Problem 1 (10 points). *Let $(X_i)_{i\in I}$ be a family of topological spaces, let $X = \prod_{i\in I} X_i$. Let $\mathfrak{B}$ be the set of subsets of X which have the form $\prod_{i\in I} U_i$, where all U_i are open in X_i and the set of $i \in I$ with $U_i \neq X_i$ is finite.*

- *Show that $\mathfrak{B}$ is the base for some topology on X, which is the topology on X to be used in the following considerations.*
- *Show that all projections $X \xrightarrow{p_i} X_i$ are continuous, and that X satisfies the following universal property[1], which characterizes it uniquely up to unique homeomorphism:*

  If T is a topological space and $(T \xrightarrow{t_i} X_i)_{i\in I}$ a family of continuous maps, then there is a unique continuous map $T \xrightarrow{\tau} X$ such that $t_i = p_i\tau$ for all $i \in I$.
- *If all X_i are Hausdorff (resp. (quasi)compact), show that X is Hausdorff (resp. (quasi)compact).*

Recall that a topological group is a group G together with a topology on the underlying set, such that the group operations

$$G \times G \xrightarrow{(g,h) \to gh} G$$

$$G \xrightarrow{g \to g^{-1}} G$$

are continuous.

Problem 2 (1 point). *Show that G is Hausdorff if and only if it is T_1.*

From now on we assume that this is the case.

Definition 1. *A sequence $(g_i)_{i=0}^{\infty}$ of elements of G (resp. a filter $\mathfrak{F}$ on G) is called a* Cauchy sequence *(resp. a* Cauchy filter*) if for every neighbourhood U of the unit element of G, there is $i \in \mathbb{N}$ such that $g_j g_k^{-1} \in U$ when $\min(j,k) \geq i$ (resp. such that there is $X \in \mathfrak{F}$ such that $gh^{-1} \in U$ when $g, k \in X$). We say that X is* complete *if every Cauchy ultrafilter has a limit.*

Remark 1. *Obviously, an equivalent characterization of completeness is that every Cauchy filter has a (unique) point of accumulation.*

Problem 3 (4 points).

- *If G is complete, show that every Cauchy sequence has a limit!*
- *If every Cauchy sequence has a limit and G has a countable neighbourhood base of 0, show that G is complete.*

[1] This is the universal property for products in the category with topological spaces as objects and continuous maps as morphisms

Problem 4 (7 points). *Let $\Gamma \subseteq G$ be a dense subgroup such that every Cauchy ultrafilter on Γ has a limit in G. Show that G is complete!*

Two of the 22 possible points from this sheet are bonus points which do not count in the calculation of the $\geq 50\%$ lower bound of points needed to pass the exercises. Solutions should be submitted in the exercises on Wednesday, January 10.