

Problem 1 (3 points). Let L be an ordered field and $K \subseteq L$ a subfield over which L is algebraic. Show that for every $l \in L$ there is $k \in K$ such that $|l| < k$ holds in K .

Remark 1. In particular, L is Archimedean if and only if K is, a remark which was made in the lecture.

Problem 2 (4 points). Let K be an ordered field such that for every $P \in K[T]$ for which there are elements a and b of K with $P(a)P(b) < 0$ there is a solution x to $P(x) = 0$. Show that K is real closed!

Remark 2. In particular, K is real closed if and only if for all elements $a < b$ of K , every polynomial P with $P(a)P(b) < 0$ has zero on $(a, b)_K$.

The remaining exercises are devoted to the remaining part of the derivation of the other points in Theorem 1 from Theorem 1.D. As fields of characteristic 0 have been dealt with on the previous sheet all fields are assumed to be of characteristic $p > 0$. It is well-known from basic Galois theory that for such a field K , the following conditions are equivalent:

- Every algebraic extension of K is separable over K .
- Every element of K is a p -th power in K .

Such a field is called *perfect*.¹

Problem 3 (5 points). Let L/K be a finite field extension such that L is perfect. Show that K is perfect!

If L/K is a finite inseparable field extension then it can be shown that $\text{Tr}_{L/K}$ vanishes identically. We will need a special case of the converse, that $L \xrightarrow{\text{Tr}_{L/K}} K$ does not vanish identically, and is therefore surjective, in the case of finite separable field extensions.

Remark 3. If $[L : K]$ is not a multiple of the characteristic p this is trivial as $\text{Tr}_{L/K}(l) = [L : K]l$ when $l \in K$.

Problem 4 (3 points). Let L/K be a separable field extension of degree p of fields of characteristic p , and let $x \in L$ be a primitive element. Show that there is $y \in \mathbb{F}_p$ such that $\text{Tr}_{L/K}(1 - (xy)) \neq 0$.

Remark 4. From this result one can derive the non-vanishing of $\text{Tr}_{L/K}$ for Galois extensions with $[L : K] \in p^{\mathbb{N}}$ by induction on the degree. By Sylow theory and 3, the case of general finite Galois extensions follows.

¹The usual convention is that all fields of characteristic 0 are considered perfect.

This case is normally dealt with by using the first of the two important formulæ

$$(1) \quad \begin{aligned} \mathrm{Tr}_{L/K}(l) &= \sum_{\sigma \in \mathrm{Gal}(L/K)} l^\sigma \\ \mathrm{N}_{L/K}(l) &= \prod_{\sigma \in \mathrm{Gal}(L/K)} l^\sigma \end{aligned}$$

and the L -linear independence of the Galois group elements shown in basic Galois theory.

The surjectivity of $\mathrm{Tr}_{L/K}$ for general finite separable extensions can be established by extending L to a Galois extension M/K and using Problem 6 on sheet 1 and the previously established surjectivity of $\mathrm{Tr}_{M/K}$.

A similar surjectivity result for $\mathrm{N}_{L/K}$ holds when the number of elements of L is finite but not in general. For extensions of number fields and the related extensions of local fields this is one of the problems studied by Class Field Theory. For instance, the fact that $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ has two elements is related to the fact the the image of $\mathbb{C}^\times \xrightarrow{\mathrm{N}_{\mathbb{C}/\mathbb{R}}} \mathbb{R}^\times$ has index 2.

Problem 5 (3 points). Let L/K be a Galois extension of fields of characteristic p such that $\mathrm{Gal}(L/K)$ is cyclic with generator σ . Show that the sequence

$$0 \rightarrow K \rightarrow L \xrightarrow{l \mapsto l^\sigma - l} L \xrightarrow{\mathrm{Tr}_{L/K}} K \rightarrow 0$$

is exact!

For elements x of a field of characteristic $p > 0$, consider the Artin-Schreier operator

$$\mathrm{AS}(x) = x^p - x.$$

Problem 6 (3 points). In the situation of the previous exercise, show that there is $x \in L \setminus K$ such that $\mathrm{AS}(x) \in K$!

Remark 5. For cyclic extensions of degree p in characteristic p this plays a similar role to the role played by Kummer theory for cyclic extensions of a degree indivisible by the characteristic and with ground fields having the needed roots of 1.

Solutions should be submitted in the lecture on Friday, November 10. One of the 21 possible points from this sheet is a bonus point, which does not count in the calculation of the $\geq 50\%$ lower bound of points needed to pass the exercises.