Exercises to „Algebraic geometry I“, 5

Exercise 1 (4 points). *Let $\mathcal{A} \xrightarrow{L} \mathcal{B}$ be left adjoint to the covariant functor $\mathcal{B} \xrightarrow{R} \mathcal{A}$. Show the following:*

- *L preserves initial objects in the sense that when I is an initial object of $\mathcal{A}$, $L(I)$ is an initial object of $\mathcal{B}$.*
- *R preserves equalizers in the sense that whenever $K \xrightarrow{\kappa} X$ is an equalizer of $X \underset{b}{\overset{a}{\rightrightarrows}} Y$ in $\mathcal{B}$, then $R(K) \xrightarrow{R(\kappa)} R(X)$ is an equalizer of $R(X) \underset{R(b)}{\overset{R(a)}{\rightrightarrows}} Y$ in $\mathcal{A}$.*

Moreover, formulate and show the fact that R preserves arbitrary finite or infinite products in $\mathcal{A}$.

Remark 1. Dually, R preserves final objects and L preserves coequalizers and arbitrary *coproducts* (i. e., products in $\mathcal{A}^{\mathrm{op}}$).

Throughout the remaining exercises, let $\mathfrak{A}$ be one of the categories of sets, commutative or not necessarily commutative rings, or abelian groups or not necessarily abelian groups.

Exercise 2 (3 points). *Let $\mathfrak{S}$ be the category of $\mathfrak{A}$-valued sheaves on X and $\mathcal{F} \underset{b}{\overset{a}{\rightrightarrows}} \mathcal{G}$ be a pair of morphisms with common source and target in $\mathfrak{A}$. Show that an equalizer $\mathcal{K} \to \mathcal{F}$ of this pair is given by*

$$\mathcal{K}(U) = \big\{ f \in \mathcal{F}(U) \bigm| a(f) = b(f) \big\},$$

equipped with the restriction(s) of the required group or ring operation(s).

Definition 1. Let $X \xrightarrow{\xi} S \xleftarrow{\upsilon} Y$ be morphisms in a category $\mathcal{A}$. A *fibre product of X and Y over S* is an object P of $\mathcal{A}$ with a pair of morphisms $X \xleftarrow{\pi_X} P \xrightarrow{\pi_Y} Y$ in $\mathcal{A}$ satisfying $\xi\pi_X = \upsilon\pi_Y$ and enjoing the universal property for such objects: If T is any object of $\mathcal{A}$ together with morphisms $X \xleftarrow{\tau_X} T \xrightarrow{\tau_Y} Y$ such that $\xi\tau_X = \upsilon\tau_Y$, then there is a unique morphism $T \xrightarrow{t} P$ such that $\tau_X = \pi_X t$ and $\tau_Y = \pi_Y t$. Fibre products are often denoted $P = X \times_S Y$.

Exercise 3 (2 points). *Show that fibre products in $\mathfrak{A}$ exist and are given by*

$$X \underset{S}{\times} Y = \big\{ (x,y) \in X \times Y \bigm| \xi(x) = \upsilon(y) \big\}$$

where the required algebraic operation(s) is/are given by

$$(x,y) \star (\xi, \upsilon) = (x \star \xi, y \star \upsilon).$$

EXERCISE 4 (2 points).
- *Show that a morphism $X \to Y$ in a category $\mathcal{A}$ is a monomorphism if and only if*

$$X \xleftarrow{\mathrm{Id}_X} X \xrightarrow{\mathrm{Id}_X} X$$

is a fibre product of X with itself over Y.
- *Let $\mathcal{A} \xrightarrow{F} \mathcal{B}$ be a covariant functor preserving fibre products: If $X \xleftarrow{\pi_X} P \xrightarrow{\pi_Y} Y$ is a fibre product of X and Y over P in $\mathcal{A}$, then $F(X) \xleftarrow{F(\pi_X)} F(P) \xrightarrow{F(\pi_Y)} F(Y)$ is a fibre product of $F(X)$ and $F(Y)$ over $F(P)$ in $\mathcal{B}$.*

  Show that F preserves monomorphisms.

EXERCISE 5 (6 points). *Let $\mathfrak{S}$ be the category of $\mathfrak{A}$-valued sheaves on X. Show that fibre products in $\mathfrak{S}$ exist and are given by*

$$(\mathcal{X} \underset{\mathcal{S}}{\times} \mathcal{Y})(U) = \mathcal{X}(U) \underset{\mathcal{S}(U)}{\times} \mathcal{Y}(U),$$

where the fibre product on the right hand side is taken in the target category. Moreover, show that the stalk functors $\mathfrak{S} \to \mathfrak{A}$, $\mathcal{F} \to \mathcal{F}_x$ preserve fibre products.

REMARK 2.
- Note that it can be shown as in the first exercise that right adjoint functors preserve fibre products. But the stalk functor typically has no left adjoint.
- It can be shown in a similar way that the stalk functors preserve equalizers.
- If follows from the previous two exercises that a monomorphism in $\mathfrak{S}$ induces monomorphisms on stalks. Together with the result proved in the lecture this shows that the property of being a monomorphism in $\mathfrak{S}$ may be tested on stalks.
- Note that a morphism in $\mathfrak{A}$ is a monomorphism if and only if the underlying map is injective. This can be seen in various ways, for instance by applying exercises 4 and 3 or by first verifying it for sets, then using exercise 3 on the previous sheet combined with the fact that the fact that the forgetful functor from $\mathfrak{A}$ to sets has a left adjoint given by the functors of forming the free abelian group, free group, and (commutative or non-commutative) polynomial ring over $\mathbb{Z}$ in the appropriate set of generators or variables.

EXERCISE 6 (1 point). *Let $R \overset{a}{\underset{b}{\rightrightarrows}} S$ be a pair of local ring morphisms with common source and target. Show that $\mathrm{Coeq}(a, b)$ is a local ring if and only if a and b define the same morphism on residue fields.*

REMARK 3. Note that $S \to \mathrm{Coeq}(a, b)$ is a local ring morphism in this case, for instance by fact 1.3.3 of the lecture, as it is surjective.

EXERCISE 7 (2 points). *Let X be a locally ringed space and $U \xrightarrow{j} X$ the inclusion of an open subset of X. Show that we have a bijection*

$$\begin{aligned}\mathrm{Hom}_{\mathrm{LRS}}(T, U) &\to \left\{ g \in \mathrm{Hom}_{\mathrm{LRS}}(T, X) \mid g_{\mathrm{Top}}(T) \subseteq U \right\}\\ f &\to g = jf,\end{aligned}$$

where LRS *is the category of locally ringed spaces.*

Solutions should be submitted Friday, November 24 in the lecture.