Recall the Artin-Schreier operator

$$\mathrm{AS}(x) = x^p - x$$

for elements of fields of characteristic $p > 0$.

Problem 1 (2 points). *If L/K is a finite field extension in characteristic $p > 0$, $\mathrm{AS}\big(\mathrm{Tr}_{L/K}(l)\big) = \mathrm{Tr}_{L/K}\big(\mathrm{AS}(l)\big)$.*

Together with results from the previous exercise sheet, this can be used to prove

Problem 2 (6 points). *Let L/K be a field extension in characteristic $p > 0$ such that L is algebraically closed and $[L : K]$ is finite. Show that $L = K$.*

Remark 1.
- *As the case of characteristic 0 has been dealt with on sheet 3, this finishes the proof the Theorem 1 of the lecture, proving the equivalence of Theorem 1D with Theorem 1A–C.*
- *The same proof also shows L/K if the extension is finite and separable and L separably closed.*

From now on until the end of this sheet, all fields are assumed to be real closed.

Problem 3 (3 points). *Formulate and show a version of Proposition 1.4.2 which applies to elements of $K(T)$, elements of the field of rational functions in one variable.*

Remark 2. *Every such function has a unique representation $f = \frac{P}{Q}$ where $Q \in K[T]$ is a normed polynomial and $\gcd(P, Q) = 1$ in $K[T]$. The zeroes of Q are called the* poles *of f and we consider f to be undefined there, as no reasonable way to give $f(t)$ a value of $\pm\infty$ exists for poles of odd order. The formulation of the desired result must take care of this.*

Problem 4 (6 points). *Let $a < b$ be elements of K and let $f, g \in K(T)$ be rationals functions without poles on $[a, b]_K$ and such that $g(a) \neq g(b)$. Show that there is $t \in (a, b)_K$ such that*

$$\frac{f(a) - f(b)}{g(a) - g(b)} = \frac{f'(t)}{g'(T)}.$$

Problem 5 (5 points). *Let $P \in K[T]$ be a non-constant polynomial and let $-\infty \leq a < b \leq \infty$ be elements of $K \cup \{\pm\infty\}$ such that $P(t) > 0$ for all $t \in [a, b]_{K\cup\{\pm\infty\}}$.*

- *Show that $m = \min_{t\in[a,b]_K} P(t)$ exists.*

- *Show that there are only finitely many* $t \in [a,b]_K$ *with* $P(t) = m$*, and show that all such* t *are zeroes of* P'*, provided that they are elements of* $(a,b)_K$*.*

Remark 3.
- *It is easy to derive Corollary 1.4.3 of the lecture from this result. However, the corollary could also be shown by verifying it for the individual prime factors of* P *in* $K[T]$*.*
- *The result also holds for rational functions without poles in* $K \cap [a,b]_{K\cup\{\pm\infty\}}$*, provided that the values at* $\pm\infty$ *of such functions are defined in a reasonable way.*

Solutions should be submitted in the lecture on Wednesday, November 17. Two of the 22 possible points from this sheet are bonus points, which do not count in the calculation of the $\geq 50\%$ lower bound of points needed to pass the exercises.