Exercise Sheet 7

For the following exercise, you may use the well-known fact from set theory that the set of real numbers can be constructed as the set of all Dedekind cuts on $\mathbb{Q}$.

Problem 7.1 (4 points)

Let k be an Archimedian-ordered field. Then there exists a unique, order-preserving homomorphism $k \to \mathbb{R}$.

Problem 7.1 (2 points)

Every non-Archimedian-ordered field has a free Dedekind cut.

Definition 7.1
Let $k[[^P X]]$ be the ring of formal series of the form

$$\sum_{l \in \mathbb{Q}, l \geq 0} f_l X^l$$

*such that there is a natural number $n \in \mathbb{N}$ with $f_l = 0$ whenever $n \cdot l \notin \mathbb{N}$. We call this ring the ring of **Puiseux series**.*
Similarly, let $k[[^P X]][X^{-1}]$ be the ring of formal series of the form form

$$\sum_{l \in \mathbb{Q}} f_l X^l$$

such that there are natural numbers $m, n \in \mathbb{N}$ with $f_l = 0$ whenever $n \cdot l \notin \mathbb{N}$ or $l \leq m$.

The aim of the rest of this sheet is to prove the following theorem.

Theorem 7.2
Let K be an algebraically closed field of characteristic 0, then $K[[^P X]][X^{-1}]$ is algebraically closed.

Definition 7.3
We can define a valuation on $k[[^P X]][X^{-1}]$. Namely, let $f \in k[[^P X]][X^{-1}]$, then

$$\upsilon(f) = \min\{l \mid f_l \neq 0\}.$$

Having this we can define the Newton polygons.

Definition 7.4
*The **Newton-Polygon** of $P \in k[[^P X]][X^{-1}][T] \setminus \{0\}$ with $P(0) \in k[[^P X]][X^{-1}] \setminus \{0\}$ and $\deg_T(P) := d$ is the (unique) function $n_P : [0, d] \to \mathbb{R}$ with the following properties*

1. *There are integers* $0 = i_0 < \cdots < i_k = d$ *s.t.* $n_P(t) = n_P(i_j) + s_{j+1} \cdot (t - i_j)$ *for* $t \in [i_j, i_{j+1}]$ *with rational numbers* $s_0 < \cdots < s_k$.
2. $\upsilon(P_i) \geq n_P(i)$ *for* $i \in \mathbb{N} \cap [0, d]$ *with equality whenever* $i \in \{i_0, ..., i_k\}$.

Here P_i *is the i-th coefficient of P. The numbers* s_i *are called the slope and* $w_j = i_j - i_{j-1}$ *the width of the* j*-th segment.* $[i_0, i_1]$ *is called the initial segment.*

Problem 7.3 (2 points)

Show that for all $P \in k[[^PX]][X^{-1}][T] \setminus \{0\}$ with $P(0) \in k[[^PX]][X^{-1}] \setminus \{0\}$ the Newton-Polygon exists and is unique.

Problem 7.4 (3 points)

Assume $0 < w := i_1 < i_2 < \cdots < i_k = l$ and $\sigma < s_2 < s_3 < \cdots < s_k$.

1. $\upsilon(p_j) \geq \upsilon(p_{i_{k-1}}) + s_k(j - i_{k-1})$ for $2 \leq j \leq k$ and $i_{l-1} \leq l \leq i_j$.
2. $\upsilon(p_l) \geq \upsilon(p_w) - \sigma(l - w)$ for $0 \leq l \leq w$.

Then

1. The width of the initial segment of n_p is $\leq w$, its slope is $\leq \sigma$.
2. On $[i_{k-1}, i_k]$, n_p is given by the RHS in 1. and $n_p(d) = r(p_d)$.

Problem 7.5 (3 points)

Let $P \in k[[X]][X^{-1}, T]$ of degree $d > 0$ such that the slope σ of the initial segment of n_p is an integer. Furthermore, write $P(T, P) = \sum_{i=0}^{d} \sum_{j=r(p_i)}^{\infty} p_{i,j} T^i X^j$. Define

$$\widehat{P}(T) = \sum_{i=0}^{w} p_{i,r(p_j)+\sigma i} T^j \in k[T].$$

Let l/k be a field extension and let $\lambda \in l$ be a root of $\widehat{P}$ of multiplicity v. Let $Q(T, X) = P(T + \lambda T^{-s}, X) \in l[[X]][X^{-1}, T]$. Then the width of the initial segment of the Newton polygon of Q is $\leq v$ and its slope is $< d$.

Problem 7.6 (3 points)

Let $w \in \mathbb{N}$ such that for any field k of characteristic 0, any $Q \in k[[^pX]][X^{-1}, T]$ for which the width of the initial slope of the n_Q is $< w$ has a zero in $l[[^pX]][X^{-1}, T]$ for a finite extension l/k. Let k have characteristic 0 and $P \in k[[X]][X^{-1}, T]$ such that the initial segment of n_P has slope $s \in \mathbb{Z}$ and width w. Assume that there is no finite field extension l of k such that P has a zero in in $l[[^PX]][X^{-1}]$.

1. All $(\upsilon(p_i))_{i=0}^{w}$ are on segment: $\upsilon(p_i) = \upsilon(p_0) + si = n_P(i)$.
2. There is $\lambda \in k$ such that the width of the initial segment of n_Q, where $Q(T, X) = P(T + \lambda X^{-s}, X)$, is equal to w, the slope of that segment is an integer $< s$, and $n_Q = n_P$ on $[w, d]$, $d = \deg_T P = \deg_T Q$.

Problem 7.7 (3 points)

Let k be a field of characteristic 0 and $P \in k[[{}^{p}X]][X^{-1}, T]$ be a polynomial of T-degree $d > 0$. Then there is a field extension l of k such that P has a zero in $l[[{}^{p}X]][X^{-1}]$.

Problem 7.8 (1 point)

Prove Theorem 7.2.

Solutions should be submitted in the exercise session on Wednesday, November 29. One of the 21 possible points from this sheet is a bonus point, which does not count in the calculation of the $\geq 50\%$ lower bound of points needed to pass the exercises.