Exercises to „Algebraic geometry I", 2

Let $\mathfrak{k}$ be an algebraically closed field. We will abbreviate $\mathbb{A}^n(\mathfrak{k}) = \mathfrak{k}^n$ to $\mathbb{A}^n$ and $\mathbb{P}^n(\mathfrak{k})$ to $\mathbb{P}^n$. Recall the definitions of structure sheaves in the projective and affine cases: For locally closed $X \subseteq \mathfrak{k}^n$, we put

$$\begin{aligned}\mathcal{O}_X^{(\mathrm{Aff})}(X) = \Big\{ f : X \to \mathfrak{k} \;\Big|\; &\text{For every } x \in X, \text{ there exist an open} \\ &\text{neighbourhood } V \text{ of } x \text{ in } V \text{ and polynomials} \\ &p, q \in \mathfrak{k}[X_1, \ldots, X_n] \text{ such that } f(y) = \frac{p(y)}{q(y)} \text{ holds} \\ &\text{for all } y \in V \Big\}\end{aligned}$$

whereas

$$\begin{aligned}\mathcal{O}_X^{(\mathrm{Proj})}(X) = \Big\{ f : X \to \mathfrak{k} \;\Big|\; &\text{For every } x \in X, \text{ there exist an open} \\ &\text{neighbourhood } V \text{ of } x \text{ in } V \text{ and homogenuous} \\ &\text{polynomials } p, q \in \mathfrak{k}[X_0, \ldots, X_n] \text{ such that} \\ &\deg p = \deg q \text{ and such that } f([y_0, \ldots, y_n]) = \frac{p(y)}{q(y)} \\ &\text{holds for all } y \in \mathfrak{k}^n \setminus \{0\} \text{ with } [y_0, \ldots, y_n] \in V \Big\}\end{aligned}$$

for locally closed $X \subseteq \mathbb{P}^n$. In both cases, the condition on f is meant to include non-vanishing of the denominator.

For the following exercise, we keep the above superscripts (Aff) and (Proj) to avoid confusions. Normally these superscripts are omitted.

Exercise 1 (5 points). *Let $\mathbb{A}^n \xrightarrow{j} \mathbb{P}^n$ be the homeomorphism onto an open subset studied in exercise 4 of the previous sheet. Show that when X is contained in the image of j, the pull-back j^* of functions with respect to j defines an isomorpism*

$$\mathcal{O}_X^{(\mathrm{Proj})}(X) \cong \mathcal{O}_{j^{-1}X}^{(\mathrm{Aff})}(j^{-1}X).$$

Exercise 2 (7 points). *Let X be a prevariety over $\mathfrak{k}$ in the sense of Definition 1.1.2 and let $Z \subseteq X$ be an irreducible closed subset. Let*

$$\begin{aligned}(1)\quad \mathcal{O}_Z(U) = \Big\{ f : U \to \mathfrak{k} \;\Big|\; &\textit{For every } x \in U, \textit{ there exist an open} \\ &\textit{neighourhood } V \textit{ of } x \textit{ in } X \textit{ and } \phi \in \mathcal{O}_X(V) \textit{ such} \\ &\textit{that } \phi \textit{ and } f \textit{ cooincide on } V \cap U. \Big\}\end{aligned}$$

Show that $(Z, \mathcal{O}_Z)$ is a prevariety!

Remark 1. Conditions of the type occuring on the right hand sides of the previous three displayed multi-line equations are often called *coherence conditions*. There are often slight modifications to them resulting in equivalent conditions. The following is an example.

Exercise 3 (1 point). *Show that the definition* (1) *is not changed if the condition on f is replaced by the following:*

> *For every $x \in U$, there exist an open neighourhood V of x in X and $\phi \in \mathcal{O}_X(V)$ such that $V \cap Z \subseteq U$ and ϕ and f cooincide on $V \cap Z$.*

Exercise 4 (7 points). *Decide which of the following categories have initial (resp. final) objects. Confirm your decision by giving an initial (resp. final) object and showing that it has the desired property, or by showing that no such object exists.*

- *The category of abelian groups.*
- *The category of rings.*
- *The category of pre-varieties over $\mathfrak{k}$.*

Remark 2. Rings are supposed to be commutative and with 1, as in the lecture. For a morphism $A \xrightarrow{f} B$ this means that f must satisfy $f(a \circ b) = f(a) \circ f(b)$ where $\circ$ is $+$ or $\cdot$ and that $f(1) = 1$.

Solutions should be submitted Tuesday, October 31, in the lecture.