Exercises to „Algebraic geometry I“, 6

DEFINITION 1. Let $X \xleftarrow{\xi} R \xrightarrow{\upsilon} Y$ be morphisms in a category $\mathcal{A}$. We say that an object P of $\mathcal{A}$ with morphisms $X \xrightarrow{i_X} P \xleftarrow{i_Y} Y$ is a *cofibre product* of X and Y with respect to R if $i_X\xi = i_Y\upsilon$ and for every pair of morphisms $X \xrightarrow{t_X} T \xleftarrow{t_Y} Y$ such that $t_X\xi = t_Y\upsilon$, there is a unique morphism $P \xrightarrow{\tau} T$ such that $t_X = \tau i_X$ and $t_Y = \tau i_Y$.

REMARK 1.
- In other words, cofibre products are dual fibre products or fibre products in $\mathcal{A}^{\mathrm{op}}$.
- In topological situations where at least one of the i-maps is a closed immersion, cofibre products are pushouts. For this reason they are sometimes encountered under this name.

EXERCISE 1 (6 points). *Let R be a ring, A and B (commutative) R-algebras. Show that there is a unique ring structure on $A \otimes_R B$ such that $(a \otimes b) \times (\alpha \otimes \beta) = (a\alpha) \otimes (b\beta)$.*

It follows from this formula that $A \xrightarrow{i_A} A \otimes_R B \xleftarrow{i_B} B$, $i_A(a) = a \otimes 1$, $i_B(b) = 1 \otimes b$ are ring homomorphisms.

EXERCISE 2 (4 points). *Show that $A \xrightarrow{i_A} A \otimes_R B \xleftarrow{i_B} B$ is a cofibre product of A and B with respect to R in the category of rings.*

REMARK 2. In both exercises, the universal properties of the tensor product for R-bilinear maps $A \times B \to T$ or for $\mathbb{Z}$-bilinear maps $A \times B \xrightarrow{f} T$ such that $f(ra, b) = f(a, rb)$ holds for all $r \in R$, $a \in A$, $b \in B$ should be used as the characterization of tensor products. It follows from the first universal property that elements of the form $a \otimes b$ generate $A \otimes_R B$ as an R-module. Since $r \cdot (a \otimes b) = (ra) \otimes b = a \otimes (rb)$, or because of the uniqueness part of the second universal property, they also generate $A \otimes B$ as an abelian group. These consequences of the universal properties may also be used. Because of this, many of the desired properties of the products or of ring homomorphisms only have to be verified when the arguments are pure tensor products. As I typically do in the lecture, it is sufficient to point this out out after completely working it out in a representative special case.

DEFINITION 2. Let $X \xrightarrow{p} Y$ be a morphism in a category $\mathcal{A}$. We say that p is an effective epimorphism if

$$\begin{aligned}\mathrm{Hom}_{\mathcal{A}}(Y, T) \to \big\{g \in \mathrm{Hom}_{\mathcal{A}}(X, T) \;\big|\; & ga = gb \text{ whenever } S \underset{b}{\overset{a}{\rightrightarrows}} X \text{ is a} \\ & \text{pair of morphisms in } \mathcal{A} \text{ with } pa = pb\big\} \\ f \to fp \qquad & \end{aligned}$$

is bijective for any object T of $\mathcal{A}$.

REMARK 3.
- It follows from the uniqueness part of the above bijection that effective epimorphisms are epimorphisms.
- If p is a monomorphism, then the condition on the pair of morphisms $S \overset{a}{\underset{b}{\rightrightarrows}} X$ implies $a = b$, and the condition on g becomes trivial. By Yoneda it follows that a monomorphism is an effective epimorphism if and only if it is an isomorphism. The notion thus cures some of the problems with mono- and epimorphisms: In some categories (eg, the category of rings or the category of Banach spaces with bounded linear maps as morphisms) a morphism may be a monomorphism and an epimorphism but fail to be an isomorphism.

EXERCISE 3 (5 points).
- *Show that coequalizers are effective epimorphisms.*
- *Show that in a category where any diagram $X \xrightarrow{\xi} S \xleftarrow{\upsilon} Y$ has a fibre product, a morphism is an effective epimorphism if and anly if it can be represented as a coequalizer of a pair of morphisms.*
- *Show that the effective epimorphisms in the category of rings are the surjective ring homomorphisms.*

REMARK 4. Of course, the dual notion of "effective monomorphism" also exists and the dual assertions to the first two points also hold.

EXERCISE 4 (5 points). *Let $\mathcal{A}$ be a category where any diagram $A \xrightarrow{\alpha} S \xleftarrow{\beta} B$ has a fibre product. Show that any pair $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ of morphisms in $\mathcal{A}$ for which there exists a morphism $Y \xrightarrow{\upsilon} S$ in $\mathcal{A}$ such that $\upsilon f = \upsilon g$ has an equalizer in $\mathcal{A}$.*

Solutions should be submitted Friday, December 1 in the lecture.