Definition 1. *A subset S of a ring R is called* multiplicative *if every finite (possibly empty) product of elements of S is an element of S. For a ring R and a multiplicative subset S of R, the localization R_S is the set of equivalence classes of pairs (r, s) where $r \in R$, $s \in S$ and $(r, s) \sim (\rho, \sigma)$ if there is $t \in S$ with $tr\sigma = ts\rho$. The ring structure on R_S is given by*

$$[(r,s)/\sim] + [(\rho,\sigma)/\sim] = (r\sigma + s\rho, s\sigma)/\sim$$
$$[(r,s)/\sim] \cdot [(\rho,\sigma)/\sim] = (r\rho, s\sigma)/\sim$$

and there is a ring homomorphism $R \to R_S$, $r \to (r, 1)/\sim$.

The theory of the localization is quite similar to the construction of the field of quotients, which is the localization of a domain R with respect to $S = R \setminus \{0\}$. However, there are a few subtle differences. For instance, if S contains zero-divisors in R the homomorphism $R \to R_S$ will fail to be injective, and the introduction of t in the above equivalence relation is necessary.

Despite of these differences, the similarities with the case of the field of quotients abund, and it may thus be acceptable to limit proofs to the following

Problem 1 (2 points)**.** *Show that $\sim$ is indeed an equivalence relation.*

As in the case of the field of quotients we will from now on write $\frac{r}{s}$ for $(r, s)/\sim$.

Remark 1. *One easily verifies that R_S is indeed a ring and $R \xrightarrow{\iota} R_S$ a ring homomorphism. It enjoys the follwing universal property: If $s \in S$ then $\iota(s) \in R_S^\times$, and if $R \xrightarrow{t} T$ is a ring homomorphism with $t(S) \subseteq T^\times$ then there is a unique homomorphism $R_S \xrightarrow{\tau} T$ with $t = \tau\iota$, and τ is given by $\tau\left(\frac{r}{s}\right) = \frac{t(r)}{t(s)}$.*

Problem 2 (7 points)**.** *Let R be a ring and S a multiplicative subset of R. The S-saturation of an ideal $I \subseteq R$ is the ideal*

$$J = \left\{r \in R \;\middle|\; rs \in I \text{ for some } s \in S\right\}.$$

We call I S-saturated if $I = J$. If I is any ideal in R, let

$$I_S = \left\{\frac{r}{s} \;\middle|\; r \in S, s \in S\right\} \subseteq R_S.$$

It is easy to see that this is an ideal in R_S.

- *Show that the inverse image under $R \to R_S$ of I_S is the S-saturation of I.*

- *Show that we get a bijection between S-saturated ideals $I \subseteq R$ and ideals of R_S, sending an S-saturated $I \subseteq R$ to I_S and an ideal in R_S to its preimage in R.*
- *Show that this restricts to a bijection*

$$\mathrm{Spec} R_S \cong \left\{ \mathfrak{p} \in \mathrm{Spec} R \;\middle|\; \mathfrak{p} \cap S = \emptyset \right\} = \mathrm{Spec} R \setminus \bigcup_{s \in S} V(s) \tag{1}$$

which is a homeomorphism if the right hand is equipped with the topology induced from the Zariski topology on $\mathrm{Spec} R$.

Remark 2. *For the sake of brevity, it is not necessary to prove that I_S is an ideal in R_S, or that the saturation of an ideal in R is an ideal in R.*

Problem 3 (3 points)**.** *Let R be a ring and $I \subseteq R$ an ideal with $\sqrt{I} = I$. Show that I is a prime ideal if and only if $V(I)$ is an irreducible subset of* $\mathrm{Spec} R$.

Problem 4 (2 points)**.** *Show that for a filter $\mathfrak{F}$ on a set X, the following conditions are equivalent:*

- *For every subset $A \subseteq X$, $A \in \mathfrak{F}$ or $X \setminus A \in \mathfrak{F}$.*
- *If $\mathfrak{G}$ is a filter on X with $\mathfrak{F} \subseteq \mathfrak{G}$ then $\mathfrak{G} = \mathfrak{F}$.*

Remark 3. *Recall that such filters are called ultrafilters. Another equivalent condition would be that $A \cup B \in \mathfrak{F}$ implies $A \in \mathfrak{F}$ or $B \in \mathfrak{F}$.*

Problem 5 (2 points)**.** *Show that every filter is contained in some ultrafilter.*

Problem 6 (3 points)**.** *Let X be a set, K a field and R the ring of K-valued functions on X. Show that an ideal in R is a prime ideal if and only if it is maximal, and show that we have a bijection between* $\mathrm{Spec} R$ *and the set of ultrafilters on X, sending $\mathfrak{p}$ to*

$$\mathfrak{U} = \left\{ A \subseteq X \;\middle|\; f\,|_A = 0 \text{ for all } f \in \mathfrak{p} \right\}$$

and the ultrafilter $\mathfrak{U}$ to

$$\mathfrak{p} = \left\{ f \in R \;\middle|\; \{x \in X | f(x) = 0\} \in \mathfrak{U} \right\}.$$

The notion of limit of ultrafilters can be generalized to arbitrary filters, but then it is necessary to distinguish between points of accumulation and limits.

Definition 2. *We call $x \in X$ a* limit *of a filter $\mathfrak{F}$ if every neighbourhood of x is an element of $\mathfrak{F}$. We call x a* point of accumulation *if every neighbourhood of x has a non-empty intersection with every element of $\mathfrak{F}$.*

While the notion of "limit" for ultrafilters is almost certainly the same throughout the relevant literature, this may be different for "point of accumulation" and for filters which are not ultrafilters.

Problem 7 (3 points)**.** *Let $\mathfrak{F}$ be a filter on the set underlying a topological space X.*

- *If $x \in X$ is a limit of $\mathfrak{F}$, show that x is a point of accumulation of $\mathfrak{F}$.*
- *If $x \in X$ is a point of accumulation of $\mathfrak{F}$, show that x is a limit of some ultrafilter containing $\mathfrak{F}$.*

In particular, for ultrafilters the notions of "limit" and of "point of accumulation" coincide.

Problem 8 (2 points)**.** *Show that a topological space is Hausdorff if and only if every ultrafilter has at most one limit.*

Five of the 25 possible points from this sheet are bonus points which do not count in the calculation of the $\geq 50\%$ lower bound of points needed to pass the exercises. Solutions should be submitted in the lecture on Friday, December 8 as there are no exercises on the *dies academicus*, Wednesday, December 6.