The Extended Affine Lie Algebra Associated with a Connected Non-negative Unit Form

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Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a connected non-negative unit form unit form.

We associate with q a matrix C given by

$$C_{ij} = q(c_i + c_j) - q(c_i) - q(c_j).$$

Where $\{c_1, \ldots, c_n\}$ denotes the standard basis of $\mathbb{Z}^n$.

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Construction [Barot, Kussin, Lenzing]

Let FL be the free Lie algebra with $3n$ generators

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The Algebra $\tilde{G}(q)$

Let $\operatorname{corank} q \in \mathbb{Z}_{\geqslant}0$ and let $G(q)$ be the quotient of FL by the ideal generated by the following generalized Serre relations:

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whenever $q(\sum_{k=1}^{t} \varepsilon_k c_k) > 1$, for $\varepsilon_k = \pm 1$ and $i_k \in \{1, \ldots, n\}$.

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One cannot define a *non-degenerate* symmetric invariant bilinear form on $\tilde{G}(q)$.

How do we fix it?

The algebra $\tilde{G}(q)$ is an H^*-graded H-module, hence it contains a unique maximal ideal I with respect to $I \cap H = \{0\}$.

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Main result

Theorem

Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a connected non-negative unit form with associated root system R. Then the Lie algebra $E(q)$ is a centerless tame EALA with root system R.

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Remark

In order to show that $E(q)$ is an EALA it is useful to introduce an alternative construction of this algebra.

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Construct a Lie algebra $\hat{E}(q)$

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$$\hat{E}(q) \xleftarrow{\;p\;} \tilde{G}(q)$$

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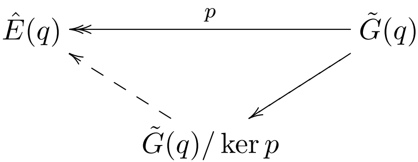

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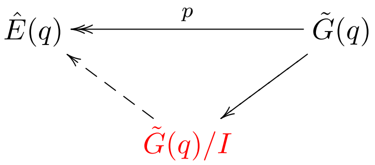

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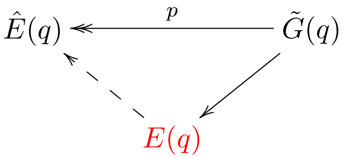

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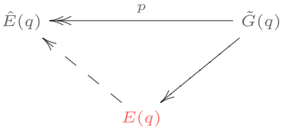

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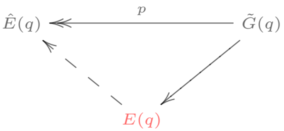

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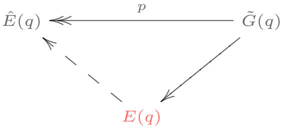

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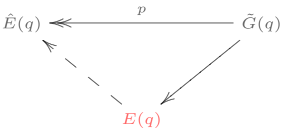

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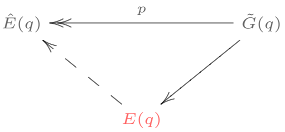

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The Algebra $\hat{E}(q)$

For $\alpha \in R$ we let

$$\hat{E}(q)_\alpha = \begin{cases} \mathbb{C}e_\alpha & \text{if } \alpha \in R^{\times}, \\ \mathbb{C}^n / \operatorname{rad} q & \text{if } \alpha \in R^0 \setminus \{0\}, \\ \mathbb{C}^n \oplus (\operatorname{rad} q)^* & \text{if } \alpha = 0. \end{cases}$$

An let $\hat{E}(q) = \bigoplus_{\alpha \in R} \hat{E}(q)_\alpha$ as a vector space.

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Thanks for your attention!

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