Third exercise sheet "Class field theory" summer term 2025.

Problem 1 (3 points). *Give the precise formulation of Theorem 1Bc of the lecture!*

Problem 2 (3 points). *Let R be a PID, $I \subsetneq R$ a proper ideal and $A = R/I$. Show that every free A-module (in particular, A itself) is an injective object of the category of A-modules!*

Remark 1. *It is easy to see that the same holds when R is a Dedekind domain as in this case there are a PID $\tilde{R}$ and a non-zero ideal $\tilde{I}$ such that R/I and $\tilde{R}/\tilde{I}$ are isomorphic.*

Problem 3 (2 points). *In the situation of the previous problem, show that every projective A-module is injective!*

Problem 4 (3 points). *In the situation of the previous two problems, assume moreover that R is a discrete valuaton ring (DVR), I an injective A-module and $x \in I \setminus \{0\}$. Show that I contains a submodule which is isomorphic to A and contains x!*

Problem 5 (3 points). *Let $A = R/I$ where R is a DVR and I a non-zero ideal. For an A-module M, show that the following conditions are equivalent:*

- *M is projective.*
- *M is injective.*
- *M is free.*

Remark 2.

- *In view of the previous results and Example 1.1.3 from the lecture, it is sufficient to derive the freeness of M from its injectivity. This can be done by a Zorn lemma argument somewhat similar to the existence of bases of arbitrary vector spaces over fields.*
- *If A is of the general form as in Remark 1 then it is of the form $\bigoplus_{i=1}^{n} A_i$ where the previous problem can be applied to A_i. Thus it still holds that M is projective if and only if M is injective. However unless $n = 1$ this is not equivalent to the freeness of M.*

Let L/K be an algebraic (but not necessarily finite) field extension which is a Galois extension in the sense that it is normal (it satisfies the conditions of sheet 1 Problem 5) and separable (in the sense that every element of L is separable over K, or equivalently such that it can be generated as a field extension of K by elements separable over K). Let $\mathrm{Gal}(L/K)$ be the group of K-linear automorphisms of the field L. If $\overline{L}$ is an algebraic closure of L then $\mathrm{Gal}(L/K)$ coincides with

$\mathrm{Emb}_K(L, \overline{L})$ from the end of the previous sheet, which was equipped with a topology and shown to be compact.

Problem 6 (2 points). *Show that* $\mathrm{Gal}(L/K)$ *is a topological group in the sense of Problem 4 of the previous sheet!*

Problem 7 (2 points). *Show that* K *is the fixed field of* $\mathrm{Gal}(L/K)$*!*

Problem 8 (1 point). *If* M *is an intermediate field between* K *and* L *then show that* $\mathrm{Gal}(L/M)$ *is a closed subgroup of* $\mathrm{Gal}(L/K)$.

Problem 9 (3 points). *In the situation of the previous problem show that* L/M *is a Galois extension!*

Two of the 22 points from this sheet are bonus points which do not count in the calculation of the 50%-limit for passing the exercises module. Solutions should be submitted to the tutor by e-mail before Tuesday April 29 24:00.