Eleventh exercise sheet "Class field theory" summer term 2025. Let p be an odd prime. It is interesting to derive a reciprocity law for p-th powers from the Problem 8 on the previous sheet. As was already stated in the first lecture of this term and is clear from the assumptions of that Problem 8, the ground field must contain all $\sqrt[p]{1}$, hence the smallest reasonable ground field is $K = \mathbb{Q}(\mu_p)$. Then (unlike the case $p = 2$) the only Archimedean prime of K does not contribute to the product formula $\prod_v (x, y)_{K_v,p} = 1$ as the local field $K_v \cong \mathbb{C}$ in that case. The factors $(x, y)_{K_v,p}$ with $|p|_v = 1$ can be calculated using the tame symbol as in Problem 5 of the previous sheet. Thus it remains to deal with the unique prime v such that $|p|_v < 1$.

We therefore change notations and put $K = \mathbb{Q}_p(\mu_p)$ and $(x, y) = (x, y)_{K,p}$. We will only derive a partial result and state the remaining formulas for (x, y) without proof.

Let ζ be a generator of the cyclic group $\mu_p(K)$. It is well known that $\pi = 1 - \zeta$ is a uniformizer of the discrete valuation ring $\mathcal{O}_K$ and that $\pi^{p-1}\mathcal{O}_K = p\mathcal{O}_K$. We also put $\eta_i = 1 - \pi^i$.

Problem 1 (2 points)**.** *Show that* $(\pi, \eta_i) = 1$ *when* $0 < i < p$.

Problem 2 (4 points)**.** *Show that* $K(\sqrt[p]{x})/K$ *is unramified when* $x \in 1 + \pi^p\mathcal{O}_K$ *and that* $x \in K^p$ *when* $x \in 1 + \pi^{p+1}\mathcal{O}_K$*!*

Problem 3 (3 points)**.** *Show that* $(\pi, \eta_p) = \zeta^{-1}$*!*

Let $S = \mathrm{Tr}_{K/\mathbb{Q}_p}$. An easy calculation using the binomial theorem for $\pi^m = (1 - \zeta)^m$ gives

Problem 4 (4 points)**.** *Show that we have* $S(\zeta) = -1$, $S(\zeta\pi^{p-1}) = p$ *and* $S(\zeta\pi^m) = 0$ *when* $1 \le m < p - 1$*!*

Problem 5 (2 points)**.** *If* $l \in \mathbb{Z}$ *and* $x \in \pi^{1+(l-1)(p-1)}\mathcal{O}_K$*, show that* $S(x) \in p^l\mathbb{Z}_p$.

Problem 6 (5 points)**.** *For* $x \in 1 + \pi\mathcal{O}_K$*, let* $\log(x) \in K$ *be given by the convergent power series* $\log(1 - t) = -\sum_{m=1}^{\infty} \frac{t^m}{m}$*. Show that*

$$(\pi, x) = \zeta^{\frac{S((\zeta/\pi)\log(x))}{p}} \tag{1}$$

where the exponent is $\in \mathbb{Z}_p$*!*

Remark 1. *The formula* $\log(xy) = \log(x) + \log(y)$ *for* $x, y \in 1 + \pi\mathcal{O}_K$ *can be used without proof.*

Remark 2. *The result* (1) *is due to Artin and Hasse. It can be considered as a version of the formala* $(2, x)_{\mathbb{Q}_2,2} = (-1)^{\frac{x^2-1}{8}}$ *from the previous* *sheet (which however is obviously not a special case of* (1)*). The remaining formulas, also due to Artin and Hasse, are*

$$(y,x) = \zeta^{\frac{S(\log y D \log x)}{p}} \tag{2}$$

$$(\zeta,x) = \zeta^{\frac{S(\log x)}{p}} \tag{3}$$

when $x \in 1+\pi\mathcal{O}_K$ *and* $y \in 1+\pi^2\mathcal{O}_K$*. In this, the term* $D\log x$ *needs some explanation. If* $x \in \mathcal{O}_K$ *then* $x = P(\pi)$ *for some* $P \in \mathbb{Z}_p[\pi]$*. We put*

$$D\log x = \frac{P'(\pi)}{P(\pi)} = \frac{P'(\pi)}{x}.$$

If $x = Q(\pi)$ *then the minimal Polynomial* M *of* π *divides* $Q-P$*,* $Q-P = MR$ *and replacing* P *by* Q *changes* $D\log x$ *by the addition of*

$$\frac{M'(\pi)R(\pi)}{x} \in \pi^{p-2}\mathcal{O}_K.$$

Since $\log y \in \pi^2\mathcal{O}_K$ *and* $S(\pi^p\mathcal{O}_K) \subseteq p^2\mathbb{Z}_p$ *this means that the right hand side of* (2) *does not change. Actually, power series are also allowed for* P *in the typical expositions of this formula.*

The supplementary laws (1) *and* (3) *were published in the 1920s in a joint paper of Artin and Hasse while they published* (2) *independently of each other in the 1950s. Also after World War II was the publication of a paper of Shafarewitch giving a different solution to the problem of explicit reciprocity laws. A very general formula, working for arbitrary prime powers* q *and assuming only* $\mu_q \subseteq K$*, was given by Brückner in the 1960s. Also in the 1960s and after Brückner, Iwasawa proved a generalization of the Artin-Hasse formula to prime powers.*

Since the Artin-Hasse and Iwasawa formulas involve logarithms which are related to the multiplicative group, the work of Lubin and Tate describing abelian extensions of ultrametric local fields in terms of formal groups makes it natural to try generalize the Artin-Hasse and Iwasawa formulas to division by powers of a prime p in a Lubin-Tate style formal group. Such a result was derived by Wiles using work of Coates-Wiles in the 1970s and needs a completely new proof as the result of Problem 5 from sheet 9 is no longer available. This result of Wiles has seen extensive applications like a partial proof of the Birch and Swinnerton-Dyer conjecture and of the finiteness of the Shafarewitch group of elliptic curves. Further work on explicit reciprocity laws was be Henniart and de Shalit.

The Artin-Hasse formulas are a special case of the Iwasawa formulas, which in turn can be derived from the results of Wiles. Apart from this it is not easy to derive one of these formulas from another.

A proof of (2) *and* (3) *along the lines of our arguments leading to* (1) *can be found in the Artin-Tate book on class field theory. Actually it seems to provide a wrong formula for* (1) *lacking the denominator* π *in* $S\big((\zeta/\pi)\log x\big)$ *but it is the correct formula which is later on shown in the text. Their proof works by proving*

$$(\eta_i, \eta_j) = (\eta_i, \eta_{i+j})(\eta_{i+j}, \eta_j)(\pi, \eta_{i+j}) \tag{4}$$

reducing the calculation of (η_i, η_j) *to the results of this sheet as the right hand side only involves* (η_k, η_l) *with* $\max(k, l) > \max(i, j)$ *and since* $(\eta_k, \eta_l) = 1$ *when* $\max(k, l) \geq p$. *The proof of* (4) *uses* $\eta_{i+k} = \eta_i + \pi^i \eta_k$ *together with the result of Problem 5 from Sheet 9. See also the exercises provided in the Cassels-Fröhlich volume. They do not describe the derivation of the final formulas using* (4). *This is, however, only moderately difficult even without such help.*

Because of the immense importance of the Wiles explicit reciprocity law, the classical proof of the Artin-Hasse formulas is nowadays of historical interest only.[1] *This, besides the lack of a remaining exercise sheet, is one reason for my describing only a proof of the easiest Artin-Hasse formula* (1). *A modern introduction to explicit reciprocity laws is at the end of the textbook of Iwasawa on local class field theory.*

Solutions should be submitted to the tutor by e-mail before Tuesday July 8 24:00.

[1] Safe for the fact that the classical explicit reciprocity laws for the multiplicative group are related to K_2 of fields.