Fifth exercise sheet "Class field theory" summer term 2025. When solving the following problem it is allowed to tacitly assume that the abelian category under consideration is of a kind well-known to us, like G-modules or left or right modules over an arbitrary ring.

Problem 1 (7 points). *For a short exact sequence $0 \to A \to B \to C \to 0$ in an abelian category $\mathcal{A}$ and injective resolutions $A' \xrightarrow{\alpha} I^0 \to I^1 \to \dots$ and $0 \to C \xrightarrow{\gamma} K^0 \to K^1 \to \dots$, show the existence of an injective resolution $0 \to B \xrightarrow{\beta} J^0 \to J^1 \to dots$ and a commutative diagram*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \cdots \\
0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d] & I^0 \arrow[r] \arrow[d] & I^1 \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & B \arrow[r, "\beta"] \arrow[d] & J^0 \arrow[r] \arrow[d] & J^1 \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & C \arrow[r, "\gamma"] \arrow[d] & K^0 \arrow[r] \arrow[d] & K^1 \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 & \cdots
\end{tikzcd}$$

Problem 2 (3 points). *In the Definition 1.3.4 of the functor*

$$\mathfrak{Ho}_G \xrightarrow{[1]} \mathfrak{Ho}_G,$$

show that the identity $(fg)[1] = (f[1])(g[1])$ actually holds.

If L/K is a finite Galois extension with Galois group G, we say that $l \in L$ defines a normal base of L/K if $\{\sigma(x) \mid \sigma \in G\}$ is a base of L/K.

Problem 3 (4 points). *Let L/K be a Galois extension of degree p of fields of characteristic $p > 0$, and let σ be a generator of the cyclic group $\mathrm{Gal}(L/K)$. Recall from exercise sheet 1 that there is $l \in L$ with $\sigma(l) = l + 1$. For such l, show that l^{p-1} defines a normal base for L/K!*

Let L/K be an algebraic field extension which is normal and separable and let G be its Galois group equipped with its *Krull topology*, i. e., the topology from exercise sheet 3.

Problem 4 (1 point). *Show that a neighbourhood base of the neutral element of G is given by the subgroups $\mathrm{Gal}(L/F)$ where $K \subseteq F \subseteq L$ is an intermediate field which is a finite Galois extension of K.*

Problem 5 (5 points). *If $H \subseteq G$ is a subgroup and L^H its fixed field, show that $\mathrm{Gal}(L/L^H) \subseteq G$ is the closure of H in G!*

Combined with the results from sheet 3 we get a bijection between the closed subgroups $H \subseteq G$ and the intermediate fields between K and L.

Problem 6 (2 points). *Show that under this bijection the intermediate fields which are Galoid extensions of K correspond to the closed subgroups H which are normal divisors of G!*

It is also easy to see that in this case $\mathrm{Gal}(L^H/K) \cong G/H$ where G/H is equipped with the quotient topology, a subset being open iff its inverse image in G is.

Problem 7 (2 points). *Show that under the previous bijection the intermediate fields which are finite over K correspond to the open subgroups $H \subseteq G$.*

The arguments for this also show that $[G : H] = [L^H : K]$. With this we have finished a brief exposition of the Galois theory of not necessarily finite Galois extensions, which plays an important role in modern class field theory and Galois cohomology.

Four of the 24 points from this sheet are bonus points which do not count in the calculation of the 50%-limit for passing the exercises module. Solutions should be submitted to the tutor by e-mail before Tuesday May 13 24:00.