Exercises to „Algebraic geometry I“, 10

Exercise 1 (3 points). *Give an example of a sheaf of ideals on* $\operatorname{Spec}\mathbb{Z}$ *which is not quasi-coherent.*

Exercise 2 (1 point). *For a prescheme X and a morphism $\mathcal{M} \xrightarrow{f} \mathcal{N}$ of $\mathcal{O}_X$-modules where X and Y are quasi-coherent, show that the cokernel of f is quasi-coherent.*

Exercise 3 (4 points). *Let R be a ring and I an ideal in it. Show that the morphism* $\operatorname{Spec}(R/I) \to \operatorname{Spec} R$ *obtained by applying the contravariant functor* Spec *to the ring morphism $R \to R/I$ is a closed immersion and that the sheaf of ideals associated to it equals the (isomorphic) image of $\tilde{I}$ in $\mathcal{O}_{\operatorname{Spec} R}$.*

Exercise 4 (9 points). *Let $X \xrightarrow{f} Y$ be a morphism of ringed spaces.*

- *Construct a functor* $\operatorname{Mod}_{\mathcal{O}_Y} \xrightarrow{f^*} \operatorname{Mod}_{\mathcal{O}_X}$ *which is left adjoint to* $\operatorname{Mod}_{\mathcal{O}_X} \xrightarrow{f_*} \operatorname{Mod}_{\mathcal{O}_Y}$ *and show that the stalk of $f^*\mathcal{M}$ at $x \in X$ is given by $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{M}_{f(x)}$.*
- *Show that $(f\,|_{f^{-1}U})^*\big(\mathcal{M}\,|_U\big)$ is isomorphic to $f^*\mathcal{M}\,|_{f^{-1}U}$.*
- *If $X = \operatorname{Spec} R$ and $Y = \operatorname{Spec} S$ and f is given by a ring morphism $S \to R$, describe $f^*\tilde{M}$ for an S-module M.*
- *If f is a morphism of preschemes and $\mathcal{M}$ quasi-coherent, show that $f^*\mathcal{M}$ is quasi-coherent.*

Remark 1. It may be used that for a ring R and an R-algebra S, $M \to S \otimes_R M$ is left adjoint to the forgetful functor from S-modules to R-modules.

Let $\mathcal{R}$ be a sheaf of rings and $\mathcal{M}$ an $\mathcal{R}$-module. A derivation of $\mathcal{R}$ with values in $\mathcal{M}$ is a morphism $\mathcal{R} \xrightarrow{d} \mathcal{M}$ of sheaves of abelian groups such that $d(rs) = rd(s) + sd(r)$ in $\mathcal{M}(U)$ when r and s are elements of $\mathcal{R}(U)$. In the case where a morphism $X \xrightarrow{\xi} S$ of locally ringed spaces is given, a derivation $\mathcal{O}_X \xrightarrow{d} \mathcal{M}$ is called $\mathcal{O}_S$-linear if $d(f^*\lambda) = 0$ in $\mathcal{M}(f^{-1}U)$ when $\lambda \in \mathcal{O}_S(U)$.

Exercise 5 (8 points). *Let $X \xrightarrow{f} S$ be a morphism of preschemes.*

- *Show that there is an $\mathcal{O}_X$-module $\Omega_{X/S}$ with an $\mathcal{O}_S$-linear derivation $\mathcal{O}_X \xrightarrow{d_{X/S}} \Omega_{X/S}$ such that for any $\mathcal{O}_X$-linear derivation $\mathcal{O}_X \xrightarrow{d} \mathcal{M}$ there is a unique morphism $\Omega_{X/S} \xrightarrow{\delta} \mathcal{M}$ such that $d = \delta d_{X/S}$.*

- *Show that* $\Omega_{X/S}$ *is quasi-coherent, that the stalk of* $d_{X/S}$ *at* $x \in X$ *is isomorphic to* $\mathcal{O}_{X,x} \xrightarrow{d_{\mathcal{O}_{X,x}/\mathcal{O}_{S,f(x)}}} \Omega_{\mathcal{O}_{X,x}/\mathcal{O}_{S,f(x)}}$ *and that for any pair* (U, V) *of affine open subsets of* X *and* S *such that* $f(U) \subseteq V$, $\mathcal{O}_X(U) \xrightarrow{d_{X/S}} \Omega_{X/S}(U)$ *is isomorphic to* $\mathcal{O}_X(U) \xrightarrow{d_{\mathcal{O}_X(U)/\mathcal{O}_S(V)}} \Omega_{\mathcal{O}_X(U)/\mathcal{O}_S(V)}$.

Remark 2. Of course, the universal property of $d_{X/S}$ determines it and the *sheaf of Kähler differentials* $\Omega_{X/S}$ uniquely up to unique isomorphism. The required facts about Kähler differentials of rings (eg, existence, compatibility with localization and base-change) may all be used, provided they are correct.

Of the 25 points for this exercise sheet, 5 are bonus points which do not count for the calculation of the 50%-limit for passing the exercises.

Solutions should be submitted Friday, January 12 in the lecture.