspension functor of the triangulated structure will be the suspension functor $\Sigma\colon \mathbb{D}(e) \longrightarrow \mathbb{D}(e)$ we constructed already. Thus, let us construct the class of distinguished triangles. For this purpose, let K denote the poset:

$$\begin{tikzcd} (0,0) \arrow[r] \arrow[d] & (1,0) \arrow[r] & (2,0) \\ (0,1) & & \end{tikzcd}$$

Moreover, let $i_0\colon [1] \longrightarrow K$ be the map classifying the left horizontal arrow and let $i_1\colon K \longrightarrow [2] \times [1]$ be the obvious inclusion. Let us denote the composition by $i\colon [1] \xrightarrow{i_0} K \xrightarrow{i_1} [2] \times [1]$. Again, since i_0 is an open immersion, i_{0*} gives us an extension by zero functor. Let us consider:

$$T\colon\ \mathbb{D}([1]) \xrightarrow{i_{0*}} \mathbb{D}(K) \xrightarrow{i_{1!}} \mathbb{D}([2] \times [1])$$

Lemma 4.19. *Let $\mathbb{D}$ be a stable derivator and consider an object $f \in \mathbb{D}([1])$ with underlying diagram $f\colon X \longrightarrow Y$. The squares $d_0T(f)$, $d_1T(f)$, and $d_2T(f) \in \mathbb{D}(\square)$ are then biCartesian. Moreover, we have canonical isomorphisms $T(f)_{2,1} \cong \Sigma X$ and $T(f)_{1,1} \cong \mathsf{C}(f)$.*

Proof. By Proposition 4.10, it is enough to show the biCartesianness of $d_0T(f)$ and $d_2T(f)$. This can be done by two applications of the detection result Proposition 3.5 to $i_1\colon K \longrightarrow J = [2] \times [1]$. It is easy to check (using Lemma 3.6 in one of the cases) that the assumptions of that proposition are satisfied. Since i_0 is an open immersion, the underlying diagram of $d_1T(f)$ looks like:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & T(f)_{2,1} \end{tikzcd}$$

Moreover, by the proof of Proposition 4.8, $d_1T(f)$ lies in the essential image of

$$\mathbb{D}(e) \xrightarrow{(0,0)_*} \mathbb{D}(\ulcorner) \xrightarrow{\mathrm{i}_{\ulcorner}{}_!} \mathbb{D}(\square).$$

Hence, we have a canonical isomorphism $T(f)_{2,1} \cong \Sigma X$. Similarly, the underlying diagram of $d_2T(f)$ looks like

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & T(f)_{1,1} \end{tikzcd}$$

Let again $j\colon [1] \longrightarrow \mathrm{i}_{\ulcorner}$ denote the functor classifying the upper horizontal morphism. Then, the square $d_2T(f)$ lies in the essential image of

$$\mathbb{D}([1]) \xrightarrow{j_*} \mathbb{D}(\ulcorner) \xrightarrow{\mathrm{i}_{\ulcorner}{}_!} \mathbb{D}(\square).$$

Hence, we also have a canonical isomorphism $T(f)_{1,1} \cong \mathsf{C}(f)$ concluding the proof. $\square$

Thus, for $f \in \mathbb{D}([1])$, by first restricting $T(f)$ to $[3]$ in the expected way and then forming the underlying diagram in $\mathbb{D}(e)$, we obtain a triangle (T_f) in $\mathbb{D}(e)$ which is of the following form:

$$(T_f): \quad X \longrightarrow Y \longrightarrow \mathsf{C}(f) \longrightarrow \Sigma X$$

Call a triangle in $\mathbb{D}(e)$ *distinguished* if it is isomorphic to (T_f) for some $f \in \mathbb{D}([1])$. We are now in position to state the following important theorem.

Theorem 4.20. *Let $\mathbb{D}$ be a stable derivator and let J be a category. Endowed with the suspension functor $\Sigma\colon \mathbb{D}(J) \longrightarrow \mathbb{D}(J)$ and the above class of distinguished triangles, $\mathbb{D}(J)$ is a triangulated category.*

The fact that this triangulated structure is compatible with the restriction and homotopy Kan extension functors will be discussed in Corollary 4.24. For easier reference to the axioms of a triangulated category we include a definition. For more background on this theory cf. for example [Nee01] or to [Sch07]. The form of the octahedron axiom given here is sufficient in order to obtain the usual form of the octahedron axiom. This observation was made in [KV87] (for a proof of it see [Sch07]).

Definition 4.21. Let $\mathcal{T}$ be an additive category with a self-equivalence $\Sigma\colon \mathcal{T} \longrightarrow \mathcal{T}$ and a class of so-called distinguished triangles $X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$. The pair consisting of Σ and the class of distinguished triangles determines a *triangulated structure* on $\mathcal{T}$ if the following four axioms are satisfied. In this case, the triple consisting of the category, the endofunctor, and the class of distinguished triangles is called a *triangulated category.*

(T1) For every $X \in \mathcal{T}$, the triangle $X \xrightarrow{\mathrm{id}} X \longrightarrow 0 \longrightarrow \Sigma X$ is distinguished. Every morphism in $\mathcal{T}$ occurs as the first morphism in a distinguished triangle and the class of distinguished triangles is replete, i.e., is closed under isomorphisms.

(T2) A triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is distinguished if and only if the rotated triangle $Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-f} \Sigma Y$ is.

(T3) Given two distinguished triangles and a commutative solid arrow diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "u"] & Y \arrow[r] \arrow[d, "v"] & Z \arrow[r] \arrow[d, dashed, "w"] & \Sigma X \arrow[d, "\Sigma u"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X'
\end{tikzcd}$$

there exists a dashed arrow $w\colon Z \longrightarrow Z'$ as indicated such that the extended diagram commutes.

(T4) For every pair of composable arrows $f_3\colon X \xrightarrow{f_1} Y \xrightarrow{f_2} Z$ there is a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f_1"] \arrow[d, equal] & Y \arrow[r, "g_1"] \arrow[d, "f_2"'] & C_1 \arrow[r, "h_1"] \arrow[d] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "f_3"'] & Z \arrow[r, "g_3"'] \arrow[d, "g_2"'] & C_3 \arrow[r, "h_3"'] \arrow[d] & \Sigma X \\
 & C_2 \arrow[r, equal] \arrow[d, "h_2"'] & C_2 \arrow[d, "\Sigma g_1 \circ h_2"] & \\
 & \Sigma Y \arrow[r, "\Sigma g_1"'] & \Sigma C_1 &
\end{tikzcd}$$

in which the rows and columns are distinguished triangles.

We will now give the proof of the theorem.

Proof. (of Theorem 4.20)
It suffices to do this for the case $J = e$. The additivity of $\mathbb{D}(e)$ is already given by Corollary 4.18. Moreover, in this stable setting, the suspension functor Σ is an equivalence.

(T1): The first part of axiom (T1) is settled by Lemma 4.9 and the second part is settled using the assumed strongness. The last part of (T1) holds by definition of the class of distinguished triangles.

(T3): Axiom (T3) is settled similarly by reducing first to the situation of triangles of the form (T_f) for $f \in \mathbb{D}([1])$ and then applying the strongness again.

(T2): Before we give the actual proof of axiom (T2) we recall that the axioms of a triangulated category as given here are not in a minimal form. In fact, if one has already established axioms (T1) and (T3) it suffices to give a proof of one half of the rotation axiom as indicated in the next claim (cf. again to [Sch07] for this fact).

Claim: let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ be a distinguished triangle in $\mathbb{D}(e)$, then also $Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$ is distinguished.

We can again reduce to the case where the given distinguished triangle is (T_f) for some $f \in \mathbb{D}([1])$. Let us consider the category J given by the following full subposet of $[2] \times [2]$

$$\begin{tikzcd}
(0,0) \arrow[r] \arrow[d] & (1,0) \arrow[r] \arrow[dd] & (2,0) \\
(0,1) \arrow[dr, bend right] & & \\
& (1,2) &
\end{tikzcd}$$

and let $i\colon [1] \longrightarrow J$ be the functor classifying the upper left horizontal morphism. Then i is an open immersion and i_* gives us thus an extension by zero functor. Moreover, let us denote by j the canonical inclusion of J in $K = [2] \times [2] - \{(0,2)\}$. For a given $f \in \mathbb{D}([1])$ let us consider $j_! i_*(f)$. Again, by a repeated application of Proposition 3.5 all squares in $j_! i_*(f)$ are biCartesian. If the diagram of f is $f\colon X \longrightarrow Y$ then the underlying diagram of $j_! i_*(f)$ looks like:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[r] \arrow[d, "g"] & 0 \arrow[d] \\
0 \arrow[r] & \mathsf{C}f \arrow[r, "h"] \arrow[d] & \Sigma X \arrow[d] \\
& 0 \arrow[r] & \Sigma Y
\end{tikzcd}$$

In fact, the inclusion $(d^1 \times d^2)\colon \ulcorner \longrightarrow K$ allows us to identify the value at $(2,1)$ with ΣX while the inclusion $(d^0 \times d^1)\colon \ulcorner \longrightarrow K$ gives us an identification of the lower right corner with ΣY. However, this last inclusion differs from the usual one by the automorphism $\sigma\colon \ulcorner \longrightarrow \ulcorner$. By Proposition 4.16, the induced map $\sigma^*\colon \Sigma Y \longrightarrow \Sigma Y$ is $-\mathrm{id}_{\Sigma Y}$. Hence, using moreover the unique natural transformation of the two inclusions $(d^0 \times d^1) \longrightarrow (d^1 \times d^2)\colon \ulcorner \longrightarrow K$, we can identify the morphism $\Sigma X \longrightarrow \Sigma Y$ as $-\Sigma f$ and this shows that the triangle (T_g) is as stated in the claim.

(T4): It remains to give a proof of the octahedron axiom. The proof of this will be split

into two parts.
i) In the first part, given an object $F \in \mathbb{D}([2])$, we construct an associated octahedron diagram in $\mathbb{D}(e)$. The pattern of this part of the proof is by now quite familiar. Consider the category J given by the following full subposet of $[4] \times [2]$

$$\begin{tikzcd}
(0,0) \arrow[r] \arrow[d] & (1,0) \arrow[r] \arrow[dd] & (2,0) \arrow[r] & (3,0) \arrow[dr, bend left] & \\
(0,1) \arrow[rrrr] \arrow[dr, bend right] & & & & (4,1) \\
& (1,2) & & &
\end{tikzcd}$$

and let $i\colon [2] \longrightarrow J$ classify the two composable upper left morphisms. Moreover, let

$$j\colon J \longrightarrow K = [4] \times [2] - \{(4,0),(0,2)\}$$

be the canonical inclusion. Since i is an open immersion, the homotopy right Kan extension functor i_* is an extension by zero functor. For $F \in \mathbb{D}([2])$ let us consider $D = j_! i_*(F) \in \mathbb{D}(K)$. If the underlying diagram of F is $X \xrightarrow{f_1} Y \xrightarrow{f_2} Z$ then the underlying diagram of D is

$$\begin{tikzcd}
X \arrow[r, "f_1"] \arrow[d] & Y \arrow[r, "f_2"] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \widehat{C_1} \arrow[r] \arrow[d] & \widehat{C_3} \arrow[r] \arrow[d] & SX \arrow[r] \arrow[d] & 0 \arrow[d] \\
& 0 \arrow[r] & \widehat{C_2} \arrow[r] & SY \arrow[r] & S\widehat{C_1}
\end{tikzcd}$$

A repeated application of Proposition 3.5 guarantees that all squares in D are biCartesian. Hence the same is also true for all compound squares one can find in D. This allows us to find canonical isomorphisms $\widehat{C_k} \cong \mathsf{C}(f_k)$ if we set $f_3 = f_2 \circ f_1$. More precisely, the cone functor C has of course to be applied to $f_1 = d_2(F)$, $f_2 = d_0(F)$, and $f_3 = d_1(F) \in \mathbb{D}([1])$. Similarly, we obtain isomorphisms $SX \cong \Sigma X$, $SY \cong \Sigma Y$, and $S\widehat{C_1} \cong \Sigma \widehat{C_1}$. Thus, one can extract an octahedron diagram in $\mathbb{D}(e)$ from the object D.
ii) In this part, we show that every 'first half of an octahedron diagram' comes up to iso from an object $F \in \mathbb{D}([2])$. Let us restrict attention to the upper left square

$$\begin{tikzcd}
X \arrow[r, "f_1"] \arrow[d, equal] & Y \arrow[d, "f_2"] \\
X \arrow[r, "f_3"'] & Z
\end{tikzcd}$$

of such a diagram. By (Der6), there is an object $F_1 \in \mathbb{D}([1])$ such that $\mathsf{dia}\, F_1 \cong (f_1 \colon X \longrightarrow Y)$. Moreover, let us consider $p^*Z \in \mathbb{D}([1])$, where $p = p_{[1]} \colon [1] \longrightarrow e$ is the unique functor. Then, we obtain a morphism $\phi \colon F_1 \longrightarrow p^*Z$ by following f_2 through the following sequence of natural isomorphisms:

$$\begin{aligned} \mathsf{hom}_{\mathbb{D}(e)}(Y, Z) &\cong \mathsf{hom}_{\mathbb{D}(e)}((F_1)_1, Z) \\ &\cong \mathsf{hom}_{\mathbb{D}(e)}(\mathrm{Hocolim}_{[1]} F_1, Z) \\ &\cong \mathsf{hom}_{\mathbb{D}([1])}(F_1, p^*Z) \end{aligned}$$

The second isomorphism in this sequence is obtained by an application of Lemma 1.29. Considering this map $\phi \colon F_1 \longrightarrow p^*Z$ as an object of $\mathbb{D}([1])^{[1]}$, a further application of (Der6) guarantees the existence of an object $Q \in \mathbb{D}(\square)$ such that $\mathsf{dia}_{[1],[1]}\, Q \cong (\phi \colon F_1 \longrightarrow p^*Z)$:

$$\mathsf{dia}\, Q : \quad \begin{tikzcd} X \arrow[r, "\phi_0"] \arrow[d, "f_1"'] & Z \arrow[d] \\ Y \arrow[r, "\phi_1"'] & Z \end{tikzcd}$$

If $i \colon [2] \longrightarrow \square$ classifies the non-degenerate pair of composable arrows passing through the lower left corner $(0, 1)$ then let us set $F = i^*Q \in \mathbb{D}([2])$. This F does the job. $\square$

From now on, whenever we consider the values of a stable derivator as triangulated categories we will always mean the triangulated structure of Theorem 4.20. The next aim is to show that the functors belonging to a stable derivator can be canonically made into exact functors with respect to these structures. In the stable setting, Corollary 4.18 induces immediately the following one.

Corollary 4.22. *Let $F \colon \mathbb{D} \longrightarrow \mathbb{D}'$ be a morphism of stable derivators, then:*

$$F \text{ is left exact} \iff F \text{ is exact} \iff F \text{ is right exact}$$

In particular, the components $F_J \colon \mathbb{D}(J) \longrightarrow \mathbb{D}'(J)$ of an exact morphism are additive functors.

Exact morphisms are the 'correct' morphisms for stable derivators. Some evidence for this is given by the next result.

Proposition 4.23. *Let $F \colon \mathbb{D} \longrightarrow \mathbb{D}'$ be an exact morphism of stable derivators and let J be a category. The functor $F_J \colon \mathbb{D}(J) \longrightarrow \mathbb{D}'(J)$ can be canonically endowed with the structure of an exact functor of triangulated categories.*

Proof. By Proposition 4.7, we can assume without loss of generality that $J = e$. Moreover, by definition, F preserves zero objects and coCartesian squares. In particular, coCartesian squares such that the two off-diagonal entries vanish are preserved by F. This gives us the canonical isomorphism $F \circ \Sigma \cong \Sigma \circ F$. Similarly, F preserves composites of two coCartesian squares. In particular, among the composites those which vanish at $(2, 0)$ and $(0, 1)$ are preserved. These were used to define the class of distinguished triangles in the canonical triangulated structures from where it follows that F preserves distinguished triangles. □

This result can now be applied to Example 3.14. In particular, we can deduce that the functors belonging to a stable derivator respect the canonical triangulated structures we just constructed.

Corollary 4.24. *Let $\mathbb{D}$ be a stable derivator and let $u\colon J \longrightarrow K$ be a functor. The induced functors $u^*\colon \mathbb{D}(K) \longrightarrow \mathbb{D}(J)$ and $u_!,\ u_*\colon \mathbb{D}(J) \longrightarrow \mathbb{D}(K)$ can be canonically endowed with the structure of exact functors.*

Proof. Since we have adjunctions $(u_!, u^*)$ and (u^*, u_*), it suffices to show that u^* can be canonically endowed with the structure of an exact functor (cf. [Mar83, p.463]). But this functor u^* can be considered as $u^*\colon \mathbb{D}_K(e) \longrightarrow \mathbb{D}_J(e)$ and hence the result follows by a combination of the last proposition and Example 3.14. □

Remark 4.25. Theorem 4.20 and Proposition 4.23 reveal certain advantages of the language of stable derivators over the language of triangulated categories. A triangulated category $\mathcal{T}$ is, by the very definition, a triple consisting of a category $\mathcal{T}$ together with a functor $\Sigma\colon \mathcal{T} \longrightarrow \mathcal{T}$ and a class of distinguished triangle as additionally *specified structure*. These are then subject to a long list of axioms. One advantage of the stable derivators is that this structure does not have to be specified but instead is canonically available. Once the derivator is stable, i.e., has some easily motivated *properties*, triangulated structures can be canonically constructed. In particular, the octahedron axiom does not have to be made explicit.

Similarly, the fact that a morphism F of triangulated categories is exact means, by the very definition, that the functor is endowed with an *additional structure* given by a natural isomorphism $\sigma\colon F\circ\Sigma \longrightarrow \Sigma\circ F$ which behaves nicely with respect to the two chosen classes of distinguished triangles. But, in fact, the exactness of such a morphism should only be a property and not a structure. In most applications, the exact functors under consideration are 'derived functors' of functors defined 'on certain models in the background'. And in this situation, the exactness then reflects the fact that this functor preserves (certain) finite homotopy (co)limits. In the setting of stable derivators this is precisely the notion of an exact morphism. In particular, the exactness of a morphism is again a property and not the specification of an additional structure.

These same advantages are also shared by stable $\infty-$categories as studied in detail in Lurie's [Lur11]. A short introduction to that theory can be found in [Gro10b, Section 5].

We are now basically done with the development of the theory of (stable) derivators. So let us analyze what conditions on a 2-subcategory $\mathsf{Dia} \subseteq \mathsf{Cat}$ have to be imposed in order to be able to also deduce the same results for (stable) derivators of type Dia . By the very definition of a derivator, we need that the empty category and the terminal category belong to Dia. Moreover, it has to be closed under finite coproducts to give sense to axiom (Der1). Furthermore, we frequently reduced situations to the case of the underlying category by using the passage from $\mathbb{D}$ to $\mathbb{D}_J$. Thus, Dia has also to be closed under products. We also used various finite posets as admissible shapes in the proofs of this section so we should ask axiomatically for a sufficient supply of them. Finally, Dia has to be closed under the slice construction since we impose axiomatically Kan's formula. There is the following definition of a diagram category which we cite from [CN08]. In particular, this notion has the closure properties we used in the development of the theory.

Definition 4.26. A full 2-subcategory $\mathsf{Dia} \subseteq \mathsf{Cat}$ is called a *diagram category* if it satisfies the following axioms:
- All finite posets considered as categories belong to Dia.
- For every $J \in \mathsf{Dia}$ and every $j \in J$, the slice constructions $J_{j/}$ and $J_{/j}$ belong to Dia.
- If $J \in \mathsf{Dia}$ then also $J^{\mathrm{op}} \in \mathsf{Dia}$.
- For every Grothendieck fibration $u\colon J \longrightarrow K$, if all fibers J_k, $k \in K$, and the base K belong to Dia then also J lies in Dia.

With this notion one can now define prederivators and (pointed, stable) derivators of type Dia as 2-functors $\mathsf{Dia}^{\mathrm{op}} \longrightarrow \mathsf{CAT}$ satisfying the corresponding axioms. We leave it to the reader to check that all results we established so far can also be proved in that more general situation.

Example 4.27. The full 2-subcategory of finite posets is the smallest diagram category, Cat itself is the largest one. Further examples are given by the full 2-subcategories spanned by the finite categories or the finite-dimensional categories. Moreover, the intersection of a family of diagram categories is again a diagram category.

4.3. **Recollements of triangulated categories.** In this short subsection, we mainly mention that open and closed immersions give us in the stable situation recollements of triangulated categories. This can be used to reprove (in the stable case) that the (co)exceptional inverse image functors show up for free. We begin with a very short recap of the theory of recollements of triangulated categories. For classical examples of recollements in algebraic geometry cf. [BBD82], for a very nice modern treatment cf. also to the thesis of Heider [Hei07]. Recollements capture axiomatically the situation in which we are given three triangulated categories $\mathcal{T}'$, $\mathcal{T}$, and $\mathcal{T}''$ such that every object of $\mathcal{T}$ can be obtained as an extension of an object of $\mathcal{T}''$ by an object of $\mathcal{T}'$ and vice-versa. More precisely, there is the following definition.

Definition 4.28. A *recollement of triangulated categories* is a diagram of triangulated categories and exact functors

$$\begin{tikzcd} \mathcal{T}' \arrow[r, "i_!"] & \mathcal{T} \arrow[l, bend right, "i^?"'] \arrow[l, bend left, "i^*"] \arrow[r, "j^*"] & \mathcal{T}'' \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

such that the following properties hold:
- the pairs $(i^?, i_!)$, $(i_!, i^*)$, $(j_!, j^*)$, and (j^*, j_*) are adjunctions
- $j^* i_! = 0$

- the functors $i_!$, $j_!$, and j_* are fully-faithful and
- every object $X \in \mathcal{T}$ sits in two distinguished triangles of the form

$$i_!i^*X \longrightarrow X \longrightarrow j_*j^*X \longrightarrow \Sigma i_!i^*X, \qquad j_!j^*X \longrightarrow X \longrightarrow i_!i^?X \longrightarrow \Sigma j_!j^*X$$

where in both triangles the first two arrows are the respective adjunction morphisms.

One can show that in this situation $\mathcal{T}' = \ker j^*$ and that $\mathcal{T}''$ is the Verdier quotient $\mathcal{T}/\mathcal{T}'$ ([Hei07]). The latter follows immediately from the first since by definition a recollement gives us a reflective localization and a coreflective colocalization ([Kra10]). Let us remark further that this definition is not given in a minimal form but is overdetermined. Recall from classical category theory that if a functor admits an adjoint on both sides then if one of the adjoints is fully-faithful then this is also the case for the other one ([Bor94a, Prop. 3.4,2]). And, even more interesting for us, it suffices to only have the right half of a recollement. More precisely, there is the following result ([Hei07, Prop. 1.14]).

Proposition 4.29. *Consider a diagram of triangulated categories and exact functors*

$$\begin{tikzcd} \mathcal{T} \arrow[r, "j^*"] & \mathcal{T}'' \arrow[l, bend right=40, "j_!"'] \arrow[l, bend left=40, "j_*"] \end{tikzcd}$$

such that $(j_!, j^*)$ and (j^*, j_*) are adjunctions and one of the two functors $j_!$, j_* is fully-faithful. If we denote by $\mathcal{T}'$ the kernel of j^* and by $i_!\colon \mathcal{T}' \longrightarrow \mathcal{T}$ the inclusion then the above diagram can be extended to a recollement:

$$\begin{tikzcd} \mathcal{T}' \arrow[r, "i_!"] & \mathcal{T} \arrow[l, bend right=40, "i^?"'] \arrow[l, bend left=40, "i^*"] \arrow[r, "j^*"] & \mathcal{T}'' \arrow[l, bend right=40, "j_!"'] \arrow[l, bend left=40, "j_*"] \end{tikzcd}$$

In the context of a stable derivator, there is the following class of examples.

Example 4.30. Let $\mathbb{D}$ be a stable derivator and consider an open immersion $j\colon U \longrightarrow X$. Moreover, let Z be the full subcategory of X spanned by the objects which are not in the image of j. Then the inclusion $i\colon Z \longrightarrow X$ is a closed immersion. Moreover, by the fully-faithfulness of homotopy Kan extensions along fully-faithful functors and by Proposition 1.40, the last proposition gives us the following recollements:

$$\begin{tikzcd} \mathbb{D}(U) \arrow[r, "j_*"] & \mathbb{D}(X) \arrow[l, bend right=40, "j^*"'] \arrow[l, bend left=40, "j^!"] \arrow[r, "i^*"] & \mathbb{D}(Z) \arrow[l, bend right=40, "i_!"'] \arrow[l, bend left=40, "i_*"] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{D}(Z) \arrow[r, "i_!"] & \mathbb{D}(X) \arrow[l, bend right=40, "i^?"'] \arrow[l, bend left=40, "i^*"] \arrow[r, "j^*"] & \mathbb{D}(U) \arrow[l, bend right=40, "j_!"'] \arrow[l, bend left=40, "j_*"] \end{tikzcd}$$

This example shows that for an open immersion $j\colon U \longrightarrow X$ resp. for a closed immersion $i\colon Z \longrightarrow X$ the the additional adjoint functor $j^!\colon \mathbb{D}(X) \longrightarrow \mathbb{D}(U)$ resp. $i^?\colon \mathbb{D}(X) \longrightarrow \mathbb{D}(Z)$ shows up for free in the above recollements. Thus, this example reproves, in the stable case, that a pointed derivator is also strongly pointed.

Appendix A. Derivators and Grothendieck (op)fibrations

In this appendix, we quickly recall the definition of Grothendieck (op)fibrations. Moreover, we will show that the base change axioms for derivators expressing Kan's formulas can be replaced by alternative 'base change axioms'. As was pointed out to the author by Maltsiniotis, this observation was already in [Gro]. Using these observations, we will, in particular, be able to finish the proof that with a derivator $\mathbb{D}$ also $\mathbb{D}_M$ is a derivator.

One way to motivate the theory of Grothendieck (op)fibrations is the following. Given a functor $u\colon J \longrightarrow K$ and an object $k \in K$, let us denote the fiber of u above k by J_k, i.e., we form the following pullback of categories:

$$\begin{tikzcd} J_k \arrow[r] \arrow[d] & J \arrow[d, "u"] \\ e \arrow[r, "k"'] & K \end{tikzcd}$$

The theory of Grothendieck opfibrations resp. fibrations is an axiomatic framework using so-called u-coCartesian resp. u-Cartesian arrows to obtain a 'covariant resp. contravariant functorial dependence of the fiber on the basepoint' [Vis05, Bor94b]. The theory of these fibration allows one, in particular, to encode many 'coherent systems of functors' in a very convenient way. For an example of this perspective in the context of (symmetric) monoidal $(\infty-)$categories see [Gro10b].

However, from this theory, we only need that the following proposition is true (cf. [Qui73]). Before we can state it, let us quickly remark that there are canonical functors:

$$c\colon J_k \longrightarrow J_{k/}\colon j \longmapsto (j, k \xrightarrow{\mathrm{id}} u(j)) \qquad \text{and} \qquad c\colon J_k \longrightarrow J_{/k}\colon j \longmapsto (j, u(j) \xrightarrow{\mathrm{id}} k)$$

Proposition A.1. *Let $u\colon J \longrightarrow K$ be a functor, then:*
i) *If the functor u is a Grothendieck fibration then the canonical functor $c\colon J_k \longrightarrow J_{k/}$ has a right adjoint for all $k \in K$.*
ii) *If the functor u is a Grothendieck opfibration then the canonical functor $c\colon J_k \longrightarrow J_{/k}$ has a left adjoint for all $k \in K$.*

Example A.2. • Let $u\colon J \longrightarrow K$ be a functor and let $k \in K$. Then the functor $\mathrm{pr}\colon J_{k/} \longrightarrow J$ is a Grothendieck opfibration while the functor $\mathrm{pr}\colon J_{/k} \longrightarrow J$ is a Grothendieck fibration. Moreover, the fibers are in both cases discrete categories, i.e., sets considered as categories.
• Let J and K be arbitrary categories then the projection functor $J \times K \longrightarrow K$ is both a Grothendieck fibration and a Grothendieck opfibration.

Proposition A.3. *Let $\mathbb{D}$ be a derivator and consider a pullback diagram in* Cat*:*

$$\begin{tikzcd} J_1 \arrow[r, "v"] \arrow[d, "u_1"'] & J_2 \arrow[d, "u_2"] \\ K_1 \arrow[r, "w"'] & K_2 \end{tikzcd}$$

If u_2 is a Grothendieck fibration or if w is a Grothendieck opfibration then the base change morphism $w^*u_{2*} \xrightarrow{\beta} u_{1*}v^*$ associated to this diagram is an isomorphism of functors $\mathbb{D}(J_2) \longrightarrow \mathbb{D}(K_1)$.

Proof. We give a proof in the case where u_2 is a Grothendieck fibration. It is enough to check that the base change morphism is an isomorphism at every point $k_1 \in K_1$. In our situation, the following functors and categories are involved:

$$\begin{tikzcd}
(J_1)_{k_1} \arrow[rr, "c_1"] \arrow[dr, bend right, "i_1"'] & & J_{1k_1/} \arrow[dl, bend left, "\mathrm{pr}_1"] & (J_2)_{w(k_1)} \arrow[rr, "c_2"] \arrow[dr, bend right, "i_2"'] & & J_{2w(k_1)/} \arrow[dl, bend left, "\mathrm{pr}_2"] \\
& J_1 \arrow[rrr, "v"] \arrow[d, "u_1"'] & & & J_2 \arrow[d, "u_2"] & \\
& K_1 \arrow[rrr, "w"'] & & & K_2 &
\end{tikzcd}$$

The horizontal morphisms labeled c_1 and c_2 are the canonical functors from the respective fibers to the respective slice constructions. Since u_2 and hence also u_1 is a Grothendieck fibration these canonical functors c are left adjoint functors. Moreover, we have the two natural transformations α_1 resp. α_2 as in:

$$\begin{tikzcd}
J_{1k_1/} \arrow[r, "\mathrm{pr}_1"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\Nearrow"] & J_1 \arrow[d, "u_1"] \\
e \arrow[r, "k_1"'] & K_1
\end{tikzcd} \qquad \text{resp.} \qquad \begin{tikzcd}
J_{2w(k_1)/} \arrow[r, "\mathrm{pr}_2"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\Nearrow"] & J_2 \arrow[d, "u_2"] \\
e \arrow[r, "w(k_1)"'] & K_2
\end{tikzcd}$$

The base change morphism β occurring in the statement fits into the following diagram:

$$\begin{tikzcd}
k_1^*w^*u_{2*} \arrow[r, "\beta"] \arrow[d, "\beta_2"', "\cong"] & k_1u_{1*}v^* \arrow[d, "\cong"', "\beta_1"] \\
\mathrm{Holim}_{J_{2w(k_1)/}}\, \mathrm{pr}_2^* \arrow[d, "\cong"] & \mathrm{Holim}_{J_{1k_1/}}\, \mathrm{pr}_1^*\, v^* \arrow[d, "\cong"'] \\
\mathrm{Holim}_{(J_2)_{w(k_1)}}\, i_2^* \arrow[r, "\cong"] & \mathrm{Holim}_{(J_1)_{k_1}}\, i_1^*v^*
\end{tikzcd}$$

In this diagram the unlabeled vertical isomorphisms are given by an application of Lemma 1.28 to the canonical functors c_1 and c_2, while the horizontal isomorphism at the bottom is given by the fact that we started with a pullback diagram. Finally, the isomorphisms β_1 and β_2 are given by Kan's formula. We claim that this diagram commutes which will then imply that β is an isomorphism. Unraveling definitions we thus have to show that the following diagram commutes:

$$\begin{tikzcd}
k_1^* w^* u_{2*} \arrow[r, "\eta"] \arrow[d, "\eta"'] & k_1^* u_{1*} u_1^* w^* u_{2*} \arrow[r, "="] & k_1^* u_{1*} v^* u_2^* u_{2*} \arrow[r, "\epsilon"] & k_1^* u_{1*} v^* \arrow[d, "\eta"] \\
p_{2*} p_2^* k_1^* w^* u_{2*} \arrow[d, "\alpha_2^*"'] & & & p_{1*} p_1^* k_1^* u_{1*} v^* \arrow[d, "\alpha_1^*"] \\
p_{2*} \operatorname{pr}_2^* u_2^* u_{2*} \arrow[d, "\epsilon"'] & & & p_{1*} \operatorname{pr}_1^* u_1^* u_{1*} v^* \arrow[d, "\epsilon"] \\
p_{2*} \operatorname{pr}_2^* \arrow[r, "\cong"] & \operatorname{Holim}_{(J_2)_{w(k_1)}} i_2^* \arrow[r, "\cong"] & \operatorname{Holim}_{(J_1)_{k_1}} i_1^* v^* & p_1^* \operatorname{pr}_1^* v^* \arrow[l, "\cong"']
\end{tikzcd}$$

Let us focus on the morphism $k_1^* w^* u_{2*} \longrightarrow p_1^* \operatorname{pr}_1^* v^*$ passing through the upper right corner. This can be rewritten as follows:

$$\begin{tikzcd}
k_1^* w^* u_{2*} \arrow[r, "\eta"] \arrow[d, "\eta"'] & k_1^* u_{1*} u_1^* w^* u_{2*} \arrow[r, "="] \arrow[d, "\eta"'] & k_1^* u_{1*} v^* u_2^* u_{2*} \arrow[r, "\epsilon"] \arrow[d, "\eta"] & k_1^* u_{1*} v^* \arrow[d, "\eta"] \\
p_{1*} p_1^* k_1^* w^* u_{2*} \arrow[r, "\eta"] \arrow[d, "\alpha_1^*"'] & p_{1*} p_1^* k_1^* u_{1*} u_1^* w^* u_{2*} \arrow[r, "="] \arrow[d, "\alpha_1^*"'] & p_{1*} p_1^* k_1^* u_{1*} v^* u_2^* u_{2*} \arrow[r, "\epsilon"] \arrow[d, "\alpha_1^*"] & p_{1*} p_1^* k_1^* u_{1*} v^* \arrow[d, "\alpha_1^*"] \\
p_{1*} \operatorname{pr}_1^* u_1^* w^* u_{2*} \arrow[r, "\eta"] \arrow[dr, bend right, "\mathrm{id}"'] & p_{1*} \operatorname{pr}_1^* u_1^* u_{1*} u_1^* w^* u_{2*} \arrow[r, "="] \arrow[d, "\epsilon"'] & p_{1*} \operatorname{pr}_1^* u_1^* u_{1*} v^* u_2^* u_{2*} \arrow[r, "\epsilon"] \arrow[d, "\epsilon"] & p_{1*} \operatorname{pr}_1^* u_1^* u_{1*} v^* \arrow[d, "\epsilon"] \\
& p_{1*} \operatorname{pr}_1^* u_1^* w^* u_{2*} \arrow[r, "="] & p_{1*} \operatorname{pr}_1^* v^* u_2^* u_{2*} \arrow[r, "\epsilon"] & p_1^* \operatorname{pr}_1^* v^*
\end{tikzcd}$$

In this diagram, all squares commute by naturality and the triangle does by the triangular identities for adjunctions. Hence, to conclude the proof, it suffices to show that the following diagram commutes. In that diagram, the two morphisms we want to show to be equal form the boundary of the outer rectangle:

$$
\begin{tikzcd}
k_1^*w^*u_{2*} \arrow[r, "="] \arrow[d, "\eta"'] & k_1^*w^*u_{2*} \arrow[r, "="] \arrow[d, "\eta"'] & k_1^*w^*u_{2*} \arrow[r, "="] \arrow[d, "\eta"] & k_1^*w^*u_{2*} \arrow[d, "\eta"] \\
p_{2*}p_2^*k_1^*w^*u_{2*} \arrow[r, "\cong"] \arrow[d, "\alpha_2^*"'] & \mathrm{Holim}_{(J_2)_{w(k_1)}} c_2^*p_2^*k_1^*w^*u_{2*} \arrow[r, "\cong"] \arrow[d, "c_2^*\alpha_2^*"'] & \mathrm{Holim}_{(J_1)_{k_1}} c_1^*p_1^*k_1^*w^*u_{2*} \arrow[d, "c_1^*\alpha_1^*w^*"] & p_{1*}p_1^*k_1^*w^*u_{2*} \arrow[l, "\cong"'] \arrow[d, "\alpha_1^*"] \\
p_{2*}\,\mathrm{pr}_2^*\,u_2^*u_{2*} \arrow[r, "\cong"] \arrow[dd, "\epsilon"'] & \mathrm{Holim}_{(J_2)_{w(k_1)}} c_2^*\,\mathrm{pr}_2^*\,u_2^*u_{2*} \arrow[d, "="'] & \mathrm{Holim}_{(J_1)_{k_1}} c_1^*\,\mathrm{pr}_1^*\,u_1^*w^*u_{2*} \arrow[d, "="] & p_{1*}\,\mathrm{pr}_1^*\,u_1^*w^*u_{2*} \arrow[l, "\cong"'] \arrow[d, "="] \\
 & \mathrm{Holim}_{(J_2)_{w(k_1)}} i_2^*u_2^*u_{2*} \arrow[r, "\cong"] \arrow[d, "\epsilon"'] & \mathrm{Holim}_{(J_1)_{k_1}} c_1^*\,\mathrm{pr}_1^*\,v^*u_2^*u_{2*} \arrow[d, "\epsilon"] & p_{1*}\,\mathrm{pr}_1^*\,v^*u_2^*u_{2*} \arrow[l, "\cong"'] \arrow[d, "\epsilon"] \\
p_{2*}\,\mathrm{pr}_2^* \arrow[r, "\cong"] & \mathrm{Holim}_{(J_2)_{w(k_1)}} i_2^* \arrow[r, "\cong"] & \mathrm{Holim}_{(J_1)_{k_1}} i_1^*v^* & p_{1*}\,\mathrm{pr}_1^*\,v^* \arrow[l, "\cong"']
\end{tikzcd}
$$

Only the commutativity of the rectangle in the middle does not follow immediately by naturality. But using the relation $w\alpha_1 c_1 = \alpha_2 c_2 v$ and the fact that the horizontal isomorphisms are induced by v one sees that also that rectangle commutes. Hence, the initial diagram commutes and our base change morphism is an isomorphism. $\square$

Next, we show that the property of the last proposition and its dual can be taken as a replacement for axiom (Der4). Let $\mathbb{D}$ be a prederivator which admits homotopy Kan extensions on both sides. Let us agree to say $\mathbb{D}$ *satisfies base change for Grothendieck (op)fibrations* if the conclusion of the last proposition and its dual hold for $\mathbb{D}$. There is a basic case which holds in broad generality. Let K be a category and $k \in K$. Then let us consider the canonical natural transformation α as in:

$$
\begin{tikzcd}
K_{k/} \arrow[r, "\mathrm{pr}"] \arrow[d, "p"'] \arrow[dr, phantom, "\nearrow"] & K \arrow[d, "\mathrm{id}"] \\
e \arrow[r, "k"'] & K
\end{tikzcd}
$$

In this situation, we have the next lemma.

Lemma A.4. *For any right adjoint p_* of p^*, the associated base change morphism $k^* \longrightarrow p_*\,\mathrm{pr}^*$ is an isomorphism.*

Proof. The slice category $K_{k/}$ admits an initial object $(k, \mathrm{id}\colon k \longrightarrow k)$. Hence, by Lemma 1.29, we have a natural isomorphism $p_* \cong i^*$ compatible with the units, where i classifies the initial object. It follows that the base change morphism β of the statement fits into the following composition $\mathrm{id}\colon k^* \xrightarrow{\beta} p_*\,\mathrm{pr}^* \cong i^*\,\mathrm{pr}^* = k^*$ and is hence an isomorphism. $\square$

Proposition A.5. *Let $\mathbb{D}$ be a prederivator satisfying all axioms of a derivator but possibly the base change axiom. If $\mathbb{D}$ satisfies base change for Grothendieck (op)fibrations then $\mathbb{D}$ satisfies the base change axiom, i.e., is a derivator.*

Proof. Let $u\colon J \longrightarrow K$ be a functor and let $k \in K$. We want to show that $\mathbb{D}$ satisfies base change along u. We will give a proof for the case of homotopy right Kan extensions. Consider the following factorization of the transformation α_2 occurring in the base change formula:

$$\begin{tikzcd}
J_{k/} \arrow[r, "\mathrm{pr}"] \arrow[d, "u_{k/}"'] & J \arrow[d, "u"] \\
K_{k/} \arrow[r, "\mathrm{pr}"] \arrow[d, "p"'] \arrow[dr, phantom, "\nearrow"] & K \arrow[d, "\mathrm{id}"] \\
e \arrow[r, "k"'] & K
\end{tikzcd}$$

Let us denote the 2−cell of this diagram by α_1 so that we have $\alpha_1 u_{k/} = \alpha_2$. With this preparation we claim now that the base change morphism for the homotopy right Kan extension along u can be calculated as a composition of the following two base change morphisms:

$$\beta\colon k^*u_* \xrightarrow{\cong} p_* \operatorname{pr}^* u_* \xrightarrow{\cong} p_*(u_{k/})_* \operatorname{pr}^* \cong (p \circ u_{k/})_* \operatorname{pr}^*$$

In this sequence, the second morphism is the base change isomorphism guaranteed by our assumptions on $\mathbb{D}$ since pr is a Grothendieck opfibration and since the upper square is a pullback diagram. The first morphism is the base change isomorphism of the last lemma and the last isomorphism is guaranteed by the uniqueness of right adjoints up to natural isomorphism. That we really have this factorization of β can be seen using the following diagram:

$$\begin{tikzcd}
k^*u_* \arrow[r, "\eta"] \arrow[d, "\eta"'] & p_*p^*k^*u_* \arrow[d, "\eta"'] \arrow[dr, bend left, "\alpha_1^*"] & \\
(pu_{k/})_*(pu_{k/})^*k^*u_* \arrow[r, "\cong"] \arrow[d, "\alpha_2^*"'] & p_*u_{k/*}u_{k/}{}^*p^*k^*u_* \arrow[d, "\alpha_2^*"'] \arrow[dr, bend left, "\alpha_1^*"] & p_* \operatorname{pr}^* u_* \arrow[d, "\eta"] \\
(pu_{k/})_* \operatorname{pr}^* u^*u_* \arrow[r, "\cong"] \arrow[d, "\epsilon"'] & p_*u_{k/*} \operatorname{pr}^* u^*u_* \arrow[r, "="] \arrow[d, "\epsilon"'] & p_*u_{k/*}u_{k/}{}^* \operatorname{pr}^* u_* \arrow[dl, bend left, "\epsilon"] \\
(pu_{k/})_* \operatorname{pr}^* \arrow[r, "\cong"] & p_*u_{k/*} \operatorname{pr}^* &
\end{tikzcd}$$

□

We are now in position to finish the proof of Proposition 1.18.

Lemma A.6. *Let $\mathbb{D}$ be a derivator and let M be a category. The prederivator $\mathbb{D}_M$ satisfies the base change axiom.*

Proof. We only give a sketch of the proof since it is very similar to the last one. So, let $u\colon J \longrightarrow K$ be a functor and let $k \in K$. Let us consider the following diagram:

$$\begin{tikzcd}
M \times J_{k/} \arrow[r, "\mathrm{pr}"] \arrow[d, "u_{k/}"'] & M \times J \arrow[d, "u"] \\
M \times K_{k/} \arrow[r, "\mathrm{pr}"] \arrow[d, "p"'] \arrow[dr, phantom, "\nearrow\!\!\!\!\nearrow"] & M \times K \arrow[d, "\mathrm{id}"] \\
M \times e \arrow[r, "k"'] & M \times K
\end{tikzcd}$$

Since pr is a Grothendieck opfibration we can conclude that the two respective base change morphisms of this diagram are isomorphisms. Then one checks that up to natural isomorphism the composition of these is the base change morphism of Kan's formula occurring in the calculation of the homotopy right Kan extension $u_*\colon \mathbb{D}_M(J) \longrightarrow \mathbb{D}_M(K)$. $\square$

Appendix B. Some technical proofs and constructions

B.1. **Proof of Proposition 2.12.**

Proof. We want to show that the following canonical morphism is an isomorphism:

$$\begin{tikzcd}
(\mathrm{id}_M \times u)_! F_{M\times J} \arrow[rr, "\beta"] \arrow[d, "\eta"'] & & F_{M\times K}(\mathrm{id}_M \times u)_! \\
(\mathrm{id}_M \times u)_! F_{M\times J}(\mathrm{id}_M \times u)^*(\mathrm{id}_M \times u)_! \arrow[r, "\gamma"'] & (\mathrm{id}_M \times u)_!(\mathrm{id}_M \times u)^* F_{M\times K}(\mathrm{id}_M \times u)_! \arrow[ur, "\epsilon"'] &
\end{tikzcd}$$

We will show that this natural transformation is pointwise an isomorphism. So, let $m \in M$ be an object and let us consider the following 'main' diagram:

$$\begin{tikzcd}
(m\times \mathrm{id}_K)^*(\mathrm{id}_M \times u)_! F_{M\times J} \arrow[r, "\beta"] & (m\times \mathrm{id}_K)^* F_{M\times K}(\mathrm{id}_M \times u)_! \\
u_!(m\times \mathrm{id}_J)^* F_{M\times J} \arrow[u, "\beta", "\cong"'] & F_K(m\times \mathrm{id}_K)^*(\mathrm{id}_M \times u)_! \arrow[u, "\cong", "\gamma"'] \\
u_! F_J(m\times \mathrm{id}_J)^* \arrow[u, "\gamma", "\cong"'] \arrow[r, "\cong", "\beta"'] & F_K u_!(m\times \mathrm{id}_J)^* \arrow[u, "\cong", "\beta"']
\end{tikzcd}$$

In this diagram, the vertical base change morphisms labeled by β are isomorphisms since, by Proposition 2.9, $m^*\colon \mathbb{D}_M \longrightarrow \mathbb{D}$ preserves homotopy left Kan extensions. The bottom horizontal base change morphism is an isomorphism by our assumption that F commutes with homotopy left Kan extensions along u. Thus, it suffices to show that this diagram commutes. We will do this in the following steps. Unraveling the definition of the base change morphisms, we easily see that the diagram consisting of the left column and the respective first morphism of the two horizontal base change morphisms can be extended to the following diagram:

$$
\begin{tikzcd}
(m\times \mathrm{id}_K)^*(\mathrm{id}_M \times u)_! F_{M\times J} \arrow[r, "\eta"] & (m\times \mathrm{id}_K)^*(\mathrm{id}_M \times u)_! F_{M\times J}(\mathrm{id}_M \times u)^*(\mathrm{id}_M \times u)_! \\
u_! u^*(m\times \mathrm{id}_K)^*(\mathrm{id}_M \times u)_! F_{M\times J} \arrow[r, "\eta"] \arrow[u, "\epsilon"] & u_! u^*(m\times \mathrm{id}_K)^*(\mathrm{id}_M \times u)_! F_{M\times J}(\mathrm{id}_M \times u)^*(\mathrm{id}_M \times u)_! \arrow[u, "\epsilon"'] \\
u_!(m\times \mathrm{id}_J)^*(\mathrm{id}_M \times u)^*(\mathrm{id}_M \times u)_! F_{M\times J} \arrow[r, "\eta"] \arrow[u, "\gamma", "="'] & u_!(m\times \mathrm{id}_J)^*(\mathrm{id}_M \times u)^*(\mathrm{id}_M \times u)_! F_{M\times J}(\mathrm{id}_M \times u)^*(\mathrm{id}_M \times u)_! \arrow[u, "=", "\gamma"'] \\
u_!(m\times \mathrm{id}_J)^* F_{M\times J} \arrow[r, "\eta"] \arrow[u, "\eta"] & u_!(m\times \mathrm{id}_J)^* F_{M\times J}(\mathrm{id}_M \times u)^*(\mathrm{id}_M \times u)_! \arrow[u, "\eta"'] \\
u_! F_J (m\times \mathrm{id}_J)^* \arrow[r, "\eta"] \arrow[u, "\gamma"] \arrow[dd, equal] & u_! F_J (m\times \mathrm{id}_J)^*(\mathrm{id}_M \times u)^*(\mathrm{id}_M \times u)_! \arrow[u, "\gamma"'] \\
 & u_! F_J u^*(m\times \mathrm{id}_K)^*(\mathrm{id}_M \times u)_! \arrow[u, "=", "\gamma"'] \\
u_! F_J (m\times \mathrm{id}_J)^* \arrow[r, "\eta"] & u_! F_J u^* u_! (m\times \mathrm{id}_J)^* \arrow[u, "\beta"']
\end{tikzcd}
$$

That this diagram commutes follows by naturality for all but the bottom quadrilateral. But the one at the bottom commutes by Lemma 2.10. Similarly, the right column of the 'main' diagram and the respective last morphism of the two horizontal base change morphisms can be extended to the commutative diagram:

$$
\begin{tikzcd}
(m\times \mathrm{id}_K)^*(\mathrm{id}_M \times u)_!(\mathrm{id}_M \times u)^*F_{M\times K}(\mathrm{id}_M \times u)_! \arrow[r, "\epsilon"] & (m\times \mathrm{id}_K)^*F_{M\times K}(\mathrm{id}_M \times u)_! \\
u_!(m\times \mathrm{id}_J)^*(\mathrm{id}_M \times u)^*F_{M\times K}(\mathrm{id}_M \times u)_! \arrow[u, "\beta"] & \\
u_!u^*(m\times \mathrm{id}_K)^*F_{M\times K}(\mathrm{id}_M \times u)_! \arrow[u, "\gamma", "="'] \arrow[r, "\epsilon"] & (m\times \mathrm{id}_K)^*F_{M\times K}(\mathrm{id}_M \times u)_! \arrow[uu, equal] \\
u_!u^*F_K(m\times \mathrm{id}_K)^*(\mathrm{id}_M \times u)_! \arrow[u, "\gamma"] \arrow[r, "\epsilon"] & F_K(m\times \mathrm{id}_K)^*(\mathrm{id}_M \times u)_! \arrow[u, "\gamma"'] \\
u_!u^*F_Ku_!u^*(m\times \mathrm{id}_K)^*(\mathrm{id}_M \times u)_! \arrow[u, "\epsilon"] \arrow[r, "\epsilon"] & F_Ku_!u^*(m\times \mathrm{id}_K)^*(\mathrm{id}_M \times u)_! \arrow[u, "\epsilon"'] \\
u_!u^*F_Ku_!(m\times \mathrm{id}_J)^*(\mathrm{id}_M \times u)^*(\mathrm{id}_M \times u)_! \arrow[u, "\gamma", "="'] \arrow[r, "\epsilon"] & F_Ku_!(m\times \mathrm{id}_J)^*(\mathrm{id}_M \times u)^*(\mathrm{id}_M \times u)_! \arrow[u, "=", "\gamma"'] \\
u_!u^*F_Ku_!(m\times \mathrm{id}_J)^* \arrow[u, "\eta"] \arrow[r, "\epsilon"] & F_Ku_!(m\times \mathrm{id}_J)^* \arrow[u, "=", "\eta"']
\end{tikzcd}
$$

To conclude the proof we have to show now that the left column of the last auxiliary diagram and the right column of the first auxiliary diagram can be compared by the respective missing morphisms γ occurring in the horizontal base change morphisms of our 'main' diagram. But, the composition of the three bottom morphisms in the left column of the last auxiliary diagram is precisely the canonical morphism expressing that m^* preserves homotopy left Kan extensions. Similar remarks apply to the composition of the three upper most arrows in the right column of the first auxiliary diagram. Thus, with this observation the two columns together with the two missing morphisms γ fit into the following diagram:

$$
\begin{tikzcd}
(m\times \mathrm{id}_K)^*(\mathrm{id}_M \times u)_!F_{M\times J}(\mathrm{id}_M \times u)^*(\mathrm{id}_M \times u)_! \arrow[r, "\gamma"] & (m\times \mathrm{id}_K)^*(\mathrm{id}_M \times u)_!(\mathrm{id}_M \times u)^*F_{M\times K}(\mathrm{id}_M \times u)_! \\
u_!(m\times \mathrm{id}_J)^*F_{M\times J}(\mathrm{id}_M \times u)^*(\mathrm{id}_M \times u)_! \arrow[u, "\beta"] \arrow[r, "\gamma"] & u_!(m\times \mathrm{id}_J)^*(\mathrm{id}_M \times u)^*F_{M\times K}(\mathrm{id}_M \times u)_! \arrow[u, "\beta"'] \\
u_!F_J(m\times \mathrm{id}_J)^*(\mathrm{id}_M \times u)^*(\mathrm{id}_M \times u)_! \arrow[u, "\gamma"] & u_!u^*(m\times \mathrm{id}_K)^*F_{M\times K}(\mathrm{id}_M \times u)_! \arrow[u, "=", "\gamma"'] \\
u_!F_Ju^*(m\times \mathrm{id}_K)^*(\mathrm{id}_M \times u)_! \arrow[u, "\gamma", "="'] \arrow[r, "\gamma"] & u_!u^*F_K(m\times \mathrm{id}_J)^*(\mathrm{id}_M \times u)_! \arrow[u, "\gamma"'] \\
u_!F_Ju^*u_!(m\times \mathrm{id}_J)^* \arrow[u, "\beta"] \arrow[r, "\gamma"] & u_!u^*F_Ku_!(m\times \mathrm{id}_J)^* \arrow[u, "\beta"']
\end{tikzcd}
$$

In this diagram, the two smaller quadrilaterals commute by naturality while the larger one does by definition of a morphism of derivators. Thus, we have proved that our 'main' diagram commutes which finally implies that the base change morphism

$$\beta\colon (\mathrm{id}_M \times u)_! F_{M\times J} \longrightarrow F_{(M\times K)}(\mathrm{id}_M \times u)_!$$

is an isomorphism. □

B.2. **Proof of Proposition 3.5.**

Proof. We give a proof of i). By assumption on f, f factors as $K \xrightarrow{\bar{f}} J - i(1,1) \xrightarrow{j} J$ and gives us thus a natural isomorphism $f_! \cong j_! \bar{f}_!$. In particular, $X = f_!(Y) \cong j_! \bar{f}_!(Y)$ lies in the essential image of $j_!$. Since j is fully-faithful, the same holds for $j_!$ and $\epsilon\colon j_! j^*(X) \longrightarrow X$ is hence an isomorphism. Our setup is summarized by:

$$\begin{tikzcd}
 & \big(J - i(1,1)\big)_{/i(1,1)} \arrow[d, "\mathrm{pr}"] & \\
\ulcorner \arrow[ur, "\tilde{i}"] \arrow[r] \arrow[d, "\mathrm{i}_\ulcorner"'] & J - i(1,1) \arrow[d, "j"] & K \arrow[l, "\bar{f}"'] \arrow[dl, "f"] \\
\square \arrow[r, "i"'] & J &
\end{tikzcd}$$

We claim that the following diagram is commutative:

$$\begin{tikzcd}
\mathrm{Hocolim}_{\ulcorner_{/(1,1)}} \mathrm{pr}^* \, \mathrm{i}_\ulcorner{}^* \, i^*(X) \arrow[r, "\beta"] \arrow[d, "\cong"'] & \mathrm{i}_{\ulcorner !} \, \mathrm{i}_\ulcorner{}^* \, i^*(X)_{(1,1)} \arrow[ddd, "\epsilon"] \\
\mathrm{Hocolim}_\ulcorner \, \mathrm{i}_\ulcorner{}^* i^*(X) \arrow[d, equal] & \\
\mathrm{Hocolim}_\ulcorner \, \tilde{i}^* \, \mathrm{pr}^* \, j^*(X) \arrow[d, "\cong"'] & \\
\mathrm{Hocolim}_{(J-i(1,1))_{/i(1,1)}} \mathrm{pr}^* \, j^*(X) \arrow[d, "\beta"'] & i^*(X)_{(1,1)} \arrow[d, equal] \\
j_! j^*(X)_{i(1,1)} \arrow[r, "\epsilon"'] & X_{i(1,1)}
\end{tikzcd}$$

Here, the morphisms denoted by β are base change isomorphisms and the adjunction counit at the bottom is an isomorphism by the above. Finally, the lower isomorphism in the left column is given by Lemma 1.28 which applies since $\tilde{i}$ is a right adjoint by assumption. Thus, if we succeed to show that the diagram commutes, then the other counit morphism is also an isomorphism which ensures that $i^*(X)$ is coCartesian. So, let us show that the two possible ways from the upper left corner to the lower right corner are equal. By Lemma 3.4, the way passing through the upper right corner just gives the following expression where we use an index 1 in α_1 to distinguish this α from the α_2 occurring in the other base change morphism:

$$\mathrm{Hocolim}_{\ulcorner_{/(1,1)}} \mathrm{pr}^* \, \mathrm{i}_{\ulcorner}{}^* \, i^*(X) \xrightarrow{\alpha_1^*} \mathrm{Hocolim}_{\ulcorner_{/(1,1)}} p^*_{\ulcorner_{/(1,1)}} (1,1)^* i^*(X) \xrightarrow{\epsilon} i^*(X)_{(1,1)}$$

An application of the same lemma to the other base change morphism allows us to remark that the above diagram commutes if and only if the following one does:

$$\begin{tikzcd}
\mathrm{Hocolim}_{\ulcorner_{/(1,1)}} \mathrm{pr}^* \, \mathrm{i}_{\ulcorner}{}^* \, i^*(X) \arrow[r, "\alpha_1^*"] \arrow[d, "\cong"'] & \mathrm{Hocolim}_{\ulcorner_{/(1,1)}} p^*_{\ulcorner_{/(1,1)}} (1,1)^* i^*(X) \arrow[d, "\epsilon"] \\
\mathrm{Hocolim}_{\ulcorner} \mathrm{i}_{\ulcorner}{}^* i^*(X) \arrow[d, equal] & (1,1)^* i^*(X) \arrow[d, equal] \\
\mathrm{Hocolim}_{\ulcorner} \tilde{i}^* \, \mathrm{pr}^* \, j^*(X) \arrow[d, "\cong"'] & X_{i(1,1)} \\
\mathrm{Hocolim}_{(J-i(1,1))_{/i(1,1)}} \mathrm{pr}^* \, j^*(X) \arrow[r, "\alpha_2^*"'] & \mathrm{Hocolim}_{(J-i(1,1))_{/i(1,1)}} p^*_{(J-i(1,1))_{/i(1,1)}} X_{i(1,1)} \arrow[u, "\epsilon"']
\end{tikzcd}$$

But this diagram can be completed to:

$$\begin{tikzcd}
\mathrm{Hocolim}_{\ulcorner_{/(1,1)}} \mathrm{pr}^* \, \mathrm{i}_{\ulcorner}{}^* \, i^*(X) \arrow[r, "\alpha_1^*"] \arrow[dd, "\cong"'] & \mathrm{Hocolim}_{\ulcorner_{/(1,1)}} p^*_{\ulcorner_{/(1,1)}} (1,1)^* i^*(X) \arrow[r, "\epsilon"] \arrow[d, equal] & X_{i(1,1)} \arrow[dd, equal] \\
 & \mathrm{Hocolim}_{\ulcorner_{/(1,1)}} p^*_{\ulcorner_{/(1,1)}} X_{i(1,1)} \arrow[d, "\cong"] & \\
\mathrm{Hocolim}_{\ulcorner} \mathrm{i}_{\ulcorner}{}^* i^*(X) \arrow[d, equal] & \mathrm{Hocolim}_{\ulcorner} p_{\ulcorner}^* X_{i(1,1)} \arrow[r, "\epsilon"] \arrow[d, equal] & X_{i(1,1)} \arrow[dd, equal] \\
\mathrm{Hocolim}_{\ulcorner} \tilde{i}^* \, \mathrm{pr}^* \, j^*(X) \arrow[r, "\alpha_2^*"] \arrow[d, "\cong"'] & \mathrm{Hocolim}_{\ulcorner} \tilde{i}^* p^*_{(J-i(1,1))_{/i(1,1)}} X_{i(1,1)} \arrow[d, "\cong"] & \\
\mathrm{Hocolim}_{(J-i(1,1))_{/i(1,1)}} \mathrm{pr}^* \, j^*(X) \arrow[r, "\alpha_2^*"'] & \mathrm{Hocolim}_{(J-i(1,1))_{/i(1,1)}} p^*_{(J-i(1,1))_{/i(1,1)}} X_{i(1,1)} \arrow[r, "\epsilon"'] & X_{i(1,1)}
\end{tikzcd}$$

In this diagram, it is immediate that the top square on the right and the bottom square on the left commute. The bottom square on the right commutes by Lemma 1.28 so it remains only to check the top square on the left. But this square also commutes since, under the identification $\ulcorner \cong \ulcorner_{/(1,1)}$, we have the relation $\alpha_2 \circ \tilde{i} = i \circ \alpha_1$. $\square$

B.3. **Proof of Proposition 3.9.**

Proof. We give a proof of i). For this purpose, let J resp. K be the posets

$$\begin{tikzcd}
(0,0) \arrow[r] \arrow[d] & (1,0) \arrow[r] & (2,0) \\
(0,1) & &
\end{tikzcd}
\qquad \text{resp.} \qquad
\begin{tikzcd}
(0,0) \arrow[r] \arrow[d] & (1,0) \arrow[r] \arrow[d] & (2,0) \\
(0,1) \arrow[r] & (1,1) &
\end{tikzcd}$$

and denote the fully-faithful inclusion functors by $i\colon J \xrightarrow{i_2} K \xrightarrow{i_1} [2] \times [1]$.

1): By Lemma 1.31, $\epsilon\colon i_! i^*(X) \longrightarrow X$ is an isomorphism if and only if $\epsilon_{1,1}$ and $\epsilon_{2,1}$ are isomorphisms. We want to reformulate this in terms of conditions on the squares $d_2(X)$ and $d_1(X)$.

For the case of $(1,1)$, consider the following diagram:

$$\begin{tikzcd}
i_! i^*(X)_{1,1} \arrow[d, "\epsilon"'] & & \mathrm{Hocolim}_{J_{/1,1}} \mathrm{pr}^* \, i^*(X) \arrow[ll, "\beta"'] \\
X_{1,1} = d_2(X)_{1,1} & & \\
\mathrm{i}_{\ulcorner !} \, i^*_{\ulcorner} d_2(X)_{1,1} \arrow[u, "\epsilon_{\ulcorner}"] & & \mathrm{Hocolim}_{\ulcorner} \, d_2 i^*(X) \arrow[uu, "\cong"'] \\
\mathrm{Hocolim}_{\ulcorner_{/1,1}} \mathrm{pr}^* \, \mathrm{i}_{\ulcorner}{}^* \, d_2(X) \arrow[u, "\beta_{\ulcorner}"] \arrow[rr, "\cong"] & & \mathrm{Hocolim}_{\ulcorner} \, \mathrm{i}_{\ulcorner}{}^* \, d_2(X) \arrow[u, equal]
\end{tikzcd}$$

In this diagram, the morphisms β and $\beta_{\ulcorner}$ denote base change isomorphisms. Using Lemma 3.4, one checks that the diagram commutes. Thus, the counit $\epsilon_{1,1}\colon i_! i^*(X)_{1,1} \longrightarrow X_{1,1}$ is an isomorphism if and only if $d_2(X)$ is coCartesian. Hence, under the assumption that $d_2(X)$ is coCartesian, $\epsilon\colon i_! i^*(X) \longrightarrow X$ is an isomorphism if and only if $\epsilon_{2,1}\colon i_! i^*(X)_{2,1} \longrightarrow X_{2,1}$ is an isomorphism. A calculation of this last morphism leads to the following diagram:

$$\begin{tikzcd}
i_! i^*(X)_{2,1} \arrow[d, "\epsilon"'] & \mathrm{Hocolim}_{J_{/2,1}} \mathrm{pr}^* \, i^*(X) \arrow[l, "\beta"'] \arrow[r, "\cong"] & \mathrm{Hocolim}_J \, i^*(X) \arrow[d, "\cong"] \\
X_{2,1} = d_1(X)_{1,1} & & \mathrm{Hocolim}_{\ulcorner} \, d_1 i^*(X) \\
\mathrm{i}_{\ulcorner !} \, \mathrm{i}_{\ulcorner}{}^* \, d_1(X)_{1,1} \arrow[u, "\epsilon_{\ulcorner}"] & \mathrm{Hocolim}_{\ulcorner_{/1,1}} \mathrm{pr}^* \, \mathrm{i}_{\ulcorner}{}^* \, d_1(X) \arrow[l, "\beta_{\ulcorner}"'] \arrow[r, "\cong"'] & \mathrm{Hocolim}_{\ulcorner} \, \mathrm{i}_{\ulcorner}{}^* \, d_1(X) \arrow[u, equal]
\end{tikzcd}$$

In this diagram, the morphisms denoted by β and $\beta_{\ulcorner}$ are base change isomorphisms. Moreover, the isomorphism in the column on the right follows from an application of Lemma 1.28 to the adjunction $(s^1, d^1)\colon J \longrightarrow \ulcorner$. Again, by an application of Lemma 3.4 and using the relation $\alpha \circ d^1 = d^1 \circ \alpha_{\ulcorner}$ in the obvious notation, one can check that this diagram commutes. From this we deduce, that $\epsilon_{2,1}\colon \mathrm{i}_{\ulcorner !}\, \mathrm{i}_{\ulcorner}{}^*(X)_{2,1} \longrightarrow X_{2,1}$ is an isomorphism if and only if $d_1(X)$ is coCartesian.

2): By a further application of Lemma 1.31, we deduce that the counit $i_{1!}i_1^*(X) \longrightarrow X$ is an isomorphism if and only if $i_{1!}i_1^*(X)_{2,1} \longrightarrow X_{2,1}$ is an isomorphism. We now want to reformulate this condition in terms of a condition on the square $d_0(X)$. To calculate this last map, consider the following diagram:

$$\begin{tikzcd}
i_{1!}i_1^*(X)_{2,1} \arrow[d, "\epsilon"'] & \mathrm{Hocolim}_{K_{/2,1}} \mathrm{pr}^*\, i_1^*(X) \arrow[l, "\beta"'] \arrow[r, "\cong"] & \mathrm{Hocolim}_K\, i_1^*(X) \arrow[d, "\cong"] \\
X_{2,1} = d_0(X)_{1,1} & & \mathrm{Hocolim}_{\ulcorner}\, d_0 i_1^*(X) \arrow[d, equal] \\
\mathrm{i}_{\ulcorner !}\, \mathrm{i}_{\ulcorner}{}^*\, d_0(X)_{1,1} \arrow[u, "\epsilon_{\ulcorner}"] & \mathrm{Hocolim}_{\ulcorner_{/1,1}} \mathrm{pr}^*\, \mathrm{i}_{\ulcorner}{}^*\, d_0(X) \arrow[l, "\beta_{\ulcorner}"'] \arrow[r, "\cong"'] & \mathrm{Hocolim}_{\ulcorner}\, \mathrm{i}_{\ulcorner}{}^*\, d_0(X)
\end{tikzcd}$$

Again, the morphisms β and $\beta_{\ulcorner}$ denote base change isomorphisms. An application of Lemma 1.28 to the adjunction (s^0, d^0) yields the isomorphism in the right column. Using Lemma 3.4 and the relation $d^0 \circ \alpha_{\ulcorner} = \alpha \circ d^0$, one can check that the above diagram commutes. We thus deduce that the counit $i_{1!}i_1^*(X)_{2,1} \longrightarrow X_{2,1}$ is an isomorphism if and only if $d_0(X)$ is coCartesian.

If we were now able to show that ϵ as in 1) is an isomorphism if and only if ϵ as in 2) is an isomorphism we were done.

3): Under the assumption that $d_2(X)$ is coCartesian, we have the following

$$\epsilon_1\colon i_{1!}i_1^*(X) \longrightarrow X \text{ is an isomorphism} \quad \Longleftrightarrow \quad \epsilon\colon i_! i^*(X) \longrightarrow X \text{ is an isomorphism.}$$

By the above and since isomorphisms can be tested pointwise, it is enough to show that this is the case for the point $(2,1)$.

Step1): $(\epsilon_2)_{i_1^*(X)}\colon i_{2!}i^*(X) = i_{2!}i_2^*i_1^*(X) \longrightarrow i_1^*(X)$ is an isomorphism.

In fact, to prove this, by Lemma 1.31, we only have to check this at the point $(1,1)$. To calculate this map, let us consider the following diagram:

$$\begin{tikzcd}
i_{2!}i_2^*i_1^*(X)_{1,1} \arrow[d, "\epsilon_2"'] & \mathrm{Hocolim}_{J_{/1,1}}\,\mathrm{pr}^*\,i_2^*i_1^*(X) \arrow[l, "\beta"'] \arrow[d, "\cong"] \\
i_1^*(X)_{1,1} \arrow[d, equal] & \mathrm{Hocolim}_{\ulcorner_{/1,1}}\,\mathrm{pr}^*\,\mathrm{i}_{\ulcorner}{}^*\,d_2(X) \arrow[d, "\beta_{\ulcorner}"] \\
d_2(X)_{1,1} & \mathrm{i}_{\ulcorner\,!}\,\mathrm{i}_{\ulcorner}{}^*\,d_2(X)_{1,1} \arrow[l, "\epsilon_{\ulcorner}"]
\end{tikzcd}$$

Using Lemma 3.4 and the relation $d^2 \circ \alpha_{\ulcorner} = \alpha \circ d^2$, one can show that this diagram commutes. Since $d_2(X)$ is coCartesian, it follows that the counit is an isomorphism.
Step2): The above claim holds. We have to check this claim at the point $(2,1)$. For this purpose, recall that we have $i = i_1 \circ i_2$ from where we obtain an isomorphism $i_! \cong i_{1!} \circ i_{2!}$ which is compatible with the counits. Using this isomorphism, we consider the following diagram

$$\begin{tikzcd}
i_!i^*(X)_{2,1} \arrow[d, "\epsilon"'] & i_{1!}i_{2!}i^*(X)_{2,1} \arrow[l, "\cong"'] \arrow[d, "(\epsilon_2)_{i_1^*(X)}"] \\
X_{2,1} & i_{1!}i_1^*(X)_{2,1} \arrow[l, "\epsilon_1"]
\end{tikzcd}$$

computing the counit of a composite adjunction. Thus, using Step1), we obtain that ϵ is an isomorphism if and only if ϵ_1 is an isomorphism.

Finally, we only have to put the above results together to obtain the following chain of equivalences under the assumption that $d_2(X)$ is coCartesian:

$$\begin{aligned}
& d_1(X) \text{ is coCartesian} \\
\overset{1)}{\iff}\quad & \epsilon\colon i_!i^*(X)_{2,1} \xrightarrow{\cong} X_{2,1} \\
\iff\quad & \epsilon\colon i_!i^*(X) \xrightarrow{\cong} X \\
\overset{3)}{\iff}\quad & \epsilon_1\colon i_{1!}i_1^*(X) \xrightarrow{\cong} X \\
\iff\quad & \epsilon_1\colon i_{1!}i_1^*(X)_{2,1} \xrightarrow{\cong} X_{2,1} \\
\overset{2)}{\iff}\quad & d_0(X) \text{ is coCartesian}
\end{aligned}$$

□

B.4. **Some properties of the construction $P_\bullet$.** We want to show that P_n is functorial in the object $\langle n\rangle \in \mathcal{F}in$. So, let us define its behavior on morphisms. For a morphism

$f\colon \langle k\rangle \longrightarrow \langle n\rangle$ in $\mathcal{F}in$ let us consider the base change morphism β_f associated to

$$
\begin{tikzcd}
\lrcorner_k \arrow[r, "\lrcorner_f"] \arrow[d, "p_k=p_{\lrcorner_k}"'] & \lrcorner_n \arrow[d, "p_n=p_{\lrcorner_n}"] \\
e \arrow[r, equal] & e
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{D}(\lrcorner_k) \arrow[d, "\mathrm{Holim}_{\lrcorner_k}"'] & \mathbb{D}(\lrcorner_n) \arrow[l, "\lrcorner_f^*"'] \arrow[d, "\mathrm{Holim}_{\lrcorner_n}"] \arrow[dl, phantom, "\Nwarrow"] \\
\mathbb{D}(e) \arrow[r, equal] & \mathbb{D}(e)
\end{tikzcd}
$$

which by definition is the natural transformation:

$$
\begin{tikzcd}
\mathrm{Holim}_{\lrcorner_n} \arrow[rr, "\beta_f"] \arrow[d, "\eta_k"'] & & \mathrm{Holim}_{\lrcorner_k}\, \lrcorner_f^* \\
\mathrm{Holim}_{\lrcorner_k}\, p_k^*\, \mathrm{Holim}_{\lrcorner_n} \arrow[r, equal] & \mathrm{Holim}_{\lrcorner_k}\, \lrcorner_f^*\, p_n^*\, \mathrm{Holim}_{\lrcorner_n} \arrow[ur, "\epsilon_n"'] &
\end{tikzcd}
$$

An application of β_f to $(1,1)_!X$ thus yields $P_nX = \mathrm{Holim}_{\lrcorner_n}(1,1)_!X \longrightarrow \mathrm{Holim}_{\lrcorner_k}\, \lrcorner_f^*(1,1)_!X$. Let us now consider the diagram

$$
\begin{tikzcd}
\mathbb{D}(e) \arrow[r, "(1,1)_!"] \arrow[d, equal] & \mathbb{D}(\lrcorner_n) \arrow[r, "\mathrm{Holim}_{\lrcorner_n}"] \arrow[d, "\lrcorner_f^*"] & \mathbb{D}(e) \arrow[d, equal] \\
\mathbb{D}(e) \arrow[r, "(1,1)_!"'] & \mathbb{D}(\lrcorner_k) \arrow[r, "\mathrm{Holim}_{\lrcorner_k}"'] & \mathbb{D}(e)
\end{tikzcd}
$$

where the left square commutes up to the following natural isomorphism:

$$(1,1)_!^k \xrightarrow{\eta} (1,1)_!^k(1,1)^{n*}(1,1)_!^n \xrightarrow{=} (1,1)_!^k(1,1)^{k*}\, \lrcorner_f^*(1,1)_!^n \xrightarrow{\epsilon} \lrcorner_f^*(1,1)_!^n.$$

In order to not get confused by the different maps $(1,1)\colon e \longrightarrow \lrcorner_n$, we distinguished notationally between them by adding an upper index n. Here, η is a natural isomorphism by the fully-faithfulness of $(1,1)^n$ and ϵ is an isomorphism since it its applied to an object of the essential image of $(1,1)_!^k$. An application of $\mathrm{Holim}_{\lrcorner_k}$ to this composite isomorphism gives us then an isomorphism $\gamma_f\colon P_kX \longrightarrow \mathrm{Holim}_{\lrcorner_k}\, \lrcorner_f^*(1,1)_!X$. Putting β_f and γ_f^{-1} together, we obtain the desired map:

$$P_fX = \alpha_f = \gamma_f^{-1} \circ \beta_f\colon P_nX \longrightarrow P_kX$$

We can now give the proof of Lemma 4.13.

Proof. (of Lemma 4.13) By the triangular identities for adjunctions, the assignment behaves well with identities. So, we only have to consider the behavior with respect to compositions. Let $\langle l\rangle \xrightarrow{g} \langle m\rangle \xrightarrow{f} \langle n\rangle$ be a composable pair of morphisms in $\mathcal{F}in$. For this proof, let us denote the target $\mathrm{Holim}_{\lrcorner_k}\, \lrcorner_f^*(1,1)_!X$ of β_f by $Q_f(X)$ and similarly for other morphisms.

We want to show that the following two compositions give the same map $P_nX \longrightarrow P_lX$:

$$\begin{tikzcd}
P_nX \arrow[rr, "\alpha_f"] \arrow[dr, "\beta_f"'] & & P_mX \arrow[rr, "\alpha_g"] \arrow[dl, "\gamma_f", "\cong"'] \arrow[dr, "\beta_g"'] & & P_lX \arrow[dl, "\gamma_g", "\cong"'] \\
& Q_fX & & Q_gX &
\end{tikzcd}
\qquad
\begin{tikzcd}
P_nX \arrow[rr, "\alpha_{fg}"] \arrow[dr, "\beta_{fg}"'] & & P_lX \arrow[dl, "\gamma_{fg}", "\cong"'] \\
& Q_{fg}X &
\end{tikzcd}$$

We do so by extending the left diagram to the following larger commutative diagram:

$$\begin{tikzcd}
P_nX \arrow[rr, "\alpha_f"] \arrow[dr, "\beta_f"'] \arrow[ddrr, bend right, "\beta_{fg}"'] & & P_mX \arrow[rr, "\alpha_g"] \arrow[dl, "\gamma_f", "\cong"'] \arrow[dr, "\beta_g"'] & & P_lX \arrow[dl, "\gamma_g", "\cong"'] \arrow[ddll, bend left, "\gamma_{fg}", "\cong"'] \\
& Q_fX \arrow[dr, dashed] & & Q_gX \arrow[dl, dashed] & \\
& & Q_{fg}X & &
\end{tikzcd}$$

To obtain a minor simplification we drop the object X from notation. It is quite formal and not particularly enlightening to check that the maps we are about to construct do have the claimed properties. So, we only give the two new maps and let the reader check that they do the job.

i) The map $Q_fX \longrightarrow Q_{fg}X$ is given by:

$$\begin{tikzcd}
Q_f \arrow[rr] \arrow[d, equal] & & Q_{fg} \arrow[d, equal] \\
\mathrm{Holim}_{\lrcorner_m} \lrcorner_f^*(1,1)_! \arrow[d, "\eta_l"'] & & \mathrm{Holim}_{\lrcorner_l} \lrcorner_g^* \lrcorner_f^*(1,1)_! \\
\mathrm{Holim}_{\lrcorner_l} p_l^* \,\mathrm{Holim}_{\lrcorner_m} \lrcorner_f^*(1,1)_! \arrow[r, equal] & \mathrm{Holim}_{\lrcorner_l} \lrcorner_g^* p_m^* \,\mathrm{Holim}_{\lrcorner_m} \lrcorner_f^*(1,1)_! \arrow[ur, "\epsilon_m"'] &
\end{tikzcd}$$

ii) The map $Q_g \longrightarrow Q_{fg}$ is constructed using the adjunctions $\big((1,1)_!, (1,1)^*\big)$:

$$\begin{tikzcd}
Q_g \arrow[rr] \arrow[d, equal] & & Q_{fg} \arrow[d, equal] \\
\mathrm{Holim}_{\lrcorner_l} \lrcorner_g^*(1,1)_!^m \arrow[d, "\eta^n"'] & & \mathrm{Holim}_{\lrcorner_l} \lrcorner_g^* \lrcorner_f^*(1,1)_!^n \\
\mathrm{Holim}_{\lrcorner_l} \lrcorner_g^*(1,1)_!^m(1,1)^{n,*}(1,1)_!^n \arrow[r, equal] & \mathrm{Holim}_{\lrcorner_l} \lrcorner_g^*(1,1)_!^m(1,1)^{m,*} \lrcorner_f^*(1,1)_!^n \arrow[ur, "\epsilon^m"'] &
\end{tikzcd}$$

□

We close this subsection with a proof of Lemma 4.14.

Proof. (of Lemma 4.14) By induction on n and by the functoriality of $P_\bullet X$, it is enough to check this for $n = 2$. Let J be the poset obtained from $\lrcorner_2$ by adding two new elements ω_0 and ω_1 such that $\omega_0 \leq e_0, e_1$ and $\omega_1 \leq e_1, e_2$. Moreover, let us denote the resulting inclusion by $j\colon\ \lrcorner_2 \longrightarrow J$. Under the obvious isomorphism $J \cong [1] \times \lrcorner$, we can consider the adjunction $(d^1 \times \mathrm{id}, s^0 \times \mathrm{id})\colon\ \lrcorner \rightharpoonup [1] \times \lrcorner$ as an adjunction $(L, R)\colon\ \lrcorner \rightharpoonup J$. Hence, we can consider the following diagram

$$\begin{tikzcd}
\mathbb{D}(e) \arrow[r, "(1,1)_!"] & \mathbb{D}(\lrcorner_2) \arrow[r, "j_*"] \arrow[dr] & \mathbb{D}(J) \arrow[r, "L^*"] \arrow[d] & \mathbb{D}(\lrcorner) \arrow[dl] \\
 & & \mathbb{D}(e) &
\end{tikzcd}$$

where the unlabeled arrows are the respective homotopy limit functors. Using Lemma 1.28, we obtain thus isomorphisms:

$$P_2X = \mathrm{Holim}_{\lrcorner_2}(1,1)_!X \cong \mathrm{Holim}_J\, j_*(1,1)_!X \cong \mathrm{Holim}_{\lrcorner}\, L^*j_*(1,1)_!X$$

To calculate this map more precisely, let us remark that we have

$$(L^*j_*(1,1)_!X)_{1,0} = j_*\big((1,1)_!X\big)_{\omega_1} \cong \mathrm{Holim}_{\lrcorner}(1,1)_!X = \Omega X$$

where the isomorphism is induced from a base change isomorphism. Similar calculations yield $(L^*j_*(1,1)_!X)_{0,1} \cong \Omega X$ and $(L^*j_*(1,1)_!X)_{1,1} \cong 0$. Hence the above map induces an isomorphism $P_2X \xrightarrow{\cong} \Omega X \oplus \Omega X$. From this one checks that the Segal map is an isomorphism which concludes the proof. $\square$

Part 2. Monoidal and enriched derivators

0. Introduction

In this paper, we develop the monoidal (cf. also to [Cis08]) and enriched aspects of the theory of derivators. As we saw in the companion paper [Gro10a], two important classes of derivators are given by the derivators associated to combinatorial model categories and the derivators represented by bicomplete categories. Both classes of examples can be refined to give corresponding statements about situations where the 'input is suitably monoidal, tensored, cotensored, or enriched'. We formalize the notions of monoidal, tensored, cotensored, and enriched (pre)derivators and make these statements precise. The author is not aware of a place in the literature where linear structures on a derivator are considered. It is for this purpose that we introduce the notion of an additive derivator and of the center of a derivator. This latter notion gives a compact definition of a derivator which is linear over some ring (and a graded variant thereof in the stable situation).

It is well-known that the homotopy categories of (combinatorial) monoidal model categories (in the sense of Hovey [Hov99], as opposed to the slightly different notion of [SS00]) can be canonically endowed with monoidal structures and similarly for suitably monoidal Quillen adjunctions. These statements are truncations of more structured results as we will see below. Once we define a monoidal derivator as a monoidal object in the Cartesian monoidal 2-category Der of derivators we will show that the derivator associated to a combinatorial monoidal model category can be canonically endowed with a monoidal structure. These results generalize to model categories which are suitably (co)tensored over a monoidal model category. As a consequence of the general theory, we show that the 2-categories of prederivators and derivators are Cartesian closed monoidal. As expected, we see that the internal hom $\mathsf{HOM}(\mathbb{D},\mathbb{D})$ coming from this Cartesian structure gives the universal example of a derivator acting on $\mathbb{D}$.

Since the passage from combinatorial model categories to derivators respects monoidal structures we obtain a very conceptual explanation for the existence of linear structures on certain naturally occurring derivators. In fact, the linear structures are obtained by specializing to a small part of the structure available on a derivator $\mathbb{D}$ which is left tensored over a monoidal derivator $\mathbb{E}$: under suitable additivity assumptions the left action restricts to an algebra structure on the center $\mathsf{Z}(\mathbb{D})$. We also have a corresponding result in the stable situation where one has to add some exactness assumptions and where the outcome is a graded-linear structure on the stable derivator. As special cases we obtain, e.g., that the derivator of spectra is linear over the stable homotopy groups of spheres and that the derivator of chain complexes is linear over the ground ring. It is easy to extend this in both cases to modules over commutative monoids. We also have such a result for modules over non-commutative monoids. In these cases the derivators are linear over the homotopy groups of the topological Hochschild cohomology of the ring spectrum resp. over the Hochschild cohomology of the differential-graded algebra. These examples were our original motivation for studying these questions.

Since we are restricting attention to combinatorial model categories, hence presentable categories, the special adjoint functor theorem (SAFT) of Freyd can in certain situations be applied to deduce that there is canonically more structure available. For example, let us assume that we have a combinatorial model category $\mathcal{M}$ which is left tensored over a combinatorial monoidal model category $\mathcal{N}$ such that the action preserves colimits separately in each variable. Then it is a consequence of SAFT that the category $\mathcal{M}$ is also cotensored and enriched over $\mathcal{N}$. To capture these additional structures at the level of derivators, we also introduce the notion of cotensored and enriched derivators and establish the relevant examples.

Along the development of the theory, we will see that the notions introduced below extend the corresponding ones from classical category theory. The 2-functor which sends a bicomplete category to the represented derivator is a faithful 2-functor and one should guarantee that the notions introduced here are compatible with the ones from classical category theory. For example, it is straightforward to see that a bicomplete category is additive if and only if the represented derivator is additive. There are similar observations for the other notions we discuss in this paper.

We now turn to a description of the content by sections. In Section 1, we begin by considering the Cartesian monoidal 2-categories of derivators and prederivators. We introduce the notion of bimorphisms between derivators and remark that the Cartesian product corepresents this bimorphism functor. We then consider the basic notions of monoidal (pre)derivators, monoidal morphisms, and monoidal transformations between such which are organized in the 2-categories $\mathsf{MonPDer}$ and MonDer. We define a monoidal prederivator by making explicit the notion of a monoidal object in the Cartesian 2-category of prederivators. One then remarks that monoidal prederivators can be identified with 2-functors $\mathsf{Cat}^{\mathrm{op}} \longrightarrow \mathsf{CAT}$ which factor over the 2-category MonCAT of monoidal categories. Since the 2-functor which sends a category to the associated represented prederivator preserves 2-products, we obtain an induced 2-functor $\mathsf{MonCAT} \longrightarrow \mathsf{MonPDer}$. We then show that derivators associated to combinatorial monoidal model categories can be canonically endowed with monoidal structures. This is done by showing, more generally, that a Brown functor between model categories (cf. Definition 1.15) induces a morphism of associated derivators. Some relevant examples related to simplicial sets, chain complexes, and symmetric spectra are given, before we turn, in the last subsection, to the center $\mathsf{Z}(\mathbb{D})$ of a derivator. This notion allows for a convenient formalization of linear structures on a derivator. We establish the result that suitably additive monoidal derivators are linear over the ring of self-maps of the monoidal unit of the underlying monoidal category.

In Section 2, we turn to prederivators tensored or cotensored over monoidal prederivators. We begin with a short technical subsection in which we construct the 2-Grothendieck fibration of tensored categories. In the next subsection, we introduce the notions of tensored and cotensored derivators as certain module objects. We show that the Cartesian monoidal 2-categories PDer resp. Der of prederivators resp. derivators are closed and that the internal hom $\mathsf{HOM}(\mathbb{D},\mathbb{D})$ together with the canonical action on $\mathbb{D}$ provides the universal example of a module structure on $\mathbb{D}$. The latter part is, in fact, a special case of a general 2-categorical statement which we prove as Theorem B.11. In the last subsection we give some interesting examples. We show that if a combinatorial model category $\mathcal{M}$ is tensored over a combinatorial monoidal model category $\mathcal{N}$, then the derivator $\mathbb{D}_{\mathcal{M}}$ associated to $\mathcal{M}$ is canonically tensored over $\mathbb{D}_{\mathcal{N}}$. The result of Section 1 on the linear structures on suitably additive monoidal derivators can be generalized to the situation of a suitably tensored additive derivator.

In Section 3, we introduce the notion of derivators enriched over a monoidal derivator. In order to have a compact definition of such a gadget we start by considering the 2-Grothendieck opfibration of enriched categories. Elaborating a bit on the fact that enriched category theory admits base change along monoidal functors we obtain the 2-category of enriched categories. In the next subsection, we use this to give a compact definition of an enriched derivator. Our main source of examples of enriched derivators is the following result (Theorem 3.10): an action of a monoidal derivator $\mathbb{E}$ on a derivator $\mathbb{D}$ which is part of an adjunction of two variables exhibits the derivator $\mathbb{D}$ as being canonically enriched over $\mathbb{E}$. This is, in particular, the case for closed monoidal derivators. In the last subsection, we show that if a combinatorial model category $\mathcal{M}$ is suitably tensored over a combinatorial monoidal model category $\mathcal{N}$ then the associated derivator $\mathbb{D}_{\mathcal{M}}$ is canonically enriched over $\mathbb{D}_{\mathcal{N}}$. We close by mentioning some derivators related to chain complexes and symmetric spectra as more specific examples of enriched derivators.

Finally, in the appendices we recall and establish some 2-categorical notions and results. In Appendix A, we quickly recall the classical Grothendieck construction associated to a category-valued functor. We then give a variant thereof in the 2-categorical setting. These constructions are used in Section 2 and Section 3. Appendix B has two subsections. In the first one, we recall the notions of monoidal objects and modules in monoidal 2-categories and construct the 2-category of all modules using the 2-categorical Grothendieck construction. In the second subsection, we show that in a closed monoidal 2-category the canonical actions of internal endomorphism objects give us the terminal module structures (in a bicategorical sense, cf. Theorem B.11). This allows us to put the results on the linear structures of Section 1 and Section 2 into perspective.

Before we begin with the proper content of this paper let us make two more comments. The first comment concerns set-theoretical issues. In what follows we will frequently consider the 'category of categories' and similar gadgets. Strictly speaking these are not honest categories in the sense that they would be locally small, i.e., have hom-sets as opposed to more general hom-classes. These problems could be circumvented by a use of Grothendiecks language of universes. Since we do not wish to add an additional technical layer to the exposition by keeping track of the different universes we decided to ignore these issues.

The second remark concerns the different kinds of 'hom-objects' which will show up frequently. Let $\mathcal{C}$ be a category and let $X,\, Y \in \mathcal{C}$ be two objects. The set of categorical morphisms from X to Y will be denoted by $\mathsf{hom}_{\mathcal{C}}(X,Y)$. If the category $\mathcal{C}$ is enriched over a monoidal category $\mathcal{D}$, we will usually write $\mathsf{Hom}_{\mathcal{C}}(X,Y) \in \mathcal{D}$ for the enriched hom-objects. Finally, in the case of a closed monoidal category $\mathcal{D}$ and two objects $X, Y \in \mathcal{D}$, the internal hom will be denoted by $\mathsf{HOM}_{\mathcal{D}}(X,Y) \in \mathcal{D}$. The author is aware of the fact that these three situations are of course not disjoint but we will apply these conventions as a rule of thumb.

1. Monoidal derivators

1.1. **The Cartesian monoidal 2-categories Der and PDer.** For the basic notions of the theory of 2-categories we refer to [Bor94a, ML98, Kel05b] which will be used more systematically here than in the companion paper [Gro10a]. In order to establish some notation we begin by quickly recalling the definitions of a prederivator and morphisms of prederivators. By contrast, the notion of a derivator will not be recalled and we refer instead to [Gro10a]. Original references for derivators are [Gro, Hel88]. Other references for the theory of derivators and stable derivators include [Fra96, Kel91, Mal07a, Mal01, CN08].

We recall that a *prederivator* is a 2-functor $\mathbb{D}\colon \mathsf{Cat}^{\mathrm{op}} \longrightarrow \mathsf{CAT}$ where Cat denotes the 2-category of small categories and CAT denotes the 2-category of (not necessarily small) categories. Spelling out this definition, we thus have for every small category J an associated category $\mathbb{D}(J)$, for a functor $u\colon J \longrightarrow K$ an induced functor $\mathbb{D}(u) = u^*\colon \mathbb{D}(K) \longrightarrow \mathbb{D}(J)$ and for a natural transformation $\alpha\colon u \longrightarrow v$ of two such functors a natural transformation $\mathbb{D}(\alpha) = \alpha^*\colon u^* \longrightarrow v^*$ as indicated in the following diagram:

$$\begin{tikzcd} J \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] \arrow[r, phantom, "\Downarrow\alpha"] & K, \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathbb{D}(K) \arrow[r, bend left, "u^*"] \arrow[r, bend right, "v^*"'] \arrow[r, phantom, "\Downarrow\alpha^*"] & \mathbb{D}(J). \end{tikzcd}$$

These associations are compatible with compositions and units in a strict sense, i.e., we have equalities of the respective expressions. One can of course also consider 2-functors which are only defined on certain 2-subcategories $\mathsf{Dia} \subseteq \mathsf{Cat}$ (for example finite categories, finite and finite-dimensional categories or posets) subject to certain closure properties (cf. Section 4 of [Gro10a]). This would then lead to the notion of a (pre)derivator *of type* Dia. For simplicity, we will stick to the case of all small categories but everything that we do in this paper can also be done for prederivators of type Dia.

A *morphism* $F\colon \mathbb{D} \longrightarrow \mathbb{D}'$ *of prederivators* is a pseudo-natural transformation of 2-functors. Thus, such a morphism consists of a family of functors $F_J\colon \mathbb{D}(J) \longrightarrow \mathbb{D}'(J)$ together with specified isomorphisms $\gamma_u^F\colon u^* \circ F_K \longrightarrow F_J \circ u^*$ for each functor $u\colon J \longrightarrow K$. These isomorphisms have to be suitably compatible with compositions and identities. More precisely, given a pair of composable functors $J \xrightarrow{u} K \xrightarrow{v} L$ and a natural transformation $\alpha\colon u_1 \longrightarrow u_2\colon J \longrightarrow K$, we then have the following relation resp. commutative diagrams:

$$\gamma_{\mathrm{id}_J} = \mathrm{id}_{F_J} \qquad \begin{tikzcd} u^*v^*F \arrow[r, "\gamma_v"] \arrow[dr, bend right, "\gamma_{vu}"'] & u^*Fv^* \arrow[d, "\gamma_u"] \\ & Fu^*v^* \end{tikzcd} \qquad \begin{tikzcd} u_1^*F \arrow[r, "\alpha^*"] \arrow[d, "\gamma_{u_1}"'] & u_2^*F \arrow[d, "\gamma_{u_2}"] \\ Fu_1^* \arrow[r, "\alpha^*"'] & Fu_2^* \end{tikzcd}$$

Here, we suppressed the indices of F and the upper indices of the natural transformation γ (as we will frequently do in the sequel) to avoid awkward notation. Moreover, we will not distinguish notationally between the natural transformations γ and their inverses. If all the components γ_u^F are identities then F will be called a *strict morphism.*

We will later introduce the notion of an adjunction of two variables between prederivators and in that context it will be important that we also have a lax version of morphisms. So, let us call a lax natural transformation $F\colon \mathbb{D} \longrightarrow \mathbb{D}'$ a *lax morphism of prederivators*. Thus, such a lax morphism consists of a similar datum as a morphism satisfying the same coherence conditions with the difference that the natural transformations $\gamma_u^F\colon u^*\circ F \longrightarrow F\circ u^*$ are not necessarily invertible. For simplicity we will also apply the same terminology for 'extranatural' variants thereof as in the context of adjunctions of two variables (cf. Lemma 1.11).

Finally, let $F,G\colon \mathbb{D} \longrightarrow \mathbb{D}'$ be two morphisms of prederivators. A *natural transformation* $\tau\colon F \longrightarrow G$ is a family of natural transformations $\tau_J\colon F_J \longrightarrow G_J$ which are compatible with the coherence isomorphisms belonging to the functors F and G. Thus, for every functor $u\colon J \longrightarrow K$ the following diagram commutes

$$\begin{tikzcd} u^*F \arrow[r, "\tau"] \arrow[d, "\gamma"'] & u^*G \arrow[d, "\gamma"] \\ Fu^* \arrow[r, "\tau"'] & Gu^*. \end{tikzcd}$$

One checks that a natural transformation is precisely the same as a *modification* of pseudo-natural transformations (see [Bor94a, Definition 7.5.3]). Given two parallel morphisms F and G of prederivators let us denote by $\mathsf{nat}(F,G)$ the natural transformations from F to G. Thus, with prederivators as objects, morphisms as 1-cells and natural transformations as 2-cells we obtain the 2-category PDer of prederivators. In fact, this is just a special case of the 2-category of 2-functors, pseudo-natural transformations and modifications. The full sub-2-category spanned by the derivators is denoted by Der. Given two (pre)derivators $\mathbb{D}$ and $\mathbb{D}'$ let us denote the category of morphisms by $\mathsf{Hom}(\mathbb{D},\mathbb{D}')$ while we will write $\mathsf{Hom}^{\mathsf{strict}}(\mathbb{D},\mathbb{D}')$ for the full subcategory spanned by the strict morphisms.

Example 1.1. The Yoneda embedding $y\colon \mathsf{CAT} \longrightarrow \mathsf{PDer}$ sends a category $\mathcal{C}$ to the represented prederivator $y(\mathcal{C})\colon J \longmapsto \mathsf{Fun}(J,\mathcal{C})$. Here, $\mathsf{Fun}(J,\mathcal{C})$ denotes the category of functors from J to $\mathcal{C}$. The 2-categorical Yoneda lemma implies that for an arbitrary prederivator $\mathbb{D}$ we have a natural isomorphism of categories

$$Y\colon\ \mathsf{Hom}^{\mathsf{strict}}_{\mathsf{PDer}}(y(J),\mathbb{D}) \xrightarrow{\cong} \mathbb{D}(J).$$

For simplicity, we will sometimes drop the embedding y from notation and again just write $\mathcal{C}$ for the prederivator represented by a category $\mathcal{C}$.

In every 2-category we have the notion of adjoint 1-morphisms, equivalences, and Kan extensions (see Sections 1 and 2 of [Str72]). Let us consider the first two notions in the 2-categories PDer and Der. So, let $L\colon \mathbb{D} \longrightarrow \mathbb{D}'$ and $R\colon \mathbb{D}' \longrightarrow \mathbb{D}$ be two morphisms of (pre)derivators and let $\eta\colon \mathrm{id}_{\mathbb{D}} \longrightarrow R\circ L$ and $\epsilon\colon L\circ R \longrightarrow \mathrm{id}_{\mathbb{D}'}$ be two natural transformations. Then one can check that the 4-tuple (L,R,η,ϵ) defines an adjunction $\mathbb{D} \rightharpoonup \mathbb{D}'$ resp. an equivalence $\mathbb{D} \xrightarrow{\sim} \mathbb{D}'$ if and only if for each category J we obtain an adjunction $\mathbb{D}(J) \rightharpoonup \mathbb{D}'(J)$ resp. an equivalence $\mathbb{D}(J) \xrightarrow{\sim} \mathbb{D}'(J)$ by evaluation. Given such an adjunction $(L,R)\colon \mathbb{D} \longrightarrow \mathbb{D}'$ the adjunctions at the different levels are compatible in the sense that for a functor $u\colon J \longrightarrow K$ we obtain the following commutative diagram:

$$
\begin{tikzcd}
\mathsf{hom}_{\mathbb{D}'(K)}(L_K X, Y) \arrow[r, "\cong"] \arrow[d, "u^*"'] & \mathsf{hom}_{\mathbb{D}(K)}(X, R_K Y) \arrow[d, "u^*"] \\
\mathsf{hom}_{\mathbb{D}'(J)}(u^* L_K X, u^* Y) \arrow[d, "\gamma^L"'] & \mathsf{hom}_{\mathbb{D}(J)}(u^* X, u^* R_K Y) \arrow[d, "\gamma^R"] \\
\mathsf{hom}_{\mathbb{D}'(J)}(L_J u^* X, u^* Y) \arrow[r, "\cong"'] & \mathsf{hom}_{\mathbb{D}(J)}(u^* X, R_J u^* Y)
\end{tikzcd}
$$

Here, the morphisms γ^L resp. γ^R are the natural transformations which belong to the morphisms L resp. R.

The fact that adjoint morphisms of derivators behave in the expected way with respect to homotopy Kan extensions is the content of the following lemma. Recall that given a functor $u\colon J \longrightarrow K$ and a derivator $\mathbb{D}$, one of the axioms of a derivator guarantees that the *restriction functor* $u^*\colon \mathbb{D}(K) \longrightarrow \mathbb{D}(J)$ has an adjoint on either side. We denote any left resp. right adjoint functor of u^* by $u_!\colon \mathbb{D}(J) \longrightarrow \mathbb{D}(K)$ resp. $u_*\colon \mathbb{D}(J) \longrightarrow \mathbb{D}(K)$ and call such a functor a *homotopy left* resp. *homotopy right Kan extension functor.*

Lemma 1.2. *Let $(L, R)\colon \mathbb{D} \rightharpoonup \mathbb{D}'$ be an adjunction of derivators. Then L preserves homotopy left Kan extensions and R preserves homotopy right Kan extensions.*

Proof. By duality it suffices to give the proof for homotopy left Kan extensions. Let $u\colon J \longrightarrow K$ be a functor between small categories and let u^* denote the induced functors in $\mathbb{D}$ and $\mathbb{D}'$. Similarly, let us denote both respective homotopy left Kan extension functors by $u_!$. For objects $X \in \mathbb{D}(J)$ and $Y \in \mathbb{D}'(K)$ we have the following chain of natural isomorphisms:

$$
\begin{aligned}
\mathsf{hom}_{\mathbb{D}'(K)}(u_! L_J(X), Y) &\cong \mathsf{hom}_{\mathbb{D}'(J)}(L_J(X), u^*(Y)) \\
&\cong \mathsf{hom}_{\mathbb{D}(J)}(X, R_J u^*(Y)) \\
&\cong \mathsf{hom}_{\mathbb{D}(J)}(X, u^* R_K(Y)) \\
&\cong \mathsf{hom}_{\mathbb{D}(K)}(u_!(X), R_K(Y)) \\
&\cong \mathsf{hom}_{\mathbb{D}'(K)}(L_K u_!(X), Y)
\end{aligned}
$$

By the Yoneda lemma, this natural isomorphism is induced by an isomorphism between the corepresenting objects. Taking $Y = L_K u_!(X)$ and tracing the map $\mathrm{id}\colon L_K u_!(X) \longrightarrow L_K u_!(X)$ through this chain of isomorphisms we obtain a natural isomorphism $\beta\colon u_! L_J \longrightarrow L_K u_!$. But this natural isomorphism is easily identified with a base change morphism occurring in the definition of a homotopy left Kan extension preserving morphism of derivators (cf. Section 3 of [Gro10a]). This concludes the proof. $\square$

Let us define the ('internal') product of two prederivators. Thus, let $\mathbb{D}$ and $\mathbb{D}'$ be prederivators, then their product $\mathbb{D} \times \mathbb{D}'$ is defined to be the composition of the 2-functors

$$\mathsf{Cat}^{\mathrm{op}} \xrightarrow{\Delta} \mathsf{Cat}^{\mathrm{op}} \times \mathsf{Cat}^{\mathrm{op}} \xrightarrow{\mathbb{D} \times \mathbb{D}'} \mathsf{CAT} \times \mathsf{CAT} \xrightarrow{\times} \mathsf{CAT}$$

where Δ denotes the diagonal. The product of morphisms of prederivators and natural transformations is defined similarly and this gives us the 2-product in the 2-category PDer of prederivators. Recall from [Gro10a] that we also have the notions of a pointed resp. stable derivator.

Lemma 1.3. *Let $\mathbb{D}$ and $\mathbb{D}'$ be derivators. Then $\mathbb{D} \times \mathbb{D}'$ is again a derivator. Moreover, if $\mathbb{D}$ and $\mathbb{D}'$ are in addition pointed, resp. stable then the product $\mathbb{D} \times \mathbb{D}'$ is also pointed, resp. stable.*

Proof. Since isomorphisms in product categories are detected pointwise and since a product of two functors is an adjoint functor resp. an equivalence if and only if this is the case for the two factors the axioms (Der1)-(Der3) are immediate. Also the base change axiom holds since the base change morphism in $\mathbb{D} \times \mathbb{D}'$ can be taken to be the product of the base change morphisms in $\mathbb{D}$ and $\mathbb{D}'$ which are isomorphisms by assumption. Thus, with $\mathbb{D}$ and $\mathbb{D}'$ also the product $\mathbb{D} \times \mathbb{D}'$ is a derivator. Similarly, since the product of pointed categories is again pointed we obtain the result for pointed derivators. For stable derivators, note that $\mathbb{D} \times \mathbb{D}'$ is strong since the product of two full resp. essentially surjective functors is again a full resp. essentially surjective functor. Finally, an object $X = (Y, Y') \in \mathbb{D}(\square) \times \mathbb{D}'(\square)$ is (co)Cartesian if and only if the components $Y \in \mathbb{D}(\square)$ and $Y' \in \mathbb{D}'(\square)$ are (co)Cartesian. Hence, if $\mathbb{D}$ and $\mathbb{D}'$ are stable, the product $\mathbb{D} \times \mathbb{D}'$ is also stable. $\square$

The product endows the 2-categories PDer and Der with the structure of a symmetric monoidal 2-category, called the Cartesian monoidal structure. The unit e of the monoidal structure is the prederivator with constant value the terminal category e (consisting of one object and its identity morphism only) and the symmetry constraint is given by the twist morphism $T\colon \mathbb{D} \times \mathbb{D}' \longrightarrow \mathbb{D}' \times \mathbb{D}$. To simplify notation we will suppress the canonical associativity isomorphisms and hence also brackets from notation. In the next subsection, we will introduce monoidal (pre)derivators as monoidal objects in the respective 2-categories.

Before we turn to monoidal derivators let us introduce *bimorphisms* between (pre)derivators. Since the product of two derivators is the 2-categorical product we understand morphisms into them. But also maps out of a product of two derivators are easy to describe: up to an equivalence of categories these are just the bimorphisms as we will define them now.

Definition 1.4. Let $\mathbb{D}$, $\mathbb{E}$, and $\mathbb{F}$ be prederivators. A *bimorphism B from* $(\mathbb{D}, \mathbb{E})$ *to* $\mathbb{F}$, denoted $B\colon (\mathbb{D}, \mathbb{E}) \longrightarrow \mathbb{F}$, consists of a family of functors

$$B_{J_1,J_2}\colon \mathbb{D}(J_1) \times \mathbb{E}(J_2) \longrightarrow \mathbb{F}(J_1 \times J_2), \qquad J_1,\ J_2 \in \mathsf{Cat},$$

and for each pair of functors $(u_1, u_2)\colon (J_1, J_2) \longrightarrow (K_1, K_2)$ a natural isomorphism $\gamma^B_{u_1,u_2}$ as indicated in:

$$\begin{tikzcd}
\mathbb{D}(K_1) \times \mathbb{E}(K_2) \arrow[r, "B_{K_1,K_2}"] \arrow[d, "u_1^* \times u_2^*"'] \arrow[dr, phantom, "\Leftarrow"] & \mathbb{F}(K_1 \times K_2) \arrow[d, "(u_1 \times u_2)^*"] \\
\mathbb{D}(J_1) \times \mathbb{E}(J_2) \arrow[r, "B_{J_1,J_2}"'] & \mathbb{F}(J_1 \times J_2)
\end{tikzcd}$$

These data have to satisfy the following coherence conditions. Given a pair of composable pairs $(u_1, u_2)\colon (J_1, J_2) \longrightarrow (K_1, K_2)$ and $(v_1, v_2)\colon (K_1, K_2) \longrightarrow (L_1, L_2)$ and a pair of natural transformations $(\alpha_1, \alpha_2)\colon (u_1, u_2) \longrightarrow (u_1', u_2')$ we have $\gamma_{\mathrm{id}_{J_1},\mathrm{id}_{J_2}} = \mathrm{id}_{B_{J_1,J_2}}$ and the commutativity of the following two diagrams:

$$\begin{tikzcd}
(u_1 \times u_2)^*(v_1 \times v_2)^*B \arrow[r, "\gamma"] \arrow[dr, bend right, "\gamma"'] & (u_1 \times u_2)^*B(v_1^* \times v_2^*) \arrow[d, "\gamma"] \\
& B(u_1^* \times u_2^*)(v_1^* \times v_2^*)
\end{tikzcd}
\qquad
\begin{tikzcd}
(u_1 \times u_2)^*B \arrow[r] \arrow[d, "\gamma"'] & (u_1' \times u_2')^*B \arrow[d, "\gamma"] \\
B(u_1^* \times u_2^*) \arrow[r] & B(u_1'^* \times u_2'^*)
\end{tikzcd}$$

Now, given two parallel bimorphism $B, B'\colon (\mathbb{D}, \mathbb{E}) \longrightarrow \mathbb{F}$, a natural transformation $\tau\colon B \longrightarrow B'$ of bimorphisms consists of natural transformations $\tau_{J_1,J_2}\colon B_{J_1,J_2} \longrightarrow B'_{J_1,J_2}$. These have to be compatible in the sense that given a pair of functors $(u_1, u_2)\colon (J_1, J_2) \longrightarrow (K_1, K_2)$ the following diagram commutes:

$$\begin{tikzcd}
(u_1 \times u_2)^*B \arrow[r, "\tau"] \arrow[d, "\gamma"'] & (u_1 \times u_2)^*B' \arrow[d, "\gamma"] \\
B(u_1^* \times u_2^*) \arrow[r, "\tau"'] & B'(u_1^* \times u_2^*)
\end{tikzcd}$$

Given three prederivators $\mathbb{D}$, $\mathbb{E}$, and $\mathbb{F}$ we obtain a category of bimorphism from $(\mathbb{D}, \mathbb{E})$ to $\mathbb{F}$ which we denote by $\mathsf{BiHom}((\mathbb{D}, \mathbb{E}), \mathbb{F})$. In fact, given three such prederivators we can consider the *exterior product* $\mathbb{D} \boxtimes \mathbb{E}$ *of* $\mathbb{D}$ and $\mathbb{E}$ and the 2-functor $\mathbb{F} \circ (- \times -)$ which are defined by

$$(\mathbb{D} \boxtimes \mathbb{E})(J_1, J_2) = \mathbb{D}(J_1) \times \mathbb{E}(J_2) \qquad \text{and} \qquad (\mathbb{F} \circ (- \times -))(J_1, J_2) = \mathbb{F}(J_1 \times J_2).$$

Then, we have an equality of categories

$$\mathsf{BiHom}((\mathbb{D}, \mathbb{E}), \mathbb{F}) = \mathsf{PsNat}(\mathbb{D} \boxtimes \mathbb{E}, \mathbb{F} \circ (- \times -))$$

where $\mathsf{PsNat}(-,-)$ denotes the category of pseudo-natural transformations and modifications. This observation shows that $\mathsf{BiHom}((-,-),-)$ is functorial in all three arguments. Let us now show that $\mathsf{BiHom}((-,-),-)$ is corepresentable. For two prederivators $\mathbb{D}$ and $\mathbb{E}$, the universal bimorphism $(\mathbb{D}, \mathbb{E}) \longrightarrow \mathbb{D} \times \mathbb{E}$ has components induced by the projections:

$$\mathbb{D}(J_1) \times \mathbb{E}(J_2) \xrightarrow{\mathrm{pr}_1^* \times \mathrm{pr}_2^*} \mathbb{D}(J_1 \times J_2) \times \mathbb{E}(J_1 \times J_2)$$

This bimorphism induces the right adjoint in the following proposition.

Proposition 1.5. *For prederivators $\mathbb{D},\mathbb{E}$, and $\mathbb{F}$ we have a natural equivalence of categories*

$$\mathsf{BiHom}((\mathbb{D},\mathbb{E}),\mathbb{F}) \xrightarrow{\simeq} \mathsf{Hom}(\mathbb{D}\times\mathbb{E},\mathbb{F}).$$

Proof. Let us begin by defining a natural functor $l\colon\ \mathsf{BiHom}((\mathbb{D},\mathbb{E}),\mathbb{F}) \longrightarrow \mathsf{Hom}(\mathbb{D}\times\mathbb{E},\mathbb{F})$ so let us consider a bimorphism $B\colon (\mathbb{D},\mathbb{E}) \longrightarrow \mathbb{F}$. The component $l(B)_J$ of $l(B)\colon\ \mathbb{D}\times\mathbb{E}\longrightarrow\mathbb{F}$ is defined by:

$$l(B)_J\colon\ \mathbb{D}(J)\times\mathbb{E}(J) \xrightarrow{B_{J,J}} \mathbb{F}(J\times J) \xrightarrow{\Delta_J^*} \mathbb{F}(J)$$

The structure morphism belonging to $u\colon J\longrightarrow K$ is defined by $\gamma_u^{l(B)} = \Delta_J^*\gamma_{u,u}^B$:

$$\begin{tikzcd}
\mathbb{D}(K)\times\mathbb{E}(K) \arrow[r, "B_{K,K}"] \arrow[d, "u^*\times u^*"'] \arrow[dr, phantom, "\Swarrow"] & \mathbb{F}(K\times K) \arrow[r, "\Delta_K^*"] \arrow[d, "(u\times u)^*"] & \mathbb{F}(K) \arrow[d, "u^*"] \\
\mathbb{D}(J)\times\mathbb{E}(J) \arrow[r, "B_{J,J}"'] & \mathbb{F}(J\times J) \arrow[r, "\Delta_J^*"'] & \mathbb{F}(J)
\end{tikzcd}$$

It is immediate that $l(B)$ is in fact a morphism of prederivators and one checks that this assignment can be completed to the definition of a functor l.

We now construct a functor $r\colon\ \mathsf{Hom}(\mathbb{D}\times\mathbb{E},\mathbb{F}) \longrightarrow \mathsf{BiHom}((\mathbb{D},\mathbb{E}),\mathbb{F})$ so let us consider a morphism $G\colon\ \mathbb{D}\times\mathbb{E}\longrightarrow\mathbb{F}$. The component $r(G)_{J_1,J_2}$ is defined to be the following composition:

$$r(G)_{J_1,J_2}\colon\ \mathbb{D}(J_1)\times\mathbb{E}(J_2) \xrightarrow{\mathrm{pr}_1^*\times\mathrm{pr}_2^*} \mathbb{D}(J_1\times J_2)\times\mathbb{E}(J_1\times J_2) \xrightarrow{G_{J_1\times J_2}} \mathbb{F}(J_1\times J_2)$$

For a pair of functors $(u_1,u_2)\colon (J_1,J_2)\longrightarrow(K_1,K_2)$ we set $\gamma_{(u_1,u_2)}^{r(G)} = \gamma_{u_1\times u_2}^G(\mathrm{pr}_1^*\times\mathrm{pr}_2^*)$:

$$\begin{tikzcd}
\mathbb{D}(K_1)\times\mathbb{E}(K_2) \arrow[r, "\mathrm{pr}_1^*\times\mathrm{pr}_2^*"] \arrow[d, "u_1^*\times u_2^*"'] & \mathbb{D}(K_1\times K_2)\times\mathbb{E}(K_1\times K_2) \arrow[r, "G_{K_1\times K_2}"] \arrow[d, "(u_1\times u_2)^*\times(u_1\times u_2)^*"'] \arrow[dr, phantom, "\Leftarrow"] & \mathbb{F}(K_1\times K_2) \arrow[d, "(u_1\times u_2)^*"] \\
\mathbb{D}(J_1)\times\mathbb{E}(J_2) \arrow[r, "\mathrm{pr}_1^*\times\mathrm{pr}_2^*"'] & \mathbb{D}(J_1\times J_2)\times\mathbb{E}(J_1\times J_2) \arrow[r, "G_{J_1\times J_2}"'] & \mathbb{F}(J_1\times J_2)
\end{tikzcd}$$

This completes the definition of a bimorphism $r(G)\colon (\mathbb{D},\mathbb{E})\longrightarrow\mathbb{F}$. One checks again that this assignment can be extended to a functor r as intended.

Let us next show that the composition $r\circ l$ is naturally isomorphic to the identity. For a bimorphism $B\colon (\mathbb{D},\mathbb{E})\longrightarrow\mathbb{F}$ the component of $(r\circ l)(B)$ at (J_1,J_2) is given by $\Delta^*\circ B\circ(\mathrm{pr}_1^*\times\mathrm{pr}_2^*)$ as depicted in:

$$\begin{tikzcd}
\mathbb{D}(J_1)\times\mathbb{E}(J_2) \arrow[r, "B"] \arrow[d, "\mathrm{pr}_1^*\times\mathrm{pr}_2^*"'] \arrow[dr, phantom, "\Swarrow"] & \mathbb{F}(J_1\times J_2) \arrow[d, "(\mathrm{pr}_1\times\mathrm{pr}_2)^*"] & \\
\mathbb{D}(J_1\times J_2)\times\mathbb{E}(J_1\times J_2) \arrow[r, "B"'] & \mathbb{F}(J_1\times J_2\times J_1\times J_2) \arrow[r, "\Delta_{J_1\times J_2}^*"'] & \mathbb{F}(J_1\times J_2)
\end{tikzcd}$$

Since $(\mathrm{pr}_1 \times \mathrm{pr}_2)\circ\Delta_{J_1\times J_2} = \mathrm{id}$ this diagram shows that we have an isomorphism from B_{J_1,J_2} to $(r\circ l)(B)_{J_1,J_2}$ given by $\tau_{J_1,J_2} = \Delta^*_{J_1\times J_2}\gamma^B_{\mathrm{pr}_1,\mathrm{pr}_2}$. Let us check that these assemble into an isomorphism of bimorphisms $\tau\colon B \longrightarrow rl(B)$. Thus, let us consider a pair of functors $(u_1,u_2)\colon (J_1,J_2) \longrightarrow (K_1,K_2)$ and let us check that the following diagram commutes:

$$\begin{tikzcd}
(u_1\times u_2)^* B_{K_1,K_2} \arrow[r, "\tau_{K_1,K_2}"] \arrow[d, "\gamma^B_{u_1,u_2}"'] & (u_1\times u_2)^* rl(B)_{K_1,K_2} \arrow[d, "\gamma^{rl(B)}_{u_1,u_2}"] \\
B_{J_1,J_2}(u_1^*\times u_2^*) \arrow[r, "\tau_{J_1,J_2}"'] & rl(B)_{J_1,J_2}(u_1^*\times u_2^*)
\end{tikzcd}$$

Unraveling definitions we see that this diagram can be rewritten in the following form where we omit the indices of B for simplicity:

$$\begin{tikzcd}
(u_1\times u_2)^*\Delta^*_{K_1\times K_2}(\mathrm{pr}_1\times\mathrm{pr}_2)^*B \arrow[r, "\gamma^B"] \arrow[d, "="'] & (u_1\times u_2)^*\Delta^*_{K_1\times K_2}B(\mathrm{pr}_1^*\times\mathrm{pr}_2^*) \arrow[d, "="] \\
\Delta^*_{J_1\times J_2}(u_1\times u_2\times u_1\times u_2)^*(\mathrm{pr}_1\times\mathrm{pr}_2)^*B \arrow[r, "\gamma^B"] \arrow[d, "="'] & \Delta^*_{J_1\times J_2}(u_1\times u_2\times u_1\times u_2)^*B(\mathrm{pr}_1^*\times\mathrm{pr}_2^*) \arrow[d, "\gamma^B"] \\
\Delta^*_{J_1\times J_2}(\mathrm{pr}_1\times\mathrm{pr}_2)^*(u_1\times u_2)^*B \arrow[d, "\gamma^B"'] & \Delta^*_{J_1\times J_2}B((u_1\times u_2)^*\times(u_1\times u_2)^*)(\mathrm{pr}_1^*\times\mathrm{pr}_2^*) \arrow[d, "="] \\
\Delta^*_{J_1\times J_2}(\mathrm{pr}_1\times\mathrm{pr}_2)^*B(u_1^*\times u_2^*) \arrow[r, "\gamma^B"'] & \Delta^*_{J_1\times J_2}B(\mathrm{pr}_1^*\times\mathrm{pr}_2^*)(u_1^*\times u_2^*)
\end{tikzcd}$$

But by the coherence property of the bimorphism B we deduce that this diagram commutes and thus that we have constructed an isomorphism of bimorphisms $\tau\colon B \longrightarrow rl(B)$.

Finally, let us construct a natural isomorphism $l\circ r \longrightarrow \mathrm{id}$. So, let G be a morphism $\mathbb{D}\times\mathbb{E} \longrightarrow \mathbb{F}$. The component $(l\circ r)(G)_J$ is given by $\Delta^*\circ G\circ(\mathrm{pr}_1^*\times\mathrm{pr}_2^*)$ as in:

$$\begin{tikzcd}
\mathbb{D}(J)\times\mathbb{E}(J) \arrow[r, "\mathrm{pr}_1^*\times\mathrm{pr}_2^*"] & \mathbb{D}(J\times J)\times\mathbb{E}(J\times J) \arrow[r, "G"] \arrow[d, "\Delta_J^*\times\Delta_J^*"'] \arrow[dr, phantom, "\Leftarrow"] & \mathbb{F}(J\times J) \arrow[d, "\Delta_J^*"] \\
 & \mathbb{D}(J)\times\mathbb{E}(J) \arrow[r, "G"'] & \mathbb{F}(J)
\end{tikzcd}$$

By the equality $(\Delta_J^*\times\Delta_J^*)\circ(\mathrm{pr}_1^*\times\mathrm{pr}_2^*) = \mathrm{id}$ it follows that this diagram gives us an isomorphism $\sigma_J = \gamma^G_{\Delta_J}(\mathrm{pr}_1^*\times\mathrm{pr}_2^*)\colon lr(G)_J \longrightarrow G_J$. Let us check that these isomorphisms assemble into a natural isomorphism $lr(G) \longrightarrow G$. Thus, let us consider a functor $u\colon J \longrightarrow$

K. Unraveling definitions we have to show this time that the following square commutes:

$$
\begin{tikzcd}
u^*\Delta_K^* G_{K\times K}(\mathrm{pr}_1^*\times\mathrm{pr}_2^*) \arrow[r, "\gamma^G"] \arrow[d, "="'] & u^* G_K(\Delta_K^*\times\Delta_K^*)(\mathrm{pr}_1^*\times\mathrm{pr}_2^*) \arrow[d, "\gamma^G"] \\
\Delta_J^*(u\times u)^* G_{K\times K}(\mathrm{pr}_1^*\times\mathrm{pr}_2^*) \arrow[d, "\gamma^G"'] & G_J(u^*\times u^*)(\Delta_K^*\times\Delta_K^*)(\mathrm{pr}_1^*\times\mathrm{pr}_2^*) \arrow[d, "="] \\
\Delta_J^* G_{J\times J}((u\times u)^*\times(u\times u)^*)(\mathrm{pr}_1^*\times\mathrm{pr}_2^*) \arrow[r, "\gamma^G"'] & G_J(\Delta_J^*\times\Delta_J^*)((u\times u)^*\times(u\times u)^*)(\mathrm{pr}_1^*\times\mathrm{pr}_2^*)
\end{tikzcd}
$$

But this diagram is commutative by the coherence conditions of the 2-cells belonging to a morphism of prederivators. Thus σ is an isomorphism and we can now conclude that the functor $\mathsf{BiHom}((\mathbb{D},\mathbb{E}),-)$ is corepresentable by $\mathbb{D}\times\mathbb{E}$. □

The proof shows that these natural equivalences give us natural isomorphisms of categories if we restrict to strict bimorphisms (in the sense that all 2-cells are identities) on the left-hand-side and to strict morphisms on the right-hand-side.

1.2. **Monoidal prederivators, monoidal morphisms, and monoidal transformations.** Emphasizing similarity to the fact that a monoidal category ([EK66] or [ML98]) is just a monoidal object (called a pseudo-monoid in [DS97]) in the Cartesian 2-category CAT, we could just say that a monoidal prederivator is a monoidal object in the Cartesian 2-category PDer (cf. Appendix B). We prefer to make this more explicit:

Definition 1.6. A *monoidal structure on a prederivator* $\mathbb{D}$ is a 5-tuple $(\otimes,\mathbb{S},a,l,r)$ consisting of two morphisms of prederivators

$$\otimes\colon\ \mathbb{D}\times\mathbb{D}\longrightarrow\mathbb{D}\qquad\text{and}\qquad\mathbb{S}\colon e\longrightarrow\mathbb{D}$$

and natural isomorphisms l, a, and r as indicated in the diagrams:

$$
\begin{tikzcd}
e\times\mathbb{D} \arrow[r, "\mathbb{S}\times\mathrm{id}"] \arrow[dr, bend right, "\cong"'] \arrow[dr, phantom, "\Nearrow"] & \mathbb{D}\times\mathbb{D} \arrow[d, "\otimes"] \\
& \mathbb{D}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{D}\times\mathbb{D}\times\mathbb{D} \arrow[r, "\mathrm{id}\times\otimes"] \arrow[d, "\otimes\times\mathrm{id}"'] \arrow[dr, phantom, "\Longleftarrow"] & \mathbb{D}\times\mathbb{D} \arrow[d, "\otimes"] \\
\mathbb{D}\times\mathbb{D} \arrow[r, "\otimes"'] & \mathbb{D},
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{D}\times\mathbb{D} \arrow[d, "\otimes"'] & \mathbb{D}\times e \arrow[l, "\mathrm{id}\times\mathbb{S}"'] \arrow[dl, bend left, "\cong"] \arrow[dl, phantom, "\Nwarrow"] \\
\mathbb{D} &
\end{tikzcd}
$$

This structure has to satisfy the usual coherence conditions as in Appendix B. A *symmetric monoidal structure on* $\mathbb{D}$ is a 6-tuple $(\otimes,\mathbb{S},a,l,r,t)$ where $(\otimes,\mathbb{S},a,l,r)$ is a monoidal structure and t is a natural isomorphism as in

$$
\begin{tikzcd}
\mathbb{D}\times\mathbb{D} \arrow[r, "T"] \arrow[dr, bend right, "\otimes"'] \arrow[dr, phantom, "\Swarrow"] & \mathbb{D}\times\mathbb{D} \arrow[d, "\otimes"] \\
& \mathbb{D}
\end{tikzcd}
$$

satisfying additional coherence conditions as specified in [Bor94b]. A *monoidal* resp. *symmetric monoidal prederivator* is a prederivator endowed with a monoidal resp. symmetric monoidal structure.

We will often denote a monoidal prederivator simply by $(\mathbb{D},\otimes,\mathbb{S})$ or even by $\mathbb{D}$. Moreover, we apply the same terminology for derivators, i.e., a derivator is monoidal if and only if the underlying prederivator is monoidal. The prederivator e can also be considered as the prederivator represented by the terminal category e. So, the 2-categorical Yoneda lemma provides a natural isomorphism of categories

$$Y\colon\ \mathsf{Hom}^{\mathsf{strict}}_{\mathsf{PDer}}(e,\mathbb{D}) \xrightarrow{\cong} \mathbb{D}(e).$$

The left-hand-side denotes the full subcategory of $\mathsf{Hom}_{\mathsf{PDer}}(e,\mathbb{D})$ spanned by the strict morphisms of derivators, i.e., those morphisms for which the coherence isomorphisms γ are identities. Thus, in particular, a strict morphism $e\longrightarrow\mathbb{D}$ amounts to the choice of an object in $\mathbb{D}(e)$. A not necessarily strict morphism $e\longrightarrow\mathbb{D}$ contains more information but see Lemma 1.46 (this reflects the fact that we should work with the *bicategorical* Yoneda lemma as opposed to the 2-*categorical* one since we are working with pseudo-natural transformations instead of the more restrictive 2-natural transformations).

Let $\mathbb{D}$ be a (symmetric) monoidal prederivator and let J be a category. Then, by definition, we have a functor $\otimes\colon\ \mathbb{D}(J)\times\mathbb{D}(J)\longrightarrow\mathbb{D}(J)$, an object $\mathbb{S}(J)\in\mathbb{D}(J)$, and also natural transformations which endow $\mathbb{D}(J)$ with the structure of a (symmetric) monoidal category. Moreover, for a functor $u\colon J\longrightarrow K$ we have an induced natural isomorphism $\gamma^{\otimes}$ as indicated in:

$$\begin{tikzcd}
\mathbb{D}(K)\times\mathbb{D}(K) \arrow[r, "\otimes"] \arrow[d, "u^*\times u^*"'] \arrow[dr, phantom, "\Nearrow"] & \mathbb{D}(K) \arrow[d, "u^*"] \\
\mathbb{D}(J)\times\mathbb{D}(J) \arrow[r, "\otimes"'] & \mathbb{D}(J)
\end{tikzcd}$$

Similarly, since $\mathbb{S}\colon e\longrightarrow\mathbb{D}$ is a morphism of derivators we have a canonical natural isomorphism $\gamma^{\mathbb{S}}$ as in the following diagram:

$$\begin{tikzcd}
e \arrow[r, "\mathbb{S}(K)"] \arrow[dr, bend right, "\mathbb{S}(J)"'] \arrow[dr, phantom, "\Nearrow"] & \mathbb{D}(K) \arrow[d, "u^*"] \\
 & \mathbb{D}(J)
\end{tikzcd}$$

It is easy to check that these two natural isomorphisms endow $u^*\colon\ \mathbb{D}(K)\longrightarrow\mathbb{D}(J)$ with the structure of a strong (symmetric) monoidal functor. For example, the definition of a natural transformation between morphisms of prederivators implies that the following diagram commutes:

$$\begin{tikzcd}
(\otimes\circ(\otimes\times\mathrm{id}))\circ u^* \arrow[r, "a"] \arrow[d, "\gamma"'] & (\otimes\circ(\mathrm{id}\times\otimes))\circ u^* \arrow[d, "\gamma"] \\
u^*\circ(\otimes\circ(\otimes\times\mathrm{id})) \arrow[r, "a"'] & u^*\circ(\otimes\circ(\mathrm{id}\times\otimes))
\end{tikzcd}$$

Evaluating this at three objects X, Y, and $Z\in\mathbb{D}(K)$ gives us precisely the first coherence condition as imposed on a strong (symmetric) monoidal structure on a functor:

$$\begin{tikzcd}
(u^*X\otimes u^*Y)\otimes u^*Z \arrow[r, "a"] \arrow[d, "\gamma"'] & u^*X\otimes(u^*Y\otimes u^*Z) \arrow[d, "\gamma"] \\
u^*(X\otimes Y)\otimes u^*Z \arrow[d, "\gamma"'] & u^*X\otimes u^*(Y\otimes Z) \arrow[d, "\gamma"] \\
u^*((X\otimes Y)\otimes Z) \arrow[r, "a"'] & u^*(X\otimes(Y\otimes Z))
\end{tikzcd}$$

The other coherence axioms are checked similarly. Moreover, there is a corresponding result for natural transformations. Let $\alpha\colon u\longrightarrow v$ be a natural transformation of functors $J\longrightarrow K$. Then it follows immediately that $\alpha^*\colon u^*\longrightarrow v^*$ is a monoidal transformation with respect to the canonical monoidal structures. For example the fact that $\mathbb{S}\colon e\longrightarrow\mathbb{D}$ is a morphism of prederivators encodes that α^* is compatible with the unitality constraints of u^* and v^*. In fact, the commutative square on the left reduces to the triangle on the right:

$$\begin{tikzcd}
\mathbb{S}_J\,u^* \arrow[r, "\gamma"] \arrow[d, "\alpha^*"'] & u^*\,\mathbb{S}_K \arrow[d, "\alpha^*"] \\
\mathbb{S}_J\,v^* \arrow[r, "\gamma"'] & v^*\,\mathbb{S}_K
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathbb{S}(J) \arrow[r] \arrow[dr, bend right] & u^*(\mathbb{S}(K)) \arrow[d, "\alpha^*"] \\
 & v^*(\mathbb{S}(K))
\end{tikzcd}$$

Thus, a monoidal prederivator resp. a symmetric monoidal prederivator $\mathbb{D}$ factors canonically as

$$\mathbb{D}\colon\ \mathsf{Cat}^{\mathrm{op}}\longrightarrow\mathsf{MonCAT}\longrightarrow\mathsf{CAT}\qquad\text{resp.}\qquad\mathbb{D}\colon\ \mathsf{Cat}^{\mathrm{op}}\longrightarrow\mathsf{sMonCAT}\longrightarrow\mathsf{CAT}\,.$$

Here, MonCAT resp. $\mathsf{sMonCAT}$ denotes the 2-category of monoidal resp. symmetric monoidal categories with strong (symmetric) monoidal functors and monoidal transformations. Note that the dual $\mathbb{D}^{\mathrm{op}}$ of a monoidal prederivator $\mathbb{D}$ is also canonically endowed with a monoidal structure. Before we turn to some interesting examples, let us quickly give the adapted classes of morphisms and natural transformations. Again, the same terminology will also apply for derivators.

Definition 1.7. Let $\mathbb{D}$ and $\mathbb{D}'$ be monoidal prederivators. A *monoidal structure on a morphism* $F\colon \mathbb{D} \longrightarrow \mathbb{D}'$ *of prederivators* is a pair of natural transformations

$$\begin{tikzcd} \mathbb{D}\times\mathbb{D} \arrow[r, "\otimes"] \arrow[d, "F\times F"'] \arrow[dr, phantom, "\nearrow"] & \mathbb{D} \arrow[d, "F"] \\ \mathbb{D}'\times\mathbb{D}' \arrow[r, "\otimes"'] & \mathbb{D}' \end{tikzcd} \qquad \begin{tikzcd} e \arrow[r, "\mathbb{S}"] \arrow[dr, bend right, "\mathbb{S}"'] \arrow[dr, phantom, "\nearrow"] & \mathbb{D} \arrow[d, "F"] \\ & \mathbb{D}' \end{tikzcd}$$

such that the coherence diagrams of Appendix B are satisfied. A monoidal structure is called *strong* if these natural transformations are isomorphisms. A *(strong) monoidal morphism* $F\colon \mathbb{D} \longrightarrow \mathbb{D}'$ between monoidal prederivators is a morphism endowed with a (strong) monoidal structure.

There is an obvious variant for the case of symmetric monoidal prederivators [Bor94b] which demands for an additional coherence property but which again will not be made precise. For completeness we include the definition of a monoidal natural transformation.

Definition 1.8. Let $\mathbb{D}$ and $\mathbb{D}'$ be monoidal prederivators and let $F,\ G\colon \mathbb{D} \longrightarrow \mathbb{D}'$ be monoidal morphisms. A natural transformation $\phi\colon F \longrightarrow G$ is called *monoidal* if the following two diagrams commute:

$$\begin{tikzcd} \otimes\circ(F\times F) \arrow[r] \arrow[d, "\phi\times\phi"'] & F\circ\otimes \arrow[d, "\phi"] \\ \otimes\circ(G\times G) \arrow[r] & G\circ\otimes \end{tikzcd} \qquad \begin{tikzcd} \mathbb{S} \arrow[r] \arrow[dr, bend right] & F\circ\mathbb{S} \arrow[d, "\phi"] \\ & G\circ\mathbb{S} \end{tikzcd}$$

As in classical category theory, there is no additional assumption on a monoidal transformation of symmetric monoidal functors. Thus, with these notions we have the 2-categories of (symmetric) monoidal (pre)derivators together with the strong monoidal morphisms and monoidal transformations, which are denoted by:

$$\mathsf{MonPDer},\qquad \mathsf{sMonPDer},\qquad \mathsf{MonDer}\qquad \text{resp.}\qquad \mathsf{sMonDer}$$

For a summary, let us use the following notation of Appendix B: Given a monoidal 2-category $\mathcal{C}$, let us denote by $\mathsf{Mon}(\mathcal{C})$ the 2-category of monoidal objects in $\mathcal{C}$. For the case of the Cartesian monoidal 2-category CAT we have $\mathsf{Mon}(\mathsf{CAT}) = \mathsf{MonCAT}$, the 2-category of monoidal categories. Thus, we may summarize our discussion as the following isomorphism of 2-categories:

$$\mathsf{MonPDer} = \mathsf{Mon}(\mathsf{CAT}^{\mathsf{Cat}^{\mathrm{op}}}, \times, e) \cong \mathsf{Mon}(\mathsf{CAT}, \times, e)^{\mathsf{Cat}^{\mathrm{op}}} = (\mathsf{MonCAT})^{\mathsf{Cat}^{\mathrm{op}}}$$

There is an analogous variant for symmetric monoidal prederivators.

Using the equivalence of categories $\mathsf{BiHom}((\mathbb{D},\mathbb{D}),\mathbb{D}) \simeq \mathsf{Hom}(\mathbb{D}\times\mathbb{D},\mathbb{D})$ of Proposition 1.5 a monoidal structure on a prederivator $\mathbb{D}$ induces a bimorphism $\otimes\colon (\mathbb{D},\mathbb{D}) \longrightarrow \mathbb{D}$. This bimorphism is then also coherently associative and unital, which gives us the associated exterior version of the monoidal structure.

Similarly to the theory of ordinary derivators, also in the monoidal context there are the two important classes of examples coming from categories and model categories. Let us recall that given a small category J and a category $\mathcal{C}$ we denote the associated functor category by $\mathsf{Fun}(J,\mathcal{C})$.

Example 1.9. The 2-functor $y\colon \mathsf{CAT} \longrightarrow \mathsf{PDer}$ sending a category $\mathcal{C}$ to the represented prederivator $\mathcal{C}$ defined by $\mathcal{C}\colon J \longmapsto \mathsf{Fun}(J,\mathcal{C})$ preserves 2-products and hence monoidal objects. Thus, we obtain induced 2-functors

$$y\colon \mathsf{MonCAT} \longrightarrow \mathsf{MonPDer} \qquad \text{and} \qquad y\colon \mathsf{sMonCAT} \longrightarrow \mathsf{sMonPDer}\,.$$

The monoidal structure on the prederivator represented by a monoidal category $\mathcal{C}$ sends two objects $X,Y \in \mathsf{Fun}(J,\mathcal{C})$ to the composition

$$J \xrightarrow{\Delta} J \times J \xrightarrow{X\times Y} \mathcal{C}\times\mathcal{C} \xrightarrow{\otimes} \mathcal{C},$$

where Δ is the diagonal functor. The monoidal unit is given by

$$J \longrightarrow e \xrightarrow{\mathbb{S}} \mathcal{M}.$$

If the monoidal category $\mathcal{C}$ is bicomplete, then we obtain a corresponding monoidal derivator. There is a similar result for symmetric monoidal categories.

The second class of examples of monoidal derivators comes from combinatorial monoidal model categories and will be treated in Subsection 1.4. But let us first extend the last example to include results about biclosed monoidal categories. This will be done in the next subsection.

1.3. **Adjunctions of two variables and closed monoidal derivators.** Our next aim is to introduce the notion of an *adjunction of two variables between prederivators* (which will, in particular, allow us to talk about *closed* monoidal derivators). Similarly to the theory of monoidal structures this can be given in two equivalent ways: there is an exterior version using bimorphisms and an interior version using morphisms of two variables. Let us give the details for the exterior version.

We begin by recalling the following. Let $\mathcal{D}$, $\mathcal{E}$, and $\mathcal{F}$ be categories and let us agree that we call a bifunctor $\otimes\colon \mathcal{D}\times\mathcal{E} \longrightarrow \mathcal{F}$ a left adjoint of two variables if there are functors Hom_l and Hom_r and natural isomorphisms as in:

$$\mathsf{hom}_{\mathcal{F}}(X\otimes Y, Z) \quad\cong\quad \mathsf{hom}_{\mathcal{E}}(Y, \mathsf{Hom}_l(X,Z)) \quad\cong\quad \mathsf{hom}_{\mathcal{D}}(X, \mathsf{Hom}_r(Y,Z))$$

Let now $\mathbb{D}$, $\mathbb{E}$, and $\mathbb{F}$ be prederivators and let $\otimes\colon (\mathbb{D},\mathbb{E}) \longrightarrow \mathbb{F}$ be a bimorphism of prederivators. The minimum we expect from the notion of an adjunction of two variables is the following. For two categories J_1 and J_2 we would like to obtain an adjunction of two variables by evaluation. Thus, for $X \in \mathbb{D}(J_1)$, $Y \in \mathbb{E}(J_2)$, and $Z \in \mathbb{F}(J_1\times J_2)$ we would expect to have natural isomorphisms

$$\mathsf{hom}_{\mathbb{F}(J_1\times J_2)}(X\otimes Y, Z) \quad\cong\quad \mathsf{hom}_{\mathbb{E}(J_2)}(Y, \mathsf{Hom}_l(X,Z)) \quad\cong\quad \mathsf{hom}_{\mathbb{D}(J_1)}(X, \mathsf{Hom}_r(Y,Z)).$$

Here, $\mathsf{Hom}_l(-,-)$ is a functor $\mathsf{Hom}_l(-,-)\colon \mathbb{D}(J_1)^{\mathrm{op}}\times\mathbb{F}(J_1\times J_2) \longrightarrow \mathbb{E}(J_2)$ and similarly for Hom_r . Moreover, these natural isomorphisms should be compatible with restriction of diagrams in the following sense. Let us focus on $\mathsf{Hom}_l(-,-)$ but similar reasonings apply to $\mathsf{Hom}_r(-,-)$. Thus, for a pair of functors $(u_1,u_2)\colon (J_1,J_2)\longrightarrow (K_1,K_2)$ we would like to have a commutative diagram of the following form:

$$\begin{tikzcd}
\mathsf{hom}_{\mathbb{F}(K_1\times K_2)}(X\otimes Y,Z) \arrow[r, "\cong"] \arrow[d] & \mathsf{hom}_{\mathbb{E}(K_2)}(Y,\mathsf{Hom}_l(X,Z)) \arrow[d] \\
\mathsf{hom}_{\mathbb{F}(J_1\times J_2)}(u_1^*X\otimes u_2^*Y,(u_1\times u_2)^*Z) \arrow[r, "\cong"] & \mathsf{hom}_{\mathbb{E}(J_2)}(u_2^*Y,\mathsf{Hom}_l(u_1^*X,(u_1\times u_2)^*Z))
\end{tikzcd}$$

Here, the left vertical morphism is obtained by an application of the restriction of diagram functor $(u_1\times u_2)^*\colon\ \mathbb{F}(K_1\times K_1)\longrightarrow\mathbb{F}(J_1\times J_2)$ followed by a map which is induced by the structure *isomorphism* of the bimorphism $\otimes:(\mathbb{D},\mathbb{E})\longrightarrow\mathbb{F}$:

$$\gamma^{\otimes}_{u_1,u_2}\colon u_1^*X\otimes u_2^*Y\longrightarrow (u_1\times u_2)^*(X\otimes Y)$$

Now, if we want to construct the vertical map on the right-hand-side we would certainly start by applying $u_2^*\colon\ \mathbb{E}(K_2)\longrightarrow\mathbb{E}(J_2)$. But then we are in the situation that we need a map

$$\mathsf{hom}_{\mathbb{E}(J_2)}(u_2^*Y,u_2^*\,\mathsf{Hom}_l(X,Z))\longrightarrow\mathsf{hom}_{\mathbb{E}(J_2)}(u_2^*Y,\mathsf{Hom}_l(u_1^*X,(u_1\times u_2)^*Z))$$

which would most naturally be induced by a morphism $u_2^*\,\mathsf{Hom}_l(X,Z)\longrightarrow\mathsf{Hom}_l(u_1^*X,(u_1\times u_2)^*Z)$. Let us check that such a map can be canonically constructed from the structure morphisms belonging to the bimorphism $\otimes$. By adjointness, a map

$$\gamma^{\mathsf{Hom}_l}_{u_1,u_2}\colon u_2^*\,\mathsf{Hom}_l(X,Z)\longrightarrow\mathsf{Hom}_l(u_1^*X,(u_1\times u_2)^*Z)$$

is equivalently given by a map $u_1^*X\otimes u_2^*\,\mathsf{Hom}_l(X,Z)\longrightarrow(u_1\times u_2)^*Z$. Using the adjunction counit $\epsilon^K\colon X\otimes\mathsf{Hom}_l(X,Z)\longrightarrow Z$ we can consider the map:

$$u_1^*X\otimes u_2^*\,\mathsf{Hom}_l(X,Z)\xrightarrow{\gamma^{\otimes}}(u_1\times u_2)^*(X\otimes\mathsf{Hom}_l(X,Z))\xrightarrow{\epsilon^K}(u_1\times u_2)^*Z$$

The adjoint of this map is taken as the definition of $\gamma^{\mathsf{Hom}_l}_{u_1,u_2}$, i.e., we set:

$$\begin{tikzcd}
u_2^*\,\mathsf{Hom}_l(X,Z) \arrow[r, "\eta^J"] \arrow[d, "\gamma^{\mathsf{Hom}_l}_{u_1,u_2}"'] & \mathsf{Hom}_l(u_1^*X,u_1^*X\otimes u_2^*\,\mathsf{Hom}_l(X,Z)) \arrow[d, "\gamma^{\otimes}_{u_1,u_2}"] \\
\mathsf{Hom}_l(u_1^*X,(u_1\times u_2)^*Z) & \mathsf{Hom}_l\left(u_1^*X,(u_1\times u_2)^*(X\otimes\mathsf{Hom}_l(X,Z))\right) \arrow[l, "\epsilon^K"]
\end{tikzcd}$$

We now claim that this map can be used to show that the adjunctions at the different levels are compatible. Thus, we have to show that the following diagram is commutative:

$$\begin{tikzcd}
\mathsf{hom}_{\mathbb{F}(K_1\times K_2)}(X\otimes Y, Z) \arrow[r, "\simeq"] \arrow[d, "(u_1\times u_2)^*"'] & \mathsf{hom}_{\mathbb{E}(K_2)}(Y, \mathsf{Hom}_l(X,Z)) \arrow[d, "u_2^*"] \\
\mathsf{hom}_{\mathbb{F}(J_1\times J_2)}((u_1\times u_2)^*(X\otimes Y), (u_1\times u_2)^*Z) \arrow[d, "\gamma^{\otimes}"'] & \mathsf{hom}_{\mathbb{E}(J_2)}(u_2^*Y, u_2^*\,\mathsf{Hom}_l(X,Z)) \arrow[d, "\gamma^{\mathsf{Hom}_l}"] \\
\mathsf{hom}_{\mathbb{F}(J_1\times J_2)}(u_1^*X\otimes u_2^*Y, (u_1\times u_2)^*Z) \arrow[r, "\simeq"'] & \mathsf{hom}_{\mathbb{E}(J_2)}(u_2^*Y, \mathsf{Hom}_l(u_1^*X, (u_1\times u_2)^*Z))
\end{tikzcd}$$

For a map $f\colon X\otimes Y\longrightarrow Z$ let us denote by $\phi_1(f)$ resp. $\phi_2(f)$ the image of f under the path passing through the upper right resp. lower left corner. By definition $\phi_1(f)$ is the map $\epsilon\gamma^{\otimes}\eta f$ as depicted in the next diagram precomposed by $\eta^K\colon u_2^*Y\longrightarrow u_2^*\,\mathsf{Hom}_l(X,X\otimes Y)$:

$$\begin{tikzcd}
u_2^*\,\mathsf{Hom}_l(X,X\otimes Y) \arrow[r, "f"] \arrow[d, "\eta^J"'] & u_2^*\,\mathsf{Hom}_l(X,Z) \arrow[d, "\eta^J"] \\
\mathsf{Hom}_l(u_1^*X, u_1^*X\otimes u_2^*\,\mathsf{Hom}_l(X,X\otimes Y)) \arrow[r, "f"] \arrow[d, "\gamma^{\otimes}"'] & \mathsf{Hom}_l(u_1^*X, u_1^*X\otimes u_2^*\,\mathsf{Hom}_l(X,Z)) \arrow[d, "\gamma^{\otimes}"] \\
\mathsf{Hom}_l\left(u_1^*X, (u_1\times u_2)^*(X\otimes\mathsf{Hom}_l(X,X\otimes Y))\right) \arrow[r, "f"] \arrow[d, "\epsilon^K"'] & \mathsf{Hom}_l\left(u_1^*X, (u_1\times u_2)^*(X\otimes\mathsf{Hom}_l(X,Z))\right) \arrow[d, "\epsilon^K"] \\
\mathsf{Hom}_l(u_1^*X, (u_1\times u_2)^*(X\otimes Y)) \arrow[r, "f"'] & \mathsf{Hom}_l(u_1^*X, (u_1\times u_2)^*Z)
\end{tikzcd}$$

Since the above diagram is commutative we can calculate

$$\phi_1(f) \quad = \quad f\epsilon^K\gamma^{\otimes}\eta^J\eta^K \quad = \quad f\epsilon^K\eta^K\gamma^{\otimes}\eta^J \quad = \quad f\gamma^{\otimes}\eta^J \quad = \quad \phi_2(f).$$

Here we used once more the triangular identity and the naturality of the transformations as expressed by the commutativity of the following diagram

$$\begin{tikzcd}
u_2^*Y \arrow[r, "\eta^K"] \arrow[d, "\eta^J"'] & u_2^*\,\mathsf{Hom}_l(X,X\otimes Y) \arrow[d, "\eta^J"] \\
\mathsf{Hom}_l(u_1^*X, u_1^*X\otimes u_2^*Y) \arrow[r, "\eta^K"] \arrow[d, "\gamma^{\otimes}"'] & \mathsf{Hom}_l(u_1^*X, u_1^*X\otimes u_2^*\,\mathsf{Hom}_l(X,X\otimes Y)) \arrow[d, "\gamma^{\otimes}"] \\
\mathsf{Hom}_l(u_1^*X, (u_1\times u_2)^*(X\otimes Y)) \arrow[r, "\eta^K"] \arrow[dr, bend right, "\mathrm{id}"'] & \mathsf{Hom}_l\left(u_1^*X, (u_1\times u_2)^*(X\otimes\mathsf{Hom}_l(X,X\otimes Y))\right) \arrow[d, "\epsilon^K"] \\
 & \mathsf{Hom}_l(u_1^*X, (u_1\times u_2)^*(X\otimes Y))
\end{tikzcd}$$

together with the fact that ϕ_2 sends f in a short-hand-notation to $f\gamma^{\otimes}\eta^J$.

Thus, in order to express the compatibility of the adjunction isomorphisms with restriction of diagrams we have constructed a natural transformation $\gamma^{\mathsf{Hom}_l}_{u_1,u_2}$ as indicated in:

$$\begin{tikzcd} \mathbb{D}(K_1)^{\mathrm{op}} \times \mathbb{F}(K_1 \times K_2) \arrow[r, "\mathsf{Hom}_l(-,-)_{K_1,K_2}"] \arrow[d, "u_1^* \times (u_1\times u_2)^*"'] \arrow[dr, phantom, "\Leftarrow"] & \mathbb{E}(K_2) \arrow[d, "u_2^*"] \\ \mathbb{D}(J_1)^{\mathrm{op}} \times \mathbb{F}(J_1 \times J_2) \arrow[r, "\mathsf{Hom}_l(-,-)_{J_1,J_2}"'] & \mathbb{E}(J_2) \end{tikzcd}$$

But this time –as the examples will show– it is important to note that this natural transformation is not necessarily invertible! Moreover, these natural transformations satisfy certain coherence conditions which are very similar to the ones in the case of a bimorphism (we will show this below in the case of an internal adjunction of two variables (see Lemma 1.11) since this will be used in Section 3). Said differently the functors $\mathsf{Hom}_l(-,-)_{K_1,K_2}$ together with the natural transformations $\gamma^{\mathsf{Hom}_l}_{u_1,u_2}$ assemble into a lax dinatural transformation $\mathsf{Hom}_l(-,-)$.

In the case of $\mathsf{Hom}_r(-,-)$ a similar reasoning leads to the conclusion that we can construct natural transformations $\gamma^{\mathsf{Hom}_r}_{u_1,u_2}\colon u_1^* \circ \mathsf{Hom}_r(-,-)_{K_1,K_2} \longrightarrow \mathsf{Hom}_r(-,-)_{J_1,J_2} \circ (u_2^* \times (u_1 \times u_2)^*)$ which satisfy suitable coherence conditions. Again it is important to note that these natural transformations are not necessarily invertible. Thus, also the $\mathsf{Hom}_r(-,-)_{K_1,K_2}$ together with the natural transformations $\gamma^{\mathsf{Hom}_r}_{u_1,u_2}$ assemble into a lax dinatural transformation $\mathsf{Hom}_r(-,-)$.

Definition 1.10. Let $\mathbb{D}$, $\mathbb{E}$, and $\mathbb{F}$ be prederivators. A *left adjoint of two variables from* $(\mathbb{D},\mathbb{E})$ *to* $\mathbb{F}$ is a bimorphism $\otimes\colon(\mathbb{D},\mathbb{E}) \longrightarrow \mathbb{F}$ such that for all pairs of categories (K_1,K_2) the functor $\otimes_{K_1,K_2}\colon \mathbb{D}(K_1)\times\mathbb{E}(K_2) \longrightarrow \mathbb{F}(K_1\times K_2)$ is a left adjoint of two variables.

The discussion preceding the definition thus guarantees the following. Given a left adjoint of two variables $\otimes\colon(\mathbb{D},\mathbb{E}) \longrightarrow \mathbb{F}$, we can find functors $\mathsf{Hom}_l(-,-)$ and $\mathsf{Hom}_r(-,-)$ and natural isomorphisms:

$$\mathsf{hom}_{\mathbb{F}(K_1\times K_2)}(X\otimes Y,Z) \quad\cong\quad \mathsf{hom}_{\mathbb{E}(K_2)}(Y,\mathsf{Hom}_l(X,Z)) \quad\cong\quad \mathsf{hom}_{\mathbb{D}(K_1)}(X,\mathsf{Hom}_r(Y,Z))$$

Moreover, the functors $\mathsf{Hom}_l(-,-)$ and $\mathsf{Hom}_r(-,-)$ can both be extended to lax dinatural transformations which in turn can be used to show that the adjunctions of the different levels are compatible. Let us denote an adjunction of two variables by $\otimes\colon(\mathbb{D},\mathbb{E}) \rightharpoonup \mathbb{F}$ or by $(\otimes,\mathsf{Hom}_l,\mathsf{Hom}_r)\colon(\mathbb{D},\mathbb{E}) \longrightarrow \mathbb{F}$.

The internal version of a left adjoint of two variables is completely parallel. A morphism $\otimes\colon \mathbb{D}\times\mathbb{E} \longrightarrow \mathbb{F}$ is called a *left adjoint of two variables* if for all categories J the induced functor $\otimes\colon \mathbb{D}(J)\times\mathbb{E}(J) \longrightarrow \mathbb{F}(J)$ is a left adjoint of two variables. By similar arguments as in the exterior case this implies that we have compatible adjunctions of two variables at the different levels. For example let $\mathsf{Hom}_r(-,-)\colon \mathbb{E}(J)^{\mathrm{op}}\times\mathbb{F}(J) \longrightarrow \mathbb{D}(J)$ be chosen levelwise right adjoints to $-\otimes-$ then we can define natural transformations $\gamma^{\mathsf{Hom}_r}_u$ by the following diagram:

$$\begin{tikzcd}
u^* \mathsf{Hom}_r(Y,Z) \arrow[r, "\eta"] \arrow[d, "\gamma_u^{\mathsf{Hom}_r}"'] & \mathsf{Hom}_r(u^*Y, u^* \mathsf{Hom}_r(Y,Z) \otimes u^*Y) \arrow[d, "\gamma_u^{\otimes}"] \\
\mathsf{Hom}_r(u^*Y, u^*Z) & \mathsf{Hom}_r(u^*Y, u^*(\mathsf{Hom}_r(Y,Z) \otimes Y)) \arrow[l, "\epsilon"]
\end{tikzcd}$$

Next, we want to show that these data assemble into a lax morphism. Let us allow ourselves to commit a slight abuse of notation and write Hom_r as a lax morphism $\mathbb{E}^{\mathrm{op}} \times \mathbb{F} \longrightarrow \mathbb{D}$ although, strictly speaking, this is not correct since $\mathbb{E}^{\mathrm{op}}(K) = \mathbb{E}(K^{\mathrm{op}})^{\mathrm{op}} \neq \mathbb{E}(K)^{\mathrm{op}}$.

Lemma 1.11. *Let $\mathbb{D}$, $\mathbb{E}$, and $\mathbb{F}$ be prederivators and let $\otimes\colon\, \mathbb{D} \times \mathbb{E} \longrightarrow \mathbb{F}$ be a left adjoint of two variables. The functors Hom_l resp. Hom_r together with the natural transformations $\gamma_u^{\mathsf{Hom}_l}$ resp. $\gamma_u^{\mathsf{Hom}_r}$ define a lax morphism of prederivators:*

$$\mathsf{Hom}_l(-,-)\colon\, \mathbb{D}^{\mathrm{op}} \times \mathbb{F} \longrightarrow \mathbb{E} \qquad \textit{resp.} \qquad \mathsf{Hom}_r(-,-)\colon\, \mathbb{E}^{\mathrm{op}} \times \mathbb{F} \longrightarrow \mathbb{D}$$

Proof. Let us give the proof in the case of Hom_r . It is easy to see that $\gamma_{\mathrm{id}}^{\mathsf{Hom}_r} = \mathrm{id}$ since this reduces to a triangular identity of adjunctions. So, let us consider two composable functors $J \xrightarrow{v} K \xrightarrow{u} L$. Using the fact that $\otimes\colon\, \mathbb{D} \times \mathbb{E} \longrightarrow \mathbb{F}$ is a morphism of derivators it is easy to verify that the following diagram commutes for arbitrary objects $X \in \mathbb{D}(L)$, $Y \in \mathbb{E}(L)$, and $Z \in \mathbb{F}(L)$:

$$\begin{tikzcd}
\mathsf{hom}_{\mathbb{F}(L)}(X \otimes Y, Z) \arrow[r, "u^*"] \arrow[d, "(uv)^*"'] & \mathsf{hom}_{\mathbb{F}(K)}(u^*(X \otimes Y), u^*Z) \arrow[r, "\gamma^{\otimes}"] & \mathsf{hom}_{\mathbb{F}(K)}(u^*X \otimes u^*Y, u^*Z) \arrow[d, "v^*"] \\
\mathsf{hom}_{\mathbb{F}(J)}((uv)^*(X \otimes Y), (uv)^*Z) \arrow[d, "\gamma^{\otimes}"'] & & \mathsf{hom}_{\mathbb{F}(J)}(v^*(u^*X \otimes u^*Y), v^*u^*Z) \arrow[d, "\gamma^{\otimes}"] \\
\mathsf{hom}_{\mathbb{F}(J)}((uv)^*X \otimes (uv)^*Y, (uv)^*Z) \arrow[rr, "\mathrm{id}"] & & \mathsf{hom}_{\mathbb{F}(J)}(v^*u^*X \otimes v^*u^*Y, v^*u^*Z)
\end{tikzcd}$$

Using the fact that we have levelwise adjunctions and that these adjunctions are compatible we obtain the corresponding result for the 'right-hand-side of the adjunction'. By this we mean that also the following diagram commutes in which we use H resp. γ^{H} as abbreviations for Hom_r resp. γ^{Hom_r}:

$$\begin{tikzcd}
\mathsf{hom}_{\mathbb{D}(L)}(X, \mathsf{H}(Y,Z)) \arrow[r, "u^*"] \arrow[d, "(uv)^*"'] & \mathsf{hom}_{\mathbb{D}(K)}(u^*X, u^* \mathsf{H}(Y,Z)) \arrow[r, "\gamma^{\mathsf{H}}"] & \mathsf{hom}_{\mathbb{D}(K)}(u^*X, \mathsf{H}(u^*Y, u^*Z)) \arrow[d, "v^*"] \\
\mathsf{hom}_{\mathbb{D}(J)}((uv)^*X, (uv)^* \mathsf{H}(Y,Z)) \arrow[d, "\gamma^{\mathsf{H}}"'] & & \mathsf{hom}_{\mathbb{D}(J)}(v^*u^*X, v^* \mathsf{H}(u^*Y, u^*Z)) \arrow[d, "\gamma^{\mathsf{H}}"] \\
\mathsf{hom}_{\mathbb{D}(J)}((uv)^*X, \mathsf{H}((uv)^*Y, (uv)^*Z)) \arrow[rr, "\mathrm{id}"] & & \mathsf{hom}_{\mathbb{D}(J)}(v^*u^*X, \mathsf{H}(v^*u^*Y, v^*u^*Z))
\end{tikzcd}$$

Choosing $X = \mathsf{Hom}_r(Y,Z)$ and tracing the identity $\mathrm{id}\colon \mathsf{Hom}_r(Y,Z) \longrightarrow \mathsf{Hom}_r(Y,Z)$ through the two possible ways to the lower right corner we obtain the second coherence condition of a lax morphism.

Finally, we also have to show a certain compatibility with 2-cells. So, let us consider a natural transformation $\alpha\colon u_1 \longrightarrow u_2$ between parallel functors $J \longrightarrow K$. We have to show that the following diagram commutes:

$$\begin{tikzcd}
u_1^* \circ \mathsf{H} \arrow[rr, "\alpha^*"] \arrow[d, "\gamma^{\mathsf{H}}_{u_1}"'] & & u_2^* \circ \mathsf{H} \arrow[d, "\gamma^{\mathsf{H}}_{u_2}"] \\
\mathsf{H} \circ (u_1^{*\mathrm{op}} \times u_1^*) \arrow[r, "\alpha^*"'] & \mathsf{H} \circ (u_1^{*\mathrm{op}} \times u_2^*) & \mathsf{H} \circ (u_2^{*\mathrm{op}} \times u_2^*) \arrow[l, "\alpha^{*\mathrm{op}}"]
\end{tikzcd}$$

But unraveling definitions we can see that the above diagram can be extended to the following one. For simplicity of notation we drop the 'op' in $u_1^{*\mathrm{op}}$ and $u_2^{*\mathrm{op}}$:

$$\begin{tikzcd}
u_1^* \,\mathsf{H}(Y,Z) \arrow[rr, "\alpha^*"] \arrow[d, "\eta_1"'] & & u_2^* \,\mathsf{H}(Y,Z) \arrow[dl, bend right, "\eta_1"'] \arrow[d, "\eta_2"] \\
\mathsf{H}(u_1^*Y, u_1^* \,\mathsf{H}(Y,Z) \otimes u_1^*Y) \arrow[r, "\alpha^*"] \arrow[dd, "\gamma^{\otimes}_{u_1}"'] & \mathsf{H}(u_1^*Y, u_2^* \,\mathsf{H}(Y,Z) \otimes u_1^*Y) \arrow[d, "\alpha^*"] & \mathsf{H}(u_2^*Y, u_2^* \,\mathsf{H}(Y,Z) \otimes u_2^*Y) \arrow[dl, bend right, "\alpha^{*\mathrm{op}}"'] \arrow[dd, "\gamma^{\otimes}_{u_2}"] \\
 & \mathsf{H}(u_1^*Y, u_2^* \,\mathsf{H}(Y,Z) \otimes u_2^*Y) \arrow[d, "\gamma^{\otimes}_{u_2}"] & \\
\mathsf{H}(u_1^*Y, u_1^*(\mathsf{H}(Y,Z) \otimes Y)) \arrow[r, "\alpha^*"'] \arrow[d, "\epsilon"'] & \mathsf{H}(u_1^*Y, u_2^*(\mathsf{H}(Y,Z) \otimes Y)) \arrow[d, "\epsilon"] & \mathsf{H}(u_2^*Y, u_2^*(\mathsf{H}(Y,Z) \otimes Y)) \arrow[l, "\alpha^{*\mathrm{op}}"] \arrow[d, "\epsilon"] \\
\mathsf{H}(u_1^*Y, u_1^*Z) \arrow[r, "\alpha^*"'] & \mathsf{H} \circ (u_1^{*\mathrm{op}} \times u_2^*) & \mathsf{H} \circ (u_2^{*\mathrm{op}} \times u_2^*) \arrow[l, "\alpha^{*\mathrm{op}}"]
\end{tikzcd}$$

In this diagram, the upper right quadrilateral commutes by the extranaturality of the adjunction unit in the context of an adjunction with parameters (cf. [ML98, Section IX.4]) while the center left square commutes since $\otimes$ is a morphism of prederivators. The remaining part of the diagram commutes by naturality which concludes the proof. □

The interior and the exterior version of adjunctions of two variables are compatible. If we have a bimorphism $\otimes\colon (\mathbb{D},\mathbb{E}) \longrightarrow \mathbb{F}$ and a morphism of two variables $\otimes\colon \mathbb{D}\times\mathbb{E} \longrightarrow \mathbb{F}$ which correspond to each other under the equivalence of Proposition 1.5 then the bimorphism $\otimes$ is a left adjoint of two variables if and only if this is the case for the morphism $\otimes$.

We now turn to examples in the context of represented (pre)derivators. Let $\otimes\colon \mathcal{C}\times\mathcal{D} \longrightarrow \mathcal{E}$ be a functor of two variables. We can extend $\otimes$ to a (strict) bimorphism $\otimes\colon (\mathcal{C},\mathcal{D}) \longrightarrow \mathcal{E}$ of the associated represented prederivators. In fact, for a pair of categories (J_1, J_2) let us define $\otimes_{J_1,J_2}\colon \mathcal{C}^{J_1} \times \mathcal{D}^{J_2} \longrightarrow \mathcal{E}^{J_1 \times J_2}$ by sending a pair (X,Y) to:

$$X \otimes Y\colon J_1 \times J_2 \xrightarrow{X\times Y} \mathcal{C} \times \mathcal{D} \xrightarrow{\otimes} \mathcal{E}$$

Let us call this bimorphism $\otimes$ the *bimorphism represented by* $\otimes$.

Proposition 1.12. *Let $\mathcal{C}$, $\mathcal{D}$ be complete categories, $\mathcal{E}$ a category and $\otimes\colon \mathcal{C}\times\mathcal{D}\longrightarrow\mathcal{E}$ a left adjoint of two variables. The represented bimorphism $\otimes\colon(\mathcal{C},\mathcal{D})\longrightarrow\mathcal{E}$ is then also a left adjoint of two variables. In particular, adjunctions of two variables between bicomplete categories induce adjunctions of two variables between represented derivators.*

Proof. Let us content ourselves by giving the construction of $\mathsf{Hom}_l(-,-)$ and the natural isomorphism expressing one half of the fact that we have an adjunction of two variables. So, let us consider a pair of categories (J_1,J_2) and let us construct a right adjoint

$$\mathsf{Hom}_l(-,-)\colon (\mathcal{C}^{J_1})^{\mathrm{op}} \times \mathcal{E}^{J_1\times J_2} \longrightarrow \mathcal{D}^{J_2}.$$

Using $(\mathcal{C}^{J_1})^{\mathrm{op}} = (\mathcal{C}^{\mathrm{op}})^{J_1^{\mathrm{op}}}$, as an intermediate step we can associate a pair (X,Z) to the functor

$$\mathsf{Hom}_l(-,-)\circ(X\times Z)\colon J_1^{\mathrm{op}}\times J_1\times J_2 \longrightarrow \mathcal{C}^{\mathrm{op}}\times\mathcal{E}\longrightarrow\mathcal{D}.$$

Here, Hom_l is a functor expressing the fact that $\otimes$ is an adjunction of two variables. Forming the end over the category J_1 we can define $\mathsf{Hom}_l(X,Z)\colon J_2\longrightarrow\mathcal{D}$ by:

$$\mathsf{Hom}_l(X,Z)(-) = \int_{j_1} \mathsf{Hom}_l(X(j_1), Z(j_1\times -))$$

Let us check that this gives us the desired adjunction. For this purpose let us consider a functor $Y\in\mathcal{D}^{J_2}$. Using the fact that natural transformations give an example of a further end construction we can make the following calculation:

$$\begin{aligned}
\mathsf{hom}_{\mathcal{E}^{J_1\times J_2}}(X\otimes Y, Z) &\cong \int_{(j_1,j_2)} \mathsf{hom}_{\mathcal{E}}(X(j_1)\otimes Y(j_2), Z(j_1,j_2)) \\
&\cong \int_{(j_1,j_2)} \mathsf{hom}_{\mathcal{D}}\big(Y(j_2), \mathsf{Hom}_l(X(j_1), Z(j_1,j_2))\big) \\
&\cong \int_{j_2} \mathsf{hom}_{\mathcal{D}}\big(Y(j_2), \int_{j_1}\mathsf{Hom}_l(X(j_1), Z(j_1,j_2))\big) \\
&= \int_{j_2} \mathsf{hom}_{\mathcal{D}}\big(Y(j_2), \mathsf{Hom}_l(X,Z)(j_2)\big) \\
&\cong \mathsf{hom}_{\mathcal{D}^{J_2}}\big(Y, \mathsf{Hom}_l(X,Z)\big)
\end{aligned}$$

The third isomorphism follows from Fubini's theorem for ends and the fact that corepresented functors are end preserving, the second one is the adjunction isomorphism at the level of categories, while the first and the last one are given by the fact that natural transformations can be expressed as ends. This concludes the construction of an adjunction of two variables. $\square$

We use this example to illustrate that the structure maps belonging to the right adjoints are not necessarily isomorphisms, i.e., that we only obtain *lax* dinatural transformations as opposed to *pseudo* dinatural transformations. So, let us consider a pair of functors $(u_1,u_2)\colon (J_1,J_2)\longrightarrow(K_1,K_2)$, two diagrams $X\colon K_1\longrightarrow\mathcal{C}$ and $Z\colon K_1\times K_2\longrightarrow\mathcal{E}$ and let us have a look at the diagram:

$$\begin{tikzcd}
\int_{k_1}\mathsf{Hom}_l\big(X(k_1),Z(k_1,u_2(-))\big) \arrow[d, dashed] \arrow[dr, bend left, "\mathrm{pr}_{u_1(j_1'),u_1(j_1'')}"] & \\
\int_{j_1}\mathsf{Hom}_l\big(X(u_1(j_1)),Z(u_1(j_1),u_2(-))\big) \arrow[r, "\mathrm{pr}_{j_1',j_1''}"'] & \mathsf{Hom}_l\big(X(u_1(j_1')),Z(u_1(j_1''),u_2(-))\big)
\end{tikzcd}$$

The upper left object is $u_2^*\,\mathsf{Hom}_l(X,Z)$ and the lower left one is $\mathsf{Hom}_l(u_1^*X,(u_1\times u_2)^*Z)$. The horizontal morphism belongs to the universal wedge of the lower end construction while the diagonal morphism is part of the universal wedge belonging to the upper end construction. By the universal property of the lower wedge there is a unique dashed arrow as indicated which is compatible with all projection morphisms. If we take these dashed arrows as a definition of $\gamma_{u_1,u_2}^{\mathsf{Hom}_l(-,-)}\colon u_2^*\,\mathsf{Hom}_l(-,-)\longrightarrow\mathsf{Hom}_l(u_1^*(-),(u_1\times u_2)^*(-))$ one can check that Hom_l becomes a lax dinatural transformation. The fact that the adjunctions at the different levels are compatible with the restriction functors is expressed by the commutativity of the following diagram. In this diagram, we drop the arguments to simplify notation:

$$\begin{tikzcd}
\int_{K_1\times K_2}\mathsf{hom}_{\mathcal{E}}(X\otimes Y,Z) \arrow[r] \arrow[d] & \int_{K_2}\mathsf{hom}_{\mathcal{D}}\big(Y,\int_{K_1}\mathsf{Hom}_l(X,Z)\big) \arrow[d] \\
\int_{J_1\times J_2}\mathsf{hom}_{\mathcal{E}}((u_1\times u_2)^*(X\otimes Y),(u_1\times u_2)^*Z) \arrow[d] & \int_{J_2}\mathsf{hom}_{\mathcal{D}}\Big(u_2^*Y,\int_{K_1}\mathsf{Hom}_l\big(X,(\mathrm{id}\times u_2)^*Z\big)\Big) \arrow[d] \\
\int_{J_1\times J_2}\mathsf{hom}_{\mathcal{E}}(u_1^*X\otimes u_2^*Y,(u_1\times u_2)^*Z) \arrow[r] & \int_{J_2}\mathsf{hom}_{\mathcal{D}}\Big(u_2^*Y,\int_{J_1}\mathsf{Hom}_l\big(u_1^*X,(u_1\times u_2)^*Z\big)\Big)
\end{tikzcd}$$

This diagram commutes by the universal property of end constructions.

Let us recall from Example 1.9 that prederivators represented by monoidal categories can be canonically endowed with a monoidal structure. We can now obtain a similar result for biclosed monoidal categories under an additional completeness assumption. Of course there is a similar result for closed monoidal categories, i.e., symmetric biclosed monoidal categories.

Definition 1.13. Let $(\mathbb{D},\otimes,\mathbb{S})$ be a monoidal prederivator. The monoidal prederivator or the monoidal structure is called *biclosed* if the morphism $\otimes\colon\ \mathbb{D}\times\mathbb{D}\longrightarrow\mathbb{D}$ is a left adjoint of two variables. A symmetric monoidal prederivator having this additional property is called a *closed monoidal prederivator.*

Corollary 1.14. *Let $\mathcal{C}$ be a (bi)closed monoidal, complete category. The represented monoidal structure on the represented prederivator $\mathcal{C}$ is then also (bi)closed. In particular, derivators represented by (bi)closed monoidal, bicomplete categories are canonically (bi)closed monoidal.*

1.4. **Monoidal model categories induce monoidal derivators.** Before we turn to monoidal model categories let us make some more comments on the derivator $\mathbb{D}_{\mathcal{M}}$ associated to a combinatorial model category $\mathcal{M}$ (cf. [Gro10a]). Recall that combinatorial model categories as introduced by Smith are cofibrantly generated model categories which have an underlying presentable category (for the theory of presentable categories cf. the original source [GU71] but also [AR94, MP89]). In the construction of the derivator $\mathbb{D}_{\mathcal{M}}$ we use the fact that the diagram categories $\mathcal{M}^J$ can be endowed both with the injective and the projective model structure. The existence of the projective model structure follows from a general lifting result of cofibrantly generated model structures along a left adjoint functor ([Hir03]) while the existence of the injective model structure is, for example, shown in [Lur09]. Since both model structures have the same class of weak equivalences, it is not important which one we use in the definition of the value $\mathbb{D}_{\mathcal{M}}(J)$ as they have canonically isomorphic homotopy categories:

$$\mathsf{Ho}(\mathcal{M}^J_{\mathsf{proj}}) \cong \mathsf{Ho}(\mathcal{M}^J_{\mathsf{inj}})$$

Now, for a functor $u\colon J \longrightarrow K$, the induced precomposition functor $u^*\colon \mathcal{M}^K \longrightarrow \mathcal{M}^J$ preserves weak equivalences with respect to both structures. Hence, by the universal properties of the localization functors $\gamma\colon \mathcal{M}^J \longrightarrow \mathsf{Ho}(\mathcal{M}^J)$ and $\gamma\colon \mathcal{M}^K \longrightarrow \mathsf{Ho}(\mathcal{M}^K)$ we obtain a unique induced functor u^* at the level of the homotopy categories such that the following diagram commutes on the nose:

$$\begin{tikzcd} \mathcal{M}^K \arrow[r, "u^*"] \arrow[d, "\gamma"'] & \mathcal{M}^J \arrow[d, "\gamma"] \\ \mathsf{Ho}(\mathcal{M}^K) \arrow[r, "u^*"'] & \mathsf{Ho}(\mathcal{M}^J) \end{tikzcd}$$

By definition, this induced functor is taken as the value $\mathbb{D}_{\mathcal{M}}(u)$.

Alternatively, one could also form the left derived functor $\mathbb{L}u^*$ with respect to the injective model structures or the right derived functor $\mathbb{R}u^*$ with respect to the projective model structures. Recall that these are functors endowed with natural transformations which turn $\mathbb{L}u^*$ into a right Kan extension of $\gamma \circ u^*$ along γ while $\mathbb{R}u^*$ becomes a left Kan extension of $\gamma \circ u^*$ along γ :

$$\begin{tikzcd} \mathcal{M}^K \arrow[r, "u^*"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\Nearrow"] & \mathcal{M}^J \arrow[d, "\gamma"] \\ \mathsf{Ho}(\mathcal{M}^K) \arrow[r, "\mathbb{L}u^*"'] & \mathsf{Ho}(\mathcal{M}^J) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{M}^K \arrow[r, "u^*"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\Swarrow"] & \mathcal{M}^J \arrow[d, "\gamma"] \\ \mathsf{Ho}(\mathcal{M}^K) \arrow[r, "\mathbb{R}u^*"'] & \mathsf{Ho}(\mathcal{M}^J) \end{tikzcd}$$

Since the localization functor $\gamma\colon \mathcal{M}^K \longrightarrow \mathsf{Ho}(\mathcal{M}^K)$ is a 2-localization, we obtain, in particular, an isomorphism of categories:

$$\gamma^*\colon \mathsf{Ho}(\mathcal{M}^J)^{\mathsf{Ho}(\mathcal{M}^K)} \longrightarrow \mathsf{Ho}(\mathcal{M}^J)^{(\mathcal{M}^K,W)}$$

Here, the right-hand-side is the full subcategory of $\mathsf{Ho}(\mathcal{M}^J)^{\mathcal{M}^K}$ spanned by the functors which invert the weak equivalences. For an arbitrary functor $F\colon \mathsf{Ho}(\mathcal{M}^K) \longrightarrow \mathsf{Ho}(\mathcal{M}^J)$ this gives us the following two bijections

$$\mathsf{nat}(F,u^*) \xrightarrow{\gamma^*} \mathsf{nat}(F\circ\gamma, u^*\circ\gamma), \qquad \mathsf{nat}(u^*,F) \xrightarrow{\gamma^*} \mathsf{nat}(u^*\circ\gamma, F\circ\gamma).$$

But these bijections express that the induced functor $u^*\colon \mathsf{Ho}(\mathcal{M}^K) \longrightarrow \mathsf{Ho}(\mathcal{M}^J)$ is simultaneously also a right Kan extension and a left Kan extension of $\gamma\circ u^*$ along γ. We thus obtain natural isomorphisms

$$\mathbb{L}u^* \quad\cong\quad u^* \quad\cong\quad \mathbb{R}u^*.$$

This observation will be useful in the construction of the monoidal derivator underlying a combinatorial monoidal model category. More generally, it allows for the construction of morphisms of derivators induced by Brown functors and hence, in particular, by Quillen functors or Quillen bifunctors. One motivation for the notion of Brown functors is the following. In order to form the derived functor of a –say– left Quillen functor not all of the defining properties of a left Quillen functor are needed as already emphasized in [Hov99, Hir03, Mal07b]. Thus, sometimes the following definition is useful (cf. also to [DHKS04] and [Shu11] where these are called deformable functors and derivable functors, respectively).

Definition 1.15. Let $\mathcal{M}$ and $\mathcal{N}$ be model categories and let $F\colon \mathcal{M} \longrightarrow \mathcal{N}$ be a functor. F is a *left Brown functor* if F preserves weak equivalences between cofibrant objects. Dually, F is a *right Brown functor* if F preserves weak equivalences between fibrant objects.

As one sees from the constructions in [Hov99, Hir03], this suffices to obtain the respective derived functors which again will have the universal property of the respective Kan extensions. In what follows, we will only state and prove the results for left Brown functors (and left Quillen (bi)functors), but also the dual statements hold true.

Proposition 1.16. *Let $\mathcal{M}$ and $\mathcal{N}$ be combinatorial model categories and let $F\colon \mathcal{M} \longrightarrow \mathcal{N}$ be a left Brown functor. Then by forming left derived functors we obtain a morphism of derivators $\mathbb{L}F\colon \mathbb{D}_{\mathcal{M}} \longrightarrow \mathbb{D}_{\mathcal{N}}$. In particular, this is the case for left Quillen functors.*

Proof. Let J be a category and let us consider the induced functor $F\colon \mathcal{M}^J \longrightarrow \mathcal{N}^J$. With respect to the injective model structures, this is again a left Brown functor. Hence, given a functor $u\colon J \longrightarrow K$ we have the following commutative diagram of left Brown functors:

$$\begin{tikzcd}
\mathcal{M}^K_{\mathsf{inj}} \arrow[r, "F"] \arrow[d, "u^*"'] & \mathcal{N}^K_{\mathsf{inj}} \arrow[d, "u^*"] \\
\mathcal{M}^J_{\mathsf{inj}} \arrow[r, "F"'] & \mathcal{N}^J_{\mathsf{inj}}
\end{tikzcd}$$

Passing to left derived functors for the horizontal arrows and to the induced functors on the localizations for the vertical arrows gives us the following diagram which commutes up to a canonical natural isomorphism γ_u:

$$\begin{tikzcd} \mathbb{D}_{\mathcal{M}}(K) \arrow[r, "\mathbb{L}F"] \arrow[d, "u^*"'] \arrow[dr, phantom, "\not\swarrow"] & \mathbb{D}_{\mathcal{N}}(K) \arrow[d, "u^*"] \\ \mathbb{D}_{\mathcal{M}}(J) \arrow[r, "\mathbb{L}F"'] & \mathbb{D}_{\mathcal{N}}(J) \end{tikzcd}$$

It is easy to check that these natural isomorphisms γ_u, $u\colon J \longrightarrow K$, endow the functors $\mathbb{L}F$ with the structure of a morphism of derivators. $\square$

Since adjunctions and equivalences of derivators are detected levelwise we immediately obtain the following corollary.

Corollary 1.17. *Let $(F,U)\colon \mathcal{M} \rightharpoonup \mathcal{N}$ be a Quillen adjunction of combinatorial model categories. Then we obtain a derived adjunction $(\mathbb{L}F,\mathbb{R}U)\colon \mathbb{D}_{\mathcal{M}} \rightharpoonup \mathbb{D}_{\mathcal{N}}$. If (F,U) is a Quillen equivalence then $(\mathbb{L}F,\mathbb{R}U)$ is an equivalence of derivators.*

There is a further important class of Brown functors, namely the Quillen bifunctors. These are central to many notions of homotopical algebra.

Definition 1.18. Let $\mathcal{M}$, $\mathcal{N}$, and $\mathcal{P}$ be model categories. A functor $\otimes\colon \mathcal{M}\times\mathcal{N} \longrightarrow \mathcal{P}$ is a *left Quillen bifunctor* if it preserves colimits separately in each variable and has the following property: For every cofibration $f\colon X_1 \longrightarrow X_2$ in $\mathcal{M}$ and every cofibration $g\colon Y_1 \longrightarrow Y_2$ in $\mathcal{N}$ the pushout-product map

$$f\Box g = (X_2 \otimes g) \amalg (f \otimes Y_2)\colon \quad X_2 \otimes Y_1 \amalg_{X_1\otimes Y_1} X_1 \otimes Y_2 \longrightarrow X_2 \otimes Y_2$$

is a cofibration which is acyclic if in addition f or g is acyclic.

Here, the map $f\Box g$ is the unique map induced by the fact that we are given a bifunctor $\otimes$. There is the dual notion of a right Quillen bifunctor $\mathsf{Hom}\colon \mathcal{M}^{op} \times \mathcal{N} \longrightarrow \mathcal{P}$. In that case one considers the induced maps

$$\mathsf{Hom}_{\Box}(f,g)\colon\ \mathsf{Hom}(X_2,Y_1) \longrightarrow \mathsf{Hom}(X_1,Y_1) \times_{\mathsf{Hom}(X_1,Y_2)} \mathsf{Hom}(X_2,Y_2).$$

The following is immediate.

Lemma 1.19. *Let $\otimes\colon \mathcal{M}\times\mathcal{N} \longrightarrow \mathcal{P}$ be a left Quillen bifunctor and let $X \in \mathcal{M}$ resp. $Y \in \mathcal{N}$ be cofibrant objects. The functors $X \otimes -\colon \mathcal{N} \longrightarrow \mathcal{P}$ and $- \otimes Y\colon \mathcal{M} \longrightarrow \mathcal{P}$ are then left Quillen functors. In particular, $\otimes\colon \mathcal{M} \times \mathcal{N} \longrightarrow \mathcal{P}$ is a left Brown functor when we endow $\mathcal{M} \times \mathcal{N}$ with the product model structure.*

Thus Proposition 1.16 can be applied to Quillen bifunctors. Under the canonical isomorphism $\mathbb{D}_{\mathcal{M}\times\mathcal{N}} \cong \mathbb{D}_{\mathcal{M}} \times \mathbb{D}_{\mathcal{N}}$ we obtain that a Quillen bifunctor $\otimes\colon \mathcal{M} \times \mathcal{N} \longrightarrow \mathcal{P}$ induces a morphism of derivators $\mathbb{D}_{\mathcal{M}} \times \mathbb{D}_{\mathcal{N}} \longrightarrow \mathbb{D}_{\mathcal{P}}$. Let us not distinguish notationally between this morphism and the associated bimorphism (cf. Proposition 1.5) and let us denote both by

$$\overset{\mathbb{L}}{\otimes}\colon\ \mathbb{D}_{\mathcal{M}}\times\mathbb{D}_{\mathcal{N}}\longrightarrow\mathbb{D}_{\mathcal{P}}\qquad\text{and}\qquad\overset{\mathbb{L}}{\otimes}\colon(\mathbb{D}_{\mathcal{M}},\mathbb{D}_{\mathcal{N}})\longrightarrow\mathbb{D}_{\mathcal{P}}\,.$$

The bimorphism can also be obtained without invoking Proposition 1.5. The bifunctor $\otimes$ induces a strict bimorphism of represented derivators $\otimes\colon(\mathcal{M},\mathcal{N})\longrightarrow\mathcal{P}$. For each morphism of pairs $(u_1,u_2)\colon(J_1,J_2)\longrightarrow(K_1,K_2)$ we have a commutative diagram of left Brown functors as follows if all model categories are endowed with the injective model structures:

$$\begin{tikzcd}
\mathcal{M}^{K_1}\times\mathcal{N}^{K_2} \arrow[r, "\otimes"] \arrow[d, "u_1^*\times u_2^*"'] & \mathcal{P}^{K_1\times K_2} \arrow[d, "(u_1\times u_2)^*"] \\
\mathcal{M}^{J_1}\times\mathcal{N}^{J_2} \arrow[r, "\otimes"'] & \mathcal{P}^{J_1\times J_2}
\end{tikzcd}$$

Forming derived functors at the different levels and taking the natural isomorphisms induced by these diagrams we obtain again the bimorphism $(\mathbb{D}_{\mathcal{M}},\mathbb{D}_{\mathcal{N}})\longrightarrow\mathbb{D}_{\mathcal{P}}$.

In the context of combinatorial model categories, we get a stronger statement. Recall that the adjoint functor theorem of Freyd takes the following form in the context of presentable categories: a functor between presentable categories is a left adjoint if and only if it preserves colimits. For example, in the context of combinatorial model categories a monoidal structure which preserves colimits in each variable is always a biclosed monoidal structure, i.e., we have an adjunction of two variables $(\otimes,\mathsf{Hom}_l,\mathsf{Hom}_r)$.

Now, let $\mathcal{M}$, $\mathcal{N}$, and $\mathcal{P}$ be combinatorial model categories. Then given a left Quillen bifunctor $\otimes\colon\mathcal{M}\times\mathcal{N}\longrightarrow\mathcal{P}$ we obtain an adjunction of two variables $(\otimes,\mathsf{Hom}_l,\mathsf{Hom}_r)$. This adjunction is expressed by natural isomorphisms

$$\mathsf{hom}_{\mathcal{P}}(X\otimes Y,Z)\quad\cong\quad\mathsf{hom}_{\mathcal{M}}(X,\mathsf{Hom}_r(Y,Z))\quad\cong\quad\mathsf{hom}_{\mathcal{N}}(Y,\mathsf{Hom}_l(X,Z))$$

for certain functors

$$\mathsf{Hom}_l(-,-)\colon\mathcal{M}^{\mathrm{op}}\times\mathcal{P}\longrightarrow\mathcal{N}\qquad\text{and}\qquad\mathsf{Hom}_r(-,-)\colon\mathcal{N}^{\mathrm{op}}\times\mathcal{P}\longrightarrow\mathcal{M}.$$

Lemma 1.20. *Let $\mathcal{M}$, $\mathcal{N}$, and $\mathcal{P}$ be model categories and let $(\otimes,\mathsf{Hom}_l,\mathsf{Hom}_r)\colon\mathcal{M}\times\mathcal{N}\rightharpoonup\mathcal{P}$ be an adjunction of two variables. If we endow $\mathcal{M}^{\mathrm{op}}$ resp. $\mathcal{N}^{\mathrm{op}}$ with the dual model structures we have the following equivalent statements: $\otimes$ is a left Quillen bifunctor if and only if Hom_l is a right Quillen bifunctor if and only if Hom_r is a right Quillen bifunctor.*

By the above discussion, we know that a left Quillen bifunctor $\otimes\colon\mathcal{M}\times\mathcal{N}\longrightarrow\mathcal{P}$ between combinatorial model categories extends to an adjunction of two variables. By Proposition 1.12 or again by the special adjoint functor theorem, we deduce that this adjunction induces adjunctions of two variables between represented derivators $\otimes\colon(\mathcal{M},\mathcal{N})\longrightarrow\mathcal{P}$. By the last lemma, we have thus adjunctions of two variables consisting of Quillen bifunctors which induce derived adjunctions of two variables $\mathbb{D}_{\mathcal{M}}(J_1)\times\mathbb{D}_{\mathcal{N}}(J_2)\longrightarrow\mathbb{D}_{\mathcal{P}}(J_1\times J_2)$. This shows that the morphism

$$\overset{\mathbb{L}}{\otimes}\colon(\mathbb{D}_{\mathcal{M}},\mathbb{D}_{\mathcal{N}})\longrightarrow\mathbb{D}_{\mathcal{P}}$$

is a left adjoint of two variables. We have thus established the following result.

Corollary 1.21. *Let $\mathcal{M}$, $\mathcal{N}$, and $\mathcal{P}$ be combinatorial model categories and let $\otimes\colon \mathcal{M}\times\mathcal{N}\longrightarrow \mathcal{P}$ be a left Quillen bifunctor. Then, by forming derived functors, we obtain an adjunction of two variables at the level of associated derivators:*

$$(\overset{\mathbb{L}}{\otimes}, \mathbb{R}\mathsf{Hom}_l, \mathbb{R}\mathsf{Hom}_r)\colon (\mathbb{D}_{\mathcal{M}}, \mathbb{D}_{\mathcal{N}}) \rightharpoonup \mathbb{D}_{\mathcal{P}}$$

For later reference let us quickly introduce the notion of Quillen homotopies.

Definition 1.22. Let F, $G\colon \mathcal{M}\longrightarrow\mathcal{N}$ be left Brown functors. A natural transformation $\tau\colon F\longrightarrow G$ is called a *(left) Quillen homotopy* if the components τ_X are weak equivalences for all cofibrant objects X.

Lemma 1.23. *Let F, $G\colon \mathcal{M}\longrightarrow\mathcal{N}$ be left Brown functors between combinatorial model categories and let $\tau\colon F\longrightarrow G$ be a left Quillen homotopy. Then we obtain a natural isomorphism*

$$\mathbb{L}\tau\colon\ \mathbb{L}F \xrightarrow{\cong} \mathbb{L}G$$

of induced morphisms $\mathbb{L}F$, $\mathbb{L}G\colon\ \mathbb{D}_{\mathcal{M}}\longrightarrow\mathbb{D}_{\mathcal{N}}$.

With these preparations we can now turn to monoidal model categories. We use the following definition of a monoidal model category, which is close to the original one in [Hov99].

Definition 1.24. A *monoidal model category* is a model category $\mathcal{M}$ endowed with a monoidal structure such that the monoidal pairing $\otimes\colon \mathcal{M}\times\mathcal{M}\longrightarrow\mathcal{M}$ is a Quillen bifunctor and such that a (and hence any) cofibrant replacement $Q\mathbb{S}\longrightarrow\mathbb{S}$ of the monoidal unit has the property that the induced natural transformations $Q\,\mathbb{S}\otimes- \longrightarrow \mathbb{S}\otimes-$ and $-\otimes Q\,\mathbb{S}\longrightarrow -\otimes\mathbb{S}$ are Quillen homotopies.

Theorem 1.25. *Let $\mathcal{M}$ be a combinatorial monoidal model category. The associated derivator $\mathbb{D}_{\mathcal{M}}$ inherits canonically the structure of a biclosed monoidal derivator. If the monoidal structure on $\mathcal{M}$ is symmetric, then this is also the case for the induced structure on $\mathbb{D}_{\mathcal{M}}$.*

Proof. We only have to put the above results together and care about the unit. The injective model structures on the diagram categories $\mathcal{M}^J_{\mathsf{inj}}$ have the property that the natural transformations $Q\,\mathbb{S}\otimes- \longrightarrow \mathbb{S}\otimes-$ and $-\otimes Q\,\mathbb{S}\longrightarrow -\otimes\mathbb{S}$ are again Quillen homotopies since everything is defined levelwise. Thus, at each stage we can apply the corresponding result of [Hov99] to obtain a monoidal structure on $\mathsf{Ho}(\mathcal{M}^J)$. Moreover, by Corollary 1.21 these fit together to define a biclosed monoidal structure on $\mathbb{D}_{\mathcal{M}}$ since the left Quillen bifunctor $\otimes$ induces a derived adjunction of two variables. $\square$

There is a similar result for monoidal left Quillen functors. Recall from [Hov99] that a monoidal left Quillen functor is a left Quillen functor which is strong monoidal and satisfies an additional unitality condition. This extra condition ensures that the derived functor will respect the monoidal unit at the level of homotopy categories. We omit the proof that such a monoidal left Quillen functor between combinatorial model categories induces a monoidal morphism of associated derivators. After having given the following central examples we will shortly consider the situation of weakly monoidal Quillen adjunctions.

Example 1.26. Let k be a commutative ring and let $\mathsf{Ch}(k)$ be the category of unbounded chain complexes over k. This category can be equipped with the combinatorial (so-called projective) model structure where the weak equivalences are the quasi-isomorphisms and the fibrations are the surjections ([Hov99]). The tensor product of chain complexes endows this category with the structure of a closed monoidal model category. The unit object is given by $k[0]$, which denotes the chain complex concentrated in degree zero where it takes the value k. Thus, the associated stable *derivator of chain complexes*

$$\mathbb{D}_k = \mathbb{D}_{\mathsf{Ch}(k)}$$

is a closed monoidal derivator. More generally, let C be a commutative monoid in $\mathsf{Ch}(k)$, i.e., let C be a commutative differential-graded algebra. Then, the category $C-\mathsf{Mod}$ of differential-graded left C-modules inherits a stable, combinatorial model structure ([SS00]). Moreover, forming the tensor product over C endows $C-\mathsf{Mod}$ with the structure of a closed monoidal model category. We deduce that the associated stable *derivator of differential-graded $C-$modules*

$$\mathbb{D}_C = \mathbb{D}_{C-\mathsf{Mod}}$$

is also closed monoidal.

Example 1.27. Let Set_Δ denote the presentable category of simplicial sets. If we endow it with the homotopy-theoretic Kan model structure ([Qui67], [GJ99, Chapter 1]) we obtain a Cartesian closed monoidal model category $\mathsf{Set}_\Delta^{\mathsf{Kan}}$. Recall that the cofibrations are the monomorphisms, the weak equivalences are the maps which become homotopy equivalences after geometric realization and the fibrations are the Kan fibrations. Since this model structure is combinatorial, we obtain a closed monoidal *derivator of simplicial sets*:

$$\mathbb{D}_{\mathsf{Set}_\Delta} = \mathbb{D}_{\mathsf{Set}_\Delta^{\mathsf{Kan}}}$$

But, there is also the Joyal model structure on the category of simplicial sets (see for example [Joy08b], [Lur09], and also [Gro10b]). This cofibrantly generated model structure is Cartesian so that we again have a Cartesian closed monoidal model category $\mathsf{Set}_\Delta^{\mathsf{Joyal}}$ where the underlying model category is combinatorial. Thus, we obtain a further closed monoidal derivator, the *derivator of $\infty-$categories*:

$$\mathbb{D}_{\infty-\mathsf{Cat}} = \mathbb{D}_{\mathsf{Set}_\Delta^{\mathsf{Joyal}}}$$

Example 1.28. Let Sp^Σ be the category of symmetric spectra based on simplicial sets as introduced in [HSS00]. This presentable category carries a symmetric monoidal structure given by the smash product $\wedge$ where the monoidal unit is given by the sphere spectrum $\mathbb{S}$. It is shown in [HSS00] that Sp^Σ endowed with the stable model structure is a cofibrantly generated, stable, symmetric monoidal model category in which the unit object is cofibrant. We obtain hence an associated stable, closed monoidal *derivator of spectra*:

$$\mathbb{D}_{\mathsf{Sp}} = \mathbb{D}_{\mathsf{Sp}^\Sigma}$$

Moreover, let us denote by $E-\mathsf{Mod}$ the category of left $E-$module spectra for a commutative symmetric ring spectrum $E \in \mathsf{Sp}^\Sigma$. The category $E-\mathsf{Mod}$ can be endowed with the projective model structure, i.e., the weak equivalences and the fibrations are reflected by the forgetful functor $E-\mathsf{Mod} \longrightarrow \mathsf{Sp}^{\Sigma}$. This model category is a combinatorial monoidal model category when endowed with the smash product over E and hence gives rise to the stable, closed monoidal derivator of $E-$module spectra:

$$\mathbb{D}_E = \mathbb{D}_{E-\mathsf{Mod}}$$

We will now consider weakly monoidal Quillen adjunctions as introduced by Schwede and Shipley in [SS03a] and illustrate them by an example. This example will also reveal a technical advantage derivators do have when compared to model categories. Before we get to that let us give the following result (cf. [Kel74]). Let us consider an adjunction $(L,R)\colon \mathcal{C} \rightharpoonup \mathcal{D}$ where both categories $\mathcal{C}$ and $\mathcal{D}$ are monoidal. Moreover, let us assume that we are given a lax monoidal structure on the right adjoint:

$$m\colon RX\otimes RY \longrightarrow R(X\otimes Y) \qquad \text{and} \qquad u\colon \mathbb{S}\longrightarrow R\mathbb{S}$$

The map u is adjoint to a map $u'\colon L\mathbb{S}\longrightarrow \mathbb{S}$ while we can define $m'\colon L(X\otimes Y)\longrightarrow LX\otimes LY$ to be the map adjoint to

$$X\otimes Y \xrightarrow{\eta\otimes\eta} RLX\otimes RLY \xrightarrow{m} R(LX\otimes LY).$$

It is now a lengthy formal calculation to show that the pair (m',u') defines a lax comonoidal structure on L. Similarly, if we start with a lax comonoidal structure on L given by

$$m'\colon L(X\otimes Y)\longrightarrow L(X\otimes Y) \qquad \text{and} \qquad u'\colon L\mathbb{S}\longrightarrow \mathbb{S},$$

we can consider the map $u\colon \mathbb{S}\longrightarrow R\mathbb{S}$ which is adjoint to u'. Moreover, let $m\colon RX\otimes RY\longrightarrow R(X\otimes Y)$ be the map adjoint to

$$L(RX\otimes RY)\xrightarrow{m'} LRX\otimes LRY \xrightarrow{\epsilon\otimes\epsilon} X\otimes Y.$$

This will then define a lax monoidal structure on the right adjoint R. With these preparations we can formulate the next lemma.

Lemma 1.29. *Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories and let $(L,R)\colon \mathcal{C}\rightharpoonup\mathcal{D}$ be an adjunction. The above constructions define a bijection between lax monoidal structures on R and lax comonoidal structures on L. Moreover, if (L,R) is an equivalence then we have a bijection between strong monoidal structures on L and strong monoidal structures on R.*

Proof. We have to show that the two constructions are inverse to each other. Let us consider the case where we start with a lax monoidal structure (m,u) on R. From this we can form the lax comonoidal structure (m',u') on L and again a lax monoidal structure (m'',u'') on R. It is immediate that we have $u=u''$, so it remains to show that we also have $m=m''$. By the definition of the morphisms m' and m'', we have the following commutative diagram:

$$\begin{tikzcd}
RX\otimes RY \arrow[r, "\eta"] \arrow[d, "m''"'] & RL(RX\otimes RY) \arrow[r, "\eta\otimes\eta"] \arrow[d, "m'"'] & RL(RLRX\otimes RLRY) \arrow[d, "m"] \\
R(X\otimes Y) & R(LRX\otimes LRY) \arrow[l, "\epsilon\otimes\epsilon"] & RLR(LRX\otimes LRY) \arrow[l, "\epsilon"]
\end{tikzcd}$$

Using $(\eta \otimes \eta) \circ \eta = \eta \circ (\eta \otimes \eta)$, we can deduce the following commutative diagram

$$\begin{tikzcd}
RX \otimes RY \arrow[r, "\eta\otimes\eta"] \arrow[dd, "m''"'] & RLRX \otimes RLRY \arrow[r, "\eta"] \arrow[d, "m"'] & RL(RLRX \otimes RLRY) \arrow[d, "m"] \\
 & R(LRX \otimes LRY) \arrow[r, "\eta"] \arrow[d, "="'] & RLR(LRX \otimes LRY) \arrow[d, "="] \\
R(X \otimes Y) & R(LRX \otimes LRY) \arrow[l, "\epsilon\otimes\epsilon"] & RLR(LRX \otimes LRY) \arrow[l, "\epsilon"]
\end{tikzcd}$$

where the lower right square commutes by a triangular identity. Using the triangular identity again, we can conclude by the following calculation:

$$m'' = (\epsilon \otimes \epsilon) \circ m \circ (\eta \otimes \eta) = m \circ (\epsilon \otimes \epsilon) \circ (\eta \otimes \eta) = m$$

The second statement for the case of an equivalence of monoidal categories is immediate since in that case the adjunction unit and counit are natural isomorphisms. □

Let us now recall the following definition of [SS03a].

Definition 1.30. Let $\mathcal{M}$ and $\mathcal{N}$ be monoidal model categories. A *weak monoidal Quillen adjunction* $\mathcal{M} \rightharpoonup \mathcal{N}$ is a Quillen adjunction (F, U) together with a lax monoidal structure (m, u) on the right adjoint U such that the following two properties are satisfied:
i) The natural transformation $m' \colon F \circ \otimes \longrightarrow \otimes \circ (F \times F)$ which is part of the induced lax comonoidal structure on F is a left Quillen homotopy.
ii) For any cofibrant replacement $Q\,\mathbb{S} \longrightarrow \mathbb{S}$ of the monoidal unit $\mathbb{S}$ of $\mathcal{M}$ the map $FQ\,\mathbb{S} \longrightarrow F\,\mathbb{S} \xrightarrow{u'} \mathbb{S}$ is a weak equivalence.
We call such a datum a *weak monoidal Quillen equivalence* if the underlying Quillen adjunction (F, U) is a Quillen equivalence.

In the context of combinatorial monoidal model categories one checks that weak monoidal Quillen adjunctions (resp. equivalences) can be extended to weak monoidal Quillen adjunctions (resp. equivalences) at the level of diagram categories with respect to the injective model structures.

Proposition 1.31. *Let $(F, U) \colon \mathcal{M} \rightharpoonup \mathcal{N}$ be a weak monoidal Quillen adjunction between combinatorial model categories. Then the left derived morphism $\mathbb{L}F \colon \mathbb{D}_{\mathcal{M}} \longrightarrow \mathbb{D}_{\mathcal{N}}$ carries canonically the structure of a strong monoidal morphism while $\mathbb{R}U \colon \mathbb{D}_{\mathcal{N}} \longrightarrow \mathbb{D}_{\mathcal{M}}$ is canonically lax monoidal. If (F, U) is a weak monoidal Quillen equivalence then both $\mathbb{L}F$ and $\mathbb{R}U$ carry canonically a strong monoidal structure.*

Proof. By our assumption the natural transformation $m' \colon F \circ \otimes \longrightarrow \otimes \circ (F \times F)$ is a Quillen homotopy. By the additional compatibility assumption of the induced map $u' : F\,\mathbb{S} \longrightarrow \mathbb{S}$ we can use m' and u' in order to obtain a strong comonoidal structure on $\mathbb{L}F \colon \mathbb{D}_{\mathcal{M}} \longrightarrow \mathbb{D}_{\mathcal{N}}$. Since there is an obvious bijection between strong comonoidal and strong monoidal structures, we end up with a strong monoidal structure on $\mathbb{L}F$. If (F, U) is actually a weak monoidal Quillen equivalence, we can apply a variant of Lemma 1.29 for derivators to also construct a strong monoidal structure on $\mathbb{R}U$. □

Corollary 1.32. *Let $\mathcal{M}$, $\mathcal{N}$ be combinatorial monoidal model categories which are Quillen equivalent through a zigzag of weakly monoidal Quillen equivalences between combinatorial monoidal model categories. Then we obtain a strongly monoidal equivalence of derivators* $\mathbb{D}_{\mathcal{M}} \xrightarrow{\simeq} \mathbb{D}_{\mathcal{N}}$.

As an illustration we want to apply this to the situation described in [Shi07]. In that paper, Shipley constructs a zigzag of three weak monoidal Quillen equivalences between the category of unbounded chain complexes of abelian groups and the category of $H\mathbb{Z}$-module spectra. To be more specific, the monoidal model for spectra is chosen to be the category of symmetric spectra ([HSS00]) and $H\mathbb{Z}$ denotes the integral Eilenberg-MacLane spectrum. The chain of weak monoidal Quillen equivalence passes through the following intermediate model categories

$$H\mathbb{Z} - \mathsf{Mod} \simeq_Q \mathsf{Sp}^{\mathbf{\Sigma}}(\mathsf{sAb}) \simeq_Q \mathsf{Sp}^{\mathbf{\Sigma}}(\mathsf{Ch}^+) \simeq_Q \mathsf{Ch}\,.$$

Here, Ch^+ is the category of non-negatively graded chain complexes of abelian groups, sAb is the category of simplicial abelian groups and $\mathsf{Sp}^{\mathbf{\Sigma}}(-)$ denotes Hovey's stabilization process by forming symmetric spectra internal to a sufficiently nice model category ([Hov01]). There is a similar such chain of weak monoidal Quillen equivalences if we replace the integers by an arbitrary commutative ground ring k. Since all the four model categories occurring in that chain are combinatorial we can apply the last corollary in order to obtain the following example.

Example 1.33. For a commutative ring k let us denote by Hk the symmetric Eilenberg-MacLane ring spectrum. Then we have a strong monoidal equivalence of derivators

$$\mathbb{D}_k \simeq \mathbb{D}_{Hk}\,.$$

1.5. **Additive derivators, the center of a derivator, and linear structures.** For a derivator $\mathbb{D}$ and a category J it is immediate that $\mathbb{D}(J)$ has initial and final objects as well as finite coproducts and finite products (cf. Subsection 1.1 of [Gro10a]). A pointed derivator is a derivator such that every initial object of the underlying category $\mathbb{D}(e)$ is also final. It follows then that all values $\mathbb{D}(J)$ are pointed. For additive derivators, it also suffices to impose the additivity assumption on the underlying category. For us the notion of an additive category does *not* include an enrichment in abelian groups. The additional *structure* given by the enrichment in abelian groups can be uniquely reconstructed using the exactness *properties* of an additive category. Thus, the category $\mathbb{D}(e)$ is assumed to be pointed and the canonical map from the coproduct of two objects to the product of them is to be an isomorphism. Moreover, for every object there is a self-map which 'behaves as an additive inverse of the identity'. For a precise formulation of this axiom, compare to Definition 8.2.8 of [KS06]. Alternatively, one can demand the shear map

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \colon X \sqcup X \longrightarrow X \times X$$

to be an isomorphism for each object X.

Definition 1.34. A derivator $\mathbb{D}$ is *additive* if the underlying category $\mathbb{D}(e)$ is additive.

Proposition 1.35. *If a derivator $\mathbb{D}$ is additive, then all categories $\mathbb{D}(J)$ are additive and for any functor $u\colon J \longrightarrow K$ the induced functors u^*, $u_!$, and u_* are additive.*

Proof. Let us assume $\mathbb{D}$ to be additive and let us consider an arbitrary category J. We already know that $\mathbb{D}(J)$ is pointed. Since isomorphisms in $\mathbb{D}(J)$ can be tested pointwise and since the evaluation functors have adjoints on both sides it is easy to see that finite coproducts and finite products in $\mathbb{D}(J)$ are canonically isomorphic. Similarly, let $X \in \mathbb{D}(J)$ be an arbitrary object and let us consider the shear map $\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)\colon X \sqcup X \longrightarrow X \times X$. This map is an isomorphism if and only if this is the case when evaluated at all objects $j \in J$. But $j^*\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$ can be canonically identified with the shear map of $j^*X \in \mathbb{D}(e)$ which is an isomorphism by assumption. Finally, given a functor $u\colon J \longrightarrow K$, the induced functors u^*, $u_!$, and u_* are all additive since each of them has an adjoint on at least one side. $\square$

In contrast to the above definition, let us call a prederivator additive if all values and all precomposition functors are additive.

Example 1.36. i) Let $\mathcal{C}$ be a category. Then the prederivator $\mathcal{C}$ represented by $\mathcal{C}$ is additive if and only if the category $\mathcal{C}$ is additive.
ii) Let $\mathbb{D}$ be a stable derivator. Then we showed in Section 4 of [Gro10a] that $\mathbb{D}$ is also an additive derivator. So, this is, in particular, the case for derivators associated to stable (combinatorial) model categories.

Definition 1.37. Let $\mathbb{D}$ be a prederivator. The *center* $\mathsf{Z}(\mathbb{D})$ *of* $\mathbb{D}$ is the set of natural transformations

$$\mathsf{Z}(\mathbb{D}) = \mathsf{nat}(\mathrm{id}_{\mathbb{D}}, \mathrm{id}_{\mathbb{D}}).$$

Thus, an element of $\mathsf{Z}(\mathbb{D})$ is a natural transformation $\tau\colon \mathrm{id}_{\mathbb{D}} \longrightarrow \mathrm{id}_{\mathbb{D}}$, i.e., a family of natural transformations $\tau_J\colon \mathrm{id}_{\mathbb{D}(J)} \longrightarrow \mathrm{id}_{\mathbb{D}(J)}$ which behave well with the precomposition functors u^*. The composition of natural transformations endows $\mathsf{Z}(\mathbb{D})$ with the structure of a (commutative) monoid.

Lemma 1.38. *Let $\mathbb{D}$ be an additive derivator. The center $\mathsf{Z}(\mathbb{D})$ of $\mathbb{D}$ is then a commutative ring.*

Proof. The multiplication on $\mathsf{Z}(\mathbb{D})$ is given by composition. For two elements τ, $\sigma \in \mathsf{Z}(\mathbb{D})$, a category K and an element $X \in \mathbb{D}(K)$, by naturality we have the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "(\tau_K)_X"] \arrow[d, "(\sigma_K)_X"'] & X \arrow[d, "(\sigma_K)_X"] \\ X \arrow[r, "(\tau_K)_X"'] & X \end{tikzcd}$$

Thus we have $\sigma\tau = \tau\sigma$, i.e., the multiplication is commutative. Since the precomposition functors u^* are additive, the sum $\tau + \sigma$ of two elements $\tau,\ \sigma \in \mathsf{Z}(\mathbb{D})$ lies again in the center. Finally, the biadditivity of the composition in the additive situation concludes the proof. □

This commutative ring $\mathsf{Z}(\mathbb{D})$ can be used to endow an additive derivator with $k-$linear structures as follows.

Definition 1.39. Let $\mathbb{D}$ be an additive derivator and let k be a commutative ring. A *$k-$linear structure on* $\mathbb{D}$ is a ring homomorphism

$$\sigma\colon k \longrightarrow \mathsf{Z}(\mathbb{D}).$$

A pair $(\mathbb{D},\sigma)$ consisting of an additive derivator $\mathbb{D}$ and a $k-$linear structure σ on $\mathbb{D}$ is a *$k-$linear derivator.*

As emphasized in the definition, $k-$linearity of an additive derivator is additional *structure* (contrary to the additivity of an additive derivator which is a *property*). Nevertheless, we will drop σ from notation and speak of a $k-$linear additive derivator $\mathbb{D}$. Every additive derivator is canonically endowed with a $\mathbb{Z}$-linear structure.

Now, let $\mathbb{D}$ be an additive derivator. Evaluation at a category J induces a ring homomorphism $\mathsf{Z}(\mathbb{D}) \longrightarrow \mathsf{Z}(\mathbb{D}(J))$, where $\mathsf{Z}(\mathbb{D}(J))$ denotes the usual center of the additive category $\mathbb{D}(J)$, i.e., the commutative ring of natural transformations $\mathrm{id}_{\mathbb{D}(J)} \longrightarrow \mathrm{id}_{\mathbb{D}(J)}$. Thus, a $k-$linear structure on an additive derivator induces $k-$linear structures on all its values. Moreover, these $k-$linear structures are preserved by the precomposition functors. Recall for example from [KS06] that for a morphism $f\colon X \longrightarrow Y$ in $\mathbb{D}(K)$ and a ring element $s \in k$ the morphism $sf\colon X \longrightarrow Y$ is given by the diagonal in the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "f"'] & X \arrow[d, "f"] \\ Y \arrow[r, "s"'] & Y \end{tikzcd}$$

Here, we simplified notation by writing s for $(\sigma(s)_K)_X$ resp. $(\sigma(s)_K)_Y$. Now, since $\sigma(s) \in \mathsf{Z}(\mathbb{D})$ we have an equality of natural transformations $u^*s = su^*\colon u^* \longrightarrow u^*$ for an arbitrary functor $u\colon J \longrightarrow K$. For a morphism $f\colon X \longrightarrow Y$ in $\mathbb{D}(K)$ this equality implies $su^*(f) = u^*(sf)$, i.e., the $k-$linearity of u^*. Conversely, $k-$linear structures on the values of an additive derivator such that the precomposition functors are $k-$linear give a $k-$linear structure on the additive derivator. This gives the first part of the following proposition.

Proposition 1.40. *Let $\mathbb{D}$ be an additive derivator. A k-linear structure on $\mathbb{D}$ is equivalently given by a k-linear structure on $\mathbb{D}(J)$ for each category J such that the precomposition functors are k-linear. Moreover, in that case also the homotopy Kan extension functors $u_!, u_*\colon \mathbb{D}(J) \longrightarrow \mathbb{D}(K)$ associated to an arbitrary functor $u\colon J \longrightarrow K$ are $k-$linear.*

Proof. It remains to give a proof of the second statement and, by duality, it suffices to treat the case of homotopy left Kan extensions. Let X, Y be objects of $\mathbb{D}(J)$ and let $s \in k$. Let us consider the following commutative diagram in which the horizontal isomorphisms are the adjunction isomorphisms:

$$\begin{tikzcd}
\mathsf{hom}_{\mathbb{D}(K)}(u_!X, u_!Y) \arrow[r, "\cong"] \arrow[d, "s_*"'] & \mathsf{hom}_{\mathbb{D}(J)}(X, u^*u_!Y) \arrow[d, "(u^*(s))_*"] \\
\mathsf{hom}_{\mathbb{D}(K)}(u_!X, u_!Y) & \mathsf{hom}_{\mathbb{D}(J)}(X, u^*u_!Y) \arrow[l, "\cong"]
\end{tikzcd}$$

The vertical map on the left sends $u_!(f)\colon u_!X \longrightarrow u_!Y$ to $su_!(f)$. So let us calculate the image of $u_!(f)$ under the composition of the three maps. Let us remark first that $(u^*(s))_* = s_*$ since u^* is $k-$linear. Thus, the image of $u_!(f)$ under the composition of the first two maps is the composition of the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "\eta"] \arrow[d, "s"'] & u^*u_!X \arrow[r, "f"] \arrow[d, "s"'] & u^*u_!Y \arrow[d, "s"] \\
X \arrow[r, "\eta"'] & u^*u_!X \arrow[r, "f"'] & u^*u_!Y
\end{tikzcd}$$

But, using the triangular identities, this composition is sent by the second adjunction isomorphism to $u_!(f)u_!(s) = u_!(sf)$. Hence, we obtain the intended relation $su_!(f) = u_!(sf)$ expressing the $k-$linearity of $u_!$. $\square$

We finish by giving the notion of $k-$linear morphisms of $k-$linear derivators. Let us note that an additive morphism $F\colon \mathbb{D} \longrightarrow \mathbb{D}'$ of additive derivators induces ring maps $F_*\colon \mathsf{Z}(\mathbb{D}) \longrightarrow \mathsf{nat}(F,F)$ and $F^*\colon \mathsf{Z}(\mathbb{D}') \longrightarrow \mathsf{nat}(F,F)$.

Definition 1.41. Let $\mathbb{D}$ and $\mathbb{D}'$ be $k-$linear derivators with respective $k-$linear structures σ and σ'. An additive morphism $F\colon \mathbb{D} \longrightarrow \mathbb{D}'$ is *$k-$linear* if $F_* \circ \sigma = F^* \circ \sigma'\colon k \longrightarrow \mathsf{nat}(F,F)$. With all natural transformations as 2-morphisms we thus obtain the 2-category $\mathsf{Der}^{\mathsf{add},k}$ of $k-$linear derivators.

In particular, we have $\mathsf{Der}^{\mathsf{add},\mathbb{Z}} = \mathsf{Der}^{\mathsf{add}}$. It is easy to see that an additive morphism $F\colon \mathbb{D} \longrightarrow \mathbb{D}'$ of $k-$linear derivators is $k-$linear if and only if all components $F_K\colon \mathbb{D}(K) \longrightarrow \mathbb{D}'(K)$ are $k-$linear functors. Thus we obtain the following example – more specific examples of linear structures will be given at the end of this subsection.

Example 1.42. Let $\mathbb{D}$ and $\mathbb{D}'$ be additive derivators. Then a $\mathbb{Z}-$linear morphism $F\colon \mathbb{D} \longrightarrow \mathbb{D}'$ is the same as a coproduct-preserving morphism. In particular, all exact morphisms between stable derivators are $\mathbb{Z}$-linear. This is, for example, the case for all morphisms $u^*\colon \mathbb{D}_K \longrightarrow \mathbb{D}_J$ induced by the precomposition functors of a stable derivator $\mathbb{D}$. Recall that $\mathbb{D}_J$ is the derivator which sends a category L to $\mathbb{D}(J \times L)$.

In the case of a stable derivator $\mathbb{D}$ there is the following graded variant of the center. Recall from Section 4 of [Gro10a] that the suspension functor $\Sigma\colon \mathbb{D}(J) \longrightarrow \mathbb{D}(J)$ is defined as the following composition:

$$\Sigma\colon\ \mathbb{D}(J) \xrightarrow{(0,0)_*} \mathbb{D}(J \times \ulcorner) \xrightarrow{\mathrm{i}_{\ulcorner !}} \mathbb{D}(J \times \square) \xrightarrow{(1,1)^*} \mathbb{D}(J)$$

Since the morphisms of derivators $u^*\colon \mathbb{D}_K \longrightarrow \mathbb{D}_J$ preserve homotopy left and homotopy right Kan extensions, the above suspension functors can be taken together to define a self-equivalence $\Sigma\colon \mathbb{D} \longrightarrow \mathbb{D}$ of the derivator, the *suspension morphism*. More precisely, we use Lemma 2.9 twice which states that the homotopy Kan extensions at the different levels assemble into a morphism of derivators.

Let us consider the values of a stable derivator as graded categories in the following way. For a category J and two objects $X,Y \in \mathbb{D}(J)$, the graded abelian groups $\mathsf{hom}_{\mathbb{D}(J)}(X,Y)_\bullet$ and $\mathsf{hom}_{\mathbb{D}(J)}(X,Y)^\bullet$ are defined to be

$$\mathsf{hom}_{\mathbb{D}(J)}(X,Y)_n = \mathsf{hom}_{\mathbb{D}(J)}(X,Y)^{-n} = \mathsf{hom}_{\mathbb{D}(J)}(\Sigma^n X,Y), \qquad n \in \mathbb{Z}.$$

Here, we used that the suspension is invertible in the stable situation in order to define the $\mathbb{Z}-$graded abelian groups.

Example 1.43. For a commutative ring k and k-modules M and N we have the following identification:

$$\mathsf{hom}_{\mathbb{D}_k(e)}(\Sigma^{-n}M,N) = \mathsf{Ext}^n_k(M,N).$$

For a functor $u\colon J \longrightarrow K$ the induced functors u^*, $u_!$, and u_* are graded since they are exact with respect to the canonical triangulated structures [Gro10a]. Let us now come to a graded-commutative variant of the center for stable derivators.

Definition 1.44. Let $\mathbb{D}$ be a stable derivator and let $\Sigma\colon \mathbb{D} \longrightarrow \mathbb{D}$ be the suspension morphism. Then the *graded center* $\mathsf{Z}_\bullet(\mathbb{D})$ *of* $\mathbb{D}$ is the $\mathbb{Z}-$graded abelian group which in degree n is the subgroup $\mathsf{Z}_n(\mathbb{D}) = \mathsf{Z}^{-n}(\mathbb{D})$ of $\mathsf{nat}(\Sigma^n, \mathrm{id}_{\mathbb{D}})$ given by the natural transformations τ that commute with the suspension up to a sign, i.e., satisfy $\Sigma\tau = (-1)^n \tau\Sigma\colon \Sigma^{n+1} \longrightarrow \Sigma$.

It is immediate to see that the composition of elements of the center endows $\mathsf{Z}_\bullet(\mathbb{D})$ with the structure of a graded-commutative ring. Similarly to the unstable case, we can now talk about graded-linear structures. A *graded-linear structure on a stable derivator* is a map $\sigma\colon R_\bullet \longrightarrow \mathsf{Z}_\bullet(\mathbb{D})$ of graded rings. Similarly to the ungraded case, it follows that also the homotopy Kan extensions are linear over $R_\bullet$.

Lemma 1.45. *Let $\mathbb{D}$ be a stable derivator endowed with a linear structure over the graded ring $R_\bullet$ and let $u\colon J \longrightarrow K$ be a functor. The graded category $\mathbb{D}(K)$ is then canonically $R_\bullet$-linear. Moreover, the induced graded functors $u^*\colon \mathbb{D}(K) \longrightarrow \mathbb{D}(J)$ and $u_!,\ u_*\colon \mathbb{D}(J) \longrightarrow \mathbb{D}(K)$ are linear over $R_\bullet$.*

Let us now turn towards the linear structures which are canonically available for suitable additive, monoidal derivators. The 2-categorical Yoneda lemma gives us for every monoidal prederivator $\mathbb{D}$ the strict morphism $\kappa_{\mathbb{S}_e}\colon e \longrightarrow \mathbb{D}$ corresponding to the monoidal unit $\mathbb{S}_e$ of the underlying monoidal category $\mathbb{D}(e)$.

Lemma 1.46. *Let $\mathbb{D}$ be a monoidal prederivator. Then the unit morphism $\mathbb{S}\colon e \longrightarrow \mathbb{D}$ and the strict morphism $\kappa_{\mathbb{S}_e}\colon e \longrightarrow \mathbb{D}$ are naturally isomorphic.*

Proof. Recall from the proof of the 2-Yoneda lemma that the value of $\kappa_{\mathbb{S}_e}$ at a category K is just the element $p_K^*(\mathbb{S}_e)$ where $p_K\colon K \longrightarrow e$ is the unique functor to the terminal category

e. Moreover, for a functor $u\colon J \longrightarrow K$ the induced functors $u^*\colon \mathbb{D}(K) \longrightarrow \mathbb{D}(J)$ are canonically monoidal functors. In particular, there is a canonical isomorphism $u^*(\mathbb{S}_K) \longrightarrow \mathbb{S}_J$ which is, as we saw in the first subsection, the structure isomorphism γ_u belonging to the morphism of prederivators $\mathbb{S}\colon e \longrightarrow \mathbb{D}$. Applied to the canonical functor p_K, this gives us an isomorphism

$$\tau_K = \gamma_{p_K}\colon (\kappa_{\mathbb{S}_e})_K = p_K^*(\mathbb{S}_e) \longrightarrow \mathbb{S}_K\,.$$

These τ_K assemble to a natural isomorphism $\tau\colon \kappa_{\mathbb{S}_e} \longrightarrow \mathbb{S}$. In fact, we just have to check that the following diagram commutes:

$$\begin{tikzcd}
u^*p_K^*(\mathbb{S}_e) \arrow[r, "u^*\tau_K=u^*\gamma_{p_K}"] \arrow[d, equal] & u^*\,\mathbb{S}_K \arrow[d, "\gamma_u"] \\
p_J^*(\mathbb{S}_e) \arrow[r, "\tau_J u^*=\gamma_{p_J}u^*"'] & \mathbb{S}_J
\end{tikzcd}$$

But this is just a special case of the coherence properties of the isomorphisms belonging to the morphisms of prederivators $\mathbb{S}\colon e \longrightarrow \mathbb{D}$. □

We can also give a more conceptual proof of this lemma. For this purpose, let us recall the *bicategorical* Yoneda lemma. For a general introduction to the theory of bicategories cf. [Bén67]. Although we are only concerned with 2-categories, let us quickly mention that the basic idea with bicategories is that one wants to relax the notion of 2-categories in the sense that one only asks for a composition law which is unital and associative up to specified natural coherent isomorphisms. Given two 2-categories $\mathcal{C}$ and $\mathcal{D}$ and two parallel 2-functors $F, G\colon \mathcal{C} \longrightarrow \mathcal{D}$ one can now consider the category $\mathsf{PsNat}(F,G)$ of pseudo-natural transformations where the morphisms are given by the modifications. As a special case, let us take $\mathcal{D} = \mathsf{CAT}$, let us fix an object $X \in \mathcal{C}$ and let us consider the corepresented 2-functor $y(X) = \mathsf{Hom}(X,-) : \mathcal{C} \longrightarrow \mathsf{CAT}$. If we are given in addition a CAT-valued 2-functor $F\colon \mathcal{C} \longrightarrow \mathsf{CAT}$ then we can consider the category $\mathsf{PsNat}(y(X), F)$. The bicategorical Yoneda lemma states that the evaluation at the identity of X induces a natural *equivalence of categories*:

$$Y\colon\ \mathsf{PsNat}(y(X), F) \xrightarrow{\simeq} F(X)$$

The bicategorical Yoneda lemma in the more general situation of homomorphisms of bicategories can be found in [Str80].

Now, given two prederivators $\mathbb{D}$ and $\mathbb{D}'$ the category of morphisms from $\mathbb{D}$ to $\mathbb{D}'$ is given by $\mathsf{Hom}(\mathbb{D},\mathbb{D}') = \mathsf{PsNat}(\mathbb{D},\mathbb{D}')$. In the situation of the last lemma, the bicategorical Yoneda lemma hence gives us an equivalence of categories

$$Y\colon\ \mathsf{Hom}(e,\mathbb{D}) = \mathsf{Hom}(y(e),\mathbb{D}) \xrightarrow{\simeq} \mathbb{D}(e).$$

Both morphisms $\mathbb{S}$ and $\kappa_{\mathbb{S}_e}$ are mapped to $\mathbb{S}_e$ under Y showing that they must be isomorphic.

Let now $\mathbb{D}$ be a monoidal, additive derivator and let us assume that the monoidal structure preserves coproducts. The natural isomorphism of the last lemma induces a ring map

$$\mathsf{hom}_{\mathbb{D}(e)}(\mathbb{S}_e, \mathbb{S}_e) \longrightarrow \mathsf{nat}(\kappa_{\mathbb{S}_e} \otimes -, \kappa_{\mathbb{S}_e} \otimes -) \xrightarrow{\cong} \mathsf{nat}(\mathbb{S} \otimes -, \mathbb{S} \otimes -).$$

A final conjugation with the coherence isomorphism $l\colon \mathbb{S} \otimes - \cong \mathrm{id}$ thus gives us a ring map

$$\mathsf{hom}_{\mathbb{D}(e)}(\mathbb{S}_e, \mathbb{S}_e) \longrightarrow \mathsf{Z}(\mathbb{D}),$$

i.e., the derivator $\mathbb{D}$ is endowed with a linear structure over the endomorphisms of $\mathbb{S}$. Thus, we have proved the following result.

Corollary 1.47. *Let $\mathbb{D}$ be an additive, monoidal derivator with an additive monoidal structure. Then $\mathbb{D}$ is canonically endowed with a linear structure over $\mathsf{hom}_{\mathbb{D}(e)}(\mathbb{S}_e, \mathbb{S}_e)$. In particular, let $\mathcal{M}$ be a combinatorial, closed monoidal model category with unit object $\mathbb{S}$ such that the associated derivator is additive. The derivator $\mathbb{D}_{\mathcal{M}}$ is then canonically endowed with a linear structure over $\mathsf{hom}_{\mathsf{Ho}(\mathcal{M})}(\mathbb{S}, \mathbb{S})$.*

Note that there is a certain asymmetry in the construction of the linear structures. We only used the coherence isomorphism $\mathbb{S} \otimes - \cong \mathrm{id}$. As a consequence, a similar result concerning the existence of linear structures can also be established for suitably left-tensored derivators (cf. Section 2). Furthermore, there is a graded variant of this result for stable, monoidal derivators if the monoidal structure has certain exactness properties. But before we come to that we want to mention that the existence of these linear structures is only the shadow of a much more structured result. In the next section we will see that a derivator has an associated derivator of endomorphisms denoted $\mathsf{END}(\mathbb{D})$. Using Theorem B.11 of Appendix B we deduce that $\mathsf{END}(\mathbb{D})$ is canonically monoidal and that associated to a monoidal derivator $\mathbb{D}$ there is a monoidal morphism $\mathbb{D} \longrightarrow \mathsf{END}(\mathbb{D})$. In the additive context, the ring map of the last result is just a shadow of this monoidal morphism. We will prove such a result in the more general context of tensored derivators in Section 2.

Recall, e.g. from [HPS97, Definition A.2.1] and [May01, Section 4], that there are notions of when a closed monoidal structure on a triangulated category is *compatible* with the triangulation. In the context of stable derivators the 'triangulation' is not an additional structure (cf. [Gro10a]) but we nevertheless want to introduce a similar notion here. Before we introduce it, let us assume we were given a stable, monoidal derivator $\mathbb{D}$ such that the monoidal structure $\otimes$ commutes with the suspension in both variables. Then, for arbitrary s, t we can consider the following possibly non-commutative diagram in which the left vertical arrow is given by the symmetry constraint:

$$\begin{tikzcd}
\Sigma^s \mathbb{S} \otimes \Sigma^t \mathbb{S} \arrow[r, "\cong"] \arrow[d, "\cong"'] & \Sigma^{s+t} \mathbb{S} \arrow[d, "(-1)^{st}"] \\
\Sigma^t \mathbb{S} \otimes \Sigma^s \mathbb{S} \arrow[r, "\cong"'] & \Sigma^{t+s} \mathbb{S}
\end{tikzcd}$$

Definition 1.48. A derivator $\mathbb{D}$ is *compatibly stable and closed monoidal* if $\mathbb{D}$ is stable, closed monoidal, and if the above diagram commutes for all r and s.

We could have given the same definition in the more general context of a stable derivator with a monoidal structure which commutes with the suspension and is additive in both variables. However, to be closer to the situations as considered in [HPS97, May01] we assumed the derivator to be closed monoidal which is anyhow fulfilled by all examples we are considering here.

Let $\mathbb{D}$ be a compatibly stable and closed monoidal derivator and let $\mathbb{S}_e$ be the monoidal unit of the underlying monoidal category $\mathbb{D}(e)$. It follows that the graded abelian group of self-maps $\mathsf{hom}_{\mathbb{D}(e)}(\mathbb{S}_e, \mathbb{S}_e)_\bullet$ is a graded-commutative ring. In fact, as a special case of the composition in the graded category $\mathbb{D}(e)$, the composition of $g\colon \Sigma^n \mathbb{S}_e \longrightarrow \mathbb{S}_e$ and $f\colon \Sigma^m \mathbb{S}_e \longrightarrow \mathbb{S}_e$ is given by:

$$g \circ f\colon \quad \Sigma^{n+m} \mathbb{S}_e \xrightarrow{\Sigma^n f} \Sigma^n \mathbb{S}_e \xrightarrow{g} \mathbb{S}_e$$

The graded-commutativity of this composition follows now from the following diagram which uses the fact that we are given a *compatibly* stable and closed monoidal derivator:

$$\begin{tikzcd}
\Sigma^m \mathbb{S}_e \arrow[ddd, bend right, "f"'] & \Sigma^{m+n} \mathbb{S}_e \arrow[l, "\Sigma^m g"'] \arrow[r, "(-1)^{mn}"] & \Sigma^{n+m} \mathbb{S}_e \arrow[r, "\Sigma^n f"] & \Sigma^n \mathbb{S}_e \arrow[ddd, bend left, "g"] \\
\Sigma^m \mathbb{S}_e \otimes \mathbb{S}_e \arrow[u, "\cong"'] \arrow[dddr, bend right, "f \otimes \mathrm{id}"] & \Sigma^m \mathbb{S}_e \otimes \Sigma^n \mathbb{S}_e \arrow[l, "\mathrm{id} \otimes g"] \arrow[u, "\cong"] \arrow[r, "\cong"'] \arrow[d, "f \otimes \mathrm{id}"'] & \Sigma^n \mathbb{S}_e \otimes \Sigma^m \mathbb{S}_e \arrow[u, "\cong"'] \arrow[r, "\mathrm{id} \otimes f"'] \arrow[d, "g \otimes \mathrm{id}"] & \Sigma^n \mathbb{S}_e \otimes \mathbb{S}_e \arrow[u, "\cong"] \arrow[dddl, bend left, "g \otimes \mathrm{id}"'] \\
 & \mathbb{S}_e \otimes \Sigma^n \mathbb{S}_e \arrow[d, "\mathrm{id} \otimes g"'] & \mathbb{S}_e \otimes \Sigma^m \mathbb{S}_e \arrow[d, "\mathrm{id} \otimes f"] & \\
\mathbb{S}_e \arrow[r, "\cong"] & \mathbb{S}_e \otimes \mathbb{S}_e \arrow[r, "\cong"] & \mathbb{S}_e \otimes \mathbb{S}_e & \mathbb{S}_e \arrow[l, "\cong"']
\end{tikzcd}$$

Here, the composition of the bottom line just gives $\mathrm{id}_{\mathbb{S}_e}$ by one of the coherence axioms for a symmetric monoidal category.

Proposition 1.49. *A compatibly stable and closed monoidal derivator $\mathbb{D}$ is canonically endowed with a linear structure over the graded-commutative ring $\mathsf{hom}_{\mathbb{D}(e)}(\mathbb{S}_e, \mathbb{S}_e)_\bullet$, i.e., we have a morphism of graded rings*

$$\mathsf{hom}_{\mathbb{D}(e)}(\mathbb{S}_e, \mathbb{S}_e)_\bullet \longrightarrow \mathsf{Z}_\bullet(\mathbb{D}).$$

Proof. We only give a sketch of the proof. Using the same notation as in the unstable case, we obtain a map $\mathsf{hom}_{\mathbb{D}(e)}(\mathbb{S}_e, \mathbb{S}_e)_n = \mathsf{hom}_{\mathbb{D}(e)}(\Sigma^n \mathbb{S}_e, \mathbb{S}_e) \longrightarrow \mathsf{nat}(\kappa_{\Sigma^n \mathbb{S}_e} \otimes -, \kappa_{\mathbb{S}_e} \otimes -)$ which can be composed with the following chain of identifications:

$$\begin{aligned}
\mathsf{nat}(\kappa_{\Sigma^n \mathbb{S}_e} \otimes -, \kappa_{\mathbb{S}_e} \otimes -) \quad &\cong \quad \mathsf{nat}(\Sigma^n \circ (\kappa_{\mathbb{S}_e} \otimes -), \kappa_{\mathbb{S}_e} \otimes -) \\
&\cong \quad \mathsf{nat}(\Sigma^n \circ (\mathbb{S} \otimes -), \mathbb{S} \otimes -) \\
&\cong \quad \mathsf{nat}(\Sigma^n, \mathrm{id}_{\mathbb{D}}) \\
&= \quad \mathsf{Z}_n(\mathbb{D})
\end{aligned}$$

These assemble together to define the intended map of graded rings $\mathsf{hom}_{\mathbb{D}(e)}(\mathbb{S}_e, \mathbb{S}_e)_\bullet \longrightarrow \mathsf{Z}_\bullet(\mathbb{D})$. □

This can be applied to interesting derivators which are associated to certain combinatorial, stable, monoidal model categories. We take up again two of the examples of Subsection 1.4 which give rise to compatibly stable and closed monoidal derivators. In both contexts, the differential-graded and the spectral one, it is well-known that they satisfy the compatibility condition.

Example 1.50. Let us consider the projective model structure on the category $\mathsf{Ch}(k)$ of chain complexes over k. The ring of endomorphisms of the monoidal unit $k[0]$ in the homotopy category, i.e., in the derived category $D(k)$ of the ring k, is just the ground ring, i.e., we have $\mathsf{hom}_{D(k)}(k[0],k[0]) \cong k$. Thus, the derivator $\mathbb{D}_k$ is canonically endowed with a k-linear structure. Furthermore, the projective model structure on unbounded chain complexes is a stable model structure so that we obtain even a linear structure over a graded ring by the last corollary. But, since the graded ring of endomorphisms $\mathsf{hom}_{D(k)}(k[0],k[0])_\bullet$ is concentrated in degree zero, we gain no additional structure by considering the graded ring map

$$\mathsf{hom}_{D(k)}(k[0],k[0])_\bullet \longrightarrow \mathsf{Z}_\bullet(\mathbb{D}_k).$$

But, if we consider a commutative differential-graded algebra C over k we have the associated closed monoidal derivator $\mathbb{D}_C$ of $C-$modules. The monoidal unit in this case is C itself and the ring of graded self-maps in $\mathbb{D}_C(e) = \mathsf{Ho}(\mathsf{Mod}-C)$ is canonically isomorphic to the homology $H_\bullet(C)$. Thus, $\mathbb{D}_C$ is endowed with a linear structure over the graded ring $H_\bullet(C)$ via a map of graded rings

$$H_\bullet(C) \longrightarrow \mathsf{Z}_\bullet(\mathbb{D}_C).$$

Example 1.51. Let us consider the absolute projective stable model structure on the category Sp^Σ. The endomorphisms of the sphere spectrum in the homotopy category, i.e., in *the stable homotopy category* SHC, are the integers, i.e., we have $\mathsf{hom}_{\mathsf{SHC}}(\mathbb{S},\mathbb{S}) \cong \mathbb{Z}$. Thus, the derivator $\mathbb{D}_{\mathsf{Sp}}$ is endowed with a $\mathbb{Z}-$linear structure what we already knew since $\mathbb{D}_{\mathsf{Sp}}$ is stable. But there is even more structure in this case: the graded self-maps of the sphere spectrum in SHC form the graded ring $\pi_\bullet^S$ given by the stable homotopy groups of spheres. Thus, the derivator $\mathbb{D}_{\mathsf{Sp}}$ is endowed with a $\pi_\bullet^S-$linear structure, i.e., we have a map of graded rings

$$\pi_\bullet^S \longrightarrow \mathsf{Z}_\bullet(\mathbb{D}_{\mathsf{Sp}}).$$

In particular, all categories $\mathbb{D}_{\mathsf{Sp}}(K)$ are $\pi_\bullet^S-$linear categories and all induced functors u^*, $u_!$, and u_* preserve these linear structures. Similarly, if E is a commutative ring spectrum, then the derivator $\mathbb{D}_E$ of right $E-$module spectra is canonically endowed with a linear structure over the graded ring of self-maps of E in the homotopy category $\mathsf{Ho}(\mathsf{Mod}-E)$. Thus, we obtain a canonical morphism of graded rings

$$\pi_\bullet(E) \longrightarrow \mathsf{Z}_\bullet(\mathbb{D}_E)$$

where $\pi_\bullet(E)$ denotes the graded-commutative ring of homotopy groups of E.

2. Derivators tensored or cotensored over a monoidal derivator

2.1. **The 2-Grothendieck fibration of tensored categories.** Let us motivate the construction of this subsection by an analogy with algebra. For a ring R, we denote by $R-\mathsf{Mod}$ the category of left R-modules. Moreover, given a ring homomorphism $f\colon R \longrightarrow S$ we denote the associated restriction of scalar functor by $f^*\colon S-\mathsf{Mod} \longrightarrow R-\mathsf{Mod}$. Since restricting scalars is functorial we can use the two assignments $R \longmapsto R-\mathsf{Mod}$ and $f \longmapsto f^*$ to obtain a functor

$$(-)-\mathsf{Mod}\colon\ \mathsf{Ring}^{\mathrm{op}} \longrightarrow \mathsf{CAT}\,.$$

In Appendix A, we recall that in the context of a category-valued functor there is the so-called Grothendieck construction [Bor94b, Vis05]: it turns such a functor into a Grothendieck fibration or a Grothendieck opfibration depending on its variance over the domain of the original functor. The basic idea behind this construction is to glue the different values together in order to obtain a single category which memorizes for each object that it lived in the image category of a certain object. Applied to our situation of the functor $\mathsf{Ring}^{\mathrm{op}} \longrightarrow \mathsf{CAT}$, the Grothendieck construction gives us the category Mod of modules. An object in this category is a pair (R,M) consisting of a ring R and an R-module M. A morphism $(R,M) \longrightarrow (S,N)$ is a pair (f,h) consisting of a map of rings $f\colon R \longrightarrow S$ and a map of R-modules $h\colon M \longrightarrow f^*N$. This category is endowed with the Grothendieck fibration

$$p\colon\ \mathsf{Mod} \longrightarrow \mathsf{Ring}$$

which projects an object (R,M) resp. a morphism (f,h) onto the first component. There is a further canonical functor associated to Mod, namely the functor $U\colon\ \mathsf{Mod} \longrightarrow \mathsf{Ab}$ which sends a module to the underlying abelian group. Using the restriction of scalars, this functor sends (R,M) to c_R^*M where $c_R\colon \mathbb{Z} \longrightarrow R$ is the characteristic of the ring. We thus have the following diagram in which the vertical arrow is a Grothendieck fibration:

$$\begin{tikzcd} \mathsf{Mod} \arrow[r, "U"] \arrow[d, "p"'] & \mathsf{Ab} \\ \mathsf{Ring} & \end{tikzcd}$$

For a ring R the fiber of p over R, i.e., the left pullback below, is canonically isomorphic to the category $R-\mathsf{Mod}$. Given an abelian group A, we can consider the category $\mathsf{Mod}(A) = U^{-1}(A)$ of *module structures on* A which is defined to be the pullback on the right-hand-side:

$$\begin{tikzcd} R-\mathsf{Mod} \arrow[r] \arrow[d] & \mathsf{Mod} \arrow[d, "p"] \\ e \arrow[r, "R"'] & \mathsf{Ring} \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Mod}(A) \arrow[r] \arrow[d] & \mathsf{Mod} \arrow[d, "U"] \\ e \arrow[r, "A"'] & \mathsf{Ab} \end{tikzcd}$$

The universal example of a ring acting on A is given by the ring $\mathsf{end}(A) = \mathsf{hom}_{\mathbb{Z}}(A,A)$ of $\mathbb{Z}$-linear endomorphisms together with the action by evaluation. The universal property of this action is precisely the fact that the pair consisting of the ring $\mathsf{end}(A)$ and this action is a terminal object of the category $\mathsf{Mod}(A)$.

In Subsection 2.3, we want to redo the same reasoning where we replace the closed monoidal category of abelian groups by the Cartesian closed monoidal 2-category Der of derivators. In particular, given a derivator $\mathbb{D}$ we want to construct a derivator $\mathsf{END}(\mathbb{D})$ of endomorphisms and show that it satisfies the 2-categorical version of this universal property.

Recall that we have defined a monoidal derivator $\mathbb{E}$ as a monoidal object in Der. We have then observed that such an $\mathbb{E}$ is, in particular, a 2-functor $\mathbb{E}\colon \mathsf{Cat}^{\mathrm{op}} \longrightarrow \mathsf{CAT}$ which factors over the 2-category MonCAT of monoidal categories. To have a similar 'pointwise description' for derivators which are tensored over a monoidal derivator, it is convenient to consider the 2-category ModCAT of tensored categories. This 2-category comes up naturally as a 2-categorical Grothendieck construction as we describe it now. In fact, this is just a special case of results from Appendix B applied to the Cartesian monoidal 2-category CAT. Since the details for the case of an arbitrary monoidal 2-category are given in that appendix we allow ourselves to be sketchy here.

Let us recall that a monoidal category $\mathcal{C}$ is just a monoidal object in the Cartesian monoidal 2-category CAT. Associated to such a monoidal category $\mathcal{C}$ there is the 2-category $\mathcal{C}-\mathsf{Mod}^{\mathsf{lax}}$ of categories which are left $\mathcal{C}$-modules. An object of this 2-category is a category $\mathcal{D}$ endowed with a left action $\otimes\colon \mathcal{C}\times\mathcal{D} \longrightarrow \mathcal{D}$ and certain specified coherence isomorphisms expressing adequate multiplicativity and unitality conditions. There are the notions of lax $\mathcal{C}-$module morphisms and $\mathcal{C}-$module transformations between two such so that we indeed obtain a 2-category $\mathcal{C} - \mathsf{Mod}^{\mathsf{lax}}$.

Moreover, given a monoidal functor $f\colon \mathcal{C}_1 \longrightarrow \mathcal{C}_2$, we obtain an induced restriction of scalars functor $f^*\colon \mathcal{C}_2 - \mathsf{Mod}^{\mathsf{lax}} \longrightarrow \mathcal{C}_1 - \mathsf{Mod}^{\mathsf{lax}}$ and similar observations can be made for monoidal transformations between monoidal functors. Thus, summarizing these associations, we obtain a 2-functor

$$(-) - \mathsf{Mod}^{\mathsf{lax}}\colon\ \mathsf{MonCAT}^{\mathrm{op}} \longrightarrow \mathsf{2\text{-}CAT}\,.$$

Here, $\mathsf{2\text{-}CAT}$ denotes the 2-category of large 2-categories. An application of the 2-categorical Grothendieck construction (cf. Appendix A) gives us the *2-Grothendieck fibration of tensored categories* $p\colon \mathsf{ModCAT}^{\mathsf{lax}} \longrightarrow \mathsf{MonCAT}$. Here, $\mathsf{ModCAT}^{\mathsf{lax}}$ is the 2-category where the objects are pairs $(\mathcal{C},\mathcal{D})$ consisting of a monoidal category $\mathcal{C}$ and a category $\mathcal{D}$ which is left-tensored over $\mathcal{C}$. A morphism $(\mathcal{C}_1,\mathcal{D}_1) \longrightarrow (\mathcal{C}_2,\mathcal{D}_2)$ is a pair (f,h) where $f\colon \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ is a monoidal functor and $h\colon \mathcal{D}_1 \longrightarrow f^*\mathcal{D}_2$ is a lax morphism of $\mathcal{C}_1$-modules. We will not make the 2-morphisms explicit here since this is done in more generality in Appendix A. Let us form the 2-subcategory $\mathsf{ModCAT} \subseteq \mathsf{ModCAT}^{\mathsf{lax}}$ given by all objects, the strong module morphisms and all 2-cells. Thus, the morphisms are expected to be multiplicative up to specified natural isomorphism. The projection on the second component defines a 2-functor $U\colon \mathsf{ModCAT} \longrightarrow \mathsf{CAT}$ which will be used in the next subsection to express the universal property of the (pre)derivator of endomorphisms. Thus, as an upshot we obtain the following diagram of 2-categories in which the vertical arrow is again called the

2-Grothendieck fibration of tensored categories:

$$\begin{tikzcd} \mathsf{ModCAT} \arrow[r, "U"] \arrow[d, "p"'] & \mathsf{CAT} \\ \mathsf{MonCAT} & \end{tikzcd}$$

2.2. **Tensors and cotensors on derivators.** In this subsection, we want to formalize actions of monoidal derivators on other derivators. Recall that a monoidal derivator is just a monoidal object in the Cartesian 2-category Der. As it is the case for every monoidal 2-category, there is thus the derived notion of a module over a monoidal derivator. Since the coherence conditions are harder to find in the literature we include them in Appendix B. In contrast, we allow ourselves to be a bit sketchy here in setting up the 2-category of left module derivators over a monoidal derivator.

So, let $(\mathbb{E},\otimes,\mathbb{S})$ be a monoidal derivator and let $\mathbb{D}$ be a derivator. A *(left)* $\mathbb{E}$ *−module structure on* $\mathbb{D}$ is a triple $(\otimes, m, u)$ consisting of a morphism of derivators $\otimes\colon\ \mathbb{E}\times\mathbb{D}\longrightarrow\mathbb{D}$ together with natural isomorphisms m and u as indicated in the following diagrams:

$$\begin{tikzcd} \mathbb{E}\times\mathbb{E}\times\mathbb{D} \arrow[r, "\mathrm{id}\times\otimes"] \arrow[d, "\otimes\times\mathrm{id}"'] \arrow[dr, phantom, "\Swarrow"] & \mathbb{E}\times\mathbb{D} \arrow[d, "\otimes"] \\ \mathbb{E}\times\mathbb{D} \arrow[r, "\otimes"'] & \mathbb{D} \end{tikzcd} \qquad \begin{tikzcd} e\times\mathbb{D} \arrow[r, "\mathbb{S}"] \arrow[dr, bend right, "\cong"'] \arrow[dr, phantom, "\Nearrow"] & \mathbb{E}\times\mathbb{D} \arrow[d, "\otimes"] \\ & \mathbb{D} \end{tikzcd}$$

These natural isomorphisms expressing the multiplicativity and unitality of the action are subject to certain coherence axioms. Given two $\mathbb{E}$ −modules $(\mathbb{D}_i,\otimes,m_i,u_i),\ i=1,2$, a *lax* $\mathbb{E}$ *−module morphism* $\mathbb{D}_1\longrightarrow\mathbb{D}_2$ is a morphism $F\colon\ \mathbb{D}_1\longrightarrow\mathbb{D}_2$ of derivators together with a natural transformation m_F

$$\begin{tikzcd} \mathbb{E}\times\mathbb{D}_1 \arrow[r, "\otimes"] \arrow[d, "F"'] \arrow[dr, phantom, "\Nearrow"] & \mathbb{D}_1 \arrow[d, "F"] \\ \mathbb{E}\times\mathbb{D}_2 \arrow[r, "\otimes"'] & \mathbb{D}_2 \end{tikzcd}$$

which again have to satisfy certain coherence conditions. If the 2-cell m_F belonging to such a morphism is invertible we speak of a *strong morphism* or simply of a *morphism of* $\mathbb{E}$*-modules*. Finally, given two $\mathbb{E}$ −module morphisms (F,m_F) and (G,m_G), a natural transformation $\phi\colon\ F\longrightarrow G$ is an $\mathbb{E}$ *−module transformation* if the following diagram commutes:

$$\begin{tikzcd} -\otimes F(-) \arrow[r, "\phi"] \arrow[d, "m_F"'] & -\otimes G(-) \arrow[d, "m_G"] \\ F(-\otimes -) \arrow[r, "\phi"'] & G(-\otimes -) \end{tikzcd}$$

With these notions, we obtain the 2-category $\mathbb{E}-\mathsf{Mod}^{\mathsf{lax}}$ of $\mathbb{E}$-modules, lax $\mathbb{E}$-module morphisms and $\mathbb{E}$ −module transformations. There is a similar 2-category $\mathbb{E}-\mathsf{Mod}$ if one only takes the strong $\mathbb{E}$-module morphisms. Moreover, there are different flavors of actions like exact actions in the stable case, colimit-preserving actions and so on. We do not give explicit definitions for all of these but, nevertheless, allow ourselves to use these notions. Moreover, using the Cartesian 2-category PDer instead of Der we obtain corresponding notions for prederivators. Since the dual of a monoidal derivator is again a monoidal derivator we can make the following definition.

Definition 2.1. Let $\mathbb{E}$ be a monoidal derivator. A derivator is *tensored over* $\mathbb{E}$ if it is a left module over $\mathbb{E}$ and is *cotensored over* $\mathbb{E}$ if it is a right module over $\mathbb{E}^{\mathrm{op}}$. A left $\mathbb{E}$-module $\mathbb{D}$ is called a *closed module* if the action map $\otimes\colon\ \mathbb{E}\times\mathbb{D}\longrightarrow\mathbb{D}$ is a left adjoint of two variables.

Given a monoidal derivator $\mathbb{E}$ we just constructed the 2-category $\mathbb{E}-\mathsf{Mod}^{\mathsf{lax}}$ of $\mathbb{E}$ −modules. We leave it to the reader to check that a monoidal morphism of derivators $F\colon\ \mathbb{E}_1\longrightarrow\mathbb{E}_2$ induces a restriction of scalars 2-functor $F^*\colon\ \mathbb{E}_2-\mathsf{Mod}^{\mathsf{lax}}\longrightarrow\mathbb{E}_1-\mathsf{Mod}^{\mathsf{lax}}$. The assignment $F\longmapsto F^*$ is functorial and there is a similar observation for monoidal transformations so that we end up with a 2-functor

$$(-)-\mathsf{Mod}^{\mathsf{lax}}\colon\ \mathsf{MonDer}^{\mathrm{op}}\longrightarrow 2\text{-}\mathsf{CAT}\,.$$

Thus, we can again apply the 2-categorical Grothendieck construction of Appendix A in order to obtain the *2-Grothendieck fibration of tensored derivators* $p\colon\ \mathsf{ModDer}^{\mathsf{lax}}\longrightarrow\mathsf{MonDer}$. If we form the 2-subcategory consisting of all objects, the strong module morphisms only and all 2-cells then we obtain the 2-category ModDer. Moreover, it is easy to verify that there is a 2-functor $U\colon\ \mathsf{ModDer}\longrightarrow\mathsf{Der}$ which sends an object, i.e., a pair consisting of a monoidal derivator and a module over it to the derivator underlying the module derivator. Thus we are in the situation of the following diagram:

$$\begin{tikzcd}
\mathsf{ModDer} \arrow[r, "U"] \arrow[d, "p"'] & \mathsf{Der} \\
\mathsf{MonDer} &
\end{tikzcd}$$

Given a derivator $\mathbb{D}$ let us call the 2-category $\mathsf{Mod}(\mathbb{D})=U^{-1}(\mathbb{D})$ the *2-category of module structures on* $\mathbb{D}$.

Before we give some immediate examples let us mention the 'pointwise description' of tensored derivators. Let $\mathbb{D}$ be an $\mathbb{E}$ −module derivator and let J be a category. Then it is immediate that $\mathbb{D}(J)$ is canonically an $\mathbb{E}(J)$−module. Moreover, let us consider a functor $u\colon J\longrightarrow K$. By the reasoning in Section 1 the functor $u^*_{\mathbb{E}}\colon\ \mathbb{E}(K)\longrightarrow\mathbb{E}(J)$ is canonically endowed with a strong monoidal structure. Thus, we have the induced restriction of scalars functor $\mathbb{E}(u)^*\colon\ \mathbb{E}(J)-\mathsf{Mod}\longrightarrow\mathbb{E}(K)-\mathsf{Mod}$. In particular, we can consider the $\mathbb{E}(K)$−module $\mathbb{E}(u)^*\,\mathbb{D}(J)$ for which the action map is given by

$$\mathbb{E}(K)\times\mathbb{D}(J)\xrightarrow{u^*_{\mathbb{E}}\times\mathrm{id}}\mathbb{E}(J)\times\mathbb{D}(J)\xrightarrow{\otimes}\mathbb{D}(J).$$

Since $\otimes\colon \mathbb{E}\times\mathbb{D}\longrightarrow\mathbb{D}$ is a morphism of derivators we have a natural isomorphism $\gamma_u^{\otimes}$ which can be rewritten as:

$$\begin{tikzcd}
\mathbb{E}(K)\times\mathbb{D}(K) \arrow[rr, "\otimes"] \arrow[d, "\mathrm{id}\times u_{\mathbb{D}}^*"'] & & \mathbb{D}(K) \arrow[d, "u_{\mathbb{D}}^*"] \\
\mathbb{E}(K)\times\mathbb{D}(J) \arrow[r, "u_{\mathbb{E}}^*\times\mathrm{id}"'] & \mathbb{E}(J)\times\mathbb{D}(J) \arrow[u, phantom, "\Longrightarrow"] \arrow[r, "\otimes"'] & \mathbb{D}(J)
\end{tikzcd}$$

Similarly to the case of a monoidal derivator one can check that the pair $(u_{\mathbb{D}}^*,\gamma_u^{\otimes})$ defines a morphism $\mathbb{D}(K)\longrightarrow\mathbb{E}(u)^*\,\mathbb{D}(J)$ in $\mathbb{E}(K)-\mathsf{Mod}$. Said differently, we have a morphism

$$u^*=(u_{\mathbb{E}}^*,(u_{\mathbb{D}}^*,\gamma_u^{\otimes}))\colon (\mathbb{E}(K),\mathbb{D}(K))\longrightarrow(\mathbb{E}(J),\mathbb{D}(J))$$

in ModCAT and one can make similar observations for natural transformations. Using the 2-Grothendieck fibration of left-tensored categories and the corresponding forgetful functor as indicated in

$$\begin{tikzcd}
\mathsf{ModCAT} \arrow[r, "U"] \arrow[d, "p"'] & \mathsf{CAT} \\
\mathsf{MonCAT}
\end{tikzcd}$$

we can hence give the following 'pointwise description'. A left-tensored prederivator is a 2-functor $\mathbb{D}\colon \mathsf{Cat}^{\mathrm{op}}\longrightarrow\mathsf{ModCAT}$. Such a 2-functor has an underlying prederivator $U\circ\mathbb{D}\colon \mathsf{Cat}^{\mathrm{op}}\longrightarrow\mathsf{CAT}$ and this prederivator is then left-tensored over the monoidal prederivator $\mathbb{E}=p\circ\mathbb{D}\colon \mathsf{Cat}^{\mathrm{op}}\longrightarrow\mathsf{MonCAT}$. A left-tensored derivator is a left-tensored prederivator such that the underlying prederivator is a derivator.

Example 2.2. The 2-functor $y\colon \mathsf{CAT}\longrightarrow\mathsf{PDer}\colon \mathcal{C}\longmapsto y(\mathcal{C})$ sending a category to the represented prederivator preserves 2-products and hence monoidal objects and modules. It follows that with a monoidal category $\mathcal{C}$ also the represented prederivator $\mathcal{C}$ is canonically monoidal and there is a similar remark for $\mathcal{C}$-modules. Thus, we have induced 2-functors

$$y\colon \mathcal{C}-\mathsf{Mod}\longrightarrow y(\mathcal{C})-\mathsf{Mod}\qquad\text{and}\qquad y\colon \mathsf{ModCAT}\longrightarrow\mathsf{ModPDer}\,.$$

Example 2.3. Every monoidal derivator is canonically a left and a right module over itself. In particular, this is the case for the monoidal derivators associated to combinatorial monoidal model categories.

In the last section we proved that an additive, monoidal derivator with an additive monoidal structure is canonically endowed with a linear structure over the ring of endomorphisms of the monoidal unit of the underlying monoidal category. Recall from the proof of that result that we only used the fact that the monoidal derivator is left-tensored over itself. Thus, we obtain immediately the following more general result.

Corollary 2.4. *Let $\mathbb{E}$ be an additive, monoidal derivator with an additive monoidal structure. Any additive left $\mathbb{E}-$module $\mathbb{D}$ is canonically endowed with a linear structure over* $\mathsf{hom}_{\mathbb{E}(e)}(\mathbb{S}_e,\mathbb{S}_e)$.

In this context, an additive $\mathbb{E}-$module $\mathbb{D}$ is of course an $\mathbb{E}-$module $\mathbb{D}$ such that the underlying derivator and the action are additive. We give more specific examples in Subsection 2.4 where we also apply this last corollary. But before that let us develop a bit more of the general theory and show that the Cartesian monoidal 2-categories PDer and Der are closed. This will also put into perspective the above corollary in that the ring map giving the linear structure is only a shadow of the fact that there is monoidal morphism of derivators in the background.

2.3. **The closedness of the Cartesian monoidal 2-categories PDer and Der.** Recall from classical category theory that given a category $\mathcal{D}$ there is the monoidal category of endomorphisms of $\mathcal{D}$. This is the universal example of a monoidal category acting from the left on $\mathcal{D}$. The corresponding result is also true in the world of $\infty-$categories as is shown by Lurie in Chapter 6 of [Lur11]. If one wants to give a corresponding result in the world of prederivators one should at first show that the 2-category PDer is Cartesian closed in a sense which is to be specified.

For this purpose, let us recall from [Gro10a] that PDer is right-tensored over $\mathsf{Cat}^{\mathrm{op}}$. In fact, for every prederivator $\mathbb{D}$ and every small category J we have the prederivator

$$\mathbb{D}_J=\mathbb{D}(J\times -)\colon\ \mathsf{Cat}^{\mathrm{op}}\xrightarrow{J\times -}\mathsf{Cat}^{\mathrm{op}}\xrightarrow{\mathbb{D}}\mathsf{CAT}\,.$$

This gives us an induced 2-functor $(-)_{(-)}\colon\ \mathsf{PDer}\times\mathsf{Cat}^{\mathrm{op}}\longrightarrow\mathsf{PDer}$ which turns PDer into a right $\mathsf{Cat}^{\mathrm{op}}-$module.

The aim is now to show that the Cartesian monoidal 2-category PDer is closed in the *bicategorical* sense. Thus, given three prederivators $\mathbb{D}$, $\mathbb{D}'$, and $\mathbb{D}''$ we want to construct a prederivator $\mathsf{HOM}(\mathbb{D}',\mathbb{D}'')$ of morphisms and a natural equivalence of categories

$$\mathsf{Hom}(\mathbb{D}\times\mathbb{D}',\mathbb{D}'')\xrightarrow{\simeq}\mathsf{Hom}(\mathbb{D},\mathsf{HOM}(\mathbb{D}',\mathbb{D}'')).$$

In more formal terms, we are looking for a *biadjunction* (see [Gra74], [Fio06, Chapter 9] or Appendix B.2). Note that we have $\mathsf{Hom}(-,-)=\mathsf{PsNat}(-,-)$ in our situation. For a category J let us again denote the represented prederivator by $y(J)$. If we now assume that we were given such a construction of an internal hom $\mathsf{HOM}(-,-)$ then for an arbitrary category J we would deduce the following chain of natural equivalences of categories:

$$\begin{aligned}\mathsf{HOM}(\mathbb{D},\mathbb{D}')(J)&\simeq\mathsf{PsNat}(y(J),\mathsf{HOM}(\mathbb{D},\mathbb{D}'))\\&\simeq\mathsf{PsNat}(y(J)\times\mathbb{D},\mathbb{D}')\\&\simeq\mathsf{PsNat}(\mathbb{D},\mathsf{HOM}(y(J),\mathbb{D}'))\end{aligned}$$

The equivalences are given by the bicategorical Yoneda lemma and the assumed closedness property. So, we have reduced the problem to giving an identification of $\mathsf{HOM}(y(J),\mathbb{D}')$ for a category J and a prederivator $\mathbb{D}'$. By similar arguments and for a category K we obtain natural equivalences of categories as follows:

$$\begin{aligned}\mathsf{HOM}(y(J),\mathbb{D}')(K) &\simeq \mathsf{PsNat}(y(K),\mathsf{HOM}(y(J),\mathbb{D}'))\\ &\simeq \mathsf{PsNat}(y(K)\times y(J),\mathbb{D}')\\ &\simeq \mathsf{PsNat}(y(J\times K),\mathbb{D}')\\ &\simeq \mathbb{D}'(J\times K)=\mathbb{D}'_J(K)\end{aligned}$$

Putting these chains of natural equivalences together we would obtain as an upshot the following equivalence which motivates the next definition

$$\mathsf{HOM}(\mathbb{D},\mathbb{D}')(J)\simeq \mathsf{PsNat}(\mathbb{D},\mathbb{D}'_J)=\mathsf{Hom}(\mathbb{D},\mathbb{D}'_J).$$

Definition 2.5. The *prederivator hom* HOM is the 2-functor $\mathsf{HOM}\colon \mathsf{PDer}^{\mathrm{op}}\times\mathsf{PDer}\longrightarrow \mathsf{PDer}$ which is adjoint to

$$\mathsf{PDer}^{\mathrm{op}}\times\mathsf{PDer}\times\mathsf{Cat}^{\mathrm{op}}\xrightarrow{\mathrm{id}\times(-)_{(-)}}\mathsf{PDer}^{\mathrm{op}}\times\mathsf{PDer}\xrightarrow{\mathsf{Hom}}\mathsf{CAT}\,.$$

Given two prederivators $\mathbb{D}$ and $\mathbb{D}'$ the prederivator $\mathsf{HOM}(\mathbb{D},\mathbb{D}')$ is called the *prederivator of morphisms from* $\mathbb{D}$ *to* $\mathbb{D}'$. Moreover, for a single prederivator $\mathbb{D}$ we set

$$\mathsf{END}(\mathbb{D})=\mathsf{HOM}(\mathbb{D},\mathbb{D})\in\mathsf{PDer}$$

and call this the *prederivator of endomorphisms of* $\mathbb{D}$. For derivators $\mathbb{D}$ and $\mathbb{D}'$ we define $\mathsf{HOM}(\mathbb{D},\mathbb{D}')$ and $\mathsf{END}(\mathbb{D})$ using the underlying prederivators.

More explicitly, for two prederivators $\mathbb{D}$, $\mathbb{D}'$, and a small category J we have thus

$$\mathsf{HOM}(\mathbb{D},\mathbb{D}')(J)=\mathsf{Hom}(\mathbb{D},\mathbb{D}'_J)\in\mathsf{CAT}\,.$$

Our next aim is to show that the bifunctor HOM defines an internal hom in the bicategorical sense (cf. Appendix B.2) for the Cartesian monoidal 2-category PDer. As a preparation for that result let us construct pseudo-natural transformations which will be used in order to define the adjunction.

Lemma 2.6. *For $\mathbb{D}$, $\mathbb{D}'\in\mathsf{PDer}$ there are canonical morphisms $\eta\colon\mathbb{D}\longrightarrow\mathsf{HOM}(\mathbb{D}',\mathbb{D}\times\mathbb{D}')$ and $\epsilon\colon\mathsf{HOM}(\mathbb{D},\mathbb{D}')\times\mathbb{D}\longrightarrow\mathbb{D}'$. Moreover, η resp. ϵ is pseudo-natural in $\mathbb{D}$ resp. $\mathbb{D}'$.*

Proof. Let us begin with the construction of $\eta\colon\mathbb{D}\longrightarrow\mathsf{HOM}(\mathbb{D}',\mathbb{D}\times\mathbb{D}')$. For a category K we thus have to construct a functor $\eta_K\colon\mathbb{D}(K)\longrightarrow\mathsf{Hom}(\mathbb{D}',(\mathbb{D}\times\mathbb{D}')_K)$. For an arbitrary category J, let us define the component $\eta_K(-)_J\colon\mathbb{D}(K)\longrightarrow\mathsf{Fun}(\mathbb{D}'(J),\mathbb{D}(K\times J)\times\mathbb{D}'(K\times J))$ to be adjoint to the functor

$$(\mathrm{pr}_1^*,\mathrm{pr}_2^*)\colon\ \mathbb{D}(K)\times\mathbb{D}'(J)\longrightarrow\mathbb{D}(K\times J)\times\mathbb{D}'(K\times J),$$

i.e., we set $\eta_K(X)_J(Y)=(\mathrm{pr}_1^*(X),\mathrm{pr}_2^*(Y))$ for $X\in\mathbb{D}(K)$ and $Y\in\mathbb{D}'(J)$. For a functor $u\colon J_1\longrightarrow J_2$ we can use the diagram

$$\begin{tikzcd} K \arrow[d, "="'] & K\times J_1 \arrow[l, "\mathrm{pr}_1"'] \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{id}\times u"] & J_1 \arrow[d, "u"] \\ K & K\times J_2 \arrow[l, "\mathrm{pr}_1"] \arrow[r, "\mathrm{pr}_2"'] & J_2 \end{tikzcd}$$

to convince ourselves that the maps $\{\eta_K(-)_J\}_J$ assemble to define a strict morphism of prederivators $\eta_K(-)\colon \mathbb{D}' \longrightarrow (\mathbb{D}\times\mathbb{D}')_K$. A similar reasoning shows that also the $\{\eta_K\}_K$ together define a morphism of prederivators $\eta\colon \mathbb{D} \longrightarrow \mathsf{HOM}(\mathbb{D}',\mathbb{D}\times\mathbb{D}')$. Let us show now that η is pseudo-natural in $\mathbb{D}$. For this purpose, let $F\colon \mathbb{D}_1 \longrightarrow \mathbb{D}_2$ be a morphism of prederivators, let K be a category and let us consider the following non-commutative diagram:

$$\begin{tikzcd}
\mathbb{D}_1(K) \arrow[r, "\eta_K"] \arrow[d, "F_K"'] & \mathsf{Hom}(\mathbb{D}',(\mathbb{D}_1\times\mathbb{D}')_K) \arrow[d, "(F\times\mathrm{id})_{K*}"] \\
\mathbb{D}_2(K) \arrow[r, "\eta_K"'] & \mathsf{Hom}(\mathbb{D}',(\mathbb{D}_2\times\mathbb{D}')_K)
\end{tikzcd}$$

For $X\in\mathbb{D}_1(K)$ and $Y\in\mathbb{D}'(J)$ we can use the natural isomorphisms belonging to the morphism F to deduce the following one:

$$\begin{aligned}
\big((F\times\mathrm{id})_{K*}\circ\eta_K\big)(X)_J(Y) &= (F\times\mathrm{id})_K\circ(\mathrm{pr}_1^*(X),\mathrm{pr}_2^*(Y))\\
&= (F_{K\times J}\,\mathrm{pr}_1^*(X),\mathrm{pr}_2^*(Y))\\
&\cong (\mathrm{pr}_1^*\,F_K(X),\mathrm{pr}_2^*(Y))\\
&= (\eta_K\circ F_K)(X)_J(Y)
\end{aligned}$$

One checks that these isomorphisms can be used to obtain a pseudo-natural transformation η as intended.

Let us now construct a morphism $\epsilon\colon \mathsf{HOM}(\mathbb{D},\mathbb{D}')\times\mathbb{D} \longrightarrow \mathbb{D}'$. Thus, for a category K we have to define a functor $\epsilon_K\colon \mathsf{Hom}(\mathbb{D},\mathbb{D}'_K)\times\mathbb{D}(K) \longrightarrow \mathbb{D}'(K)$. This is defined to be the following composition

$$\begin{tikzcd}
\mathsf{Hom}(\mathbb{D},\mathbb{D}'_K)\times\mathbb{D}(K) \arrow[rr, "\epsilon_K"] \arrow[d, "\mathrm{pr}\times\mathrm{id}"'] & & \mathbb{D}'(K) \\
\mathsf{Fun}(\mathbb{D}(K),\mathbb{D}'_K(K))\times\mathbb{D}(K) \arrow[r, "\mathsf{ev}"'] & \mathbb{D}'_K(K)=\mathbb{D}'(K\times K) \arrow[ur, "\Delta_K^*"'] &
\end{tikzcd}$$

i.e., for $F\in\mathsf{Hom}(\mathbb{D},\mathbb{D}'_K)$ and $X\in\mathbb{D}(K)$ we set $\epsilon_K(F,X)=\Delta_K^*F_K(X)$. To see that these ϵ_K assemble to define a morphism of prederivators let us consider a functor $u\colon J\longrightarrow K$ and let us check that there is a canonical natural isomorphism γ_u^ϵ as in:

$$\begin{tikzcd}
\mathsf{Hom}(\mathbb{D},\mathbb{D}'_K)\times\mathbb{D}(K) \arrow[r, "\epsilon_K"] \arrow[d, "\mathbb{D}'_{u*}\times\mathbb{D}(u)"'] \arrow[dr, phantom, "\cong"] & \mathbb{D}'(K) \arrow[d, "\mathbb{D}(u)"] \\
\mathsf{Hom}(\mathbb{D},\mathbb{D}'_J)\times\mathbb{D}(J) \arrow[r, "\epsilon_J"'] & \mathbb{D}'(J)
\end{tikzcd}$$

But evaluated at $F \in \mathsf{Hom}(\mathbb{D}, \mathbb{D}'_K)$ and $X \in \mathbb{D}(K)$ we can use the natural isomorphisms belonging to F to obtain

$$\begin{aligned}
\mathbb{D}(u) \circ \epsilon_K(F, X) &= u^*\Delta_K^* F_K(X) \\
&= \Delta_J^*(u \times \mathrm{id})^*(\mathrm{id} \times u)^* F_K(X) \\
&\cong \Delta_J^*(u \times \mathrm{id})^* F_J(u^*X) \\
&= \Delta_J^*(\mathbb{D}'_{u*}F)_J(u^*X) \\
&= \epsilon_J \circ (\mathbb{D}'_{u*} \times \mathbb{D}(u))(F, X).
\end{aligned}$$

Thus, slightly sloppy we have $(\gamma_u^\epsilon)_{F,X} = (\gamma_u^F)_X$. Again one checks that these isomorphisms assemble together to define a morphism of prederivators $\epsilon\colon \mathsf{HOM}(\mathbb{D}, \mathbb{D}') \times \mathbb{D} \longrightarrow \mathbb{D}'$. In order to show that ϵ is pseudo-natural let us consider a morphism of prederivators $G\colon \mathbb{D}' \longrightarrow \mathbb{D}''$ and let us construct a natural isomorphism as in:

$$\begin{tikzcd}
\mathsf{HOM}(\mathbb{D}, \mathbb{D}') \times \mathbb{D} \arrow[r, "\epsilon"] \arrow[d, "G_* \times \mathrm{id}"'] \arrow[dr, phantom, "\cong"] & \mathbb{D}' \arrow[d, "G"] \\
\mathsf{HOM}(\mathbb{D}, \mathbb{D}'') \times \mathbb{D} \arrow[r, "\epsilon"'] & \mathbb{D}''
\end{tikzcd}$$

Using the natural isomorphisms belonging to G and a similar calculation as above we obtain for $F \in \mathsf{Hom}(\mathbb{D}, \mathbb{D}'_K)$ and $X \in \mathbb{D}(K)$ the following isomorphism:

$$\begin{aligned}
(\epsilon \circ (G_* \times \mathrm{id}))_K(F, X) &= \Delta_K^* G_{K \times K} F_K(X) \\
&\cong G_K \Delta_K^* F_K(X) \\
&= (G \circ \epsilon)_K(F, X)
\end{aligned}$$

These isomorphisms give us the desired natural isomorphisms turning ϵ into a pseudo-natural transformation which concludes the proof. □

With this preparation we can now give the following desired result.

Proposition 2.7. *The prederivator of morphisms defines an internal hom in the Cartesian monoidal* 2*-category* PDer*, i.e., for three prederivators* $\mathbb{D}, \mathbb{D}'$*, and* $\mathbb{D}''$ *we have pseudo-natural equivalences of categories:*

$$\mathsf{Hom}_{\mathsf{PDer}}(\mathbb{D} \times \mathbb{D}', \mathbb{D}'') \simeq \mathsf{Hom}_{\mathsf{PDer}}(\mathbb{D}, \mathsf{HOM}(\mathbb{D}', \mathbb{D}''))$$

Proof. We use the pseudo-natural transformations of the last lemma to define functors l and r by $l = \eta^* \circ \mathsf{HOM}(\mathbb{D}', -)$ and $r = \epsilon_* \circ (- \times \mathbb{D}')$ as depicted in:

$$\begin{tikzcd}
\mathsf{Hom}(\mathbb{D} \times \mathbb{D}', \mathbb{D}'') \arrow[r, "\mathsf{HOM}(\mathbb{D}', -)"] & \mathsf{Hom}(\mathsf{HOM}(\mathbb{D}', \mathbb{D} \times \mathbb{D}'), \mathsf{HOM}(\mathbb{D}', \mathbb{D}'')) \arrow[d, "\eta^*"] \\
\mathsf{Hom}(\mathbb{D} \times \mathbb{D}', \mathsf{HOM}(\mathbb{D}', \mathbb{D}'') \times \mathbb{D}') \arrow[u, "\epsilon_*"] & \mathsf{Hom}(\mathbb{D}, \mathsf{HOM}(\mathbb{D}', \mathbb{D}'')) \arrow[l, "- \times \mathbb{D}'"]
\end{tikzcd}$$

Let us check that these are inverse equivalences of categories and let us begin by showing that we have a natural isomorphism $r \circ l \cong \mathrm{id}$. For this purpose, let us consider a morphism

$F\colon \mathbb{D}\times\mathbb{D}' \longrightarrow \mathbb{D}''$. We claim that we have the following diagram which commutes up to a natural isomorphism:

$$\begin{tikzcd}
\mathbb{D}\times\mathbb{D}' \arrow[r, "\eta\times 1"] \arrow[dr, bend right, "\mathrm{id}"'] & \mathsf{HOM}(\mathbb{D}',\mathbb{D}\times\mathbb{D}')\times\mathbb{D}' \arrow[r, "F_*\times\mathrm{id}"] \arrow[d, "\epsilon"] \arrow[dr, phantom, "\cong"] & \mathsf{HOM}(\mathbb{D}',\mathbb{D}'')\times\mathbb{D}' \arrow[d, "\epsilon"] \\
 & \mathbb{D}\times\mathbb{D}' \arrow[r, "F"'] & \mathbb{D}''
\end{tikzcd}$$

By the lemma we only have to check that the triangle commutes. But using the explicit formulas of the last proof we can calculate for $X\in\mathbb{D}(K)$ and $Y\in\mathbb{D}'(K)$ the following:

$$(\epsilon\circ(\eta\times 1))(X,Y)=\epsilon(\eta(X),Y)=\Delta_K^*\eta_K(X)_K(Y)=\Delta_K^*(\mathrm{pr}_1^*X,\mathrm{pr}_2^*Y)=(X,Y)$$

Since the longer boundary path from $\mathbb{D}\times\mathbb{D}'$ to $\mathbb{D}''$ calculates $r\circ l(F)$ we conclude $r\circ l\cong\mathrm{id}$.

Let us show that we also have $l\circ r\cong\mathrm{id}$. Thus, let us consider a morphism $G\colon\mathbb{D}\longrightarrow\mathsf{HOM}(\mathbb{D}',\mathbb{D}'')$ and let us show that the following diagram commutes up to natural isomorphisms:

$$\begin{tikzcd}
\mathsf{HOM}(\mathbb{D}',\mathbb{D}\times\mathbb{D}') \arrow[r, "\mathsf{HOM}(\mathbb{D}',G\times 1)"] \arrow[dr, phantom, "\cong"] & \mathsf{HOM}(\mathbb{D}',\mathsf{HOM}(\mathbb{D}',\mathbb{D}'')\times\mathbb{D}') \arrow[r, "\mathsf{HOM}(\mathbb{D}',\epsilon)"] & \mathsf{HOM}(\mathbb{D}',\mathbb{D}'') \\
\mathbb{D} \arrow[u, "\eta"] \arrow[r, "G"'] & \mathsf{HOM}(\mathbb{D}',\mathbb{D}'') \arrow[u, "\eta"'] \arrow[ur, bend right, "\mathrm{id}"'] \arrow[ur, phantom, "\cong"] &
\end{tikzcd}$$

By the last lemma it remains to show that the triangle commutes up to a natural isomorphism. But for $F\in\mathsf{Hom}(\mathbb{D}',\mathbb{D}''_K)$ and $Y\in\mathbb{D}'_J$ we can again use the formulas of the last proof to make the following calculation:

$$\begin{array}{rcl}
((\epsilon_*)_K\circ\eta_K)(F)_J(Y) & = & \epsilon_{K\times J}(\mathrm{pr}_1^*F,\mathrm{pr}_2^*Y)\\
 & = & \Delta_{K\times J}^*(\mathrm{pr}_1^*)_{K\times J}F_{K\times J}pr_2^*Y\\
 & \cong & \Delta_{K\times J}^*(\mathrm{pr}_1\times 1\times 1)^*(1\times\mathrm{pr}_2)^*F_J(Y)\\
 & = & F_J(Y).
\end{array}$$

Here, we used the natural isomorphism belonging to F and the commutativity of the following diagram in which the composition of the bottom row is just the identity:

$$\begin{tikzcd}
\mathbb{D}''_K(J) \arrow[r, "\mathrm{pr}_2^*"] \arrow[d, "="'] & \mathbb{D}''_K(K\times J) \arrow[r, "(\mathrm{pr}_1^*)_{(K\times J)}"] \arrow[d, "="'] & \mathbb{D}''_{K\times J}(K\times J) \arrow[d, "="] & \\
\mathbb{D}''(K\times J) \arrow[r, "(1\times\mathrm{pr}_2)^*"'] & \mathbb{D}''(K\times K\times J) \arrow[r, "(\mathrm{pr}_1\times 1\times 1)^*"'] & \mathbb{D}''(K\times J\times K\times J) \arrow[r, "\Delta_{K\times J}^*"'] & \mathbb{D}''(K\times J)
\end{tikzcd}$$

It follows that the triangle in the previous diagram also commutes up to natural isomorphism. Again, the longer path passing through the boundary from $\mathbb{D}$ to $\mathsf{HOM}(\mathbb{D}',\mathbb{D}'')$ is $l\circ r(G)$ and we can thus deduce that we have a natural isomorphism $l\circ r\cong\mathrm{id}$. This concludes the proof of the proposition. $\square$

From classical category theory we know that a functor category $\mathsf{Fun}(J,\mathcal{C})$ is (co)complete as soon as this is the case for the target category $\mathcal{C}$. The corresponding result for derivators also holds true as we will show now.

Proposition 2.8. *If $\mathbb{D}$ is a prederivator and $\mathbb{D}'$ a derivator then $\mathsf{HOM}(\mathbb{D},\mathbb{D}')$ is a derivator. If $\mathbb{D}'$ is in addition pointed resp. additive then this is also the case for $\mathsf{HOM}(\mathbb{D},\mathbb{D}')$. Thus, the 2-categories Der, Der^*, and $\mathsf{Der}^{\mathsf{add}}$ are Cartesian closed 2-categories.*

Proof. The axiom (Der1) is immediate. For axiom (Der2), let us consider a map $\phi\colon F\longrightarrow G$ in $\mathsf{HOM}(\mathbb{D},\mathbb{D}')(K)$. Then ϕ is an isomorphism if and only if $\phi_J\colon F_J\longrightarrow G_J$ is an isomorphism in $\mathsf{nat}(\mathbb{D}(J),\mathbb{D}'(K\times J))$ for all categories J. The fact that isomorphisms in $\mathbb{D}'$ are detected pointwise shows that this is equivalent to all $(\phi_J)_k=(\phi_k)_J$ being isomorphisms. Thus, ϕ is an isomorphism if and only if all ϕ_k are isomorphisms. For axiom (Der3), let us consider a functor $u\colon J\longrightarrow K$. We will prove in Lemma 2.9 that u induces an adjunction $(u_!,u^*)\colon\ \mathbb{D}'_J\rightharpoonup\mathbb{D}'_K$ of derivators. Then, since $\mathsf{Hom}(\mathbb{D},-)$ preserves adjunctions, we obtain the intended adjunction $(u_!,u^*)\colon\ \mathsf{HOM}(\mathbb{D},\mathbb{D}')(J)\rightharpoonup\mathsf{HOM}(\mathbb{D},\mathbb{D}')(K)$. One proceeds similarly for homotopy right Kan extensions. For the base change axiom, let $u\colon J\longrightarrow K$ be a functor and $k\in K$ an object. Then, we have to show that the base change morphism in the square on the right-hand-side induced by the natural transformation on the left-hand-side is an isomorphism:

$$
\begin{tikzcd}
J_{/k} \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow"] & J \arrow[d] \\
e \arrow[r] & K
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathsf{Hom}(\mathbb{D},\mathbb{D}'_{J_{/k}}) \arrow[d] \arrow[dr, phantom, "\searrow"] & \mathsf{Hom}(\mathbb{D},\mathbb{D}'_J) \arrow[l] \arrow[d] \\
\mathsf{Hom}(\mathbb{D},\mathbb{D}'_e) & \mathsf{Hom}(\mathbb{D},\mathbb{D}'_K) \arrow[l]
\end{tikzcd}
$$

Evaluation of this base change morphism is just given by postcomposition with the base change morphism belonging to $\mathbb{D}'$. But this one is an isomorphism because $\mathbb{D}'$ is a derivator by assumption. Thus, this together with a dual reasoning for homotopy right Kan extensions implies (Der4) for $\mathsf{HOM}(\mathbb{D},\mathbb{D}')$. Since homotopy Kan extensions are calculated pointwise it follows that $\mathsf{HOM}(\mathbb{D},\mathbb{D}')$ is pointed resp. additive if this is the case for $\mathbb{D}'$. $\square$

Lemma 2.9. *Let $\mathbb{D}$ be a derivator and let $u\colon J\longrightarrow K$ be a functor. Then we obtain an induced adjunction of derivators $(u_!,u^*)\colon\ \mathbb{D}_J\rightharpoonup\mathbb{D}_K$.*

Proof. The morphism $u^*\colon\ \mathbb{D}_K\longrightarrow\mathbb{D}_J$ has an adjoint at least levelwise: for a category M, an adjoint to $(u^*)_M$ is given by $(u_!)_M=(u\times\mathrm{id}_M)_!\colon\ \mathbb{D}(J\times M)\longrightarrow\mathbb{D}(K\times M)$. We thus have to check that these can be canonically assembled into a morphism of derivators. So, let $f\colon M\longrightarrow N$ be a functor and let us consider the following diagram:

$$
\begin{tikzcd}
\mathbb{D}(J\times N) \arrow[r, "(\mathrm{id}\times f)^*"] \arrow[d, "(u\times\mathrm{id})_!"'] & \mathbb{D}(J\times M) \arrow[d, "(u\times\mathrm{id})_!"] \\
\mathbb{D}(K\times N) \arrow[r, "(\mathrm{id}\times f)^*"'] & \mathbb{D}(K\times M)
\end{tikzcd}
$$

But, $f^*\colon \mathbb{D}(-\times N) \longrightarrow \mathbb{D}(-\times M)$ preserves homotopy Kan extensions by Proposition 2.8 of [Gro10a] from where we obtain that the base change morphism β_f given by

$$\begin{tikzcd}
u_! f^* \arrow[r, "\beta_f"] \arrow[d, "\eta"'] & f^* u_! \\
u_! f^* u^* u_! \arrow[r, "="'] & u_! u^* f^* u_! \arrow[u, "\epsilon"']
\end{tikzcd}$$

is a natural isomorphism. The claim is that these base change morphisms together with the levelwise left adjoints $(u\times \mathrm{id})_!$ define a morphism $u_!\colon \mathbb{D}_J \longrightarrow \mathbb{D}_K$ of derivators. The compatibility with respect to the identities reduces to one of the triangular identities for adjunctions. The behavior with respect to compositions is a bit more technical and is checked by the following diagram. In that diagram, everything commutes by naturality besides probably the 'circle at the bottom' which commutes again by a triangular identity:

$$\begin{tikzcd}
u_! f^* g^* \arrow[rr, "\beta_f g^*"] \arrow[d, equal] \arrow[dr, "\eta"] & & f^* u_! g^* \arrow[rr, "f^*\beta_g"] \arrow[dr, "\eta"] & & f^* g^* u_! \arrow[d, equal] \\
u_!(gf)^* \arrow[ddddr, bend right, "\eta"'] & u_! f^* u^* u_! g^* = u_! u^* f^* u_! g^* \arrow[ur, "\epsilon"] \arrow[dd, "\eta"'] \arrow[dr, "\eta"] & & f^* u_! g^* u^* u_! = f^* u_! u^* g^* u_! \arrow[ur, "\epsilon"] & (gf)^* u_! \\
& & u_! u^* f^* u_! g^* u^* u_! \arrow[ur, "\epsilon"] \arrow[dl, dash, "="] \arrow[dr, dash, "="'] & & \\
& u_! f^* u^* u_! g^* u^* u_! \arrow[dr, dash, "="'] & & u_! u^* f^* u_! u^* g^* u_! \arrow[uu, "\epsilon"'] \arrow[dl, dash, "="] \arrow[dd, "\epsilon"] & \\
& & u_! f^* u^* u_! u^* g^* u_! \arrow[d, bend left, "\epsilon"] & & \\
& u_!(gf)^* u^* u_! \arrow[uu, "\eta"] \arrow[r, equal] & u_! f^* u^* g^* u_! \arrow[u, bend left, "\eta"] \arrow[r, equal] & u_! u^* (gf)^* u_! \arrow[uuuur, bend right, "\epsilon"'] &
\end{tikzcd}$$

The long composition of morphisms through the bottom line gives $\beta_{gf}\colon u_!(gf)^* \longrightarrow (gf)^* u_!$ and we obtain hence the intended relation $\beta_{gf} = (f^*\beta_g)(\beta_f g^*)$. Thus, we have indeed constructed a morphism of derivators $u_!\colon \mathbb{D}_J \longrightarrow \mathbb{D}_K$ which is left adjoint to $u^*\colon \mathbb{D}_K \longrightarrow \mathbb{D}_J$. □

Remark 2.10. In the case of a stable derivator $\mathbb{D}'$ there is the following comment concerning the internal derivator hom $\mathsf{HOM}(\mathbb{D},\mathbb{D}')$. By the last proposition we know that this gives us a pointed (even additive) derivator. Moreover, since homotopy Kan extensions are calculated pointwise it follows that this derivator has the additional property that the classes of Cartesian and coCartesian squares coincide and hence that the suspension functor is invertible. But it is not known yet whether the internal hom $\mathsf{HOM}(\mathbb{D},\mathbb{D}')$ is again strong, i.e., if the partial underlying diagram functors associated to the ordinal $[1] = (0 \longrightarrow 1)$

$$\mathsf{HOM}(\mathbb{D}, \mathbb{D}')(J \times [1]) \longrightarrow \mathsf{HOM}(\mathbb{D}, \mathbb{D}')(J)^{[1]}$$

are full and essentially surjective. Thus, we cannot deduce that the 2-category $\mathsf{Der}^{\mathsf{ex}}$ of stable derivators is closed monoidal with respect to the Cartesian structure. This is a certain drawback of the notion of a stable derivator. In fact, the notion of a stable derivator can be thought of as a 'minimal notion' which guarantees that one can construct the canonical triangulated structures on all of its values and the induced functors. However, the 'correct notion' of a *stable* derivator has probably still to be found. At least to the knowledge of the author, all known stable derivators are derivators associated to stable ∞-categories. This is not of a surprise since examples of triangulated categories which are neither algebraic nor topological were only constructed recently (cf. [MSS07]). So, – although one certainly does not want this– one could include this as an axiom in the notion of a stable derivator. Whatever the final notion of a stable derivator will be, it should, in particular, have the additional property that it gives us a Cartesian closed 2-category. Once we have this good notion of stable derivators it would then correct two more of the typical drawbacks of the theory of triangulated categories, namely the absence of both products and functor categories inside the world of triangulated categories. The related notion of stable $\infty-$categories as developed by Joyal and Lurie has all these nice properties. An exposition of that theory can be found e.g. in [Lur11], while an introduction is given in [Gro10b].

By Proposition 2.7 and Proposition 2.8 the 2-categories PDer and Der are Cartesian closed monoidal 2-categories. We are thus in the context of Appendix B and can, in particular, apply Theorem B.11. This gives us the following result which we formulate for the 2-category of derivators. Recall also Definition B.9 of terminal objects in 2-categories from that appendix.

Theorem 2.11. *For a derivator $\mathbb{D}$ the derivator $\mathsf{END}(\mathbb{D})$ of endomorphisms can be canonically endowed with a monoidal structure. Moreover, $\mathbb{D}$ can canonically be turned into a left $\mathsf{END}(\mathbb{D})-$module and this module structure defines a terminal object in the 2-category $\mathsf{Mod}(\mathbb{D})$ of module structures on $\mathbb{D}$.*

In fact, the action map belonging to the module structure is just given by the pseudo-natural transformation $\mathsf{ev}\colon \mathsf{END}(\mathbb{D}) \times \mathbb{D} \longrightarrow \mathbb{D}$ of Lemma 2.6. The monoidal structure on $\mathsf{END}(\mathbb{D})$ is derived from this map using the biadjunction. For the details of this structure see the constructive proof of Theorem B.11.

We can use this theorem to put Corollary 2.4 into perspective. Namely, let $\mathbb{E}$ be a monoidal derivator and let $(\mathbb{D}, a\colon \mathbb{E} \times \mathbb{D} \longrightarrow \mathbb{D})$ be an $\mathbb{E}$-module. By the theorem, there is an essentially unique morphism $(\mathbb{E}, a) \longrightarrow (\mathbb{D}, \mathsf{ev})$ in $\mathsf{Mod}(\mathbb{D})$. Thus we obtain a pair consisting of a monoidal morphism $\mathbb{E} \longrightarrow \mathsf{END}(\mathbb{D})$ and a natural isomorphism as in:

$$\begin{tikzcd} \mathbb{E} \times \mathbb{D} \arrow[d] \arrow[drr, bend left, "a"] & & \\ \mathsf{END}(\mathbb{D}) \times \mathbb{D} \arrow[rr, "\mathsf{ev}"'] \arrow[urr, phantom, "\Nearrow"] & & \mathbb{D} \end{tikzcd}$$

Now, a monoidal morphism of derivators induces monoidal functors at all values so that we obtain, in particular, a monoidal functor $\mathbb{E}(e) \longrightarrow \mathsf{END}(\mathbb{D})(e) = \mathsf{Hom}(\mathbb{D}, \mathbb{D})$. If we consider from this functor only the induced map between the endomorphisms of the respective monoidal units then we obtain a map of monoids

$$\mathsf{hom}_{\mathbb{E}(e)}(\mathbb{S}_e, \mathbb{S}_e) \longrightarrow \mathsf{nat}(\mathrm{id}, \mathrm{id}) = \mathsf{Z}(\mathbb{D}).$$

In the context of an additive action this gives us back the ring map of Corollary 2.4. Thus, the canonical linear structure over the ring of self-maps of the monoidal unit of the underlying monoidal category is only the shadow of the fact that we have a monoidal morphism $\mathbb{E} \longrightarrow \mathsf{END}(\mathbb{D})$ of monoidal derivators.

2.4. **Examples coming from model categories.**

Definition 2.12. Let $\mathcal{M}$ be a monoidal model category. A model category $\mathcal{N}$ is a *left* $\mathcal{M}-$*module as a model category* if $\mathcal{N}$ is a left $\mathcal{M}-$module via a Quillen bifunctor $\otimes\colon \mathcal{M} \times \mathcal{N} \longrightarrow \mathcal{N}$ which has the following additional property: For any cofibrant replacement $Q\,\mathbb{S} \longrightarrow \mathbb{S}$ of the monoidal unit of $\mathcal{M}$ the induced natural transformation $Q\,\mathbb{S} \otimes - \longrightarrow \mathbb{S} \otimes -$ is a Quillen homotopy.

We want to emphasize that there is automatically more structure available in the setting of combinatorial model categories, i.e., it will follow that $\mathcal{N}$ is a left $\mathcal{M}-$module as a model category if and only if it is an $\mathcal{M}-$model category in the sense of [DS07b]. The first part of this observation is a purely categorical one. So, let $\mathcal{C}$ and $\mathcal{D}$ be presentable categories, such that $\mathcal{C}$ is monoidal and $\mathcal{D}$ is a left $\mathcal{C}-$module via an action $\otimes\colon \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{D}$ which preserves colimits separately in each variable. Then, using the special form of the Freyd adjoint functor theorem for presentable categories, we obtain an adjunction of two variables

$$(\otimes, \mathsf{Hom}_l, \mathsf{Hom}_r)\colon \mathcal{C} \times \mathcal{D} \rightharpoonup \mathcal{D}$$

for certain functors

$$\mathsf{Hom}_l(-,-)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \longrightarrow \mathcal{D} \qquad \text{and} \qquad \mathsf{Hom}_r(-,-)\colon \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \longrightarrow \mathcal{C}.$$

In order to better distinguish these functors notationally let us write from now on $X^K = \mathsf{Hom}_l(K, X)$ for $X \in \mathcal{D},\ K \in \mathcal{C}$, and $\mathsf{Hom}(X, Y) = \mathsf{Hom}_r(X, Y)$ for $X, Y \in \mathcal{D}$. With these notations the adjunction of two variables takes the familiar form

$$\mathsf{hom}_{\mathcal{D}}(X, Y^K) \cong \mathsf{hom}_{\mathcal{D}}(K \otimes X, Y) \cong \mathsf{hom}_{\mathcal{C}}(K, \mathsf{Hom}(X, Y)).$$

Once one has this adjunction of two variables it can be used to endow $\mathcal{D}$ with an enrichment over $\mathcal{C}$. This works in full generality, i.e., without any presentability assumption on the categories involved. The enriched mapping object for two objects $X, Y \in \mathcal{D}$ is of course $\mathsf{Hom}(X,Y) \in \mathcal{C}$. The enriched identity $i_X\colon \mathbb{S} \longrightarrow \mathsf{Hom}(X,X)$ is the map which is adjoint to the left unitality constraint $\lambda\colon \mathbb{S}\otimes X \longrightarrow X$. Finally, we only need to specify a composition law. This is constructed in two steps. First, for objects X, $Y \in \mathcal{D}$ we obtain an evaluation map $\mathsf{ev}_{X,Y}\colon \mathsf{Hom}(X,Y)\otimes X \longrightarrow Y$ given by the counit of the adjunction $(-\otimes X, \mathsf{Hom}(X,-))\colon \mathcal{C} \rightharpoonup \mathcal{C}$. Thus, we set $\mathsf{ev}_{X,Y} = (\epsilon_X)_Y$ if ϵ_X is the adjunction counit. With these evaluation maps we can construct a composition map associated to three objects X, Y, and $Z \in \mathcal{D}$. The composition $\circ$ is defined to be the map which is adjoint to the following composition:

$$\begin{tikzcd}
(\mathsf{Hom}(Y,Z)\otimes \mathsf{Hom}(X,Y))\otimes X \arrow[r, "\alpha"] \arrow[d] & \mathsf{Hom}(Y,Z)\otimes(\mathsf{Hom}(X,Y)\otimes X) \arrow[d, "\mathsf{ev}"] \\
Z & \mathsf{Hom}(Y,Z)\otimes Y \arrow[l, "\mathsf{ev}"]
\end{tikzcd}$$

It is straightforward but lengthy to see that this defines an enrichment in $\mathcal{C}$. Thus, in the context of presentable categories, one only has to specify a colimit-preserving action $\otimes\colon \mathcal{C}\times\mathcal{D} \longrightarrow \mathcal{D}$ in order to obtain that $\mathcal{D}$ is also canonically cotensored and enriched over $\mathcal{C}$.

Let us now switch back to the homotopical setting. So, let us assume $\mathcal{M}$ to be a combinatorial monoidal model category and $\mathcal{N}$ a combinatorial model category which is a left $\mathcal{M}$-module as a model category. The adjunction of two variables

$$(\otimes, (-)^{(-)}, \mathsf{Hom})\colon \mathcal{M}\times\mathcal{N} \rightharpoonup \mathcal{N}$$

has then the additional property that the involved functors are Quillen bifunctors. By the results of Section 1 (in particular, Corollary 1.21) we thus obtain the following.

Theorem 2.13. *Let $\mathcal{M}$ be a combinatorial monoidal model category and let $\mathcal{N}$ be a combinatorial model category which is a left $\mathcal{M}$-module as a model category. Then we have an adjunction of two variables at the level of derivators*

$$(\overset{\mathbb{L}}{\otimes}, \mathbb{R}(-)^{(-)}, \mathbb{R}\mathsf{Hom})\colon\ \mathbb{D}_{\mathcal{M}}\times\mathbb{D}_{\mathcal{N}} \rightharpoonup \mathbb{D}_{\mathcal{N}}$$

exhibiting $\mathbb{D}_{\mathcal{N}}$ as a closed $\mathbb{D}_{\mathcal{M}}$-module. In particular, $\mathbb{D}_{\mathcal{N}}$ is tensored and cotensored over $\mathbb{D}_{\mathcal{M}}$.

It follows, in particular, that for each category K the value $\mathbb{D}_{\mathcal{N}}(K)$ is canonically tensored, cotensored, and enriched over $\mathbb{D}_{\mathcal{M}}(K)$. We come back to this enrichment issue in Section 3. As a special case we can apply this to the case of a combinatorial monoidal model category $\mathcal{M}$ which is in a canonical way a left $\mathcal{M}$-module as a model category. This reproves then the corresponding results of the last section. But, in addition, we see that there are the cotensors and that at each level we have a canonical enrichment of $\mathbb{D}_{\mathcal{M}}(K)$ over itself.

In the additive context, there is moreover the following result about canonical linear structures.

Corollary 2.14. *Let $\mathcal{M}$ be a combinatorial monoidal model category and let $\mathcal{N}$ be a combinatorial model category with additive associated derivators. If $\mathcal{N}$ is a left $\mathcal{M}-$module as a model category then the associated derivator $\mathbb{D}_{\mathcal{N}}$ is canonically endowed with a linear structure over the ring $\mathsf{hom}_{\mathsf{Ho}(\mathcal{M})}(\mathbb{S},\mathbb{S})$.*

If both $\mathcal{M}$ and $\mathcal{N}$ are, in addition, stable and if the induced derivator $\mathbb{D}_{\mathcal{M}}$ is compatibly stable and closed monoidal, we obtain a linear structure over the graded-commutative ring $\mathsf{hom}_{\mathsf{Ho}(\mathcal{M})}(\mathbb{S},\mathbb{S})_\bullet$. Thus depending on the context we have a canonical map of (graded) rings

$$\mathsf{hom}_{\mathsf{Ho}(\mathcal{M})}(\mathbb{S},\mathbb{S}) \longrightarrow \mathsf{Z}(\mathbb{D}_{\mathcal{N}}) \qquad \text{resp.} \qquad \mathsf{hom}_{\mathsf{Ho}(\mathcal{M})}(\mathbb{S},\mathbb{S})_\bullet \longrightarrow \mathsf{Z}_\bullet(\mathbb{D}_{\mathcal{N}}).$$

Let us now give three important classes of situations to which these results can be applied. Note that the category Set_Δ of simplicial sets, the category $\mathsf{Ch}(k)$ of chain complexes over some commutative ground ring k, and the category Sp^Σ of symmetric spectra based on simplicial sets are all examples of presentable categories. Moreover, the model structures mentioned in the last section have the property that the respective monoidal units (i.e., the zero-simplex Δ^0, the ground ring $k[0]$, and the sphere spectrum $\mathbb{S}$ respectively) are cofibrant. Thus, the unit condition for modules over these model categories is for free. The class of model categories for which the usual definition of a simplicial, spectral, or differential-graded model category can be simplified is a bit larger than the class of combinatorial ones: it suffices that the underlying category is presentable. In particular, this is the case for the class of presentable model categories in the sense of Dugger [Dug06] (which turns out to be the closure of the class of combinatorial model categories under Quillen equivalences).

Proposition 2.15. *Let $\mathcal{N}$ be a model category with an underlying presentable category.*
i) *The model category $\mathcal{N}$ is a* simplicial model category *if and only if it is a left $\mathsf{Set}_\Delta-$module via a Quillen bifunctor $\otimes\colon\ \mathsf{Set}_\Delta \times \mathcal{N} \longrightarrow \mathcal{N}$.*
ii) *The model category $\mathcal{N}$ is a* spectral model category *if and only if it is a left $\mathsf{Sp}^\Sigma-$module via a Quillen bifunctor $\otimes\colon\ \mathsf{Sp}^\Sigma \times \mathcal{N} \longrightarrow \mathcal{N}$.*
iii) *The model category $\mathcal{N}$ is a* dg model category *if and only if it is a left $\mathsf{Ch}(k)-$module for some commutative ground ring k via a Quillen bifunctor $\otimes\colon\ \mathsf{Ch}(k) \times \mathcal{N} \longrightarrow \mathcal{N}$.*

Theorem 2.13 applied to a combinatorial simplicial, spectral, resp. differential-graded model category $\mathcal{N}$ thus gives us that the associated derivator $\mathbb{D}_{\mathcal{N}}$ is canonically tensored and cotensored over the derivator of simplicial sets, spectra, resp. chain complexes. Moreover, the category $\mathbb{D}_{\mathcal{N}}(K)$ is canonically enriched over the category $\mathsf{Ho}(\mathsf{Set}_\Delta{}^K)$, $\mathsf{Ho}((\mathsf{Sp}^\Sigma)^K)$, resp. $\mathsf{Ho}(\mathsf{Ch}(k)^K)$. To illustrate the applicability of this result let us recall from [Dug06] that every stable combinatorial model category is Quillen equivalent to a spectral model category. An alternative set of sufficient conditions for this conclusion can be found in [SS03b, Theorem 3.8.2]. Let us now give more specific examples for the differential-graded and the spectral setting.

Let A be a differential-graded algebra over a ground ring k. As we recalled already in the context of a commutative differential-graded algebra the category $\mathsf{Mod}-A$ of right modules over A can be endowed with the projective model structure. A map in this category is a weak equivalence resp. a fibration if and only if the induced map of underlying chain complexes is a quasi-isomorphisms resp. an epimorphism. This model structure is stable and combinatorial so that we can consider the associated stable derivator $\mathbb{D}_{A^{\mathrm{op}}} = \mathbb{D}_{\mathsf{Mod}-A}$. Moreover, the usual tensor product $\otimes_k\colon \mathsf{Ch}(k) \times \mathsf{Mod}-A \longrightarrow \mathsf{Mod}-A$ turns $\mathsf{Mod}-A$ into a differential-graded model category.

Example 2.16. For a differential-graded algebra A we have an adjunction of two variables

$$\otimes_k^{\mathbb{L}}\colon \mathbb{D}_k \otimes \mathbb{D}_{A^{\mathrm{op}}} \longrightarrow \mathbb{D}_{A^{\mathrm{op}}}$$

exhibiting $\mathbb{D}_{A^{\mathrm{op}}}$, in particular, as an additive left $\mathbb{D}_k$-module. Thus, the derivator $\mathbb{D}_{A^{\mathrm{op}}}$ is canonically endowed with a $k-$linear structure induced by a ring map

$$k = \hom_{D(k)}(k,k) \longrightarrow \mathsf{Z}(\mathbb{D}_{A^{\mathrm{op}}}).$$

A bit more general, let us consider three differential-graded algebras A, B, and C. For more details about module categories in this one object case and also in the more object case we refer to [Hei07, Appendix A]. The tensor product over B gives us a functor

$$\otimes_B\colon (A-\mathsf{Mod}-B) \times (B-\mathsf{Mod}-C) \longrightarrow A-\mathsf{Mod}-C.$$

Here, $A-\mathsf{Mod}-B$ denotes the category of left A-, right B-modules, i.e., of left $A\otimes B^{\mathrm{op}}-$modules. The functors $\otimes_B$ are coherently associative and unital in the obvious sense. Moreover, each of them is part of an adjunction of two variables. In fact, the adjunctions look like

$$(\otimes_B, \mathsf{Hom}_C, \mathsf{Hom}_A)\colon (A-\mathsf{Mod}-B) \times (B-\mathsf{Mod}-C) \rightharpoonup A-\mathsf{Mod}-C.$$

If we now endow the bimodule categories with the projective model structures then one checks that $\otimes_B$ is a left Brown functor as soon as the underlying chain complex of B is cofibrant in $\mathsf{Ch}(k)$. Thus, in that case we obtain an adjunction of two variables

$$\mathbb{D}_{A\otimes B^{\mathrm{op}}} \times \mathbb{D}_{B\otimes C^{\mathrm{op}}} \rightharpoonup \mathbb{D}_{A\otimes C^{\mathrm{op}}}$$

which by the closedness of Der induces a morphism of derivators

$$\mathbb{D}_{A\otimes B^{\mathrm{op}}} \longrightarrow \mathsf{HOM}(\mathbb{D}_{B\otimes C^{\mathrm{op}}}, \mathbb{D}_{A\otimes C^{\mathrm{op}}}).$$

In case we take C to be the monoidal unit $k[0]$ we get a map

$$\mathbb{D}_{A\otimes B^{\mathrm{op}}} \longrightarrow \mathsf{HOM}(\mathbb{D}_B, \mathbb{D}_A)$$

from the derivator of bimodules to the derivator of morphisms. Specializing further to the situation of $A = B$ we obtain an action of the monoidal derivator $\mathbb{D}_{A\otimes A^{\mathrm{op}}}$ on $\mathbb{D}_A$. By Theorem 2.11, this action induces a *monoidal* morphism of derivators

$$\mathbb{D}_{A\otimes A^{\mathrm{op}}} \longrightarrow \mathsf{HOM}(\mathbb{D}_A, \mathbb{D}_A).$$

Example 2.17. Let A be a differential-graded algebra over k which is cofibrant as an object of $\mathsf{Ch}(k)$. Then we have an adjunction of two variables

$$\otimes_A^{\mathbb{L}}\colon \mathbb{D}_{A\otimes A^{\mathrm{op}}} \times \mathbb{D}_A \rightharpoonup \mathbb{D}_A$$

exhibiting $\mathbb{D}_A$ as an additive left $\mathbb{D}_{A\otimes A^{\mathrm{op}}}$-module. In particular, the derivator $\mathbb{D}_A$ can canonically be endowed with a linear structure over the Ext-algebra of A via a ring map

$$\hom^\bullet_{D(A\otimes A^{\mathrm{op}})}(A,A)\longrightarrow \mathsf{Z}^\bullet(\mathbb{D}_A).$$

Under our cofibrancy condition (cf. [Kel94, Example 6.6]) one can identify this Ext-algebra with $HH^\bullet(A,A)$, the Hochschild cohomology of A. Thus, in that case we obtain that the derivator $\mathbb{D}_A$ is canonically linear over the Hochschild cohomology of A.

This example can still be generalized if one sticks to the 'many objects versions' ([Mit72]) of differential-graded algebras, i.e., to small differential-graded categories. What we are about to do can be done axiomatically with $\mathsf{Ch}(k)$ replaced by a sufficiently nice closed monoidal model category but we prefer to give some details in the case of $\mathsf{Ch}(k)$. In the corresponding examples where the role of $\mathsf{Ch}(k)$ is taken by the category Sp^Σ of symmetric spectra (based on simplicial sets) we will be much shorter.

Recall that every biclosed monoidal category is canonically enriched over itself (this is a special case of the result that an adjunction of two variables such that the action is part of a module structure induces a canonical enrichment on the module). In particular, the category $\mathsf{Ch}(k)$ is canonically enriched over itself. Thus, given a small dg-category J it makes sense to consider the dg-functors from J to $\mathsf{Ch}(k)$. Spelling out this definition, such a dg-functor X associates to each object $j\in J$ a chain complex $X(j)\in\mathsf{Ch}(k)$ together with action maps

$$\mathsf{Hom}_J(j_1,j_2)\otimes_k X(j_1)\longrightarrow X(j_2).$$

These maps are supposed to be coherently associative and unital. Taking as morphisms of such dg-functors the dg-natural transformations we obtain the category $J-\mathsf{Mod}$ of *J-modules*. The category $A-\mathsf{Mod}$ can be considered as a special case of this situation: Given a differential-graded algebra A we can associate a dg-category J_A with one object and A as endomorphism object. It is easy to see that in that case $J_A-\mathsf{Mod}$ is canonically isomorphic to $A-\mathsf{Mod}$. As a special case of [SS03a, Theorem 6.1] we deduce that $J-\mathsf{Mod}$ can be endowed with the projective model structure. Moreover, that same theorem guarantees that the model structure is cofibrantly generated and it is also true that the category $J-\mathsf{Mod}$ is presentable. Thus, we have the combinatorial model category $J-\mathsf{Mod}$ and can consequently consider the associated derivator

$$\mathbb{D}_J=\mathbb{D}(J-\mathsf{Mod}).$$

The passage to bimodules involves the following additional bit of enriched category theory which can be done as soon as the enrichment level is symmetric. Given two dg-categories J and K we can form the dg-category $J\otimes K$ as follows. The objects are given by pairs (j,k) consisting of an object $j\in J$ and an object $k\in K$. Given two such pairs, the chain complex of morphisms is defined to be

$$\mathsf{Hom}_{J\otimes K}((j_1,k_1),(j_2,k_2))=\mathsf{Hom}_J(j_1,j_2)\otimes\mathsf{Hom}_K(k_1,k_2).$$

The symmetry constraint allows one to define a composition law. Similarly, one can use the symmetry constraint in order to define the opposite of a dg-category. The opposite J^{op}

of J has the same objects as J but the morphism complexes are defined by

$$\mathsf{Hom}_{J^{\mathrm{op}}}(j_1,j_2)=\mathsf{Hom}_J(j_2,j_1).$$

Given two small dg-categories J and K, we can now combine these two constructions and define the category of $J-K$-bimodules as

$$J-\mathsf{Mod}-K=(J\otimes K^{\mathrm{op}})-\mathsf{Mod}\,.$$

The next step is to give a generalization of the tensor product over a dga. So, let us consider three small dg-categories J, K, and L and two bimodules $X\in J-\mathsf{Mod}-K$, $Y\in K-\mathsf{Mod}-L$. Then we define $X\otimes_K Y\in J-\mathsf{Mod}-L$ by the following enriched coend construction (cf. [Dub70, Section 1.3]). For $j\in J$ and $l\in L$ we define $X\otimes_K Y$ evaluated at (j,l) to be the coequalizer of

$$\bigoplus\nolimits_{k_1,k_2\in K} X(j,k_2)\otimes K(k_1,k_2)\otimes Y(k_1,l)\overset{\lambda}{\underset{\rho}{\rightrightarrows}}\bigoplus\nolimits_{k\in K} X(j,k)\otimes Y(k,l).$$

Here, the two morphisms are induced by the action maps given by the K-module structures. This tensor product $\otimes_K$ is part of an adjunction of two variables

$$\otimes_K\colon J-\mathsf{Mod}-K\times K-\mathsf{Mod}-L\rightharpoonup J-\mathsf{Mod}-L.$$

It can be shown ([Hei07, Appendix A]) that $\otimes_K$ is a left Brown functor as soon as K is locally cofibrant, i.e., if all mapping objects of K are cofibrant. If we assume this we can specialize as in the one object case and obtain a morphism of derivators

$$\mathbb{D}_{J\otimes K^{\mathrm{op}}}\longrightarrow\mathsf{HOM}(\mathbb{D}_K,\mathbb{D}_J).$$

In the case $J=K$ this morphism is monoidal and we thus obtain the next example. Recall that the monoidal unit in the category $K-\mathsf{Mod}-K$ of bimodules is given by the hom-functor K itself.

Example 2.18. Let K be a small dg-category which is locally cofibrant then we have an adjunction of two variables

$$\otimes_K^{\mathbb{L}}\colon\ \mathbb{D}_{K\otimes K^{\mathrm{op}}}\times\mathbb{D}_K\rightharpoonup\mathbb{D}_K$$

exhibiting $\mathbb{D}_K$ as a left $\mathbb{D}_{K\otimes K^{\mathrm{op}}}$-module. Thus, $\mathbb{D}_K$ is canonically endowed with a linear structure over $\hom^\bullet_{D(K\otimes K^{\mathrm{op}})}(K,K)$. Again, under our cofibrancy condition this can be identified with the *Hochschild-Mitchell cohomology* $HH^\bullet(K,K)$ of the small dg-category K.

Similar examples are obtained if we replace chain complexes by symmetric spectra. For this purpose let us endow Sp^Σ with the absolute projective stable model structure which interacts nicely with the smash product. Then, for a ring spectrum R the category $\mathsf{Mod}-R$ of right R-modules is a stable, combinatorial model category when endowed with the projective model structure giving rise to the stable derivator $\mathbb{D}_{R^{\mathrm{op}}}$. Moreover, the smash product

$$\wedge\colon\ \mathsf{Sp}\times\mathsf{Mod}-R\longrightarrow\mathsf{Mod}-R$$

turns $\mathsf{Mod}-R$ into a spectral model category.

Example 2.19. For a symmetric ring spectrum R, we have an adjunction of two variables

$$\mathbb{D}_{\mathsf{Sp}} \times \mathbb{D}_{R^{\mathrm{op}}} \rightharpoonup \mathbb{D}_{R^{\mathrm{op}}}$$

turning $\mathbb{D}_{R^{\mathrm{op}}}$ into an additive left $\mathbb{D}_{\mathsf{Sp}}$-module. In particular, $\mathbb{D}_{R^{\mathrm{op}}}$ is canonically endowed with a graded linear structure over the stable homotopy groups of spheres

$$\pi_\bullet^S \longrightarrow \mathsf{Z}_\bullet(\mathbb{D}_{R^{\mathrm{op}}}).$$

The same reasoning as in the differential-graded context leads to results about bimodules. Given two symmetric ring spectra R and S such that the underlying spectrum of S is cofibrant we obtain a morphism of derivators

$$\mathbb{D}_{R\wedge S^{\mathrm{op}}} \longrightarrow \mathsf{HOM}(\mathbb{D}_S, \mathbb{D}_R).$$

Let us emphasize that we are working with the *flat* stable model structure so that this cofibrancy condition is not an empty condition. In the case of $R = S$ we can again apply Theorem 2.11 in order to obtain a monoidal morphism of derivators $\mathbb{D}_{R\wedge R^{\mathrm{op}}} \longrightarrow \mathsf{END}(\mathbb{D}_R)$. This can be specialized to the following result.

Example 2.20. Let R be a symmetric ring spectrum such that the underlying spectrum is cofibrant. Then we have an adjunction of two variables

$$\wedge_R^{\mathbb{L}} \colon \mathbb{D}_{R\wedge R^{\mathrm{op}}} \times \mathbb{D}_R \rightharpoonup \mathbb{D}_R$$

which turns $\mathbb{D}_R$ into an additive left $\mathbb{D}_{R\wedge R^{\mathrm{op}}}$-module. In particular, $\mathbb{D}_R$ is canonically endowed with a linear structure over $\mathsf{hom}^\bullet_{D(R\wedge R^{\mathrm{op}})}(R,R)$. By [DS07c, 4.4] this graded ring can be identified with the graded homotopy groups of the Topological Hochschild cohomology spectrum $THH(R,R)$ of R.

For completeness, let us quickly mention the many object variant thereof. Given small spectral categories J and K such that K is locally cofibrant, the smash product over K induces a morphism of derivators

$$\mathbb{D}_{J\wedge K^{\mathrm{op}}} \longrightarrow \mathsf{HOM}(\mathbb{D}_K, \mathbb{D}_J).$$

Specializing to $J = K$ we finally get

Example 2.21. Let J be a locally cofibrant spectral category. We then have an adjunction of two variables

$$\wedge_J^{\mathbb{L}} \colon \mathbb{D}_{J\wedge J^{\mathrm{op}}} \times \mathbb{D}_J \rightharpoonup \mathbb{D}_J$$

endowing $\mathbb{D}_J$ with the structure of an additive left $\mathbb{D}_{J\wedge J^{\mathrm{op}}}$-module. In particular, this induces a graded linear structure on $\mathbb{D}_J$:

$$\mathsf{hom}^\bullet_{D(J\wedge J^{\mathrm{op}})}(J,J) \longrightarrow Z^\bullet(\mathbb{D}_J)$$

3. Enriched derivators

3.1. **The 2-Grothendieck opfibration of enriched categories.** To motivate the construction of this subsection let us quickly recall the following. In Section 1 we defined a monoidal prederivator as a monoidal object in PDer. In Section 2 we defined a left-tensored prederivator as a module object in the same 2-category. In both cases we saw that the respective notion can be equivalently defined as a 2-functor

$$\mathbb{D}\colon \mathsf{Cat}^{\mathrm{op}} \longrightarrow \mathsf{MonCAT} \qquad \text{resp.} \qquad \mathbb{D}\colon \mathsf{Cat}^{\mathrm{op}} \longrightarrow \mathsf{ModCAT}\,.$$

In this subsection we want to construct a target 2-category ECAT of enriched categories which will then be used in the definition of an enriched prederivator as a 2-functor

$$\mathbb{D}\colon \mathsf{Cat}^{\mathrm{op}} \longrightarrow \mathsf{ECAT}\,.$$

Recall that given a monoidal category $\mathcal{M}$ we have the notion of categories enriched over $\mathcal{M}$. An $\mathcal{M}-$enriched category $\mathcal{C}$ consists of the following. First, we are given a class of objects $\mathcal{C}_0$ and for two such objects $X,\ Y \in \mathcal{C}_0$ we have a mapping object $\mathsf{Hom}_{\mathcal{C}}(X,Y) \in \mathcal{M}$. Moreover, for each object X there is a 'unit map' specified by a morphism $\mathbb{S} \longrightarrow \mathsf{Hom}_{\mathcal{C}}(X,X)$ and for each triple X, Y, and Z of objects we have a composition morphism $\mathsf{Hom}_{\mathcal{C}}(Y,Z) \otimes \mathsf{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathsf{Hom}_{\mathcal{C}}(X,Z)$. These data are subject to the expected unitality and associativity conditions. For details see for example [Kel05a, Bor94b]. There is also a notion of enriched functors and enriched natural transformations over a fixed monoidal category $\mathcal{M}$ so that we have in fact the 2-category $\mathcal{M} - \mathsf{CAT}$ of $\mathcal{M}$-enriched categories. In the special case where the monoidal category is given by the Cartesian monoidal category Set of sets enriched category theory reduces to classical category theory.

Enriched category theory has the nice feature that given a monoidal functor $F\colon \mathcal{M} \longrightarrow \mathcal{N}$ we obtain an induced *base change* 2*-functor*

$$F_*\colon \mathcal{M} - \mathsf{CAT} \longrightarrow \mathcal{N} - \mathsf{CAT}\,.$$

For convenience let us quickly recall the construction at least on objects. Given an $\mathcal{M}$-enriched category $\mathcal{C}$ then the $\mathcal{N}$-enriched category $F_*\mathcal{C}$ is defined to have the same class of objects. The mapping objects are given by $\mathsf{Hom}_{F_*\mathcal{C}}(X,Y) = F(\mathsf{Hom}_{\mathcal{C}}(X,Y))$. Since F is a monoidal functor we obtain unit maps in $F_*\mathcal{C}$ by taking

$$\mathbb{S}_{\mathcal{N}} \longrightarrow F(\mathbb{S}_{\mathcal{M}}) \longrightarrow F(\mathsf{Hom}_{\mathcal{C}}(X,X)).$$

The first map is given by the monoidal structure on F while the second one is the image under F of the unit map of X in the $\mathcal{M}$-enriched category $\mathcal{C}$. Similarly, using the other part of the monoidal structure on F one defines a composition law in $F_*\mathcal{C}$ and it is straightforward to check that this defines an $\mathcal{N}$-enriched category $F_*\mathcal{C}$.

Example 3.1. Let $\mathcal{M}$ be a monoidal category and $\mathbb{S}$ the unit object. The *functor of elements* $\mathcal{M} \longrightarrow \mathsf{Set}\colon M \longmapsto \mathsf{hom}_{\mathcal{M}}(\mathbb{S}, M)$ can be canonically endowed with the structure of a monoidal functor. Thus, we obtain an induced 2-functor

$$U = U_{\mathcal{M}}\colon \mathcal{M} - \mathsf{CAT} \longrightarrow \mathsf{CAT}$$

which sends an $\mathcal{M}-$enriched category to its *underlying category.*

Let us also give the following well-known more specific examples.

Example 3.2. i) Let k be a commutative ring and let us consider the categories $\mathsf{Mod}(k)$, $\mathsf{grMod}(k)$, and $\mathsf{Ch}(k)$ of k-modules, $\mathbb{Z}-$graded k-modules, and unbounded chain complexes over k respectively. The homology functor $H_*\colon \mathsf{Ch}(k) \longrightarrow \mathsf{grMod}(k)$ and the evaluation at zero functor $\mathsf{grMod}(k) \longrightarrow \mathsf{Mod}(k)$ are canonically monoidal functors. Thus, for every dg-category $\mathcal{C}$ [Kel06b, Toë07] we have two associated homology categories, namely the graded one $H_*\mathcal{C}$ and the ungraded one $H_0\mathcal{C}$.
ii) The functor $\pi_0\colon \mathsf{Set}_\Delta \longrightarrow \mathsf{Set}$ sending a simplicial set to its set of path components preserves products. We thus have a base change functor which sends a simplicial category $\mathcal{C}$ to its ('naive') homotopy category $\pi_0\mathcal{C}$. For a more concrete example, let $\mathcal{C}$ be a simplicial model category and let us denote by $\mathcal{C}_{cf}$ the full simplicial subcategory of $\mathcal{C}$ spanned by the bifibrant objects. Since for maps between bifibrant objects the left homotopy, the right homotopy, and the simplicial homotopy relations coincide we have that $\pi_0\mathcal{C}_{cf}$ is the classical homotopy category of $\mathcal{C}$ as described in [DS95].

Note that there is also a similar base change construction for monoidal transformations of monoidal functors. So, let $F, G\colon \mathcal{M} \longrightarrow \mathcal{N}$ be monoidal functors such that we have induced base change 2-functors $F_*, G_*\colon \mathcal{M}-\mathsf{CAT} \longrightarrow \mathcal{N}-\mathsf{CAT}$. If we have in addition a monoidal transformation $\beta\colon F \longrightarrow G$ we can construct a 2-natural transformation $\beta_*\colon F_* \longrightarrow G_*$ as follows. For an $\mathcal{M}$-enriched category $\mathcal{C}$ we obtain an $\mathcal{N}$-enriched functor $\beta_*\colon F_*\mathcal{C} \longrightarrow G_*\mathcal{C}$ which is the identity on objects by setting

$$(\beta_*)_{X,Y} = \beta_{\mathsf{Hom}_\mathcal{C}(X,Y)}\colon F\,\mathsf{Hom}_\mathcal{C}(X,Y) \longrightarrow G\,\mathsf{Hom}_\mathcal{C}(X,Y).$$

The coherence conditions imposed on a monoidal natural transformation guarantee that β_* is in fact an $\mathcal{N}$-enriched functor. For example, the compatibility with the composition is ensured by the following commutative diagram in which the left square commutes since β is monoidal:

$$\begin{tikzcd}
F\,\mathsf{Hom}_\mathcal{C}(Y,Z)\otimes F\,\mathsf{Hom}_\mathcal{C}(X,Y) \arrow[r] \arrow[d, "\beta\otimes\beta"'] & F(\mathsf{Hom}_\mathcal{C}(Y,Z)\otimes \mathsf{Hom}_\mathcal{C}(X,Y)) \arrow[r] \arrow[d, "\beta"] & F\,\mathsf{Hom}_\mathcal{C}(X,Z) \arrow[d, "\beta"] \\
G\,\mathsf{Hom}_\mathcal{C}(Y,Z)\otimes G\,\mathsf{Hom}_\mathcal{C}(X,Y) \arrow[r] & G(\mathsf{Hom}_\mathcal{C}(Y,Z)\otimes \mathsf{Hom}_\mathcal{C}(X,Y)) \arrow[r] & G\,\mathsf{Hom}_\mathcal{C}(X,Z)
\end{tikzcd}$$

These constructions taken together give the following result.

Proposition 3.3. *The assignments $\mathcal{M} \longmapsto \mathcal{M}-\mathsf{CAT}$, $F \longmapsto F_*$, and $\beta \longmapsto \beta_*$ define a 2-functor*

$$(-)-\mathsf{CAT}\colon \mathsf{MonCAT} \longrightarrow 2\text{-}\mathsf{CAT}.$$

Thus, the 2-categorical Grothendieck construction of Appendix A can be applied to this 2-functor and yields a single 2-category ECAT of enriched categories together with a projection functor $p\colon \mathsf{ECAT} \longrightarrow \mathsf{MonCAT}$. Let us call p the 2-*Grothendieck opfibration of enriched categories.*

Let us describe ECAT in some more detail since this will be helpful in the remainder of this section. The objects of ECAT are given by pairs $(\mathcal{M},\mathcal{C})$ where $\mathcal{M}$ is a monoidal category and $\mathcal{C}$ is an $\mathcal{M}$-enriched category. Given two such objects $(\mathcal{M},\mathcal{C})$ and $(\mathcal{N},\mathcal{D})$, a morphism $(\mathcal{M},\mathcal{C}) \longrightarrow (\mathcal{N},\mathcal{D})$ is a pair (u,f) where $u\colon \mathcal{M} \longrightarrow \mathcal{N}$ is a monoidal functor and $f\colon u_*\mathcal{C} \longrightarrow \mathcal{D}$ is an $\mathcal{N}$-enriched functor. The first component of the composition of two such composable morphisms $(u,f)\colon (\mathcal{M},\mathcal{C}) \longrightarrow (\mathcal{N},\mathcal{D})$ and $(v,g)\colon (\mathcal{N},\mathcal{D}) \longrightarrow (\mathcal{P},\mathcal{E})$ is $vu\colon \mathcal{M} \longrightarrow \mathcal{P}$. The second component is given by the $\mathcal{P}$-enriched functor

$$v_*u_*\mathcal{C} \xrightarrow{v_*f} v_*\mathcal{D} \xrightarrow{g} \mathcal{E}.$$

It is obvious that the identity of an object $(\mathcal{M},\mathcal{C})$ with respect to this composition is given by the morphism $(\mathrm{id}_{\mathcal{M}},\mathrm{id}_{\mathcal{C}})$. Now, let us turn to 2-morphisms. So let us assume (u,f) and (v,g) to be a pair of parallel morphisms $(\mathcal{M},\mathcal{C}) \longrightarrow (\mathcal{N},\mathcal{D})$. A 2-morphism $(u,f) \longrightarrow (v,g)$ is a pair (β,α) where $\beta\colon u \longrightarrow v$ is a monoidal natural transformation and $\alpha\colon f \longrightarrow g\circ(\beta_*)_{\mathcal{C}}$ is an $\mathcal{N}$-enriched natural transformation as indicated in:

$$\begin{tikzcd} u_*\mathcal{C} \arrow[d, "\beta_*"'] \arrow[dr, bend left, "f"] \arrow[dr, phantom, "\swarrow"] & \\ v_*\mathcal{C} \arrow[r, "g"'] & \mathcal{D} \end{tikzcd}$$

Note that we have just seen that β_* is an $\mathcal{N}$-enriched functor so that this definition makes sense. Moreover, given an object $(\mathcal{M},\mathcal{C})$, a morphism (u,f), or a 2-morphism (β,α) in ECAT we can project onto the first component $\mathcal{M}$, u, or β in order to obtain a monoidal category, a monoidal functor or a monoidal natural transformation respectively. This describes the 2-Grothendieck opfibration $p\colon \mathsf{ECAT} \longrightarrow \mathsf{MonCAT}$.

The remaining aim of this subsection is to show that we can elaborate on Example 3.1 in a way that the formation of underlying categories defines a 2-functor $U\colon \mathsf{ECAT} \longrightarrow \mathsf{CAT}$. To begin with let $(\mathcal{M},\mathcal{C})$ be an object of ECAT then we associate to it the underlying category $U(\mathcal{M},\mathcal{C}) = U_{\mathcal{M}}(\mathcal{C})$. Now, let $(u,f)\colon (\mathcal{M},\mathcal{C}) \longrightarrow (\mathcal{N},\mathcal{D})$ be a morphism in ECAT. Then, we obtain a functor

$$U(u,f)\colon U_{\mathcal{M}}\mathcal{C} \longrightarrow U_{\mathcal{N}}\mathcal{D}$$

as the composition of the following two (unenriched) functors

$$U_{\mathcal{M}}\mathcal{C} \longrightarrow U_{\mathcal{N}}(u_*\mathcal{C}) \xrightarrow{U_{\mathcal{N}}f} U_{\mathcal{N}}\mathcal{D}.$$

Here, the first arrow is the functor which is the identity on objects and is given on morphism sets by the composition

$$\mathsf{hom}_{\mathcal{M}}(\mathbb{S},\mathsf{Hom}_{\mathcal{C}}(X,Y)) \xrightarrow{u} \mathsf{hom}_{\mathcal{N}}(u\,\mathbb{S},\mathsf{Hom}_{u_*\mathcal{C}}(X,Y)) \xrightarrow{\cong} \mathsf{hom}_{\mathcal{N}}(\mathbb{S},\mathsf{Hom}_{u_*\mathcal{C}}(X,Y)).$$

Using the monoidal structure on u one checks that this indeed defines a functor. The isomorphism in this composition is of course given by the monoidal structure on u. It remains only to define the value of U on 2-morphisms in ECAT. So, let (u,f) and (v,g) be parallel morphisms $(\mathcal{M},\mathcal{C}) \longrightarrow (\mathcal{N},\mathcal{D})$ and let $(\beta,\alpha)\colon (u,f) \longrightarrow (v,g)$ be such a 2-morphism. Recall that we hence have, in particular, an $\mathcal{N}$-natural transformation $\alpha\colon f \longrightarrow$

$g \circ \beta_*$. Let us consider the following diagram in which the rows are given by the value of U at (u, f) resp. (v, g) and where the 2-morphism is given by $U_{\mathcal{N}}\alpha$:

$$\begin{tikzcd} U_{\mathcal{M}}\mathcal{C} \arrow[r] \arrow[d, equal] & U_{\mathcal{N}}(u_*\mathcal{C}) \arrow[r, "U_{\mathcal{N}}f"] \arrow[d, "U_{\mathcal{N}}\beta_*"'] \arrow[dr, phantom, "\swarrow"] & U_{\mathcal{N}}\mathcal{D} \arrow[d, equal] \\ U_{\mathcal{M}}\mathcal{C} \arrow[r] & U_{\mathcal{N}}(v_*\mathcal{C}) \arrow[r, "U_{\mathcal{N}}g"] & U_{\mathcal{N}}\mathcal{D} \end{tikzcd}$$

Using the fact that β is a monoidal transformation it follows that the left square commutes. So, let us define the value of $U\colon \mathsf{ECAT} \longrightarrow \mathsf{CAT}$ on the 2-morphism (β, α) to be the composite natural transformation of this diagram.

Proposition 3.4. *The above constructions define a* 2*-functor* $U\colon \mathsf{ECAT} \longrightarrow \mathsf{CAT}$.

Proof. We will not give the details of the proof since it is quite lengthy but essentially straightforward. Let us only show that U preserves the composition of morphisms. For that purpose let $(\mathcal{M}, \mathcal{C}) \xrightarrow{(u,f)} (\mathcal{N}, \mathcal{D}) \xrightarrow{(v,g)} (\mathcal{P}, \mathcal{E})$ be a pair of composable morphisms in ECAT . By definition of the composition of morphisms in ECAT we have $(v, g)\circ(u, f) = (vu, g\circ v_*f)$. Both functors $U(v, g) \circ U(u, f)$ and $U((v, g) \circ (u, f))$ send an object $X \in U(\mathcal{M}, \mathcal{C})$ to $gf(X) \in U(\mathcal{P}, \mathcal{E})$. Thus, it remains to show that both functors have the same behavior on morphisms. The functor $U((v, g) \circ (u, f))$ sends a morphism $\phi\colon \mathbb{S} \longrightarrow \mathsf{Hom}_{\mathcal{C}}(X, Y)$ in $U_{\mathcal{M}}\mathcal{C}$ to the composition

$$\mathbb{S} \cong v(\mathbb{S}) \cong vu(\mathbb{S}) \xrightarrow{vu\phi} vu\,\mathsf{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{vf} v\,\mathsf{Hom}_{\mathcal{D}}(fX, fY) \xrightarrow{g} \mathsf{Hom}_{\mathcal{E}}(gfX, gfY).$$

On the other hand, $U(u, f)$ maps such a morphism ϕ to

$$\mathbb{S} \cong u(\mathbb{S}) \xrightarrow{u\phi} u\,\mathsf{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{f} \mathsf{Hom}_{\mathcal{D}}(fX, fY)$$

which is then sent to $U((v, g) \circ (u, f))\phi$ by $U(v, g)$. □

Thus, the upshot of this subsection is that we have constructed the 2-category ECAT of enriched categories together with two 2-functors:

$$\begin{tikzcd} \mathsf{ECAT} \arrow[r, "U"] \arrow[d, "p"'] & \mathsf{CAT} \\ \mathsf{MonCAT} & \end{tikzcd}$$

This will allow us to give compact definitions of enriched (pre)derivators in the next subsection.

3.2. **Enriched derivators.** After the preparations of the last subsection we can immediately give the following definition.

Definition 3.5. An *enriched prederivator* $\mathbb{D}$ is a 2-functor $\mathbb{D}\colon \mathsf{Cat}^{\mathrm{op}} \longrightarrow \mathsf{ECAT}$. Given such a $\mathbb{D}$ it is said to be *enriched over the monoidal prederivator* $\mathbb{E} = p \circ \mathbb{D}$ while the prederivator $U \circ \mathbb{D}$ is called the *underlying prederivator.*

Note that by the very definition an enriched prederivator is not a prederivator but – parallel to classical enriched category theory– it canonically has an underlying prederivator.

Let us unravel this definition a bit. A prederivator $\mathbb{D}$ enriched over a monoidal prederivator $\mathbb{E}$ gives us for each category $K \in \mathsf{Cat}$ a category $\mathbb{D}(K)$ enriched over $\mathbb{E}(K)$. Moreover, for a functor $u\colon J \longrightarrow K$, the monoidal prederivator $\mathbb{E}$ induces a monoidal functor $\mathbb{E}(u)\colon \mathbb{E}(K) \longrightarrow \mathbb{E}(J)$ which has an associated base change 2-functor. Then, the enriched prederivator assigns to the functor u a morphism

$$(\mathbb{E}(u), \mathbb{D}(u))\colon (\mathbb{E}(K), \mathbb{D}(K)) \longrightarrow (\mathbb{E}(J), \mathbb{D}(J))$$

in ECAT. Thus, we have an $\mathbb{E}(J)$-enriched functor $\mathbb{D}(u)\colon \mathbb{E}(u)_*\mathbb{D}(K) \longrightarrow \mathbb{D}(J)$. There are similar assignments for natural transformations and these satisfy certain coherence relations and all this is nicely hidden by the construction of ECAT and the associated 2-functors.

From now on, given a prederivator $\mathbb{D}$ enriched over a monoidal prederivator $\mathbb{E}$ we will commit a slight abuse of notation and write u^* for both $\mathbb{E}(u)$ and $\mathbb{D}(u)$ and similarly for natural transformations. It will always be clear from the context which one of the two is meant.

There is also the notion of an *enriched derivator* which is an enriched prederivator such that the underlying prederivator is a derivator. Similarly, an *enrichment of a derivator* $\mathbb{D}$ over a monoidal prederivator $\mathbb{E}$ is given by an $\mathbb{E}$-enriched derivator $\mathbb{D}'$ and an isomorphism $U \circ \mathbb{D}' \cong \mathbb{D}$.

Let us give an immediate example. Given a commutative ring k the monoidal category $k-\mathsf{Mod}$ of k-modules gives us the constant monoidal prederivator $\mathsf{Cat}^{\mathrm{op}} \longrightarrow e \xrightarrow{k-\mathsf{Mod}} \mathsf{CAT}$.

Example 3.6. Let $\mathbb{D}$ be an additive derivator (e.g. a stable derivator). Then there is a canonical enrichment of $\mathbb{D}$ over the monoidal prederivator with constant value $\mathbb{Z}-\mathsf{Mod}$. Similarly, let $\mathbb{D}$ be an additive derivator and let $\sigma\colon k \longrightarrow \mathsf{Z}(\mathbb{D})$ be a k-linear structure on it. Then there is a canonical enrichment of $\mathbb{D}$ over the monoidal prederivator with constant value $k-\mathsf{Mod}$.

Example 3.7. Let $\mathcal{M}$ be a bicomplete monoidal closed category and let us also denote by $\mathcal{M}$ the associated constant monoidal derivator. Moreover, let $\mathcal{C}$ be a category enriched over $\mathcal{M}$. Recall that for a small category J, the ordinary functor category $\mathsf{Fun}(J,\mathcal{C})$ can be enriched over $\mathcal{M}$. Given two functors $F, G\colon J \longrightarrow \mathcal{C}$ there is an object $\mathsf{Nat}(F,G) \in \mathcal{M}$ of natural transformations defined by the following end formula:

$$\mathsf{Nat}(F,G) = \int_J \mathsf{Hom}(Fj, Gj)$$

In fact, since we assumed $\mathcal{M}$ to have coproducts we can consider the free $\mathcal{M}$-enriched category $\mathbb{S}\,J$ on the ordinary category J. Then this construction is just a special case of an $\mathcal{M}$-enriched category of $\mathcal{M}$-enriched functors. These $\mathcal{M}$-enriched functor categories assemble to define an $\mathcal{M}$-enriched prederivator which provides us with an enrichment of the prederivator represented by the underlying category of $\mathcal{C}$. For the corresponding statement in enriched category theory cf. [Kel05a, Section 2.5].

Definition 3.8. Let $\mathbb{E}$ be a monoidal prederivator and let $\mathbb{D}$ and $\mathbb{D}'$ be $\mathbb{E}$-enriched prederivators. A *morphism of* $\mathbb{E}$*-enriched prederivators* $\mathbb{D} \longrightarrow \mathbb{D}'$ is a pseudo-natural transformation $F\colon \mathbb{D} \longrightarrow \mathbb{D}'$ of 2-functors $\mathsf{Cat}^{\mathrm{op}} \longrightarrow \mathsf{ECAT}$ such that $p \circ F = \mathrm{id}_{\mathbb{E}}$.

Unraveling the definition such a morphism consists of an $\mathbb{E}(K)$-enriched functor for each category K and an $\mathbb{E}(J)$-natural isomorphism as indicated in

$$\begin{tikzcd} (u^*)_*\,\mathbb{D}(K) \arrow[r] \arrow[d] \arrow[dr, phantom, "\not\swarrow"] & (u^*)_*\,\mathbb{D}'(K) \arrow[d] \\ \mathbb{D}(J) \arrow[r] & \mathbb{D}'(J) \end{tikzcd}$$

for each functor $u\colon J \longrightarrow K$. Here, the vertical morphisms are the structure morphisms of the enriched prederivators while the horizontal ones belong to the morphism of enriched prederivators. These data have to satisfy certain coherence properties which are precisely the same as in the case of a morphism of unenriched prederivators. As in the unenriched case the direction of the above natural isomorphism is not important since we can always pass to its inverse. With a similar notion of $\mathbb{E}$-enriched natural transformations we obtain thus the 2-category of $\mathbb{E}$-enriched prederivators and the full 2-subcategory spanned by the $\mathbb{E}$-enriched derivators which are denoted by

$$\mathbb{E} - \mathsf{PDer} \qquad \text{resp.} \qquad \mathbb{E} - \mathsf{Der}\,.$$

We now give an analog in the theory of derivators of the following result from category theory. Let us consider an adjunction of two variables

$$(\otimes, \mathsf{Hom}_l, \mathsf{Hom}_r)\colon \mathcal{C} \times \mathcal{D} \rightharpoonup \mathcal{D}$$

exhibiting $\mathcal{D}$ as a closed $\mathcal{C}$-module. Then, there is a canonical enrichment of $\mathcal{D}$ over $\mathcal{C}$ where the mapping objects are given by Hom_r. We will now establish the corresponding result for derivators which will then be used to give important examples of enriched derivators. As a preparation for the proof let us give the following lemma in which we only state the results for Hom_r. Similar results are also valid for Hom_l.

Lemma 3.9. *Let* $\mathbb{D}$, $\mathbb{E}$, *and* $\mathbb{F}$ *be prederivators and let us consider an adjunction of two variables* $(\otimes, \mathsf{Hom}_l, \mathsf{Hom}_r)\colon \mathbb{D} \times \mathbb{E} \rightharpoonup \mathbb{F}$. *The adjunction units and counits at the different levels are compatible in the following sense. For a functor* $u\colon J \longrightarrow K$ *and objects* $X \in$

$\mathbb{D}(K)$, $Y \in \mathbb{E}(K)$, *and* $Z \in \mathbb{F}(K)$ *the following diagrams commute:*

$$\begin{tikzcd}
u^*X \arrow[r, "u^*\eta"] \arrow[d, "\eta"'] & u^*\,\mathsf{Hom}_r(Y, X\otimes Y) \arrow[d, "\gamma^{\mathsf{Hom}_r}"] \\
\mathsf{Hom}_r(u^*Y, u^*X\otimes u^*Y) \arrow[r, "\gamma^{\otimes}"'] & \mathsf{Hom}_r(u^*Y, u^*(X\otimes Y))
\end{tikzcd}$$

$$\begin{tikzcd}
u^*\,\mathsf{Hom}_r(Y,Z)\otimes u^*Y \arrow[r, "\gamma^{\mathsf{Hom}_r}"] \arrow[d, "\gamma^{\otimes}"'] & \mathsf{Hom}_r(u^*Y, u^*Z)\otimes u^*Y \arrow[d, "\epsilon"] \\
u^*(\mathsf{Hom}_r(Y,Z)\otimes Y) \arrow[r, "u^*\epsilon"'] & u^*Z
\end{tikzcd}$$

Proof. Let us begin with the statement about the adjunction units. Recall from Subsection 1.3 that in the context of an adjunction of two variables the adjunctions at the different levels are compatible with each other. This is expressed by the commutativity of the upper rectangle in the next diagram for the special case where we chose $Z = X\otimes Y$:

$$\begin{tikzcd}
\mathsf{hom}_{\mathbb{F}(K)}(X\otimes Y, X\otimes Y) \arrow[r] \arrow[d, "u^*"'] & \mathsf{hom}_{\mathbb{D}(K)}(X, \mathsf{Hom}_r(Y, X\otimes Y)) \arrow[d, "u^*"] \\
\mathsf{hom}_{\mathbb{F}(J)}(u^*(X\otimes Y), u^*(X\otimes Y)) \arrow[d, "\gamma^{\otimes}"'] & \mathsf{hom}_{\mathbb{D}(J)}(u^*X, u^*\,\mathsf{Hom}_r(Y, X\otimes Y)) \arrow[d, "\gamma^{\mathsf{Hom}_r}"] \\
\mathsf{hom}_{\mathbb{F}(J)}(u^*X\otimes u^*Y, u^*(X\otimes Y)) \arrow[r] \arrow[d, "\gamma^{\otimes}"'] & \mathsf{hom}_{\mathbb{D}(J)}\big(u^*X, \mathsf{Hom}_r(u^*Y, u^*(X\otimes Y))\big) \arrow[d, "\gamma^{\otimes}"] \\
\mathsf{hom}_{\mathbb{F}(J)}(u^*X\otimes u^*Y, u^*X\otimes u^*Y) \arrow[r] & \mathsf{hom}_{\mathbb{D}(J)}\big(u^*X, \mathsf{Hom}_r(u^*Y, u^*X\otimes u^*Y)\big)
\end{tikzcd}$$

Starting with the identity in the upper left corner and comparing its two images in the bottom right corner we obtain the compatibility statement about the units. The corresponding result for the adjunction counits is obtained in a very similar manner. For this purpose, let us take $X = \mathsf{Hom}_r(Y,Z)$ and let us consider the following commutative diagram:

$$\begin{tikzcd}
\mathsf{hom}_{\mathbb{F}(K)}(\mathsf{Hom}_r(Y,Z)\otimes Y, Z) \arrow[d, "u^*"'] & \mathsf{hom}_{\mathbb{D}(K)}(\mathsf{Hom}_r(Y,Z), \mathsf{Hom}_r(Y,Z)) \arrow[l] \arrow[d, "u^*"] \\
\mathsf{hom}_{\mathbb{F}(J)}(u^*(\mathsf{Hom}_r(Y,Z)\otimes Y), u^*Z) \arrow[d, "\gamma^{\otimes}"'] & \mathsf{hom}_{\mathbb{D}(J)}(u^*\,\mathsf{Hom}_r(Y,Z), u^*\,\mathsf{Hom}_r(Y,Z)) \arrow[d, "\gamma^{\mathsf{Hom}_r}"] \\
\mathsf{hom}_{\mathbb{F}(J)}(u^*\,\mathsf{Hom}_r(Y,Z)\otimes u^*Y, u^*Z) & \mathsf{hom}_{\mathbb{D}(J)}(u^*\,\mathsf{Hom}_r(Y,Z), \mathsf{Hom}_r(u^*Y,u^*Z)) \arrow[l] \\
\mathsf{hom}_{\mathbb{F}(J)}(\mathsf{Hom}_r(u^*Y,u^*Z)\otimes u^*Y, u^*Z) \arrow[u, "\gamma^{\mathsf{Hom}_r}"] & \mathsf{hom}_{\mathbb{D}(J)}(\mathsf{Hom}_r(u^*Y,u^*Z), \mathsf{Hom}_r(u^*Y,u^*Z)) \arrow[l] \arrow[u, "\gamma^{\mathsf{Hom}_r}"']
\end{tikzcd}$$

Since both identities –the one in the upper right and the one in the lower right corner– have the same image in $\mathsf{hom}_{\mathbb{D}(J)}(u^*\,\mathsf{Hom}_r(Y,Z), \mathsf{Hom}_r(u^*Y,u^*Z))$ they are both mapped onto the same element of $\mathsf{hom}_{\mathbb{F}(J)}(u^*\,\mathsf{Hom}_r(Y,Z)\otimes u^*Y, u^*Z)$. Using the commutativity of this diagram one sees immediately that this is precisely the compatibility statement about the adjunction counits. □

Using the last lemma we can now establish the following theorem on the existence of enrichments in the context of adjunctions of two variables. Since this theorem is our main source for enriched derivators we will give a fairly complete proof.

Theorem 3.10. *Let $\mathbb{E}$ be a monoidal derivator and let us consider an adjunction of two variables*

$$(\otimes, \mathsf{Hom}_l, \mathsf{Hom}_r)\colon\ \mathbb{E}\times\mathbb{D} \rightharpoonup \mathbb{D}$$

exhibiting $\mathbb{D}$ as a closed $\mathbb{E}$-module. The derivator $\mathbb{D}$ can then be canonically enriched over $\mathbb{E}$ and is naturally cotensored over $\mathbb{E}$.

Proof. Since we have an adjunction of two variables of derivators, by evaluation at a category K we obtain a corresponding adjunction of two variables which we will write as

$$(\otimes, (-)^{(-)}, \mathsf{Hom}_{\mathbb{D}(K)})\colon\ \mathbb{E}(K)\times\mathbb{D}(K) \rightharpoonup \mathbb{D}(K).$$

Let us agree that we use the short-hand-notation H for the functor Hom_r and also γ^{H} instead of γ^{Hom_r}.

Using the corresponding result from category theory, we obtain thus that the category $\mathbb{D}(K)$ can be canonically enriched over $\mathbb{E}(K)$ where the enrichment is given by the functor $\mathsf{H} = \mathsf{Hom}_{\mathbb{D}(K)}$. The composition law $\circ_{\mathbb{D}(K)}\colon\ \mathsf{Hom}_{\mathbb{D}(K)}(Y,Z)\otimes\mathsf{Hom}_{\mathbb{D}(K)}(X,Y) \longrightarrow \mathsf{Hom}_{\mathbb{D}(K)}(X,Z)$ is given by the map which is adjoint to the following composition:

$$\begin{tikzcd}
\big(\mathsf{Hom}_{\mathbb{D}(K)}(Y,Z)\otimes\mathsf{Hom}_{\mathbb{D}(K)}(X,Y)\big)\otimes X \arrow[r] \arrow[d, "\cong"'] & Z \\
\mathsf{Hom}_{\mathbb{D}(K)}(Y,Z)\otimes\big(\mathsf{Hom}_{\mathbb{D}(K)}(X,Y)\otimes X\big) \arrow[r, "\mathsf{id}\otimes\mathsf{ev}"'] & \mathsf{Hom}_{\mathbb{D}(K)}(Y,Z)\otimes Y \arrow[u, "\mathsf{ev}"']
\end{tikzcd}$$

Here, ev is an adjunction counit, i.e., a map adjoint to the identity on $\mathsf{Hom}_{\mathbb{D}(K)}(X,Y)$ and similarly for Y,Z. The unit morphism for an object X is the map adjoint to the left unitality constraint $\lambda\colon \mathbb{S}\otimes X \longrightarrow X$. We will not distinguish notationally between the ordinary category $\mathbb{D}(K)$ and the $\mathbb{E}(K)$-enriched version.

Now, for a functor $u\colon J \longrightarrow K$ we have to construct a morphism $(\mathbb{E}(K),\mathbb{D}(K)) \longrightarrow (\mathbb{E}(J),\mathbb{D}(J))$ in ECAT. The first component is of course given by the monoidal functor $\mathbb{E}(u)\colon \mathbb{E}(K) \longrightarrow \mathbb{E}(J)$. Let us recall from Subsection 1.2 that the monoidal structure on $\mathbb{E}(u)$ is given by the 2-cells belonging to the morphisms $\otimes$ and $\mathbb{S}$. It remains hence to construct an $\mathbb{E}(J)$-enriched functor

$$\mathbb{E}(u)_*\,\mathbb{D}(K) \longrightarrow \mathbb{D}(J).$$

Since the base change 2-functor $\mathbb{E}(u)\colon \mathbb{E}(K)-\mathsf{CAT} \longrightarrow \mathbb{E}(J)-\mathsf{CAT}$ sends an $\mathbb{E}(K)$-enriched category to an $\mathbb{E}(J)$-enriched category with the same objects, we can define our would-be enriched functor to have the same behavior on objects as the unenriched functor $u^*\colon \mathbb{D}(K) \longrightarrow \mathbb{D}(J)$ given by the derivator $\mathbb{D}$. Now, for two objects $X,\ Y \in \mathbb{E}(u)_*\,\mathbb{D}(K)$, we have to specify a map on morphism objects

$$\alpha_u\colon\ \mathsf{Hom}_{\mathbb{E}(u)_*\,\mathbb{D}(K)}(X,Y) = u^*\,\mathsf{Hom}_{\mathbb{D}(K)}(X,Y) \longrightarrow \mathsf{Hom}_{\mathbb{D}(J)}(u^*X,u^*Y).$$

We take α_u to be the morphisms which belong to

$$\mathsf{H} = \mathsf{Hom}_r\colon\ \mathbb{D}^{\mathrm{op}}\times\mathbb{D} \longrightarrow \mathbb{E},$$

i.e., we set $\alpha_u = \gamma_u^{\mathsf{H}}$. Let us now check that these definitions assemble to give the intended $\mathbb{E}(J)$-enriched functor.

We begin by the unitality condition so let us fix an object X of $\mathbb{E}(u)_*\,\mathbb{D}(K)$ which is hence simultaneously an object of $\mathbb{D}(K)$. For this purpose let us consider the following diagram:

$$\begin{tikzcd}
\mathbb{S} \arrow[r, "\eta"] \arrow[d] & \mathsf{Hom}(u^*X,\mathbb{S}\otimes u^*X) \arrow[d] \arrow[ddd, bend left=60, "\lambda"] \\
u^*\,\mathbb{S} \arrow[r, "\eta"] \arrow[d, "\eta"'] & \mathsf{Hom}(u^*X,u^*\,\mathbb{S}\otimes u^*X) \arrow[d, "\gamma^{\otimes}"] \\
u^*\,\mathsf{Hom}(X,\mathbb{S}\otimes X) \arrow[r, "\gamma^{\mathsf{H}}"'] \arrow[d, "\lambda"'] & \mathsf{Hom}(u^*X,u^*(\mathbb{S}\otimes X)) \arrow[d, "\lambda"] \\
u^*\,\mathsf{Hom}(X,X) \arrow[r, "\gamma^{\mathsf{H}}"'] & \mathsf{Hom}(u^*X,u^*X)
\end{tikzcd}$$

In this diagram, the identity of $X \in \mathbb{E}(u)_*\,\mathbb{D}(K)$ is given by the left column while the identity of $u^*X \in \mathbb{D}(J)$ is given by $\lambda\circ\eta$. The right part of the diagram commutes since u^* is a monoidal functor and that part is precisely one of the coherence conditions for a monoidal functor. Moreover, the two outer squares commute by naturality while the last one does by Lemma 3.9. Thus, the diagram commutes and we can hence conclude that our would-be enriched functor is unital.

It remains to show that the maps on morphism objects are compatible with composition. So, let us consider three objects X, Y, and Z of $\mathbb{E}(u)_*\,\mathbb{D}(K)$. Spelling out the definition of the composition laws we thus have to check the commutativity of the following diagram:

$$
\begin{tikzcd}
u^*\,\mathsf{H}(Y,Z)\otimes u^*\,\mathsf{H}(X,Y) \arrow[r, "\gamma^{\mathsf{H}}\otimes\gamma^{\mathsf{H}}"] \arrow[d, "\gamma^{\otimes}"'] & \mathsf{H}(u^*Y,u^*Z)\otimes\mathsf{H}(u^*X,u^*Y) \arrow[d, "\eta"] \\
u^*(\mathsf{H}(Y,Z)\otimes\mathsf{H}(X,Y)) \arrow[d, "\eta"'] & \mathsf{H}(u^*X,(\mathsf{H}(u^*Y,u^*Z)\otimes\mathsf{H}(u^*X,u^*Y))\otimes u^*X) \arrow[d, "a"] \\
u^*\,\mathsf{H}(X,(\mathsf{H}(Y,Z)\otimes\mathsf{H}(X,Y))\otimes X) \arrow[d, "a"'] & \mathsf{H}(u^*X,\mathsf{H}(u^*Y,u^*Z)\otimes(\mathsf{H}(u^*X,u^*Y)\otimes u^*X)) \arrow[d, "\epsilon"] \\
u^*\,\mathsf{H}(X,\mathsf{H}(Y,Z)\otimes(\mathsf{H}(X,Y)\otimes X)) \arrow[d, "\epsilon"'] & \mathsf{H}(u^*X,\mathsf{H}(u^*Y,u^*Z)\otimes u^*Y) \arrow[d, "\epsilon"] \\
u^*\,\mathsf{H}(X,\mathsf{H}(Y,Z)\otimes Y) \arrow[d, "\epsilon"'] & \mathsf{H}(u^*X,u^*Z) \arrow[d, "="] \\
u^*\,\mathsf{H}(X,Z) \arrow[r, "\gamma^{\mathsf{H}}"'] & \mathsf{H}(u^*X,u^*Z)
\end{tikzcd}
$$

Let us consider the composition of morphisms from the upper left corner to the bottom right corner which passes through $u^*\,\mathsf{H}(X,Z)$ and remark that it can be rewritten as:

$$
\begin{tikzcd}
u^*\,\mathsf{H}(Y,Z)\otimes u^*\,\mathsf{H}(X,Y) \arrow[r, "\eta"] \arrow[d, "\gamma^{\otimes}"'] & \mathsf{H}(u^*X,(u^*\,\mathsf{H}(Y,Z)\otimes u^*\,\mathsf{H}(X,Y))\otimes u^*X) \arrow[d, "\gamma^{\otimes}"] \\
u^*(\mathsf{H}(Y,Z)\otimes\mathsf{H}(X,Y)) \arrow[r, "\eta"] \arrow[d, "\eta"'] & \mathsf{H}(u^*X,u^*(\mathsf{H}(Y,Z)\otimes\mathsf{H}(X,Y))\otimes u^*X) \arrow[d, "\gamma^{\otimes}"] \\
u^*\,\mathsf{H}(X,(\mathsf{H}(Y,Z)\otimes\mathsf{H}(X,Y))\otimes X) \arrow[r, "\gamma^{\mathsf{H}}"] \arrow[d, "a"'] & \mathsf{H}(u^*X,u^*\big((\mathsf{H}(Y,Z)\otimes\mathsf{H}(X,Y))\otimes X\big)) \arrow[d, "a"] \\
u^*\,\mathsf{H}(X,\mathsf{H}(Y,Z)\otimes(\mathsf{H}(X,Y)\otimes X)) \arrow[r, "\gamma^{\mathsf{H}}"'] \arrow[d, "\epsilon"'] & \mathsf{H}(u^*X,u^*\big(\,\mathsf{H}(Y,Z)\otimes(\mathsf{H}(X,Y)\otimes X)\big)) \arrow[d, "\epsilon"] \\
u^*\,\mathsf{H}(X,\mathsf{H}(Y,Z)\otimes Y) \arrow[r, "\gamma^{\mathsf{H}}"'] \arrow[d, "\epsilon"'] & \mathsf{H}(u^*X,u^*(\mathsf{H}(Y,Z)\otimes Y)) \arrow[d, "\epsilon"] \\
u^*\,\mathsf{H}(X,Z) \arrow[r, "\gamma^{\mathsf{H}}"'] & \mathsf{H}(u^*X,u^*Z)
\end{tikzcd}
$$

An application of the last lemma guarantees the commutativity of the second square while the remaining ones are commutative by naturality. Now the right column of this diagram itself can be rewritten as follows:

$$\begin{tikzcd}
\mathsf{H}(u^*X,(u^*\mathsf{H}(Y,Z)\otimes u^*\mathsf{H}(X,Y))\otimes u^*X) \arrow[r, "a"] \arrow[d, "\gamma^{\otimes}"'] & \mathsf{H}(u^*X,u^*\mathsf{H}(Y,Z)\otimes(u^*\mathsf{H}(X,Y)\otimes u^*X)) \arrow[ddd, "\gamma^{\otimes}"] \\
\mathsf{H}(u^*X,u^*(\mathsf{H}(Y,Z)\otimes\mathsf{H}(X,Y))\otimes u^*X) \arrow[d, "\gamma^{\otimes}"'] & \\
\mathsf{H}(u^*X,u^*\big((\mathsf{H}(Y,Z)\otimes\mathsf{H}(X,Y))\otimes X\big)) \arrow[d, "a"'] & \\
\mathsf{H}(u^*X,u^*\big(\mathsf{H}(Y,Z)\otimes(\mathsf{H}(X,Y)\otimes X)\big)) \arrow[d, "\epsilon"'] & \mathsf{H}(u^*X,u^*\mathsf{H}(Y,Z)\otimes u^*(\mathsf{H}(X,Y)\otimes X)) \arrow[l, "\gamma^{\otimes}"] \arrow[d, "\epsilon"] \\
\mathsf{H}(u^*X,u^*(\mathsf{H}(Y,Z)\otimes Y)) \arrow[d, "\epsilon"'] & \mathsf{H}(u^*X,u^*\mathsf{H}(Y,Z)\otimes u^*Y) \arrow[l, "\gamma^{\otimes}"] \arrow[d, "\gamma^{\mathsf{H}}"] \\
\mathsf{H}(u^*X,u^*Z) & \mathsf{H}(u^*X,\mathsf{H}(u^*Y,u^*Z)\otimes u^*Y) \arrow[l, "\epsilon"]
\end{tikzcd}$$

In this diagram, the upper square commutes since for a monoidal derivator the restriction functors u^* are canonically monoidal and that square just expresses one of the coherence axioms for a monoidal functor. The second square commutes by naturality while the bottom square does by Lemma 3.9. Now, this new composition $\epsilon\circ\gamma^{\mathsf{H}}\circ\epsilon\circ\gamma^{\otimes}\circ a$ can again be rewritten as $\epsilon\circ\epsilon\circ\gamma^{\otimes}\circ\gamma^{\mathsf{H}}\circ a$ as depicted in the next diagram:

$$\begin{tikzcd}
\mathsf{H}(u^*X,(u^*\mathsf{H}(Y,Z)\otimes u^*\mathsf{H}(X,Y))\otimes u^*X) \arrow[r, "\gamma^{\mathsf{H}}\otimes\gamma^{\mathsf{H}}"] \arrow[d, "a"'] & \mathsf{H}(u^*X,(\mathsf{H}(u^*Y,u^*Z)\otimes\mathsf{H}(u^*X,u^*Y))\otimes u^*X) \arrow[d, "a"] \\
\mathsf{H}(u^*X,u^*\mathsf{H}(Y,Z)\otimes(u^*\mathsf{H}(X,Y)\otimes u^*X)) \arrow[r, "\gamma^{\mathsf{H}}\otimes\gamma^{\mathsf{H}}"] \arrow[d, "\gamma^{\mathsf{H}}"'] & \mathsf{H}(u^*X,\mathsf{H}(u^*Y,u^*Z)\otimes(\mathsf{H}(u^*X,u^*Y)\otimes u^*X)) \arrow[d, "="] \\
\mathsf{H}(u^*X,\mathsf{H}(u^*Y,u^*Z)\otimes(u^*\mathsf{H}(X,Y)\otimes u^*X)) \arrow[r, "\gamma^{\mathsf{H}}"] \arrow[d, "\gamma^{\otimes}"'] & \mathsf{H}(u^*X,\mathsf{H}(u^*Y,u^*Z)\otimes(\mathsf{H}(u^*X,u^*Y)\otimes u^*X)) \arrow[d, "\epsilon"] \\
\mathsf{H}(u^*X,\mathsf{H}(u^*Y,u^*Z)\otimes u^*\big(\mathsf{H}(X,Y)\otimes X\big)) \arrow[r, "\epsilon"'] & \mathsf{H}(u^*X,\mathsf{H}(u^*Y,u^*Z)\otimes u^*Y) \arrow[d, "\epsilon"] \\
 & \mathsf{H}(u^*X,u^*Z)
\end{tikzcd}$$

In this diagram the bottom square commutes by a further application of Lemma 3.9 while the upper two squares do by naturality. Now, by the commutativity of the square

$$\begin{tikzcd}
u^* \mathsf{H}(Y,Z) \otimes u^* \mathsf{H}(X,Y) \arrow[r, "\gamma^{\mathsf{H}} \otimes \gamma^{\mathsf{H}}"] \arrow[d, "\eta"'] & \mathsf{H}(u^*Y, u^*Z) \otimes \mathsf{H}(u^*X, u^*Y) \arrow[d, "\eta"] \\
\mathsf{H}(u^*X, (u^* \mathsf{H}(Y,Z) \otimes u^* \mathsf{H}(X,Y)) \otimes u^*X) \arrow[r] & \mathsf{H}(u^*X, (\mathsf{H}(u^*Y, u^*Z) \otimes \mathsf{H}(u^*X, u^*Y)) \otimes u^*X)
\end{tikzcd}$$

we can conclude that we have constructed an $\mathbb{E}(J)$-enriched functor. In fact, putting the above diagrams together we get the following chain of equalities

$$\begin{aligned}
\gamma^{\mathsf{H}} \circ \epsilon \circ \epsilon \circ a \circ \eta \circ \gamma^{\otimes} &= \epsilon \circ \epsilon \circ a \circ \gamma^{\otimes} \circ \gamma^{\otimes} \circ \eta \\
&= \epsilon \circ \gamma^{\mathsf{H}} \circ \epsilon \circ \gamma^{\otimes} \circ a \circ \eta \\
&= \epsilon \circ \epsilon \circ \gamma^{\otimes} \circ \gamma^{\mathsf{H}} \circ a \circ \eta \\
&= \epsilon \circ \epsilon \circ a \circ (\gamma^{\mathsf{H}} \otimes \gamma^{\mathsf{H}}) \circ \eta \\
&= \epsilon \circ \epsilon \circ a \circ \eta \circ (\gamma^{\mathsf{H}} \otimes \gamma^{\mathsf{H}})
\end{aligned}$$

expressing the compatibility of our functor with enriched composition laws. This concludes the proof that the above constructions assemble to an $\mathbb{E}(J)$-enriched functor $\mathbb{E}(u)_* \mathbb{D}(K) \longrightarrow \mathbb{D}(J)$. The assignment which sends a functor u to the enriched functors we just constructed is itself functorial by Lemma 1.11.

The final part of the proof consists of the construction of enriched natural transformations associated to 2-cells in Cat. More precisely, given a natural transformation $\alpha\colon u \longrightarrow v$ between functors $u, v\colon J \longrightarrow K$ we want to construct a 2-cell in ECAT between the induced morphisms $(\mathbb{E}(K), \mathbb{D}(K)) \longrightarrow (\mathbb{E}(J), \mathbb{D}(J))$. Unraveling definitions we have to construct an $\mathbb{E}(J)$-enriched natural transformation between the $\mathbb{E}(J)$-enriched functors

$$\mathbb{E}(u)_* \mathbb{D}(K) \xrightarrow{u^*} \mathbb{D}(J) \qquad \text{and} \qquad \mathbb{E}(u)_* \mathbb{D}(K) \xrightarrow{\mathbb{E}(\alpha)_*} \mathbb{E}(v)_* \mathbb{D}(K) \xrightarrow{v^*} \mathbb{D}(J).$$

Evaluated at an object X we take the component $\alpha^X\colon \mathbb{S} \longrightarrow \mathsf{Hom}(u^*X, v^*X)$ of our would-be enriched natural transformation to be the map adjoint to $\mathbb{S} \otimes u^*X \longrightarrow u^*X \longrightarrow v^*X$. In order to show that this defines an enriched natural transformation we have to check that for any pair X, Y of objects the following square commutes:

$$\begin{tikzcd}
u^* \mathsf{H}(X,Y) \arrow[r, "\cong"] \arrow[d, "\cong"'] & u^* \mathsf{H}(X,Y) \otimes \mathbb{S} \arrow[r] & v^* \mathsf{H}(X,Y) \otimes \mathbb{S} \arrow[d, "\gamma^{\mathsf{H}} \otimes \alpha^X"] \\
\mathbb{S} \otimes u^* \mathsf{H}(X,Y) \arrow[d, "\alpha^Y \otimes \gamma^{\mathsf{H}}"'] & & \mathsf{H}(v^*X, v^*Y) \otimes \mathsf{H}(u^*X, v^*X) \arrow[d, "\circ"] \\
\mathsf{H}(u^*Y, v^*Y) \otimes \mathsf{H}(u^*X, u^*Y) \arrow[rr, "\circ"'] & & \mathsf{H}(u^*X, v^*Y)
\end{tikzcd}$$

Let us show that the two maps $u^* \mathsf{H}(X,Y) \longrightarrow \mathsf{H}(u^*X, v^*Y)$ are sent by the adjunction to the same maps $u^* \mathsf{H}(X,Y) \otimes u^*X \longrightarrow v^*Y$. We begin by calculating this adjoint morphism for the map obtained by passing through the bottom left corner. By naturality of the associativity constraint this can be written as:

$$\begin{tikzcd}
u^*\,\mathsf{H}(X,Y)\otimes u^*X \arrow[r, "\cong"] \arrow[ddd] & \mathbb{S}\otimes(u^*\,\mathsf{H}(X,Y)\otimes u^*X) \arrow[d, "\alpha^Y"] \\
& \mathsf{H}(u^*Y,v^*Y)\otimes(u^*\,\mathsf{H}(X,Y)\otimes u^*X) \arrow[d, "\gamma_u^{\mathsf{H}}"] \\
& \mathsf{H}(u^*Y,v^*Y)\otimes(\mathsf{H}(u^*X,u^*Y)\otimes u^*X) \arrow[d, "\mathsf{ev}"] \\
v^*Y & \mathsf{H}(u^*Y,v^*Y)\otimes u^*Y \arrow[l, "\mathsf{ev}"]
\end{tikzcd}$$

From the construction of γ_u^{H} in Subsection 1.3 we know that the composition $\mathsf{ev}\circ\gamma_u^{\mathsf{H}}$ is just the map

$$u^*\,\mathsf{H}(X,Y)\otimes u^*X \xrightarrow{\gamma^{\otimes}} u^*(\mathsf{H}(X,Y)\otimes X) \xrightarrow{\mathsf{ev}} u^*Y.$$

Thus we can rewrite the above diagram as

$$\begin{tikzcd}
u^*\,\mathsf{H}(X,Y)\otimes u^*X \arrow[r, "\cong"] \arrow[d, "\gamma^{\otimes}"'] & \mathbb{S}\otimes(u^*\,\mathsf{H}(X,Y)\otimes u^*X) \arrow[d, "\gamma^{\otimes}"] \\
u^*(\mathsf{H}(X,Y)\otimes X) \arrow[r, "\cong"] \arrow[d, "\mathsf{ev}"'] & \mathbb{S}\otimes u^*(\mathsf{H}(X,Y)\otimes X) \arrow[d, "\mathsf{ev}"] \\
u^*Y \arrow[r, "\cong"] \arrow[d, "\alpha^*"'] & \mathbb{S}\otimes u^*Y \arrow[d, "\alpha^Y"] \\
v^*Y & \mathsf{H}(u^*Y,v^*Y)\otimes u^*Y \arrow[l, "\mathsf{ev}"]
\end{tikzcd}$$

and conclude that the adjoint map is just given by $\alpha^*\circ\mathsf{ev}\circ\gamma^{\otimes}$.

Let us now calculate the morphism $u^*\,\mathsf{H}(X,Y)\otimes u^*X\longrightarrow v^*Y$ which is adjoint to the map obtained by passing through the upper right corner. Using again the naturality of the associativity constraint we can identify the morphism as

$$\begin{tikzcd}
(u^* \mathsf{H}(X,Y) \otimes \mathbb{S}) \otimes u^*X \arrow[r, "\alpha^*\otimes\alpha^X"] & (v^* \mathsf{H}(X,Y) \otimes \mathsf{H}(u^*X, v^*X)) \otimes u^*X \arrow[d, "a"] \\
u^* \mathsf{H}(X,Y) \otimes u^*X \arrow[u, "\cong"] \arrow[ddd] & v^* \mathsf{H}(X,Y) \otimes (\mathsf{H}(u^*X, v^*X) \otimes u^*X) \arrow[d, "\mathsf{ev}"] \\
 & v^* \mathsf{H}(X,Y) \otimes v^*X \arrow[d, "\gamma_v^{\mathsf{H}}"] \\
v^*Y & \mathsf{H}(v^*X, v^*Y) \otimes v^*X \arrow[l, "\mathsf{ev}"]
\end{tikzcd}$$

This description can be simplified by using again the relation $\mathsf{ev} \circ \gamma_v^{\mathsf{H}} = \mathsf{ev} \circ \gamma_v^{\otimes}$ and the definition of α^X as the adjoint map of $\mathbb{S} \otimes u^*X \cong u^*X \longrightarrow v^*X$. So, the map under consideration is given by:

$$\begin{tikzcd}
u^* \mathsf{H}(X,Y) \otimes (\mathbb{S} \otimes u^*X) \arrow[r, "\alpha^X"] & u^* \mathsf{H}(X,Y) \otimes (\mathsf{H}(u^*X, v^*X) \otimes u^*X) \arrow[d, "\mathsf{ev}"] \\
u^* \mathsf{H}(X,Y) \otimes u^*X \arrow[u, "\cong"] \arrow[r, "\alpha^*"] \arrow[ddd] & u^* \mathsf{H}(X,Y) \otimes v^*X \arrow[d, "\alpha^*"] \\
 & v^* \mathsf{H}(X,Y) \otimes v^*X \arrow[d, "\gamma_v^{\otimes}"] \\
v^*Y & v^*(\mathsf{H}(X,Y) \otimes X) \arrow[l, "\mathsf{ev}"]
\end{tikzcd}$$

Hence, we have calculated the second adjoint morphism as $\mathsf{ev} \circ \gamma_v^{\otimes} \circ (\alpha^* \otimes \alpha^*)$.

With these descriptions of the two adjoint morphisms it is easy to see that they coincide since both fit into the following commutative diagram:

$$\begin{tikzcd}
u^* \mathsf{H}(X,Y) \otimes u^*X \arrow[r, "\gamma_u^{\otimes}"] \arrow[d, "\alpha^*\otimes\alpha^*"'] & u^*(\mathsf{H}(X,Y) \otimes X) \arrow[r, "\mathsf{ev}"] \arrow[d, "\alpha^*"] & u^*Y \arrow[d, "\alpha^*"] \\
v^* \mathsf{H}(X,Y) \otimes v^*X \arrow[r, "\gamma_v^{\otimes}"'] & v^*(\mathsf{H}(X,Y) \otimes X) \arrow[r, "\mathsf{ev}"'] & v^*Y
\end{tikzcd}$$

Here the left square commutes since $\alpha^*\colon u^* \longrightarrow v^*$ is a monoidal transformation while the right square does by naturality. Thus, we have shown that the family $\{\alpha^X\colon \mathbb{S} \longrightarrow \mathsf{Hom}(u^*X, v^*X)\}_X$ defines an $\mathbb{E}(J)$-enriched natural transformation as intended. We omit the details verifying that this assignment is compatible with identities and horizontal and vertical compositions, which then concludes the proof. □

Corollary 3.11. *Let $\mathbb{D}$ be a biclosed monoidal derivator. Then $\mathbb{D}$ is canonically tensored, cotensored, and enriched over itself.*

The first class of examples of enriched (pre)derivators is obtained by an application of Theorem 3.10 to represented prederivators. Further examples coming from model categories will be given in the next subsection.

Example 3.12. Let $\mathcal{C}$ be a complete monoidal category and let $\mathcal{D}$ be a left $\mathcal{C}$-module such that the underlying category is complete. If the action map $\otimes\colon \mathcal{C}\times\mathcal{D}\longrightarrow\mathcal{D}$ is a left adjoint of two variables then Proposition 1.12 implies that we obtain an adjunction of two variables $\otimes\colon y(\mathcal{C})\times y(\mathcal{D})\rightharpoonup y(\mathcal{D})$ exhibiting $y(\mathcal{D})$ as a closed left $y(\mathcal{C})$-module. Thus, the prederivator $y(\mathcal{D})$ is canonically enriched over $y(\mathcal{C})$.

3.3. **Enriched model categories induce enriched derivators.** We now only have to put together the above results in order to obtain the second important class of enriched derivators as guaranteed by the following theorem.

Theorem 3.13. *Let $\mathcal{M}$ and $\mathcal{N}$ be combinatorial model categories and let $\mathcal{M}$ be in addition a monoidal model category. If $\mathcal{N}$ is a left $\mathcal{M}$-module as a model category, then the derivator $\mathbb{D}_{\mathcal{N}}$ is canonically tensored, cotensored, and enriched over $\mathbb{D}_{\mathcal{M}}$.*

Proof. By the discussion preceding Theorem 2.13, we know that the action $\otimes\colon\mathcal{M}\times\mathcal{N}\longrightarrow\mathcal{M}$ is part of an adjunction of two variables

$$(\otimes,(-)^{(-)},\mathsf{Hom})\colon \mathcal{M}\times\mathcal{N}\rightharpoonup\mathcal{N}.$$

That theorem implies that this adjunction of two variables induces an adjunction of two variables at the level of the associated derivators:

$$(\overset{\mathbb{L}}{\otimes},\mathbb{R}(-)^{(-)},\mathbb{R}\mathsf{Hom})\colon\ \mathbb{D}_{\mathcal{M}}\times\mathbb{D}_{\mathcal{N}}\rightharpoonup\mathbb{D}_{\mathcal{N}}$$

Moreover, this adjunction exhibits $\mathbb{D}_{\mathcal{M}}$ as a left $\mathbb{D}_{\mathcal{N}}$-module. Thus it suffices to apply Theorem 3.10 to deduce that $\mathbb{D}_{\mathcal{M}}$ can be canonically enriched over $\mathbb{D}_{\mathcal{N}}$. Recall from the proof of that result that the enrichment is actually given by $\mathbb{R}\mathsf{Hom}$. □

Let us take up again our three classes of examples.

Corollary 3.14. *Let $\mathcal{M}$ be a combinatorial model category.*
i) *If $\mathcal{M}$ is a simplicial model category, then the associated derivator $\mathbb{D}_{\mathcal{M}}$ is canonically tensored, cotensored, and enriched over the derivator $\mathbb{D}_{\mathsf{Set}_\Delta}$ of simplicial sets.*
ii) *If $\mathcal{M}$ is a spectral model category, then the associated derivator $\mathbb{D}_{\mathcal{M}}$ is canonically tensored, cotensored, and enriched over the derivator $\mathbb{D}_{\mathsf{Sp}}$ of spectra.*
iii) *If $\mathcal{M}$ is a dg model category over the ground ring k, then the associated derivator $\mathbb{D}_{\mathcal{M}}$ is canonically tensored, cotensored, and enriched over the derivator $\mathbb{D}_k$ of chain complexes over k.*

We close by mentioning the more specific examples of the earlier sections. Again, these examples are completely parallel and could be given for any nice monoidal, combinatorial model category. We only stick to the cases of chain complexes and spectra.

Example 3.15. Let k be a commutative ground ring. The derivator $\mathbb{D}_k$ of chain complexes over k is canonically tensored, cotensored, and enriched over itself. More generally, let C be a commutative differential graded algebra such that C is cofibrant as a chain complex, then the derivator $\mathbb{D}_C$ of differential-graded C-modules is canonically tensored, cotensored, and enriched over itself. For a non-monoidal example, let us consider a non-commutative differential-graded algebra A. Then we can deduce that the associated derivator $\mathbb{D}_A$ of differential-graded A-modules is canonically tensored, cotensored, and enriched over $\mathbb{D}_k$. Moreover, if A is cofibrant as a chain complex then $\mathbb{D}_A$ is also canonically tensored, cotensored, and enriched over $\mathbb{D}_{A\otimes A^{\mathrm{op}}}$.

Example 3.16. The derivator $\mathbb{D}_{\mathsf{Sp}}$ of spectra is canonically tensored, cotensored, and enriched over itself. More generally, let us consider a commutative symmetric ring spectrum E which has a cofibrant underlying symmetric spectrum. The derivator $\mathbb{D}_E$ of E-module spectra is also canonically tensored, cotensored, and enriched over itself. If we are considering a symmetric ring spectrum R which is not necessarily commutative then we still obtain that the associated derivator $\mathbb{D}_R$ of R-module spectra is canonically tensored, cotensored, and enriched over $\mathbb{D}_{\mathsf{Sp}}$. Finally, if R is cofibrant as a symmetric spectrum then $\mathbb{D}_R$ is also canonically tensored, cotensored, and enriched over $\mathbb{D}_{R\wedge R^{\mathrm{op}}}$.

Appendix A. The 2-categorical Grothendieck construction

In this appendix we will give a short description of the Grothendieck construction in the setting of 2-categories. As an input for that construction, one starts with a 2-functor $F\colon I \longrightarrow \mathsf{2\text{-}CAT}$where $\mathsf{2\text{-}CAT}$ denotes the 2-category of 2-categories. Remark that we ignore the modifications [Bor94a] which would give us a 3-category of 2-categories. The basic idea behind the Grothendieck construction is that one wants to glue the different 2-categories $F(i)$ together to obtain a single new 2-category $\int F$. This is done in a way that an object 'remembers in which category $F(i)$ it lived before': $\int F$ will be canonically endowed with a 'projection 2-functor' $p\colon \int F \longrightarrow I$. Before we give the actual construction in our 2-categorical situation, let us begin with a short recap of two 'lower dimensional' cases. For this purpose we let I be a category and replace 2-categories first by sets and then by categories.

Example A.1. (two dimensions less: the category of elements)
Let us consider a set-valued functor $F\colon I \longrightarrow \mathsf{Set}$. Then one can construct the *category* $el(F)$ *of elements of* F. An object in $el(F)$ is a pair (i, X) consisting of an object $i \in I$ and an element $X \in F(i)$. Given two such objects, a morphism $(i, X) \longrightarrow (j, Y)$ in $el(F)$ is a morphism $f\colon i \longrightarrow j$ in I such that the induced map $F(f)\colon F(i) \longrightarrow F(j)$ maps X to Y. In the special case where the indexing category is the simplicial index category, i.e., if $I = \Delta^{\mathrm{op}}$, we are starting with a simplicial set $F\colon \Delta^{\mathrm{op}} \longrightarrow \mathsf{Set}$. In that case the category $el(F)$ is just the category ΔF of simplices of F (cf. [GJ99] for the importance of this construction). Note that there is a canonical functor $p\colon el(F) \longrightarrow I$ sending an object (i, X) to i and keeping the morphisms. This functor has the property that we have a canonical bijection $p^{-1}(i) \cong F(i)$ where we identified the discrete category $p^{-1}(i)$ with its set of objects.

Climbing up the dimension ladder by one, let us now consider categories instead of sets.

Example A.2. (one dimension less: the classical Grothendieck construction)
Let us consider a category-valued functor $F\colon I \longrightarrow \mathsf{CAT}$. The Grothendieck construction $\int F$ of F is the following category. An object of $\int F$ is a pair (i, X) consisting of an object $i \in I$ and an object $X \in F(i)$. The fact that our functor F takes values in categories allows for a more general notion of morphisms than in the last example. So, let (i, X) and (j, Y) be two objects of $\int F$. A morphism $(i, X) \longrightarrow (j, Y)$ is a pair (f, u) consisting of a morphism $f\colon i \longrightarrow j$ in I and a morphism $u\colon F(f)X \longrightarrow Y$ in $F(j)$. Given two composable morphisms $(f, u)\colon (i, X) \longrightarrow (j, Y)$ and $(g, v)\colon (j, Y) \longrightarrow (k, Z)$, their composition is defined to be $(g \circ f, v \circ F(g)(u))$. It is immediate to check that this is a category with the obvious identity morphisms. Again, we have a canonical projection functor $p\colon \int F \longrightarrow I$. By definition, p sends an object (i, X) to i and a morphism (f, u) to f. Moreover, let us note that we have a canonical isomorphism of categories $p^{-1}(i) \cong F(i)$.

These projection functors p are not arbitrary functors but have particularly nice properties. In fact, they are examples of Grothendieck opfibrations [Vis05] and we will comment shortly on this after the next construction.

Having recalled these two classical cases the 2-categorical version will now go as expected. However, we also increase the dimension of the domain of F by one, so let us consider a 2-category-valued 2-functor $F\colon I \longrightarrow 2\text{-}\mathsf{CAT}$. The 2-*categorical Grothendieck construction* $\int F$ is the following 2-category. The underlying category of $\int F$ will be as in the last example, so that we will only make explicit the 2-morphisms and the vertical and horizontal composition laws. Thus, let $(f,u),(g,v)\colon (i,X) \longrightarrow (j,Y)$ be two parallel morphisms in $\int F$. A 2-morphism $(\alpha,\phi)\colon (f,u) \longrightarrow (g,v)$ is a pair consisting of a 2-morphism $\alpha\colon f \longrightarrow g$ in I and a 2-cell ϕ in $F(j)$ as indicated in:

$$\begin{tikzcd}
f_*X \arrow[r, "u"] \arrow[dr, bend right, "\alpha_*X"'] \arrow[dr, phantom, "\swArrow" near end, shift left=3] & Y \\
& g_*X \arrow[u, "v"']
\end{tikzcd}$$

Given three parallel morphisms (f,u), (g,v) and (h,w) in $\int F$ and two vertically composable 2-morphisms $(\alpha,\phi)\colon (f,u) \longrightarrow (g,v)$ and $(\beta,\psi)\colon (g,v) \longrightarrow (h,w)$, their vertical composition is defined by

$$(\beta,\psi)\cdot(\alpha,\phi) = (\beta\cdot\alpha, \psi\alpha_*\cdot\phi).$$

Finally, let us consider two horizontally composable 2-morphisms $(\alpha,\phi)\colon (f_1,u_1) \longrightarrow (f_2,u_2)$ and $(\beta,\psi)\colon (g_1,v_1) \longrightarrow (g_2,v_2)$ as in:

$$\begin{tikzcd}
(i,X) \arrow[r, bend left, "{(f_1,u_1)}"] \arrow[r, bend right, "{(f_2,u_2)}"'] \arrow[r, phantom, "\Downarrow"] & (j,Y) \arrow[r, bend left, "{(g_1,v_1)}"] \arrow[r, bend right, "{(g_2,v_2)}"'] \arrow[r, phantom, "\Downarrow"] & (k,Z)
\end{tikzcd}$$

Then their horizontal composition is defined by the following formula

$$(\beta,\psi)*(\alpha,\phi) = (\beta*\alpha, \psi*g_{1*}\phi).$$

The corresponding diagram in $F(k)$ looks like:

$$\begin{tikzcd}
g_{1*}f_{1*}X \arrow[r, "u_1"] \arrow[d, "\alpha_*"'] \arrow[dr, phantom, "\swArrow"] & g_{1*}Y \arrow[r, "v_1"] \arrow[d, "\beta_*"] \arrow[dr, phantom, "\swArrow"] & Z \\
g_{1*}f_{2*}X \arrow[ur, bend right, "u_2"'] \arrow[d, "\beta_*"'] & g_{2*}Y \arrow[ur, bend right, "v_2"'] \\
g_{2*}f_{2*}X \arrow[ur, bend right, "u_2"'] &
\end{tikzcd}$$

It is now a straightforward calculation to verify that these two composition laws satisfy the interchange law, i.e., that $\int F$ is indeed a 2-category. Let us note that the projection on the first variable gives us a 2-functor $p\colon \int F \longrightarrow I$ such that we have canonical isomorphisms $p^{-1}(i) \cong F(i)$ of 2-categories.

Already in the 1-dimensional case, the functor $p\colon \int F \longrightarrow I$ is not an arbitrary functor but is a Grothendieck opfibration. A similar Grothendieck construction can be applied to a contravariant category-valued functor in which case the projection functor would be a Grothendieck fibration. Recall that a Grothendieck opfibration is by definition a functor $p\colon \mathcal{C} \longrightarrow I$ which allows for a sufficient supply of p-coCartesian arrows (cf. [Bor94b, Vis05]). These p-coCartesian arrows are morphisms satisfying a universal property which is expressed by the bijectivity of a certain canonical map of sets. In the 2-categorical picture, there are now different ways of introducing a notion of coCartesian arrows: the role of the canonical map of sets is taken by a canonical functor and one could demand this functor to be an *isomorphism* or an *equivalence.* Since we will not need these p-(co)Cartesian arrows we will not get into this. Nevertheless, depending on the variance with respect to 1-morphisms of the 2-category-valued 2-functor F we started with, we will call the associated projection functor p the 2-*Grothendieck (op)fibration associated to* F.

Let us close this appendix by remarking that Grothendieck (op)fibrations and also the Grothen-dieck construction were generalized to the setting of ∞-categories by Joyal and Lurie ([Lur09]). A short introduction to these notions is given in [Gro10b].

Appendix B. Monoidal and closed monoidal 2-categories

B.1. **The 2-categories of monoidal objects and modules in a monoidal 2-category.** In this subsection, let $\mathcal{C} = (\mathcal{C}, \otimes, \mathbb{S}, \alpha, \lambda, \rho)$ be a monoidal 2-category which is given by a 2-category $\mathcal{C}$, a 2-functor $\otimes\colon \mathcal{C}\times\mathcal{C} \longrightarrow \mathcal{C}$ called the monoidal pairing, a monoidal unit $\mathbb{S}\in\mathcal{C}$, and invertible 2-natural transformations α, λ and ρ called the associativity constraint and the unitality constraints respectively. The coherence conditions are the same as in the 1-categorical case. Again, we will not distinguish notationally between these 2-natural transformations and their respective inverses. We will quickly recall the notions of monoidal objects and modules in $\mathcal{C}$. This is done since we need some details about these notions in the construction of certain 2-categories of modules which are important in the next subsection.

So, let us begin with the monoidal objects. A *monoidal object* X in $\mathcal{C}$ is a sixtuple $(X, \mu_X, u_X, \alpha_X, \lambda_X, \rho_X)$ consisting of an object $X \in \mathcal{C}$, morphisms $\mu_X\colon X\otimes X \longrightarrow X$ and $u_X\colon \mathbb{S} \longrightarrow X$ and invertible 2-cells λ_X, α_X and ρ_X as indicated in:

$$\begin{tikzcd}
\mathbb{S}\otimes X \arrow[r, "u_X"] \arrow[dr, bend right, "\lambda"'] \arrow[dr, phantom, "\Nearrow"] & X\otimes X \arrow[d, "\mu_X"] \\
& X
\end{tikzcd}
\quad
\begin{tikzcd}
(X\otimes X)\otimes X \arrow[rr, "\alpha"] \arrow[d, "\mu_X"'] & \arrow[d, phantom, "\Leftarrow"] & X\otimes(X\otimes X) \arrow[d, "\mu_X"] \\
X\otimes X \arrow[r, "\mu_X"'] & X & X\otimes X \arrow[l, "\mu_X"]
\end{tikzcd}
\quad
\begin{tikzcd}
X\otimes X \arrow[d, "\mu_X"'] & X\otimes\mathbb{S} \arrow[l, "u_X"'] \arrow[dl, bend left, "\rho"] \arrow[dl, phantom, "\Nwarrow"] \\
X &
\end{tikzcd}$$

These data are subject to certain coherence conditions which are the same as in [ML98, pp.162-163] suitably adapted to the context of a general monoidal 2-category. Let us express these coherence conditions as conditions on 2-cells in $\mathsf{Hom}_{\mathcal{C}}(X^{\otimes n}, X)$ where we denote by $X^{\otimes n}$ the n-fold tensor power of X w.r.t. $\otimes$ where we moved all brackets as far to the left as possible. For this purpose, let us denote the map μ_X by $(-)\cdot(-)$, the identity id_X by $(-)$, u_X by itself and similarly for combinations of these maps. In this notation the 2-cells $\lambda_X\lambda^{-1}$, α_X resp. $\rho_X\rho^{-1}$ are hence denoted by:

$$(-) \longrightarrow u_X\cdot(-), \qquad (-)\cdot((-)\cdot(-)) \longrightarrow ((-)\cdot(-))\cdot(-) \qquad \text{resp.} \qquad (-) \longrightarrow (-)\cdot u_X$$

The coherence conditions on the above 2-cells are given by the commutativity of the pentagon

$$\begin{tikzcd}
\big(((-)\cdot(-))\cdot(-)\big)\cdot(-) \arrow[rr] \arrow[d] & & ((-)\cdot(-))\cdot((-)\cdot(-)) \arrow[d] \\
\big((-)\cdot((-)\cdot(-))\big)\cdot(-) \arrow[r] & (-)\cdot\big(((-)\cdot(-))\cdot(-)\big) \arrow[r] & (-)\cdot\big((-)\cdot((-)\cdot(-))\big),
\end{tikzcd}$$

the equality of the two 2-cells $u_X\cdot u_X \Longrightarrow u_X$ and the commutativity of:

$$\begin{tikzcd}
(u_X\cdot(-))\cdot(-) \arrow[r] \arrow[d] & (-)\cdot(-) \\
u_X\cdot((-)\cdot(-)) \arrow[ur, bend right] &
\end{tikzcd}$$

Let now M and N be two monoidal objects in $\mathcal{C}$. A *monoidal morphism* $f\colon M \longrightarrow N$ is a triple (f, m_f, u_f) consisting of a morphism $f\colon M \longrightarrow N$ in $\mathcal{C}$ and two invertible 2-cells m_f and u_f as indicated in:

$$\begin{tikzcd} M\otimes M \arrow[r, "\mu_M"] \arrow[d, "f\otimes f"'] \arrow[dr, phantom, "\nearrow"] & M \arrow[d, "f"] \\ N\otimes N \arrow[r, "\mu_N"'] & N \end{tikzcd} \qquad \begin{tikzcd} \mathbb{S} \arrow[r, "u_M"] \arrow[dr, bend right, "u_N"'] & M \arrow[d, "f"] \arrow[dl, phantom, "\nearrow"] \\ & N \end{tikzcd}$$

Using a similar notation as in the previous case the coherence conditions on such a triple are given by the commutativity of the following two diagrams:

$$\begin{tikzcd} (f(-)\cdot f(-))\cdot f(-) \arrow[r] \arrow[d] & f(-)\cdot(f(-)\cdot f(-)) \arrow[d] \\ f((-)\cdot(-))\cdot f(-) \arrow[d] & f(-)\cdot f((-)\cdot(-)) \arrow[d] \\ f\big(((-)\cdot(-))\cdot(-)\big) \arrow[r] & f\big((-)\cdot((-)\cdot(-))\big) \end{tikzcd} \qquad \begin{tikzcd} u_N\cdot f(-) \arrow[r] \arrow[d] & f(-) \arrow[dd, "="] & f(-)\cdot u_N \arrow[l] \arrow[d] \\ f(u_M)\cdot f(-) \arrow[d] & & f(-)\cdot f(u_M) \arrow[d] \\ f(u_M\cdot(-)) \arrow[r] & f(-) & f((-)\cdot u_M) \arrow[l] \end{tikzcd}$$

The composition of monoidal morphisms is defined by composition of the underlying morphisms and by splicing of 2-cells. Thus, for a pair (g, f) of composable monoidal morphisms we have

$$(g, m_g, u_g)\circ(f, m_f, u_f) \quad = \quad (gf, m_{gf}, u_{gf}) \quad = \quad (gf, gm_f\cdot m_g(f\otimes f), gu_g\cdot u_g).$$

Here, we use the central dot $\cdot$ to denote the vertical composition of 2-cells in $\mathcal{C}$.

Finally, a *monoidal 2-morphism* $\phi\colon f \longrightarrow g$ between two parallel monoidal morphisms (f, m_f, u_f) and (g, m_g, u_g) is a 2-morphism $\phi\colon f \longrightarrow g$ in $\mathcal{C}$ making the following two diagrams commute:

$$\begin{tikzcd} \mu_N\circ(f\otimes f) \arrow[r, "m_f"] \arrow[d, "\phi\otimes\phi"'] & f\circ\mu_M \arrow[d, "\phi"] \\ \mu_N\circ(g\otimes g) \arrow[r, "m_g"'] & g\circ\mu_M \end{tikzcd} \qquad \begin{tikzcd} u_N \arrow[r, "u_f"] \arrow[dr, bend right, "u_g"'] & f\circ u_M \arrow[d, "\phi"] \\ & g\circ u_M \end{tikzcd}$$

The vertical and the horizontal composition of monoidal 2-morphisms is the same as the corresponding one in $\mathcal{C}$. It is straightforward to check that we obtain a 2-category this way so let us make the following definition.

Definition B.1. Let $\mathcal{C}$ be a monoidal 2-category. The *2-category* $\mathsf{Mon}(\mathcal{C})$ *of monoidal objects in* $\mathcal{C}$ consists of the monoidal objects together with the monoidal morphisms and the monoidal 2-morphisms.

Example B.2. i) For the Cartesian 2-category $\mathcal{C} = \mathsf{CAT}$ we have $\mathsf{Mon}(\mathsf{CAT}) = \mathsf{MonCAT}$, the 2-category of monoidal categories.

ii) For the Cartesian 2-category $\mathcal{C} = \mathsf{PDer}$ resp. $\mathcal{C} = \mathsf{Der}$ we have $\mathsf{Mon}(\mathsf{PDer}) = \mathsf{MonPDer}$ resp. $\mathsf{Mon}(\mathsf{Der}) = \mathsf{MonDer}$, the 2-category of monoidal prederivators resp. monoidal derivators.

Let us now turn to modules. A *(left) module* X over a monoidal object $M \in \mathsf{Mon}(\mathcal{C})$ is a quadruple (X, a_X, m_X, u_X) consisting of an object $X \in \mathcal{C}$, an action map $a_X\colon M \otimes X \longrightarrow X$ and two invertible 2-cells u_X and m_X as indicated in:

$$\begin{tikzcd}
\mathbb{S} \otimes X \arrow[r, "u_M"] \arrow[dr, bend right, "\lambda"'] \arrow[dr, phantom, "\Nearrow"] & M \otimes X \arrow[d, "a_X"] \\
 & X
\end{tikzcd}
\qquad
\begin{tikzcd}
(M \otimes M) \otimes X \arrow[rr, "\alpha"] \arrow[d, "\mu_M"'] & & M \otimes (M \otimes X) \arrow[d, "a_X"] \\
M \otimes X \arrow[r, "a_X"'] & X \arrow[u, phantom, "\Leftarrow"] & M \otimes X \arrow[l, "a_X"]
\end{tikzcd}$$

These are again subject to certain coherence conditions which we will depict using a similar notation as in the case of monoidal objects. Note, that in this context the outer right central dot corresponds to the action while the other ones correspond to multiplications on the monoid. The coherence conditions for a module consist again of a pentagon diagram as in the case of monoids and also the following two triangles:

$$\begin{tikzcd}
u_M \cdot ((-) \cdot (-)) \arrow[r] & (-) \cdot (-) \\
(u_M \cdot (-)) \cdot (-) \arrow[u] \arrow[ur, bend right] &
\end{tikzcd}
\qquad
\begin{tikzcd}
(-) \cdot (u_M \cdot (-)) \arrow[r] & (-) \cdot (-) \\
((-) \cdot u_M) \cdot (-) \arrow[u] \arrow[ur, bend right] &
\end{tikzcd}$$

Given two M-modules X and Y, a *lax morphism of modules* $f\colon X \longrightarrow Y$ is a pair (f, m_f) consisting of an underlying morphism $f\colon X \longrightarrow Y$ in $\mathcal{C}$ and a (not necessarily invertible) 2-cell m_f as in:

$$\begin{tikzcd}
M \otimes X \arrow[r, "a_X"] \arrow[d, "f"'] \arrow[dr, phantom, "\Nearrow"] & X \arrow[d, "f"] \\
M \otimes Y \arrow[r, "a_Y"'] & Y
\end{tikzcd}$$

This 2-cell is subject to the following two coherence conditions:

$$\begin{tikzcd}
((-) \cdot (-)) \cdot f(-) \arrow[r] \arrow[d] & (-) \cdot ((-) \cdot f(-)) \arrow[r] & (-) \cdot f((-) \cdot (-)) \arrow[d] \\
f\big(((-) \cdot (-)) \cdot (-)\big) \arrow[rr] & & f\big((-) \cdot ((-) \cdot (-))\big)
\end{tikzcd}
\qquad
\begin{tikzcd}
u_M \cdot f(-) \arrow[r] \arrow[d] & f(-) \\
f(u_M \cdot (-)) \arrow[ur, bend right] &
\end{tikzcd}$$

It is important that we allow m_f to be a non-invertible 2-cell here since this will be needed in the construction of the 2-category $\mathsf{Mod}(\mathcal{C})^{\mathsf{lax}}$ of modules via the 2-categorical Grothendieck construction. If the 2-cell m_f is invertible then let us call f a *strong morphism of modules* (or simply a *morphism of modules*). The composition of lax module morphisms is again defined by composition of the underlying morphisms and by splicing of 2-cells. Thus, for two composable lax morphisms g and f we set:

$$(g, m_g) \circ (f, m_f) \quad = \quad (gf, m_{gf}) \quad = \quad (gf, gm_f \cdot m_g(1 \otimes f))$$

Finally, given two lax morphisms $f,\, g\colon X \longrightarrow Y$ a *2-morphism of modules* $\phi\colon f \longrightarrow g$ is just such a 2-cell ϕ in $\mathcal{C}$. This 2-cell has to satisfy the coherence condition:

$$\begin{tikzcd}
a_Y \circ (1 \otimes f) \arrow[r, "\phi"] \arrow[d, "m_f"'] & a_Y \circ (1 \otimes g) \arrow[d, "m_g"] \\
f \circ a_X \arrow[r, "\phi"'] & g \circ a_X
\end{tikzcd}$$

Thus, the 2-cell ϕ has to satisfy the following equation:

$$m_g \cdot a_Y(1 \otimes \phi) \quad = \quad \phi a_X \cdot m_f$$

It is again immediate that these definitions can be assembled to give us a 2-category.

Definition B.3. Let $\mathcal{C}$ be a monoidal 2-category and let M be a monoidal object in $\mathcal{C}$. The *2-category $M - \mathsf{Mod}^{\mathsf{lax}}$ of (left) M-modules* is given by the M-modules, the lax M-module morphisms and the 2-morphisms of M-modules.

We now want to show that the association which sends a monoidal object $M \in \mathsf{Mon}(\mathcal{C})$ to the 2-category $M - \mathsf{Mod}^{\mathsf{lax}}$ is 2-functorial. This allows us then to apply the 2-categorical Grothendieck construction of Appendix A in order to obtain the 2-category $\mathsf{Mod}(\mathcal{C})^{\mathsf{lax}}$ of modules in $\mathcal{C}$.

Let us begin by defining the behavior of the 2-functor on morphisms of monoids. So, let us consider a morphism $f\colon M \longrightarrow N$ in $\mathsf{Mon}(\mathcal{C})$ and let us construct the associated 2-functor $f^*\colon N - \mathsf{Mod}^{\mathsf{lax}} \longrightarrow M - \mathsf{Mod}^{\mathsf{lax}}$ which basically is a restriction of scalar 2-functor. For this purpose, let $X = (X, a_X, m_X, u_X)$ be an N-module. The underlying object of f^*X is again just X while a_{f^*X} and u_{f^*X} are defined by the following diagrams:

$$\begin{tikzcd}
M \otimes X \arrow[r, "f \otimes 1"] \arrow[dr, bend right, "a_{f^*X}"'] & N \otimes X \arrow[d, "a_X"] \\
 & X
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{S} \otimes X \arrow[r, "u_M"] \arrow[dr, "u_N"'] \arrow[ddr, bend right, "\lambda"'] \arrow[dr, phantom, shift left=3, "\Nearrow"] & M \otimes X \arrow[d, "f"] \\
\arrow[r, phantom, shift right=2, "\Nearrow"] & N \otimes X \arrow[d, "a_X"] \\
 & X
\end{tikzcd}$$

The upper unlabeled 2-cell is given by $u_f \otimes 1$ while the lower one is u_X. Thus, in formulas we are setting:

$$a_{f^*X} = a_X(f \otimes 1) \qquad \text{and} \qquad u_{f^*X} = a_X(u_f \otimes 1) \cdot u_X$$

Finally, in order to construct m_{f^*X} let us consider the following diagram in which the left 2-cell is induced by m_f while the other one is just m_X :

$$\begin{tikzcd}
& (M\otimes M)\otimes X \arrow[r, "\alpha"] \arrow[ddl, bend right, "\mu_M"'] \arrow[dd, "f\otimes f"'] & M\otimes(M\otimes X) \arrow[d, "f"] \arrow[dr, bend left, "a_{f^*X}"] & & \\
& & M\otimes(N\otimes X) \arrow[r, "a_X"'] \arrow[d, "f"] & M\otimes X \arrow[d, "f"] \arrow[dr, bend left, "a_{f^*X}"] & \\
M\otimes X \arrow[r, phantom, "\overset{m_f}{\Leftarrow}"] \arrow[dr, bend right, "f"'] & (N\otimes N)\otimes X \arrow[r, "\alpha"'] \arrow[d, "\mu_N"] & N\otimes(N\otimes X) \arrow[r, "a_X"'] \arrow[d, phantom, "\overset{m_X}{\Leftarrow}"] & N\otimes X \arrow[r, "a_X"'] & X \\
& N\otimes X \arrow[urrr, bend right, "a_X"'] & & &
\end{tikzcd}$$

The 2-cell obtained by splicing from this diagram is taken to be m_{f^*X}, i.e., we set

$$m_{f^*X} = a_X(m_f \otimes 1) \cdot m_X((f\otimes f)\otimes 1).$$

This concludes the definition of f^* on objects. Let us now define its behavior on morphisms. So, for a morphism $h = (h, m_h)\colon X \longrightarrow Y$ in $N - \mathsf{Mod}^{\mathsf{lax}}$ let us set:

$$f^*h = (h, m_{f^*h}) = (h, m_h(f\otimes 1))$$

Finally, given a 2-morphism $\phi\colon h_1 \longrightarrow h_2$ of morphisms $h_1,\ h_2\colon X \longrightarrow Y$ of modules, let f^* just map ϕ to itself. Then, in order to check that $f^*\phi\colon f^*h_1 \longrightarrow f^*h_2$ has the necessary coherence property let us consider the following chain of equalities.

$$\begin{array}{rcl}
m_{f^*h_2}\cdot a_{f^*Y}(1\otimes f^*\phi) & = & m_{h_2}(f\otimes 1)\cdot a_Y(f\otimes 1)(1\otimes\phi) \\
& = & (m_{h_2}\cdot a_Y(1\otimes\phi))(f\otimes 1) \\
& = & (\phi a_X \cdot m_{h_1})(f\otimes 1) \\
& = & \phi a_X(f\otimes 1)\cdot m_{h_1}(f\otimes 1) \\
& = & f^*\phi a_{f^*X}\cdot m_{f^*h_1}
\end{array}$$

Here, the third equation uses the fact that ϕ is a 2-cell in $N - \mathsf{Mod}^{\mathsf{lax}}$ while the composite equality precisely says that $f^*\phi = \phi$ is also a 2-cell $f^*h_1 \longrightarrow f^*h_2$ in $M - \mathsf{Mod}^{\mathsf{lax}}$. This concludes the definition of f^* and it is easy to verify that it in fact defines a 2-functor. Now, given two composable morphisms $M \xrightarrow{f} N \xrightarrow{g} P$ in $\mathsf{Mon}(\mathcal{C})$ we want to check that we have an equality $f^*g^* = (gf)^*$ of 2-functors $P - \mathsf{Mod}^{\mathsf{lax}} \longrightarrow M - \mathsf{Mod}^{\mathsf{lax}}$. But this is obvious for their behavior on morphisms and 2-morphisms and hence also for their behavior on objects.

Finally, let us consider a 2-cell ψ in $\mathsf{Mon}(\mathcal{C})$ as in:

$$\begin{tikzcd}
M \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\psi"] & N
\end{tikzcd}$$

We want to associate a 2-natural transformation $\psi^*\colon g^* \longrightarrow f^*$ to ψ. So, let us consider an N-module X and the associated M-modules f^*X, g^*X. We claim that the pair $(\mathrm{id}_X, a_X(\psi\otimes \mathrm{id}_X))$ defines a morphism $g^*X \longrightarrow f^*X$ in $M-\mathsf{Mod}^{\mathsf{lax}}$. Let us only check the unitality condition, i.e., let us consider the following diagram:

$$
\begin{tikzcd}
 & \mathbb{S}\otimes X \arrow[r, "\mathrm{id}"] \arrow[d, "u_M"'] \arrow[dd, bend right=70, "\lambda"'] & \mathbb{S}\otimes X \arrow[d, "u_M"] \arrow[dd, bend left=70, "\lambda"] & \\
\overset{u_{g^*X}}{\Longrightarrow} & M\otimes X \arrow[r, "\mathrm{id}"] \arrow[d, "a_{g^*X}"'] \arrow[dr, phantom, "\Leftarrow"] & M\otimes X \arrow[d, "a_{f^*X}"] & \overset{u_{f^*X}}{\Longleftarrow} \\
 & X \arrow[r, "\mathrm{id}"'] & X &
\end{tikzcd}
$$

Here, the unlabeled 2-cell is given by $a_X(\psi\otimes \mathrm{id}_X)$. But the unitality coherence condition satisfied by ψ as a 2-morphism in $\mathsf{Mon}(\mathcal{C})$ implies that we have the equality:

$$u_{g^*X} \quad = \quad a_X(\psi\otimes \mathrm{id}_X)(u_M\otimes 1)\cdot u_{f^*X}$$

Thus, $\psi^*_X = (\mathrm{id}_X, a_X(\psi\otimes \mathrm{id}_X))$ defines a morphism $g^*X \longrightarrow f^*X$ in $M-\mathsf{Mod}^{\mathsf{lax}}$. It is now easy to verify that these ψ^*_X assemble to define a 2-natural transformation $\psi^*\colon g^* \longrightarrow f^*$.

This concludes the construction of our 2-functor. Before we can summarize the construction by the following proposition let us quickly recall that given an arbitrary 2-category $\mathcal{D}$, the 2-category obtained from $\mathcal{D}$ by inverting both the direction of the 1-cells and of the 2-cells is denoted by $\mathcal{D}^{\mathrm{op,co}}$.

Proposition B.4. *Let $\mathcal{C}$ be a monoidal 2-category and let us consider a 2-cell $\psi\colon f \longrightarrow g\colon M \longrightarrow N$ in $\mathsf{Mon}(\mathcal{C})$. The following assignments define a 2-category valued 2-functor $(-)-\mathsf{Mod}^{\mathsf{lax}}$:*

$$M \longmapsto M-\mathsf{Mod}^{\mathsf{lax}}, \qquad f\longmapsto f^*, \qquad \textit{and} \qquad \psi\longmapsto \psi^*$$

Having established this proposition we can now apply the 2-categorical Grothendieck construction of Appendix A to the 2-category-valued 2-functor $(-)-\mathsf{Mod}^{\mathsf{lax}}$. This gives us the *2-category* $\mathsf{Mod}(\mathcal{C})^{\mathsf{lax}}$ *of modules in* $\mathcal{C}$. Let us be a bit more specific about this 2-category. An object is a pair (M,X) consisting of a monoidal object M and an M-module X. Similarly, a morphism $(f,u)\colon (M,X) \longrightarrow (N,Y)$ is a pair consisting of monoidal morphism $f\colon M \longrightarrow N$ and a lax morphism of M-modules $u\colon X \longrightarrow f^*Y$. Finally, given two parallel such morphisms (f,u) and (g,v), a 2-cell $(\beta,\phi)\colon (f,u) \longrightarrow (g,v)$ is a monoidal 2-cell $\beta\colon f \longrightarrow g$ together with a 2-cell ϕ of M-modules as in:

$$
\begin{tikzcd}
X \arrow[r, "u"] \arrow[dr, bend right, "v"'] \arrow[dr, phantom, shift left=2, "\Swarrow"] & f^*Y \\
 & g^*Y \arrow[u, "\beta^*"']
\end{tikzcd}
$$

This 2-category is endowed with a projection functor $p\colon \mathsf{Mod}(\mathcal{C})^{\mathsf{lax}} \longrightarrow \mathsf{Mon}(\mathcal{C})$ which we call the *2-Grothendieck fibration of modules in* $\mathcal{C}$.

Of more importance in the next subsection is the 2-subcategory $\mathsf{Mod}(\mathcal{C}) \subseteq \mathsf{Mod}(\mathcal{C})^{\mathsf{lax}}$. By definition this consists of all objects (M, X), the morphisms (f, u) such that u is a strong morphism of modules and all 2-cells between such morphisms. The inclusion endows $\mathsf{Mod}(\mathcal{C})$ with a projection functor $p\colon \mathsf{Mod}(\mathcal{C}) \longrightarrow \mathsf{Mon}(\mathcal{C})$ which we still call the 2-Grothendieck fibration of modules in $\mathcal{C}$.

Example B.5. i) For the Cartesian 2-category $\mathcal{C} = \mathsf{CAT}$ we have $\mathsf{Mod}(\mathsf{CAT}) = \mathsf{MonCAT}$, the 2-category of left-tensored categories.
ii) For the Cartesian 2-category $\mathcal{C} = \mathsf{PDer}$ resp. $\mathcal{C} = \mathsf{Der}$ we have $\mathsf{Mod}(\mathsf{PDer}) = \mathsf{ModPDer}$ resp. $\mathsf{Mod}(\mathsf{Der}) = \mathsf{ModDer}$, the 2-category of left-tensored prederivators resp. derivators.

In addition to this 2-functor p, we also have a canonical 2-functor $U\colon \mathsf{Mod}(\mathcal{C}) \longrightarrow \mathcal{C}$. This 2-functor sends an object (M, X) to the underlying object of X and a morphism $(f, u)\colon (M, X) \longrightarrow (N, Y)$ to the underlying morphism of $u\colon X \longrightarrow f^*Y$. Similarly, U sends a 2-cell (β, ϕ) just to the underlying 2-cell $\phi\colon u \longrightarrow \beta^*v$ in $\mathcal{C}$. It is immediate that this defines a 2-functor U. As an upshot, we thus obtain the following diagram of 2-categories:

$$\begin{tikzcd} \mathsf{Mod}(\mathcal{C}) \arrow[r, "U"] \arrow[d, "p"'] & \mathcal{C} \\ \mathsf{Mon}(\mathcal{C}) \end{tikzcd}$$

Now, given an object $X \in \mathcal{C}$ let $\mathsf{Mod}(X) = U^{-1}(X)$ denote the fiber of U over X, i.e., it is the 2-category defined by the following pullback diagram:

$$\begin{tikzcd} \mathsf{Mod}(X) \arrow[r] \arrow[d] & \mathsf{Mod}(\mathcal{C}) \arrow[d, "U"] \\ e \arrow[r, "X"'] & \mathcal{C} \end{tikzcd}$$

Here, e denotes the terminal 2-category and by abuse of notation X denotes at the same time the object X and the unique 2-functor $e \longrightarrow \mathcal{C}$ classifying the object X. Let us call the 2-category $\mathsf{Mod}(X)$ the *2-category of module structures on* X. Since the underlying object of an arbitrary object in this 2-category is X let us agree that we denote such an object by (M, a_X) where a_X is the action belonging to the M-module structure on X. In Subsection B.2 we will show that for *closed* monoidal 2-categories $\mathcal{C}$ this 2-category of module structures always has a terminal object in a suitable bicategorical sense.

B.2. **Closed monoidal 2-categories.** The main aim of this subsection is to give a 2-categorical analog of the following result about closed monoidal categories. Let $\mathcal{C}$ be a closed monoidal category with symmetric monoidal pairing $\otimes$, monoidal unit $\mathbb{S}$ and internal homomorphism functor HOM. Plugging in twice the same object $X \in \mathcal{C}$ into HOM we obtain internal endomorphism objects $\mathsf{END}(X) \in \mathcal{C}$. Since we assumed the monoidal structure to be closed we have natural isomorphisms

$$\mathsf{hom}_{\mathcal{C}}(X \otimes Y, Z) \cong \mathsf{hom}_{\mathcal{C}}(X, \mathsf{HOM}(Y, Z)).$$

The adjunction counit gives us in particular a map $\epsilon = \mathsf{ev}\colon \mathsf{END}(X)\otimes X \longrightarrow X$ which we call an *evaluation map.* A combination of this map with the associativity constraint of the monoidal structure gives us the following map:

$$(\mathsf{END}(X)\otimes \mathsf{END}(X))\otimes X \xrightarrow{\alpha} \mathsf{END}(X)\otimes(\mathsf{END}(X)\otimes X) \xrightarrow{\mathsf{ev}} \mathsf{END}(X)\otimes X \xrightarrow{\mathsf{ev}} X$$

Let us denote the associated adjoint map by $\circ_X\colon\ \mathsf{END}(X)\otimes\mathsf{END}(X)\longrightarrow\mathsf{END}(X)$. Moreover, let us write $\iota_X\colon\ \mathbb{S}\longrightarrow\mathsf{END}(X)$ for the map which is adjoint to the unitality constraint $\lambda\colon\ \mathbb{S}\otimes X\ \longrightarrow\ X$. Finally, similar to the last subsection one can construct the category $\mathsf{Mod}(X)$ of module structures on X in the context of a monoidal category.

Proposition B.6. *Let $\mathcal{C}$ be a closed monoidal category and let $X\in\mathcal{C}$ be an object. The triple $(\mathsf{END}(X),\circ_X,\iota_X)$ is a monoid in $\mathcal{C}$ and the evaluation map $\mathsf{ev}\colon\ \mathsf{END}(X)\otimes X\longrightarrow X$ turns X into a module over $\mathsf{END}(X)$. The pair $(\mathsf{END}(X),\mathsf{ev})$ is the terminal object in the category $\mathsf{Mod}(X)$ of module structures on X.*

We want to give a similar result in the setting of 2-categories which will be applied in Subsection 2.2 to the Cartesian closed 2-category Der of derivators. Besides working with 2-categories, there is an additional technical difficulty resulting from the following fact. Before we can formulate this let us recall that there are two different notions of adjointness for 2-functors. The stricter one of these notions which we shall call a 2-*adjunction* is just a special case of an enriched adjunction. In such a situation the adjointness is expressed by the fact that we have natural *isomorphisms* between the respective categories of morphisms. A more general and –morally speaking– more correct notion is the notion of a *biadjunction.* In this case we instead have natural *equivalences* of categories of morphisms. To mention only one difference between biadjunctions and 2-adjunctions, note that a biadjunction does in general not induce an adjunction on underlying 1-categories. Now, the additional technical difficulty results from the fact that the closedness of the Cartesian monoidal 2-category Der given by Proposition 2.7 is expressed by a special instance of a biadjunction. With this example in mind, let us make the following definition.

Definition B.7. Let $\mathcal{C}$ be a symmetric monoidal 2-category. The monoidal structure is *closed* if the functor $X\otimes -\colon\ \mathcal{C}\longrightarrow\mathcal{C}$ has a right biadjoint $\mathsf{HOM}(X,-)$ for each object $X\in\mathcal{C}$, i.e., if there are natural equivalences of categories

$$\mathsf{Hom}_{\mathcal{C}}(X\otimes Y,Z)\simeq\mathsf{Hom}_{\mathcal{C}}(X,\mathsf{HOM}(Y,Z)).$$

For the rest of this subsection let $\mathcal{C}$ be a closed monoidal 2-category and let us choose inverse equivalences of categories:

$$\mathsf{Hom}_{\mathcal{C}}(X\otimes Y,Z)\underset{r}{\overset{l}{\rightleftarrows}}\mathsf{Hom}_{\mathcal{C}}(X,\mathsf{HOM}(Y,Z))$$

As for any biadjunction one can describe the equivalences l and r by the unit η and the counit ϵ. In fact, we have isomorphisms $r\cong\epsilon_*\circ(-\otimes Y)$ and $l\cong\eta^*\circ\mathsf{HOM}(-,Y)$.

By precisely the same formulas as in the 1-categorical case we can now use these equivalences and the constraints of the monoidal structure in order to obtain morphisms $\circ_X$, ι_X and ev. In the proof of the next proposition we will use EX as an abbreviation for $\mathsf{END}(X)$.

Proposition B.8. *Let $\mathcal{C}$ be a closed monoidal 2-category and let $X \in \mathcal{C}$ be an object. The triple $(\mathsf{END}(X), \circ_X, \iota_X)$ can be extended into a monoidal object in $\mathcal{C}$ and the map* $\mathsf{ev}\colon \mathsf{END}(X)\otimes X \longrightarrow X$ *is part of an* $\mathsf{END}(X)$*-module structure on* X.

Proof. We will only construct the 2-cells which will turn EX into a monoidal object and X into a module over EX. We will leave it to the reader to check the necessary coherence conditions.

So, let us begin by the unitality of the action, i.e., we want to construct an invertible 2-cell u_X as indicated in:

$$\begin{tikzcd}
\mathbb{S}\otimes X \arrow[r, "\iota\otimes X"] \arrow[dr, bend right, "\lambda"'] & \mathsf{EX}\otimes X \arrow[d, "\mathsf{ev}"] \arrow[dl, phantom, "\cong"] \\
& X
\end{tikzcd}$$

But this is given by $\mathsf{ev}\circ(\iota\otimes X) = \mathsf{ev}\circ(l(\lambda)\otimes X) \cong r(l(\lambda)) \cong \lambda$. The construction of the –say– left unitality of the multiplication $\circ_X$ on EX is slightly more complicated. We have to show that there is an invertible 2-cell u_{EX} as in:

$$\begin{tikzcd}
\mathbb{S}\otimes \mathsf{EX} \arrow[r, "\iota\otimes \mathsf{EX}"] \arrow[dr, bend right, "\lambda"'] & \mathsf{EX}\otimes \mathsf{EX} \arrow[d, "\circ_X"] \arrow[dl, phantom, "\cong"] \\
& \mathsf{EX}
\end{tikzcd}$$

If we can show that the images of these two compositions under r are isomorphic we can use the fully-faithfulness of r to conclude that this isomorphism comes from a unique isomorphism of morphisms $\mathbb{S}\otimes\mathsf{EX} \longrightarrow \mathsf{EX}$. To calculate the image of $\circ_X\circ(\iota\otimes\mathsf{EX})$ under r let us consider the left diagram:

$$\begin{tikzcd}
(\mathbb{S}\otimes \mathsf{EX})\otimes X \arrow[r, "\iota"] \arrow[d, "\alpha"'] & (\mathsf{EX}\otimes \mathsf{EX})\otimes X \arrow[d, "\alpha"] \\
\mathbb{S}\otimes(\mathsf{EX}\otimes X) \arrow[r, "\iota"] \arrow[d, "\mathsf{ev}"'] & \mathsf{EX}\otimes(\mathsf{EX}\otimes X) \arrow[d, "\mathsf{ev}"] \\
\mathbb{S}\otimes X \arrow[r, "\iota"] \arrow[dr, bend right, "\lambda"'] & \mathsf{EX}\otimes X \arrow[d, "\mathsf{ev}"] \arrow[dl, phantom, "\cong"] \\
& X
\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathbb{S}\otimes \mathsf{EX})\otimes X \arrow[d, "\alpha"'] \arrow[dr, bend left, "\lambda"] & \\
\mathbb{S}\otimes(\mathsf{EX}\otimes X) \arrow[r, "\lambda"'] \arrow[d, "\mathsf{ev}"'] & \mathsf{EX}\otimes X \arrow[d, "\mathsf{ev}"] \\
\mathbb{S}\otimes X \arrow[r, "\lambda"'] & X
\end{tikzcd}$$

The invertible 2-cell in the left diagram is u_X. Using the commutative diagram on the right we have thus obtained an isomorphism as intended:

$$r(\circ_X\circ(\iota\otimes\mathsf{EX})) \cong \mathsf{ev}\circ\mathsf{ev}\circ\alpha\circ\iota \cong \lambda\circ(\mathbb{S}\otimes\mathsf{ev})\circ\alpha = \mathsf{ev}\circ(\lambda\otimes X) \cong r(\lambda)$$

Let us now turn to the associativity of $\circ_X$ and the multiplicativity of ev. Again we will begin with the action ev since that 2-cell will be used in the construction of the 2-cell expressing the associativity of the multiplication on EX. So, let us show that there is an invertible 2-cell m_X:

$$\begin{tikzcd}
(\mathsf{EX}\otimes\mathsf{EX})\otimes X \arrow[rr, "\alpha"] \arrow[d, "\circ_X"'] & & \mathsf{EX}\otimes(\mathsf{EX}\otimes X) \arrow[d, "\mathsf{ev}"] \\
\mathsf{EX}\otimes X \arrow[r, "\mathsf{ev}"'] & X \arrow[u, phantom, "\Leftarrow"] & \mathsf{EX}\otimes X \arrow[l, "\mathsf{ev}"]
\end{tikzcd}$$

But this is again just the observation that we have invertible 2-cells

$$\mathsf{ev}\circ(\circ_X\otimes X)\cong r(\circ_X)=r(l(\mathsf{ev}\circ\mathsf{ev}\circ\alpha))\cong\mathsf{ev}\circ\mathsf{ev}\circ\alpha.$$

Thus, let us now show that the multiplication $\circ_X$ is associative, i.e., let us construct an invertible 2-cell α_{EX} as in:

$$\begin{tikzcd}
(\mathsf{EX}\otimes\mathsf{EX})\otimes\mathsf{EX} \arrow[rr, "\alpha"] \arrow[d, "\circ_X\otimes 1"'] & & \mathsf{EX}\otimes(\mathsf{EX}\otimes\mathsf{EX}) \arrow[d, "1\otimes\circ_X"] \\
\mathsf{EX}\otimes\mathsf{EX} \arrow[r, "\circ_X"'] & \mathsf{EX} \arrow[u, phantom, "\Leftarrow"] & \mathsf{EX}\otimes\mathsf{EX} \arrow[l, "\circ_X"]
\end{tikzcd}$$

Similarly to the proof of the unitality, let us show that the two morphisms $(\mathsf{EX}\otimes\mathsf{EX})\otimes\mathsf{EX}\longrightarrow\mathsf{EX}$ have isomorphic images under r. For this purpose, let us consider the following diagram:

$$\begin{tikzcd}
 & (\mathsf{EX}\otimes(\mathsf{EX}\otimes\mathsf{EX}))\otimes X \arrow[dl, bend right, "\circ_X"'] \arrow[d, "\cong"', "\alpha"] & ((\mathsf{EX}\otimes\mathsf{EX})\otimes\mathsf{EX})\otimes X \arrow[l, "\cong"', "\alpha"] \arrow[d, "\alpha"', "\cong"] \arrow[dr, bend left, "\circ_X"] & \\
(\mathsf{EX}\otimes\mathsf{EX})\otimes X \arrow[d, "\alpha"', "\cong"] & \mathsf{EX}\otimes((\mathsf{EX}\otimes\mathsf{EX})\otimes X) \arrow[dl, bend right, "\circ_X"] \arrow[d, "\cong"', "\alpha"] & (\mathsf{EX}\otimes\mathsf{EX})\otimes(\mathsf{EX}\otimes X) \arrow[dl, bend left, "\alpha"', "\cong"] \arrow[d, "\mathsf{ev}"] \arrow[dr, bend left, "\circ_X"'] & (\mathsf{EX}\otimes\mathsf{EX})\otimes X \arrow[d, "\cong"', "\alpha"] \arrow[ddd, bend left=60, "\circ_X"] \\
\mathsf{EX}\otimes(\mathsf{EX}\otimes X) \arrow[ddr, bend right, "\mathsf{ev}"'] \arrow[dr, phantom, "\cong"] & \mathsf{EX}\otimes(\mathsf{EX}\otimes(\mathsf{EX}\otimes X)) \arrow[d, "\mathsf{ev}"] & (\mathsf{EX}\otimes\mathsf{EX})\otimes X \arrow[dl, bend left, "\cong", "\alpha"'] \arrow[d, "\circ_X"] & \mathsf{EX}\otimes(\mathsf{EX}\otimes X) \arrow[dl, bend left, "\mathsf{ev}"] \arrow[dd, phantom, "\cong"] \\
 & \mathsf{EX}\otimes(\mathsf{EX}\otimes X) \arrow[d, "\mathsf{ev}"] \arrow[dr, phantom, "\cong"] & \mathsf{EX}\otimes X \arrow[d, "\mathsf{ev}"] & \\
 & \mathsf{EX}\otimes X \arrow[r, "\mathsf{ev}"'] & X & \mathsf{EX}\otimes X \arrow[l, "\mathsf{ev}"]
\end{tikzcd}$$

Up to an implicit use of the invertible 2-cell m_X the two possible paths through the boundary leading from $((\mathsf{EX}\otimes\mathsf{EX})\otimes\mathsf{EX})\otimes X$ to X are the images under r of the maps which we want to compare. In this diagram there are three more instances of the invertible 2-cell m_X. The remaining part commutes on the nose by naturality in four cases and by the coherence property of the associativity constraint α in the last case. The resulting invertible 2-cell gives us the intended 2-cell expressing the associativity of $\circ_X$. This concludes the extension of $\circ_X$ to a monoidal structure on EX and of the evaluation ev to a module structure on X and hence the proof of this proposition. □

In the notation of the last subsection, this proposition shows that in the context of a closed monoidal 2-category the 2-category $\mathsf{Mod}(X)$ of module structures on an object X contains the object $(\mathsf{END}(X), \mathsf{ev})$. The remaining aim of this subsection is to show that this object is a terminal object in the following sense.

Definition B.9. Let $\mathcal{D}$ be a 2-category. An object $X \in \mathcal{D}$ is *terminal* if for all objects Y the category $\mathsf{Hom}_{\mathcal{D}}(Y, X)$ of morphisms from Y to X is equivalent to the category e.

Proposition B.10. *Let $\mathcal{C}$ be a closed monoidal 2-category and let X be an object of $\mathcal{C}$. The canonical action $(\mathsf{END}(X), \mathsf{ev})$ is a terminal object of the 2-category $\mathsf{Mod}(X)$ of module structures on X.*

Proof. Let us begin by showing the following. For an arbitrary object (M, a_X) of $\mathsf{Mod}(X)$ there is a morphism $(M, a_X) \longrightarrow (\mathsf{END}(X), \mathsf{ev})$ in $\mathsf{Mod}(X)$. Similar to the proof of Proposition B.8 we only give the construction of the 1-cells and the 2-cells of the morphism and do not check the necessary coherence conditions. So, our aim is to construct a pair $(f, u)\colon (M, a_X) \longrightarrow (\mathsf{END}(X), \mathsf{ev})$ where $f = (f, m_f, u_f)\colon M \longrightarrow \mathsf{END}(X)$ is a morphism in $\mathsf{Mon}(\mathcal{C})$ and $u = (u, m_u)\colon (X, a_X) \longrightarrow f^*(X, \mathsf{ev})$ is a morphism in $M - \mathsf{Mod}$. Since the morphism (f, u) has to lie in $\mathsf{Mod}(X)$ we have $u = (\mathrm{id}_X, m_u)$.

We construct the monoidal part first. By closedness, $a_X\colon M \otimes X \longrightarrow X$ corresponds to a unique map $f = l(a_X)\colon M \longrightarrow \mathsf{END}(X)$. Here, l again denotes a natural equivalence of categories $l\colon \mathsf{Hom}(M \otimes X, X) \longrightarrow \mathsf{Hom}(M, \mathsf{END}(X))$ given by the biadjunction. The construction of u_f is similar. As part of the M-module structure on X we have the following invertible 2-cell u_X:

$$\begin{tikzcd} \mathbb{S} \otimes X \arrow[r, "u_M \otimes 1"] \arrow[dr, bend right, "\lambda"'] & M \otimes X \arrow[d, "a_X"] \arrow[dl, phantom, "\Uparrow"] \\ & X \end{tikzcd}$$

Recall from the proof of the last proposition that the unit $u_{\mathsf{END}(X)}\colon \mathbb{S} \longrightarrow \mathsf{END}(X)$ is given by $l(\lambda)$. Thus, by adjointness, the 2-cell u_X gives us an invertible 2-cell u_f as in:

$$\begin{tikzcd} \mathbb{S} \arrow[r, "u_M"] \arrow[dr, bend right, "u_{\mathsf{end}(X)}"'] & M \arrow[d, "f"] \arrow[dl, phantom, "\Uparrow"] \\ & \mathsf{END}(X) \end{tikzcd}$$

In the construction of m_f we will of course use the multiplicativity constraint of the module structure on X. So, let us consider the invertible 2-cell m_X :

$$\begin{tikzcd} (M \otimes M) \otimes X \arrow[rr, "\alpha"] \arrow[d, "\mu_M \otimes 1"'] & & M \otimes (M \otimes X) \arrow[d, "1 \otimes a_X"] \\ X \otimes X \arrow[r, "a_X"'] & X \arrow[u, phantom, "\Leftarrow"] & X \otimes X \arrow[l, "a_X"] \end{tikzcd}$$

The 1-cell $l(a_X \circ (\mu_M \otimes 1)) = l(a_X) \circ \mu_M = f \circ \mu_M \colon M \otimes M \longrightarrow \mathsf{END}(X)$ gives us already the target of m_f. We now want to identify $l(a_X \circ (1 \otimes a_X) \circ \alpha)$ with the source $\circ_X \circ (f \otimes f)$ of m_f. For this let us recall from the proof of the last proposition that the multiplication $\circ_X$ is given by $l(\mathsf{ev} \circ \mathsf{ev} \circ \alpha)$. A calculation of $r(\circ_X \circ (f \otimes f))$ thus leads to the following diagram:

$$\begin{tikzcd}
(M \otimes M) \otimes X \arrow[r, "(f\otimes f)\otimes 1"] \arrow[d, "\alpha"'] & (\mathsf{END}(X) \otimes \mathsf{END}(X)) \otimes X \arrow[r, "\circ_X"] \arrow[d, "\alpha"'] & \mathsf{END}(X) \otimes X \arrow[r, "\mathsf{ev}"] \arrow[d, phantom, "\cong"] & X \\
M \otimes (M \otimes X) \arrow[r, "f\otimes(f\otimes 1)"'] & \mathsf{END}(X) \otimes (\mathsf{END}(X) \otimes X) \arrow[r, "\mathsf{ev}"'] & \mathsf{END}(X) \otimes X \arrow[ur, bend right, "\mathsf{ev}"'] &
\end{tikzcd}$$

Here, the 2-cell is the invertible 2-cell m'_X expressing the multiplicativity of ev as constructed in the last proof. But the composition $\mathsf{ev} \circ (1 \otimes \mathsf{ev}) \circ (f \otimes (f \otimes 1))$ can be rewritten as:

$$\begin{tikzcd}
M \otimes (M \otimes X) \arrow[r, "f"] \arrow[rr, bend right, "1\otimes a_X"', "\cong"] & M \otimes (\mathsf{END}(X) \otimes X) \arrow[r, "\mathsf{ev}"] & M \otimes X \arrow[r, "f"] \arrow[rr, bend right, "a_X"', "\cong"] & \mathsf{END}(X) \otimes X \arrow[r, "\mathsf{ev}"] & X
\end{tikzcd}$$

The two invertible 2-cells are both instances of $\mathsf{ev} \circ (f \otimes 1) = r(f) = r(l(a_X) \cong a_X$. Thus, splicing these three invertible 2-cells together we obtain an isomorphism $r(\circ_X \circ (f \otimes f)) \cong a_X \circ (1 \otimes a_X) \circ \alpha$. This allows us to construct the invertible 2-cell m_f as the following composite

$$\circ_X \circ (f \otimes f) \cong l(r(\circ_X \circ (f \otimes f))) \cong l(a_X \circ (1 \otimes a_X) \circ \alpha) \cong l(a_X \circ (\mu_M \otimes 1)) = f \circ \mu_M.$$

The last invertible 2-cell in this composition is given by $l(m_X)$. The construction of the monoidal morphism $f \colon M \longrightarrow \mathsf{END}(X)$ is complete.

Let us now construct the morphism $u = (\mathrm{id}, m_u) \colon (X, a_X) \longrightarrow f^*(X, \mathsf{ev})$ in $M - \mathsf{Mod}$. But this means that we only have to construct an invertible 2-cell m_u as in:

$$\begin{tikzcd}
M \otimes X \arrow[r, "a_X"] \arrow[d, "f"'] \arrow[dr, phantom, "\Nearrow"] & X \arrow[d, "\mathrm{id}"] \\
\mathsf{END}(X) \otimes X \arrow[r, "\mathsf{ev}"'] & X
\end{tikzcd}$$

We take this to be $\mathsf{ev} \circ (f \otimes 1) = r(f) = r(l(a_X)) \cong a_X$. One can now check that this pair (f, u) defines a morphism $(M, a_X) \longrightarrow (\mathsf{END}(X), \mathsf{ev})$ in $\mathsf{Mod}(X)$.

Now, given two parallel morphisms (f, u), $(g, v) \colon (M, a_X) \longrightarrow (\mathsf{END}(X), \mathsf{ev})$ in $\mathsf{Mod}(X)$ it remains to show that there is a unique 2-cell $(f, u) \longrightarrow (g, v)$. Thus, we have to construct a pair (β, ϕ) consisting of a monoidal 2-cell $\beta \colon f \longrightarrow g$ and an invertible 2-cell ϕ in $M - \mathsf{Mod}$

as in:

$$\begin{tikzcd} (X, a_X) \arrow[r, "u"] \arrow[dr, bend right, "v"'] \arrow[dr, phantom, "\Downarrow"] & f^*(X, \mathsf{ev}) \\ & g^*(X, \mathsf{ev}) \arrow[u, "\beta^*"'] \end{tikzcd}$$

But from the module morphisms u and v we obtain invertible 2-cells $m_u\colon \mathsf{ev}\circ(f\otimes 1) \longrightarrow a_X$ and $m_v\colon \mathsf{ev}\circ(g\otimes 1) \longrightarrow a_X$. These can be combined to the invertible 2-cell

$$m_v^{-1}\circ m_u\colon r(f) = \mathsf{ev}\circ(f\otimes 1) \longrightarrow \mathsf{ev}\circ(g\otimes 1) = r(g).$$

Thus, we obtain a unique invertible 2-cell $\beta\colon f \longrightarrow g$ with $r(\beta) = m_v^{-1}\circ m_u$. One checks now that this β is a monoidal 2-cell and that the pair $(\beta, \phi = \mathrm{id}_{\mathrm{id}_X})$ gives us the intended unique 2-cell. This concludes the proof that $(\mathsf{END}(X), \mathsf{ev})$ is terminal in $\mathsf{Mod}(X)$. $\square$

As a summary of this subsection we have thus established the following theorem.

Theorem B.11. *Let $\mathcal{C}$ be a closed monoidal 2-category and let X be an object in $\mathcal{C}$. The internal endomorphism object $\mathsf{END}(X)$ can be canonically made into a monoidal object and the adjunction counit $\mathsf{ev}\colon \mathsf{END}(X)\otimes X \longrightarrow X$ is part of a canonical module structure on X. Moreover, the pair $(\mathsf{END}(X), \mathsf{ev})$ is a terminal object in the 2-category $\mathsf{Mod}(X)$ of module structures on X.*

Part 3. On the derivator associated to a differential-graded algebra over a field

0. Introduction

In this part we take a closer look at the derivator $\mathbb{D}_A$ associated to a differential-graded algebra A over a field. We give a first step towards a result which compares two different ways of encoding the homotopy type of such a dga. Let us begin by describing these two ways.

Given a dga A it is, in general, not possible to recover the dga from the associated homology algebra $H_\bullet A$. However, there is a classical result due to Kadeishvili which guarantees that, if we are working over a field, then this is possible in a certain sense: we have to enlarge our category of dgas in order to recover the homotopy type of A. More precisely, Kadeishvili ([Kad82]) established the result that given a dga A over a field one can endow the homology $H_\bullet A$ with the structure of a minimal A_∞-algebra. Moreover, we have a quasi-isomorphism of A_∞-algebras $H_\bullet A \longrightarrow A$ between this minimal A_∞-algebra and our dga (which we consider as an A_∞-algebra in a trivial way). Using this faithful functor from dgas to the category of A_∞-algebras, we can summarize the result of Kadeishvili as follows: once we pass from the world of dgas to the world of A_∞-algebras, the homotopy type of a dga A over a field can be encoded by the associated homology $H_\bullet A$ if the latter is endowed with the above minimal A_∞-algebra structure.

There is an alternative way of encoding the homotopy type of a dga, which, in fact, works much more generally. So, let $\mathcal{M}$ be a combinatorial model category. Let us recall (e.g. from [Gro10a]) that the systematic formation of homotopy categories of diagram categories yields an associated derivator $\mathbb{D}_{\mathcal{M}}$. This passage from combinatorial model categories to associated derivators does not result in an essential loss of information. More precisely, Renaudin ([Ren09]) has established the following: Once we localize a certain 2-category of combinatorial model categories at the class of Quillen equivalences the pseudo-functor which associates the underlying derivator to such a model category factors over this localization. Moreover, this induced pseudo-functor is a bi-equivalence if we restrict it to its essential image. Thus, this result tells us that given a combinatorial model category $\mathcal{M}$ we can non-canonically reconstruct the Quillen equivalence type of $\mathcal{M}$ from the associated derivator $\mathbb{D}_{\mathcal{M}}$.

In particular, if A is a dga over a field, we can consider the model category $A-\mathsf{Mod}$ of differential-graded A-modules. This gives us an example of a combinatorial model category to which the theorem of Renaudin thus applies. Hence, we can (non-canonically) reconstruct the Quillen equivalence class of this model category from the associated derivator $\mathbb{D}_A$.

Now, a combination of the results of Kadeishvili and Renaudin indicates that it should, in principle, be possible to reconstruct the derivator $\mathbb{D}_A$ from the minimal model on $H_\bullet A$ and vice-versa. Such a reconstruction is interesting because it might give us a better control over the 'reconstructed model'. The proof of the result of Renaudin is not very constructive. In fact, it relies heavily on the result of Dugger that combinatorial model categories have presentations which in turn is proved in a very abstract way.

In this part, we give a first step of such a comparison result (cf. Section 2). Let us now come to a description of the content by sections. In Section 1, we recall the aforementioned results of Kadeishvili and Renaudin. We give a partial proof of the minimal model theorem since some details of the construction will be important in the next section.

In Section 2 we construct certain Hochschild-Mitchell extensions which show that the first higher multiplication map m_3 belonging to the minimal model on $H_\bullet A$ can be used in order to partially reconstruct $\mathbb{D}_A([1])$ from the underlying category of the derivator. This is achieved by first giving an alternative description of $\mathbb{D}_A([1])$ via 'coherent diagrams'. Here, an object is a morphism in $A - \mathsf{Mod}$ while a morphism is a square which commutes up to a specified homotopy. In the first subsection, we give a careful analysis of the model structures on certain diagram categories in $A - \mathsf{Mod}$. This allows us to construct an equivalence of categories between $\mathbb{D}_A([1])$ and a 'homotopy category of coherent diagrams' in the next subsection. Finally, this alternative description is used in the last subsection to establish our Hochschild-Mitchell extension and to give the partial reconstruction result.

In the last section, we indicate two possible directions to further develop the theory. In the first subsection, we remark that our categories of coherent diagrams are closely related to certain A_∞-categories of A_∞-functors. This indicates that there might be a close relation between the derivator $\mathbb{D}_A$ and a second 2-functor $\mathbb{D}'_A$. This 2-functor is obtained by first applying a many-object version of the minimal model theorem to the differential-graded category $(A - \mathsf{Mod})_{dg}$ in order to obtain an A_∞-category $(A - \mathsf{Mod})_{A_\infty}$. The 2-functor $\mathbb{D}'_A$ systematically forms homology categories of A_∞-categories of A_∞-functors with values in (possibly a subcategory of) $(A - \mathsf{Mod})_{A_\infty}$. In the second subsection, we give a more general construction of our Hochschild-Mitchell extension. Using the notions of [Gro11] we can talk about derivators which are linear over a field k. Given an arbitrary k-linear stable derivator $\mathbb{D}$, we can build on results from [Gro10a] in order to construct an analog of the above Hochschild-Mitchell extension. One might hope that a more general comparison result holds true for stable k-linear derivators (satisfying certain smallness conditions).

1. The minimal model of Kadeishvili and the result of Renaudin

1.1. **A_∞-algebras and the minimal model of Kadeishvili.** We begin this section by quickly recalling the central notions of A_∞-algebras and their morphisms. For this section and the next one it will be sufficiently general to work within the framework of a module category $k-\mathsf{Mod}$ for some commutative ground ring k. However, it is convenient to note that the notion of an A_∞-*algebra* is just an incorporation of the concept of an A_∞-*monoid* in the monoidal, abelian category $\mathsf{gr}(\mathrm{k}) = \mathsf{gr}(k-\mathsf{Mod})$ of $\mathbb{Z}$-graded k-modules. If one replaces this 'base category' suitably one is lead to the notion of an A_∞-*category* which will be of interest to us in the last section. The notion of an A_∞-algebra was introduced by Stasheff in [Sta63]. An introduction to this theory can be found in [Kel01], while A_∞-categories are treated in [LH, Kel06a, BLM08].

Recall that we have the monoidal structure on $\mathsf{gr}(\mathrm{k})$ given by the tensor product of graded K-modules. For $A,\ B \in \mathsf{gr}(\mathrm{k})$ define $A \otimes_k B = A \otimes B \in \mathsf{gr}(\mathrm{k})$ by

$$(A \otimes B)_n = \bigoplus_{p+q=n} A_p \otimes B_q.$$

The isomorphisms defined on homogeneous elements by $a \otimes b \longmapsto (-1)^{|a||b|} b \otimes a$ show that the monoidal structure is symmetric. Iterated tensor powers of an object $A \in \mathsf{gr}(\mathrm{k})$ will be denoted by $A^{\otimes n}$ and similarly for maps.

Definition 1.1. Let $A = \{A_n,\ n \in \mathbb{Z}\} \in \mathsf{gr}(\mathrm{k})$ be a $\mathbb{Z}$-graded k-module. An A_∞-*algebra structure on* A consists of a family of morphisms

$$m_n \colon A^{\otimes n} \longrightarrow A, \qquad n \geq 1,$$

of degree $n-2$. These maps have to satisfy for each $n \geq 1$ the relation

$$\sum_{\substack{r+s+t=n \\ r,t\geq 0,\ s\geq 1}} (-1)^{r+st} m_{r+1+t} \circ (\mathrm{id}^{\otimes r} \otimes m_s \otimes \mathrm{id}^{\otimes t}) \quad = \quad 0.$$

An A_∞-*algebra* is a pair consisting of a graded k-module together with an A_∞-algebra structure $(m_n)_{n\in\mathbb{N}}$ on it.

The relation for $n = 1$ says precisely that the degree minus one map $m_1 \colon A \longrightarrow A$ is a differential. Thus, by neglect of structure every A_∞-algebra has an underlying chain complex (A, m_1). Moreover, we have the following special cases of A_∞-algebras showing that there are many well-known examples.

Example 1.2. Let $A \in \mathsf{gr}(\mathrm{k})$ be a graded k-module.
i) An A_∞-algebra structure on A such that all m_n vanish except possibly the map m_2 amounts to the same as an (graded, non-unital) algebra structure on A.
ii) An A_∞-algebra structure on A such that all m_n vanish except possibly the map m_1 is just a differential on A turning it into a chain complex.
iii) An A_∞-algebra structure on A such that only possibly m_1 and m_2 are non-trivial is the same as the structure of a (non-unital) dga on A.

Definition 1.3. A *morphism* $f\colon A \longrightarrow B$ *of* A_∞*-algebras* consists of a family of morphisms $f_n\colon A^{\otimes n} \longrightarrow B,\ n \geq 1$, of degree $n-1$. These maps have to satisfy for each $n \geq 1$ the following relation

$$\sum_{\substack{r+s+t=n\\ r,t\geq 0,\ s\geq 1}} (-1)^{r+st} f_{r+1+t} \circ (\mathrm{id}^{\otimes r} \otimes m_s \otimes \mathrm{id}^{\otimes t}) \quad = \quad \sum_{\substack{i_1+\ldots+i_q=n\\ q\geq 1,\ i_j\geq 1}} (-1)^{\epsilon} m_q \circ (f_{i_1} \otimes \ldots f_{i_q})$$

where the exponent ϵ is given by $\epsilon = (q-1)(i_1-1)+(q-2)(i_2-1)+\ldots+(i_{q-1}-1)$. A morphism is *strict* if all components $f_n,\ n \geq 2$, vanish.

One can check that this defines a category $A_\infty - \mathsf{Alg}$, *the category of* A_∞*-algebras*. Of course there is a ground ring in the background but we preferred to not indicate it in the notation for this category.

If one spells out the defining relation for a morphism in the case of $n=1$ one notes that it is equivalent to the fact that the degree zero map $f_1\colon A \longrightarrow B$ is a chain map with respect to the differentials $d = m_1$. Hence, we have a forgetful functor $A_\infty - \mathsf{Alg} \longrightarrow \mathsf{ch}(\mathrm{k})$ which is defined on objects resp. morphisms by

$$(A, \{m_n\}_n) \longmapsto (A, m_1) \qquad \text{resp.} \qquad f = \{f_n\}_n \longmapsto f_1.$$

Definition 1.4. A morphism $f\colon A \longrightarrow B$ of A_∞-algebras is called a *quasi-isomorphism* if the underlying map $f_1\colon (A, m_1) \longrightarrow (B, m_1)$ is a quasi-isomorphism of chain complexes.

More generally, there is also the notion of an A_n*-algebra* which are truncated variants of A_∞-algebras. Such an A_n-algebra consists of a $\mathbb{Z}$-graded module endowed with maps $m_l,\ 1 \leq l \leq n$, as above which have to satisfy the same equations as in the case of A_∞-algebras. Similarly, a morphism of A_n-algebras is just a truncated version of a morphism of A_∞-algebras. If we denote the resulting category of A_n-algebras by $A_n - \mathsf{Alg}$ then we obtain a chain of forgetful functors:

$$\ldots \longrightarrow A_n - \mathsf{Alg} \longrightarrow A_{n-1} - \mathsf{Alg} \longrightarrow \ldots \longrightarrow A_2 - \mathsf{Alg} \longrightarrow \mathsf{ch}(\mathrm{k}) \longrightarrow \mathsf{gr}(\mathrm{k})$$

The limit of this diagram gives us the category $A_\infty - \mathsf{Alg}$ of A_∞-algebras.

Remark 1.5. • There is the so-called *bar construction* which allows us to pass from this 'explicit form' to a more 'implicit form' of A_∞-algebras. The multiplication maps $m_n,\ n \geq 1$, of an A_∞-algebra A can also be encoded as a coderivation on a certain conilpotent tensor coalgebra ([LH]). The pair consisting of that tensor coalgebra together with the coderivation is called the bar construction BA of A. It then turns out that a morphism $A_1 \longrightarrow A_2$ of A_∞-algebras as introduced above corresponds precisely to a morphism $BA_1 \longrightarrow BA_2$ of coalgebras which preserves the differentials. In particular, it follows that we have a category $A_\infty - \mathsf{Alg}$.
• The identifications of the previous point are not canonical but depend on some choices. This implies that the signs occurring in the above formulas are not universal so that there are different conventions for the explicit form. However, the choice whether one works homologically or cohomologically does not affect these signs.
• Using the language of operads, A_∞-algebras are precisely the algebras over a certain

A_∞-operad. One has to be a bit careful with the morphisms of A_∞-algebras. The *operadic morphisms* correspond to the strict morphisms as introduced above. However, there is also the notion of ∞-*morphisms* and that class of ∞-morphisms coincides with the class of morphisms given in Definition 1.3 (cf. [LV, Chapter 9 and 10]).

From the above example we know that a dga A can be extended into an A_∞-algebra in a trivial way and a similar remark applies to morphisms. Thus, if we denote by dga the category of differential-graded algebras over k, we deduce that there is a functor $\mathsf{dga} \longrightarrow A_\infty - \mathsf{Alg}$. This functor is faithful but not full.

In general it is not possible to reconstruct a differential-graded algebra A from its homology algebra $H_\bullet A$. However, the following remarkable theorem due to Kadeishvili ([Kad82]) guarantees that over a field k the quasi-isomorphism type of A is determined by $H_\bullet A$ once one passes from the world of differential-graded algebras to the world of A_∞-algebras. There are more general versions of the theorem (cf. [Kel01] for a more general form and additional references) but we state it in a version which is sufficient for our purposes. Let us recall that an A_∞-algebra is called *minimal* if the differential vanishes, i.e., if we have $m_1 = 0$.

Theorem 1.6. *Let A be a differential-graded algebra over a field k. The homology algebra $H_\bullet A$ can be endowed with the structure of a minimal A_∞-algebra and a quasi-isomorphism $f\colon H_\bullet A \longrightarrow A$ of A_∞-algebras such that f induces the identity on homology.*

We sketch a proof of this theorem since we need some details about both the A_∞-structure on $H_\bullet A$ and the quasi-isomorphism f in the next section. The basic strategy is to climb up inductively the limit diagram for $A_\infty - \mathsf{Alg}$. By Example 1.2, A can be endowed with the structure of an A_∞-algebra such that $m_n = 0$, $n \geq 3$, while m_1 is the differential d and m_2 the multiplication map. Let us now construct inductively both the A_∞-algebra structure on $H_\bullet A$ and the quasi-isomorphism f of A_∞-algebras.

We begin in degree $n = 1$. Since the claim is that there is a minimal such structure on $H_\bullet A$ we have to take $m_1 = 0$. Moreover, $f_1\colon H_\bullet A \longrightarrow A$ is to be a chain map inducing the identity on homology. Thus, we let f_1 be a map of the form

$$f_1\colon H_\bullet A \longrightarrow Z_\bullet A \longrightarrow A.$$

Here, the first map is a *chosen* k-linear section of the projection $Z_\bullet A \xrightarrow{p} H_\bullet A$ which exists since we are working over a field. In the case of a unital differential-graded algebra we note that f_1 can be chosen unital.

In degree $n = 2$ we have again no choice concerning the definition of m_2 since we want to extend the usual homology algebra to an A_∞-algebra. Thus, m_2 is the induced multiplication on $H_\bullet A$. If one now spells out the defining relation of a morphism of A_∞-algebras in degree 2 then one obtains the relation

$$f_1 \circ m_2 \quad = \quad m_1 \circ f_2 + m_2 \circ (f_1 \otimes f_1).$$

Since m_2 on $H_\bullet A$ is the induced multiplication we can find such a map $f_2\colon H_\bullet A \otimes H_\bullet A \longrightarrow A$ of degree 1. Said differently, the map f_2 is a specified homotopy showing that f_1 is compatible with the products up to homotopy:

$$f_2\colon m_2 \circ (f_1 \otimes f_1) \longrightarrow f_1 \circ m_2$$

In case of a unital differential-graded algebra this map can be normalized in the sense that it vanishes as soon as one argument equals 1.

Let us now assume inductively that we have constructed the structure of an A_{n-1}-algebra on $H_\bullet A$ and a quasi-isomorphism $f\colon H_\bullet A \longrightarrow A$ of A_{n-1}-algebras satisfying the conclusion of the theorem. Moreover, let us rewrite the condition on a hypothetical extension to a morphism of A_n-algebras given by two maps f_n and m_n. The left-hand-side of the defining relation of an A_n-morphism reduces in our situation to

$$f_1 \circ m_n + \sum_{s=2}^{n-1} \sum_{r+t=n-s} (-1)^{r+st} f_{r+1+t} \circ (\mathrm{id}^{\otimes r} \otimes m_s \otimes \mathrm{id}^{\otimes t}).$$

The right-hand-side of that equation largely simplifies since in our situation only summands for $q \leq 2$ contribute to that sum. Thus, that side of the equation reads as

$$m_1 \circ f_n + \sum_{l=1}^{n-1} (-1)^{l-1} m_2 \circ (f_l \otimes f_{n-l}).$$

The equation is hence equivalent to the fact that $f_1 \circ m_n - m_1 \circ f_n$ equals

$$\sum_{l=1}^{n-1} (-1)^{l-1} m_2 \circ (f_l \otimes f_{n-l}) - \sum_{s=2}^{n-1} \sum_{r+t=n-s} (-1)^{r+st} f_{r+1+t} \circ (\mathrm{id}^{\otimes r} \otimes m_s \otimes \mathrm{id}^{\otimes t}).$$

If we define $\psi_n\colon (H_\bullet A)^{\otimes n} \longrightarrow A$ to be the degree $n-2$ map given by these two sums then one can check that it takes values in the cycles of A. We can thus make the following definition of m_n :

$$m_n\colon (H_\bullet A)^{\otimes n} \xrightarrow{\psi_n} Z_\bullet A \xrightarrow{p} H_\bullet A$$

Since ψ_n and $f_1 \circ m_n$ have pointwise homologous values, we can find a degree $n-1$ map $f_n\colon (H_\bullet A)^{\otimes n} \longrightarrow A$ such that $m_1 \circ f_n = f_1 \circ m_n - \psi_n$, i.e., f_n is a specified (higher) homotopy:

$$f_n\colon \psi_n \longrightarrow f_1 \circ m_n$$

This finishes the inductive construction and hence completes the proof of the theorem (modulo the claim about ψ_n).

As we will need them later on, let us make explicit the formulas for $n = 3$. The degree 1 map ψ_3 is given by $\psi_3 = m_2 \circ (f_1 \otimes f_2) - f_2 \circ (m_2 \otimes \mathrm{id}) + f_2 \circ (\mathrm{id} \otimes m_2) - m_2 \circ (f_2 \otimes f_1)$. Note that the evaluation of such an expression on specific classes results possibly in signs since one uses the symmetry constraint of the monoidal structure. In our situation, given three homogeneous classes $x,\ y,\ z \in H_\bullet A$ we get:

$$\psi_3(x, y, z) = (-1)^{|x|} f_1(x) f_2(y, z) - f_2(xy, z) + f_2(x, yz) - f_2(x, y) f_1(z)$$

Up to a sign the first higher multiplication m_3 gives us a representing Hochschild cocycle for the canonical class γ_A associated to a dga (cf. [BKS04, Corollary 5.7]).

We can give a different description of the role of ψ_3. Recall that the map f_2 shows that f_1 preserves products up to homotopy. The map ψ_3 compares the two resulting homotopies expressing that f_1 preserves threefold products up to homotopy which we obtain by iterating f_2. In fact, ψ_3 is the boundary of the following diagram:

$$\begin{tikzcd} f_1(-)f_1(-)f_1(-) \arrow[r, "f_1(-)f_2(-,-)"] \arrow[d, "f_2(-,-)f_1(-)"'] & f_1(-)f_1(-\cdot -) \arrow[d, "f_2(-,-\cdot-)"] \\ f_1(-\cdot -)f_1(-) \arrow[r, "f_2(-\cdot-,-)"'] & f_1(-\cdot - \cdot -) \end{tikzcd}$$

1.2. **Renaudin's result and the derivator associated to a dga.** In the previous subsection we saw that the minimal model theorem of Kadeishvili gives us a way of encoding the homotopy type of a differential-graded algebra over a field. There is an alternative way of doing so which we will recall in this section. In fact, this works quite generally. Given a combinatorial model category $\mathcal{M}$, let us recall from [Gro10a] that the 2-functor

$$\mathbb{D}_{\mathcal{M}}\colon \mathsf{Cat}^{op} \longrightarrow \mathsf{CAT}\colon \quad J \longmapsto \mathsf{Ho}(\mathsf{Fun}(J,\mathcal{M}))$$

defines a derivator. Moreover, given a Quillen adjunction $(F,U)\colon \mathcal{M} \rightharpoonup \mathcal{N}$ between combinatorial model categories we obtain a derived adjunction between the associated derivators:

$$(\mathbb{L}F, \mathbb{R}U)\colon \mathbb{D}_{\mathcal{M}} \rightharpoonup \mathbb{D}_{\mathcal{N}}$$

One now could wonder if there is an essential difference between the theory of combinatorial model categories and the theory of associated derivators. For this purpose, let us follow Renaudin [Ren09] and denote by ModQ^c the 2-category of combinatorial model categories. A morphism $F\colon \mathcal{M} \longrightarrow \mathcal{N}$ is given by a Quillen adjunction $(F,U)\colon \mathcal{M} \rightharpoonup \mathcal{N}$ while a 2-cell $\alpha\colon (F,U) \longrightarrow (F',U')$ is a natural transformation $\alpha\colon F \longrightarrow F'$ of the left adjoints. Similarly, we also have the 2-category $\mathsf{Der}^{\mathsf{ad}}$ of derivators, adjunctions, and natural transformations between the left adjoints. Using these notations the above assignment can be summarized by saying that we have a pseudo-functor $\mathbb{D}_{(-)}$:

$$\mathbb{D}_{(-)}\colon \mathsf{ModQ}^c \longrightarrow \mathsf{Der}^{\mathsf{ad}}\colon \quad \mathcal{M} \longmapsto \mathbb{D}_{\mathcal{M}}$$

Since $\mathbb{D}_{(-)}$ sends Quillen equivalences to equivalences between associated derivators, $\mathbb{D}_{(-)}$ would factor over a (pseudo-)localization of ModQ^c at the class W of Quillen equivalences. Renaudin shows that such a (pseudo-)localization $\mathsf{ModQ}^c[W^{-1}]$ in fact exists. He describes a model for this which is basically obtained by inverting the Quillen homotopies in all morphism categories. Now, using this localization we obtain an induced pseudo-functor which we still denote by $\mathbb{D}_{(-)}$:

$$\mathbb{D}_{(-)}\colon \mathsf{ModQ}^c[W^{-1}] \longrightarrow \mathsf{Der}^{\mathsf{ad}}$$

If we want this pseudo-functor to have a chance to be a bi-equivalence we should try to restrict it to its essential image. In order to obtain an intrinsic description of this image let us recall the result of Dugger ([Dug01a]) stating that every 'combinatorial model category has a presentation'. By this he means that up to Quillen equivalence such a model category can be written as a left Bousfield localization of the projective model structure on a certain simplicial presheaf category. A left Bousfield localization $\mathcal{M} \rightharpoonup \mathcal{M}[S^{-1}]$ descends to a reflective localization at the level of homotopy categories, i.e., we have a derived adjunction with a fully-faithful right adjoint:

$$\mathsf{Ho}(\mathcal{M}) \rightharpoonup \mathsf{Ho}(\mathcal{M}[S^{-1}])$$

This notion of a reflective localization can be extended to derivators by asking for an adjunction in Der such that the adjunction counit is an isomorphism. Now, motivated by this result of Dugger, Renaudin turns this theorem into a definition for a *derivator of small presentation*. Before we can give this definition, let us introduce a final bit of notation. Given a small category K let us denote by $U(K)$ the category of simplicial presheaves on K endowed with the projective model structure. This model category can be thought of as the universal homotopy theory generated by K (cf. [Dug01b]). With this notation, a derivator is called to be of *small presentation* if it can be written as a nice reflective localization of the derivator $\mathbb{D}_{U(K)}$ for some small category K. Let us denote by $\mathsf{Der}^{\mathsf{ad,p}}$ the full 2-subcategory of $\mathsf{Der}^{\mathsf{ad}}$ spanned by the derivators of small presentation. From the result of Dugger it follows that the pseudo-functor $\mathbb{D}_{(-)}$ has its image in this 2-category. There is the following stronger result which is due to Renaudin ([Ren09]).

Theorem 1.7. *The pseudo-functor* $\mathbb{D}_{(-)} \colon \mathsf{ModQ}^c[W^{-1}] \longrightarrow \mathsf{Der}^{\mathsf{ad}}$ *induces a bi-equivalence of* 2*-categories:*

$$\mathbb{D}_{(-)} \colon \mathsf{ModQ}^c[W^{-1}] \xrightarrow{\simeq} \mathsf{Der}^{\mathsf{ad,p}}$$

Thus, this theorem guarantees that a combinatorial model category $\mathcal{M}$ can be non-canonically reconstructed from the associated derivator $\mathbb{D}_{\mathcal{M}}$. The proof of this theorem is not really constructive. It is based on the proof of the result of Dugger that combinatorial model structures admit presentations and that proof in turn is not constructive. The presentation which is guaranteed by that theorem cannot be given very explicitly.

Now, let us again consider a dga A and let $\mathsf{Mod}-A$ be the associated category of (right) differential-graded A-modules. This category can be endowed with the so-called projective model structure and we will denote by $\mathbb{D}_A$ the derivator associated to this combinatorial model category. The result of Renaudin, interpreted in this situation, motivates that the homotopy type of the dga A should be encoded by the associated derivator $\mathbb{D}_A$. Recall from [DS07a] that the derived category $D(A) = \mathbb{D}_A(e)$ alone does not suffice to encode the homotopy type of A. In fact, building on a result of Schlichting [Sch02], Dugger and Shipley construct two differential-graded algebras which are not quasi-isomorphic but still have the property that their derived categories are equivalent as triangulated categories. Let us mention also that this is a phenomenon which cannot occur for rings. It is shown in [DS04] that two rings which have triangulated equivalent derived categories have already Quillen equivalent model categories of chain complexes. Thus a dga has more 'higher homotopical information' then only its derived category and the result of Renaudin motivates that this is encoded by the associated derivator.

We now have two abstract ways of encoding the homotopy type of a dga A, namely by forming the associated derivator $\mathbb{D}_A$ and by endowing the homology $H_\bullet A$ with the minimal model of Kadeishvili. So it should in principle be possible to construct the derivator out of the minimal model and vice-versa. In the next section, we give a first step into this direction. We show that the first higher multiplication map m_3 on $H_\bullet A$ can be used to construct an extension giving rise to a certain subcategory of $\mathbb{D}_A([1])$. It is likely that this can be modified to an extension which gives us back the entire category $\mathbb{D}_A([1])$.

2. A partial reconstruction of $\mathbb{D}_A([1])$ using m_3

2.1. **The projective model structure on $A - \mathsf{Mod}^{[n]}$.** In this subsection we quickly recall some facts about the projective model structures on $\mathsf{ch}(\mathrm{k})$, $A - \mathsf{Mod}$ for a dga A, and the diagram categories mentioned in the title. We will use this to fix some notation but also to prepare the ground for the next subsection.

Let k be a commutative ring and let $\mathsf{ch}(\mathrm{k})$ be the category of unbounded chain complexes over k. This category can be endowed with the projective model structure, where the weak equivalences are the quasi-isomorphisms and the fibrations are the epimorphisms. The cofibrations are characterized by the left lifting property with respect to the acyclic fibrations, i.e., to the surjective quasi-isomorphisms. However, they can be described more explicitly. The cofibrations are the degreewise split monomorphisms with cofibrant cokernel.

In order to obtain an explicit description of the cofibrant objects it is convenient to recall that the model structure is cofibrantly-generated. Before we can specify the generating (acyclic) cofibrations we have to fix some notation. Let $\mathbb{S}^n$ be the chain complex concentrated in degree n, where it takes the value k and let $\mathbb{D}^n$ be the chain complex concentrated in degrees n and $n-1$ such that the only non-zero differential is the map $\mathrm{id}_k\colon k \longrightarrow k$. As a motivation for the notation let us remark that we have the following natural isomorphisms:

$$\mathbb{S}^1 \otimes - \cong \Sigma\colon \mathsf{ch}(\mathrm{k}) \longrightarrow \mathsf{ch}(\mathrm{k}) \qquad \text{and} \qquad \mathbb{D}^1 \otimes - \cong \mathsf{C}\colon \mathsf{ch}(\mathrm{k}) \longrightarrow \mathsf{ch}(\mathrm{k})$$

Moreover, we have $\mathbb{D}^n \cong \mathsf{C}(\mathbb{S}^{n-1})$ where C denotes the cone functor. The object $\mathbb{S}^n$ corepresents the nth cycle functor and $\mathbb{D}^n$ corepresents the nth chain functor, i.e., we have natural isomorphisms

$$\mathsf{hom}_{\mathsf{ch}(\mathrm{k})}(\mathbb{S}^n, X) \cong Z_n X \qquad \text{and} \qquad \mathsf{hom}_{\mathsf{ch}(\mathrm{k})}(\mathbb{D}^n, X) \cong X_n.$$

The differentials in degree n define a natural transformation $X_n \longrightarrow Z_{n-1}X$ which, by the Yoneda lemma, induces a map $i_n\colon \mathbb{S}^{n-1} \longrightarrow \mathbb{D}^n$ of corepresenting objects. Of course, this map can be described directly as the map where the unique non-zero component is the identity. The set of generating cofibrations I and the set of generating acyclic cofibrations J are given by

$$I = \{i_n\colon \mathbb{S}^{n-1} \longrightarrow \mathbb{D}^n \mid n \in \mathbb{Z}\} \qquad \text{resp.} \qquad J = \{j_n\colon 0 \longrightarrow \mathbb{D}^n \mid n \in \mathbb{Z}\}.$$

From the theory of cofibrantly-generated model categories it follows that an arbitrary cofibration is a retract of a –possibly infinite– composition of pushouts of coproducts of maps in I. It turns out that a chain complex X is cofibrant if and only if it admits an exhaustive filtration

$$0 = X_0 \subseteq X_1 \subseteq \ldots \subseteq X_n \subseteq \ldots\ , \qquad \bigcup X_n = X,$$

such that the filtration quotients X_n/X_{n-1} consist of projective modules and have zero differentials.

The projective model structure on $A - \mathsf{Mod}$ is created by the adjunction

$$(- \otimes A, U)\colon \mathsf{ch}(\mathrm{k}) \longrightarrow A - \mathsf{Mod}$$

in the sense that a morphism f in $A - \mathsf{Mod}$ is a weak equivalence resp. a fibration if and only if Uf is a weak equivalence resp. a fibration in $\mathsf{ch}(\mathrm{k})$. Thus, a weak equivalence resp. a fibration is again just a quasi-isomorphism resp. an epimorphism. The model structure is cofibrantly-generated and the generating (acyclic) cofibrations are obtained by applying the left adjoint $- \otimes A$ to the generating (acyclic) cofibrations of $\mathsf{ch}(\mathrm{k})$. In this model structure all A-modules are fibrant and the cofibrant ones can be characterized as the modules admitting a similar filtration as above. In this case, the condition on the exhaustive filtration is that the filtration quotients are retracts of coproducts of shifted free A-modules.

Let us now give an explicit description of the left homotopy relation in $\mathsf{ch}(\mathrm{k})$ and $A - \mathsf{Mod}$. This will be used afterwards in order to obtain a similar description for the diagram categories $A - \mathsf{Mod}^{[n]}$. Recall from classical homological algebra that given a chain complex X we have the cylinder $\mathsf{cyl}(X) \in \mathsf{ch}(\mathrm{k})$ on X. As graded module, the cylinder is given by $\mathsf{cyl}(X)_n = X_n \oplus X_{n-1} \oplus X_n$. With respect to this decomposition the differential d is given by the matrix:

$$\begin{pmatrix} d & -\mathrm{id} & \\ & -d & \\ & \mathrm{id} & d \end{pmatrix}$$

The natural chain map $\iota\colon X \oplus X \longrightarrow \mathsf{cyl}(X)\colon (x_1, x_2) \longmapsto (x_1, 0, x_2)$ yields a short exact sequence:

$$0 \longrightarrow X \oplus X \xrightarrow{\iota} \mathsf{cyl}(X) \longrightarrow \Sigma X \longrightarrow 0$$

If X is cofibrant then this is also the case for ΣX and the map i is hence a cofibration. Moreover, we have the chain map $t\colon \mathsf{cyl}(X) \longrightarrow X\colon (x_1, s, x_2) \longmapsto x_1 + x_2$ which is easily checked to be an acyclic fibration. Thus, for a cofibrant X we have a cylinder object in the sense of homotopical algebra:

$$\nabla\colon\ X \oplus X \xrightarrow{\iota} \mathsf{cyl}(X) \xrightarrow{t} X$$

The cylinder object can also be constructed using the chain complex $\mathbb{D}^1_+$. By definition, this chain complex is non-trivial only in degrees 1 resp. 0 where it is free on one generator s resp. on two generators e_0, e_1. The only non-trivial differential sends s to $e_1 - e_0$. One now checks easily that there is a natural isomorphism

$$\mathbb{D}^1_+ \otimes - \cong \mathsf{cyl}\colon\ \mathsf{ch}(\mathrm{k}) \longrightarrow \mathsf{ch}(\mathrm{k}),$$

which motivates the notation for $\mathbb{D}^1_+$.

Using this cylinder object, it is easy to check that the left homotopy relation is the same as the usual chain homotopy relation. In fact, given two chain maps $f,\ g\colon X \longrightarrow Y$ and a degree zero map $H\colon \Sigma X \longrightarrow Y$ we can construct a graded map $H'\colon \mathsf{cyl}(X) \longrightarrow Y$ by the assignments

$$e_0 \otimes x \longmapsto f(x), \qquad e_1 \otimes x \longmapsto g(x), \qquad \text{and} \qquad s \otimes x \longmapsto H(x).$$

Then it is immediate that H' is a chain map if and only if H is a homotopy $H\colon f \simeq g$, i.e., a degree one map such that $d(H) = g - f$. From now on, we will no longer notationally distinguish between H and H'.

Remark 2.1. Dually, one can also construct a path object $\mathsf{path}(Y)$ for an arbitrary chain complex Y which in degree n is given by $\mathsf{path}(Y)_n = Y_n \oplus Y_{n+1} \oplus Y_n$. With respect to this direct sum decomposition the differential is obtained by transposing the above matrix. For an arbitrary $Y \in \mathsf{ch}(\mathrm{k})$ the maps $y \longmapsto (y, 0, y)$ and $(y_1, s, y_2) \longmapsto (y_1, y_2)$ define a path object in the sense of homological algebra:

$$\Delta\colon\ Y \longrightarrow \mathsf{path}(Y) \longrightarrow Y \oplus Y$$

The right homotopy relation with respect to this path object is also easily seen to coincide with the chain homotopy relation. Thus, under no additional (co)fibrancy assumptions the chain homotopy relation and the right homotopy relation are equal.

These constructions are also available for the model category $A - \mathsf{Mod}$. Since we are working with *right* A-modules the category $A - \mathsf{Mod}$ is *left*-tensored over $\mathsf{ch}(\mathrm{k})$. In particular, the shift functor Σ and the cylinder functor cyl on $A - \mathsf{Mod}$ are again just given by $\Sigma \cong \mathbb{S}^1 \otimes -$ resp. $\mathsf{cyl} \cong \mathbb{D}^1_+ \otimes -$. As we are only using the associativity constraints of the monoidal structure there are no signs involved here if one writes down the A-action in terms of elements (recall that the signs are caused by the symmetry constraint which would show up in the case of left modules). Thus, by precisely the same formulas as in the case of $\mathsf{ch}(\mathrm{k})$ we see that for maps in $A - \mathsf{Mod}$ with a cofibrant domain the chain homotopy relation and the left homotopy relation coincide.

Before we continue with the diagram categories let us include a technical result on the mapping cylinder and the mapping cone constructions which will be used in the next subsection. Let us recall that for a map $u\colon X_0 \longrightarrow X_1$ in $A - \mathsf{Mod}$ the *mapping cylinder* $\mathsf{cyl}(u) \in A - \mathsf{Mod}$ is defined by the following pushout diagram

$$\begin{tikzcd} X_0 \arrow[r, "u"] \arrow[d, "\iota_1"'] & X_1 \arrow[d] \\ \mathsf{cyl}(X_0) \arrow[r] & \mathsf{cyl}(u) \end{tikzcd}$$

where ι_1 picks the generator e_1. The map $\iota_0\colon X_0 \longrightarrow \mathsf{cyl}(X_0)$ selecting the generator e_0 induces a map $i\colon X_0 \longrightarrow \mathsf{cyl}(u)$. Similarly, the *(mapping) cone* $\mathsf{C}u$ of a morphism u is calculated by the following pushout:

$$\begin{tikzcd} X_0 \arrow[r, "u"] \arrow[d] & X_1 \arrow[d] \\ \mathsf{C}X_0 \arrow[r] & \mathsf{C}u \end{tikzcd}$$

Thus, in degree n we have $\mathsf{C}u_n = (X_1)_n \oplus (X_0)_{n-1}$ and with respect to this direct sum decomposition the differential reads as:

$$\begin{pmatrix} d & u \\ & -d \end{pmatrix}$$

Let us remark that these definitions only use the fact that $A-\mathsf{Mod}$ is tensored over $\mathsf{ch}(\mathrm{k})$.

Lemma 2.2. *Let $u\colon X_0 \longrightarrow X_1$ be a morphism in $A-\mathsf{Mod}$ between cofibrant objects X_0 and X_1. The inclusion $i\colon X_0 \longrightarrow \mathsf{cyl}(u)$ and the map $X_1 \longrightarrow \mathsf{C}u$ are then cofibrations between cofibrant objects.*

Proof. The tensor product $\otimes\colon \mathsf{ch}(\mathrm{k}) \times A-\mathsf{Mod} \longrightarrow A-\mathsf{Mod}$ is a Quillen bifunctor since the monoidal unit $\mathbb{S} \in \mathsf{ch}(\mathrm{k})$ is cofibrant. Let us denote by $\partial\,\mathbb{D}^1_+$ the subchain complex of $\mathbb{D}^1_+$ consisting of the degree 0 part only. It is obvious that the inclusion $\partial\,\mathbb{D}^1_+ \longrightarrow \mathbb{D}^1_+$ is a cofibration in $\mathsf{ch}(\mathrm{k})$. We can thus deduce that for cofibrant $X_0 \in A-\mathsf{Mod}$ the map

$$X_0 \oplus X_0 \cong \partial\,\mathbb{D}^1_+ \otimes X_0 \longrightarrow \mathbb{D}^1_+ \otimes X_0 \cong \mathsf{cyl}(X_0)$$

is a cofibration. Let us now consider the following diagram consisting of two pushout squares:

$$\begin{tikzcd}
X_0 \arrow[r, "u"] \arrow[d, "\iota_1"'] & X_1 \arrow[d, "\iota_1"] \\
X_0 \oplus X_0 \arrow[r, "\mathrm{id}\oplus u"] \arrow[d, tail, "\iota"'] & X_0 \oplus X_1 \arrow[d, tail, "\iota'"] \\
\mathsf{cyl}(X_0) \arrow[r] & \mathsf{cyl}(u)
\end{tikzcd}$$

Since X_1 is a cofibrant object the inclusion $X_0 \xrightarrow{\iota_0} X_0 \oplus X_1$ is a cofibration. It follows that the inclusion of X_0 in $\mathsf{cyl}(u)$ is the composition of two cofibrations $i\colon X_0 \xrightarrow{\iota_0} X_0 \oplus X_1 \xrightarrow{\iota'} \mathsf{cyl}(u)$.

For the second part it suffices to remark that the map $X_0 \longrightarrow \mathsf{C}X_0$ is a cofibration since it is the image of the generating cofibration $\mathbb{S}^0 \longrightarrow \mathbb{D}^1$ under the left Quillen functor $-\otimes X_0\colon \mathsf{ch}(\mathrm{k}) \longrightarrow A-\mathsf{Mod}$. By the stability of cofibrations under pushouts, the map $X_1 \longrightarrow \mathsf{C}u$ is also a cofibration. $\square$

The proof shows that the same result also holds if we replace $A-\mathsf{Mod}$ by an arbitrary $\mathsf{ch}(\mathrm{k})$-model category.

We now turn to the diagram categories $A-\mathsf{Mod}^{[n]}$. The index category $[n]$ is a special case of a directed category. Recall, e.g. from [Hov99], that in the case of a directed category J and an arbitrary model category $\mathcal{M}$ the diagram category $\mathcal{M}^J$ admits the projective model structure. A very convenient fact about this situation is that there is an explicit description of the projective cofibrations using the so-called latching objects. For each object $j \in J$ there is an associated j-th latching object functor $L_j\colon \mathcal{M}^J \longrightarrow \mathcal{M}$ which we describe explicitly in the situation of $J = [n]$. This functor L_j is endowed with a natural transformation $L_j \longrightarrow j^*$ where j^* denotes the evaluation at j. Now, given a morphism $f\colon X \longrightarrow Y$ in $\mathcal{M}^J$ and an object $j \in J$ this natural transformation induces a map in $\mathcal{M}$:

$$L_j(Y) \amalg_{L_j(X)} X_j \longrightarrow Y_j$$

It can be shown that a morphism f in $\mathcal{M}^J$ is an (acyclic) projective cofibration if and only if for all objects $j \in J$ the above induced morphism is an (acyclic) cofibration in $\mathcal{M}$.

In our situation the latching objects are particularly easy to identify. Given an object $i \in [n]$ we form the category $[n]_{(i)}$ which is the full subcategory of the category $[n]_{/i}$ of objects over i spanned by all objects except the terminal object $(i, \mathrm{id}\colon i \longrightarrow i)$. If we denote the canonical projection functor $[n]_{(i)} \longrightarrow [n]$ by pr we can now recall that the i-th latching object functor is given by:

$$L_i\colon A-\mathsf{Mod}^{[n]} \longrightarrow A-\mathsf{Mod}\colon \quad X \longmapsto \operatorname{colim}_{[n]_{(i)}} X \circ pr$$

It is easy to deduce that L_0 is naturally isomorphic to the constant functor on the initial object and that $L_i \cong (i-1)^*,\ i \geq 1$. Moreover, the natural transformation $(i-1)^* \longrightarrow i^*$ is given by the structure morphisms. It follows that a morphism $f\colon X \longrightarrow Y$ is an (acyclic) projective cofibration if and only if the maps

$$X_0 \longrightarrow Y_0 \qquad \text{and} \qquad Y_{i-1} \amalg_{X_{i-1}} X_i \longrightarrow Y_i, \quad i \geq 1,$$

are (acyclic) cofibrations. In particular, an object in $A-\mathsf{Mod}^{[n]}$ is projectively cofibrant if and only if X_0 is cofibrant and all structure maps are cofibrations:

$$X_0 \rightarrowtail X_1 \rightarrowtail \ldots \rightarrowtail X_n$$

Since every object in $A-\mathsf{Mod}^{[n]}$ is fibrant, these are precisely the bifibrant objects.

Let us now give an explicit description of the left homotopy relation in $A-\mathsf{Mod}^{[n]}$ in the case of a cofibrant domain. For the sake of a simpler notation we stick to the case $n = 1$. Later we will make a short remark on which modifications are in order to also cover the case of arbitrary n. So, let us consider a cofibrant object $X \in A-\mathsf{Mod}^{[1]}$, i.e., the value X_0 is cofibrant and the structure map $u\colon X_0 \longrightarrow X_1$ is a cofibration. We then define P to be double mapping cylinder of u, i.e., the following pushout:

$$\begin{tikzcd}
X_0 \oplus X_0 \arrow[r, tail, "u\oplus u"] \arrow[d, tail, "i"'] & X_1 \oplus X_1 \arrow[d, tail] \\
\mathsf{cyl}(X_0) \arrow[r, tail] & P
\end{tikzcd}$$

Thus, in degree n the chain complex P is given by $(X_1)_n \oplus (X_0)_{n-1} \oplus (X_1)_n$ and the differential with respect to this decomposition takes the form:

$$\begin{pmatrix} d & -u & \\ & -d & \\ & u & d \end{pmatrix}$$

The maps $\mathsf{cyl}(X_0) \xrightarrow{t} X_0 \xrightarrow{u} X_1$ and $X_1 \oplus X_1 \xrightarrow{\nabla} X_1$ together induce a unique map $P \longrightarrow X_1$. This map is given by $(x_1, x_0, x_1') \longmapsto x_1 + x_1'$. By the description of the cofibrations and the acyclic fibration, any factorization of this map into a cofibration followed by an acyclic fibration will give us a cylinder object. We can explicitly give a preferred such factorization. The induced map can be written as $P \rightarrowtail \mathsf{cyl}(X_1) \xrightarrow{t} X_1$ where the first map sends (x_1, x_0, x_1') to $(x_1, u(x_0), x_1')$. This first map is a cofibration since it is a coproduct of cofibrations and we know already that the second map is an acyclic fibration. Generalized to an arbitrary n we obtain the following proposition.

Proposition 2.3. *Let* $X = (X_0 \rightarrowtail X_1 \rightarrowtail \ldots \rightarrowtail X_n) \in A-\mathsf{Mod}^{[n]}$ *be a cofibrant object, then a cylinder object on* X *is given by:*

$$\begin{tikzcd}
X_0 \oplus X_0 \arrow[r, tail] \arrow[d, tail, "j"'] & X_1 \oplus X_1 \arrow[r, tail] \arrow[d, tail, "j"'] & \ldots \arrow[r, tail] & X_n \oplus X_n \arrow[d, tail, "j"] \\
\mathsf{cyl}(X_0) \arrow[r, tail] \arrow[d, "t"'] & \mathsf{cyl}(X_1) \arrow[r, tail] \arrow[d, "t"'] & \ldots \arrow[r, tail] & \mathsf{cyl}(X_n) \arrow[d, "t"] \\
X_0 \arrow[r, tail] & X_1 \arrow[r, tail] & \ldots \arrow[r, tail] & X_n
\end{tikzcd}$$

In particular, two morphisms $f,\ g\colon X \longrightarrow Y$ *are left homotopic if and only if there is a compatible family of chain homotopies* $s_i\colon f_i \simeq g_i,\ 0 \leq i \leq n$, *in the sense that the following diagram commutes:*

$$\begin{tikzcd}
\Sigma X_0 \arrow[r, tail] \arrow[d, "s_0"'] & \Sigma X_1 \arrow[r, tail] \arrow[d, "s_1"'] & \ldots \arrow[r, tail] & \Sigma X_n \arrow[d, "s_n"'] \\
Y_0 \arrow[r, tail] & Y_1 \arrow[r, tail] & \ldots \arrow[r, tail] & Y_n
\end{tikzcd}$$

2.2. **Description of $\mathbb{D}_A([n])$ via coherent diagrams.** In this subsection, we want to construct an equivalence between the category $\mathbb{D}_A([n])$ and the homotopy category $\mathsf{Coh}([n], A-\mathsf{Mod})$ of certain coherent diagrams. This alternative description will be used in the next subsection to construct the Hochschild-Mitchell extension from ψ_3. We will first treat the case of $n = 1$ and then discuss the necessary modifications for the general case.

As an intermediate step let us consider the following category $\mathcal{C}([1], A-\mathsf{Mod})$. An object in this category is just a functor $X\colon [1] \longrightarrow A-\mathsf{Mod}$, i.e., amounts to the same as an arrow in $A-\mathsf{Mod}$:

$$X = (X_0 \xrightarrow{u_X} X_1)$$

A morphism $f\colon X \longrightarrow Y$ in $\mathcal{C}([1], A-\mathsf{Mod})$ is a triple $f = (f_0, H_f, f_1)$ consisting of morphisms $f_i\colon X_i \longrightarrow Y_i,\ i = 0,\ 1$, in $A-\mathsf{Mod}$ together with a morphism of *graded* A-modules $H_f\colon \Sigma X_0 \longrightarrow Y_1$ such that $d(H_f) = f_1 u_X - u_Y f_0$. Thus, a morphism $f\colon X \longrightarrow Y$ is pair of morphisms together with a specified homotopy $H_f\colon u_Y f_0 \longrightarrow f_1 u_X$. Such a morphism will be depicted as follows:

$$\begin{tikzcd} X_0 \arrow[r, "u_X"] \arrow[d, "f_0"'] \arrow[dr, "H_f"] & X_1 \arrow[d, "f_1"] \\ Y_0 \arrow[r, "u_Y"'] & Y_1 \end{tikzcd}$$

The composition of two such morphisms $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ is defined by the formula

$$(g_0, H_g, g_1) \circ (f_0, H_f, f_1) = (g_0 f_0, g_1 H_f + H_g f_0, g_1 f_1).$$

Thus, we form the compositions of the two components and of the homotopies:

$$u_Z g_0 f_0 \xrightarrow{H_g f_0} g_1 u_Y f_0 \xrightarrow{g_1 H_f} g_1 f_1 u_X$$

It is easy to check that this defines a category $\mathcal{C}([1], A-\mathsf{Mod})$.

We now introduce a congruence relation on $\mathcal{C}([1], A-\mathsf{Mod})$ which will be called the *homotopy relation.* Given two parallel morphisms $f,\, g\colon X \longrightarrow Y$ in $\mathcal{C}([1], A-\mathsf{Mod})$, a *homotopy* from f to g is a triple (K_0, K, K_1) consisting of homotopies $K_0\colon f_0 \longrightarrow g_0,\ K_1\colon f_1 \longrightarrow g_1$, and a higher homotopy K comparing the two ways getting from $u_Y f_0$ to $g_1 u_X$. Thus, K is a morphism of *graded* A-modules $K\colon \Sigma^2 X_0 \longrightarrow Y_1$ such that

$$d(K) = (K_1 u_X + H_f) - (H_g + u_Y K_0).$$

This equation can be expressed graphically by the following diagram:

$$\begin{tikzcd} u_Y f_0 \arrow[r, "H_f"] \arrow[d, "u_Y K_0"'] \arrow[dr, "K"] & f_1 u_X \arrow[d, "K_1 u_X"] \\ u_Y g_0 \arrow[r, "H_g"'] & g_1 u_X \end{tikzcd}$$

One checks that the homotopy relation $\simeq$ is a congruence relation. We can therefore consider the associated quotient category. As usual the homotopy class of a morphism (f_0, H_f, f_1) will be written as $[f_0, H_f, f_1]$. But, in order to obtain $\mathbb{D}_A([1])$ up to equivalence we have to impose some cofibrancy condition. Let $\mathcal{C}([1], A-\mathsf{Mod}_c)$ be the full subcategory of $\mathcal{C}([1], A-\mathsf{Mod})$ spanned by the diagrams X with cofibrant values.

Definition 2.4. The *homotopy category of coherent diagrams* $\mathsf{Coh}([1], A-\mathsf{Mod})$ is defined to be the following quotient category:

$$\mathsf{Coh}([1], A-\mathsf{Mod}) = \mathcal{C}([1], A-\mathsf{Mod}_c)/_{\simeq}$$

Let us now relate the categories $\mathbb{D}_A([1]) = \mathsf{Ho}(A-\mathsf{Mod}^{[1]})$ and $\mathsf{Coh}([1], A-\mathsf{Mod})$. For this purpose, we take the classical homotopy category as a model for $\mathbb{D}_A([1])$. Thus, the objects of $\mathbb{D}_A([1])$ are given by diagrams $X\colon [1] \longrightarrow A-\mathsf{Mod}$ such that X_0 is cofibrant and the structure map is a cofibration. The morphisms are homotopy classes of morphisms with respect to the homotopy relation described in Proposition 2.3.

We now want to construct a functor $\theta\colon \mathbb{D}_A([1]) \longrightarrow \mathsf{Coh}([1], A-\mathsf{Mod})$. The behavior on objects is easily defined. A cofibration $u\colon X_0 \longrightarrow X_1$ between cofibrant objects is sent to itself considered as a coherent diagram $X_0 \longrightarrow X_1$. Given a second such cofibration $u\colon Y_0 \longrightarrow Y_1$, any morphism $X \longrightarrow Y$ in $\mathbb{D}_A([1])$ can be represented by two morphisms f_0 and f_1 as shown in the commutative square on the left-hand-side. That morphism can also be considered as a morphism in $\mathcal{C}([1], A-\mathsf{Mod}_c)$ by adding the trivial null-homotopy of the zero map as indicated in the coherent square on the right:

$$\begin{tikzcd} X_0 \arrow[r, "u"] \arrow[d, "f_0"'] & X_1 \arrow[d, "f_1"] \\ Y_0 \arrow[r, "u"'] & Y_1 \end{tikzcd} \qquad \begin{tikzcd} X_0 \arrow[r, "u"] \arrow[d, "f_0"'] \arrow[dr, "0"] & X_1 \arrow[d, "f_1"] \\ Y_0 \arrow[r, "u"'] & Y_1 \end{tikzcd}$$

Once we check that the assignment $[f_0, f_1] \longmapsto [f_0, 0, f_1]$ is well-defined, it is obviously also functorial. So, let us consider an equality $[f_0, f_1] = [g_0, g_1]$. By Proposition 2.3 there are homotopies $s_0\colon f_0 \longrightarrow g_0$ and $s_1\colon f_1 \longrightarrow g_1$ which are compatible in the sense that the following diagram commutes:

$$\begin{tikzcd} \Sigma X_0 \arrow[r, "u"] \arrow[d, "s_0"'] & \Sigma X_1 \arrow[d, "s_1"] \\ Y_0 \arrow[r, "u"'] & Y_1 \end{tikzcd}$$

These homotopies can be used to define a homotopy of morphisms of coherent diagrams

$$(s_0, 0, s_1)\colon (f_0, 0, f_1) \longrightarrow (g_0, 0, g_1).$$

In fact, the equation to be checked for the higher homotopy in this case reads as

$$d(0) = (s_1 u + 0) - (0 + u s_0)$$

which is precisely the compatibility assumption imposed on the two homotopies s_0 and s_1. Thus, we have a well-defined functor $\theta\colon \mathbb{D}_A([1]) \longrightarrow \mathsf{Coh}([1], A-\mathsf{Mod})$.

Theorem 2.5. *The functor $\theta\colon \mathbb{D}_A([1]) \longrightarrow \mathsf{Coh}([1], A-\mathsf{Mod})$ is an equivalence of categories.*

Proof. Let us first show that θ is essentially surjective. This is done by a close examination of the mapping cylinder construction. So, let $X \in \mathsf{Coh}([1], A-\mathsf{Mod})$ be an arbitrary object, i.e., X is a morphism $u\colon X_0 \longrightarrow X_1$ between cofibrant A-modules. The mapping cylinder $\mathsf{cyl}(u)$ on u is the A-module which in degree n is given by $\mathsf{cyl}(u)_n = (X_0)_n \oplus (X_0)_{n-1} \oplus (X_1)_n$. With respect to this direct sum decomposition the differential takes the form:

$$\begin{pmatrix} d & -\mathrm{id} & \\ & -d & \\ & u & d \end{pmatrix}$$

Related to the mapping cylinder we have the canonical maps $i\colon X_0 \longrightarrow \mathsf{cyl}(u)$ and $s\colon X_1 \longrightarrow \mathsf{cyl}(u)$ which are induced by the inclusion of the first resp. last summand. We know from Lemma 2.2 that $i\colon X_0 \longrightarrow \mathsf{cyl}(u)$ is a cofibration between cofibrant objects. We will show that i is isomorphic to u in $\mathsf{Coh}([1], A-\mathsf{Mod})$. In addition to i and s, we have a map $p\colon \mathsf{cyl}(u) \longrightarrow X_1$ defined by $(x_0, x_0', x_1) \longmapsto u(x_0) + x_1$. The situation is summarized by the following (non-commutative) diagram:

$$\begin{tikzcd} X_0 \arrow[r, "i"] \arrow[dr, bend right, "u"'] & \mathsf{cyl}(u) \arrow[d, shift right, "p"'] \\ & X_1 \arrow[u, shift right, "s"'] \end{tikzcd}$$

We have $u = p \circ i$ and $\mathrm{id}_{X_1} = p \circ s$. Moreover, the graded map $K_1 \colon \Sigma\, \mathsf{cyl}(u) \longrightarrow \mathsf{cyl}(u)$ sending (x_0, x_0', x_1) to $(0, x_0, 0)$ satisfies $d(K_1)(x_0, x_0', x_1) = (-x_0, -x_0', u(x_0))$. Thus, K_1 defines a homotopy $K_1 \colon \mathrm{id}_{\mathsf{cyl}(u)} \longrightarrow s \circ p$. Similarly, the graded map $H \colon \Sigma X_0 \longrightarrow \mathsf{cyl}(u)$ which in degree n is the inclusion of the second summand $(X_0)_{n-1} \longrightarrow (X_0)_n \oplus (X_0)_{n-1} \oplus (X_1)_n$ is a homotopy $H \colon i \longrightarrow s \circ u$. Thus, we have the following diagram in $\mathsf{Coh}([1], A-\mathsf{Mod})$:

$$\begin{tikzcd} X_0 \arrow[r, "i"] \arrow[d, "\mathrm{id}"'] & \mathsf{cyl}(u) \arrow[d, "p"] \\ X_0 \arrow[r, "u"] \arrow[d, "\mathrm{id}"'] \arrow[dr, "H"] & X_1 \arrow[d, "s"] \\ X_0 \arrow[r, "i"'] \arrow[d, "\mathrm{id}"'] & \mathsf{cyl}(u) \arrow[d, "p"] \\ X_0 \arrow[r, "u"'] & X_1 \end{tikzcd}$$

The composition of the last two morphisms is $[\mathrm{id}, 0, p] \circ [\mathrm{id}, H, s] = [\mathrm{id}, pH, ps] = \mathrm{id}$ while the composition of the first two is given by $[\mathrm{id}, H, s] \circ [\mathrm{id}, 0, p] = [\mathrm{id}, H, sp]$. Using that $K_1 \circ \Sigma i = H$, we see that we have a homotopy $(0, 0, K_1) \colon (\mathrm{id}, 0, \mathrm{id}) \longrightarrow (\mathrm{id}, H, sp)$. Thus, this composition also gives the identity. As a result we have constructed an isomorphism

$$(X_0 \xrightarrow{u} X_1) \quad \cong \quad \theta(X_0 \xrightarrow{i} \mathsf{cyl}(u)) \qquad \text{in} \qquad \mathsf{Coh}([1], A-\mathsf{Mod})$$

showing that θ is essentially surjective.

Let us now show that θ is full. So, let us consider two cofibrations $X_0 \longrightarrow X_1$ and $Y_0 \longrightarrow Y_1$ between cofibrant A-modules and a morphism between them in $\mathsf{Coh}([1], A-\mathsf{Mod})$:

$$\begin{tikzcd} X_0 \arrow[r, "u"] \arrow[d, "f_0"'] \arrow[dr, "H"] & X_1 \arrow[d, "f_1"] \\ Y_0 \arrow[r, "u"'] & Y_1 \end{tikzcd}$$

We can alter f_1 by a homotopy to a map g_1 so that we obtain a commutative diagram. Let us be more specific about this rigidification. The homotopy $H \colon uf_0 \simeq f_1 u$ is equivalently given by the chain map $H\colon \mathsf{cyl}(X_0) \longrightarrow Y_1$ with the expected behavior on the boundary. The commutative diagram

$$\begin{tikzcd} X_0 \arrow[r, "u"] \arrow[d, "\iota_1"'] & X_1 \arrow[d, "f_1"] \\ \mathsf{cyl}(X_0) \arrow[r, "H"'] & Y_1 \end{tikzcd}$$

induces a map $H \sqcup f_1\colon \mathsf{cyl}(u) \longrightarrow Y_1$. Moreover, the map $\mathbb{S} \longrightarrow \mathbb{D}^1_+$ in $\mathsf{ch}(\mathrm{k})$ which classifies e_1 is an acyclic cofibration. We can hence use the fact that $\otimes\colon \mathsf{ch}(\mathrm{k}) \times A{-}\mathsf{Mod} \longrightarrow A{-}\mathsf{Mod}$ is a Quillen bifunctor to deduce that the map $\mathsf{cyl}(u) \longrightarrow \mathsf{cyl}(X_1)$ induced by the universal property of the pushout is an acyclic cofibration. Since every A-module is fibrant, we can thus find an extension K_1 as indicated in the following diagram:

$$\begin{tikzcd} \mathsf{cyl}(u) \arrow[r, "H\sqcup f_1"] \arrow[d, tail, "\simeq"'] & Y_1 \\ \mathsf{cyl}(X_1) \arrow[ur, dashed, "K_1"'] & \end{tikzcd}$$

Let us define $g_1 = K_1 \circ \iota_0$ so that $K_1\colon g_1 \longrightarrow f_1$. By construction, $g_1 u = u f_0$ and K_1 extends H in the sense that $K_1 \circ u = H$. This implies that we have a homotopy

$$(0, 0, K_1)\colon (f_0, 0, g_1) \simeq (f_0, H, f_1).$$

Thus, we obtain $[f_0, H, f_1] = [f_0, 0, g_1] = \theta[f_0, g_1]$ showing that θ is full.

It remains to show that θ is faithful. So, let us consider two morphisms $[f_0, f_1]$ and $[g_0, g_1]$ from $X_0 \rightarrowtail X_1$ to $Y_0 \rightarrowtail Y_1$ such that $\theta[f_0, f_1] = \theta[g_0, g_1]$. But this means that we can find a homotopy $(K_0, K, K_1)\colon (f_0, 0, f_1) \longrightarrow (g_0, 0, g_1)$ which can be depicted as follows:

$$\begin{tikzcd} \Sigma X_0 \arrow[r, "u"] \arrow[d, "K_0"'] \arrow[dr, "K"] & \Sigma X_1 \arrow[d, "K_1"] \\ Y_0 \arrow[r, "u"'] & Y_1 \end{tikzcd}$$

By a similar reasoning as in the previous case K_1 can be altered by a homotopy to a map K_1' such that we obtain $K_1' u = u K_0$. Thus, we have two compatible homotopies which by Proposition 2.3 give us a left homotopy $(K_0, K_1')\colon (f_0, f_1) \longrightarrow (g_0, g_1)$ showing that θ is faithful. □

A similar result can also be obtained for an arbitrary $[n]$. Let $\mathcal{C}([n], A - \mathsf{Mod})$ be the category where the objects are functors $X\colon [n] \longrightarrow A - \mathsf{Mod}$. A morphism between two such diagrams X and Y consists of morphisms $f_i\colon X_i \longrightarrow Y_i,\ 0 \le i \le n$, and homotopies

$H_i\colon u_i f_i \simeq f_{i+1} u_i,\ 0 \le i \le n-1$, as depicted in:

$$\begin{tikzcd}
X_0 \arrow[r, "u_0"] \arrow[d, "f_0"'] \arrow[dr, "H_0"] & X_1 \arrow[r, "u_1"] \arrow[d, "f_1"] \arrow[dr, "H_1"] & X_2 \arrow[r] \arrow[d, "f_2"] & \ldots\ldots \arrow[r] & X_{n-1} \arrow[r, "u_{n-1}"] \arrow[d, "f_{n-1}"'] \arrow[dr, "H_{n-1}"] & X_n \arrow[d, "f_n"] \\
Y_0 \arrow[r, "u_0"'] & Y_1 \arrow[r, "u_1"'] & Y_2 \arrow[r] & \ldots\ldots \arrow[r] & Y_{n-1} \arrow[r, "u_{n-1}"'] & Y_n
\end{tikzcd}$$

Given two such morphisms $(f_0, H_0, f_1, \ldots, f_{n-1}, H_{n-1}, f_n)$ and $(f'_0, H'_0, f'_1, \ldots, f'_{n-1}, H'_{n-1}, f'_n)$, a homotopy between them is a $(2n+1)$-tuple $(K_0, K_{01}, K_1, \ldots, K_{n-1}, K_{n-1,n}, K_n)$ such that for each $0 \le i \le n-1$ we have a homotopy $(K_i, K_{i,i+1}, K_{i+1})\colon (f_i, H_i, f_{i+1}) \simeq (f'_i, H'_i, f'_{i+1})$ as in the arrow case. The definition of $\mathsf{Coh}([n], A-\mathsf{Mod})$ is completely parallel to the previous case. By adding zero homotopies we obtain a functor $\theta\colon \mathbb{D}_A([n]) \longrightarrow \mathsf{Coh}([n], A-\mathsf{Mod})$ which again can be shown to be an equivalence of categories. For example the essential surjectivity is shown by iterating the mapping cylinder construction beginning with u_0. We omit the details.

2.3. **The Hochschild-Mitchell extension induced by m_3.** In this subsection we will use the alternative description of $\mathbb{D}_A([1])$ via coherent diagrams of the last subsection to see that the first higher multiplication map m_3 of the minimal model of A can be used to describe $\mathbb{D}_A([1])$ as a certain Hochschild-Mitchell extension of $\mathbb{D}_A(e)^{[1]}$. We will see in the next section that there is a similar extension for an arbitrary stable derivator which is linear over a field.

Proposition 2.6. *Let A be a differential-graded algebra over a field k. The underlying diagram functor induces a Hochschild-Mitchell extension:*

$$0 \longrightarrow K \longrightarrow \mathsf{Coh}([1], A-\mathsf{Mod}) \xrightarrow{\mathsf{dia}} \mathbb{D}_A(e)^{[1]} \longrightarrow 0$$

Proof. Let $X = (X_0 \xrightarrow{u} X_1)$ be an object of $\mathsf{Coh}([1], A-\mathsf{Mod})$ and let us begin by extending the morphism $f\colon RX \longrightarrow X$ given by

$$\begin{tikzcd}
X_0 \arrow[r, "(\mathrm{id},0)^t"] \arrow[d, "f_0=\mathrm{id}"'] & X_0 \oplus X_1 \arrow[d, "f_1=(u,\mathrm{id})"] \\
X_0 \arrow[r, "u"'] & X_1
\end{tikzcd}$$

to a distinguished triangle. One can think of this map as being a canonical resolution of X by free diagrams. The cone $\mathsf{C}f$ of f is given by $\mathsf{C}(X_0) \xrightarrow{u'} \mathsf{C}f_1$ where the structure map in degree n is given by

$$u'\colon (X_0)_n \oplus (X_0)_{n-1} \longrightarrow (X_1)_n \oplus (X_0)_{n-1} \oplus (X_1)_{n-1}\colon \quad (x_0, x'_0) \longmapsto (ux_0, x'_0, 0).$$

By Lemma 2.2 this is well-defined, i.e., $\mathsf{C}f$ is an object of $\mathsf{Coh}([1], A-\mathsf{Mod})$. We claim that we have an isomorphism $(\mathsf{C}X_0 \longrightarrow \mathsf{C}f_1) \cong (0 \longrightarrow \Sigma X_0)$ in $\mathsf{Coh}([1], A-\mathsf{Mod})$. So let us give the two morphisms showing this:

$$\begin{tikzcd} \mathsf{C}X_0 \arrow[r, "u'"] \arrow[d] \arrow[dr, "s"] & \mathsf{C}f_1 \arrow[d, "p"] \\ 0 \arrow[r] & \Sigma X_0 \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] \arrow[d] & \Sigma X_0 \arrow[d, "i"] \\ \mathsf{C}X_0 \arrow[r, "u'"'] & \mathsf{C}f_1 \end{tikzcd}$$

Here, $i\colon \Sigma X_0 \longrightarrow \mathsf{C}f_1$ sends x_0' to $(0, x_0', -ux_0')$ while $p\colon \mathsf{C}f_1 \longrightarrow \Sigma X_0$ is the projection on the second summand. The composition $p \circ u'$ sends an element (x_0, x_0') to x_0' so that this square does not commute on the nose. But we have a null-homotopy $s\colon 0 \longrightarrow p \circ u'$ given by

$$s\colon \Sigma\mathsf{C}X_0 \longrightarrow \Sigma X_0\colon \quad (x_0, x_0') \longmapsto x_0.$$

The composition $(0 \longrightarrow \Sigma X_0) \longrightarrow (0 \longrightarrow \Sigma X_0)$ is the identity, so let us calculate the other composition. It is given by the left square in the following diagram and we want to construct a homotopy as indicated:

$$\begin{tikzcd} \mathsf{C}X_0 \arrow[r, "u'"] \arrow[d, "0"'] \arrow[dr, "is"] & \mathsf{C}f_1 \arrow[d, "ip"] \\ \mathsf{C}X_0 \arrow[r, "u'"'] & \mathsf{C}f_1 \end{tikzcd} \qquad \simeq \qquad \begin{tikzcd} \mathsf{C}X_0 \arrow[r, "u'"] \arrow[d, "\mathrm{id}"'] & \mathsf{C}f_1 \arrow[d, "\mathrm{id}"] \\ \mathsf{C}X_0 \arrow[r, "u'"'] & \mathsf{C}f_1 \end{tikzcd}$$

The graded map $K_0\colon \Sigma\mathsf{C}X_0 \longrightarrow \mathsf{C}X_0\colon (x_0, x_0') \longmapsto (0, x_0)$ defines a homotopy $K_0\colon 0 \simeq \mathrm{id}$. Similarly, the graded map $K_1\colon \Sigma\mathsf{C}f_1 \longrightarrow \mathsf{C}f_1\colon (x_1, x_0', x_1') \longmapsto (0, 0, x_1)$ defines a homotopy $K_1\colon ip \simeq \mathrm{id}$. These are compatible in the sense that we obtain a homotopy

$$(K_0, 0, K_1)\colon (0, is, ip) \simeq (\mathrm{id}, 0, \mathrm{id}).$$

Thus, we indeed have this alternative description of $\mathsf{C}f$. Under this identification the distinguished triangle associated to f looks as follows:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & X_0 \arrow[r, "\mathrm{id}"] \arrow[d, "(\mathrm{id},0)^t"'] & X_0 \arrow[r] \arrow[d, "u"] \arrow[dr, "\mathrm{id}"] & 0 \arrow[d] \\ X_0 \arrow[r, "(-\mathrm{id},u)^t"'] & X_0 \oplus X_1 \arrow[r, "(u,\mathrm{id})"'] & X_1 \arrow[r, "0"'] & \Sigma X_0 \end{tikzcd}$$

The objects are drawn vertically this time and we rotated the triangle once. Let us consider a further object $Y = (Y_0 \xrightarrow{u} Y_1) \in \mathsf{Coh}([1], A - \mathsf{Mod})$ and let us apply the cohomological functor $\mathsf{hom}_{\mathsf{Coh}([1],A-\mathsf{Mod})}(-, Y)$ to this distinguished triangle. Using the notation $1_!X_0$ for the homotopy left Kan extension $0 \longrightarrow X_0$ of X_0 along $1\colon e \longrightarrow [1]$ the part of interest of the induced long exact sequence reads as:

$$\mathsf{hom}(1_!\Sigma X_0, Y) \longrightarrow \mathsf{hom}(X, Y) \xrightarrow{f^*} \mathsf{hom}(RX, Y) \longrightarrow \mathsf{hom}(1_!X_0, Y)$$

Let us rewrite the last map in this sequence. Since we have $RX = 0_!X_0 \oplus 1_!X_1$ we obtain a natural isomorphism $\mathsf{hom}(RX, Y) \cong \mathsf{hom}_{\mathbb{D}_A(e)}(X_0, Y_0) \oplus \mathsf{hom}_{\mathbb{D}_A(e)}(X_1, Y_1)$. Combined with the isomorphism $\mathsf{hom}(1_!X_0,Y)\cong \mathsf{hom}_{\mathbb{D}_A(e)}(X_0,Y_1)$ the last map becomes identified with

$$\begin{array}{rcl}\mathsf{hom}_{\mathbb{D}_A(e)}(X_0,Y_0)\oplus\mathsf{hom}_{\mathbb{D}_A(e)}(X_1,Y_1) & \longrightarrow & \mathsf{hom}_{\mathbb{D}_A(e)}(X_0,Y_1)\\ (g_0,g_1) & \longmapsto & g_1\circ u-u\circ g_0.\end{array}$$

Thus, the kernel of this last map is given by $\mathsf{hom}_{\mathbb{D}_A(e)^{[1]}}(\mathsf{dia}\,X,\mathsf{dia}\,Y)$. Let us now give a closer description of f^*. Given a morphism $g=[g_0,H_g,g_1]\in\mathsf{hom}(X,Y)$ its value under f^* is:

$$\begin{tikzcd} X_0 \arrow[r, "\mathrm{id}"] \arrow[d, "(\mathrm{id},0)^t"'] & X_0 \arrow[r, "g_0"] \arrow[d, "u"'] \arrow[dr, "H_g"] & Y_0 \arrow[d, "u"] \\ X_0\oplus X_1 \arrow[r, "(u,\mathrm{id})"'] & X_1 \arrow[r, "g_1"'] & Y_1 \end{tikzcd} = \begin{tikzcd} X_0 \arrow[r, "g_0"] \arrow[d, "(\mathrm{id},0)^t"'] \arrow[dr, "H_g"] & Y_0 \arrow[d, "u"] \\ X_0\oplus X_1 \arrow[r, "(g_1u,0)"'] & Y_1 \end{tikzcd} + \begin{tikzcd} X_0 \arrow[r, "0"] \arrow[d, "(\mathrm{id},0)^t"'] & Y_0 \arrow[d, "u"] \\ X_0\oplus X_1 \arrow[r, "(0,g_1)"'] & Y_1 \end{tikzcd}$$

Since we have a homotopy $(0,0,(-H_g,0))\colon (g_0,H_g,(g_1u,0))\simeq(g_0,0,(ug_0,0))$, we deduce that the image of g under f^* is the sum $[g_0,0,(ug_0,0)]+[(0,0,(0,g_1))]$. Now, under the same identification $\mathsf{hom}(RX,Y)\cong\mathsf{hom}_{\mathbb{D}_A(e)}(X_0,Y_0)\oplus\mathsf{hom}_{\mathbb{D}_A(e)}(X_1,Y_1)$ as above, we see that the map f^* becomes:

$$\begin{array}{rcl}\mathsf{hom}_{\mathsf{Coh}([1],A-\mathsf{Mod})}(X,Y) & \stackrel{\mathsf{dia}}{\longrightarrow} & \mathsf{hom}_{\mathbb{D}_A(e)}(X_0,Y_0)\oplus\mathsf{hom}_{\mathbb{D}_A(e)}(X_1,Y_1)\\ g & \longmapsto & (g_0,g_1)\end{array}$$

If we denote the kernel of this map by $K(X,Y)$ we can summarize these two identifications by saying that we have the following short exact sequence:

$$0\longrightarrow K(X,Y)\longrightarrow \mathsf{hom}_{\mathsf{Coh}([1],A-\mathsf{Mod})}(X,Y)\stackrel{\mathsf{dia}}{\longrightarrow}\mathsf{hom}_{\mathbb{D}_A(e)^{[1]}}(\mathsf{dia}\,X,\mathsf{dia}\,Y)\longrightarrow 0$$

Since $\mathbb{D}_A$ is a stable derivator, we knew already that such an epimorphism would exist, but we now have, in addition, a surjection onto the kernel. In fact, the identification $\mathsf{hom}_{\mathbb{D}_A(e)}(\Sigma X_0,Y_1)\cong\mathsf{hom}_{\mathsf{Coh}([1],A-\mathsf{Mod})}(1_!\Sigma X_0,Y)$ together with the first map of the above exact sequence gives us an epimorphism $\mathsf{hom}_{\mathbb{D}_A(e)}(\Sigma X_0,Y_1)\longrightarrow K(X,Y)$. This epimorphism sends a map H to the class $[0,H,0]\colon X\longrightarrow Y$. From this description it is immediate that the extension occurring in the statement is a square zero extension:

$$\begin{tikzcd} X_0 \arrow[r, "0"] \arrow[d, "u"'] \arrow[dr, "H"] & Y_0 \arrow[r, "0"] \arrow[d, "u"] \arrow[dr, "G"] & Z_0 \arrow[d, "u"] \\ X_1 \arrow[r, "0"'] & Y_1 \arrow[r, "0"'] & Z_1 \end{tikzcd} = \begin{tikzcd} X_0 \arrow[r, "0"] \arrow[d, "u"'] \arrow[dr, "0"] & Z_0 \arrow[d, "u"] \\ X_1 \arrow[r, "0"'] & Z_1 \end{tikzcd}$$

Moreover, the sequence $0\longrightarrow K\longrightarrow\mathsf{Coh}([1],A-\mathsf{Mod})\longrightarrow\mathbb{D}_A(e)^{[1]}\longrightarrow 0$ splits k-linearly since we are working over a field. Thus, we have indeed a Hochschild-Mitchell extension. □

The proof showed that all morphisms in $\mathsf{hom}_{\mathsf{Coh}([1],A-\mathsf{Mod})}(X,Y)$ which induce the zero morphism on underlying diagrams come in a certain sense from morphisms $\Sigma X_0\longrightarrow Y_1$. We will see in the next section that there is a similar result in the case of an arbitrary stable derivator.

Recall e.g. from [Wei94, Section 9.3] that the equivalence classes of Hochschild extensions of an algebra A by a bimodule M are in bijection with the second Hochschild cohomology $HH^2(A;M)$. The proof of this result can be adapted to also give a many object version of the aforementioned bijection. Let us quickly recall the definition of Hochschild cohomology for an (essentially) small, k-linear category A with values in an A-bimodule M. In this discussion, k is again allowed to be an arbitrary commutative ground ring and such a bimodule is just a functor $M\colon A^{\mathrm{op}}\otimes A\longrightarrow k-\mathsf{Mod}$. The *Hochschild-Mitchell cochain complex* $C^\bullet(A;M)$ in degree $n\geq 1$ is defined to be

$$C^n(A;M)=\prod_{x_0,\ldots,x_n\in A}\mathsf{hom}_k(A(x_{n-1},x_n)\otimes_k\ldots\otimes_k A(x_0,x_1),M(x_0,x_n))$$

In dimension $n=0$ one sets $C^0(A;M)=\prod_{x_0\in A}\mathsf{hom}_k(k,M(x_0,x_0))\cong\prod_{x_0\in A}M(x_0,x_0)$. The boundary operator $d\colon C^n(A;M)\longrightarrow C^{n+1}(A,M)$ is given by the usual Hochschild formula. Thus, for an element $f=(f^{x_0,\ldots x_n})$ its boundary df is defined by: $(df)(\alpha_{n+1},\ldots,\alpha_1)$ $=\alpha_{n+1}f(\alpha_n,\ldots,\alpha_1)-f(\alpha_{n+1}\alpha_n,\ldots,\alpha_1)+\ldots+(-1)^{n+1}f(\alpha_{n+1},\ldots,\alpha_2)\alpha_1$.
Here, $\alpha_i\in A(x_{i-1},x_i),\ 1\leq i\leq n+1$, and the equation holds in $M(x_0,x_{n+1})$. It is immediate from this description that $HH^0(A)=HH^0(A;A)$ is isomorphic to the center $Z(A)$ of the category A.

Now, the analogue of the above result guarantees that there is a bijection between the second Hochschild-Mitchell cohomology $HH^2(A;M)$ and the set of equivalence classes of Hochschild-Mitchell extensions. Let us recall the construction in one direction. So, let us assume to be given a Hochschild-Mitchell extension $0\longrightarrow M\longrightarrow B\xrightarrow{F}A\longrightarrow 0$. Then the associated cohomology class is represented by the following cocycle $f\in C^2(A;M)$. By definition of such an extension, given two objects $x_0,\ x_1\in ob(A)=ob(B)$ the map $A(x_0,x_1)\longrightarrow B(x_0,x_1)$ induced by F splits over k. So, let us choose such a splitting $\sigma=\sigma^{x_0,x_1}$ for each pair of objects. The cocycle $f=(f^{x_0,x_1,x_2})$ is defined to send two composable morphisms $x_0\xrightarrow{\alpha_1}x_1\xrightarrow{\alpha_2}x_2$ in A to

$$f^{x_0,x_1,x_2}(\alpha_2,\alpha_1)\quad=\quad\sigma^{x_0,x_2}(\alpha_2\alpha_1)-\sigma^{x_1,x_2}(\alpha_2)\sigma^{x_0,x_1}(\alpha_1)\quad\in M(x_0,x_2).$$

Thus, the cocyle measures how far the chosen splittings are from being multiplicative. One checks that this definition is well-defined and establishes the desired bijection.

With this preparation we can now return to our Hochschild-Mitchell extension given by Proposition 2.6. Let us denote by $\mathcal{F}^{[1]}\subseteq\mathbb{D}_A(e)^{[1]}$ resp. by $\mathcal{F}([1])\subseteq\mathsf{Coh}([1],A-\mathsf{Mod})$ the respective full subcategory spanned by morphisms $u\colon X_0\longrightarrow X_1$ such that $X_0,\ X_1$ are graded free of rank one. By restriction we hence obtain a further Hochschild-Mitchell extension

$$0\longrightarrow K\longrightarrow\mathcal{F}([1])\xrightarrow{\mathsf{dia}}\mathcal{F}^{[1]}\longrightarrow 0.$$

Let us denote the associated Hochschild-Mitchell cohomology class by $\gamma_{(A)}\in HH^2(\mathcal{F}^{[1]};K)$.

Let us shortly comment on this Hochschild-Mitchell extension. Strictly speaking, we do not have such an extension because the functor is not the identity on objects. But by restriction we could turn it into a bijection on objects. For this purpose, let us recall that we have a natural isomorphism $\mathsf{hom}_{\mathbb{D}_A(e)}(A,A)\cong H_\bullet A$. Given a homology class x

let us denote the associated morphism again by x. The behavior of the functor dia on objects is now as follows. It sends an object $z\colon A \longrightarrow A$, i.e., a cycle of A, to the morphism $[z]\colon A \longrightarrow A$. We can define a section of this association by sending an object $x\colon A \longrightarrow A$ of $\mathcal{F}^{[1]}$ to the object $f_1(x)\colon A \longrightarrow A$. Here f_1 is a chosen k-linear section of $p\colon Z_\bullet A \longrightarrow H_\bullet A$.

Recall from the inductive construction of the minimal model that the first higher multiplication map m_3 is of the form $(H_\bullet A)^{\otimes 3} \longrightarrow Z_\bullet A \xrightarrow{\psi_3} H_\bullet A$ for a certain map ψ_3.

Proposition 2.7. *Let A be a differential graded algebra over a field k. The Hochschild-Mitchell cohomology class $\gamma_{(A)} \in HH^2(\mathcal{F}^{[1]};K)$ can be constructed from the map ψ_3 representing m_3.*

Proof. The strategy of the proof is to evaluate a certain representing cocycle of the Hochschild-Mitchell class $\gamma_{(A)}$ on special pairs of composable morphisms in $\mathcal{F}^{[1]}$. As a preparation for this, let us begin by choosing k-linear sections of the maps induced by $\mathsf{dia}\colon \mathcal{F}[1] \longrightarrow \mathcal{F}^{[1]}$ on the morphism sets. For two objects $X = (x\colon A \longrightarrow A)$ and $Y = (y\colon A \longrightarrow A)$ of $\mathcal{F}^{[1]}$ an element of $\mathsf{hom}_{\mathcal{F}^{[1]}}(X,Y)$ is given by a pair of homology classes (u,v) such that $uy = xv$. We define our section

$$\mathsf{hom}_{\mathcal{F}^{[1]}}(A \xrightarrow{x} A, A \xrightarrow{y} A) \longrightarrow \mathsf{hom}_{\mathcal{F}([1])}(A \xrightarrow{f_1(x)} A, A \xrightarrow{f_1(y)} A)$$

to send such a pair (u,v) to $[f_1(u), f_2(u,y) - f_2(x,v), f_1(v)]$:

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "u"'] & A \arrow[d, "v"] \\ A \arrow[r, "y"'] & A, \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f_1(x)"] \arrow[d, "f_1(u)"'] \arrow[dr, "H"] & A \arrow[d, "f_1(v)"] \\ A \arrow[r, "f_1(y)"'] & A, \end{tikzcd} \qquad \begin{tikzcd} f_1(u)f_1(y) \arrow[r, "f_2(u,y)"] \arrow[d, "H"'] & f_1(uy) \arrow[d, equal] \\ f_1(x)f_1(v) & f_1(xv) \arrow[l, "-f_2(x,v)"] \end{tikzcd}$$

Let us recall from the proof of Theorem 1.6 that the maps f_1 and f_2 can be suitably normalized. If we are now given three homology classes x, y, $z \in H_\bullet A$ we can construct the pair of composable morphisms in $\mathcal{F}^{[1]} \subseteq \mathbb{D}_A(e)^{[1]}$ given by the left part of the following diagram:

$$\begin{tikzcd} A \arrow[r, "="] \arrow[d, "x"'] & A \arrow[d, "xy"] \\ A \arrow[r, "y"] \arrow[d, "yz"'] & A \arrow[d, "z"] \\ A \arrow[r, "="'] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "1"] \arrow[d, "f_1(x)"'] \arrow[dr, "H_1"] & A \arrow[d, "f_1(xy)"] \\ A \arrow[r, "f_1(y)"] \arrow[d, "f_1(yz)"'] \arrow[dr, "H_2"] & A \arrow[d, "f_1(z)"] \\ A \arrow[r, "1"'] & A \end{tikzcd} \qquad \begin{array}{c} H_1{=}f_2(x,y) \\[4em] H_1{=}{-f_2(y,z)} \end{array}$$

Let us evaluate the class $\gamma_{(A)}$ on this pair of composable morphisms. If we lift the two morphisms separately we obtain the right part of the above diagram. The composition of these two lifts is given by

$$[f_1(x)f_1(yz), -(-1)^{|x|}f_1(x)f_2(y,z) + f_2(x,y)f_1(z), f_1(xy)f_1(z)].$$

The sign $(-1)^{|x|}$ comes from the definition of the graded composition in $\mathsf{Coh}([1], A\,\mathsf{Mod})$. If we instead compose first and then lift the composition we obtain $[f_1(xyz), 0, f_1(xyz)]$. Using the homotopy $(-f_2(x, yz), 0, -f_2(xy, z))$ we obtain that this class can also be written as:

$$[f_1(xyz), 0, f_1(xyz)] = [f_1(x)f_1(yz), f_2(x, yz) - f_2(xy, z), f_1(xy)f_1(z)]$$

It follows that our representative of the Hochschild-Mitchell class $\gamma_{(A)}$ evaluated on the above pair of composable arrows gives:

$$[0, (-1)^{|x|} f_1(x)f_2(y, z) - f_2(xy, z) + f_2(x, yz) - f_2(x, y)f_1(z), 0] = [0, \psi_3(x, y, z), 0]$$

□

3. Perspective

3.1. **Coherent diagrams as a special case of strictly unital A_∞-functors.** Before we give the definition of an A_∞-category we describe the monoidal category which takes the role of the category $\mathsf{gr}(\mathrm{k})$ in this more general situation. Let A_0 be a set and let us consider the category $\mathsf{BiMod}(A_0) = \mathsf{Fun}(A_0^{\mathrm{op}} \times A_0, \mathsf{gr}(\mathrm{k}))$ of functors defined on the discrete category $A_0^{\mathrm{op}} \times A_0$ with values in $\mathsf{gr}(\mathrm{k})$. Given two such $X, Y \in \mathsf{BiMod}(A_0)$ one can define their external tensor product:

$$X \boxtimes Y \colon A_0^{\mathrm{op}} \times A_0 \times A_0^{\mathrm{op}} \times A_0 \xrightarrow{X\times Y} \mathsf{gr}(\mathrm{k}) \times \mathsf{gr}(\mathrm{k}) \xrightarrow{\otimes} \mathsf{gr}(\mathrm{k})$$

The (internal) tensor product $X \otimes_{A_0} Y = X \otimes Y \in \mathsf{BiMod}(A_0)$ is defined by the following coend formula:

$$(a_0, a_2) \longmapsto (X \otimes Y)(a_0, a_2) = \int^{A_0} X(-, a_2) \otimes Y(a_0, -) \in \mathsf{gr}(\mathrm{k})$$

Since A_0 is a discrete category the coend formula reduces to the direct sum

$$(X \otimes Y)(a_0, a_2) = \bigoplus_{a_1 \in A_0} X(a_1, a_2) \otimes Y(a_0, a_1)$$

Let us commit a slight abuse of notation and denote the bimodule $k\,\mathsf{hom}_{A_0}(-,-)[0]$ again simply by $A_0 \in \mathsf{BiMod}(A_0)$. From the above direct sum formula it is immediate that A_0 is unital with respect to $-\otimes-$. We thus have a symmetric monoidal, k-linear, graded category $(\mathsf{BiMod}(A_0), -\otimes-, A_0)$. An A_∞-category is now just an A_∞-monoid in a monoidal category of this form.

Definition 3.1. An A_∞-*category* $\mathcal{A} = (A_0, A)$ consists of a *set of objects* A_0 and a *bimodule of morphisms* $A \in \mathsf{BiMod}(A_0)$ together with maps of bimodules $m_n \colon A^{\otimes n} \longrightarrow A$, $n \geq 1$, of degree $n-2$. These maps have to satisfy for each $n \geq 1$ the relation:

$$\sum_{\substack{r+s+t=n \\ r,t\geq 0,\ s\geq 1}} (-1)^{r+st} m_{r+1+t} \circ (\mathrm{id}^{\otimes r} \otimes m_s \otimes \mathrm{id}^{\otimes t}) \quad = \quad 0$$

Of course one can also give a definition of A_∞-categories which have a class of objects. In that case, however, the category $\mathsf{BiMod}(A_0)$ is no more well-defined and one should instead give a more explicit definition (cf. [Kel06a]). Similarly to the situation of A_∞-algebras we have the following classes of examples.

Example 3.2. i) An A_∞-category with one object $A_0 = *$ is the same as an A_∞-algebra. In fact, in that case already the category $\mathsf{BiMod}(A_0)$ reduces to $\mathsf{gr}(\mathrm{k})$.
ii) Every linear, graded linear, or differential-graded category can be considered as an A_∞-category.
iii) Since the formation of free k-modules is a monoidal functor $k(-)\colon \mathsf{Set} \longrightarrow k-\mathsf{Mod}$, one can associate to every category J the free k-linear category on J which we again denote by J. Using part ii) we can thus consider an ordinary category as an A_∞-category which is then concentrated in degree zero. The only nontrivial structure morphism in this case is m_2 which is given by the linearization of the composition law of J.

Given an A_∞-category $\mathcal{A}$ the relation $m_1^2 = 0$ implies that we can form an associated homology category $H_\bullet\mathcal{A}$ which is a graded linear category. There is also an underlying ungraded variant $H_0\mathcal{A}$.

As a preparation for the definition of an A_∞-functor let us recall the following. Given two sets (of objects) A_0 and B_0 and a map of sets $f_0 \colon A_0 \longrightarrow B_0$ we obtain a restriction of scalars functor

$$(f_0^{\mathrm{op}} \times f_0)^* \colon \mathsf{BiMod}(B_0) \longrightarrow \mathsf{BiMod}(A_0) \colon \quad B \longmapsto B(f_0(-), f_0(-)).$$

Definition 3.3. Let $\mathcal{A} = (A_0, A)$ and $\mathcal{B} = (B_0, B)$ be two A_∞-categories. An A_∞-functor $f \colon \mathcal{A} \longrightarrow \mathcal{B}$ is a pair $f = (f_0, \{f_n\}_{n\geq 1})$ consisting of a map of objects $f_0 \colon A_0 \longrightarrow B_0$ and morphisms of bimodules $f_n \colon A^{\otimes n} \longrightarrow B(f_0(-), f_0(-))$, $n \geq 1$, of degree $n-1$. These maps have to satisfy for each $n \geq 1$ the following relation

$$\sum_{\substack{r+s+t=n \\ r,t\geq 0,\ s\geq 1}} (-1)^{r+st} f_{r+1+t} \circ (\mathrm{id}^{\otimes r} \otimes m_s \otimes \mathrm{id}^{\otimes t}) \quad = \quad \sum_{\substack{i_1+\ldots+i_q=n \\ q\geq 1, i_j\geq 1}} (-1)^{\epsilon} m_q \circ (f_{i_1} \otimes \ldots f_{i_q})$$

where the exponent ϵ is given by $\epsilon = (q-1)(i_1-1) + (q-2)(i_2-1) + \ldots + (i_{q-1}-1)$.

The same comments as in the case of morphisms of A_∞-algebras apply again. In particular, an A_∞-functor is compatible with the differentials and hence induces a functor between the homology categories. In this many objects context one can now reprove some of the result for algebras. For example, there is a many objects version of the minimal model result of Kadeishvili which can be established in a similar way as it was for algebras.

Proposition 3.4. *Let $\mathcal{A}$ be a dg-category over a field k and let $H_\bullet\mathcal{A}$ be the associated homology category. Then, $H_\bullet\mathcal{A}$ can be endowed with the structure of a minimal A_∞-category. Moreover, there is a quasi-isomorphism of A_∞-categories $f \colon H_\bullet\mathcal{A} \longrightarrow \mathcal{A}$ inducing the identity on homology categories.*

Remark 3.5. The proof of the above proposition shows that the minimal model can be chosen to be strictly unital, i.e., the maps f_n, $n \geq 2$, and m_n, $n \geq 3$, vanish as soon as one argument is an identity. Moreover, f_1 preserves identities and m_2 is unital in the usual sense. This can be checked inductively using the explicit formulas for the quasi-isomorphism of A_n-categories assuming that one has already such a morphism of A_{n-1}-categories.

It can be shown that given two A_∞-categories we can then construct an A_∞-category of A_∞-functors between them ([LH, BLM08]). Moreover, there is a similar such A_∞-category consisting of strictly unital A_∞-functors.

Example 3.6. Let us consider the special case of $\mathcal{A} = [1]$ and where $\mathcal{B}$ is the dg-category $A - \mathsf{Mod}_{dg}$. In this case, the A_∞-category $\mathsf{Fun}_{A_\infty}([1], A - \mathsf{Mod}_{dg})$ of strictly unital A_∞-functors admits a simple concrete description. An A_∞-functor $X \colon [1] \longrightarrow A - \mathsf{Mod}_{dg}$ is just a morphism of differential-graded A-modules $u \colon X_0 \longrightarrow X_1$. Given two such A_∞-functors X and Y a strictly unital A_∞-transformation $f \colon X \longrightarrow Y$ consists of three *graded* A-linear maps as indicated in

$$\begin{tikzcd} X_0 \arrow[r, "u"] \arrow[d, "f_0"'] \arrow[dr, "H"] & X_1 \arrow[d, "f_1"] \\ Y_0 \arrow[r, "u"'] & Y_1 \end{tikzcd}$$

without any commutativity assumption. If such an A_∞-transformation is a 0-cocycle then the components f_0 and f_1 are also 0-cocycles, i.e., chain maps. The diagram is then still not necessarily commutative, but H is a homotopy $H\colon u \circ f_0 \longrightarrow f_1 \circ u$. Thus, we have a morphism in $\mathcal{C}([1], A - \mathsf{Mod})$.

We close this subsection by mentioning a further strategy how one might try to recover the minimal model on $H_\bullet A$ from the derivator $\mathbb{D}_A$. Given a dga A we can consider the dg-version $(\mathsf{Mod}-A)_{dg}$ of the category of (right) A-modules. Let us apply the above proposition in order to construct a minimal (strictly unital) A_∞-category $(\mathsf{Mod}-A)_{min}$ associated to this dg-category. If we are now given a small (ordinary) category J, we can consider it by Example 3.2 iii) as an A_∞-category in a trivial way. Let us denote this A_∞-category again simply by J. If we denote the A_∞-categories of strictly unital A_∞-functors by $\mathsf{Fun}_{A_\infty}(-,-)$ then we can consider the following 2-functor:

$$\mathbb{D}'_A\colon\ \mathsf{Cat}^{op} \longrightarrow \mathsf{CAT}\colon \quad J \longmapsto H_0\big(\mathsf{Fun}_{A_\infty}(J, (\mathsf{Mod}-A)_{min})\big)$$

It is likely that there is a strong relation between this 2-functor $\mathbb{D}'_A$ and the derivator $\mathbb{D}_A$ associated to A. Moreover, since the minimal model $(\mathsf{Mod}-A)_{min}$ is used in the construction of $\mathbb{D}'_A$, it might be easier to reconstruct the minimal model on $H_\bullet A$ from this 2-functor.

3.2. **The Hochschild-Mitchell extension for a stable derivator over a field.** In this subsection we want to show that an analogue of the Hochschild-Mitchell extension of Subsection 2.3 is also available if we replace our derivator $\mathbb{D}_A$ by an arbitrary stable derivator $\mathbb{D}$ which is linear over a field k. Thus, we assume our stable derivator $\mathbb{D}$ to be endowed with a ring map $\sigma\colon k \longrightarrow Z(\mathbb{D})$ where $Z(\mathbb{D})$ denotes the center of $\mathbb{D}$ (cf. Definition 1.37 of Part 2). This ring map can be used to endow each value of the derivator with a k-linear structure in a compatible way.

Before we turn to that result let us include an alternative description of the suspension, loop, cone, and fiber functors. This will shed some more light on the fact that there is a difference between zero morphisms in a pointed derivator and morphisms which induce the zero morphism on underlying diagrams. Let us recall Corollary 1.42 of Part 1 which is valid for a pointed derivator. The homotopy left Kan extension functors $u_!\colon \mathbb{D}(J) \longrightarrow \mathbb{D}(K)$ associated to a closed immersion u has itself a left adjoint $u^?\colon \mathbb{D}(K) \longrightarrow \mathbb{D}(J)$. Dually, in the case of an open immersion u the homotopy right Kan extension functor u_* has itself a right adjoint $u^!$. The proof of that result was a constructive one and we gave precise formulas for these adjoints immediately after the corollary. Let us consider them in the special case of the closed immersion $1\colon e \longrightarrow [1]$. The functor $1_!\colon \mathbb{D}(e) \longrightarrow \mathbb{D}([1])$ is an extension by zero functor. A left adjoint to $1_!\colon \mathbb{D}(e) \longrightarrow \mathbb{D}([1])$ is given as follows. The inclusion of the complement of $u = 1\colon e \longrightarrow [1]$ is just $v = 0\colon e \longrightarrow [1]$. In this case the mapping cylinder construction is quite simple and the left adjoint $1^?$ reads as

$$1^?\colon\ \mathbb{D}([1]) \xrightarrow{j_*} \mathbb{D}(\mathrm{i}_\ulcorner, (0,1)) \xrightarrow{p_!} \mathbb{D}([1], 0) \xrightarrow{1^*} \mathbb{D}(e).$$

Here, $j\colon [1] \longrightarrow \mathrm{i}_\ulcorner$ classifies the horizontal morphism while $p\colon \ulcorner \longrightarrow [1]$ is the projection onto the first component. Using Kan's formula one calculates that for $Y \in \mathbb{D}(\ulcorner)$ we have a natural isomorphism $p_!(Y)_1 \cong \mathrm{Hocolim}_\ulcorner Y$. Precomposing this with the extension by zero functor j_* gives $1^?(X) \cong \mathsf{C}X$ for $X \in \mathbb{D}([1])$. Proceeding dually for the functor 0_* we obtain the following result.

Lemma 3.7. *Let $\mathbb{D}$ be a pointed derivator, then we have adjunctions*

$$(\mathsf{C}, 1_!)\colon\ \mathbb{D}([1]) \rightharpoonup \mathbb{D}(e) \qquad \textit{and} \qquad (0_*, \mathsf{F})\colon\ \mathbb{D}(e) \rightharpoonup \mathbb{D}([1]).$$

Thus, this lemma expresses the expected fact that for an object $f = (X_0 \longrightarrow X_1) \in \mathbb{D}([1])$ and $Y \in \mathbb{D}(e)$ we have a natural bijection between morphisms as indicated in the two diagrams:

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d, "f"'] & 0 \arrow[d] \\ X_1 \arrow[r] & Y \end{tikzcd} \qquad\qquad \mathsf{C}f \longrightarrow Y$$

It is important that the morphism in the left diagram is read as a morphism in $\mathbb{D}([1])$ and not in $\mathbb{D}(e)^{[1]}$. Said differently, this result does *not* say that the category $\mathbb{D}(e)$ has cokernels.

Using the notion of recollements of triangulated categories, we can summarize the above by saying that for a stable derivator we have the following two recollements:

$$\begin{tikzcd} \mathbb{D}(e) \arrow[r, "0_*"] & \mathbb{D}([1]) \arrow[l, bend right, "0^*"'] \arrow[l, bend left, "\mathsf{F}"] \arrow[r, "1^*"] & \mathbb{D}(e) \arrow[l, bend right, "1_!"'] \arrow[l, bend left, "1_*"] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{D}(e) \arrow[r, "1_!"] & \mathbb{D}([1]) \arrow[l, bend right, "\mathsf{C}"'] \arrow[l, bend left, "1^*"] \arrow[r, "0^*"] & \mathbb{D}(e) \arrow[l, bend right, "0_!"'] \arrow[l, bend left, "0_*"] \end{tikzcd}$$

If we apply the cone construction to objects in the image of the extension by zero functor 0_* we obtain up to natural isomorphism the suspension functor. Thus, we have the following factorization of the adjunction (Σ, Ω) :

$$\Sigma\colon\ \mathbb{D}(e) \underset{\mathsf{F}=0^!}{\overset{0_*}{\rightleftarrows}} \mathbb{D}([1]) \underset{1_!}{\overset{\mathsf{C}=1^?}{\rightleftarrows}} \mathbb{D}(e)\ :\Omega$$

This can now be used to construct non-trivial morphisms in $\mathbb{D}([1])$ which induce trivial morphisms on underlying diagrams. For two objects $X,\ Y \in \mathbb{D}(e)$ we obtain from the adjunction $(\mathsf{C}, 1_!)$ a bijection

$$\mathsf{hom}(\Sigma X, Y) \longrightarrow \mathsf{hom}(0_* X, 1_! Y).$$

Applied to the special case of $Y = \Sigma X$ we can consider the image of $\mathrm{id}_{\Sigma X}$ under this bijection which gives us a map $H_X\colon 0_*X \longrightarrow 1_!\Sigma X$. Similarly to the last section we depict H_X and also all other morphisms $0_*X \longrightarrow 1_!Y$ horizontally as:

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma X \end{tikzcd}$$

Obviously, this map H_X induces a trivial morphism on underlying diagrams but it is not trivial itself unless $\Sigma X = 0$. Moreover, this is the universal example of such a morphism in that any morphism $0_*X \longrightarrow 1_!Y$ factors uniquely over H_X :

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr] & 0 \arrow[d] \\ 0 \arrow[r] & Y \end{tikzcd} = \begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr] & 0 \arrow[d] \arrow[r] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma X \arrow[r] & Y \end{tikzcd}$$

Having these comments in mind might be helpful during the proof of the next result.

Theorem 3.8. *Let $\mathbb{D}$ be a stable derivator which is linear over a field k. Then we obtain a canonical Hochschild-Mitchell extension*

$$0 \longrightarrow K \longrightarrow \mathbb{D}([1]) \xrightarrow{\mathsf{dia}} \mathbb{D}(e)^{[1]} \longrightarrow 0.$$

Proof. Let $X \in \mathbb{D}([1])$ be given and let us denote its underlying diagram by $X_0 \longrightarrow X_1$. Using Proposition 1.40 and Lemma 1.32 of Part 1 we obtain that the underlying diagram of $0_!0^*X$ is isomorphic to $X_0 \xrightarrow{\mathrm{id}} X_0$ while the one of $1_!1^*X$ looks like $0 \longrightarrow X_1$. Taking the adjunction counits $\epsilon_0\colon 0_!0^*X \longrightarrow X$ and $\epsilon_1\colon 1_!1^*X \longrightarrow X$ we obtain a map

$$r = (\epsilon_0, \epsilon_1)\colon\ RX = 0_!0^*X \oplus 1_!1^*X \xrightarrow{\quad r \quad} X$$

which can be thought of as a *canonical resolution* of X by *free diagrams.* The underlying diagram of the exact triangle associated to $r\colon RX \longrightarrow X$ is isomorphic to the following one:

$$\begin{tikzcd} X_0 \arrow[r, "r_0"] \arrow[d, "{(\mathrm{id},0)}"'] & X_0 \arrow[r] \arrow[d, "u"'] & 0 \arrow[r] \arrow[d] & \Sigma X_0 \arrow[d, "{(\mathrm{id},0)}"] \\ X_0 \oplus X_1 \arrow[r, "r_1"'] & X_1 \arrow[r] & \mathsf{C}(r_1) \arrow[r] & \Sigma(X_0 \oplus X_1) \end{tikzcd}$$

Since r_1 is up to isomorphism given by $(u, \mathrm{id})\colon X_0 \oplus X_1 \longrightarrow X_1$, where the second component is obviously an isomorphism, it follows by a standard fact about triangulated categories that we have an isomorphism $\mathsf{C}(r_1) \cong \Sigma X_0$. Thus, by the characterization of the essential image of the homotopy left Kan extension along the closed immersion $1\colon e \longrightarrow [1]$ we have an isomorphism $\mathsf{C}r \cong 1_!\Sigma X_0$. After one rotation the triangle associated to the canonical resolution by free diagrams hence looks like:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & X_0 \arrow[r, "\mathrm{id}"] \arrow[d, "(\mathrm{id},0)"'] & X_0 \arrow[r] \arrow[d, "u"'] & 0 \arrow[d] \\ X_0 \arrow[r, "(\mathrm{id},-u)"'] & X_0 \oplus X_1 \arrow[r, "(u,\mathrm{id})"'] & X_1 \arrow[r, "0"'] & \Sigma X_0 \end{tikzcd}$$

Now, let us consider a further object $Y \in \mathbb{D}([1])$ and let us apply the cohomological functor $\mathsf{hom}_{\mathbb{D}([1])}(-,Y)$ to our distinguished triangle. The part of interest of the corresponding long exact sequence is given by:

$$\mathsf{hom}_{\mathbb{D}([1])}(\mathsf{C}r,Y) \longrightarrow \mathsf{hom}_{\mathbb{D}([1])}(X,Y) \xrightarrow{r^*} \mathsf{hom}_{\mathbb{D}([1])}(RX,Y) \longrightarrow \mathsf{hom}_{\mathbb{D}([1])}(\Omega\mathsf{C}r,Y)$$

Let us identify these maps in more explicit terms. Let $f\colon X \longrightarrow Y$ be a morphism in $\mathbb{D}([1])$ with components f_0 and f_1. Under the identification

$$\begin{aligned} \mathsf{hom}_{\mathbb{D}([1])}(RX,Y) &\cong \mathsf{hom}_{\mathbb{D}([1])}(0_!0^*X,Y) \oplus \mathsf{hom}_{\mathbb{D}([1])}(1_!1^*X,Y) \\ &\cong \mathsf{hom}_{\mathbb{D}(e)}(X_0,Y_0) \oplus \mathsf{hom}_{\mathbb{D}(e)}(X_1,Y_1) \end{aligned}$$

the map r^* becomes the map

$$\mathsf{dia}\colon \mathsf{hom}_{\mathbb{D}([1])}(RX,Y) \longrightarrow \mathsf{hom}_{\mathbb{D}(e)}(X_0,Y_0) \oplus \mathsf{hom}_{\mathbb{D}(e)}(X_1,Y_1)\colon f \longmapsto (f_0,f_1).$$

Using the same identification for $\mathsf{hom}_{\mathbb{D}([1])}(RX,Y)$ again and the chain of isomorphisms

$$\mathsf{hom}_{\mathbb{D}([1])}(\Omega\mathsf{C}r,Y) \cong \mathsf{hom}_{\mathbb{D}([1])}(1_!X_0,Y) \cong \mathsf{hom}_{\mathbb{D}(e)}(X_0,Y_1)$$

the last map of the sequence becomes identified with the map

$$\mathsf{hom}_{\mathbb{D}(e)}(X_0,Y_0) \oplus \mathsf{hom}_{\mathbb{D}(e)}(X_1,Y_1) \longrightarrow \mathsf{hom}_{\mathbb{D}(e)}(X_0,Y_1)\colon (f_0,f_1) \longmapsto u \circ f_0 - f_1 \circ u.$$

The kernel of this map is precisely $\mathsf{hom}_{\mathbb{D}(e)^{[1]}}(\mathsf{dia}\,X,\mathsf{dia}\,Y)$ so that we obtain an exact sequence

$$\mathsf{hom}_{\mathbb{D}([1])}(\mathsf{C}r,Y) \longrightarrow \mathsf{hom}_{\mathbb{D}([1])}(X,Y) \xrightarrow{\mathsf{dia}} \mathsf{hom}_{\mathbb{D}(e)^{[1]}}(\mathsf{dia}\,X,\mathsf{dia}\,Y) \longrightarrow 0.$$

Of course, by the definition of a stable derivator, we knew already that dia is full. But the point is that we now have a better control over the kernel. So, let us denote the kernel of dia by $K(X,Y) \subseteq \mathsf{hom}_{\mathbb{D}([1])}(X,Y)$ so that we obtain a short exact sequence and an epimorphism

$$0 \longrightarrow K \longrightarrow \mathbb{D}([1]) \longrightarrow \mathbb{D}(e)^{[1]} \longrightarrow 0, \qquad \mathsf{hom}_{\mathbb{D}([1])}(\mathsf{C}r,Y) \longrightarrow K(X,Y) \longrightarrow 0.$$

The last aim is to show that this short exact sequence yields a Hochschild-Mitchell extension. Since we assumed to be given a k-linear derivator where k is a field this sequence splits linearly. Let us now show that this extension is a square zero extension, i.e., that

$K^2 = 0$. First, let us consider an arbitrary element $f\colon X \longrightarrow Y$ of $K(X,Y)$. Using the above epimorphism such an f can be factored as

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & Y_0 \arrow[d] \\ X_1 \arrow[r, "0"'] & \Sigma X_0 \arrow[r] & Y_1 \end{tikzcd}$$

where the first morphism f' again induces the zero morphism on underlying diagrams. The next aim is to show that this morphism f' factors even further as indicated in the next diagram:

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X_0 \arrow[r] \arrow[d] \arrow[dr] & 0 \arrow[d] \\ X_1 \arrow[r] & 0 \arrow[r] & \Sigma X_0 \end{tikzcd}$$

This will basically be achieved by redoing what we did until now in a dual fashion. So, let us begin by forming the *canonical resolution* of $X' \in \mathbb{D}([1])$ *by cofree diagrams* having in mind $X' = (0 \longrightarrow \Sigma X_0)$ as the case of particular interest. By this we mean the map

$$q = (\eta_0, \eta_1)\colon X' \longrightarrow QX' = 0_* 0^* X' \oplus 1_* 1^* X'$$

which is given by the adjunction units. We then form an associated distinguished triangle

$$\mathsf{F}q \longrightarrow X' \xrightarrow{q} QX' \longrightarrow \Sigma \mathsf{F}q$$

which can be plugged into the homological functor $\mathsf{hom}_{\mathbb{D}([1])}(X,-)$ to induce a long exact sequence. Similar identifications as in the previous situation show that this reproduces the short exact sequence $0 \longrightarrow K \longrightarrow \mathbb{D}([1]) \longrightarrow \mathbb{D}(e)^{[1]} \longrightarrow 0$ together with a second epimorphism

$$\mathsf{hom}_{\mathbb{D}([1])}(X, \mathsf{F}q) \longrightarrow K(X,X') \longrightarrow 0.$$

In our special situation of $X' = (0 \longrightarrow \Sigma X_0)$ the above distinguished triangle reads as

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] \arrow[dr] & 0 \arrow[r] \arrow[d] & \Sigma X_0 \arrow[r] \arrow[d] & \Sigma X_0 \arrow[d] \\ 0 \arrow[r] & \Sigma X_0 \arrow[r] & \Sigma X_0 \arrow[r] & 0 \end{tikzcd}$$

so that we obtain the intended factorization of f'. If we have two morphisms $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ lying in the kernel of dia then their composition can be factored as follows:

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X_0 \arrow[r] \arrow[d] \arrow[dr] & 0 \arrow[r] \arrow[d] & Y_0 \arrow[r] \arrow[d] & Y_0 \arrow[r] \arrow[d] \arrow[dr] & 0 \arrow[r] \arrow[d] & Z_0 \arrow[d] \\ X_1 \arrow[r] & 0 \arrow[r] & \Sigma X_0 \arrow[r] & Y_1 \arrow[r] & 0 \arrow[r] & \Sigma Y_0 \arrow[r] & Z_1 \end{tikzcd}$$

But this composition is zero since

$$\mathsf{hom}_{\mathbb{D}([1])}(0 \longrightarrow \Sigma X_0, Y_0 \longrightarrow 0) \cong \mathsf{hom}_{\mathbb{D}([1])}(1_! \Sigma X_0, 0_* Y_0) \cong \mathsf{hom}_{\mathbb{D}(e)}(\Sigma X_0, 1^* 0_* Y_0) = 0.$$

Thus we have shown that $0 \longrightarrow K \longrightarrow \mathbb{D}([1]) \xrightarrow{\mathsf{dia}} \mathbb{D}(e)^{[1]} \longrightarrow 0$ is a square zero extension and hence by the above a Hochschild-Mitchell extension. □

Remark 3.9. i) The proof of the last theorem and the discussion preceding the theorem show that the family of canonical morphisms $H\colon 0_*X_0 \longrightarrow 1_!\Sigma X_0$ yields a –in a weak sense– universal family of morphisms which induce zero morphisms on underlying diagrams. In fact, every morphism $f\colon X \longrightarrow Y$ satisfying $\mathsf{dia}\, f = 0$ has a factorization

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X_0 \arrow[r] \arrow[d] \arrow[dr] & 0 \arrow[r] \arrow[d] & Y_0 \arrow[d] \\ X_1 \arrow[r] & 0 \arrow[r] & \Sigma X_0 \arrow[r] & Y_1 \end{tikzcd}$$

such that the diagonal morphism is given by the universal H. A dual statement can be given using the loop functor instead of the suspension.

ii) We saw in the proof of the theorem that the strongness of $\mathbb{D}$ is a consequence of the fact that we have the canonical resolutions by free diagrams together with the fact that represented functors are cohomological. However, this cannot be used to show that the strongness can be rederived from the other axioms. The proof that represented functors are cohomological in the case of triangulated categories uses already the fact that the cone construction is weakly functorial. And in order to show that the cone construction for the canonical triangulated structures on the values of a stable derivator is weakly functorial we already had to use the strongness.

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Curriculum vitae

Education

03/2007-09/2011	Rheinische Friedrich-Wilhelms-Universität Bonn, Ph.D. student under supervision of Prof. Schwede, thesis: 'On the theory of derivators'
08/2004-06/2005	Massachusets Institute of Technology, Cambridge, Massachusets
10/2002-12/2006	Rheinische Friedrich-Wilhelms-Universität Bonn, diploma in mathematics (grade: 'sehr gut'), diploma thesis: 'Connections for twisted vector bundles and a twisted version of the Hermite-Einstein equation'
10/2000-08/2002	Technische Universität Dresden, pre-degree in mathematics (grade: 'sehr gut')

Preprints

2011	Monoidal derivators and enriched derivators, available at the homepage www.math.uni-bonn.de/people/mgroth
2011	Derivators, pointed derivators, and stable derivators, available at the homepage www.math.uni-bonn.de/people/mgroth
2010	A short course on ∞-categories, available at the homepage www.math.uni-bonn.de/people/mgroth

Invited talks

07/2011	Universität Osnabrück, three talks on 'A short course on ∞-categories'
06/2011	Universität Regensburg, four 2-hour talks on 'A short course on ∞-categories'
09/2010	Harvard University, Cambridge, Massachusets, two talks on 'Derivators, pointed derivators, and stable derivators'

09/2010	University of Notre Dame, Indiana, four talks on 'A short course on ∞-categories'
01/2010	University of Warsaw, Poland, three talks on 'A short course on ∞-categories'

Grants

08/2010	fundings for the conference on Homotopy Theory and Derived Algebraic Geometry at the Fields Institute, Toronto
03/2007-02/2010	doctoral fellow of the Graduiertenkolleg 1150 'Homotopy and Cohomology' at the math department of the Rheinische Friedrich-Wilhelms-Universität Bonn
08/2004-06/2005	foreign exchange scholarship of the 'Studienstiftung des deutschen Volkes' for one year at the Massachusets Institute of Technology, Cambridge, Massachusets
02/2003-03/2006	fellow of the 'Studienstiftung des deutschen Volkes'

Teaching

winter 2010-2011	Seminar on model categories (joint with Prof. Schwede)
summer 2010	Seminar on p-adic analyis (joint with Dr. Welter)
winter 2009-2010	Ph.D. seminar on relative homological algebra
10/2002-02/2007	Teaching assistent in mathematics for chemistry students (calculus and linear algebra)
summer 2002	Teaching assistent in 'Differential geometry II' at the Technische Universität Dresden
winter 2001	Mentor for first-year students at the math department of the Technische Universität Dresden

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