Fourth exercise sheet "Class field theory" summer term 2025. Let $\mathfrak{k}$ be a field and $A = \mathfrak{k}[T]/T^2\mathfrak{k}[T]$. By Problem 5 from the previous sheet, the classes of injective, projective and free A-modules coincide. Let this class of A-modules be denoted $\mathfrak{K}$.

Problem 1 (2 points). *For a morphism $M \xrightarrow{f} N$ of A-modules, show the equivalence of the following assertions:*

- *There is a $\mathfrak{k}$-linear map $M \xrightarrow{s} N$ such that $f(m) = Ts(m) + s(Tm)$ for all $m \in M$.*
- *It is possible to factor f as a composition $M \to K \to N$ with $K \in \mathfrak{K}$.*

Remark 1. *This example has been included to motivate the notions of "contractible" and "homotopic" used in the lecture to construct Tate groups. If A is graded such that T has degree 1, then graded A-modules are cochain complexes. Problem 5 of the previous sheet still holds if free generators of non-zero degree are allowed in the definition of "free", and the assertion of the previous problem also holds where the $\mathfrak{k}$-linear map s must be graded of degree -1. The second condition is then equivalent to f being cochain homotopic to 0.*

Problem 2 (2 points). *Let G be a group and $I_G \subseteq \mathbb{Z}[G]$ the augmentation ideal. For $g \in G$, let $s(g) \in (I_G)_G$ be the image in $(I_G)_G$ of $\delta_g - \delta_{\mathbf{1}} \in I_G$. Show that s is a group homomorphism!*

For a G-Module M, the morphism $M \to \operatorname{Ind}_{\{\mathbf{1}_G\}}^G M$ corresponding to Id_M unter the bijection (1.1.1) from the lecture sends m to f_m where $f_m(g) = gm$. Let $Q_M = Q_{G,M}$ be the cokernel of this morphism. Every element of Q_M has a unique representative $f \in \operatorname{Ind}_{\{\mathbf{1}_G\}}^G M$ satisfying $f(\mathbf{1}_G) = 0$.

Problem 3 (2 points). *Show that this choice of representatives gives us a of Q_M^G with Z, the set of maps $G \xrightarrow{f} M$ satisfying $f(gh) = f(g) + gf(h)$!* [1]

Problem 4 (3 points). *Under the same identifications, show that the map $M = \left(\operatorname{Ind}_{\{\mathbf{1}_G\}}^G M\right)^G \to Q_M^G \cong Z$ sends $m \in M$ to $f \in Z$ given by $f(g) = m - gm$.*

The previous two problems finish the proof of Proposition 1.2.4 from the lecture. In particular this finishes the proof Hilbert's Theorem 90 and of the bijection $H^1(G, A) \cong \operatorname{Hom}(G, A)$ where A is an abelian group with trivial G-action.

[1] Such f are often called *crossed homomorphisms.*

Problem 5 (2 points)**.** *With A as before, let $H \subseteq G$ be a subgroup. Show that the following diagram commutes:*

$$\begin{array}{ccc} H^1(G,A) & \xrightarrow{\cong} & \mathrm{Hom}(G,A) \\ \Big\downarrow{\scriptstyle \mathrm{Res}} & & \Big\downarrow{\scriptstyle f \to f|_H} \\ H^1(H,A) & \xrightarrow{\cong} & \mathrm{Hom}(H,A) \end{array}$$

Problem 6 (5 points)**.** *In the situation of the previous problem let in addition $[G:H]$ be finite. Show that the following diagram commutes, where* Verl *is transfer:*

$$\begin{array}{ccc} H^1(H,A) & \xrightarrow{\cong} & \mathrm{Hom}(H_{\mathrm{ab}},A) \\ \Big\downarrow{\scriptstyle \mathrm{Cores}} & & \Big\downarrow{\scriptstyle f \to f \circ \mathrm{Verl}} \\ H^1(G,A) & \xrightarrow{\cong} & \mathrm{Hom}(G_{\mathrm{ab}},A) \end{array}$$

Remark 2. *For both problems it seems best to consider a commutative diagram of H-modules*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & \mathrm{Ind}_{\mathbf{1}}^{G} M & \longrightarrow & Q_{M,G} & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle =} & & \Big\downarrow & & \Big\downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & \mathrm{Ind}_{\mathbf{1}}^{H} M & \longrightarrow & Q_{M,H} & \longrightarrow & 0. \end{array}$$

One can then study what Res *and* Cores *do on the first row, then use the diagram to investigate what happens on the side of H-modules. It seems that Problem 6 is a bit harder.*

Problem 7 (2 points)**.** *If H is a normal divisor of G and $G \xrightarrow{\pi} G/H$ the projection, show that the following diagram commutes*

$$\begin{array}{ccc} H^1(G/H,A) & \xrightarrow{\cong} & \mathrm{Hom}(G/H,A) \\ \Big\downarrow{\scriptstyle \mathrm{Inf}} & & \Big\downarrow{\scriptstyle f \to f \circ \pi} \\ H^1(G,A) & \xrightarrow{\cong} & \mathrm{Hom}(G,A). \end{array}$$

When solving the following problem it is allowed to tacitly assume that the abelien category under consideration is of a kind well-known to us, like G-modules or left or right modules over an arbitrary ring.

Problem 8 (5 points). *Let*

$$X \xrightarrow{\xi} I^0 \to I^1 \to \dots$$

be an exact sequence of objects of an abelian category $\mathcal{A}$*, and let*

$$Y \xrightarrow{\upsilon} J^0 \to J^1 \to \dots$$

be an injective resolution of Y*. Show that every morphism* $X \xrightarrow{f} Y$ *extends to a morphsm* $I^* \xrightarrow{\phi} J^*$ *of cochain complexes such that* $\upsilon f = \phi^0 \xi$*, and show that two choices for* ϕ *are cochain homotopic!*

Three of the 23 points from this sheet are bonus points which do not count in the calculation of the 50%-limit for passing the exercises module. Solutions should be submitted to the tutor by e-mail before Tuesday May 6 24:00.