Throughout this set of exercises let always K be a real closed field.

Problem 1 (10 points)**.** *Let $\mathcal{C}_K$ be the set of decompositions $K = A \cup B$ subject to the following conditions:*

- *$\alpha < a$ and $a \in A$ implies $\alpha \in A$.*
- *$b < \beta$ and $b \in B$ implies $\beta \in B$.*
- *$A \cap B$ has at most one element.*

We have a map $\mathrm{Sper} K[T] \to \mathcal{C}_K$ sending a prime cone $\mathfrak{P}$ to the decomposition $K = A_{\mathfrak{P}} \cup B_{\mathfrak{P}}$ where

$$A_{\mathfrak{P}} = \{a \in K \mid T - a \in \mathfrak{P}\}$$
$$B_{\mathfrak{P}} = \{b \in K \mid b - T \in \mathfrak{P}\}.$$

We also have a map $\mathcal{C}_K \to \mathrm{Sper} K[T]$ sending the decomposition $K = A \cup B$ to $\mathfrak{P}_{A,B}$, the cone containing precisely those $f \in K[T]$ for which there are $a \in A \cup \{-\infty\}$ and $b \in B \cup \{\infty\}$ such that $f(t) \geq 0$ for all $t \in K$ with $a \leq t \leq b$.

Show that these maps are well-defined and inverse to each other!

The remaining problems deal with the question of whether or not the first quadrant $\mathcal{P}(X, Y)$ can be written as a union of the form $\bigcup_{i \in I} \mathcal{P}(f_i)$, for suitable polynomials f_i. The answer depends on whether the classical K^2 or $\mathrm{Sper} K[X, Y]$ is considered.

The motivation for looking at the first quadrant is the following: If it was possible to write $\mathcal{P}(X, Y) \subseteq \mathrm{Sper}\mathbb{Z}[X, Y]$ as a union $\bigcup_{i \in I} \mathcal{P}(f_i)$ then for general rings R and general $x, y \in R$ we had $\mathcal{P}(x, y) = \bigcup_{i \in I} \mathcal{P}(f_i(x, y))$. One could then use $\{\mathcal{P}(r) \mid r \in R\}$ as a topology base for $\mathrm{Sper} R$. We will however see that this is not possible.

Problem 2 (3 points)**.** *Express the first quadrant in K^2 as a union of the form*

$$\{(x, y) \in K \mid x > 0 \text{ and } y > 0\} = \bigcup_{i \in I} \{(x, y) \in K^2 \mid f_i(x, y) > 0\},$$

where $f_i \in K[X, Y]$.

The fact that an analogous representation of the first quadrant in $\mathrm{Sper} K[X, Y]$ is impossible can be demonstrated by looking at certain elements of $\mathrm{Sper} K[X, Y]$ which are in a sense inifinitesimally closed to, or infinitely far away from, the origin. To define them we consider an arbitrary number n of coordinates and orderings $\leq$ of $\mathbb{N}^n$ with the following properties, for arbitrary $\alpha, \beta, \gamma \in \mathbb{N}^n$:

- $\alpha \leq \alpha$
- $\alpha \leq \beta$ and $\beta \leq \gamma$ implies $\alpha \leq \gamma$

- If $\alpha \neq \beta$ then precisely one of $\alpha \leq \beta$ or $\beta \leq \alpha$ holds.
- If $\alpha \leq \beta$ then $\alpha + \gamma \leq \beta + \gamma$.
- If $\alpha_i \leq \beta_i$ for $1 \leq i \leq n$, then $\alpha \leq \beta$.

For instance, lexicographical orderings can be taken. Fixing such an ordering and considering a non-zero $f \in R = K[X_1, \ldots, X_n]$, $f = \sum_{\alpha \in \mathbb{N}^n} f_\alpha X^\alpha$, let the most significant $\mathrm{msc}_\leq f$ (resp. least significant $\mathrm{lsc}_\leq f$) coefficient of f be f_α, where α is $\leq$-maximal (resp. $\leq$-minimal) with $f_\alpha \neq 0$.

Problem 3 (5 points)**.** *Show that*

$$\mathfrak{P} = \{f \in K[X_1, \ldots, X_n] \mid f = 0 \textit{ or } \mathrm{lsc}(f) > 0\}$$

and

$$\mathfrak{Q} = \{f \in K[X_1, \ldots, X_n] \mid f = 0 \textit{ or } \mathrm{msc}(f) > 0\}$$

are elements of $\mathrm{Sper} R$*.*

Problem 4 (5 points)**.** *Let $R = K[X, Y]$. Show that there is no decomposition of $\mathcal{P}(X, Y) \subseteq \mathrm{Sper} R$ as*

$$\mathcal{P}(X, Y) = \bigcup_{i \in I} \mathcal{P}(f_i)$$

with all $f_i \in R$.

Remark 1. *The covering from 2 will thus fail to correctly deal with the elements of* Sper *introduced in 3 and with their reflections along the X-axis, the Y-axis or the origin. Moreover, the covering from 2 will be infinite while $\mathcal{P}(X, Y) \subseteq \mathrm{Sper} R$ is quasi-compact.*

One could try to bring the theories of K^2 and $\mathrm{Sper} R$ *closer to each other by considering certain coverings of subsets of K^2, including the one from 2, as "non-admissible." Such a machinery, for general K^n, was developed by Delfs and Knebusch. It is equivalent to the theory of the real spectrum in the sense that the sheaf categories are canonically equivalent.*

This is somewhat similar to the situation for rigid analytic geometry where a classical setting with a restricted notion of "admissible covering" was proposed by Tate and fully developed by the school around Grauert, while machineries using additional, non-classical points were developed later by Berkovich and Huber. In fact, Huber (a member of the Knebusch school) was almost certainly motivated by the example of real algebraic geometry where the two machineries were developed at about the same time, while rigid analytic geometry developed for three decades before the work of Berkovich and Huber.

Three of the 23 possible points from this sheet are bonus points which do not count in the calculation of the $\geq 50\%$ lower bound of points needed to pass the exercises. Solutions should be submitted in the exercises on Wednesay, January 17.