Remark 1. *Throughout this sheet let X be a topological space. For a subset $A \subseteq X$, the closure is denoted $\overline{A}$. In solutions, the following equivalent characterizations of $\overline{A}$ can be used as their equivalence is easily seen:*

- *The intersection of all closed subsets of X containing A.*
- *The smallest closed subset of X containing A.*
- *The unique closed subset of X containing A as a dense subset.*

Problem 1 (3 points)**.** *Let $x \in X$, and let $A \subseteq X$ be any subset.*

- *Show that $x \in \overline{A}$ if and only if every neighbourhood of x intersects A.*
- *If x has a countable neighbourhood base, show that $x \in \overline{A}$ if and only if x is the limit of some sequence of elements of A.*

Remark 2. *As an easy consequence of the first point, the "only if"-part of the second one holds even if x has no countable neighbourhood base.*

Problem 2 (1 point)**.** *Show that $x \in X$ is a point of accumulation of the filter $\mathfrak{F}$ if and only if $x \in \overline{A}$ for all $A \in \mathfrak{F}$.*

Problem 3 (3 points)**.** *Show that the following conditions are equivalent:*

- *X is quasi-compact.*
- *Every filter on X has at least one point of accumulation.*
- *Every ultrafilter on X has at least one limit.*

Remark 3.

- *In particular, X is compact iff every ultrafilter has precisely one limit.*
- *The following obviously equivalent characterization of quasi-compactness may be used in solutions:*
  - *Every open covering has a finite subcovering.*
  - *Every family of closed subsets has a non-empty intersection, provided that this holds for every finite subfamily.*

The following shows that the assumption of countable neighbourhood bases can be omitted in the second point of 1 if filters are used instead of sequences.

Problem 4 (2 points)**.** *In the situation of 1, show that $x \in \overline{A}$ if and only if x is the limit of some ultrafilter on A.*

Remark 4. *If $X \xrightarrow{f} Y$ is any map between sets and $\mathfrak{U}$ some ultrafilter on X, $f_*\mathfrak{U} = \left\{A \subseteq Y \;\middle|\; f^{-1}A \in \mathfrak{U}\right\}$ is easily seen to be a ultrafilter on X. This covariant functoriality of ultrafilters coincides* *with the one obtained as the composition of the contravariant functor from sets to rings, sending X to the ring of $\mathbb{F}_2$-valued functions on X, with the contravariant functor* Spec *from rings to topological spaces. As in Problem 6 of the previous sheet, every field can be used instead of $\mathbb{F}_2$.*

In the situation of the previous problem, one has a bijection

$$(\textit{ultrafilters } \mathfrak{U} \textit{ on } A) \cong (\textit{ultrafilters } \mathfrak{V} \textit{ on } X \textit{ with } A \in \mathfrak{V})$$
$$\mathfrak{U} \to \mathfrak{V} = i_* \mathfrak{V}$$
$$\mathfrak{U} = \{B \subseteq A \mid B \subseteq A\} \leftarrow \mathfrak{V},$$

where $A \xrightarrow{i} X$ is the inclusion. We say that x is a limit of $\mathfrak{U}$ in X if and only if it is a limit of $i_*\mathfrak{U}$.

Problem 5 (4 points)**.**
- *Let $(C_i)_{i\in I}$ be a familiy of connected subsets, and assume that $\bigcap_{i\in I} C_i \neq \emptyset$. Show that $\bigcup_{i\in I} C_i$ is a connected subset in X.*
- *Let $C \subseteq X$ be a connected subset of X. Show that the closure $\overline{C}$ of C in X is a connected subset of X.*

Problem 6 (4 points)**.** *Show that every compact (i. e., quasi-compact and Hausdorff) space is T_4.*

The following finishes the proof of the second (spectral) case of the Sura-Bura theorem from the lecture.

Problem 7 (6 points)**.** *Assume that the set $\mathfrak{B}$ of quasi-compact open subsets of X is closed under finite intersections in X and a topology base for X. Let Y be the set of quasi-components of X equipped with the quotient topology for the surjection $X \xrightarrow{q} Y$ sending every $x \in X$ to its quasi-component Q_x. Let $Q \in Y$ and assume that A and B are disjoint closed subsets of X such that $Q = A \cup B$. Show that there are disjoint open subsets $U \subseteq X$ and $V \subseteq X$ such that $A \subseteq U$, $B \subseteq V$.*

Two of the 23 possible points from this sheet are bonus points which do not count in the calculation of the $\geq 50\%$ lower bound of points needed to pass the exercises. Solutions should be submitted in the exercises on Wednesday, December 20.