Exercises to „Algebraic geometry I“, 8

Generally, by a subsheaf of $\mathcal{G}$ we mean a sheaf $\mathcal{H}$ such that $\mathcal{H}(U) \subseteq \mathcal{G}(U)$ for all open subsets U is a subset which is a subobject (subgroup, subring, sub-$\mathcal{R}(U)$-module depending on the situation) of $\mathcal{G}(U)$. We will assume that after making a choice for the direct limits $\mathcal{G}_x$ of $\mathcal{G}(U)$ over open neighbourhoods of x, which are only unique up to unique isomorphism, the similar choice for the stalks $\mathcal{H}_x$ of $\mathcal{H}$ has been made in such a way that they are subsets (subgroups, etc.) of $\mathcal{G}_x$. The kernel of a morphism of sheaves of abelian groups is

$$\operatorname{Ker}\big(\mathcal{F} \xrightarrow{\phi} \mathcal{G}\big)(U) = \operatorname{Ker}\big(\mathcal{F}(U) \xrightarrow{\phi} \mathcal{G}(U)\big).$$

Exercise 1 (1 point). *Show that* $\operatorname{Ker}(\phi)_x \cong \operatorname{Ker}\big(\mathcal{F}_x \xrightarrow{\phi_x} \mathcal{G}_x\big)$.

Note that it may be useful to use results of the fifth exercise sheet here.

Exercise 2 (6 points). *Let $\mathcal{R}$ be a sheaf of rings, $\mathcal{M}$ a sheaf of modules over $\mathcal{R}$ and $\mathcal{N} \subseteq \mathcal{M}$ a subsheaf of modules. Show that there exists a morphism $\mathcal{M} \xrightarrow{\pi} \mathcal{Q}$ of sheaves of modules over $\mathcal{R}$ inducing surjective maps on stalks and such that $\mathcal{N} = \operatorname{Ker}(\pi)$. Moreover, show that*

$$\begin{aligned} (1) \qquad \operatorname{Hom}_{\mathcal{R}-\mathrm{Mod}}(\mathcal{Q}, \mathcal{T}) &\to \big\{ g \in \operatorname{Hom}_{\mathcal{R}-\mathrm{Mod}}(\mathcal{M}, \mathcal{T}) \;\big|\; \mathcal{N} \subseteq \operatorname{Ker}(g) \big\} \\ f &\to g = f\pi \end{aligned}$$

is bijective for any such π. In particular (by Yoneda), $\mathcal{M} \xrightarrow{\pi} \mathcal{Q}$ is unique up to unique isomorphism.

The usual notation for this $\mathcal{Q}$ is $\mathcal{M}/\mathcal{N}$.

Exercise 3 (3 points). *In the case of a sheaf of ideals $\mathcal{I} \subseteq \mathcal{R}$, show that $\mathcal{R}/\mathcal{I}$ has a unique structure of a sheaf of rings such that $\mathcal{R} \xrightarrow{\pi} \mathcal{R}/\mathcal{I}$ is a morphism of sheaves of rings.*

Remark 1. In the following exercises to this lecture it may be used without proof that the version of (1) for morphisms of sheaves of rings also holds.

Exercise 4 (2 points). *Let $K \xrightarrow{k} X$ be a closed immersion of topological spaces. Show that the morphism (1.2.4) $\big(k_*\mathcal{G}\big)_{k(x)} \xrightarrow{k_x^*} \mathcal{G}_x$ constructed in Remark 1.2.3 in the lecture is an isomorphism.*

Exercise 5 (8 points). *Let X be a locally ringed space, $\mathcal{I} \subseteq \mathcal{O}_X$ a sheaf of ideals. Let $K = \mathcal{V}(\mathcal{I}) \xrightarrow{k} X$ be given as follows: As a topological space,*

$$V = \big\{ x \in X \;\big|\; 1 \notin \mathcal{I}_X \big\}$$

equipped with the induced topology, and k is the embedding of that subset. For $x \in K$, let

$$\mathcal{O}_{K,[x]} = \mathcal{O}_{X,x} \Big/ \mathcal{I}_x$$

and let $\mathcal{O}_K(U)$ be the set of all $(f_x)_{x\in U} \in \prod_{x\in U} \mathcal{O}_{K,[x]}$ such that for every $y \in U$ there are an open neighbourhood V of y in X and $\phi \in \mathcal{O}_X(V)$ such that for $x \in U \cap V$, f_x equals the image of ϕ under

$$\mathcal{O}_X(V) \to \mathcal{O}_{X,x} \to \mathcal{O}_{X,x} \Big/ \mathcal{I}_x = \mathcal{O}_{K,[x]}.$$

Moreover, let $\mathcal{O}_X(V) \xrightarrow{k^*} (k_*\mathcal{O}_K)(V) = \mathcal{O}_K(V \cap K)$ send $\phi \in \mathcal{O}_X(V)$ to the family of those images, x running over $V \cap K$. Show the following:

- *X is a locally ringed space, and $\mathcal{O}_{K,x}$ maps isomorphically to $\mathcal{O}_{K,[x]}$.*
- *k is a morphism of locally ringed spaces and there is an isomorphism $k_*\mathcal{O}_K \cong \mathcal{O}_X \big/ \mathcal{I}$ of $\mathcal{O}_X$-algebras. In particular, the kernel of $\mathcal{O}_X \xrightarrow{k^*} k_*\mathcal{O}_K$ equals $\mathcal{I}$.*
- *If T is a locally ringed space, we have a canonical bijection*

$$\begin{aligned}\mathrm{Hom}_{\mathrm{LRS}}(T,K) &\to \Big\{ g \in \mathrm{Hom}_{\mathrm{LRS}}(T,X) \,\Big|\, \mathcal{I} \subseteq \mathrm{Ker}\big(\mathcal{O}_X \xrightarrow{g^*} g_*\mathcal{O}_T\big) \Big\} \\ f &\to g = kf.\end{aligned}$$

Remark 2. It has been pointed out in the lecture that K is closed. This fact may thus be used without proving it. The proofs that $\mathcal{O}_K$ is a sheaf of rings and k^* a morphism of sheaves of rings may also be omitted. The same holds for the continuity of k.

Remark 3. The construction of $\mathcal{O}_K$ may also be formulated as follows: The functor k_* is an equivalence of categories from the category of sheaves of rings on K to the full subcategory of sheaves $\mathcal{R}$ of rings on X such that $\mathcal{R}(U)$ is the null ring when $U \cap K = \emptyset$. An inverse functor is given by the inverse image functor restricted to that subcategory. One puts $\mathcal{O}_K = k^*\mathcal{O}_X \big/ \mathcal{J}$. In the formulation of the above exercise, as well as in the lecture, I preferred a more down to the earth formulation in which a description of the stalk was given first, followed by a direct description of sections of the sheaf as families of elements of the stalk satisfying a coherence condition. However, approaches like the one described in this remark may be more elegant when they are possible and when one has reached some comfortibility with sheaf theoretic constructions.

Solutions should be submitted Friday, December 15 in the lecture.