The following exercises provide part of the proof of the equivalence of conditions A–C of Theorem 1 in the lecture with the Galois theoretic condition D. Throughout this sheet, a field K satisfying the equivalent condions A–C is called real closed. As was mentioned in the lecture it easily follows from condition C that any such field satsfies condition D, that $1 < [\overline{K} : K] < \infty$.

The proof is somewhat complicated by the need to give powers of p occuring as prime factors of $[\overline{K} : K]$ a different treatment (Artin-Schreier extension, treatment of inseparable extensions) when p is the characteristic of the fields, compared with the generic case which is done by Kummer extensions and requires a further distinction between the case $p = 2$ and the case of odd p. Throughout this sheet, p will always be a prime number.

We start with the application of the theory of Kummer extensions. Let p be a prime number and K a field of characteristic $\neq p$. We say that K has all p-th roots of 1 if the group $\mu_p(K) = \left\{\zeta \in K \;\middle|\; \zeta^p = 1\right\}$ has p elements. If $p = 2$ this is automatic by our assumption on the characteristic, but otherwise it is a non-trivial condition.

Problem 1 (3 points)**.** *Let L/K be a Galois extension of degree p, where the characteristic of the fields is $\neq p$ and L has all p-th roots of 1. Show that K hat all p-th roots of 1!*

By a well-known result of Galois theory, this implies that L is a Kummer extension, $L = K(\sqrt[p]{x})$ where $x \in K$ not a p-th power in K.

Problem 2 (2 points)**.** *Under the assumptions of the previous exercise and with x as above, show that $-x$ is in the image of $L \xrightarrow{\mathrm{N}_{L/K}} K$!*

Problem 3 (4 points)**.** *Under the assumptions of exercise 1, assume that every element of L is a p-th power in L (e. g., that L is algebraically closed). Show that $p = 2$ and $L = K(\sqrt{-1})$!*

Problem 4 (3 points)**.** *Let K be field of characteristic different from 2 such that $K(\sqrt{-1})$ is quadratically closed. Show that every sum of squares in K is a square in K!*

Problem 5 (4 points)**.** *Let L/K be a field extension of degree 2, where the characteristic of the fields is not 2 and L is algebraically closed. Show that K is real closed!*

Problem 6 (2 points)**.** *If K is real closed, then K has no subfield F with $[K : F] = 2$.*

Finally, using Sylow theory, one obtains a partial result towards our claim that condition D of Theorem 1 implies real closedness:

Problem 7 (5 points)**.** *Let K be field such that $d = [\overline{K} : K]$ is finite, > 1, and not a multiple of the characteristic of d. Show that K is real closed!*

Solutions should be submitted in the lecture on Friday, November 3. One of the 23 possible points from this sheet is a bonus point, which does not count in the calculation of the $\geq 50\%$ lower bound of points needed to pass the exercises.

All results of the first sheet about traces and norms in finite field extensions may be used. Also, the results fully proved in the lecture may be used.