Exercises to „Algebraic geometry I“, 3

Exercise 1 (3 points). Give an example of a product which does not exist in the category of finite-dimensional $\mathbb{Q}$-vector spaces.

Remark 1. An argument to show that such products must be canonically isomorphic to the products in the category of all $\mathbb{Q}$-vector spaces if they exist has been scetched in the lecture. This is easy to carry out in detail but is a bit laborious and probably merits more than 3 points. A more straightforward approach is to look at the morphism sets which may occur in the category of finite-dimensional vector spaces and to look for an example where the universal property of products would force a morphism set to be of a different type.

Exercise 2 (4 points). Give an example of a category $\mathcal{A}$ an a morphism $X \xrightarrow{f} Y$ in $\mathcal{A}$ which is a Mono- and Epimorphism but not an Isomorphism in $\mathcal{A}$.

Let $\mathfrak{k}$ be an algebraically closed field.

Exercise 3 (4 points). Give an example of algebraic varieties X and Y over $\mathfrak{k}$ and of a pair of morphisms (α, β) from X to Y which has no equalizer in the categories of varieties or prevarieties over $\mathfrak{k}$.

Exercise 4 (4 points). Let X be a prevariety over $\mathfrak{k}$. Show that the affine open subsets of X form a topology base of X.

Exercise 5 (5 points). Let $\mathfrak{B}$ be a topology base for the topological space X and $\mathcal{G}$ a sheaf of sets on $\mathfrak{B}$ satisfying the following version of the sheaf axiom:

> If $U = \bigcup_{i \in I} U_i$ is an open covering of $U \in \mathfrak{B}$ by elements of $\mathfrak{B}$, then $g \to \left(g\,|_{U_i}\right)$ maps $\mathcal{G}(U)$ bijectively onto the set of all $(g_i)_{i \in I} \in \prod_{i \in I} \mathcal{G}(U_i)$ such that $g_i\,|_W = g_j\,|_W$ when $i, j \in I$ and $W \in \mathfrak{B}$ is contained in $U_i \cap U_j$.

Show that the canonical morphism $\mathcal{G} \xrightarrow{\gamma_{\mathcal{G}}} \mathrm{Sheaf}(\mathcal{G})\,|_{\mathfrak{B}}$ is an isomorphism.

Remark 2. Of course the assertion also holds for sheaves of rings and groups, with the same proof.

Solutions should be submitted Friday, November 10 in the lecture.