Higher-dimensional Auslander algebras of type $\mathbb{A}$ and the higher-dimensional Waldhausen S*-constructions*

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Abstract. These notes are an expanded version of my talk at the ICRA 2018 and are based on an ongoing project with Tobias Dyckerhoff. In them we describe a link between Iyama's higher-dimensional Auslander algebras of type $\mathbb{A}$ and higher-dimensional versions of the Waldhausen S-construction, thus relating higher-dimensional Auslander–Reiten theory to constructions in algebraic K-theory. The results presented here can also be seen as contributions to the abstract representation theory program of Groth and Šťovíček.

Motivation: Eilenberg–Mac Lane spaces

We begin these notes by reminding the reader of the notion of an Eilenberg–Mac Lane space. Let A be an abelian group. For each positive integer m there exists a topological space $K(A,m)$ characterised up to weak homotopy equivalence[1] by the existence of a natural bijection[2]

$$[X, K(A,m)] \cong H^m(X;A), \tag{1}$$

where X is a "nice" topological space (for example a CW complex). Equivalently, the topological space $K(A,m)$ is characterised by the following two properties:

- The topological space $K(A,m)$ is path connected.
- There is an isomorphism

$$\pi_n(K(A,m)) \cong \begin{cases} A & \text{if } n = m, \\ e & \text{otherwise.} \end{cases}$$

A topological space satisfying the above properties is called an *Eilenberg–Mac Lane space*[3]. The bijection (1) exhibits Eilenberg–Mac Lane spaces as fundamental objects in algebraic topology while their alternative characterisation makes their relative simplicity manifest.

Rather surprisingly, the standard construction of the Eilenberg–Mac Lane spaces can be expressed in terms of the *higher-dimensional Auslander–Reiten theory* of Iyama's higher-dimensional Auslander algebras of type $\mathbb{A}$. This elementary observation, which we make precise below, is one of the starting points of our investigations.

Interlude: the simplex category

The modern approach to homotopy theory makes extensive use of simplicial methods and the standard construction of the Eilenberg–MacLane spaces is no exception. This requires us to recall a minimal amount of terminology from simplicial homotopy theory[4].

[1] A *weak homotopy equivalence* is a continuous map $X \to Y$ between topological spaces which induces a bijection

$$\pi_0(X) \xrightarrow{\cong} \pi_0(Y)$$

on connected components as well as group isomorphisms

$$\pi_n(X,x) \xrightarrow{\cong} \pi_n(Y, f(x))$$

for all points $x \in X$ and all $n \geq 1$. Two topological spaces are *weakly homotopy equivalent* if they are connected by a zig-zag of weak homotopy equivalences.

[2] Here $[X, K(A,m)]$ denotes the set of homotopy classes of continuous maps $X \to K(A,m)$ while $H^m(X;A)$ denotes the m-th singular cohomology group of X with coefficients in A.

[3] The symbol $K(A,m)$ is more commonly used to denote the weak homotopy equivalence class (also known as the homotopy type) of an Eilenberg–Mac Lane space. For our purposes it will be convenient to abuse the notation and use this symbol to denote a preferred representative of this equivalence class.

[4] The interested reader is referred to [GJ99] for an introduction to simplicial homotopy theory.

We remind the reader of the definition of the *simplex category*, which is commonly denoted by Δ. The objects of the simplex category are the linear posets

$$[n] := \{0 \to 1 \to \cdots \to n\},$$

where n ranges over all non-negative integers, with morphisms[5] the monotone functions $[m] \to [n]$. We denote the set of morphisms $[m] \to [n]$ in Δ by $\Delta(m,n)$. Note that the set $\Delta(m,n)$ has a natural poset structure[6]; namely, $\sigma \leq \tau$ if and only if $\sigma_i \leq \tau_i$ for all $i \in [m]$. The elements of $\Delta(m,n)$ are the *m-simplices in Δ^n*; an m-simplex is *non-degenerate* if the underlying monotone function is injective and is *degenerate* otherwise.

[5] We often identify $\Delta(m,n)$ with the set of order tuples $\sigma \in \mathbb{Z}^{m+1}$ which satisfy the inequalities

$$0 \leq \sigma_0 \leq \sigma_1 \leq \cdots \leq \sigma_m \leq n.$$

[6] In fact, the poset structure on the morphism sets allows us to promote Δ to 2-category; this perspective, implicit in parts of these notes, is essential in our work.

A *simplicial object*[7] in a category $\mathcal{C}$ is a functor $X\colon \Delta^{\mathsf{op}} \to \mathcal{C}$. It is elementary to verify, using a specific presentation of Δ in terms of generators and relations, that the data of a simplicial object in $\mathcal{C}$ is equivalent to that of a diagram in $\mathcal{C}$ of the form

[7] Depending on the target category one speaks of simplicial sets, simplicial topological spaces, simplicial abelian groups, etc..

$$\cdots \; \substack{\rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow} \; X_2 \; \substack{\rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow} \; X_1 \; \substack{\rightarrow\\ \leftarrow\\ \rightarrow} \; X_0.$$

The morphisms

$$\{d_i\colon X_n \to X_{n-1} \mid i \in [n]\},$$

called *face maps*, and the morphisms

$$\{s_i\colon X_n \to X_{n+1} \mid i \in [n]\},$$

called *degeneracy maps*, are subject to a number of relations commonly referred to as the simplicial identities[8]:

$$\begin{aligned} d_i \circ d_j &= d_{j-1} \circ d_i & i < j, \\ s_i \circ s_j &= s_j \circ s_{i-1} & i > j, \end{aligned} \qquad d_i \circ s_j = \begin{cases} s_{j-1} \circ d_i & i < j, \\ s_j \circ d_{i-1} & i > j+1, \\ 1 & \text{otherwise.} \end{cases}$$

[8] The simplicial identities allow us to think of an element of X_n as an abstract (possibly degenerate) n-simplex whose boundary consists of the $(n-1)$-simplices $\{d_i(x) \in X_{n-1} \mid i \in [n]\}$.

The reader might want to think of a simplicial object in $\mathcal{C}$ as a *representation* in the category $\mathcal{C}$ of a certain infinite quiver with (inadmissible) relations corresponding to the above presentation of Δ^{op}.

Eilenberg–Mac Lane spaces via the Dold–Kan correspondence

The Dold–Kan correspondence is an explicit adjoint equivalence of categories[9]

$$C\colon \mathsf{Ab}_\Delta \rightleftarrows \mathsf{Ch}_{\geq 0}(\mathsf{Ab})\colon N$$

[9] Passing to the $\mathbb{Z}$-linear envelope of Δ, the Dold–Kan correspondence can be interpreted as an instance of Morita theory for Ab-categories.

between the (abelian) category of simplicial abelian groups and the (abelian) category of chain complexes of abelian groups which are concentrated in non-negative degrees[10].

[10] The reader can find a detailed discussion of this equivalence in Section III.2 in [GJ99].

As usual, let us denote by $A[m]$ the chain complex of abelian groups whose only non-zero component consists of the abelian group A placed in homological degree m. The Eilenberg–Mac Lane space $K(A,m)$ is defined as

$$K(A,m) := |N(A[m])|,$$

that is as the geometric realisation[11] of the underlying simplicial set of $N(A[m])$. For our purposes it is instructive to give an explicit description of the simplicial abelian group $N(A[m])$ in the simplest case $m = 1$.

Definition. The simplicial abelian group $N(A[1])$ is defined as follows. For each $n \geq 0$ let $N(A[1])_n$ be the abelian group of upper-triangular arrays

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} & \cdots & a_{0,n-1} & a_{0n} \\ & a_{11} & a_{12} & \cdots & a_{1,n-1} & a_{1n} \\ & & \ddots & \ddots & \vdots & \vdots \\ & & & & a_{n-1,n-1} & a_{n-1,n} \\ & & & & & a_{nn} \end{pmatrix} \tag{2}$$

with entries in A such that for each $0 \leq i \leq n$ there is an equality $a_{ii} = 0$ and for all $0 \leq i < j < k \leq n$ the *Euler relation*

$$a_{ij} - a_{ik} + a_{jk} = 0 \tag{3}$$

is satisfied. The i-th face map $d_i \colon N(A[1])_n \to N(A[1])_{n-1}$ is given by deleting the i-th row and the i-th column while the i-th degeneracy map $s_i \colon N(A[1])_n \to N(A[1])_{n+1}$ is given by repeating the i-th row and the i-th column.

Remark. An element (2) of $N(A[1])_n$ is completely determined by the n-tuple $(a_{01}, a_{02}, \ldots, a_{0n})$. Indeed, the Euler relations (3) imply that for each $1 \leq j < k \leq n$ the equality

$$a_{jk} = a_{0k} - a_{0j}$$

is satisfied. We conclude that there is an isomorphism

$$N(A[1])_n \cong A^n.$$

The Eilenberg–Mac Lane space $K(A, 1)$ via Grothendieck groups of linearly oriented quivers of Dynkin type $\mathbb{A}$

Our next task is to relate the previous construction of the Eilenberg–Mac Lane space $K(A, 1)$ to the Auslander–Reiten theory of the quiver

$$\mathbb{A}_n := 1 \to 2 \to \cdots \to n.$$

Consider the Auslander–Reiten quiver of the category of finite-dimensional representations[12] of the quiver $\mathbb{A}_n$, which we depict as follows:

$$\begin{array}{ccccccccccc} M_{00} & \to & M_{01} & \to & M_{02} & \to & \cdots & \to & M_{0,n-1} & \longrightarrow & M_{0n} \\ & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ & & M_{11} & \to & M_{12} & \to & \cdots & \to & M_{1,n-1} & \longrightarrow & M_{1n} \\ & & & & \downarrow & & & & \downarrow & & \downarrow \\ & & & & \ddots & & \ddots & & \vdots & & \vdots \\ & & & & & & & & \downarrow & & \downarrow \\ & & & & & & & & M_{n-1,n-1} & \to & M_{n-1,n} \\ & & & & & & & & & & \downarrow \\ & & & & & & & & & & M_{nn} \end{array}$$

[11] The geometric realisation functor is part of an adjunction

$$|\cdot| \colon \mathsf{Set}_\Delta \leftrightarrow \mathsf{Top} \colon \mathsf{Sing}$$

between the category of simplicial sets and the category of topological spaces. In fact, this adjunction is a Quillen equivalence with respect to appropriate model category structures. Roughly speaking, this justifies the simplicial approach to homotopy theory.

[12] We work over some fixed but unspecified field.

For reasons which shall become clear shortly, we have included additional vertices $M_{ii} = 0$ in the above Auslander–Reiten quiver. The Grothendieck group $K_0(\mathbb{A}_n)$ admits a presentation by generators and relations as the quotient of the free abelian group with basis

$$\{[M_{ij}] \mid 0 \leq i \leq j \leq n\}$$

modulo the subgroup generated by the relations $[M_{ii}] = 0$, $i \in [n]$, together with the Euler relations

$$[M_{ij}] - [M_{ik}] + [M_{jk}] = 0 \tag{4}$$

corresponding to short exact sequences[13]

$$0 \to M_{ij} \to M_{ik} \to M_{jk} \to 0$$

in the abelian category of finite-dimensional representations of the quiver $\mathbb{A}_n$, where $0 \leq i < j < k \leq n$.

[13] The Euler relations can also be interpreted as being induced by biCartesian squares

$$\begin{array}{ccc} M_{ij} & \longrightarrow & M_{ik} \\ \downarrow & & \downarrow \\ M_{ii} & \longrightarrow & M_{jk} \end{array}$$

where $0 \leq i < j < k \leq n$ and $M_{ii} = 0$.

It is immediate from the above presentation of $K_0(\mathbb{A}_n)$ that there are isomorphisms

$$N(A[1])_n \cong \mathrm{Hom}_{\mathbb{Z}}(K_0(\mathbb{A}_n), A) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, A) \cong A^n, \tag{5}$$

where $n \geq 1$. We leave it to the reader to verify that the Grothendieck groups $K_0(\mathbb{A}_n)$ assemble into a co-simplicial abelian group, that is into a functor

$$K_0(\mathbb{A}) \colon \Delta \to \mathsf{Ab},$$

where $K_0(\mathbb{A})_n = K_0(\mathbb{A}_n)$ for $n \geq 1$ and $K_0(\mathbb{A})_0 = 0$. Moreover, there is a canonical isomorphism of simplicial abelian groups

$$N(A[1]) \cong \mathrm{Hom}_{\mathbb{Z}}(K_0(\mathbb{A}), A) \tag{6}$$

where $\mathrm{Hom}_{\mathbb{Z}}(K_0(\mathbb{A}), A)$ denotes the composite

$$\Delta^{\mathsf{op}} \xrightarrow{K_0(\mathbb{A})^{\mathsf{op}}} \mathsf{Ab}^{\mathsf{op}} \xrightarrow{\mathrm{Hom}_{\mathbb{Z}}(-,A)} \mathsf{Ab},$$

and $K_0(\mathbb{A})^{\mathsf{op}}$ indicates the passage to the opposite functor.

Remark. It is elementary to verify that the Euler relations (4) are generated by the *Auslander–Reiten relations*

$$[M_{ij}] - ([M_{i+1,j}] + [M_{i,j+1}]) + [M_{i+1,j+1}] = 0$$

corresponding to the almost-split sequences[14]

$$0 \to M_{ij} \to M_{i+1,j} \oplus M_{i,j+1} \to M_{i+1,j+1} \to 0,$$

where $0 \leq i < j < n$. In particular, the isomorphism (6) gives a precise relationship between the simplicial abelian group $N(A[1])$ and the Auslander–Reiten theory of the linearly oriented quivers of Dynkin type $\mathbb{A}$.

[14] Equivalently, the Auslander–Reiten relations are induced by meshes

$$\begin{array}{ccc} M_{ij} & \longrightarrow & M_{i,j+1} \\ \downarrow & & \downarrow \\ M_{i+1,j} & \longrightarrow & M_{i+1,j+1} \end{array}$$

in the Auslander–Reiten quiver, where $0 \leq i < j < n$.

The Eilenberg–Mac Lane space $K(A,m)$ via Grothendieck groups of Iyama's m-dimensional Auslander algebras of type $\mathbb{A}$

OUR PREVIOUS DISCUSSION extends to the Eilenberg–Mac Lane space $K(A,m)$ provided that we replace the linearly oriented quivers of type $\mathbb{A}$ by Iyama's m-dimensional Auslander algebras $\mathbb{A}_\ell^{(m)}$ of type $\mathbb{A}$, see [Iya11] for the definition. Indeed, the Grothendieck groups of these algebras assemble into a co-simplicial abelian group

$$K_0(\mathbb{A}^{(m)})\colon \Delta \to \mathsf{Ab},$$

where

$$K_0(\mathbb{A}^{(m)})_n = K_0(\mathbb{A}_{n-m+1}^{(m)})$$

for $n \geq m$ and $K_0(\mathbb{A}^{(m)})_n = 0$ for $n < m$. In analogy with the case $m = 1$, there is a canonical isomorphism of simplicial abelian groups

$$N(A[m]) \cong \operatorname{Hom}_{\mathbb{Z}}(K_0(\mathbb{A}^{(m)}), A).$$

In particular there are isomorphisms

$$\begin{aligned} N(A[m])_n &\cong \operatorname{Hom}_{\mathbb{Z}}(K_0(\mathbb{A}_{n-m+1}^{(m)}), A) \\ &\cong \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\binom{n}{m}}, A) \\ &\cong A^{\binom{n}{m}} \end{aligned} \tag{7}$$

for each $n \geq m$.

We emphasise that the above description of $N(A[m])$ is dependent on the presentation of the Grothendieck group $K_0(\mathbb{A}_{n-m+1}^{(m)})$ in terms of the basis

$$\{[M_\sigma] \mid \sigma \in \Delta(m,n)\}$$

labelled by the indecomposable summands of the unique basic m-cluster-tilting $\mathbb{A}_n^{(m)}$-module[15]. In analogy with the case $m = 1$, one then imposes relations $[M_\sigma] = 0$ for all degenerate m-simplices in Δ^n as well as the *higher-dimensional Euler relations*

$$\sum_{i=0}^{m} (-1)^i [M_{d_i(\sigma)}] = 0, \tag{8}$$

where σ is a non-degenerate $(m+1)$-simplex in Δ^n and

$$d_i(\sigma) = (\sigma_0, \sigma_1, \cdots, \sigma_{i-1}, \widehat{\sigma}_i, \sigma_{i+1}, \cdots, \sigma_m).$$

Finally, we note that the higher-dimensional Euler relations are generated by the *higher-dimensional Auslander–Reiten relations*

$$\sum_{v} (-1)^{|v|} [M_{\sigma+v}] = 0, \tag{9}$$

where σ is a non-degenerate m-simplex in Δ^n with $\sigma_m < n$, the tuple v ranges over all vertices of the $(m+1)$-cube

$$\underbrace{\{0,1\} \times \cdots \times \{0,1\}}_{m+1 \text{ times}},$$

and $|v| := v_0 + v_1 + \cdots + v_m$. The relations (9) are induced by m-almost-split sequences, see [Iya07] for definitions.

[15] Implicit in this discussion is the Grothendieck group of an m-cluster-tilting subcategory, which can be defined using the ideas in [BT14].

THE PURPOSE OF THESE NOTES is to describe a *categorified* version of the above discussion where the abelian group A is replaced by a refined version of a triangulated category[16], namely a *stable* ∞*-category*. As such, the simplicial objects we shall consider can be thought of as "categorified Eilenberg–Mac Lane spaces", as originally advocated in [Dyc17]. The results presented in these notes can be also seen as contributions to the abstract representation theory program of Groth and Šťovíček [GS16c]. Indeed, the results presented here are for the most part higher-dimensional versions of results in [GS16a].

[16] Triangulated categories are inadequate for this purpose as the cone of a morphisms lacks the necessary functoriality.

Rudiments of the theory of ∞*-categories*

THE LANGUAGE OF ∞-CATEGORIES, developed extensively by Joyal in [Joy02] as well as in unpublished work and by Lurie in [Lur09, Lur17], affords an adequate framework for our results. It is however impractical, and for the purpose of these notes perhaps inadvisable, to include here a formal introduction to the theory of ∞-categories[17]. Instead we aim to provide the reader with a minimal amount of intuition regarding the theory which we hope will be sufficient to follow our exposition in the subsequent section.

[17] The interested reader is referred to Chapter 1 in [Lur09] and [Gro10] for an introduction to the theory of ∞-categories.

What are ∞*-categories?*

AN ∞-CATEGORY is a mathematical structure[18] which implements the idea of a

> "higher-dimensional category with morphisms of every positive degree in which the morphisms of degree greater than 1 are invertible".

[18] More precisely, an ∞-category is a particular kind of simplicial set called a "weak Kan complex".

In other words the theory of ∞-categories is a model for the theory of $(\infty, 1)$-categories. In particular, for every pair of objects x and y in an ∞-category $\mathcal{C}$ there is a "space" of maps $\mathrm{Map}_{\mathcal{C}}(x, y)$. Within this paradigm ordinary categories are those for which the non-empty spaces of maps are discrete spaces[19]. After suitable translation, further examples of ∞-categories are provided by differential graded categories and, more generally, A_∞-categories[20].

[19] See Propositions 2.3.4.5 and 2.3.4.18 in [Lur09].

[20] See Section 1.3 in [Lur17] for the case of differential graded categories and [Fao17] for the case of A_∞-categories.

An important feature of the theory of ∞-categories is that it allows us to formalise the notion of a universal property which only holds "up to homotopy", such as the one enjoyed by the cone of a morphism in the derived category[21] of, say, a module category. Just as in ordinary category theory, these universal properties are captured by appropriate notions of (homotopy) limit and (homotopy) colimit[22]. For example, given two morphisms $x \to y'$ and $x \to y''$ in an ∞-category $\mathcal{C}$, we may define the (homotopy) pushout of the diagram[23]

$$\begin{array}{ccc} x & \longrightarrow & y' \\ \downarrow & & \\ y'' & & \end{array}$$

by means of a suitable universal property, that is as a (homotopy) colimit diagram of the form[24]

[21] More precisely, in the ∞-categorical version of the derived category.

[22] In ∞-category theory it is customary to drop the qualifier "homotopy" from the terminology as only "homotopy-invariant" notions make sense in this context.

[23] Strictly speaking, specifying a diagram in an ∞-category involves an infinite amount of coherence data. For expository purposes we nonetheless display such (coherent) diagrams as we would display diagrams in ordinary categories.

[24] We decorate the square to indicate that it is a (homotopy) pushout square.

$$\begin{array}{ccc} x & \longrightarrow & y' \\ \downarrow & \ulcorner & \downarrow \\ y'' & \longrightarrow & z \end{array}$$

The space of all possible (homotopy) (co)limit diagrams of a fixed diagram is either empty (if no (co)limit of the diagram exists in $\mathcal{C}$) or contractible[25].

[25] This means that, for the purposes of ∞-category theory, (co)limit cones of a fixed diagram in an ∞-category are essentially unique in the appropriate sense. See Proposition 1.2.12.9 and Definition 1.2.13.4 in [Lur09] for details.

A few words about stable ∞-categories

In the sequel we are mostly concerned with the following class of ∞-categories, which the reader might want to think of as refined versions of triangulated categories.

Definition. An ∞-category $\mathcal{A}$ is *stable*[26] if it has the following properties:

[26] Although we have not provided formal definitions of any the notions involved, the property of being *stable* is sufficiently intuitive for it to be worth to be included in these notes. A detailed treatment of the theory of stable ∞-categories can be found in Chapter 1 in [Lur17].

1. The ∞-category $\mathcal{A}$ is *pointed*, that is $\mathcal{A}$ has a zero object.
2. For every morphism $f\colon x \to y$ in $\mathcal{A}$ there exist squares in $\mathcal{A}$ of the form

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ \downarrow & \ulcorner & \downarrow \\ 0 & \longrightarrow & z \end{array} \qquad \textit{and} \qquad \begin{array}{ccc} w & \longrightarrow & x \\ \downarrow & \lrcorner & \downarrow f \\ 0 & \longrightarrow & y \end{array}$$

   which are a (homotopy) pushout square and a (homotopy) pullback square, respectively. The objects z and w are called the *cofibre* f and the *fibre of* f, respectively[27].

[27] The cofibre and the fibre of a morphism in a stable ∞-category are the ∞-categorical versions of the cone and the co-cone of a morphism in a triangulated category. Following the established convention, we adopt the "topological" terminology which further reminds us of the homotopical nature of these concepts.

3. A diagram in $\mathcal{A}$ of the form

$$\begin{array}{ccc} x & \longrightarrow & y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & z \end{array}$$

   is a (homotopy) pushout square if and only if it is a (homotopy) pullback square. A diagram as above which satisfies these additional properties is called a *cofibre sequence*[28].

[28] Cofibre sequences in stable ∞-categories are the ∞-categorical versions of exact triangles in triangulated categories.

Remark. Note that stable ∞-categories are defined in terms of *properties* and not in terms of additional *structure*, the latter being the case for triangulated categories. Roughly speaking, this is the reason why stable ∞-categories enjoy better *formal properties* than triangulated categories.

Every ∞-category $\mathcal{C}$ has an associated *homotopy category* $\mathrm{h}(\mathcal{C})$ which is in fact an ordinary category. The following basic result[29] relates stable ∞-categories to triangulated categories, see Theorem 1.1.2.14 in [Lur17].

[29] Proposition 1 should be compared with Happel's theorem which states that the stable category of a Frobenius exact category is a triangulated category [Hap88].

Proposition 1. *Let $\mathcal{A}$ be a stable ∞-category. Then, the homotopy category $\mathrm{h}(\mathcal{A})$ is (canonically) a triangulated category.*

Remark. The suspension $\Sigma(x)$ of an object x of a stable ∞-category $\mathcal{A}$ is characterised by the existence of a cofibre sequence

$$\begin{array}{ccc} x & \longrightarrow & 0 \\ \downarrow & \square & \downarrow \\ 0 & \longrightarrow & \Sigma(x) \end{array}$$

Similarly, exact triangles in the homotopy category $\mathrm{h}(\mathcal{A})$ are induced by diagrams in $\mathcal{A}$ of the form

$$\begin{array}{ccccc} x & \longrightarrow & y & \longrightarrow & 0 \\ \downarrow & \square & \downarrow & \square & \downarrow \\ 0 & \longrightarrow & z & \longrightarrow & \Sigma(x) \end{array}$$

in which each square is a cofibre sequence.

The satisfying formal behaviour of stable ∞-categories is illustrated by the following statement[30], see Proposition 1.1.3.1 in [Lur17].

Proposition 2. *Let $\mathcal{A}$ be a stable ∞-category and K a small ∞-category. Then, the ∞-category $\mathrm{Fun}(K, \mathcal{A})$ of functors $K \to \mathcal{A}$ is also stable.*

The higher-dimensional Waldhausen S*-construction*

In this section we establish a link between the m-dimensional Auslander algebras of type $\mathbb{A}$ and the m-dimensional Waldhausen S-construction. We begin by reminding the reader of what is known in the classical situation, that is in the case $m = 1$.

The Waldhausen S*-construction*

The following construction is due to Waldhausen [Wal85]. It is the main ingredient in the definition of the algebraic K-theory space of a stable ∞-category[31], see for example [BGT13].

Definition. Let $\mathcal{C}$ be a stable ∞-category and $n \geq 0$. We let $\mathrm{S}_n(\mathcal{C})$ be the (stable) ∞-category of diagrams of the form[32]

$$\begin{array}{ccccccccccc} X_{00} & \to & X_{01} & \to & X_{02} & \to & \cdots & \longrightarrow & X_{0,n-1} & \longrightarrow & X_{0n} \\ & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ & & X_{11} & \to & X_{12} & \to & \cdots & \longrightarrow & X_{1,n-1} & \longrightarrow & X_{1n} \\ & & & & \downarrow & & & & \downarrow & & \downarrow \\ & & & & \ddots & & \ddots & & \vdots & & \vdots \\ & & & & & & & & \downarrow & & \downarrow \\ & & & & & & & & X_{n-1,n-1} & \to & X_{n-1,n} \\ & & & & & & & & & & \downarrow \\ & & & & & & & & & & X_{nn} \end{array}$$

which satisfy the following two conditions:

- For each $i \in [n]$ the object X_{ii} is a zero object of $\mathcal{C}$.

[30] Proposition 2 should be compared with the elementary fact that the category of functors from a small category into an abelian category is again abelian. Indeed, (homotopy) limits and (homotopy) colimits in functor ∞-categories are computed point-wise, see Corollary 5.1.2.3 in [Lur09].

[31] The algebraic K-theory space is a fundamental invariant of (small) stable ∞-categories which can be seen as a substantial refinement of the Grothendieck group.

[32] Note that such a diagram is precisely a functor $X \colon \Delta(1, n) \to \mathcal{C}$, where we endow the set $\Delta(1, n)$ of monotone maps $[1] \to [n]$ with the natural partial order.

- For each $0 \leq i < j < k \leq n$ the square

$$\begin{array}{ccc} X_{ij} & \longrightarrow & X_{ik} \\ \downarrow & \square & \downarrow \\ X_{ii} & \longrightarrow & X_{jk} \end{array}$$

  is (homotopy) biCartesian[33].

[33] The exactness conditions imposed on the objects of $\mathsf{S}_n(\mathcal{C})$ should be thought of as a categorification of the Euler relations (4).

Size issues aside, the ∞-categories $\mathsf{S}_n(\mathcal{C})$ assemble into a simplicial object $\mathsf{S}_\bullet(\mathcal{C})$, called the *Waldhausen* S*-construction of* $\mathcal{C}$, which takes values in the ∞-category of stable ∞-categories and exact functors between them.

The following elementary observation can be viewed as a categorification of the isomorphism (5). Proofs can be found in Lemma 7.3 in [BGT13] and Lemma 1.2.2.4 in [Lur17]. A version in the related framework of stable derivators is proven in Theorem 4.6 in [GS16a] using a combinatorial version of the knitting algorithm.

Proposition 3 (Waldhausen)**.** *Let $\mathcal{C}$ be a stable ∞-category and $n \geq 1$. The restriction functor*

$$\mathsf{S}_n(\mathcal{C}) \to \mathrm{Fun}(01 \to 02 \to \cdots 0n, \mathcal{C}),$$

which sends an object X of $\mathsf{S}_n(\mathcal{C})$ to the sub-diagram

$$X_{01} \to X_{02} \to \cdots \to X_{0,n-1} \to X_{0n},$$

is an equivalence of (stable) ∞-categories.

The following theorem extends the foregoing proposition to arbitrary orientations of the Dynkin diagram $\mathbb{A}_n$ and can be proven using combinatorial versions of the classical reflection functors. A proof, carried out in the related framework of stable derivators, can be found in [GS16a].

Theorem 4 (Groth–Šťovíček)**.** *Let $\mathcal{C}$ be a stable ∞-category and $n \geq 1$. Let S be a slice*[34] *in the poset $\Delta(1,n)$. The restriction functor*

$$\mathsf{S}_\bullet(\mathcal{C}) \to \mathrm{Fun}(S, \mathcal{C})$$

is an equivalence of (stable) ∞-categories.

[34] As left implicit above, the Hasse quiver of $\Delta(1,n)$ can be identified with the Auslander–Reiten quiver of the quiver $\mathbb{A}_n$ (with additional degenerate vertices). Slices in $\Delta(1,n)$ can then be defined in the usual way.

Our aim in this section is to provide higher-dimensional versions of Proposition 3 and Theorem 4 expressed in terms of certain higher-dimensional versions of the Waldhausen S-construction. These were introduced in [HM15, Pog17, Dyc17] in varying levels of generality.

The higher-dimensional Waldhausen S*-construction*

Before proceeding we are required to introduce further terminology. Let $I = [1]$ be the poset $\{0 \to 1\}$ and m a non-negative integer. An *m-cube* in an ∞-category $\mathcal{C}$ is a functor $X\colon I^m \to \mathcal{C}$. The isomorphism

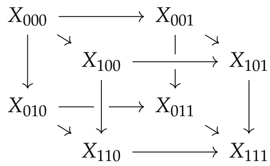
$$I^{m+1} \cong I \times I^m$$

together with the adjunction

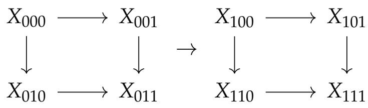
$$\mathrm{Fun}(I \times I^m, \mathcal{C}) \cong \mathrm{Fun}(I, \mathrm{Fun}(I^m, \mathcal{C}))$$

allow us to view an $(m+1)$-cube in $\mathcal{C}$ as morphism in the ∞-category $\mathrm{Fun}(I^m, \mathcal{C})$ of m-cubes in $\mathcal{C}$[35]. In the case of stable ∞-category, this identification allows for an inductive treatment of hyper-cubes as illustrated by the following definition.

[35] For example, a cube in $\mathcal{C}$ of the form



can be seen as a morphism



in the ∞-category of (coherently commutative) squares in $\mathcal{C}$.

Definition. We say that a 0-cube in a stable ∞-category $\mathcal{A}$, which is nothing but an object of $\mathcal{A}$, is *biCartesian* if it is a zero object of $\mathcal{A}$. Inductively[36], we say that an $(m+1)$-cube X in a stable ∞-category $\mathcal{A}$ is *biCartesian* if its cofibre (taken in the stable ∞-category $\mathrm{Fun}(I^m, \mathcal{A})$) is a biCartesian m-cube in $\mathcal{A}$. For example, a 1-cube in $\mathcal{A}$ is biCartesian if its underlying morphism is an equivalence[37] in $\mathcal{A}$.

[36] The notion of a biCartesian hypercube can be defined directly as a certain (homotopy) colimit diagram, see Proposition 1.2.4.13 and Lemma 1.2.4.5 in [Lur17].

[37] Equivalences are the ∞-categorical versions of isomorphisms in ordinary category theory.

We are now ready to introduce the definition of the m-dimensional Waldhausen S-construction of a stable ∞-category [HM15, Pog17, Dyc17]. At a first glance the definition might seem rather technical, but it is in fact a direct generalisation of the definition of the classical Waldausen S-construction.

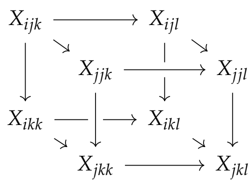
Definition. Let $\mathcal{A}$ be a stable ∞-category and m a non-negative integer. For $n \geq 0$ we denote by $\mathsf{S}^{\langle m \rangle}_\bullet(\mathcal{A})$ the full subcategory of $\mathrm{Fun}(\Delta(m,n), \mathcal{A})$ spanned by the diagrams X satisfying the following two conditions:

- For every degenerate m-simplex $\sigma \in \Delta(m,n)$ the object X_σ is a zero object of $\mathcal{A}$.
- For each non-degenerate $(m+1)$-simplex σ in Δ^n consider the $(m+1)$-cube $q\colon I^{m+1} \to \Delta(m,n)$ given by $q(v)_i = \sigma_{i+v_i}$ for each $v \in I^{m+1}$ and $i \in [m]$. Then, the induced $(m+1)$-cube

  $$X \circ q\colon I^{m+1} \to \mathcal{A}$$

  in $\mathcal{A}$ is biCartesian[38].

[38] These exactness conditions should be thought of as categorifications of the higher-dimensional Euler relations (8). For example, for $m = 2$ the cubes which are required to be biCartesian are those of the form



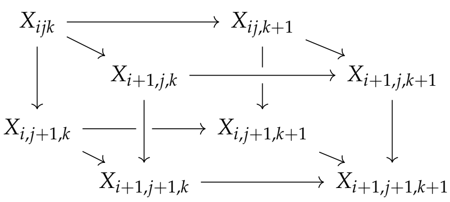
where $0 \leq i < j < k < l \leq n$; these cubes are ∞-categorical analogues of 4-term exact sequences.

Size issues aside, the ∞-categories $\mathsf{S}^{\langle m \rangle}_n(\mathcal{A})$, $n \geq 0$, assemble into a simplicial object $\mathsf{S}^{\langle m \rangle}_\bullet(\mathcal{A})$, called the *m-dimensional Waldhausen* S-*construction of* $\mathcal{A}$, which takes values in the ∞-category of stable ∞-categories and exact functors between them.

Remark. The second exactness conditions appearing in the definition of the m-dimensional Waldhausen S-construction are equivalent to the condition that the $(m+1)$-cubes $X \circ q$ are biCartesian, where

$$q(v)_i = \sigma_i + v_i, \quad i \in [m]$$

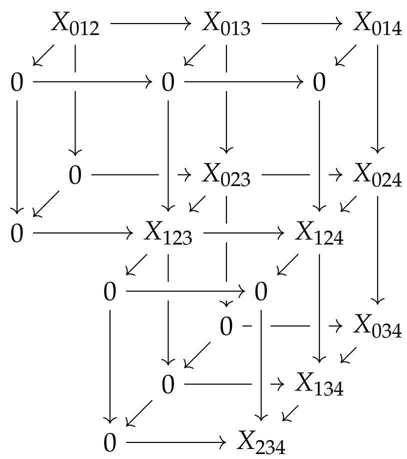
and σ ranges over all non-degenerate m-simplices in Δ^n with $\sigma_m < n$. For example, for $m = 2$ these are the cubes



where $0 \leq i < j < k < n$. These conditions should be thought of as a categorification of the higher-dimensional Auslander–Reiten relations (4).

Let us illustrate the above construction in a concrete example.

Example. Let $\mathcal{A}$ be a stable ∞-category, $m = 2$, and $n = 4$. After discarding redundant information pertaining additional zero objects, an object of $\mathsf{S}_n^{\langle 2 \rangle}(\mathcal{A})$ can be identified with a diagram of the form



(10)

in which all 'unit cubes' are biCartesian. A standard application of the theory of Kan extensions in ∞-categories shows that such a diagram is determined by its restriction to the (coherently commutative) diagram

$$\begin{array}{ccccc} X_{012} & \longrightarrow & X_{013} & \longrightarrow & X_{014} \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & X_{023} & \longrightarrow & X_{024} \\ & & \downarrow & & \downarrow \\ & & 0 & \longrightarrow & X_{034} \end{array}$$

Note that the latter diagram can be thought of as a (coherent) representation of the Auslander algebra $\mathbb{A}_3^{(2)}$ of the quiver $\mathbb{A}_3$ in the stable ∞-category $\mathcal{A}$. It is also worth noting that diagram (10) agrees with the "higher-dimensional Auslander–Reiten quiver" of the unique cluster-tilting subcategory of category of finite-dimensional $A_3^{(2)}$-modules, see for example Section 6 in [Iya11].

The following observation relates the m-dimensional Waldhausen S-construction to the m-dimensional Auslander algebras of type $\mathbb{A}$ and can be viewed as a categorification of the isomorphism (7) and is also a higher-dimensional analogue of Proposition 3. It can be proven either constructively, by means of a higher-dimensional version of the knitting algorithm[39], or using the ∞-categorical theory of Kan extensions[40].

[39] This approach is analogous to the proof of Theorem 4.6 in [GS16a]. The aforementioned knitting algorithm can be deduced, for example, from the proof of Theorem 5.27 in [IO11].

[40] This approach is analogous to the proof of Proposition 3 given for example in Lemma 1.2.2.4 in [Lur17].

Proposition 5 (Dyckerhoff–J)**.** *Let $\mathcal{A}$ be a stable ∞-category and $n \geq m \geq 1$. The restriction functor*

$$\mathsf{S}_n^{\langle m \rangle}(\mathcal{A}) \to \mathrm{Fun}_*(P(m,n),\mathcal{A})$$

is an equivalence of (stable) ∞-categories, where

$$P(m,n) := \{\sigma \in \Delta(m,n) \mid \sigma_0 = 0\}$$

and $\mathrm{Fun}_*(P(m,n),\mathcal{A})$ is the full subcategory of $\mathrm{Fun}(P(m,n),\mathcal{A})$ spanned by those functors which send degenerate m-simplices in Δ^n to zero objects in $\mathcal{A}$.

Remark. In fact, after discarding further redundant information, the poset $P(m,n)$ appearing in in Proposition 5 can be replaced by a smaller poset which models perfectly the quiver with relations of the higher-dimensional Auslander algebra $\mathbb{A}_\ell^{(m)}$, where $\ell = n - m + 1$. This fact is perhaps more transparent from the description of Iyama's higher-dimensional Auslander algebras of type $\mathbb{A}$ given in [JK16].

As explained in [IO11], the following theorem can be regarded as a higher-dimensional version of Theorem 4.

Theorem 6 (Dyckerhoff–J, Iyama–Oppermann)**.** *Let $\mathcal{A}$ be a stable ∞-category and $n \geq m \geq 1$. Let S be a slice[41] in the poset $\Delta(m,n)$. The restriction functor*

$$\mathsf{S}_n^{\langle m \rangle}(\mathcal{A}) \to \mathrm{Fun}_*(S,\mathcal{A})$$

is an equivalence of (stable) ∞-categories.

[41] The notion of a "slice" employed here is a minor modification of that introduced in [IO11] which takes into account the presence of degenerate simplices.

Remark. The proof of Theorem 6 relies on the implementation of combinatorial versions of the derived equivalences induced by higher-dimensional reflection functors[42] in the sense of [IO11]; these reflection functors rely on the operation of slice mutation also introduced in [IO11]. In slightly more detail, if S and S' are slices in $\Delta(m,n)$ which are mutation of each other, then there exists a slightly larger poset $S \diamond S'$ containing both S and S' as well as a distinguished $(m+1)$-cube (containing possibly some degenerate simplices). This larger poset allows us to realise the aforementioned reflection functors by means of equivalences of stable ∞-categories

$$\mathrm{Fun}_*(S,\mathcal{A}) \longleftarrow \mathrm{Fun}_*^{ex}(S \diamond S',\mathcal{A}) \longrightarrow \mathrm{Fun}_*(S',\mathcal{A})$$

induced by the restriction functors, where the objects of the stable ∞-category $\mathrm{Fun}_*^{ex}(S \diamond S',\mathcal{A})$ satisfy the additional requirement that their restriction along the distinguished $(m+1)$-cube in $S \diamond S'$ is a biCartesian $(m+1)$-cube in $\mathcal{A}$, see Figure 1 below. The proof of Theorem 6 is obtained by combining these equivalences with Proposition 5 and the transitivity of the mutation operation on slices proven in [IO11].

[42] Combinatorial versions of classical reflection functors are investigated in [GS16a] in the case of quivers of type $\mathbb{A}$, to which our approach is indebted, and in [GS16b] for general trees in the related framework of stable derivators.

Recollements

We conclude these notes by describing part of the rich structure of the higher-dimensional Waldhausen S-construction in terms

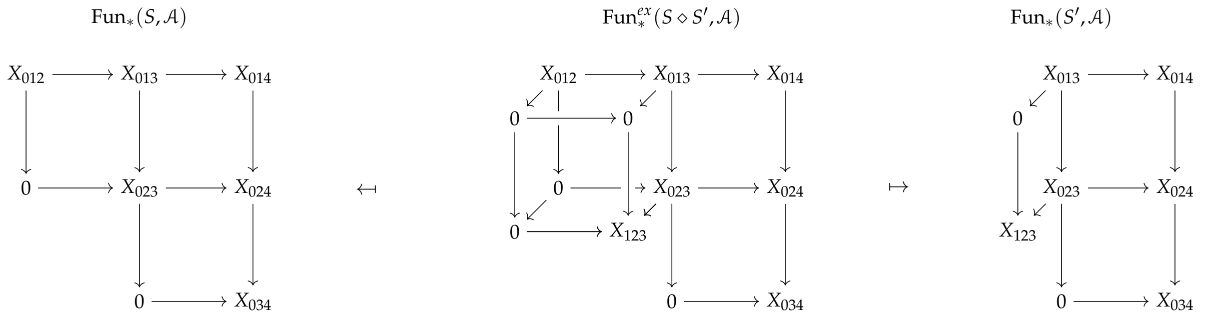


Figure 1: An example of slice mutation.

of recollements [BBD82]. We make the following general observation, which also makes the inductive nature of the m-dimensional Waldhausen S-construction of $\mathcal{A}$ readily apparent. Indeed, Proposition 7 is reminiscent of Iyama's inductive description of the higher-dimensional Auslander algebras of type $\mathbb{A}$ by means of cones of translation quivers, see Section 6 in [Iya11].

Proposition 7 (Dyckerhoff–J)**.** *Let $\mathcal{A}$ be a stable ∞-category and $n \geq m \geq 1$ integers. For each $i \in [n]$ the functor $s_i\colon \mathsf{S}^{\langle m\rangle}_n(\mathcal{A}) \to \mathsf{S}^{\langle m\rangle}_{n+1}(\mathcal{A})$ is part of a recollement of stable ∞-categories*

$$\mathsf{S}^{\langle m\rangle}_n(\mathcal{A}) \underset{\xleftarrow{\;d_{i+1}\;}}{\overset{\xleftarrow{\;d_i\;}}{\xrightarrow{\;s_i\;}}} \mathsf{S}^{\langle m\rangle}_{n+1}(\mathcal{A}) \underset{\longleftarrow}{\overset{\longleftarrow}{\longrightarrow}} \mathsf{S}^{\langle m-1\rangle}_n(\mathcal{A}).$$

In particular, the sequence of adjunctions[43]

$$d_0 \dashv s_0 \dashv d_1 \dashv s_1 \dashv \cdots \dashv s_n \dashv d_{n+1}$$

is part of a ladder of recollements in the sense of [AHKLY17].

[43] This sequence of adjunctions shows that $\mathsf{S}^{\langle m\rangle}_\bullet(\mathcal{A})$ is better thought of as a 2-simplicial stable ∞-category

$$\mathsf{S}^{\langle m\rangle}_\bullet(\mathcal{A})\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{St}_\infty$$

where $\mathbf{\Delta}$ is the 2-category obtained from Δ by taking into account the poset structure on its morphisms sets and $\mathbf{St}_\infty$ is the ∞-bicategory of stable ∞-categories, exact functors, and natural transformations, see [Dyc17] for details.

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References

[AHKLY17] Lidia Angeleri Hügel, Steffen Koenig, Qunhua Liu, and Dong Yang, *Ladders and simplicity of derived module categories*, J. Algebra **472** (2017), 15–66. MR 3584869

[BGT13] Andrew J. Blumberg, David Gepner, and Gonçalo Tabuada, *A universal characterization of higher algebraic K-theory*, Geom. Topol. **17** (2013), no. 2, 733–838. MR 3070515

[BBD82] A. A. Beĭ linson, J. Bernstein, and P. Deligne, *Faisceaux pervers*, Analysis and topology on singular spaces, I (Luminy, 1981), Astérisque, vol. 100, Soc. Math. France, Paris, 1982, pp. 5–171. MR 751966

[BT14] Petter Andreas Bergh and Marius Thaule, *The Grothendieck group of an n-angulated category*, J. Pure Appl. Algebra **218** (2014), no. 2, 354–366. MR 3120636

[Dyc17] Tobias Dyckerhoff, *A categorified Dold–Kan correspondence*, arXiv:1710.08356 (2017).

[Fao17] Giovanni Faonte, *Simplicial nerve of an A_∞-category*, Theory Appl. Categ. **32** (2017), Paper No. 2, 31–52. MR 3607208

[GJ99] Paul G. Goerss and John F. Jardine, *Simplicial homotopy theory*, Progress in Mathematics, vol. 174, Birkhäuser Verlag, Basel, 1999. MR 1711612

[Gro10] Moritz Groth, *A short course on ∞-categories*, arXiv:1007.2925 (2010).

[GS16a] Moritz Groth and Jan Šťovíček, *Abstract representation theory of Dynkin quivers of type A*, Adv. Math. **293** (2016), 856–941. MR 3474336

[GS16b] ———, *Tilting theory for trees via stable homotopy theory*, J. Pure Appl. Algebra **220** (2016), no. 6, 2324–2363. MR 3448799

[GS16c] ———, *Tilting theory via stable homotopy theory*, J. Reine Angew. Math. **Ahead of print** (2016), 1–62.

[Hap88] Dieter Happel, *Triangulated categories in the representation theory of finite-dimensional algebras*, London Mathematical Society Lecture Note Series, vol. 119, Cambridge University Press, Cambridge, 1988. MR 935124

[HM15] Lars Hesselholt and Ib Madsen, *Real algebraic k-theory*, Unpublished, April 2015.

[IO11] Osamu Iyama and Steffen Oppermann, *n-representation-finite algebras and n-APR tilting*, Trans. Amer. Math. Soc. **363** (2011), no. 12, 6575–6614. MR 2833569

[Iya07] Osamu Iyama, *Higher-dimensional Auslander-Reiten theory on maximal orthogonal subcategories*, Adv. Math. **210** (2007), no. 1, 22–50. MR 2298819

[Iya11] ———, *Cluster tilting for higher Auslander algebras*, Adv. Math. **226** (2011), no. 1, 1–61. MR 2735750

[JK16] Gustavo Jasso and Julian Külshammer, *Higher Nakayama Algebras I: Construction*, arXiv:1604.03500 (2016).

[Joy02] A. Joyal, *Quasi-categories and Kan complexes*, J. Pure Appl. Algebra **175** (2002), no. 1-3, 207–222, Special volume celebrating the 70th birthday of Professor Max Kelly. MR 1935979

[Lur09] Jacob Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. MR 2522659

[Lur17] Jacob Lurie, *Higher algebra*, May 2017, Available at the author's homepage.

[Pog17] Thomas Poguntke, *Higher segal structures in algebraic k-theory*.

[Wal85] Friedhelm Waldhausen, *Algebraic K-theory of spaces*, Algebraic and geometric topology (New Brunswick, N.J., 1983), Lecture Notes in Math., vol. 1126, Springer, Berlin, 1985, pp. 318–419. MR 802796

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