**Second exercise sheet "Class field theory" summer term 2025.**

**Problem 1** (5 points). *Let $R$ be a ring, $M$ an $R$-module such that for every ideal $I \subseteq R$ and every morphism $I \xrightarrow{\phi} M$ of $R$-modules there exists $m \in M$ such that $\phi(i) = i \cdot m$ holds for all $i \in I$. Show that $M$ is an injective object of the category of $R$-modules!*

Throughout the following let $G$ be any (discrete) group. For $G$-modules $M$, $N$ and $T$, let $\mathrm{Bil}_G(M, N; T)$ be the set of all maps $M \times N \xrightarrow{b} T$ which are $\mathbb{Z}$-bilinear and satisfy $b(gm, gn) = gb(m, n)$ for all $m \in M$, $n \in N$ and $t \in T$.

**Problem 2** (5 points). *Let $H \subseteq G$ be a subgroup, $M$ a $G$-module and $N$ an $H$-module. For $f \in \mathrm{ind}_H^G N$ and $m \in N$, let $u(f, m) \in \mathrm{ind}_H^G(N \otimes M)$ be defined by $\big(u(f, m)\big)(g) = f(g) \otimes gm$. If $T$ is a $G$-module and $b \in \mathrm{Bil}_G(\mathrm{ind}_H^G N, M; T)$, show that there is a unique morphism*

$$\mathrm{ind}_H^G(N \otimes M) \xrightarrow{\beta} T$$

*of $G$-modules such that $b(f, m) = \beta\big(u(f, m)\big)$ for all $f \in \mathrm{ind}_H^G N$ and $m \in M$.*

**Problem 3** (2 points). *Let $\mathcal{A} \xrightarrow{L} \mathcal{B}$ and $\mathcal{B} \xrightarrow{R}$ be an adjoint functor pair between arbitrary categories. If $A \xrightarrow{\alpha} A'$ is an epimorphism in $\mathcal{A}$, show that $L\alpha$ is an epimorphism in $\mathcal{B}$.*

**Problem 4** (3 points). *Let $G$ be a topological group, i. e., a group equipped with a topology such that the group operations*

$$G \times G \xrightarrow{(g,h) \to gh} G$$

$$G \xrightarrow{g \to g^{-1}} G$$

*are continuous. Show that every open subgroup of $G$ is closed and that every closed subgroup of finite index is open!*

Let $L/K$ be an algebraic field extension, $\overline{K}$ an algebraic closure of $K$ and $\mathrm{Emb}_K(L, \overline{K})$ the set of $K$-linear ring homomorphisms $L \to \overline{K}$. We equip $\mathrm{Emb}_K(L, \overline{K})$ with the topology for which a topology base is formed by the subsets of the form

$$\Omega_{F,\phi} = \big\{ f \in \mathrm{Emb}_K(L, \overline{K}) \,\big|\, f(x) = \phi(x) \text{ for all } x \in F \big\}$$

where $K \subseteq F \subseteq L$ is an intermediate field and $\phi \in \mathrm{Emb}_K(F, \overline{K})$.

**Problem 5** (6 points). *Show that $\mathrm{Emb}_K(L, \overline{K})$ is compact!*

One of the 21 points from this sheet is a bonus point which does not count in the calculation of the 50%-limit for passing the exercises module. Solutions should be submitted to the tutor by e-mail before Tuesday April 22 24:00.