Exercises to „Algebraic geometry I“, 4

Exercise 1 (2 points). *Let R be a ring. Give a left adjoint functor to the forgetful functor from the category of R-modules to the category of sets.*

The following is a straightforward application of the proof of a point of the lecture's corollary 1.2.1. It is sharper than the assertion proved in the lecture in the case of the categories of commutative or arbitrary rings as epimorphisms in these categories do not need to be surjective maps.

Exercise 2 (1 point). *Let X be a topological space, $\mathcal{A}$ one of the categories of sets, groups, abelian groups, not necessarily commutative rings, or commutative rings, and let $\mathcal{S}$ be the category of presheaves or sheaves on X with values in $\mathcal{A}$. If $\mathcal{F} \xrightarrow{\phi} \mathcal{G}$ is a morphism in $\mathcal{S}$ such that $\mathcal{F}_x \xrightarrow{\phi_x} \mathcal{G}_x$ is an epimorphism in $\mathcal{A}$ for every $x \in X$, show that ϕ is an epimorphism in $\mathcal{S}$.*

Exercise 3 (4 points). *Let the (covariant) functor $\mathcal{A} \xrightarrow{L} \mathcal{B}$ be left adjoint to the functor $\mathcal{B} \xrightarrow{R} \mathcal{A}$. Show that L preserves epimorphisms: If $X \xrightarrow{\xi} Y$ is an epimorphism in $\mathcal{A}$, then $L\xi$ is an epimorphism in $\mathcal{B}$. Also, show that R preserves monomorphisms.*

Exercise 4 (3 points). *Let X, $\mathcal{A}$ and $\mathcal{S}$ be as in exercise 2 and $x \in X$. There is a functor $S_x : \mathcal{A} \to \mathcal{S}$ of "forming the skyscraper sheaf" defined by*

$$\big(S_x(G)\big)(U) = \begin{cases} G & x \in U \\ \{0\} & \textit{otherwise.} \end{cases}$$

Restrictions are given by Id_G or the unique morphism from G to the final object $\{0\}$. A morphism $G \xrightarrow{\gamma} H$ in $\mathcal{A}$ defines a morphism $S_x(G) \to S_x(H)$ given by γ on open subsets containing x and the identity of the final object for other open subsets. Show that the stalk functor $\mathcal{S} \to \mathcal{A}$, $\mathcal{G} \to \mathcal{G}_x$ is left adjoint to S_x.

Remark 1.
- It follows from the previous two exercises that the criterion from exercise 2 is not only sufficient but also necessary.
- Note that the assertion is still valid if $\mathcal{S}$ is replaced by the category of presheaves. In particular, an epimorphism in the category of presheaves induces epimorphisms on stalks, but the opposite does not hold.

- Note that the category of $\mathcal{A}$-valued sheaves on $\{x\}$ is equivalent to $\mathcal{A}$, mapping a sheaf $\mathcal{G}$ to $\mathcal{G}(\{x\})$ and an object G of $\mathcal{A}$ to the skyscraper sheaf on $\{x\}$, and S_x is the composition of that functor with $(i_x)_*$, the functor of direct image under the continuous map $\{x\} \xrightarrow{i_x} X$ given by $i_x(x)$. Under this identification of $\mathcal{A}$ with the category of sheaves on $\{x\}$, the stalk functor is identified with the inverse image functor i_x^* on sheaves.

The reason for the usefulness of direct limits (as occurring in the definition of stalks or more general presheaf inverse images) in the construction of left adjoint functors is the fact that the direct limit is itself a left adjoint functor.

Exercise 5 (5 points). *Let $\mathcal{X}$ be a partially ordered set in which, for any finite subset $F \subseteq \mathcal{X}$, there is $x \in \mathcal{X}$ such that $f \preceq x$ for all $f \in F$. We consider $\mathcal{X}$ as a category in which there is a morphism from x to y if and only if $x \preceq y$, in which case there is precisely one morphism from x to y. For a functor F from $\mathcal{X}$ to one of the categories $\mathcal{A}$ we put*

$$L = \varinjlim_{\mathcal{X}} F = \big\{(x,f) \bigm| x \in \mathcal{X} \text{ and } f \in F(x)\big\} \big/\!\sim$$

where $(x,f) \sim (\xi,\phi)$ if there is $y \in \mathcal{X}$ dominating x and ξ (i. e., such that $x \preceq y$ and $\xi \preceq y$) and such that the images of ϕ and f in $F(y)$ coincide. An algebraic operation $\star \in \{+,\cdot\}$ required for the target category is given by

$$\big((x,f)/\sim\big) \star \big((\xi,\phi)/\sim\big) = (y, \tilde{f} \star \tilde{\phi})/\sim,$$

where $y \in \mathcal{X}$ is chosen such that it dominates x and ξ and where $\tilde{f}$ and $\tilde{\phi}$ are the images of f and ϕ in $F(y)$. Let $F(x) \xrightarrow{\lambda_x} L$ be given by $\lambda_x(f) = (x,f)/\sim$. Then the diagram

$$\begin{tikzcd} F(x) \arrow[r] \arrow[d, "\lambda_x"'] & F(y) \arrow[dl, "\lambda_y"] \\ L & \end{tikzcd} \tag{1}$$

commutes when $x \preceq y$.

Show the following: If M is an object of $\mathcal{A}$ together with families $F(x) \xrightarrow{\mu_x} M$ of morphisms such that the diagram

$$\begin{tikzcd} F(x) \arrow[r] \arrow[d, "\mu_x"'] & F(y) \arrow[dl, "\mu_y"] \\ M & \end{tikzcd}$$

commutes when $x \preceq y$, then there is a unique morphism $L \to M$ such that the diagram

$$\begin{tikzcd} F(x) \arrow[r, "\lambda_x"] \arrow[d, "\mu_x"'] & L \arrow[dl] \\ M & \end{tikzcd}$$

commutes for all $x \in \mathcal{X}$.

Remark 2. I consider it unnecessary for the written solutions to contain proofs that our construction of $\varinjlim_{\mathcal{X}} F$ indeed yields an algebraic object of the desired kind, or that (1) commutes. However, students presenting their solutions on the blackboard should be able to explain this when necessary.

Remark 3. In the above situation, the category $\mathcal{X}$ is small. When this is the case, the category $\mathrm{Fun}(\mathcal{X}, \mathcal{A})$ of (covariant) functors from $\mathcal{X}$ to $\mathcal{A}$ (with natural transformations as morphisms) is well-defined in the set theory of von Neumann, Bernays and Gödel. For any object A of $\mathcal{A}$, there is a constant functor $\mathcal{X} \to \mathcal{A}$ sending any object to A and any morphism to Id_A. The commutativity of (1) can be reformulated as stating that $(\lambda_x)_{x \in \mathcal{X}}$ is a natural transformation from F to the constant functor defined by L. The universal property of the inductive limit can thus be reformulated as expressing the fact that the functor $\varinjlim_{\mathcal{X}}$ is left adjoint to the constant functor functor.

Remark 4. For a general small category $\mathcal{X}$, it is still possible to define the *colimit* $\mathrm{colim}_{\mathcal{X}} F$ of objects F of $\mathrm{Fun}(\mathcal{X}, \mathcal{A})$ as the functor $\mathrm{Fun}(\mathcal{X}, \mathcal{A}) \to \mathcal{A}$ which is left adjoint to the constant functor functor. It can be shown to exist for the target categories $\mathcal{A}$ studied in the previous exercise, but the construction used there will only generalize if $\mathcal{X}$ has the following two properties (the second of which is trivially satisfied for partially ordered sets):

- For any finite set F of objects of $\mathcal{X}$ there is an objet y such that for every $x \in F$ there exists a morphism $x \to y$ in $\mathcal{X}$.
- For any pair of morphisms $a, b : x \to y$ in $\mathcal{X}$, there is a morphism $y \xrightarrow{e} z$ in $\mathcal{X}$ such that $ea = eb$.

Only in this situation should the colimit be denoted $\varinjlim_{\mathcal{X}} F$ and called a *direct* or *inductive* limit (direkter Limes/induktiver Limes). The direct limit of an exact sequence of diagrams abelian groups is an exact sequence, while general colimits of diagrams of abelian groups do not possess a similar exactness property.

Exercise 6 (4 points). *Let $\mathcal{X}$ be partially ordered set satisfying the assumption of the previous exercise. A subset $\mathcal{Y} \subseteq \mathcal{X}$ is called* cofinal *if for any $x \in \mathcal{X}$ there is $y \in \mathcal{Y}$ such that $x \preceq y$. Let $\mathcal{X} \xrightarrow{F} \mathcal{A}$ be a functor, where $\mathcal{A}$ is as in the previous exercise. Show that for any object L of $\mathcal{A}$, any family of morphisms $\big(F(x) \xrightarrow{\lambda_x} L\big)_{x \in \mathcal{Y}}$ with the property that* (1) *commutes when $x \preceq y$ are elements of $\mathcal{Y}$ has a unique extension to a family of morphisms $\big(F(x) \xrightarrow{\lambda_x} L\big)_{x \in \mathcal{X}}$ with the property that* (1) *commutes when $x \preceq y$ are elements of $\mathcal{X}$.*

Remark 5. If follows that $\varinjlim_{\mathcal{Y}} \big(F\,|_{\mathcal{Y}}\big)$ and $\varinjlim_{\mathcal{X}} F$ are canonically isomorphic as they are characterized by equivalent universal properties. This replacement of $\mathcal{X}$ by a cofinal subset occurred in the lecture when proving that $\mathcal{G}$ and $\mathcal{G}\,|_{\mathcal{B}}$ have isomorphic stalks.

Exercise 7 (1 point). *Let $\mathfrak{B}$ be a set of subsets of the set X. Show that $\mathfrak{B}$ is the base of a topology on X if and only if the intersection of finitely many elements of $\mathfrak{B}$ in X has a covering by subsets which are elements of $\mathfrak{B}$.*

Solutions should be submitted Friday, November 17 in the lecture.