Sixth exercise sheet "Class field theory" summer term 2025.

Problem 1 (5 points). *Let $\mathcal{A} \xrightarrow{L} \mathcal{B} \xrightarrow{R} \mathcal{A}$ be functors between categories. We consider the following two classes of data:*

- *Collections of bijections* $\mathrm{Hom}_{\mathcal{B}}(LA, B) \xrightarrow{\alpha} \mathrm{Hom}_{\mathcal{A}}(A, RB)$ *which are, for fixed $B \in \mathrm{Ob}\mathcal{B}$, morphisms of functors $\mathcal{A}^{\mathrm{op}} \to \mathfrak{Sets}$ and, for fixed $A \in \mathrm{Ob}\mathcal{A}$, morphisms of functors $\mathcal{B} \to \mathfrak{Sets}$.*
- *Pairs (c, u) where $LR \xrightarrow{c} \mathrm{Id}_{\mathcal{B}}$ and $\mathrm{Id}_{\mathcal{A}} \xrightarrow{u} RL$ are functor-morphisms satisfying the* unit-counit equations

$$\mathrm{Id}_L = L^* c L(u) \tag{1}$$

$$\mathrm{Id}_R = R(c) R^* u. \tag{2}$$

Show that we have a bijection between the two classes of data, which in one direction sends α to (c, u) where $c = \alpha^{-1}(\mathrm{Id}_R)$ and $u = \alpha(\mathrm{Id}_L)$!

In the following, let G be finite.

Problem 2 (3 points). *If $H \subseteq G$ is a subgroup, show that we have a functor of restriction $\mathfrak{Ho}_G \xrightarrow{\mathfrak{Res}} \mathfrak{Ho}_H$ of restriction to H sending a G-module M to its restriction to H and the homotopy class of a morphism of G-modules to the homotopy class of the same map viewed as a morphism of H-modules. Moreover, show that we have a functor $\mathfrak{Ho}_H \xrightarrow{\mathrm{Ind}} \mathfrak{Ho}_G$ sending an H-module N to $\mathrm{Ind}_H^G M$ and the homotopy class of a morphism $N \xrightarrow{\nu} N'$ to the homotopy class of $\mathrm{Ind}_H^G \nu$.*

Problem 3 (2 points). *In the situation of the previous problem, show that* Ind *is both left and right adjoint to* $\mathfrak{Res}$*!*

Problem 4 (5 points). *Let $\mathcal{M}: \ 0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0$ be a short exact sequence of G-modules. For $m \in m''$ such that $\mathrm{Tr}_G(m'') = 0$, let μ'' be its image in $\mathrm{Ker}\big(M''_G \xrightarrow{\mathrm{Tr}_G} M''^G\big)$, select $m \in M$ such that $m'' = p(m)$ and $m' \in M'^G$ such that $i(m') = \mathrm{Tr}_G(m)$. Show that the image of μ'' under*

$$\mathrm{Ker}\big(M''_G \xrightarrow{\mathrm{Tr}_G} M''^G\big) \cong \hat{H}^{-1}(M'') \xrightarrow{d_{\mathcal{M}}} \hat{H}^0(M') \cong \mathrm{Coker}\big(M'_G \xrightarrow{\mathrm{Tr}_G} M'^G\big) \tag{3}$$

is equal the image of m' in the right hand side! The two isomorphisms in (3) are Examples 1.3.1 and 1.3.2 from the lecture.

If K is a field and G acts on a K-vector space V, let G act on its dual space by $(g\ell)(v) = \ell(g^{-1}v)$.

Problem 5 (2 points). *If V is finite-dimensional, show that*

$$\dim_K V_G = \dim_K (V^*)^G.$$

Problem 6 (2 point)**.** *If V is finite-dimensional and $V \times V \xrightarrow{B} K$ a non-degenerate bilinear form satisfying $B(\gamma v, \gamma w) = B(v, w)$ for $\gamma \in G$ and $(v, w) \in V^2$, then show that* $\dim(V_G) = \dim(V^G)$*!*

Problem 7 (3 points)**.** *Let L/K be a finite Galois extension. Without using normal bases, show that $\hat{H}^0\big(L/K, (L, +)\big)$ and $\hat{H}^{-1}\big(L/K, (L, +)\big)$ both vanish!*

When combined with the future Proposition 1.5.5 from the lecture this will give a proof of the vanishing of $\hat{H}^*\big(L/K, (L, +)\big)$ which does not use normal bases.

Two of the 22 points from this sheet are bonus points which do not count in the calculation of the 50%-limit for passing the exercises module. Solutions should be submitted to the tutor by e-mail before Tuesday May 20 24:00.