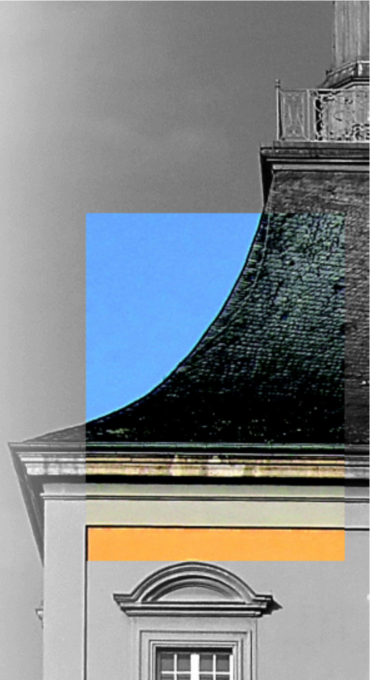
Higher-dimensional Auslander algebras of type $\mathbb{A}$ and the higher-dimensional Waldhausen S-constructions

Gustavo Jasso[1]
(joint with Tobias Dyckerhoff[2])

[1] Universität Bonn
[2] Universität Hamburg

18th International Conference on Representations of Algebras
Prague, Czech Republic, August 17, 2018



Aims for today

Relate Iyama's higher-dimensional Auslander–Reiten theory to constructions in

- algebraic topology / homotopy theory
- algebraic K-theory

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Relate Iyama's higher-dimensional Auslander–Reiten theory to constructions in

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Important perspective

Abstract representation theory in the sense of Groth and Šťovíček

The Dold-Kan nerve N($A[1]$)

$$\begin{pmatrix}
a_{00} & a_{01} & a_{02} & \cdots & a_{0,n-1} & a_{0n} \\
 & a_{11} & a_{12} & \cdots & a_{1,n-1} & a_{1n} \\
 & & \ddots & & \vdots & \vdots \\
 & & & \ddots & a_{n-2,n-1} & a_{n-2,n} \\
 & & & & a_{n-1,n-1} & a_{n-1,n} \\
 & & & & & a_{nn}
\end{pmatrix}$$

The Dold-Kan nerve $\mathrm{N}(\mathcal{A}[1])$

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} & \cdots & a_{0,n-1} & a_{0n} \\ & a_{11} & a_{12} & \cdots & a_{1,n-1} & a_{1n} \\ & & \ddots & & \vdots & \vdots \\ & & & \ddots & a_{n-2,n-1} & a_{n-2,n} \\ & & & & a_{n-1,n-1} & a_{n-1,n} \\ & & & & & a_{nn} \end{pmatrix}$$

1. For each $0 \leq i \leq n$

$$a_{ii} = 0$$

The Dold-Kan nerve $\mathrm{N}(A[1])$

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1. For each $0 \leq i \leq n$
$$a_{ii} = 0$$
2. For all $0 \leq i < j < k \leq n$
$$a_{ij} - a_{ik} + a_{jk} = 0$$
“Euler relation”

The Waldhausen S-construction

$$
\begin{array}{cccccc}
X_{00} & X_{01} & X_{02} & \cdots & X_{0,n-1} & X_{0n} \\
 & X_{11} & X_{12} & \cdots & X_{1,n-1} & X_{1n} \\
 & & \ddots & & \vdots & \vdots \\
 & & & \ddots & X_{n-2,n-1} & X_{n-2,n} \\
 & & & & X_{n-1,n-1} & X_{n-1,n} \\
 & & & & & X_{nn}
\end{array}
$$

The Waldhausen S-construction

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X_{00} & X_{01} & X_{02} & \cdots & X_{0,n-1} & X_{0n} \\
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 & & & \ddots & X_{n-2,n-1} & X_{n-2,n} \\
 & & & & X_{n-1,n-1} & X_{n-1,n} \\
 & & & & & X_{nn}
\end{array}$$

1. For all $i \in [n]$

$$X_{ii} = 0$$

The Waldhausen S-construction

$$
\begin{array}{ccccccccccc}
X_{00} & \to & X_{01} & \to & X_{02} & \to & \cdots & \longrightarrow & X_{0,n-1} & \longrightarrow & X_{0n} \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
 & & X_{11} & \to & X_{12} & \to & \cdots & \longrightarrow & X_{1,n-1} & \longrightarrow & X_{1n} \\
 & & & & \downarrow & & & & \downarrow & & \downarrow \\
 & & & & \ddots & & & & \vdots & & \vdots \\
 & & & & & & & & \big\downarrow & & \big\downarrow \\
 & & & & & & \ddots & & X_{n-2,n-1} & \to & X_{n-2,n} \\
 & & & & & & & & \downarrow & & \downarrow \\
 & & & & & & & & X_{n-1,n-1} & \to & X_{n-1,n} \\
 & & & & & & & & & & \downarrow \\
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 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
 & & X_{11} & \to & X_{12} & \to & \cdots & \to & X_{1,n-1} & \to & X_{1n} \\
 & & & & \downarrow & & & & \downarrow & & \downarrow \\
 & & & & \ddots & & & & \vdots & & \vdots \\
 & & & & & & & & \downarrow & & \downarrow \\
 & & & & & & \ddots & & X_{n-2,n-1} & \to & X_{n-2,n} \\
 & & & & & & & & \downarrow & & \downarrow \\
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 & & & & & & & & & & \downarrow \\
 & & & & & & & & & & X_{nn}
\end{array}
$$

1. For all $i \in [n]$

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2. For all $0 \leq i < j < k \leq n$

$$
\begin{array}{ccc}
X_{ij} & \longrightarrow & X_{ik} \\
\downarrow & & \downarrow \\
X_{ii} & \longrightarrow & X_{jk}
\end{array}
$$

is an exact triangle

The Waldhausen S-construction

$$
\begin{array}{ccccccccccc}
X_{00} & \to & X_{01} & \to & X_{02} & \to & \cdots & \longrightarrow & X_{0,n-1} & \longrightarrow & X_{0n} \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
 & & X_{11} & \to & X_{12} & \to & \cdots & \longrightarrow & X_{1,n-1} & \longrightarrow & X_{1n} \\
 & & & & \downarrow & & & & \downarrow & & \downarrow \\
 & & & & \ddots & & & & \vdots & & \vdots \\
 & & & & & & & & \downarrow & & \downarrow \\
 & & & & & & \ddots & & X_{n-2,n-1} & \to & X_{n-2,n} \\
 & & & & & & & & \downarrow & & \downarrow \\
 & & & & & & & & X_{n-1,n-1} & \to & X_{n-1,n} \\
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is an ~~exact triangle~~

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\downarrow & \square & \downarrow \\
X_{ii} & \longrightarrow & X_{jk}
\end{array}
$$

is an ~~exact triangle~~ cofibre sequence

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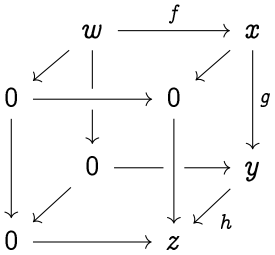
biCartesian cubes in stable ∞-categories

$$I = \{0 \to 1\} \qquad X\colon I^{m+1} \to \mathcal{A} \qquad (m+1)\text{-}\textit{cube}$$

biCartesian cubes in stable ∞-categories

$I = \{0 \to 1\}$ $\qquad$ $X \colon I^{m+1} \to \mathcal{A}$ $\qquad$ $(m+1)$-*cube*

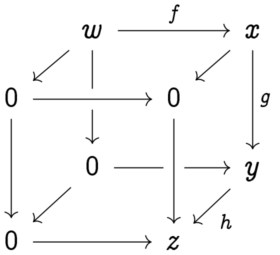


“$\Longleftrightarrow$”

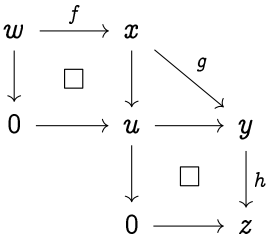
(homotopy) biCartesian

biCartesian cubes in stable ∞-categories

$I = \{0 \to 1\}$ $\quad X : I^{m+1} \to \mathcal{A}$ $\quad (m+1)$-*cube*



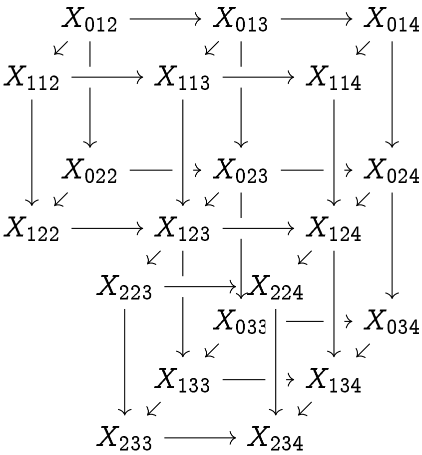
“$\Longleftrightarrow$”



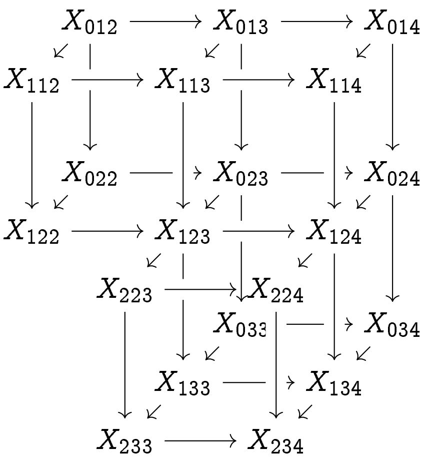
(homotopy) biCartesian

$\operatorname{cofib}(f) \cong u \cong \operatorname{fib}(h)$

The Waldhausen $S^{\langle 2\rangle}$-construction



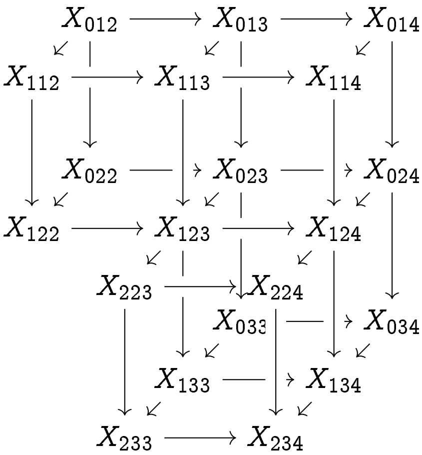
The Waldhausen $S^{\langle 2 \rangle}$-construction



1. For all $0 \leq i < j < n$

$$X_{iij} = X_{ijj} = 0$$

The Waldhausen S$^{\langle 2 \rangle}$-construction



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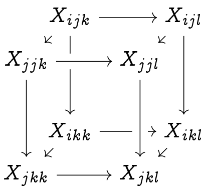
2. For all $0 \leq i < j < k < l \leq n$



is (homotopy) biCartesian.

$\mathsf{S}^{\langle m\rangle}(\mathcal{A})_n \xrightarrow{\sim} \mathsf{Fun}_*(P(m,n),\mathcal{A})$

$$
\begin{array}{ccccc}
X_{012} & \longrightarrow & X_{013} & \longrightarrow & X_{014} \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & X_{023} & \longrightarrow & X_{024} \\
 & & \downarrow & & \downarrow \\
 & & 0 & \longrightarrow & X_{034}
\end{array}
$$

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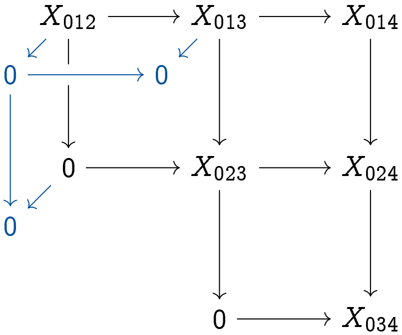
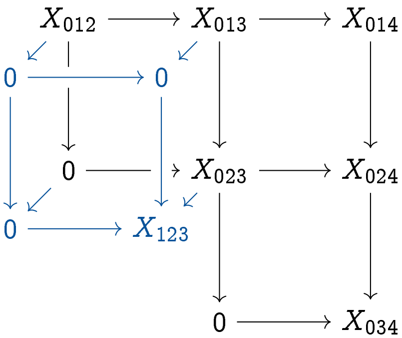
$\mathsf{S}^{\langle m\rangle}(\mathcal{A})_n \xrightarrow{\sim} \mathsf{Fun}_*(P(m,n),\mathcal{A})$



$$\mathsf{S}^{\langle m \rangle}(\mathcal{A})_n \xrightarrow{\sim} \mathsf{Fun}_*(P(m,n), \mathcal{A})$$



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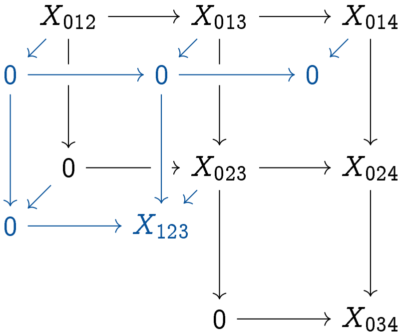

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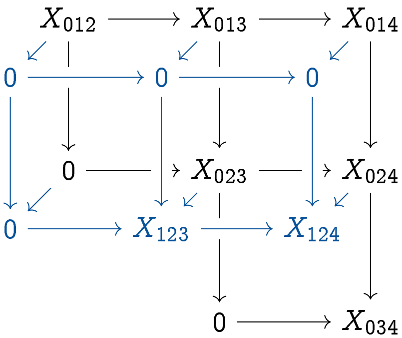

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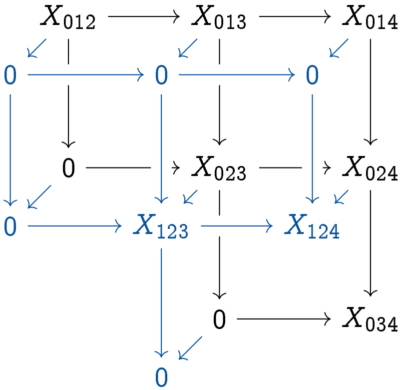

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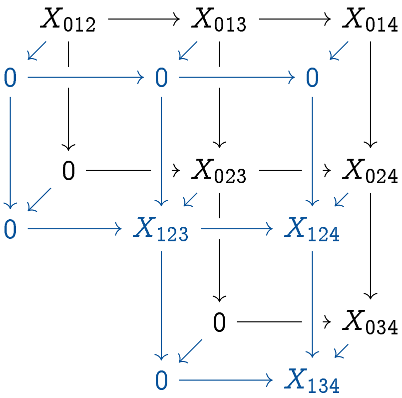

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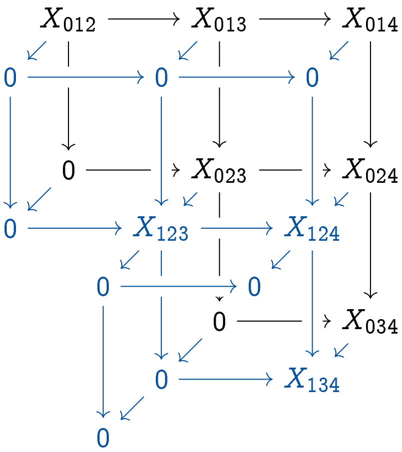
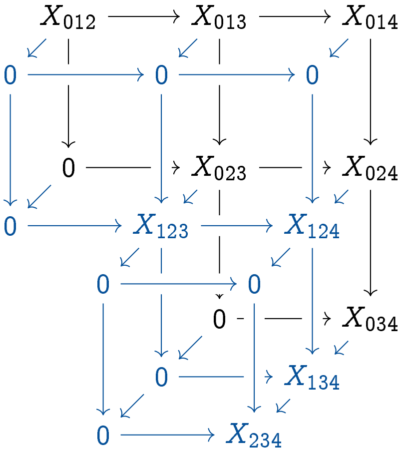
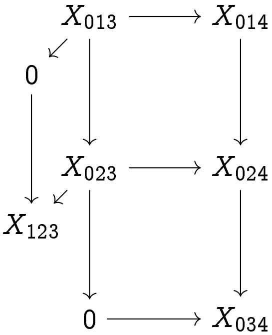


$S^{\langle m\rangle}(\mathcal{A})_n \xrightarrow{\sim} \mathrm{Fun}_*(P(m,n),\mathcal{A})$



$\mathsf{S}^{\langle m \rangle}(\mathcal{A})_n \xrightarrow{\sim} \mathsf{Fun}_*(S, \mathcal{A})$



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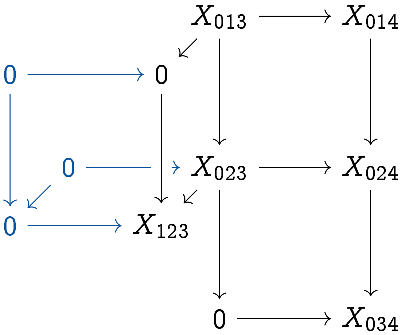

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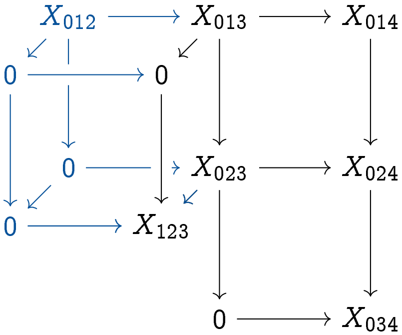

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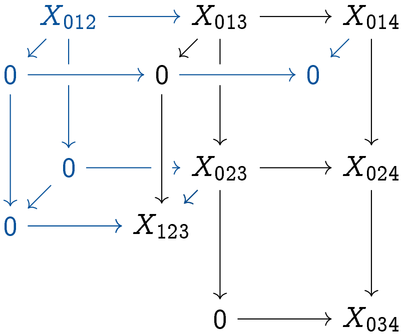

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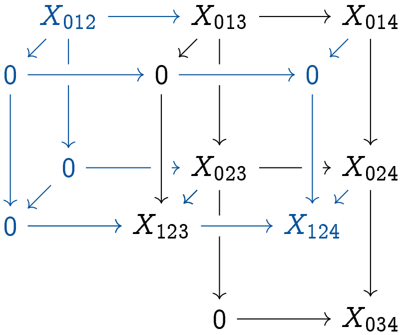

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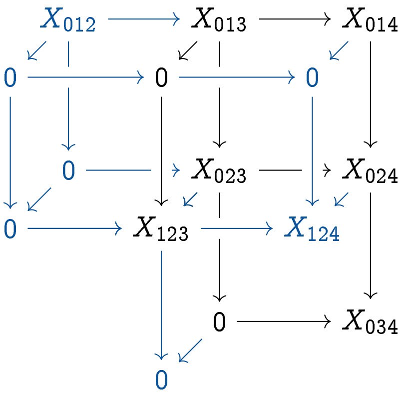

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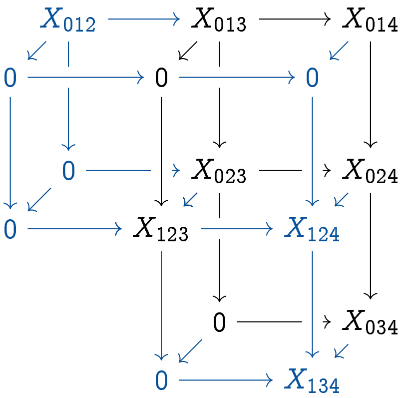

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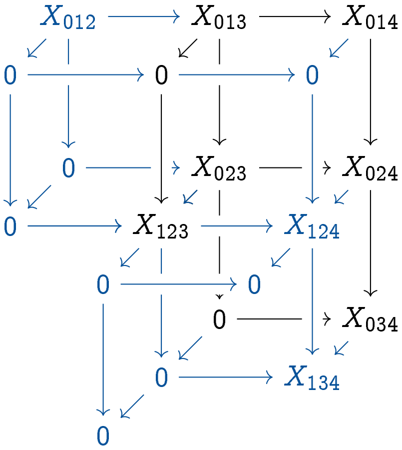

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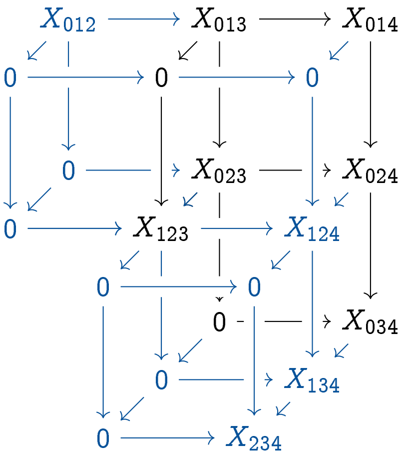

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Recollements

$$\mathsf{S}_n^{\langle m\rangle}(\mathcal{A}) \;\underset{\xleftarrow{\;d_{i+1}\;}}{\overset{\xleftarrow{\;d_i\;}}{\xrightarrow{\;s_i\;}}}\; \mathsf{S}_{n+1}^{\langle m\rangle}(\mathcal{A}) \;\underset{\longleftarrow}{\overset{\longleftarrow}{\longrightarrow}}\; \mathsf{S}_n^{\langle m-1\rangle}(\mathcal{A})$$

$$\cdots \dashv d_0 \dashv s_0 \dashv d_1 \dashv s_1 \dashv \cdots \dashv d_n \dashv s_n \dashv d_{n+1} \dashv \cdots$$

Thank you for your attention!

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