Problem 1 (10 points)**.** *Let G be a topological group. Construct a complete (in particular, Hausdorff) topological group $\hat{G}$ and a continuous homomorphism $G \overset{\iota}{\longrightarrow} \hat{G}$ with the universal property for continuous homomorphisms $G \overset{\eta}{\longrightarrow} H$ to a complete topological group H! Also, show the following facts:*

- *The image of ι is dense in $\hat{G}$.*
- *If $\overline{X}$ denotes the closure of X in $\hat{G}$, then*

$$\left\{ \overline{\iota(U)} \mid U \subseteq G \text{ is a neighbourhood of the neutral element of } G \right\}$$

   is a neighbourhood base of the neutral element of $\hat{G}$.

Remark 1. • *$\hat{G}$ is called the* completion *of G.*
- *The homomorphism ι will only be injective if G is Hausdorff (or, equivalently, T_0).*
- *The completion of normed vector spaces over $\mathbb{R}$ or $\mathbb{C}$, which may or may not be known from some basic functional analysis course, is a special case of the completion constructed in this exercise. However, in the absence of countable neighbourhood bases the completion may behave unexpectedly. For instance, it is no longer true that the completion of a surjective homomorphism $G \to H$ is surjective. In the functional analytic situation, by a result of S. Dierolf[1] every locally convex vector space is the quotient of a complete locally convex space in which every bounded subset is contained in some finite dimensional subspace.*

Problem 2 (5 points)**.** *Let K be an ordered field, equipped with the order topology, and let $\hat{K}$ denote the completion of the additive group of K. Show that multiplication in K extends to a unique continuous map $\hat{K} \times \hat{K} \to \hat{K}$, that $\hat{K}$ is a field when equipped with this multiplication, and that the closure in $\hat{K}$ of $[0, \infty)_K$ is a prime cone in $\hat{K}$.*

Remark 2. *Thus, $\hat{K}$ is an ordered field.*

Problem 3 (4 points)**.** *For a topological space X, show the equivalence of the following conditions:*

- *There is no infinite strictly descending chain $A_0 \supset A_1 \supset \ldots$ of closed subsets of X.*
- *Every non-empty set $\mathfrak{M}$ of closed subsets of X has a $\subseteq$-minimal element.*
- *Every open subset of X is quasi-compact.*

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[1] Über Quotienten vollständiger topologischer Vektorräume, Manuscripta Mathematica 17 (1975), 73-77

Definition 1. *Such a topological space is called* Noetherian.

Problem 4 (4 points)**.** *Show that a spectral space X is Noetherian if and only if every continuous map $X \xrightarrow{f} Y$ to a spectral space Y is spectral!*

Problem 5 (4 points)**.** *Describe the constructive topology on $\mathrm{Spec}\mathbb{Z}$ and the topology of the inverse spectral space $\left(\mathrm{Spec}\mathbb{Z}\right)^{*}$! Decide whether $\left(\mathrm{Spec}\mathbb{Z}\right)^{*}$ is Noetherian!*

Remark 3. *The description of topologies should consist of a description of the open or of the closed subsets which can be understood by everyone who understands what a topological space is and that*

$$\left\{ p\mathbb{Z} \mid p \text{ is a prime number or } 0\right\}$$

is a complete list of the elements of $\mathrm{Spec}\mathbb{Z}$. For instance, the following would be an acceptable description of the Zariski topology on $X = \mathrm{Spec}\mathbb{Z}$:

> *A subset is closed if and only if it equals X or if it is finite and does not contain $\{0\}$.*

Seven of the 27 possible points from this sheet are bonus points which do not count in the calculation of the $\geq 50\%$ lower bound of points needed to pass the exercises. Solutions should be submitted in the lecture on Friday, January 12.