

The following is Proposition 1.4.5, the precise formulation of which was omitted in the lecture.

**Problem 1** (6 points). *Let  $K$  be real closed,  $P \in K[T]$  a polynomial of degree  $d > 0$ ,  $\vec{\delta} = (\delta_i)_{i=0}^d$  and  $\vec{\varepsilon} = (\varepsilon_i)_{i=0}^d$  be two elements of  $\{0; \pm 1\}^{d+1}$  such that*

$$L_{\vec{\delta}}P = \{t \in K \mid \operatorname{sgn} P^{(i)}(t) = \delta_i\}$$

*and  $L_{\vec{\varepsilon}}P$  are both non-empty. Let  $k$  be the largest element of  $\mathbb{Z} \cap [0, d]$  such that  $\varepsilon_k \neq \delta_k$ . Show the following:*

- *We have  $k < d$ , and  $\varepsilon_{k+1} = \delta_{k+1}$  is not zero.*
- *If  $\delta_k \delta_{k+1} < \varepsilon_k \varepsilon_{k+1}$ , then  $s < t$  for all  $s \in L_{\vec{\delta}}P$  and all  $t \in L_{\vec{\varepsilon}}P$ . Otherwise  $s > t$  for all such  $s$  and  $t$ .*

**Problem 2** (9 points). *Determine the number of real zeroes of the Polynomial  $T^6 + 2T^3 + 6T + 1$ !*

It was already pointed out in the lecture that Sturm chains could have been used instead of the considerations about the signature of trace bilinear forms in the proof of uniqueness of the real closure. The following is a result about trace bilinear forms which gives information similar to the Sturm type Proposition 1.5.3.

**Problem 3** (5 points). *Let  $K$  be an ordered field,  $P \in K[T]$  irreducible,  $Q \in K[T]$  any polynomial. Let  $L = K[T]/PK[T]$ ,  $\tau$  the image of  $T$  in  $L$ , and let  $\mathfrak{K}$  be any real closed field containing  $K$  as a subfield, with order preserving inclusion  $K \rightarrow \mathfrak{K}$ . Show that the signature of the bilinear form*

$$(x, y) \rightarrow \operatorname{Tr}_{L/K}(Q(\tau)xy)$$

*on  $L \times L$  is equal to  $n_+ - n_-$ , where  $n_{\pm}$  is the number of zeroes  $x$  of  $P$  in  $\mathfrak{K}$  such that  $\pm Q(x) > 0$ .*

Solutions should be submitted in the exercises on Wednesday, November 22.